Integr. Equ. Oper. Theory (2021) 93:14
https://doi.org/10.1007/s00020-021-02626-8
Published online March 1, 2021


Integral Equations and Operator Theory



# Cyclicity Preserving Operators on Spaces of Analytic Functions in $\mathbb{C}^n$

Jeet Sampat

**Abstract.** For spaces of analytic functions defined on an open set in $\mathbb{C}^n$ that satisfy certain nice properties, we show that operators that preserve shift-cyclic functions are necessarily weighted composition operators. Examples of spaces for which this result holds true consist of the Hardy space $H^p(\mathbb{D}^n)\,(0 < p < \infty)$, the Drury–Arveson space $\mathcal{H}^2_n$, and the Dirichlet-type space $\mathcal{D}_\alpha\,(\alpha \in \mathbb{R})$. We focus on the Hardy spaces and show that when $1 \le p < \infty$, the converse is also true. The techniques used to prove the main result also enable us to prove a version of the Gleason–Kahane–Żelazko theorem for partially multiplicative linear functionals on spaces of analytic functions in more than one variable.

**Mathematics Subject Classification.** Primary 32A35, Secondary 47A16.

**Keywords.** Hardy spaces, Cyclic vectors, Weighted composition operators, Gleason–Kahane–Żelazko theorem, Maximal domains.

## 1. Introduction

The results presented in this paper are motivated by a number of questions about cyclic functions in the Hardy space. Fix $n \in \mathbb{N}$, and let $\mathbb{D}^n$ be the unit polydisc in $\mathbb{C}^n$. That is, $\mathbb{D}^n := \{z \in \mathbb{C}^n \,\big|\, |z_i| < 1, \forall\, 1 \le i \le n\}$. For $0 < p < \infty$ we define the Hardy space,

$$H^p(\mathbb{D}^n) := \left\{ f \in \mathrm{Hol}(\mathbb{D}^n) \;\middle|\; ||f||_p^p := \sup_{0 \le r < 1} \int_{\mathbb{T}^n} |f(rw)|^p \, d\sigma_n(w) < \infty \right\}.$$

Here, for an open set $D \subset \mathbb{C}^n$, $\mathrm{Hol}(D)$ is the set of holomorphic functions on $D$. Also, $\sigma_n$ is the normalized Lebesgue measure on the unit $n$-torus, $\mathbb{T}^n := \{z \in \mathbb{C}^n \mid |z_i| = 1, \forall\, 1 \le i \le n\}$. It is known that $H^p(\mathbb{D}^n)$ is a Banach space for all $1 \le p < \infty$ with norm $||\cdot||_p$. $f \in H^p(\mathbb{D}^n)$ is said to be cyclic if $\mathrm{S}[f] := \overline{\mathrm{span}}\,\{z^\alpha f(z) \mid \alpha \in \mathbb{Z}^+(n)\} = \overline{\mathrm{span}}\,\{pf \mid p \text{ - polynomial}\} = H^p(\mathbb{D}^n)$, where $\mathbb{Z}^+(n)$ is the set of $n$-tuples $\alpha = (\alpha_i)_{i=1}^n$ of non-negative integers, and

$z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$. We also have the space of all bounded analytic functions defined on $\mathbb{D}^n$,

$$H^\infty(\mathbb{D}^n) := \left\{ f \in \mathrm{Hol}(\mathbb{D}^n) \,\middle|\, ||f||_\infty := \sup_{w \in \mathbb{D}^n} |f(w)| < \infty \right\}.$$

Just like $H^p(\mathbb{D}^n)$ for $1 \le p < \infty$, $H^\infty(\mathbb{D}^n)$ is a Banach space with the supremum norm $|| \cdot ||_\infty$.
In $H^p(\mathbb{D})$ for $0 < p < \infty$, cyclic functions have been characterized using Beurling's theorem, and the canonical factorization theorem (see Theorem 7.4 in [4] and Theorem 4 in [5]). In the case when $n > 1$, we do not have a version of Beurling's theorem or the canonical factorization theorem (see Sect. 4.4 in [14] for more details). Several sufficient conditions for cyclicity in $H^p(\mathbb{D}^n)$ were provided by N. Nikolski (Theorems 3.3 and 3.4, [13]), but in general, not a lot is known about cyclic functions in the Hardy spaces for $n > 1$. One way of obtaining cyclic functions when $n > 1$ is through operators that preserve cyclicity, i.e. linear maps $T : H^p(\mathbb{D}^n) \to H^q(\mathbb{D}^m)$ such that $Tf$ is cyclic whenever $f$ is cyclic.

When $p = q = 2$ and $n = m = 1$, a result of P. C. Gibson, M. P. Lamoureux and G. F. Margrave shows that all such operators have to be weighted composition operators (Theorem 4, [6]). This was generalized to $H^p(\mathbb{D})$ for $0 < p \le \infty$ by J. Mashreghi and T. Ransford for *'outer-preserving' operators* in [11].

**Definition 1.1.** For $0 < p \le \infty$, a non-vanishing function $f \in H^p(\mathbb{D}^n)$ is said to be outer if

$$\log |f(0)| = \int_{\mathbb{T}^n} \log |f|.$$

In $H^p(\mathbb{D})$, using Beurling's theorem, it is known that the class of cyclic functions coincides with that of outer functions. With this in mind, the following theorem of Mashreghi and Ransford (Theorem 2.2, [11]) is a generalization of the result of P. C. Gibson, M. P. Lamoureux and G. F. Margrave.

**Theorem 1.2.** *Let $0 < p \le \infty$ and let $T : H^p(\mathbb{D}) \to Hol(\mathbb{D})$ be a linear map such that $Tg(z) \neq 0$ for all outer functions $g \in H^p(\mathbb{D})$ and all $z \in \mathbb{D}$. Then there exist holomorphic maps $\phi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{C} \setminus \{0\}$ such that*

$$Tf = \psi \cdot (f \circ \phi) \;\; (\forall\, f \in H^p(\mathbb{D})).$$

It is important to note that continuity of $T$ is not assumed in the above theorem. For more general spaces over $\mathbb{D}$ that satisy some nice properties, J. Mashreghi and T. Ransford prove a similar result. Let $X \subset \mathrm{Hol}(\mathbb{D})$ be a Banach space that satisfies the following properties :

- $(X1)$ $X$ contains the set of polynomials, and they form a dense subspace of $X$.
- $(X2)$ For each $w \in \mathbb{D}$, the evaluation map $f \mapsto f(w) : X \to \mathbb{C}$ is continuous.
- $(X3)$ $X$ is shift-invariant, i.e. $f \in X \Rightarrow zf \in X$.

We also need a subset $Y \subset X$ that satisfies the following properties.

- $(Y1)$ If $g \in X$ and $0 < \inf_{\mathbb{D}} |g| \le \sup_{\mathbb{D}} |g| < \infty$, then $g \in Y$.

$(Y2)$ If $g(z) = z - \lambda$ where $\lambda \in \mathbb{T}$, then $g \in Y$.

For these spaces, we have the following theorem (Theorem 3.2, [11]).

**Theorem 1.3.** *Suppose $X \subset Hol(\mathbb{D})$ satisfies $(X1)$–$(X3)$ and $Y \subset X$ satisfies $(Y1)$–$(Y2)$. Let $T : X \to Hol(\mathbb{D})$ be a continuous linear map such that $Tg(z) \neq 0$ for every $g \in Y$ and $z \in \mathbb{D}$. Then there exist holomorphic functions $\phi : \mathbb{D} \to \mathbb{D}$ and $\psi : \mathbb{D} \to \mathbb{C} \setminus \{0\}$ such that $Tf(z) = \psi(z) f(\phi(z))$ for each $f \in X$.*

The proof of Theorem 1.3 relies on classifying $\Lambda \in X^*$ such that $\Lambda(g) \neq 0, \forall g \in Y$ (Theorem 3.1, [11]). This is similar to a result now known as the Gleason-Kahane-Żelazko (GKŻ) theorem (see [7] and [9]), which identifies multiplicative linear functionals in a complex unital Banach algebra through its action on invertible elements (see Theorem 5.1 below). In [11], it is shown that a version of the GKŻ theorem holds for modules of a complex unital Banach algebra and it can be applied to the multiplier algebra of the space $X$ satisfying properties $(X1)$–$(X3)$ to obtain Theorem 1.3.

In [10], K. Kou and J. Liu provide a similar argument for $H^p(\mathbb{D})$ when $1 < p < \infty$. It is essentially the same as that of Theorem 1.3, but instead of the subset $Y$ they consider the set $\{e^{w \cdot z} \,|\, w \in \mathbb{C}\}$ (see Theorem 2, [10]). They also showed that the converse of Theorem 1.3 is true when $1 < p < \infty$, i.e. all weighted composition operators on $H^p(\mathbb{D})$ for $1 < p < \infty$ also preserve outer (and thus, cyclic) functions.

Using techniques similar to those in [10] and [11], we can generalize Theorem 1.3 to spaces of analytic functions in more than one variable, and also over arbitrary domains. To that end, we shall work with spaces $\mathcal{X}$ consisting of functions defined on a set $D \subset \mathbb{C}^n$ for some $n \in \mathbb{N}$, and that are holomorphic on an open subset of $D$. Furthermore, $\mathcal{X}$ satisfies the following properties.

- **Q1** The set of polynomials $\mathcal{P}$ is dense in $\mathcal{X}$.
- **Q2** The point evaluation map $\Lambda_z : \mathcal{X} \to \mathbb{C}$, defined as $\Lambda_z f := f(z)$, is a bounded linear functional on $\mathcal{X}$ for all $z \in D$. Furthermore, if for some $z \in \mathbb{C}^n$ the map $\Lambda_z p := p(z)$ defined on $\mathcal{P}$ extends to a bounded linear functional on all of $\mathcal{X}$, then $z \in D$.
- **Q3** The $i^{th}$-shift operator $S_i : \mathcal{X} \to \mathcal{X}$, defined as $S_i f(z) := z_i f(z)$ for every $(z_i)_{i=1}^n = z \in D$ and $f \in \mathcal{X}$, is bounded for every $1 \leq i \leq n$.

The domain $D$, in this case, is called the *maximal domain* of $\mathcal{X}$. Here, the maximality is with respect to bounded extension of point evaluations on the set of polynomials. We will show that the maximal domain of $H^p(\mathbb{D}^n)$ is $\mathbb{D}^n$ for all values of $p$, and provide more examples with details and references in Sect. 3.

The main results of this paper are the following theorems.

**Theorem 1.4.** *Suppose $\mathcal{X}$ satisfies* **Q1**–**Q3** *over a set $D \subset \mathbb{C}^n$. Let $\Lambda \in \mathcal{X}^*$ be such that $\Lambda(e^{w \cdot z}) \neq 0$ for every $w \in \mathbb{C}^n$. Then, there exist $a \in \mathbb{C} \setminus \{0\}$ and $b \in D$ such that $\Lambda(f) = a \cdot f(b)$.*

In Sect. 4, using Theorem 1.4, we will obtain the following generalization of Theorem 1.3.

**Theorem 1.5.** *Suppose $\mathcal{X}$ satisfies* **Q1**–**Q3** *over a set $D \subset \mathbb{C}^n$. Let $\mathcal{Y}$ be a topological vector space of functions, defined on a set $E$, such that $\Gamma_u g := g(u)$, $g \in \mathcal{Y}$ defines a continuous linear functional for all $u \in E$. Let $T : \mathcal{X} \to \mathcal{Y}$ be a continuous linear operator. Then, the following are equivalent :*

(1) *$T(e^{w\cdot z})$ is non-vanishing for every $w \in \mathbb{C}^n$.*
(2) *$Tf(u) = a(u)f(b(u))$ for some non-vanishing function $a \in \mathcal{Y}$, and a map $b : E \to D$.*

*Furthermore, $a = T1$ and $b = \frac{T(z)}{T(1)}$, where $T(z) = \big(T(z_i)\big)_{i=1}^n$.*

In Sect. 4.1, using Theorem 1.5 and some facts about Hardy spaces in several complex variables, we will prove the following generalization of Theorem 1.2, and Theorem 2 in [10].

**Theorem 1.6.** (1) *Fix $0 < p, q < \infty$ and $m, n \in \mathbb{N}$. Let $T : H^p(\mathbb{D}^n) \to H^q(\mathbb{D}^m)$ be a bounded linear operator such that $Tf$ is cyclic whenever $f$ is cyclic. Then, there exist analytic functions $a \in H^q(\mathbb{D}^m)$ and $b : \mathbb{D}^m \to \mathbb{D}^n$ such that $Tf(z) = a(z)f(b(z))$ for every $z \in \mathbb{D}^m$ and $f \in H^p(\mathbb{D}^n)$.*

*Furthermore, $a = T1$ is cyclic and $b = \frac{T(z)}{T1}$, where $T(z) = \big(T(z_i)\big)_{i=1}^n$.*
(2) *Fix $0 < p, q \leq \infty$ and $m, n \in \mathbb{N}$. Then, the conclusion of part* (1) *holds if we replace* 'cyclic' *with* 'outer'.

*For $1 \leq q < \infty$, the converse of part* (1) *is also true. That is, all bounded weighted composition operators from $H^p(\mathbb{D}^n)$ into $H^q(\mathbb{D}^m)$ also preserve cyclicity.*

In Sect. 5, as an interesting byproduct of results proved in this paper, we will prove the following version of the GKŻ theorem for Banach spaces of analytic functions.

**Theorem 1.7.** *Suppose $\mathcal{X}$ satisfies* **Q1**–**Q3** *over a set $D \subset \mathbb{C}^n$. Let $\Lambda \in \mathcal{X}^*$ such that $\Lambda(1) = 1$. Then, the following are equivalent :*

*(i)* *$\Lambda(e^{w\cdot z}) \neq 0$ for every $w \in \mathbb{C}^n$.*
*(ii)* *$\Lambda = \Lambda_z$ for some $z \in D$.*
*(iii)* *$\Lambda(fg) = \Lambda(f)\Lambda(g)$ for all $f, g \in \mathcal{X}$ such that $fg \in \mathcal{X}$.*
*(iv)* *$\Lambda(\phi f) = \Lambda(\phi)\Lambda(f)$ for all $\phi \in \mathcal{M}(\mathcal{X})$ and $f \in \mathcal{X}$.*

Here, $\mathcal{M}(\mathcal{X}) := \{\phi \in \mathrm{Hol}(D) \mid \phi f \in \mathcal{X}$ for all $f \in \mathcal{X}\}$ is the multiplier algebra of $\mathcal{X}$.

## 2. Notations and Preliminary Results

Before we consider spaces of functions defined over its maximal domain, we will work with spaces of holomorphic functions defined on an open set in $\mathbb{C}^n$ for some $n \in \mathbb{N}$. The notation is much simpler in this case, and in Sect. 3, we will show that all holomorphic function spaces that satisfy some nice properties can be identified with a space of functions defined over its maximal domain.

Fix $n \in \mathbb{N}$. For an open set $D \subset \mathbb{C}^n$, let $\mathcal{X} \subset \mathrm{Hol}(D)$ be a Banach space satisfying the following properties :

**P1** The set of polynomials $\mathcal{P}$ is dense in $\mathcal{X}$.
**P2** The point-evaluation map $\Lambda_z : \mathcal{X} \to \mathbb{C}$, defined as $\Lambda_z(f) := f(z)$ for every $f \in \mathcal{X}$, is a bounded linear functional on $\mathcal{X}$ for every $z \in D$.
**P3** The $i^{th}$-shift operator $S_i : \mathcal{X} \to \mathcal{X}$, defined as $S_i f(z) := z_i f(z)$ for every $(z_k)_{k=1}^n = z \in D$ and $f \in \mathcal{X}$, is a bounded linear operator for every $1 \leq i \leq n$.

Examples of spaces that satisfy **P1**–**P3** include the Hardy space $H^p(\mathbb{D}^n)$ for $1 \leq p < \infty$, the Drury-Arveson space $\mathcal{H}_n^2$ on the unit ball $\mathbb{B}_n := \left\{ z \in \mathbb{C}^n \,\middle|\, \sum_{i=1}^n |z_i|^2 \leq 1 \right\}$, and the Dirichlet-type spaces $\mathcal{D}_\alpha$ for $\alpha \in \mathbb{R}$.

$$\mathcal{H}_n^2 = \left\{ f \sim \sum \hat{f}(a) z^a \in \mathrm{Hol}(\mathbb{B}_n) \,\middle|\, \sum_{a \in \mathbb{Z}^+(n)} \frac{a_1!\, a_2! \cdots a_n!}{(a_1 + a_2 + \cdots + a_n)!} |\hat{f}(a)|^2 < \infty \right\}$$

$$\mathcal{D}_\alpha = \left\{ f \sim \sum \hat{f}(a) z^a \in \mathrm{Hol}(\mathbb{D}^n) \,\middle|\, \sum_{a \in \mathbb{Z}^+(n)} \big((a_1+1)\cdots(a_n+1)\big)^\alpha |\hat{f}(a)|^2 < \infty \right\}$$

The list of Dirichlet-type spaces consists of many important spaces like the usual Dirichlet space ($\alpha = 1$), the Hardy space $H^2(\mathbb{D}^n)$ ($\alpha = 0$), and also the Bergman space ($\alpha = -1$). For these spaces, we prove the following preliminary result.

**Theorem 2.1.** *Suppose $\mathcal{X}$ satisfies* **P1**–**P3** *over an open set $D \subset \mathbb{C}^n$. Let $\Lambda \in \mathcal{X}^*$ be such that $\Lambda(e^{w \cdot z}) \neq 0$ for every $w \in \mathbb{C}^n$. Then, there exist $a \in \mathbb{C} \setminus \{0\}$ and $b \in \sigma_r(S)$ such that $\Lambda p = a \cdot p(b)$ for every $p \in \mathcal{P}$. Here, $\sigma_r(S)$ is the right Harte spectrum of $S = (S_i)_{i=1}^n$.*

Recall that $\sigma_r(S)$ is the complement in $\mathbb{C}^n$ of $\rho_r(S)$, where

$$\rho_r(S) := \left\{ \lambda \in \mathbb{C}^n \,\middle|\, \exists \, \{A_i\}_{i=1}^n \subset \mathcal{B}(\mathcal{X}) \text{ such that } \sum_{i=1}^n (S_i - \lambda_i I) A_i = I \right\}.$$

Note that it is not immediate from **P1**–**P3** that $e^{w \cdot z} \in \mathcal{X}$. We address this separately as a lemma before we prove Theorem 2.1.

**Lemma 2.2.** *For each $w \in \mathbb{C}^n$, we have $e^{w \cdot z} \in \mathcal{X}$. In fact, $p_k := \sum_{|\alpha| \leq k} \frac{w^\alpha z^\alpha}{\alpha!} \to e^{w \cdot z}$ in $\mathcal{X}$ as $k \to \infty$, where $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $\alpha! := \alpha_1!\, \alpha_2! \cdots \alpha_n!$.*

*Proof.* Fix $w \in \mathbb{C}^n$. We show that $\lim_{k \longrightarrow \infty} p_k$ exists. This follows from the fact that $\mathcal{X}$ is a Banach space and

$$\sum_{\alpha \in \mathbb{Z}^+(n)} \left|\left| \frac{w^\alpha z^\alpha}{\alpha!} \right|\right| \leq \sum_{\alpha \in \mathbb{Z}^+(n)} \frac{|w|^\alpha \, ||S||^\alpha \, ||1||}{\alpha!} = ||1|| e^{|w| \cdot ||S||}$$

where $|w| := \big(|w_1|, \ldots, |w_n|\big)$ and $||S|| := \big(||S_1||, \ldots, ||S_n||\big)$. Let $g = \lim_{k \longrightarrow \infty} p_n$ in $\mathcal{X}$. Note that $p_k$ converges to $e^{w \cdot z}$ point-wise. By **P2**, this implies $g(z) = e^{w \cdot z}$. $\square$

*Proof of Theorem 2.1.* Since $\Lambda(e^{z\cdot w}) \neq 0$ for all $w \in \mathbb{C}^n$, and $\Lambda$ is continuous, we get

$$\sum_{\alpha\in\mathbb{Z}^+(n)} \frac{\Lambda(z^\alpha)w^\alpha}{\alpha!} \neq 0,\ \forall w \in \mathbb{C}^n.$$

Let $\lambda_\alpha := \Lambda(z^\alpha)$, $\forall \alpha \in \mathbb{Z}^+(n)$. Now, $|\lambda_\alpha| = ||\Lambda(z^\alpha)|| \leq ||\Lambda|| \cdot ||z^\alpha||$ implies

$$|\lambda_\alpha| \leq ||\Lambda|| \cdot ||S_1||^{\alpha_1} \cdot ||S_2||^{\alpha_2} \cdots ||S_n||^{\alpha_n} \cdot ||1|| \text{ for every } \alpha \in \mathbb{Z}^+(n).$$

Let $F(w) := \sum\limits_{\alpha\in\mathbb{Z}^+(n)} \frac{\lambda_\alpha w^\alpha}{\alpha!}$, and note that $F$ is a non-vanishing entire function such that

$$|F(w)| \leq ||\Lambda|| \cdot ||1|| \cdot e^{|w|\cdot||S||}.$$

When $n = 1$, it is well-known that all such $F$ are of the form $e^{a_0+b\cdot w}$ for some $a_0 \in \mathbb{C}$ and $b \in \mathbb{C}^n$ (see Sect. 3.2, Chapter 5 in [1]). We will show that this is true for all values of $n$. ☐

**Lemma 2.3.** *Fix $n \in \mathbb{N}$. Let $F \in Hol(\mathbb{C}^n)$ be a non-vanishing entire function for which there exist constants $A, B$ such that $|F(z)| \leq Ae^{Br^m}$ for all $z$ in $(r\mathbb{D})^n$, and for all $r > 0$. Then, there exists a polynomial $p$ with $deg(p) \leq m$ such that $F(z) = e^{p(z)}$ for all $z \in \mathbb{C}^n$.*

*Proof.* Since $F$ is non-vanishing, there exists an entire function $G$ such that $F = e^G$. Note that the hypothesis then implies $\text{Re}(G) \leq \ln A + Br^m$ in $(r\mathbb{D})^n$. We need to show that $G$ is a polynomial with $\deg(G) \leq m$. The case $n = 1$ is known (see Sect. 3.2, Chapter 5 in [1]), so assume $n > 1$. Let $G(z) = \sum G_k(z)$ be the homogeneous expansion of $G$. Fix $z \in \mathbb{C}^n$ and let $g_z(\lambda) := G(\lambda z) = \sum \lambda^k G_k(z)$ for $\lambda \in \mathbb{C}$. Notice that

$$\text{Re}\big(g_z(\lambda)\big) = \text{Re}\big(G(\lambda z)\big) \leq \ln A + B \cdot C^m|\lambda|^m$$

where $C = \sup\limits_{1\leq j\leq n} |z_j|$, since $z \in (r\mathbb{D})^n$ for every $r > C$. Thus, for $\lambda \in r\mathbb{D}$

$$\text{Re}\big(g_z(\lambda)\big) \leq \ln A + B \cdot C^m r^m.$$

Applying the one variable case to $g_z$, we get $G_k(z) = 0$ for all $k > m$. As the choice of $z \in \mathbb{C}^n$ was arbitrary, this means $G_k(z) = 0$ for all $z \in \mathbb{C}^n$ and $k > m$. Therefore, $G$ is a polynomial with $\deg(G) \leq m$ as required. ☐

*Proof of Theorem 2.1 (cont.).* By Lemma 2.3, we get that $F(w) = e^{a_0+b\cdot w}$ for some $a_0 \in \mathbb{C}$ and $b \in \mathbb{C}^n$. Using the definition of $F(w)$, and comparing power-series coefficients, we get $\lambda_\alpha = e^{a_0}b^\alpha$, $\forall \alpha \in \mathbb{Z}^+(n)$. Let $a := e^{a_0} \in \mathbb{C} \setminus \{0\}$. This means $\Lambda(z^\alpha) = a \cdot b^\alpha$, $\forall \alpha \in \mathbb{Z}^+(n)$.

Note that we have shown $\Lambda p = a \cdot p(b)$ for every polynomial $p$. It only remains to show that $b \in \sigma_r(S)$. For the sake of contradiction, suppose $b \notin \sigma_r(S)$. Therefore there exists $\{A_i\}_{i=1}^n \subset \mathcal{B}(\mathcal{X})$ such that

$$\sum_{i=1}^n (S_i - b_i I)A_i = I.$$

In particular,

$$\sum_{i=1}^{n}(z_i - b_i)A_i 1 = 1.$$

Fix an $\epsilon > 0$. Since $\mathcal{X}$ satisfies **P1**, we can pick $p_i \in \mathcal{P}$ for each $1 \le i \le n$ such that

$$||A_i 1 - p_i|| < \frac{\epsilon}{n \cdot ||\Lambda|| \cdot ||S_i - b_i I||}.$$

Note that,

$$\begin{aligned} \left|\left|1 - \sum_{i=1}^{n}(z_i - b_i)p_i\right|\right| &= \left|\left|\sum_{i=1}^{n}(z_i - b_i)(A_i 1 - p_i)\right|\right| \\ &\le \sum_{i=1}^{n} ||S_i - b_i I||\, ||A_i 1 - p_i|| < \frac{\epsilon}{||\Lambda||}. \end{aligned}$$

Based on the representation of $\Lambda$ on polynomials, we know that

$$\Lambda\left(\sum_{i=1}^{n}(z_i - b_i)p_i\right) = 0.$$

This means

$$|a| = |\Lambda 1| = \left|\Lambda 1 - \Lambda\left(\sum_{i=1}^{n}(z_i - b_i)p_i\right)\right| \le ||\Lambda|| \cdot \left|\left|1 - \sum_{i=1}^{n}(z_i - b_i)p_i\right|\right| < \epsilon.$$

As $\epsilon > 0$ was arbitrarily chosen and $a \neq 0$, we get a contradiction. Hence, $b \in \sigma_r(S)$. □

*Remark 2.4.* It would be great if we could show that $b \in D$, but that need not be the case. It is obvious that $D \subset \sigma_r(S)$, but it may not be possible to extend the domain of every function in $\mathcal{X}$ to the whole of $\sigma_r(S)$ in order to extend the functional in the theorem to all of $\mathcal{X}$.

*Example 1.* When $\mathcal{X} = H^p(\mathbb{D}^n)$, for some $1 \le p < \infty$, it is easy to check that $\sigma_r(S) = \overline{\mathbb{D}^n}$. So, $b$ obtained in Theorem 2.1 lies in $\overline{\mathbb{D}^n}$. We claim that in this case, $b$ lies in $\mathbb{D}^n$. For the sake of argument, assume $b = (b_i)_{i=1}^n \in \partial\mathbb{D}^n$ with $b_j \in \mathbb{T}$ for some $1 \le j \le n$. Consider $q(z) := z_j - b_j$. Since $z - \beta$ is cyclic in $H^p(\mathbb{D})$ for all $1 \le p < \infty$ and $\beta \notin \mathbb{D}$, $q$ is cyclic in $H^p(\mathbb{D}^n)$. This means that for any given $f \in H^p(\mathbb{D}^n)$, there exist polynomials $\{q_k\}_{k\in\mathbb{N}}$ such that $q_k q \to f$. Note that since $q(b) = 0$,

$$\Lambda(q_k q) = a \cdot q_k(b)q(b) = 0 \text{ for every } k \in \mathbb{N}.$$

Thus, $\Lambda(f) = 0$ which implies $\Lambda \equiv 0$, a contradiction. So, $b \in \mathbb{D}^n$ and $\Lambda \equiv a\Lambda_b$.

We can make a similar argument for spaces $\mathcal{X}$ that have an *envelope of cyclic polynomials over* $D$. Recall that $f \in \mathcal{X}$ is cyclic if the shift-invariant subspace $\mathrm{S}[f]$, generated by $f$, is all of $\mathcal{X}$. That is,

$$\mathrm{S}[f] = \overline{\mathrm{span}}\left\{z^\alpha f(z) \;\middle|\; \alpha \in \mathbb{Z}^+(n)\right\} = \overline{\mathrm{span}}\left\{pf \;\middle|\; p \in \mathcal{P}\right\} = \mathcal{X}.$$

By **P1**, it is easy to see that $f \in \mathcal{X}$ is cyclic if and only if $1 \in \mathrm{S}[f]$. It is also easy to see that all cyclic functions are non-vanishing.

**Definition 2.5.** $\mathcal{X}$ has an *envelope of cyclic polynomials* over $D$ if there is a family $\mathcal{F} \subset \mathcal{P}$ of cyclic polynomials such that $\widetilde{D}_{\mathcal{F}} := \bigcap_{q \in \mathcal{F}} \big(\mathbb{C}^n \setminus \mathcal{Z}(q)\big) \subseteq D$, where $\mathcal{Z}(q)$ is the zero-set of $q$.

**Proposition 2.6.** *Suppose $\mathcal{X}$ satisfies* **P1**–**P3** *over an open set $D \subset \mathbb{C}^n$, and also has an envelope of cyclic polynomials with $\mathcal{F} \subset \mathcal{P}$. Let $\Lambda \in \mathcal{X}^*$ be such that $\Lambda(e^{w \cdot z}) \neq 0$ for every $w \in \mathbb{C}^n$. Then, there exist $a \in \mathbb{C} \setminus \{0\}$ and $b \in D$, such that $\Lambda f = a \cdot f(b)$ for all $f \in \mathcal{X}$.*

*Proof.* We only need to show that $b \in D$ since in that case, we get $\Lambda \equiv a\Lambda_b$ on $\mathcal{X}$. For this, let $q \in \mathcal{F}$ be arbitrary and suppose $q(b) = 0$. Since $q$ is cyclic, for every $f \in \mathcal{X}$ we obtain a sequence of polynomials $\{q_k\}_{k \in \mathbb{N}}$ such that $q_k q \to f$. This means

$$0 = a \cdot q_k(b)q(b) = \Lambda(q_k q) \to \Lambda(f).$$

Thus, $\Lambda \equiv 0$ and we get a contradiction. So $q(b) \neq 0$ for every $q \in \mathcal{F}$, and $b \in \widetilde{D}_{\mathcal{F}} \subseteq D$. $\square$

*Example 2.* For $H^p(\mathbb{D}^n)$ when $1 \leq p < \infty$, $\{z_i - \beta \mid 1 \leq i \leq n \text{ and } \beta \notin \mathbb{D}\}$ is an envelope of cyclic polynomials over $\mathbb{D}^n$ (see Example 1). The same set of polynomials works for the Dirichlet-type spaces $\mathcal{D}_\alpha$ when $\alpha \leq 1$. For $\alpha > 1$ and $1 \leq i \leq n$, the polynomial $z_i - w$ is not cyclic in $\mathcal{D}_\alpha$ for all $w \in \mathbb{T}$, and hence the same example does not work. In fact, every $f \in \mathcal{D}_\alpha$ is continuous up to the boundary when $\alpha > 1$. Plus $\Lambda_b$ is a bounded linear functional on $\mathcal{D}_\alpha$ even when $b \in \partial\mathbb{D}^n$. Therefore $\mathcal{D}_\alpha$ cannot have an envelope of cyclic polynomials over $\mathbb{D}^n$. A detailed discussion on cyclicity of polynomials in the Dirichlet-type spaces can be found in [3].

## 3. Maximal Domains

Let us try to make sense of how big the domain of functions in a general space $\mathcal{X}$ that satisfies properties **P1**–**P3** can become without losing the structure we need.

**Definition 3.1.** Given $\mathcal{X}$ satisfying **P1**–**P3** over an open set $D \subset \mathbb{C}^n$, we define the maximal domain of functions in $\mathcal{X}$ to be the set

$$\widehat{D} := \left\{ w \in \mathbb{C}^n \;\middle|\; \Lambda_w p := p(w),\ \forall p \in \mathcal{P} \text{ has a bounded linear extension to } \mathcal{X} \right\}.$$

First, we prove an important property of the maximal domain.

**Theorem 3.2.** *Suppose $\mathcal{X}$ satisfies* **P1**–**P3** *over an open set $D \subset \mathbb{C}^n$. Then, we have*

$$D \subset \widehat{D} \subset \sigma_r(S).$$

*Proof.* $D \subset \widehat{D}$ is obvious from **P2**. To show $\widehat{D} \subset \sigma_r(S)$, let $b \in \widehat{D}$. By **P1**, we get

$$\Lambda_b(e^{w\cdot z}) = e^{w\cdot b} \neq 0 \text{ for all } w \in \mathbb{C}^n.$$

By Theorem 2.1, there exists $\hat{b} \in \sigma_r(S)$ such that $\Lambda_b|_{\mathcal{P}} \equiv \Lambda_{\hat{b}}|_{\mathcal{P}}$. Evaluating both functionals at $z_i$ for each $1 \leq i \leq n$, we get $b = \hat{b} \in \sigma_r(S)$ as needed. □

*Remark.* This shows that the maximal domain is not a very large set, since it is contained in a nice compact set. In the case of $H^p(\mathbb{D}^n)$ for $1 \leq p < \infty$ and $\mathcal{D}_\alpha$ for $\alpha \leq 1$, we saw earlier in Example 2 that $\widehat{D} = \mathbb{D}^n$. However for $\mathcal{D}_\alpha$ when $\alpha > 1$, $\widehat{D} = \overline{\mathbb{D}^n}$. Therefore, both inclusions in the theorem can be proper.

We now show that in general, $\mathcal{X}$ can be identified with a space $\widehat{\mathcal{X}}$ of functions over $\widehat{D}$, which satisfies **Q1**–**Q3**. The following discussion is similar to that of Sect. 5 in [8], where the author talks about the idea of *'algebraic consistency'* and considers a couple different notions of maximal domains. Our notion of maximal domain is different from those discussed in [8], so we will provide all the details here for the sake of completeness.
Let us begin with some notation before proving the identification. For every $f \in \mathcal{X}$, define $\hat{f}(\hat{z}) := \Lambda_{\hat{z}} f$ for every $\hat{z} \in \widehat{D}$ where, with the abuse of notation, we write $\Lambda_{\hat{z}} f$ to represent the extension of $\Lambda_{\hat{z}}|_{\mathcal{P}}$ on $\mathcal{X}$ evaluated at $f$. Notice that for $z \in D$, $\hat{f}(z) = f(z)$ for every $f \in \mathcal{X}$. This also implies $\hat{f}|_D \in \mathrm{Hol}(D)$. Also, for $p \in \mathcal{P}$, we have $\hat{p}(\hat{z}) = p(\hat{z})$ for every $\hat{z} \in \widehat{D}$. Thus, $\hat{\mathcal{P}} := \{\hat{p} \,|\, p \in \mathcal{P}\}$ is the same set as $\mathcal{P}$.
Now, let $\widehat{X} := \{\hat{f} : \widehat{D} \to \mathbb{C} \,|\, f \in \mathcal{X}\}$ and endow it with the natural vector space structure of point-wise addition and scalar multiplication. This can be done because it is obvious that $\hat{f} + \hat{g} = \widehat{f+g}$, and $\alpha\hat{f} = \widehat{\alpha f}$ for every $\alpha \in \mathbb{C}$, $f, g \in \mathcal{X}$.

Define the map $\iota : \mathcal{X} \to \widehat{\mathcal{X}}$ as $\iota(f) := \hat{f}$ for every $f \in \mathcal{X}$. $\iota$ is clearly a vector space isomorphism, and we can define $||\hat{f}||_{\widehat{\mathcal{X}}} := ||f||_{\mathcal{X}}$ for every $\hat{f} \in \widehat{\mathcal{X}}$. This implies $f_k \to f$ in $\mathcal{X}$ if and only if $\hat{f}_k \to \hat{f}$ in $\widehat{\mathcal{X}}$. So, $\widehat{\mathcal{X}}$ turns into a Banach space, and $\iota$ becomes an isometric isomorphism of Banach spaces. Note that since $\widehat{\mathcal{X}}|_D := \left\{\hat{f}|_D \,\middle|\, \hat{f} \in \widehat{\mathcal{X}}\right\} = \mathcal{X}$, we can say that $\widehat{\mathcal{X}}$ is an extension of $\mathcal{X}$ to $\widehat{D}$.

**Proposition 3.3.** *$\widehat{\mathcal{X}}$ satisfies* **Q1** *and* **Q2** *over $\widehat{D}$.*

*Proof.* In order to show **Q1**, first recall that $f_k \to f$ in $\mathcal{X}$ if and only if $\hat{f}_k \to \hat{f}$ in $\widehat{\mathcal{X}}$. Since $\mathcal{P}$ is dense in $\mathcal{X}$ by **P1**, it implies easily that the set of polynomials $\hat{\mathcal{P}}$ is dense in $\widehat{\mathcal{X}}$.
In order to show **Q2**, notice that the map $\Lambda_{\hat{z}}\hat{f} := \hat{f}(\hat{z})$ is bounded for every $\hat{z} \in \widehat{D}$ since

$$|\Lambda_{\hat{z}}\hat{f}| = |\hat{f}(\hat{z})| = |\Lambda_{\hat{z}} f| \leq ||\Lambda_{\hat{z}}||_{\mathcal{X}^*}||f|| = ||\Lambda_{\hat{z}}||_{\mathcal{X}^*}||\hat{f}||.$$

For the second part of **Q2**, suppose for some $\hat{z} \in \mathbb{C}^n$, $\Lambda_{\hat{z}}$ defined as above extends to all of $\widehat{\mathcal{X}}$. As $\mathcal{P}$ and $\hat{\mathcal{P}}$ are identical, we can evaluate $\Lambda_{\hat{z}}$ on polynomials in $\mathcal{P}$ to get

$$|\Lambda_{\hat{z}} p| = |p(\hat{z})| = |\hat{p}(\hat{z})| \leq ||\Lambda_{\hat{z}}||_{\widehat{\mathcal{X}}^*} ||\hat{p}|| \leq ||\Lambda_{\hat{z}}||_{\widehat{\mathcal{X}}^*} ||p||.$$

By **P1**, $\Lambda_{\hat{z}}$ extends to a bounded functional on $\mathcal{X}$, and by definition of $\widehat{D}$, we get $\hat{z} \in \widehat{D}$. □

Instead of showing that $\widehat{\mathcal{X}}$ satisfies **Q3** directly, we will prove a general result about multipliers. Recall that $\phi \in \mathrm{Hol}(D)$ is a multiplier of $\mathcal{X}$, if $\phi f \in \mathcal{X}$ for every $f \in \mathcal{X}$. Denote the set of multipliers by $\mathcal{M}(\mathcal{X})$. It is not difficult to check that $\mathcal{M}(\mathcal{X})$ is a Banach algebra with the norm

$$||\phi||_{\mathcal{M}(\mathcal{X})} := \sup \left\{ ||\phi f|| \;\middle|\; ||f||_{\mathcal{X}} \leq 1 \right\}.$$

As $1 \in \mathcal{X}$, we get that $\mathcal{M}(\mathcal{X}) \subset \mathcal{X}$. Using closed graph theorem, it is easy to check that $\phi$ is a multiplier if and only if multiplication by $\phi$, i.e. $M_\phi : \mathcal{X} \to \mathcal{X}$ defined as $M_\phi f := \phi f$ for every $f \in \mathcal{X}$, is a bounded linear operator on $\mathcal{X}$. Using **P2** and the above equivalence, it is easy to check that $|\phi(z)| \leq ||\phi||_{\mathcal{M}(\mathcal{X})}$ for all $z \in D$ and $\phi \in \mathcal{M}(\mathcal{X})$. Thus, $\mathcal{M}(\mathcal{X}) \subset H^\infty(D)$. With this notation, we have the following result.

**Proposition 3.4.** *$\phi \in \mathcal{M}(\mathcal{X})$ if and only if $\hat{\phi} \in \mathcal{M}(\widehat{\mathcal{X}})$.*

*Proof.* First, note that for every choice of polynomials $p, q$ we have $\widehat{pq} = \hat{p}\hat{q}$. Let $f \in \mathcal{X}$ be arbitrary, and let $\{q_k\}_{k \in \mathbb{N}}$ be a sequence of polynomials that converges to $f$ in $\mathcal{X}$. Then for every $\hat{z} \in \widehat{D}$, since $pq_k \to pf$ implies $\widehat{pq_k} \to \widehat{pf}$, we get

$$\widehat{pf}(\hat{z}) = \lim_{k \to \infty} \widehat{pq_k}(\hat{z}) = \lim_{k \to \infty} \hat{p}(\hat{z})\hat{q}_k(\hat{z}) = \hat{p}(\hat{z}) \lim_{k \to \infty} \hat{q}_k(\hat{z}) = \hat{p}(\hat{z})\hat{f}(\hat{z}).$$

Thus $\hat{p}\hat{f} = \widehat{pf} \in \mathcal{X}$ for every $p \in \mathcal{P}$, $f \in \mathcal{X}$. This implies $\hat{p} \in \mathcal{M}(\widehat{\mathcal{X}})$.

Suppose now that $\phi \in \mathcal{M}(\mathcal{X})$. We already know $\widehat{\phi q} = \hat{\phi}\hat{q}$ for every $q \in \mathcal{P}$. Let $f \in \mathcal{X}$ and suppose again that $q_k \to f$ for some polynomials $q_k$. It is now easy to see for every $\hat{z} \in \widehat{D}$,

$$\widehat{\phi f}(\hat{z}) = \lim_{k \to \infty} \widehat{\phi q_k}(\hat{z}) = \lim_{k \to \infty} \hat{\phi}(\hat{z})\hat{q}_k(\hat{z}) = \hat{\phi}(\hat{z}) \lim_{k \to \infty} \hat{q}_k(\hat{z}) = \hat{\phi}(\hat{z})\hat{f}(\hat{z}).$$

Therefore $\hat{\phi}\hat{f} = \widehat{\phi f} \in \widehat{\mathcal{X}}$ for every $\phi \in \mathcal{M}(\mathcal{X})$, $f \in \mathcal{X}$. This implies $\hat{\phi} \in \mathcal{M}(\widehat{\mathcal{X}})$. The converse is easy since $\hat{\phi}\hat{f} \in \widehat{\mathcal{X}}$ implies there exists $g \in \mathcal{X}$ such that $\hat{\phi}\hat{f} = \hat{g}$. This means $g = \hat{g}|_D = \phi f$ and so, $\phi f \in \mathcal{X}$. Thus $\phi \in \mathcal{M}(\mathcal{X})$ whenever $\hat{\phi} \in \mathcal{M}(\widehat{\mathcal{X}})$. □

**Corollary 3.5.** *$\widehat{\mathcal{X}}$ satisfies* **Q3** *over $\widehat{D}$.*

*Proof.* This follows from Proposition 3.4 as shift operators are multiplication operators. □

Now that the shift operators are bounded, we can talk about cyclic functions in $\widehat{\mathcal{X}}$. However, the way we have defined the norm in $\widehat{\mathcal{X}}$, it is obvious that $f \in \mathcal{X}$ is cyclic if and only if $\hat{f} \in \widehat{\mathcal{X}}$ is cyclic. This and the propositions above prove the following identification theorem.

**Theorem 3.6.** *Given a space $\mathcal{X}$ that satisfies* **P1–P3** *over an open set $D \subset \mathbb{C}^n$, there exists a space $\widehat{\mathcal{X}}$, consisting of functions defined over the maximal domain $\widehat{D}$ of functions in $\mathcal{X}$, that satisfies* **Q1–Q3** *and is isometrically isomorphic to $\mathcal{X}$ with the map $\iota(f) := \hat{f}$, for $f \in \mathcal{X}$.*

*Furthermore $\widehat{\mathcal{X}}|_D := \left\{ \hat{f}|_D \,\middle|\, \hat{f} \in \widehat{\mathcal{X}} \right\} = \mathcal{X}$, and $\widehat{\mathcal{X}}$ has the same set of multipliers and cyclic functions as $\mathcal{X}$. That is, $\phi \in \mathcal{M}(\mathcal{X})$ if and only if $\hat{\phi} \in \mathcal{M}(\widehat{\mathcal{X}})$, and $f$ is cyclic in $\mathcal{X}$ if and only if $\hat{f}$ is cyclic in $\widehat{\mathcal{X}}$.*

With the help of Theorems 2.1 and 3.6, we can easily prove Theorem 1.4.

*Proof of Theorem 1.4.* The proof of this theorem is the same as that of Theorem 2.1 except, by **Q2**, we directly obtain $b \in D$ instead of having to show that $b \in \sigma_r(S)$. □

It should be noted that while Theorem 1.4 is technically not a better result compared to Theorem 2.1, it shows that the point $b$ is not completely arbitrary; functions in $\mathcal{X}$ are well-behaved around $b$, and most of the structure we need can be extended to it.

## 4. Cyclicity Preserving Operators

We have now covered all the preliminaries required to identify all cyclicity preserving operators on these spaces. First, we prove Theorem 1.5.

*Proof of Theorem 1.5.* (2) ⇒ (1) is obvious.
Suppose now that (1) holds. Fix $u \in E$ and define $\Lambda := \Gamma_u \circ T \in \mathcal{X}^*$. Note that for every $w \in \mathbb{C}^n$, as $T(e^{w\cdot z})$ is non-vanishing, we get

$$\Lambda(e^{w\cdot z}) = \Gamma_u\big(T(e^{w\cdot z})\big) = T(e^{w\cdot z})(u) \neq 0.$$

By Theorem 1.4, we get that $\Lambda f = a(u)f(b(u))$ for some $a(u) \in \mathbb{C} \setminus \{0\}$, and $b(u) \in D$.
As the choice of $u \in E$ was arbitrary, we get the functions $a = T(1) \in \mathcal{Y}$ and $b = \frac{T(z)}{T(1)} : E \to D$ as desired. Also, $Tf(u) = a(u)f(b(u))$ for every $u \in E$. □

The only thing we require to identify cyclicity preserving operators is the following lemma.

**Lemma 4.1.** *$e^{w\cdot z}$ is a cyclic multiplier in $\mathcal{X}$ for every $w \in \mathbb{C}^n$.*

*Proof.* Fix $w \in \mathbb{C}^n$. We need to find polynomials $p_k$ so that $||p_k e^{w\cdot z} - 1|| \to 0$ as $k \to \infty$. Let $p_k$ be truncations of the power-series of $e^{-w\cdot z}$. By Lemma 2.2, $p_k \to e^{-w\cdot z}$ in $\mathcal{X}$.

First, we show that $e^{w\cdot z}$ is a multiplier. Let $q_k$ be truncations of the power-series of $e^{w\cdot z}$. Given $f \in \mathcal{X}$, we need to show $e^{w\cdot z}f$ lies in $\mathcal{X}$. Note that by the triangle inequality, we get

$$||q_l f - q_k f|| \leq \left( \sum_{k<|\alpha|\leq l} \frac{|w|^\alpha \, ||S||^\alpha \, ||1||}{\alpha!} \right) ||f||, \text{ for every } k \leq l.$$

Therefore $q_k f$ is a Cauchy sequence and thus, converges to some function $g \in \mathcal{X}$. As $q_k \to e^{w\cdot z}$ point-wise, by **Q2** we get that $q_k f \to e^{w\cdot z} f$, which implies $e^{w\cdot z} \in \mathcal{M}(\mathcal{X})$. This means,

$$\lim_{k\longrightarrow\infty} p_k e^{w\cdot z} = \lim_{k\longrightarrow\infty} M_{e^{w\cdot z}}(p_k) = M_{e^{w\cdot z}}(e^{-w\cdot z}) = 1.$$

That is, $p_k e^{w\cdot z} \to 1$ as $k \to \infty$ and thus, $e^{w\cdot z}$ is cyclic. □

With this in mind, the following is a trivial consequence of Theorem 1.5.

**Theorem 4.2.** (Cyclicity Preserving Operators)

*Let $m, n \in \mathbb{N}$. Suppose $\mathcal{X}$ and $\mathcal{Y}$ satisfy* **Q1**–**Q3** *over $D \subset \mathbb{C}^n$ and $E \subset \mathbb{C}^m$ respectively. Let $T : \mathcal{X} \to \mathcal{Y}$ be such that $Tf$ is cyclic whenever $f$ is cyclic. Then, there exist analytic functions $a \in \mathcal{Y}$ and $b : E \to D$ such that $Tf(u) = a(u)f(b(u))$ for every $u \in E$.*

*Moreover, $a = T(1)$ is cyclic and $b = \frac{T(z)}{T(1)}$, where $T(z) = \big(T(z_i)\big)_{i=1}^n$.*

*Remark.* One can immediately observe in Theorems 1.5 and 4.2, that the spaces $\mathcal{X}$ and $\mathcal{Y}$ may be defined for functions in different number of variables. Also note that for Theorem 4.2, we do not get a proper equivalence easily as in Theorem 1.5 since it is not at all trivial to determine when a weighted composition operator preserves cyclicity.

In the case when $\mathcal{X} = H^p(\mathbb{D}^n)$ and $\mathcal{Y} = H^q(\mathbb{D}^m)$ for some $1 \le p, q < \infty$, we get that all operators that preserve cyclicity are weighted composition operators. The same is true for the Dirichlet-type spaces $\mathcal{D}_\alpha$ when $\alpha \le 1$. When $\alpha > 1$, we need to consider the space over its maximal domain $\overline{\mathbb{D}^n}$.

### 4.1. Cyclicity Preserving Operators on Hardy Spaces

The aim of this subsection is to provide a proof of Theorem 1.6. We will start by showing that the converse of Theorem 4.2 is true whenever $\mathcal{Y} = H^q(\mathbb{D}^m)$ for some $1 \le q < \infty$. We need the following important properties of $\mathrm{S}[f]$, the shift-invariant subspace generated by a function $f \in H^p(\mathbb{D}^n)$.

**Lemma 4.3.** *Let $f \in H^p(\mathbb{D}^n)$ for some $1 \le p < \infty$. Then, $\phi f \in S[f]$ for each $\phi \in H^\infty(\mathbb{D}^n)$.*

*Proof.* For the sake of contradiction, let $\phi f \notin \mathrm{S}[f]$. By the Hahn-Banach theorem, there exists $\Gamma \in (H^p(\mathbb{D}^n))^*$ such that $\Gamma(\phi f) \ne 0$ and $\Gamma|_{\mathrm{S}[f]} \equiv 0$. Since $H^p(\mathbb{D}^n) \subset L^p(\mathbb{T}^n)$ is a closed subspace, by duality of $L^p(\mathbb{T}^n)$ there exists $h \in L^{p'}(\mathbb{T}^n)$ such that for every $g \in H^p(\mathbb{D}^n)$

$$\Gamma(g) = \int_{\mathbb{T}^n} g\overline{h},$$

where $p'$ is the exponent dual to $p$ (see Theorem 7.1 in [4] for more details).

As $\phi$ is the weak*-limit of some sequence of analytic polynomials $p_k$ in $L^\infty(\mathbb{T}^n)$ (take Fejér means, for example), and $f\overline{h} \in L^1(\mathbb{T}^n)$ for $f \in H^p(\mathbb{D}^n)$, we get that

$$\Gamma(\phi f) = \int_{\mathbb{T}^n} \phi f\overline{h} = \lim_{k\longrightarrow\infty} \int_{\mathbb{T}^n} p_k f\overline{h} = 0.$$

The last equality follows from the fact that $p_k f \in \mathrm{S}[f]$ for each $k$, and $\int_{\mathbb{T}^n} g\overline{h} = \Gamma(g) = 0$ for every $g \in \mathrm{S}[f]$. Thus, we reach a contradiction since $\Gamma$ was chosen so that $\Gamma(\phi f) \neq 0$. □

**Proposition 4.4.** *Let $f \in H^p(\mathbb{D}^n)$, $1 \leq p < \infty$. Let $\{f_k\}_{k\in\mathbb{N}} \subset H^\infty(\mathbb{D}^n)$ be such that $f_k f \to g$ for some $g \in \mathcal{X}$. Then, $g \in S[f]$. In particular, if there exists a sequence $\{f_k\}_{k\in\mathbb{N}} \subset H^\infty(\mathbb{D}^n)$ such that $f_k f \to g$ for some cyclic $g \in H^p(\mathbb{D}^n)$, then $f$ is cyclic.*

*Proof.* The first part of the proposition follows easily from Lemma 4.3, since $f_k f \in \mathrm{S}[f]$ for each $k \in \mathbb{N}$, and $\mathrm{S}[f]$ is closed implies $g = \lim_{k\to\infty} f_k f \in \mathrm{S}[f]$.
For the second part, note that $g \in \mathrm{S}[f]$ implies $\mathrm{S}[g] \subset \mathrm{S}[f]$. Since $g$ is assumed to be cyclic, $\mathrm{S}[g] = H^p(\mathbb{D}^n)$ which means $\mathrm{S}[f] = H^p(\mathbb{D}^n)$. Therefore in this case, $f$ is also cyclic. □

The following result follows easily from Theorem 4.2 and Proposition 4.4.

**Theorem 4.5.** *Suppose $\mathcal{X}$ satisfies properties* **Q1**–**Q3** *over $D \subset \mathbb{C}^n$. Let $T : \mathcal{X} \to H^q(\mathbb{D}^m)$ be a bounded linear map for some $1 \leq q < \infty$. Then, the following are equivalent :*

*(1) $T$ preserves cyclicity.*
*(2) $Tf = a \cdot (f \circ b)$, $f \in \mathcal{X}$ for some cyclic $a \in H^q(\mathbb{D}^m)$, and analytic $b : \mathbb{D}^m \to D$.*

*Proof.* (1) ⇒ (2) follows from Theorem 4.2.
For the converse, let $a \in H^q(\mathbb{D}^m)$ and $b : \mathbb{D}^m \to D$ be as in (2). We show that for every cyclic $f \in \mathcal{X}$, $Tf = a \cdot (f \circ b)$ is cyclic in $H^q(\mathbb{D}^m)$.

As $f$ is cyclic, there exist polynomials $p_k$ such that $p_k f \to 1$ in $\mathcal{X}$. Since $T$ is a bounded operator, $T(p_k f) \to T(1)$ in $H^q(\mathbb{D}^m)$. Note that $T(1) = a$ is cyclic and that

$$T(p_k f) = a \cdot (p_k \circ b) \cdot (f \circ b) = (p_k \circ b) \cdot \big(a \cdot (f \circ b)\big).$$

It is easy to see $(p_k \circ b) \in H^\infty(\mathbb{D}^m)$ for each $n$, since the image of $b$ lies in $D \subset \sigma_r(S)$ by Theorem 3.2. From the second part of Proposition 4.4, as

$$(p_k \circ b) \cdot \big(a \cdot (f \circ b)\big) \to a,$$

and $a$ is cyclic, we get that $Tf = a \cdot (f \circ b)$ is cyclic in $H^q(\mathbb{D}^m)$. Thus, (2) ⇒ (1). □

*Remark 4.6.* (*i*) The proof of (2) ⇒ (1) relies on Proposition 4.4, which further relies on the fact that the dual of $L^p(\mathbb{T}^n)$ for $1 \leq p < \infty$ is $L^{p'}(\mathbb{T}^n)$ where $1/p + 1/p' = 1$ and thus, does not translate easily to other general spaces of analytic functions.

(*ii*) Note that the proof of Theorem 1.2 and Theorem 2 in [10] uses the canonical factorization theorem for Hardy spaces on the unit disc $\mathbb{D}$ (Theorem 2.8, [4]). We do not have such a result when $n > 1$ (see Sect. 4.2 in [14]), hence a different approach was needed.

(*iii*) Recall that Theorem 1.2 does not require boundedness of $T$ for the proof of (1) ⇒ (2) to work when $\mathcal{X} = H^p(\mathbb{D})$. Plus, Theorem 1.2 is valid even for $0 < p < 1$. This is because its proof also depends on the canonical factorization theorem as mentioned above.

(*iv*) We will see later in this section that (1) $\Rightarrow$ (2) is still valid for the case when $\mathcal{X} = H^p(\mathbb{D}^n)$ and $\mathcal{Y} = H^q(\mathbb{D}^m)$ for $0 < p, q < 1$ even though they are not Banach spaces. The case $p, q = \infty$ shall be treated separately as well since $H^\infty(\mathbb{D}^n)$ is not separable and hence the standard notion of cyclicity does not make any sense.

We now show that the assumption '$T$ is a bounded operator' can be dropped in a specific case for the Hardy spaces. First, we need the following fact about boundedness of certain composition operators.

**Proposition 4.7.** *For $1 \le p < \infty$ and a given analytic function $b : \mathbb{D}^m \to \mathbb{D}$, the map $T : H^p(\mathbb{D}) \to H^p(\mathbb{D}^m)$ defined as $Tf := f \circ b$ is a well-defined bounded linear operator.*

*Proof.* First, we show that $f \circ b \in H^p(\mathbb{D}^m)$ for every $f \in H^p(\mathbb{D})$, which shows $T$ is well-defined. The linearity of $T$ is immediate after that. We use the existence of harmonic majorants for functions in the Hardy spaces and their properties for the rest of the proof. See Sect. 3.2 in [14] for more details. The argument here is inspired by the one given in the corollary of Theorem 2.12 in [4] for the case $m = 1$.
Let $U$ be the smallest harmonic majorant of $|f|^p$, i.e. the Poisson integral of $|f(e^{i\theta})|^p$,

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta - t)\, |f(e^{it})|^p\, dt, \text{ where } P(r, \theta) := \operatorname{Re}\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right).$$

Then, $|f(u)|^p \le U(u)$ for all $u \in \mathbb{D}$, which implies $|Tf(z)|^p \le U\big(b(z)\big)$ for every $z \in \mathbb{D}^m$.

Since $U$ is harmonic, $U = \operatorname{Re}(g)$ for some analytic function $g : \mathbb{D} \to \mathbb{C}$. This means that $U \circ b = \operatorname{Re}(g \circ b)$ is an $m$-harmonic function and thus, a harmonic majorant for $|Tf|^p = |f \circ b|^p$. This proves that $f \circ b \in H^p(\mathbb{D}^m)$ and so, $T$ is well-defined.
To show $T$ is bounded, observe that

$$M_p(r, f \circ b)^p \le U(b(0)) \le \left(\frac{1 + |b(0)|}{1 - |b(0)|}\right) ||f||^p,$$

where

$$M_p(r, f \circ b) := \left(\int\limits_{r\mathbb{T}^m} |f \circ b|^p d\sigma_m\right)^{\frac{1}{p}}.$$

The first inequality follows from the mean value property of $m$-harmonic functions, and the second inequality follows from the fact that $P(r, \theta) \le (1 + r)/(1 - r)$ for all values of $r$ and $\theta$. Taking supremum over $r$ in the above inequality, we get

$$||f \circ b|| \le \left(\frac{1 + |b(0)|}{1 - |b(0)|}\right)^{\frac{1}{p}} ||f|| \text{ for every } f \in H^p(\mathbb{D}).$$

Thus, $T$ is bounded. □

**Theorem 4.8.** *Fix $1 \leq p < \infty$ and let $T : H^p(\mathbb{D}) \to H^p(\mathbb{D}^m)$ be a linear map such that $T1 = 1$. Then, the following are equivalent :*

*(1) $T$ is a bounded linear map that preserves cyclicity.*

*(2) $Tf = f \circ b$, $f \in H^p(\mathbb{D})$ for some analytic $b : \mathbb{D}^m \to \mathbb{D}$.*

*Proof.* As before, (1) ⇒ (2) follows directly from Theorem 4.5.
For the converse, let $b : \mathbb{D}^m \to \mathbb{D}$ be an analytic function such that $Tf = f \circ b$ for each $f \in H^p(\mathbb{D})$. By Proposition 4.7, $T$ is a bounded linear operator. (2) ⇒ (1) in Theorem 4.5 shows that $T$ preserves cyclicity. □

*Remark 4.9.* Note that the only place we use that the domain of $H^p(\mathbb{D})$ is in one variable, is to show boundedness of $f \mapsto f \circ b$ for every $b : \mathbb{D}^m \to \mathbb{D}$. More precisely, we use the fact that any harmonic function $U$ in one variable is the real part of some holomorphic function. This is not true for $n > 1$ (see Sect. 2.4 in [14]).

As mentioned in Remark (iv) under Theorem 4.5, we now consider the cases $0 < p < 1$ and $p = \infty$. First, we address the case $\mathcal{X} = H^p(\mathbb{D}^n)$ for $0 < p < 1$.

*Example 3.* $(0 < p < 1)$ Note that $H^p(\mathbb{D}^n)$ satisfies **P1**–**P3** if we replace boundedness with continuity. The issue is that $H^p(\mathbb{D}^n)$ is not a Banach space. Even though $H^p(\mathbb{D}^n)$ is not normable, it is still a complete metric space under the metric $d_p(f, g) := ||f - g||_p^p$ where $|| \cdot ||_p$ is as defined in Sect. 1. Using this, its bounded linear functionals can be defined in the usual manner. That is, we say that $\Lambda : H^p(\mathbb{D}^n) \to \mathbb{C}$ is bounded if

$$||\Lambda|| := \sup_{||f||_p=1} |\Lambda(f)| < \infty.$$

This means that $|\Lambda(f)| \leq ||\Lambda|| \cdot ||f||$ for all bounded $\Lambda$, and $f \in H^p(\mathbb{D}^n)$. It is easy to verify that this notion of boundedness is equivalent to the continuity of $\Lambda$. Similarly, we say an operator $T : H^p(\mathbb{D}^n) \to H^q(\mathbb{D}^m)$ for some $0 < q \leq \infty$ is bounded if

$$||T|| := \sup_{||f||_p=1} ||Tf||_q < \infty.$$

As was the case with linear functionals, it is easy to verify that this notion of boundedness is equivalent to the continuity of $T$. This implies that for all bounded linear operators $T$ on $H^p(\mathbb{D}^n)$ and $f \in H^p(\mathbb{D}^n)$, $||Tf|| \leq ||T|| \cdot ||f||$. So, Lemmas 2.2 and 4.1 hold for $H^p(\mathbb{D}^n)$ even when $0 < p < 1$. In order to show that Theorem 4.2 holds for $\mathcal{X} = H^p(\mathbb{D}^n)$, we only need to show that Theorem 1.4 holds since the arguments in the proof of Theorem 4.2 do not rely on the Banach space structure of $\mathcal{X}$ except when Theorem 1.4 is applied. First, we show that the maximal domain for functions in $H^p(\mathbb{D}^n)$ when $0 < p < 1$ is also $\mathbb{D}^n$.
We will show as in Example 2 that the family $\mathcal{F} := \{z_i - \beta \mid 1 \leq i \leq n,\ \beta \notin \mathbb{D}\}$ is an envelope of cyclic polynomials in $H^p(\mathbb{D}^n)$ for $0 < p < 1$. Let $b \in \overline{\mathbb{D}^n}$ be such that $\Lambda_b|_{\mathcal{P}}$ extends to a bounded linear functional $\Lambda \in H^p(\mathbb{D}^n)$. Thus $b_j \in \mathbb{T}$ for some $1 \leq j \leq n$.

It is known that outer functions are cyclic in $H^p(\mathbb{D})$ for $0 < p < 1$ (see Theorem 4, [5]). This implies $z - b_j$ is cyclic in $H^p(\mathbb{D})$ and thus, $q(z) :=$

$z_j - b_j$ is cyclic in $H^p(\mathbb{D}^n)$. Clearly $\mathcal{F}$ defined above is then an envelope of cyclic polynomials. This means that for any given $f \in H^p(\mathbb{D}^n)$, there exists a sequence of polynomials $\{p_k\}_{k\in\mathbb{N}}$ such that $p_k q \to f$. Since $q(b) = 0$, we get

$$\Lambda(f) = \lim_{k\longrightarrow\infty} \Lambda(p_k q) = \lim_{k\longrightarrow\infty} p_k(b)q(b) = 0.$$

This means $\Lambda \equiv 0$, a contradiction. So, $b \in \mathbb{D}^n$ and we get $\widehat{D} = \mathbb{D}^n$.
Notice that the only other place we use the norm in the proof of Theorem 2.1 (and hence Theorem 1.4) is to obtain the non-vanishing entire function $F(w)$ using

$$|\Lambda(z^\alpha)| \le ||\Lambda|| \cdot ||z^\alpha|| \le ||\Lambda|| \cdot ||S_1||^{\alpha_1} \dots ||S_n||^{\alpha_n} \cdot ||1||, \text{ for every } \alpha \in \mathbb{Z}^+(n).$$

As we saw above, this should not be an issue for $H^p(\mathbb{D}^n)$ since $||\Lambda||$ makes just as much sense and $||z^\alpha|| = 1$ for all $\alpha \in \mathbb{Z}^+(n)$. This gives us $|\Lambda(z^\alpha)| \le ||\Lambda||$, which is good enough for the rest of the proof to work. Therefore Theorem 1.4 holds for $\mathcal{X} = H^p(\mathbb{D}^n)$ and $\mathcal{Y} = H^q(\mathbb{D}^m)$, and so does Theorem 4.2 even when $0 < p, q < 1$.

*Example 4.* ($p = \infty$) $H^\infty(\mathbb{D}^n)$ is different from Example 3 as it is a Banach space, but it does not satisfy **Q1** over $\mathbb{D}^n$. In fact $H^\infty(\mathbb{D}^n)$ is not separable, so cyclicity of functions does not make sense. Since outer functions (see Definition 1.1) do make sense for $n \ge 1$ and $0 < p \le \infty$, we can talk about outer functions instead of cyclic functions in this case.

Note that for $p = \infty$, the hypothesis of Theorem 1.4 does not make sense. In fact, we will completely avoid using maximal domains for $H^\infty(\mathbb{D}^n)$ since without cyclicity, we cannot even determine if $\widehat{D} \subset \sigma_r(S)$. Instead, consider $\Lambda \in \big(H^\infty(\mathbb{D}^n)\big)^*$ such that $\Lambda(f) \ne 0$ for all outer functions $f \in H^\infty(\mathbb{D}^n)$. Since $e^{w\cdot z}$ is an outer function for all $w \in \mathbb{C}^n$, we proceed as in the proof of Theorem 2.1 to obtain $\Lambda|_{\mathcal{P}} \equiv a\Lambda_b|_{\mathcal{P}}$ for some $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}^n$.
Now, proceed as in Example 2 and instead of having an envelope of cyclic polynomials, we have an envelope of outer polynomials which is the same set $\{z_i - \beta \mid 1 \le i \le n, \beta \notin \mathbb{D}\}$. Since $\Lambda(f) \ne 0$ for all outer functions $f$, we get $b_i - \beta \ne 0$ for all $1 \le i \le n$ and $\beta \notin \mathbb{D}$ which implies $b_i \in \mathbb{D}$ for every $1 \le i \le n$. Therefore, $b \in \mathbb{D}^n$ and $\Lambda f = a \cdot f(b)$ for all $f \in H^\infty(\mathbb{D}^n)$.

Thus, the conclusion of Theorem 2.1 is valid for $H^\infty(\mathbb{D}^n)$ if we consider all $\Lambda \in \big(H^\infty(\mathbb{D}^n)\big)^*$ that act on outer functions as above and so, Theorem 4.2 is valid for $\mathcal{X} = H^\infty(\mathbb{D}^n)$ if we replace cyclic functions with outer functions. A similar logic can be applied to operators that preserve outer functions in $H^p(\mathbb{D}^n)$ for $0 < p < \infty$.

This discussion about Hardy spaces above yields the proof of Theorem 1.6.

*Proof of Theorem 1.6.* (1) For $1 \le p, q < \infty$, this follows from Theorems 4.2 and 4.5. For $0 < p < 1$ or $0 < q < 1$, this follows from the discussion in Example 3.

(2) This follows from the discussion in Example 4. □

*Remark 4.10.* (*i*) Note that the proof of Proposition 4.4 above is not valid for $0 < q < 1$ or $q = \infty$ since we use the duality of $L^q(\mathbb{T}^m)$ when

$1 \leq q < \infty$. Therefore, we do not obtain a result like Theorem 4.5 when $0 < q < 1$ or $q = \infty$. Theorem 1.6 is probably the best we can expect in these cases with our techniques.

(*ii*) If all bounded weighted composition operators preserve outer functions as well, we get a kind of '*linear rigidity*' between outer and cyclic functions. The following result shows that this is not the case when $n > 1$.

**Theorem 4.11.** *Let $0 < q < 1/2$. There exists a bounded linear map $T : H^2(\mathbb{D}^2) \to H^q(\mathbb{D})$ such that it preserves cyclicity, but not outer functions.*

*Proof.* This example is from [14] but it was used in a different context; to obtain an outer function in $H^2(\mathbb{D}^2)$ which is not cyclic. We refer the reader to the discussion surrounding Theorem 4.4.8 in [14] for details on the facts mentioned below.
Fix $0 < q < 1/2$. Let $T : H^2(\mathbb{D}^2) \to H^q(\mathbb{D})$ be defined as

$$Tf(z) = f\left(\frac{1+z}{2}, \frac{1+z}{2}\right) \text{ for every } z \in \mathbb{D} \text{ and } f \in H^2(\mathbb{D}^2).$$

$T$ is a bounded linear operator that preserves cyclicity (Theorem 4.4.8 (a), [14]). Also, $f \in H^2(\mathbb{D}^2)$ defined below is outer (Theorem 4.4.8 (b), [14]), but $Tf$ is not.

$$f(z_1, z_2) = \exp\left(\frac{z_1 + z_2 + 2}{z_1 + z_2 - 2}\right).$$

$$Tf(z) = \exp\left(\frac{z+3}{z-1}\right) = \frac{1}{e} \cdot \left(\exp\left(\frac{z+1}{z-1}\right)\right)^2.$$

Therefore, $T$ does not preserve outer functions. □

It would be interesting to characterize all weighted composition operators that preserve outer functions since it might help us understand the difference between outer and cyclic functions when $n > 1$.
(*iii*) Notice that the proof of (1) $\Rightarrow$ (2) in Theorems 1.5 and 4.2 depends mostly on the properties of $\mathcal{X}$, since $\mathcal{Y}$ can be chosen to be fairly general. On the other hand, all the discussion about Hardy spaces shows that the proof of (2) $\Rightarrow$ (1) depends on the properties of $\mathcal{Y}$. In Proposition 4.4, we saw that the proof relies heavily on the properties of $H^p(\mathbb{D}^n)$ and might not work for other spaces. This shows that it is not completely obvious what properties $\mathcal{Y}$ needs to have generally in order for the converse of Theorem 4.2 to hold.

## 5. GKŻ-type theorem for spaces of analytic functions

To show some different application of the abstract results proved in Sects. 2 and 3, we conclude our discussion by proving a GKŻ-type theorem (Theorem 1.7) for spaces of analytic functions. The following result was proved independently by A. M. Gleason (Theorem 1, [7]), and J.-P. Kahane and W. Żelazko (Theorem 1, [9]) for commutative Banach algebras. Żelazko extended the result to non-commutative Banach algebras shortly after in [15].

**Theorem 5.1.** *Let $\mathcal{B}$ be a complex unital Banach algebra, and let $\Lambda \in \mathcal{B}^*$ be such that $\Lambda(1) = 1$. Then, $\Lambda(ab) = \Lambda(a)\Lambda(b)$ for every $a, b \in \mathcal{B}$ if and only if $\Lambda(a) \neq 0$ for every $a$ which is invertible in $\mathcal{B}$.*

We shall prove a similar result about *partially multiplicative linear functionals* on spaces of analytic functions as an interesting byproduct of the topics discussed in Sects. 2 and 3.

**Definition 5.2.** Suppose $\mathcal{X}$ is a space of functions that satisfies **Q1**–**Q3** over $D \subset \mathbb{C}^n$. We will consider two types of partially multiplicative linear functionals $\Lambda \in \mathcal{X}^*$ as follows.

**M1** For every $\phi \in \mathcal{M}(\mathcal{X})$, $f \in \mathcal{X}$ we have $\Lambda(\phi f) = \Lambda(\phi)\Lambda(f)$.
**M2** For every $f, g \in \mathcal{X}$ such that $fg \in \mathcal{X}$ we have $\Lambda(fg) = \Lambda(f)\Lambda(g)$.

Note that **M2** $\Rightarrow$ **M1**, but it is not obvious if the converse is true in general.

Theorem 1.7 states that when $\mathcal{X}$ satisfies **Q1**–**Q3** over its maximal domain $D \subset \mathbb{C}^n$, both **M1** and **M2** are equivalent. Not only that, but they are precisely the set of point evaluations on $D$, and can be identified by their action on a certain set of exponentials. The proof of this theorem is easy and follows from Theorem 1.4.

*Proof of Theorem 1.7.* $(i) \Rightarrow (ii)$ follows from Theorem 1.4, and $(ii) \Rightarrow (iii) \Rightarrow (iv)$ is obvious from Definition 5.2.
For the proof of $(iv) \Rightarrow (i)$, assume $\Lambda$ is **M1** and note that $e^{w\cdot z} \in \mathcal{M}(\mathcal{X})$ for every $w \in \mathbb{C}^n$. Thus, for every $w \in \mathbb{C}^n$, we get

$$\Lambda(e^{w\cdot z})\Lambda(e^{-w\cdot z}) = \Lambda(e^{w\cdot z} \cdot e^{-w\cdot z}) = \Lambda(1) = 1.$$

Therefore, $\Lambda(e^{w\cdot z}) \neq 0$ for every $w \in \mathbb{C}^n$ as required. □

This shows that all reasonable notions of partially multiplicative linear functionals align when we consider these nice spaces of analytic functions. A similar result for reproducing kernel Hilbert spaces with complete Pick property was recently proved (Corollary 3.4, [2]). It was shown that in the case of a complete Pick space, **M1** and **M2** are equivalent. It should be noted that this is not a special case of Theorem 1.7 since it covers Hilbert spaces of functions that are not necessarily analytic. On the other hand, Theorem 1.7 covers certain Banach spaces of analytic functions and not just Hilbert spaces.
It is worth mentioning that, just as we devised a maximal domain from point evaluations on polynomials that extend to $\mathcal{X}$, one can construct a different notion of maximal domain from **M1** and **M2**. We end this section by showing that our notion of maximal domain can also be identified with some form of partially multiplicative functionals.
Suppose $\mathcal{X}$ satisfies **P1**–**P3** over an open set $D \subset \mathbb{C}^n$. We say $\Lambda \in \mathcal{X}^*$ is **M0** if it satisfies the following property.

**M0** For every $p, q \in \mathcal{P}$ we have $\Lambda(pq) = \Lambda(p)\Lambda(q)$.

**Proposition 5.3.** *$\Lambda$ is* **M0** *if and only if $\Lambda|_{\mathcal{P}} \equiv \Lambda_b|_{\mathcal{P}}$ for some $b \in \widehat{D}$.*

*Proof.* If $\Lambda$ is **M0**, then $\Lambda(z_i^k) = \big(\Lambda(z_i)\big)^k$ for all $1 \leq i \leq n$ and $k \in \mathbb{N}$. Pick $b = \big(\Lambda(z_i)\big)_{i=1}^n$ and note that $\Lambda(p) = p(b)$ for all $p \in \mathcal{P}$. As $\Lambda \in \mathcal{X}^*$, and $\mathcal{X}$ satisfies **P1**, this means $\Lambda|_{\mathcal{P}}$ extends to $\mathcal{X}$. Thus $b \in \widehat{D}$, and $\Lambda|_{\mathcal{P}} \equiv \Lambda_b|_{\mathcal{P}}$ as required. The converse is trivial. □

Depending on what properties we want the extension $\widehat{\mathcal{X}}$ to have, we may want to choose between **M0**–**M2**. For more details, refer to Sect. 2 in [12], and Sect. 5 in [8].

**Acknowledgements**

This paper would not have been possible without the guidance of my PhD advisor, Greg Knese. I would like to thank Greg for providing partial financial support as well, through his NSF grant DMS-1900816. I would also like to thank John McCarthy, Brett Wick, and my colleagues Alberto Dayan and Christopher Felder for discussions on topics covered in this paper, and mathematics in general. I am also grateful for the suggestions provided by the referee, who reviewed this paper, to help improve it.

**Publisher's Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

## References

[1] Ahlfors, L.V.: Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable. New York, London, 177 (1953)

[2] Aleman, A., Hartz, M., McCarthy, J.E., Richter, S.: The Smirnov class for spaces with the complete Pick property. J. Lond. Math. Soc. **96**(1), 228–242 (2017)

[3] Bénéteau, C., Knese, G., Kosiński, Ł., Liaw, C., Seco, D., Sola, A.: Cyclic polynomials in two variables. Trans. Am. Math. Soc. **368**(12), 8737–8754 (2016)

[4] Duren, P.L.: Theory of $H^p$ Spaces. Academic Press, London (1970)

[5] Gamelin, T.W.: $H^p$ spaces and extremal functions in $H^1$. Trans. Am. Math. Soc. **124**(1), 158–167 (1966)

[6] Gibson, P.C., Lamoureux, M.P.: Constructive solutions to Pólya–Schur problems. J. Funct. Anal. **269**(10), 3264–3281 (2015)

[7] Gleason, A.M.: A characterization of maximal ideals. J. d'Analyse Math. **19**(1), 171–172 (1967)

[8] Hartz, M.: On the isomorphism problem for multiplier algebras of Nevanlinna-Pick spaces. Can. J. Math. **69**(1), 54–106 (2017)

[9] Kahane, J., Żelazko, W.: A characterization of maximal ideals in commutative Banach algebras. Studia Math. **29**, 339–343 (1968)

[10] Kou, K.I., Liu, J.: Which linear operators preserve outer functions? Indagationes Math. **28**(6), 1144–1156 (2017)

[11] Mashreghi, J., Ransford, T.: A Gleason–Kahane–Żelazko theorem for modules and applications to holomorphic function spaces. Bull. Lond. Math. Soc. **47**(6), 1014–1020 (2015)

[12] McCarthy, J.E., Shalit, O.M.: Spaces of Dirichlet series with the complete Pick property. Israel J. Math. **220**(2), 509–530 (2017)

[13] Nikolski, N.: In a shadow of the RH: Cyclic vectors of Hardy spaces on the Hilbert multidisc. Annales de l'Institut Fourier **62**(5), 1601–1626 (2012) (en)

[14] Rudin, W.: Function Theory in Polydiscs, vol. 41. WA Benjamin (1969)

[15] Żelazko, W.: A characterization of multiplicative linear functionals in complex Banach algebras. Studia Math. **1**(30), 83–85 (1968)

Jeet Sampat (✉)
Department of Mathematics and Statistics
Washington University in St. Louis
One Brookings Drive
St. Louis MO63130
USA
e-mail: sampatjeet@wustl.edu

Received: October 15, 2020.
Revised: January 27, 2021.