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A SIMPLE PROOF OF NECESSITY IN THE MCCULLOUGH-QUIGGIN THEOREM

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ABSTRACT. A short and simple proof of necessity in the McCullough-Quiggin characterization of positive semi-definite kernels with the complete Pick property is presented.

1. Introduction

Given N points $z_1, \dots, z_N$ in the unit disk $\mathbb{D} \subset \mathbb{C}$ and N "targets" $w_1, \dots, w_N \in \mathbb{C}$, when does there exist a holomorphic function $f : \mathbb{D} \to \overline{\mathbb{D}}$ that interpolates $f(z_i) = w_i$ for $i = 1, \dots, N$? Pick's theorem of 1916 says this interpolation can be done if and only if the matrix

$$\left(\frac{1 - w_i \bar{w}_j}{1 - z_i \bar{z}_j}\right)_{i,j} \qquad \text{is positive semi-definite [13].}$$

In the hundred intervening years, this theorem has been generalized and reinterpreted a number of ways; see [9]. The theorem generalizes to matrix valued functions, and it can be reinterpreted as a special property of multipliers of reproducing kernel Hilbert spaces. The remarkable McCullough-Quiggin theorem precisely describes which reproducing kernel Hilbert spaces have this property. It is not our goal to delve into motivation and background, as this is already done in the paper [1] and related book [2]. Instead, our goal is to give a short and simple proof of necessity in the McCullough-Quiggin theorem.

Let $k : X \times X \to \mathbb{C}$ be a positive semi-definite (PSD) kernel on X with associated reproducing kernel Hilbert space H. Let L_1, L_2 be auxiliary Hilbert spaces and let $B(L_1, L_2)$ denote the bounded linear operators from L_1 to L_2. A function $\Phi : X \to B(L_1, L_2)$ is a multiplier of norm at most one, i.e., belongs to $\text{Mult}_1(H \otimes L_1, H \otimes L_2)$, if and only if for every $f \in H \otimes L_1$ we have $\Phi f \in H \otimes L_2$ and $\|\Phi f\| \leq \|f\|$. This property is equivalent to the property that the operator valued kernel

$$(I - \Phi(x)\Phi(y)^*)k(x, y) \qquad \text{is PSD.}$$

Definition 1. A kernel k has the *complete Pick property* if for all natural numbers s, t, whenever $S \subset X$ is finite and $W \in \text{Mult}_1(H|_S \otimes \mathbb{C}^t, H|_S \otimes \mathbb{C}^s)$, then there exists $\Phi \in \text{Mult}_1(H \otimes \mathbb{C}^t, H \otimes \mathbb{C}^s)$ such that $\Phi|_S = W$.

Definition 2. The kernel k is irreducible if $k(x, y) \neq 0$ for all $x, y \in X$ and k_x, k_y are linearly independent for $x \neq y$.

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Theorem 1 (McCullough [10, 11], Quiggin [6]). *Let $k : X \times X \to \mathbb{C}$ be PSD and irreducible. Then, k has the complete Pick property if and only if for each $z \in X$,*

$$F_z(x, y) := 1 - \frac{k(x, z)k(z, y)}{k(z, z)k(x, y)} \quad \text{is PSD.}$$

Agler-McCarthy [1] first formulated and proved the theorem in precisely the form above. The original proofs dig into precisely what needs to be satisfied in order to extend a multiplier on a set of points to one more point ("the one point extension property"). Some sort of axiom of choice is invoked to build a multiplier on all of X. The proof of necessity given below is a short inductive proof. One innovation worth mentioning is that we do not use full irreducibility of k; we only need k to be non-vanishing. A straightforward proof of sufficiency is due to Ball-Trent-Vinnikov [4]. It proves a realization formula for multipliers that builds a multiplier directly and uses Hilbert space geometry instead of the axiom of choice. The master's thesis [8] contains a nice treatment.

While this paper is about a foundational aspect of complete Pick kernels, a lot of fascinating work has been done in recent years on more advanced aspects of these kernels and related Drury-Arveson type spaces. See [3, 5, 7] for a sampling of some recent literature. In particular, the paper [5] contains an interesting characterization of the complete Pick property in terms of completely contractive embeddings.

We shall assume knowledge of the rudiments of vector-valued reproducing kernel Hilbert spaces, including multipliers and the Schur product theorem. See [2, 12].

2. Proof of necessity

We need three basic lemmas which apply to any PSD kernel k with associated reproducing kernel Hilbert space H such that $k(z, z) \neq 0$ for some fixed $z \in X$. Set

$$k^z(x, y) := k(x, y) - \frac{k(x, z)k(z, y)}{k(z, z)}.$$

Lemma 1. *Consider the closed subspace $H_z = \{f \in H : f(z) = 0\}$. The reproducing kernel for H_z is $k^z(x, y)$.*

Proof. Note that $H = H_z \oplus \mathbb{C}k_z$. The reproducing kernel for $\mathbb{C}k_z$ is simply $k(z, z)^{-1}k_z(x)\overline{k_z(y)}$. Then, $k^z_y(x) = k^z(x, y)$ as defined above belongs to H_z and reproduces elements of H_z. □

Lemma 2. *Let L be a Hilbert space. Given $f \in H \otimes L$, we have $f(z) = \vec{0}$ if and only if $f \in H_z \otimes L$.*

Proof. Evidently, $f \in H_z \otimes L$ implies $f(z) = \vec{0}$ because the tensor project is a closed span of elements with this property. Conversely, let P be the orthogonal projection from $H \otimes L$ to $H_z \otimes L$. Let $f \in H \otimes L$ with $f(z) = \vec{0}$. Note by Lemma 1 that

$$\langle f, k^z_y \otimes v\rangle = \langle f, (k_y - k_z \frac{k(z, y)}{k(z, z)}) \otimes v\rangle = \langle f(y), v\rangle_L.$$

On the other hand, since $k^z_y \otimes v = P(k^z_y \otimes v) \in H_z \otimes L$, $\langle f, k^z_y \otimes v\rangle = \langle Pf, k^z_y \otimes v\rangle = \langle Pf(y), v\rangle_L$. So, $f = Pf \in H_z \otimes L$. □

Lemma 3. *Let L_1, L_2 be Hilbert spaces. If $\Phi \in Mult_1(H \otimes L_1, H \otimes L_2)$, then*

$$\Phi \in Mult_1(H_z \otimes L_1, H_z \otimes L_2).$$

Proof. If $f \in H_z \otimes L_1$, then $\Phi f \in H \otimes L_2$ and $\Phi(z)f(z) = \vec{0}$ so that $\Phi f \in H_z \otimes L_2$. The inequality $\|\Phi f\| \leq \|f\|$ holds for $f \in H_z \otimes L_1 \subset H \otimes L$, so

$$\Phi \in \text{Mult}_1(H_z \otimes L_1, H_z \otimes L_2).$$

□

The following corollary is the PSD kernel interpretation of the above lemma.

Corollary 1. *If $\Phi : X \to B(L_1, L_2)$ is a function such that $(I - \Phi(x)\Phi(y)^*)k(x,y)$ is PSD, then $(I - \Phi(x)\Phi(y)^*)k^z(x,y)$ is PSD.*

We now assume k is non-vanishing and that the complete Pick property holds for k. We proceed to show that F_z from the statement of Theorem 1 is PSD. For $x, z \in X$, $F_z(x,x) = 1 - \frac{|k(x,z)|^2}{k(x,x)k(z,z)} \geq 0$ by Cauchy-Schwarz. Now take any $x_1, \ldots, x_N, x_{N+1} \in X$ where $N \geq 2$. Assuming $(F_{x_N}(x_i, x_j))_{i,j=1,\ldots,N-1} \geq 0$ we will show that $(F_{x_{N+1}}(x_i, x_j))_{i,j=1,\ldots,N} \geq 0$. Note that since $F_{x_N}(x_i, x_N) = 0$, the matrix expanded with zeros $A := (F_{x_N}(x_i, x_j))_{i,j=1,\ldots,N}$ is PSD. Factor the entries of A as $A_{i,j} = v_i v_j^*$ using row vectors v_i. For simplicity we will write $k_{ij} = k(x_i, x_j)$ below. Note that

$$(1 - A_{i,j}) = (1 - v_i v_j^*) = \frac{k_{iN} k_{Nj}}{k_{NN} k_{ij}}$$

so that $(1 - v_i v_j^*) k_{ij} = \frac{k_{iN} k_{Nj}}{k_{NN}}$ is rank one and PSD. By the complete Pick property, there exists a contractive multiplier Φ with $\Phi(x_i) = v_i$. By Corollary 1 applied to $z = x_{N+1}$,

$$(1 - v_i v_j^*) \left(k_{i,j} - \frac{k_{i,N+1} k_{N+1,j}}{k_{N+1,N+1}} \right) = \frac{k_{iN} k_{Nj}}{k_{NN}} F_{x_{N+1}}(x_i, x_j) \quad \text{ is PSD.}$$

Here $i, j = 1, \ldots, N$. The matrix $\left(\frac{k_{NN}}{k_{iN} k_{Nj}} \right)_{i,j}$ is rank one and PSD, so by the Schur product theorem we see that $(F_{x_{N+1}}(x_i, x_j))_{i,j=1,\ldots,N}$ is PSD. By induction, this proves that $F_z(x,y)$ is PSD for all $z \in X$.

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