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In this paper we produce many examples of thin subgroups of special linear groups that are isomorphic to the fundamental group nonarithmetic hyperbolic manifolds. Specifically, we show that the nonarithmetic lattices in $\mathrm{SO}(n, 1, \mathbb{R})$ constructed by Gromov and Piatetski-Shapiro can be embedded into $\mathrm{SL}(n+1, \mathbb{R})$ so that their images are thin subgroups.

Introduction

Let G be a semisimple Lie group and let Γ be a finitely generated subgroup. We say that Γ is a *thin subgroup of* G if there is a lattice $\Lambda \subset G$ containing Γ such that

- Γ has infinite index in Λ,
- Γ is Zariski dense in G.

Intuitively, such groups are very sparse in the sense that they have infinite index in a lattice, but at the same time are dense in an algebraic sense. Note, that if one relaxes the first condition above, then Γ would be a lattice, so another way of thinking of thin groups is as infinite index analogues of lattices in semisimple Lie groups.

Over the last several years, thin groups have been the subject of much research, much of which has been motivated by the observation that many theorems and conjectures in number theory can be phrased in terms of counting primes in orbits of groups that are "abelian analogues of thin groups." Here are two examples. First, let $G = \mathbb{R}$, $b, m \in \mathbb{N}$ such that $(b, m) = 1$, $\Delta = \mathbb{Z}$ and $\Gamma = m\mathbb{Z}$. The orbit $b + \Gamma$ is an arithmetic progression and Dirichlet's theorem on primes in arithmetic progressions is equivalent to this orbit containing infinitely many primes. Next, let $G = \mathbb{R}^2$, $\Delta = \mathbb{Z}^2$, $\Gamma = \langle (1, 1) \rangle$ and $b = (1, 3) \in \mathbb{Z}^2$. The orbit $b + \Gamma = \{(m, m+2) \mid m \in \mathbb{Z}\}$ and the twin prime conjecture is equivalent to the statement that this orbit contains infinitely many points whose components are both prime. Note that in the first case Γ is a lattice in G, but in the second case Γ has infinite index in Δ and is an analogue of a thin group (sans Zariski density) in G.

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This orbital perspective was used by Brun to attack the twin primes conjecture using "combinatorial sieving" techniques. Although the full conjecture remains unproven these techniques did yield some powerful results. For instance, using these methods, Chen [1978] was able to prove that there are infinitely many pairs n and $n+2$ such that one is prime and the other is the product of at most 2 primes. More details of this perspective are explained in the excellent surveys of Bourgain [2014] and Lubotzky [2012].

Inspired by these results, Bourgain, Gamburd, and Sarnak [Bourgain et al. 2010] developed complementary "affine sieving" techniques to analyze thin group orbits. In this context, the thinness property of the group gives enough control of orbits to execute these counting arguments. Again, much of this is described in Lubotzky's survey [2012].

Given these connections it is desirable to produce examples of thin groups and understand what types of groups are thin. Presently, there are many constructions of thin groups. For instance, in recent work of Fuchs and Rivin [2017] it is shown that if one "randomly" selects two matrices in $\mathrm{SL}(n, \mathbb{Z})$ then with high probability, the group they generate is a thin subgroup of $\mathrm{SL}(n, \mathbb{R})$. However, the groups constructed in this way are almost always free groups. There are also several constructions that allow one to produce thin subgroups isomorphic to fundamental groups of closed surfaces in a variety of algebraic groups (see [Cooper and Futer 2019; Kahn et al. 2018; Kahn and Markovic 2012; Kahn and Wright 2018], for instance). Given these examples one may ask which isomorphism classes of groups are thin? More precisely, if G is a semisimple Lie group and H is an abstract finitely generated group then we say that H can be *realized as a thin subgroup of* G if there is an embedding $\iota : H \to G$ whose image is a thin subgroup of G. With this definition in hand we can rephrase the previous question as: given a semisimple algebraic group G, what isomorphism types of groups can be realized as thin subgroups of G? Recent work of the author and D. Long [Ballas and Long 2020] shows that there are many additional isomorphism types of groups that can arise as thin subgroups of special linear groups. More precisely, in [Ballas and Long 2020] it is shown that fundamental groups of arithmetic hyperbolic n-manifolds of "orthogonal type" can be realized as thin subgroups. In the present work, we extend the techniques of [Ballas and Long 2020] to produce infinitely many examples of nonarithmetic hyperbolic n-manifolds whose fundamental groups can be realized as thin subgroups of $\mathrm{SL}_{n+1}(\mathbb{R})$. Our main result is:

Theorem 1. *For each* $n \geq 3$, *there is an infinite collection* $\mathcal{C}_n$ *of nonarithmetic hyperbolic n-manifolds with the property that if* $M^n \in \mathcal{C}_n$ *then* $\pi_1(M)$ *can be realized as a thin subgroup of* $\mathrm{SL}_{n+1}(\mathbb{R})$. *Furthermore, the collection* $\mathcal{C}_n$ *contains representatives from infinitely many commensurability classes of both compact and noncompact manifolds.*

It should be noted that the collection $\mathcal{C}_n$ appearing in Theorem 1 can be described fairly explicitly, and roughly speaking consists of the hyperbolic manifolds coming from the nonarithmetic lattices in $\mathrm{SO}(n, 1, \mathbb{R})$ constructed by Gromov–Piatetski-Shapiro in [Gromov and Piatetski-Shapiro 1988].

Outline of paper. In Section 1 we recall the Gromov–Piatetski-Shapiro construction of nonarithmetic lattices in $\mathrm{SO}(n, 1, \mathbb{R})$ and define the collection $\mathcal{C}_n$ appearing in Theorem 1. In Section 2 we show that the fundamental group of any element of $\mathcal{C}_n$ can be embedded in several lattices in $\mathrm{SL}_{n+1}(\mathbb{R})$. Finally, in Section 3 we prove Theorem 1 by showing that the images of the previously mentioned embeddings are thin subgroups.

1. Gromov–Piatetski-Shapiro lattices

Gromov and Piatetski-Shapiro [1988], describe a method for constructing infinitely many nonarithmetic lattices in $\mathrm{SO}(n, 1, \mathbb{R})$. In this section we describe their construction and the construction of the lattices appearing in Theorem 1.

Let K be a totally real number field of degree $d+1$ with ring of integers $\mathcal{O}_K$. There are $d+1$ embeddings $\{\sigma_0, \dots, \sigma_d\}$ of K into $\mathbb{R}$. Using the embedding σ_0 we will implicitly regard K as a subset of $\mathbb{R}$. In this way, it makes sense to say that elements of F are positive or negative. Let $s_K : K^\times \to \mathbb{Z}_{\geq 0}$, where $s_K(a) = |\{i \geq 1 \mid \sigma_i(a) > 0\}|$. In other words, $s_K(a)$ counts the nonidentity embeddings for which a has positive image.

Next, let $\alpha, \beta, a_2, \dots, a_{n+1} \in \mathcal{O}_K$ be positive elements such that

- β/α is not a square in K,
- $s_K(\alpha) = s_K(\beta) = s_K(\alpha_i) = d$ for $1 \leq i \leq n$,
- $s_K(a_{n+1}) = 0$.

Next, define quadratic forms

$$J_1 = \alpha x_1^2 + \sum_{i=2}^{n} a_i x_i^2 - a_{n+1} x_{n+1}^2, \qquad J_2 = \beta x_1^2 + \sum_{i=2}^{n} a_i x_i^2 - a_{n+1} x_{n+1}^2 \tag{1-1}$$

If $A \subset \mathbb{R}$ is a subring containing 1 then we define

$$\mathrm{SO}(J_i, A) = \{B \in \mathrm{SL}_{n+1}(A) \mid J_i(Bv) = J_i(v)\ \forall v \in \mathbb{R}^{n+1}\}.$$

Using this notation, define $\Gamma_1 = \mathrm{SO}(J_1, \mathcal{O}_K)$ and $\Gamma_2 = h\,\mathrm{SO}(J_2, \mathcal{O}_K)h^{-1}$, where $h = \mathrm{Diag}(\sqrt{\beta/\alpha}, \dots, 1)$. Note that both Γ_1 and Γ_2 are lattices in $\mathrm{SO}(J_1, \mathbb{R})$, however, since β/α is not a square in K it follows from [Gromov and Piatetski-Shapiro 1988, see Corollary 2.7 and §2.9] that these lattices are not commensurable.

There is a model for hyperbolic n-space given by

$$\mathbb{H}^n = \{v \in \mathbb{R}^{n+1} \mid J_1(v) = -1,\ v_{n+1} > 0\}.$$

The identity component $\mathrm{SO}(J_1, \mathbb{R})^\circ$ of $\mathrm{SO}(J_1, \mathbb{R})$ consists of the orientation preserving isometries of $\mathbb{H}^n$ (see [Ratcliffe 2006, §3.2] for details). By passing to finite index subgroups we can assume that $\Gamma_i \subset \mathrm{SO}(J_1, \mathbb{R})^\circ$, and so $\mathbb{H}^n/\Gamma_i$ is a finite volume hyperbolic orbifold for $i = 1, 2$.

The lattice $\Gamma_2 \subset \mathrm{SO}(J_1, L)$, where $L = K(\sqrt{\beta/\alpha})$. Note that because α and β are positive and $s_K(\alpha) = s_K(\beta) = d$ it follows that L is also totally real. Furthermore, for every $\gamma \in \Gamma_2$, $\mathrm{tr}(\gamma) \in \mathcal{O}_K \subset \mathcal{O}_L$. The following lemma then shows that by passing to a subgroup of finite index we may assume that $\Gamma_2 \subset \mathrm{SO}(J_1, \mathcal{O}_L)$. This result seems well known to experts, but we include a proof for the sake of completeness.

Lemma 1.1. *Let $k \subset \mathbb{C}$ be a number field and let $\mathcal{O}_k$ be the ring of integers of k. If $\Gamma \subset \mathrm{GL}_n(k)$ acts irreducibly on $\mathbb{C}^n$ and has the property that $\mathrm{tr}(\gamma) \in \mathcal{O}_k$ for each $\gamma \in \Gamma$ then there is a finite index subgroup $\Gamma' \subset \Gamma$ such that $\Gamma' \subset \mathrm{GL}_n(\mathcal{O}_k)$.*

Proof. If $A \subset k$ is a subring then let $A\Gamma = \{\sum_i a_i\gamma_i \mid a_i \in A, \gamma_i \in \Gamma\}$. Note that in this definition all sums have finitely many terms. By [Bass 1980, Proposition 2.2], $\mathcal{O}_k\Gamma$ is an order in the central simple algebra $k\Gamma$. The order $\mathcal{O}_k\Gamma$ is contained in some maximal order $\mathcal{D}$ in $M_n(k)$ ($n \times n$ matrices over k). Let $\mathcal{D}^1 \subset \mathrm{SL}_n(k)$ be the norm 1 elements of $\mathcal{D}$. Then $M_n(\mathcal{O}_k)$ is also an order in $M_n(k)$ whose group of norm 1 elements is $\mathrm{SL}_n(\mathcal{O}_k)$. It is a standard result using restriction of scalars that groups of norm 1 elements in maximal orders of $M_n(k)$ are commensurable. Roughly speaking this is a consequence of the fact that the intersection of two orders is again an order and the unit groups of these orders are irreducible lattices in $\mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$ (see [Morris 2015, §5.1 and Example 5.1 #7]). It follows that $\mathcal{D}^1 \cap \mathrm{SL}_n(\mathcal{O}_k)$ has finite index in $\mathcal{D}^1$ and so $\Gamma \cap \mathrm{SL}_n(\mathcal{O}_k)$ has finite index in Γ. □

Note that since Γ_2 is a lattice in $\mathrm{SO}(J_1, \mathbb{R})$ it acts irreducibly on $\mathbb{C}^{n+1}$, and so by applying Lemma 1.1 we may assume that $\Gamma_2 \subset \mathrm{SO}(J_1, \mathcal{O}_L)$.

Denote by $\mathrm{SO}(n-1, 1, \mathbb{R})$ the subgroup of $\mathrm{SO}(J_1, \mathbb{R})$ that preserves both complementary components in $\mathbb{R}^{n+1}$ of the hyperplane P given by the equation $x_1 = 0$. The intersection $P \cap \mathbb{H}^n$ is a model for hyperbolic $(n-1)$-space, $\mathbb{H}^{n-1}$ and the group $\mathrm{SO}(n-1, 1, \mathbb{R})$ can be identified with the subgroup of orientation preserving isometries of $\mathbb{H}^{n-1}$. Next, let $\hat{\Gamma} = \Gamma_1 \cap \Gamma_2 \cap \mathrm{SO}(n-1, 1, \mathbb{R})$. Since each $\Gamma_i \cap \mathrm{SO}(n-1, 1, \mathbb{R})$ is sublattice of the lattice $\mathrm{SO}(n-1, 1, \mathcal{O}_L)$ in $\mathrm{SO}(n-1, 1, \mathbb{R})$, it follows that $\hat{\Gamma}$ is also a lattice in $\mathrm{SO}(n-1, 1, \mathbb{R})$. It follows that $\mathbb{H}^{n-1}/\hat{\Gamma}$ is a hyperbolic $(n-1)$-orbifold. By passing to finite index subgroups we may arrange the following properties:

(1) Γ_i is torsion-free and contained in the identity component of $\mathrm{SO}(J_1, \mathbb{R})$. This component is isomorphic to $\mathrm{Isom}^+(\mathbb{H}^n)$, and so $M_i := \mathbb{H}^n/\Gamma_i$ is a finite volume

hyperbolic manifold (apply Selberg's lemma and the fact that $\mathrm{SO}(J, \mathbb{R})^\circ$ has finite index in $\mathrm{SO}(J, \mathbb{R})$).

(2) Since $\Sigma = \mathbb{H}^{n-1}/\hat{\Gamma}$ is a totally geodesic we may assume that Σ is a hyperbolic $(n-1)$-manifold and this manifold is embedded in both M_1 and M_2 (see [Bergeron 2000, Theorem 1]).

(3) If M_i is noncompact then all cusps of M_i are diffeomorphic to an $(n-1)$-torus times an interval (apply [McReynolds et al. 2013, Theorem 3.1])

(4) The complement $\widehat{M}_i = M_i \backslash \Sigma$ is connected for $i = 1, 2$ (see [Bergeron 2000, Theorem 2]).

The manifold $\widehat{M}_i$ is a convex submanifold of M_i and so $\widehat{M}_i = V_i/\hat{\Gamma}_i$, where V_i is a component of the preimage of $\widehat{M}_i$ in $\mathbb{H}^n$ under the universal covering projection $\mathbb{H}^n \to \mathbb{H}^n/\Gamma_i = M_i$, and $\hat{\Gamma}_i$ is a subgroup of Γ_i that stabilizes V_i. The manifold $\widehat{M}_i$ is a hyperbolic manifold with totally geodesic boundary equal to two isometric copies of Σ, and so it is possible to glue $\widehat{M}_1$ and $\widehat{M}_1$ along Σ to form the finite volume hyperbolic manifold N (see [Morris 2015, §6.5] for details). The manifold N can be realized as $\mathbb{H}^n/\Delta$ where, after appropriately conjugating $\hat{\Gamma}_i$ in Γ_i, we may assume that

$$\Delta = \langle \hat{\Gamma}_1, \hat{\Gamma}_2, s \rangle. \tag{1-2}$$

Here s comes from a "graph of spaces" description of N and can thus be written as a product $s = s_2 s_1$, where s_i is the isometry corresponding to an appropriate lift to V_i a curve in M_i whose algebraic intersection with Σ is 1 (See Figure 1). In [Gromov and Piatetski-Shapiro 1988, §2.9] it is shown that Δ is a nonarithmetic lattice in $\mathrm{SO}(J_1, \mathbb{R})$. If $N = \mathbb{H}^n/\Delta$ then we call N an *interbreeding of M_1 and M_2*.

Since $\Gamma_1, \Gamma_2 \subset \mathrm{SO}(J_1, \mathcal{O}_L)$ it follows that $\Delta \subset \mathrm{SO}(J_1, \mathcal{O}_L)$. As a result, we call the field L the *field of definition of* Δ. Let $\mathcal{C}_n$ be the collection of hyperbolic n-manifolds coming from the above interbreeding construction.

We close this section by proving the following result:

Proposition 1.2. *The collection $\mathcal{C}_n$ contains representatives of infinitely many commensurability classes of both closed and noncompact hyperbolic n-manifolds satisfying the properties* (1)–(4) *from above.*

To prove this we will need the following invariant, originally due to Vinberg [1971]. Let Γ be a Zariski dense subgroup of a Lie group H with Lie algebra $\mathfrak{h}$. The adjoint action of Γ on $\mathfrak{h}$ gives a representation $\mathrm{Ad} : \Gamma \to \mathfrak{gl}(\mathfrak{h})$. In [Vinberg 1971] it is shown that the field $\mathbb{Q}(\{\mathrm{tr}(\mathrm{Ad}(\gamma)) \mid \gamma \in \Gamma\})$ is an invariant of the commensurability class of Γ in H. This field is called the *adjoint trace field of* Γ.

Next, let $N = \mathbb{H}^n/\Delta \in \mathcal{C}_n$, then Δ is a lattice in $\mathrm{SO}(J_1, \mathbb{R})$, which is Zariski dense by the Borel density theorem. The following lemma allows us to compute

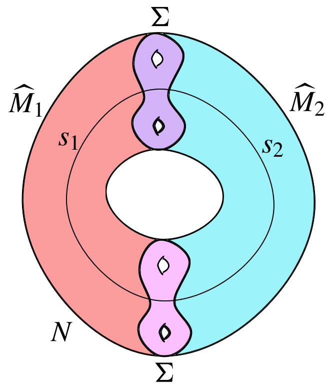

Figure 1. An graph of spaces description of the manifold N.

the adjoint trace field of Δ. It is an immediate corollary of a theorem of Mila (see [Mila 2019, Theorem 4.7]) once it is observed that L is the smallest extension of K over which the forms J_1 and J_2 are isometric.

Lemma 1.3. *Let $N = \mathbb{H}^n/\Delta \in \mathcal{C}_n$ and let L be the field of definition of Δ. Then L is the adjoint trace field of Δ.*

Proof of Proposition 1.2. From [Gromov and Piatetski-Shapiro 1988], it follows that $N = \mathbb{H}^n/\Delta$ is compact if and only if the field K used to construct Δ is not equal to $\mathbb{Q}$. For each choice of a totally real field K and a pair $\alpha, \beta \in K$ so that α/β is not a square in K we can produce an element $N \in \mathcal{C}_n$ via the interbreeding construction. By varying the choices of α and β we can produce infinitely many distinct $L = K(\sqrt{\beta/\alpha})$ for each choice of K. It follows from Lemma 1.3 that the corresponding N are representatives of infinitely many commensurability classes of both compact and noncompact hyperbolic n-manifolds. □

2. Lattices in $\mathrm{SL}_{n+1}(\mathbb{R})$

In this section we describe the lattices $\Delta \subset \mathrm{SL}_{n+1}(\mathbb{R})$ in which our thin groups will ultimately live. Let J_1 be one of the forms constructed in Section 1 and let L be the corresponding (totally real) field of definition. Let $M = L(\sqrt{r})$, where $r \in L$ is positive, square-free, and $s_L(r) = 0$. The number field M is a quadratic extension of L and we let $\tau : M \to M$ be the unique nontrivial Galois automorphism of M over L. In this context, we can extend the quadratic form J_1 on L^{n+1} to a “Hermitian” form on M^{n+1}. Let $N_{M/L} : M \to L$ given by $N_{M/L}(x) = x\tau(x)$ be the norm of the field extension M/L. Next let $x = (x_1, \dots, x_{n+1}) \in M^{n+1}$ and define $H_1 : M^{n+1} \to L$ as

$$H_1(x) = \alpha N_{M/L}(x_1) + \sum_{i=1}^{n} a_i N_{M/L}(x_i) - a_{n+1} N_{M/L}(x_{n+1}).$$

Note that this defines a Hermitian form in the sense that if $x \in M^{n+1}$ and $\lambda \in M$ then $H_1(\lambda x) = N_{M/L}(\lambda) H_1(x)$. Furthermore, since L is the fixed field of τ it follows that H_1 reduces to J_1 when restricted to L^{n+1}.

Next, we can define a unitary analogue of $\mathrm{SO}(J_1, \mathcal{O}_M)$ as

$$\mathrm{SU}(J_1, \tau, \mathcal{O}_M) = \{A \in \mathrm{SL}_{n+1}(\mathcal{O}_M) \mid H_1(Av) = H_1(v)\ \forall v \in M^{n+1}\}.$$

It is well known (see [Morris 2015, §6.8], for example) that $\mathrm{SU}(J_1, \tau, \mathcal{O}_M)$ is an arithmetic lattice in $\mathrm{SL}_{n+1}(\mathbb{R})$.

Let $N = \mathbb{H}^n/\Delta$ be one of the manifolds from $\mathcal{C}_n$. By construction, the manifold N contains the embedded totally geodesic hypersurface $\Sigma = \mathbb{H}^{n-1}/\hat{\Gamma}$, and so it is possible to deform Δ inside of $\mathrm{SL}_{n+1}(\mathbb{R})$ using the bending construction of Johnson and Millson [1987].

Specifically, let $c_t = \mathrm{Diag}(e^{-nt}, e^t, \dots, e^t) \in \mathrm{SL}_{n+1}(\mathbb{R})$. It is easy to check that c_t centralizes $\mathrm{SO}(n-1, 1, \mathbb{R})$. Since Σ is assumed to be nonseparating, we see that write Δ as an HNN extension $\Delta \cong \hat{\Delta}*_s$, where $\hat{\Delta}$ is isomorphic to the fundamental group of $N\backslash\Sigma$ and s is a free letter. In this context, we may view $\hat{\Delta} \subset \mathrm{SO}(J_1, \mathcal{O}_L)$ and $s \in \mathrm{SO}(J_1, \mathcal{O}_L)$ and observe that as a subgroup of $\mathrm{SO}(J_1, \mathcal{O}_L)$ we can write $\Delta = \langle \hat{\Delta}, s \rangle$. We now define a new family of subgroups $\Delta_t = \langle \Delta, c_t s \rangle \subset \mathrm{SL}_{n+1}(\mathbb{R})$. Using basic theory of HNN extensions, it is easy to see that, since c_t centralizes the fundamental group of Σ, as an abstract group Δ_t is a quotient of Δ. However, by using the following result due to Benoist [2005] in the compact case and Marquis [2012] in the noncompact case, we can actually say much more.

Proposition 2.1. *For each t, the group Δ_t is isomorphic to Δ.*

Next, we show for certain values of t the group Δ_t is contained in one of the unitary lattices constructed above. Specifically, if $N = \mathbb{H}^n/\Delta$ is contained in $\mathcal{C}_n$, let J_1 and L be such that $\Delta \subset \mathrm{SO}(J_1, \mathcal{O}_L)$. Recall that the field L is totally real of degree $d+1$ over $\mathbb{Q}$ and so there are $d+1$ embeddings $\{\sigma_0 = \mathrm{Id}, \dots, \sigma_d\}$ of L into $\mathbb{R}$. We can use Lemma 3.1 of [Ballas and Long 2020] to produce a unit $u \in \mathcal{O}_L^\times$ with the property that $|u| > 2$ and $0 < |\sigma_i(u)| < 1$ for $1 \le i \le d$. Let $p(x) = x^2 - ux + 1$ and let $M = L(v)$, where v is one of the roots of $p(x)$. It is easy to check that the discriminant of $p(x)$ is $u^2 - 4$ and so $M = L(\sqrt{u^2-4})$. By construction $s_L(u^2-4) = 0$, and so $\mathrm{SU}(J_1, \tau, \mathcal{O}_M)$ is an arithmetic lattice in $\mathrm{SL}_{n+1}(\mathbb{R})$, where $\tau : M \to M$ is the nontrivial Galois automorphism of M over L. The next lemma says that by carefully choosing t, we can arrange that $\Delta_t \subset \mathrm{SU}(J_1, \tau, \mathcal{O}_M)$.

Lemma 2.2. *Let u be as above. Then if $t = \log(u)$ then $\Delta_t \subset \mathrm{SU}(J_1, \tau, \mathcal{O}_M)$.*

This is basically Lemma 3.4 of [Ballas and Long 2020], but the proof is short so we include it here for the sake of completeness.

Proof. Recall from above that there is a subgroup $\hat{\Delta} \subset \mathrm{SO}(J_1, \mathcal{O}_L)$ and $s \in \mathrm{SO}(J_1, \mathcal{O}_L)$ so that $\Delta = \langle \hat{\Delta}, s \rangle$ and $\Delta_t = \langle \hat{\Delta}, c_t s \rangle$, where

$$c_t = \mathrm{Diag}(e^{-nt}, e^t, \dots, e^t) \in \mathrm{SL}_{n+1}(\mathbb{R}).$$

Since $\mathrm{SO}(J_1, \mathcal{O}_L) \subset \mathrm{SU}(J_1, \tau, \mathcal{O}_N)$ the proof will be complete if we can show that $c_t \in \mathrm{SU}(J_1, \tau, \mathcal{O}_M)$.

If $t = \log(u)$ then $c_t = \mathrm{Diag}(u^{-n}, u, \dots, u)$. Furthermore, since $\tau(u)$ is the other root of $p(x)$ it follows that $u\tau(u) = 1$, or in other words $\tau(u) = u^{-1}$. It follows that $c_t^* = \mathrm{Diag}(u^n, u^{-1}, \dots, u^{-1})$. A simple computation then shows that for each $v \in M^{n+1}$, $H_1(c_t v) = H_1(v)$, and so $c_t \in \mathrm{SU}(J_1, \tau, \mathcal{O}_M)$. ☐

By combining Lemma 2.2 and Proposition 2.1 we get the following corollary:

Corollary 2.3. *For each $N = \mathbb{H}^n/\Delta \in \mathcal{C}_n$ there are infinitely many lattices $\Lambda \subset \mathrm{SL}_{n+1}(\mathbb{R})$ that contain a subgroup Δ' isomorphic to Δ.*

3. Certifying thinness

The main goal of this section is to complete the proof of Theorem 1. The proof consist of proving that the subgroups constructed in the previous section are thin.

Proof of Theorem 1. Recall, that if $N = \mathbb{H}^n/\Delta \in \mathcal{C}_n$ from Corollary 2.3 it follows that we can find a lattice $\Lambda \subset \mathrm{SL}_{n+1}(\mathbb{R})$ and a subgroup $\Delta' \subset \Lambda$ that is isomorphic to Δ.

Since Δ' was obtained from Δ via a bending construction if follows from [Ballas and Long 2020, Proposition 4.1] that Δ' is Zariski dense in $\mathrm{SL}_{n+1}(\mathbb{R})$. The proof will be complete if we can show that Δ' has infinite index in Λ. Suppose for contradiction that this index is finite. Since Λ is a lattice in $\mathrm{SL}_{n+1}(\mathbb{R})$ this implies that Δ' is also a lattice in $\mathrm{SL}_{n+1}(\mathbb{R})$. However, Δ' is isomorphic to Δ and Δ is a lattice in the Lie group $\mathrm{SO}(n, 1)^\circ$. However, $\mathrm{SO}(n, 1)^\circ$ and $\mathrm{SL}_{n+1}(\mathbb{R})$ are not isomorphic and so this contradicts the Mostow rigidity theorem (see [Morris 2015, Theorem 15.1.2]). ☐

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