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Heat kernel analysis on diamond fractals

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Abstract

This paper presents a detailed analysis of the heat kernel on an $(\mathbb{N}\times\mathbb{N})$-parameter family of compact metric measure spaces which do not satisfy the volume doubling property. In particular, uniform bounds of the heat kernel, its Lipschitz continuity and the continuity of the corresponding heat semigroup are studied; a specific example is presented revealing a logarithmic correction. The estimates are applied to derive functional inequalities of interest in describing the convergence to equilibrium of the diffusion process.

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Keywords: Heat kernel; Diffusion process; Heat semigroup; Dirichlet form; Inverse limit space; Fractals

1. Introduction

The present paper investigates the behavior of intrinsic heat diffusion processes in *generalized diamond fractals* through the study of their associated *heat kernel*. These fractals constitute a parametric family of compact metric measure spaces that arises as a generalization of a hierarchical lattice model appearing in the physics and geometry literature [1,23,33]. With a structure reminiscent of the scale irregular fractals treated in [11], they present some additional non-standard geometric features that make them a relevant object of study. Specially because diamond fractals happen to admit a heat kernel with a rather explicit expression [2], they are most suitable to analyze non-standard model behaviors.

Due to their wide range of applications, there is an extensive literature concerning the investigation of heat kernels from different points of view [19,20,22]. In this paper, special

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attention is paid to the rich interplay between analysis, probability and geometry that comes to light through the study of functional inequalities and estimates related to them, see e.g. [7,31] and references therein. One of the main reasons to investigate this type of question in the particular setting of generalized diamond fractals is that these spaces, which may be described via inverse limits of metric measure graphs, see Fig. 1, lack regularity properties such as *volume doubling* or *uniformly bounded degree*, that are often assumed in the literature [9,16,17].

One of the aims of the paper is thus to set the starting point of a larger research program, where diamond fractals may be considered as model spaces towards a classification of inverse limit spaces in terms of their heat semigroup properties. On the one hand, this would contribute to the existing research carried out by Cheeger–Kleiner from a more purely geometric point of view in [15,16]. On the other hand, some of this analysis may transfer to direct limits of metric measure graphs, so-called *fractal quantum graphs* [4].

In order to investigate how the measure-geometric properties of diamond fractals are reflected in the analysis of the diffusion process, the Lipschitz continuity of the heat kernel p_t and the heat semigroup $\{P_t\}_{t\geq 0}$ play a central role in this paper. Heat kernel estimates were discussed in [23, Section 4] for a particular class of diamond fractals, however Lipschitz estimates remained unexplored. Dealing with this rather non-standard setting makes much of the general abstract theory not directly applicable, and being able to work with explicit expressions becomes crucial to approach its analysis. As an example, on a (regular) diamond fractal with parameters n and j, Corollary 4.2 provides the estimate

$$\frac{1}{\sqrt{4\pi}}t^{-1/2}\leq \|p_t\|_\infty\leq \frac{1}{2\pi}+\frac{1}{\sqrt{4\pi}}t^{-1/2}+C_{n,j}\,t^{-\frac{1}{2}\left(1+\frac{\log n}{\log j}\right)}$$

with a constant $C_{n,j}$ that can be explicitly bounded. Continuity estimates of the heat semigroup are deeply connected to the geometry of the underlying space, displayed for instance in so-called *Bakry–Émery type curvature conditions*. In the classical setting of a complete and connected Riemmanian manifold, such a condition can be expressed as an inequality involving the gradient of the semigroup that is known to be equivalent to a bound of the Ricci curvature of the space [6,28,30]. In recent years, a significant amount of research has been carried out to characterize curvature bounds in the context of Dirichlet spaces with sub-Gaussian heat kernel estimates or not strictly local by means of weak versions of the original Bakry–Émery condition, see e.g. [5,29,32].

This type of connection with curvature is approached in the present setting by investigating the regularity of the heat semigroup and its relation to the so-called *weak Barky–Émery nonnegative curvature condition* recently introduced in the framework of Dirichlet spaces with sub-Gaussian heat kernel bounds [3]. The most concrete computable case presented in this paper, see Theorem 5.6, reveals a logarithmic correction term

$$|P_tf(x)-P_tf(y)|\leq C\frac{|\log t|}{\sqrt{t}}d(x,y)\|f\|_\infty,\qquad 0<t<1,$$

which reflects the inhomogeneous nature of diamond fractals that allows the measure to be very different at different points. This type of phenomenon is observed in diffusion processes with multifractal structures, see e.g. [12].

The paper is organized as follows: Section 2 briefly reviews of the construction of generalized diamond fractals as inverse limits carried out in [2] and gives some basic metric properties. Section 3 investigates potential theoretical aspects of the diffusion process and its relation with the inverse limit structure in terms of the infinitesimal generator and the Dirichlet form. The main results of the paper are concentrated in Sections 4 and 5. Theorem 4.1 provides a general

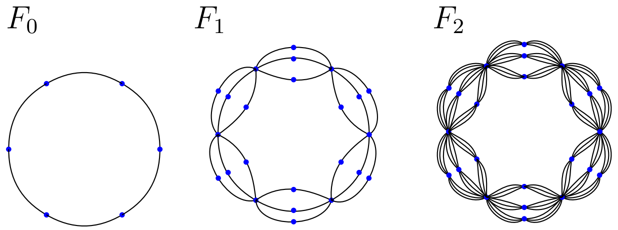


Fig. 1. Approximations of a diamond fractal with $j_1 = n_1 = 3$, $j_2 = 2, n_2 = 3$.

uniform estimate of the heat kernel, whereas Theorems 4.3 and 5.1 deal with the Lipschitz continuity of the heat kernel and the heat semigroup, respectively. To better illustrate their time-dependence, the results are applied to a class of diamond fractals for which computations become more tractable, cf. Theorem 5.6. Section 6 outlines further applications of the estimates to study logarithmic Sobolev, ultracontractivity and Poincaré inequalities. It is noteworthy to point out that generalized diamond fractals do not satisfy the *elliptic Harnack inequality*. This was proved in [23] for a (self-similar) diamond fractal and is in general a direct consequence of the fact these spaces are not *metric doubling*, see the recent result [13, Theorem 3.11] and references therein.

2. Generalized diamond fractals

This section summarizes the construction and some key results concerning the natural diffusion process associated with a generalized diamond fractal. We refer to [2] for more details. Lemma 2.3 and Theorem 2.4 restate crucial facts about the heat semigroup and the heat kernel that are essential to the analysis carried out subsequently.

2.1. Inverse limit construction

A diamond fractal arises from a sequence of metric measure graphs and is characterized by two parameter sequences $\mathcal{J} = \{j_i\}_{i\geq 0}$ and $\mathcal{N} = \{n_i\}_{i\geq 0}$ that describe its construction, see Fig. 1. Each sequence indicates, respectively, the number of new vertices added from one graph to its next generation, and the number of additional edges given to each vertex.

Definition 2.1. Let $\mathcal{J} = \{j_\ell\}_{\ell\geq 0}$, $\mathcal{N} = \{n_\ell\}_{\ell\geq 0}$ be sequences with $j_0 = 1 = n_0$ and $j_\ell, n_\ell \geq 2$ for all $\ell \geq 1$. Set $J_0 = N_0 = 1$ and define for any $0 \leq k \leq i$

$$J_{k,i} := \prod_{\ell=k}^{i} j_\ell, \qquad N_{k,i} := \prod_{\ell=k}^{i} n_\ell.$$

In particular, we write $J_i := J_{0,i}$ and $N_i := N_{0,i}$.

The inverse system associated with a diamond fractal is built upon a sequence of metric measure spaces (F_i, d_i, μ_i) that can be defined inductively in the following manner.

Definition 2.2. Let F_0 denote the unit circle and $\vartheta_0 := \{0, \pi\}$, $B_0 := \vartheta_0$. For each $i \geq 1$, set $\vartheta_i := \left\{\frac{\pi k}{J_i} \mid 0 < k < 2J_i,\ k \bmod j_i \not\equiv 0\right\}$ and

$$B_i := B_{i-1} \cup (\vartheta_i \times [n_1] \times \cdots \times [n_{i-1}]) \qquad i \geq 2,$$

$$\begin{tikzcd}
F_0 & F_1 \arrow[l] & \cdots & F_\ell \arrow[l] & \cdots & F_i \arrow[l] \arrow[ll, bend left, "\phi_{i\ell}"] & \cdots & F_k \arrow[l] \arrow[llll, bend right, "\phi_{k\ell}"'] \arrow[ll, bend left, "\phi_{ki}"] & \cdots & F_\infty \arrow[l] \arrow[ll, bend left, "\Phi_k"]
\end{tikzcd}$$

Fig. 2. Projective system structure.

where $[n_k] = \{1, \ldots, n_k\}$ and $B_1 := B_0 \cup \vartheta_1$. For each $i \geq 1$, define the quotient $F_i := F_{i-1} \times [n_i] \,/ \overset{i}{\sim}$, where $xw \overset{i}{\sim} x'w'$ if and only if $x, x' \in B_i$.

The set B_i contains the identification (branching, junction) points that yield F_i and satisfies $B_i \subseteq F_{i-1}$, see marked dots in Fig. 1. As a metric measure graph, each F_i can be regarded as the union of branches (i-cells) isomorphic to intervals of length π/J_i that suitably connect the vertices in B_{i-1}. The measure μ_i is obtained by redistributing the mass of each branch in the previous level uniformly between its "successors". The corresponding (geodesic) distance d_i on F_i coincides with the Euclidean metric on each branch.

Definition 2.2 can be used to produce a family of measurable mappings $\phi_{ik} \colon F_i \to F_k$, $0 \leq k \leq i$, such that the sequence $\{(F_i, \mu_i, \{\phi_{ik}\}_{k \leq i})\}_{i \geq 0}$ defines an inverse (projective) system of measure spaces. We refer to [2, Section 2] for a detailed construction, summarized in Fig. 2.

A generalized diamond fractal of parameters $\mathcal{J}$ and $\mathcal{N}$ arises as the inverse (projective) limit of the above-mentioned inverse system. The limit space (F_∞, μ_∞) is equipped with measurable "projection mappings", $\Phi_i \colon F_\infty \to F_i$, that play a major role in the construction of the associated diffusion process. To fully realize a diamond fractal as a metric measure space, we discuss briefly the metric that naturally comes along with the inverse limit construction.

2.2. Metric remarks

By definition, the graphs F_i are equipped with the geodesic metric d_i induced by the Euclidean on each edge. The following observation describes how metrics in different levels are related by means of the mappings ϕ_{ik} and justifies the definition of the metric on the limit space F_∞. For the ease of the notation, we write $\phi_i := \phi_{i(i-1)} \colon F_i \to F_{i-1}$ for each $i \geq 1$.

Lemma 2.1. *For any $i \geq 1$ and $x, y \in F_i$,*

(i) $d_{i-1}(\phi_i(x), \phi_i(y)) \leq d_i(x, y) \leq d_{i-1}(\phi_i(x), \phi_i(y)) + 2\pi/J_i$;

(ii) $d_k(\phi_{ik}(x), \phi_{ik}(y)) \leq d_i(x, y)$ *for any* $0 \leq k \leq i$;

(iii) *there exist* $z_1, \ldots, z_{m_{xy}} \in B_i$ *with* $1 \leq m_{xy} \leq J_i$ *and such that*

$$d_i(x, y) = d_i(x, z_1) + \sum_{\ell=1}^{m_{xy}-1} d_i(z_\ell, z_{\ell+1}) + d_i(z_{m_{xy}}, y).$$

Proof. The length of a branch in level i is π/J_i, hence (i) and (iii) follow by construction. Applying the left hand side of (i) repeatedly and using the fact that $\phi_{ik} = \phi_{k+1} \circ \cdots \circ \phi_i$ proves (ii). □

As a direct consequence of Lemma 2.1, for any $x, y \in F_\infty$ the sequence $\{d_i(\Phi_i(x), \Phi_i(y))\}_{i \geq 0}$ converges uniformly and we may thus consider

$$d_\infty(x, y) := \lim_{i \to \infty} d_i(\Phi_i(x), \Phi_i(y)) \tag{1}$$

as the natural metric that carries the inverse limit structure of F_∞.

Definition 2.3. Let $\mathcal{J} = \{j_\ell\}_{\ell\geq 0}$ and $\mathcal{N} = \{n_\ell\}_{\ell\geq 0}$ be sequences with $j_0 = n_0 = 1$ and $j_\ell, n_\ell \geq 2$. The generalized diamond fractal F_∞ of parameters $\mathcal{J}$ and $\mathcal{N}$ is the inverse limit of the system $\{(F_i, d_i, \mu_i, \{\phi_{ik}\}_{k\leq i})\}_{i\geq 0}$. If $j_\ell = j$ and $n_\ell = n$ for some $j, n \geq 2$ and all $\ell \geq 1$, we say that F_∞ is regular.

Observe that (i) together with the fact that the mappings Φ_i are surjective readily implies the convergence in the *pointed measured Gromov–Hausdorff sense* of the inverse system; cf. [16, Proposition 2.17].

Proposition 2.2. *A generalized diamond fractal $(F_\infty, d_\infty, \mu_\infty)$ is the inverse limit and the limit in the pointed measured Gromov–Hausdorff sense of $\{(F_i, d_i, \mu_i)\}_{i\geq 0}$.*

2.3. Diffusion process and heat kernel

This paragraph summarizes the results obtained in [2] whose application in the analysis of the process and its heat kernel are the main object of study in the present paper. In order to provide later on estimates that are expressible in a "classical" form, the parameter sequences $\mathcal{N} = \{n_i\}_{i\geq 0}$ and $\mathcal{J} = \{j_i\}_{i\geq 0}$ under consideration will satisfy the following weak condition.

Assumption 1. For any fixed $k \geq 0$, $\lim\limits_{\ell\to\infty} n_\ell j_\ell e^{-J^2_{k+1,\ell-1}} = 0$. In particular, the series $\sum_{\ell=k}^{\infty} N_{k,\ell} J_{k,\ell} e^{-J^2_{k+1,\ell}}$ converges and is bounded uniformly on $k \geq 0$.

The latter assumption is readily satisfied for regular sequences, see Corollary 4.2. Although weaker conditions such as

$$\lim_{i\to\infty} N_i e^{-J_i^2 t} < \infty \qquad \text{for } 0 < t < t_* < 1 \tag{2}$$

provide the existence of a jointly continuous heat kernel [2, Remark 3] and general estimates in terms of series, little about the convergence of those series can be obtained without further assumptions, see Remark 4.1.

Several results in the subsequent sections will involve the L^2-semigroup associated with the diffusion processes on F_i, $i = 1, 2, \ldots, \infty$, which we denote by $\{P_t^{F_i}\}_{t\geq 0}$. These are related to the mappings

$$\begin{aligned} \Phi_i^*\colon L^2(F_i,\mu_i) &\longrightarrow L^2(F_\infty,\mu_\infty) \\ f &\longmapsto f\circ\Phi_i, \end{aligned} \tag{3}$$

through the following intertwining property that is applied later on crucially.

Lemma 2.3 (*[2, Lemma 3, Corollary 4]*). *The family of operators $\{P_t^{F_\infty}\}_{t\geq 0}$ is a strongly continuous Markov semigroup on $L^2(F_\infty, \mu_\infty)$ that satisfies the strong Feller property. Moreover, for any $i \geq 0$,*

$$P_t^{F_\infty}\Phi_i^* f = \Phi_i^* P_t^{F_i} f \tag{4}$$

holds for any $f \in L^2(F_i, \mu_i)$.

The heat kernel associated with $\{P_t^{F_i}\}_{t\geq 0}$ turns out to be expressible in terms of the heat kernel on the circle and on intervals $[0, \pi/J_i]$ with Dirichlet boundary conditions, denoted by $p_t^{F_0}$, respectively $p_t^{[0,\pi/J_i]_D}$. Some standard estimates and facts about these are recorded in Appendix B.

Theorem 2.4. *The heat kernel associated with* $\{P_t^{F_\infty}\}_{t\ge 0}$ *is given by*

$$p_t^{F_\infty}(x,y)=p_t^{F_0}\big(\Phi_0(x),\Phi_0(y)\big)+\sum_{\ell=1}^{i_{xy}}\delta_{xy}(n_\ell)N_{\ell-1}p_t^{[0,\pi/J_\ell]_D}(\Phi_0(x),\Phi_0(y)) \tag{5}$$

for any $x,y\in F_\infty$, *where* $i_{xy}:=\max_{i\ge 0}\{\Phi_i(x),\Phi_i(y)$ *belong to the same bundle*$\}$ *and*

$$\delta_{xy}(n)=\begin{cases} n-1 & \text{if } \Phi_{i_{xy}}(x),\,\Phi_{i_{xy}}(y) \text{ same branch,}\\ -1 & \text{if } \Phi_{i_{xy}}(x),\,\Phi_{i_{xy}}(y) \text{ same bundle, different branch.}\end{cases}$$

Proof. The recursive formula in [2, Theorem 2] can be rewritten as

$$\begin{aligned} p_t^{F_i}(x,y)&=p_t^{F_0}\big(\phi_{i0}(x),\phi_{i0}(y)\big)\\ &\quad+\sum_{\ell=1}^{i_{xy}}\delta_{xy}(n_\ell)N_{\ell-1}J_\ell\Big(p^{F_0}_{J_\ell^2t}\big(\phi_{i0}(x),\phi_{i0}(y)\big)-p^{F_0}_{J_\ell^2t}\big(\phi_{i0}(x),-\phi_{i0}(y)\big)\Big),\end{aligned}$$

where $i_{xy}:=\max_{0\le k\le i}\{\phi_{ik}(x),\phi_{ik}(y)$ belong to the same bundle$\}$ and $\delta^{(i)}_{xy}(n)$ is as $\delta_{xy}(n)$ with $\phi_{i,i_{xy}}$ instead of $\Phi_{i_{xy}}$. Using (B.3) to rewrite $p^{F_0}_{J_\ell^2t}$ in terms of $p_t^{[0,\pi/J_i]_D}$ [2, Theorem 3] gives (5) after noting $\phi_{ik}(\Phi_i(x))=\Phi_k(x)$. $\square$

The results presented can be extended to further natural generalizations, however setting up formulas may result in a fairly long exercise.

3. Infinitesimal generator and Dirichlet form

As a strongly continuous Markov semigroup on $L^2(F_i,\mu_i)$, each $\{P_t^{F_i}\}_{t\ge 0}$, $i=1,\dots,\infty$ has an associated infinitesimal generator and a Dirichlet form, which we denote by L_{F_i} and $(\mathcal{E}^{F_i},\mathcal{F}^{F_i})$. In particular the Dirichlet form will appear in the functional inequalities discussed in the last section.

3.1. Liftings and projections

The mappings that provided the intertwining relation between the semigroups $\{P_t^{F_\infty}\}_{t\ge 0}$ and $\{P_t^{F_i}\}_{t\ge 0}$ from Lemma 2.3 will play a major role in the subsequent discussion. Their definition readily implies the following useful properties, see e.g. [2, Proposition 2].

Proposition 3.1. *Let* $i\ge 0$ *and* $\Phi_i^*\colon L^2(F_i,\mu_i)\to L^2(F_\infty,\mu_\infty)$ *be defined as in* (3).

(i) *For each* $i\ge 0$, Φ_i^* *is an isometry.*

(ii) *The space* $\mathcal{C}_0:=\bigcup_{i\ge 0}\Phi_i^*C(F_i)$ *is dense in* $L^2(F_\infty,\mu_\infty)$.

While Φ_i^* may be understood as a "lifting", its left inverse is in fact a projection mapping.

Proposition 3.2. *For any* $i\ge 0$ *let* $\Pi_i\colon L^2(F_\infty,\mu_\infty)\to L^2(F_i,\mu_i)$ *denote the left inverse of* Φ_i^*. *For any* $f\in L^2(F_\infty,\mu_\infty)$,

(i) $\|\varPi_i f\|_{L^2(F_i,\mu_i)} \leq \|f\|_{L^2(F_\infty,\mu_\infty)}$;
(ii) $\|f\|_{L^2(F_\infty,\mu_\infty)} = \lim_{i\to\infty} \|\varPi_i f\|^2_{L^2(F_i,\mu_i)}$.

Proof. Apply Cauchy–Schwartz and Proposition 3.1. □

In particular, (ii) in the latter proposition implies the convergence of $\{L^2(F_i,\mu_i)\}_{i\geq 0}$ to $L^2(F_\infty,\mu_\infty)$ in the sense of [26, Definition 2.5].

We finish this paragraph by analyzing the combined action of the lifting $\varPhi_i^*$, the semigroup $\{P_t^{F_i}\}_{t\geq 0}$ and the projection $\varPi_i$ through the operator $\varPhi_i^* P_t^{F_i} \varPi_i \colon L^2(F_\infty,\mu_\infty) \to L^2(F_\infty,\mu_\infty)$. This will be useful later, in particular to derive the Mosco convergence of the associated Dirichlet forms.

Lemma 3.3. *For any $t \geq 0$, the sequence of bounded operators $\{\varPhi_i^* P_t^{F_i} \varPi_i\}_{i\geq 0}$ converges strongly in $L^2(F_\infty,\mu_\infty)$ to $P_t^{F_\infty}$. In particular, the convergence is uniform in any finite time interval.*

Proof. Convergence (independent of t) follows from Lemma 2.3, the contraction property of $P_t^{F_\infty}$ and Proposition 3.2. □

3.2. Infinitesimal generator

Since the finite approximations F_i are metric graphs, for finite $i \geq 0$ the operator L_{F_i} with domain $\mathcal{D}_{F_i}$ corresponds with the standard Laplacian studied in quantum graphs/cable systems; see e.g. [8,14]. We now focus on properties of the generator L_{F_∞} and its domain $\mathcal{D}_{F_\infty}$ that can be obtained from the previous paragraph.

Theorem 3.4. *For each $i \geq 0$, let $\mathcal{D}_{F_i}$ denote the domain of infinitesimal generator L_{F_i}. The space $\mathcal{D}_0 := \bigcup_{i\geq 1} \varPhi_i^* \mathcal{D}_{F_i}$ is a core for $(L_{F_\infty}, \mathcal{D}_{F_\infty})$.*

Proof. Let $f \in \mathcal{D}_0$. Then, $f = \varPhi_i^* h$ for some $h \in \mathcal{D}_{F_i}$ and $i \geq 0$. By Lemma 2.3,

$$P_t^{F_\infty} f = P_t^{F_\infty} \varPhi_i^* h = \varPhi_i^* P_t^{F_i} h \in \varPhi_i^* \mathcal{D}_{F_i} \subseteq \mathcal{D}_0 \tag{6}$$

hence $P_t^{F_\infty} \colon \mathcal{D}_0 \to \mathcal{D}_0$. Proposition 3.1 implies that $\bigcup_{i\geq 1} \varPhi_i^* C^\infty(F_i)$, and therefore $\mathcal{D}_0$, is dense in $L^2(F_\infty,\mu_\infty)$. By virtue of [21, Section 1, Proposition 3.3] $\mathcal{D}_0$ is a core for the infinitesimal generator of $P_t^{F_\infty}$. □

Applying Lemma 3.3 to [25, Theorem 2.5] one finds the relation between the lifting and projection maps and the infinitesimal generator.

Corollary 3.5. *For each $f \in \mathcal{D}_0$, there exists $\{f_i\}_{i\geq 0}$ with $f_i \in \mathcal{D}_{F_i}$ such that*

$$\varPhi_i^* f_i \xrightarrow{i\to\infty} f \qquad \text{and} \qquad \varPhi_i^* L_{F_i} f_i \xrightarrow{i\to\infty} L_{F_\infty} f$$

hold in $L^2(F_\infty,\mu_\infty)$.

Remark 3.1. By [25, Theorem 2.5], the latter result or Lemma 3.3 yields an analogous statement for the resolvent that appears in [10, Theorem 4.3].

3.3. Dirichlet form

The Dirichlet form associated with $\{P_t^{F_\infty}\}_{t\geq 0}$ is given by

$$\mathcal{E}^{F_\infty}(f,f)=\lim_{t\to 0}\frac{1}{t}\langle f-P_t^{F_\infty}f,f\rangle_{L^2(F_\infty,\mu_\infty)}$$
$$\mathcal{F}^{F_\infty}=\{f\in L^2(F_\infty,\mu_\infty)\mid \mathcal{E}^{F_\infty}(f,f)\text{ exists and is finite}\},$$

see e. g. [7, Definition 1.7.1]. In this paragraph we prove the *generalized Mosco* convergence of the finite level Dirichlet forms to $(\mathcal{E}^{F_\infty},\mathcal{F}^{F_\infty})$. For a definition of this convergence we refer the reader e.g. to [26, Definition 2.11].

Theorem 3.6. *For the Dirichlet form* $(\mathcal{E}^{F_\infty},\mathcal{F}^{F_\infty})$ *associated with* $\{P_t^{F_\infty}\}_{t\geq 0}$ *it holds that*

(i) $(\mathcal{E}^{F_\infty},\mathcal{F}^{F_\infty})$ *is the generalized Mosco limit of* $\{(\mathcal{E}^{F_i},\mathcal{F}^{F_i})\}_{i\geq 0}$,
(ii) $\mathcal{D}_0$ *is a core for* $(\mathcal{E}^{F_\infty},\mathcal{F}^{F_\infty})$.
(iii) *For any* $i\geq 1$ *and* $h\in\mathcal{D}_{F_i}$, $\mathcal{E}^{F_\infty}(\Phi_i^*h,\Phi_i^*h)=\mathcal{E}^{F_i}(h,h)$;
(iv) *For any* $f\in\mathcal{F}^{F_\infty}$ *there is* $\{f_i\}_{i\geq 0}\subset\mathcal{D}_0$ *such that* $\mathcal{E}^{F_\infty}(f,f)=\lim_{i\to\infty}\mathcal{E}^{F_i}(f_i,f_i)$;
(v) $(\mathcal{E}^{F_\infty},\mathcal{F}^{F_\infty})$ *is local and regular.*

Proof. (i) follows from Lemma 3.3 and [25, Theorem 2.5] while (ii) from Theorem 3.4. Since $\Phi_i^*h\in\mathcal{D}_0$, Lemma 2.3 implies

$$\frac{1}{t}\langle\Phi_i^*h-P_t^{F_\infty}\Phi_i^*h,\Phi_i^*h\rangle_{L^2(F,\mu)}=\frac{1}{t}\langle h-P_t^{F_i}h,h\rangle_{L^2(F_i,\mu_i)}$$

and letting $t\to 0$ we obtain (iii). By density, (i) and (ii) yield (iv) and since $\mathcal{C}_0\subseteq\mathcal{F}^{F_\infty}\cap C(F_\infty)$, the regularity of $(\mathcal{E}^{F_\infty},\mathcal{F}^{F_\infty})$ follows from (ii). The form is also local because all $(\mathcal{E}^{F_i},\mathcal{F}^{F_i})$ are. □

4. Estimates for the heat kernel

The expression of the heat kernel in (5) will allow to obtain global estimates of the heat kernel and explicit bounds for its Lipschitz continuity. The estimates obtained in [23, Theorem 4.7] for regular diamonds with $n=j=2$ exploited the self-similarity of the space, which we avoid here. Although the (joint) continuity of $p_t^{F_\infty}$ may be derived using indirect arguments [2,23], the new estimates in Theorems 4.1 and 4.3 give a direct proof and also information about the dependence of the bounds on the parameters.

4.1. Uniform heat kernel bounds

The following estimates will be applied in later sections to study related functional inequalities.

Theorem 4.1. *There exists* $C_{\mathcal{N},\mathcal{J}}>0$ *such that for any* $t>0$,

$$\frac{1}{\sqrt{4\pi}}t^{-1/2}\leq\|p_t^{F_\infty}\|_\infty\leq\frac{1}{2\pi}+\frac{1}{\sqrt{4\pi}}t^{-1/2}+C_{\mathcal{N},\mathcal{J}}\,t^{-\frac{1}{2}(1+d(\ell_t^*))},\tag{7}$$

where $d(\ell_t^*) = \frac{\log N_{\ell_t^*-1}}{\log J_{\ell_t^*-1}}$ *and* $\ell_t^* := \inf\{\ell \geq 1 : \; J_\ell^{-2} \leq t\}$. *In particular,*

$$\frac{1}{\sqrt{4\pi}} t^{-1/2} \leq \|p_t^{F_\infty}\|_\infty \leq C_{\mathcal{N},\mathcal{J}}\, t^{-\frac{1}{2}(1+d(\ell_t^*))} \qquad \textit{for } t \in (0,1). \tag{8}$$

The exponent on the right hand side of (8) can be identified in the regular case with the *spectral dimension* of F_∞, cf. Corollary 4.2, which classically describes the short-time asymptotic behavior of the trace of the heat semigroup.

Proof of Theorem 4.1. Fix $t > 0$. With the convention $N_0 = 1$, the expression in (5), Lemmas B.1 and B.2 yield

$$p_t^{F_\infty}(x,y) \leq p_t^{F_0}(\Phi_0(x), \Phi_0(y)) + \frac{1}{\sqrt{\pi t}} \sum_{\ell=1}^{\infty} N_\ell \min\Big\{1, \frac{2}{(\pi J_\ell^2 t)^{1/2}} e^{-J_\ell^2 t}\Big\} \tag{9}$$

for any $x, y \in F_\infty$. An upper bound of the series in (9) is

$$\frac{1}{\sqrt{\pi t}} \sum_{\ell=1}^{\ell_t^*-1} N_\ell + \frac{2}{\sqrt{\pi t}} \sum_{\ell=\ell_t^*}^{\infty} \frac{N_\ell}{(J_\ell^2 t)^{1/2}} e^{-J_\ell^2 t} =: \frac{1}{\sqrt{\pi t}} S_1 + \frac{2}{\sqrt{\pi t}} S_2, \tag{10}$$

where $\ell_t^* := \inf\{\ell \geq 1 : \; J_\ell^{-2} \leq t\}$. For the first term, one can bound S_1 by

$$N_{\ell_t^*-1}\Big(1 + \sum_{\ell=1}^{\ell_t^*-2} n_{\ell+1}^{-1} \cdots n_{\ell_t^*-1}^{-1}\Big) \leq N_{\ell_t^*-1} \sum_{k=0}^{\infty} 2^{-k} = 2N_{\ell_t^*-1}. \tag{11}$$

For the second term in (10), using the notation from Definition 2.1, and the fact that $J_{\ell_t^*}^{-2} \leq t < J_{\ell_t^*-1}^{-2}$, we have

$$S_2 = N_{\ell_t^*-1} \sum_{\ell=\ell_t^*}^{\infty} \frac{N_\ell}{N_{\ell_t^*-1}} \frac{J_{\ell_t^*}}{J_\ell (J_{\ell_t^*}^2 t)^{1/2}} e^{-\frac{J_\ell^2}{J_{\ell_t^*}^2} J_{\ell_t^*}^2 t} \leq N_{\ell_t^*-1} \sum_{\ell=\ell_t^*}^{\infty} N_{\ell_t^*,\ell}\, e^{-J^2_{\ell_t^*+1,\ell}}. \tag{12}$$

The latter series converges and can be bounded independently of t by Assumption 1. Thus, there is $C_{\mathcal{N},\mathcal{J}} > 0$ such that

$$S_1 + S_2 \leq C_{\mathcal{N},\mathcal{J}} N_{\ell_t^*-1} = C_{\mathcal{N},\mathcal{J}} (J^2_{\ell_{t^*-1}})^{\frac{1}{2}d(\ell_t^*)} \leq C_{\mathcal{N},\mathcal{J}} t^{-\frac{1}{2}d(\ell_t^*)},$$

where $d(\ell_t^*) := \frac{\log N_{\ell_{t^*-1}}}{\log J_{\ell_{t^*-1}}}$. The last inequality follows from the choice of ℓ_t^*. Applying Lemma B.3 to the first term of (9) and the previous estimates to the second term yield the upper bound in (7). The lower bound readily follows from the expression (5) and (B.1). $\square$

Remark 4.1. Replacing Assumption 1 by (2), one concludes from (10) estimates of the type

$$S_1 \leq \ell_t^* t^{-\frac{1}{2}d(\ell_t^*)} \qquad \text{and} \qquad S_2 \leq C_t t^{-\frac{1}{2}d(\ell_t^*)}$$

for some $C_t > 0$ that bounds the series in (12). In the same way as before, these would now provide

$$p_t^{F_\infty}(x,y) \leq \frac{1}{2\pi} + \frac{1}{\sqrt{4\pi}} t^{-1/2} + t^{-\frac{1}{2}(1+d(\ell_t^*))}(C\ell_t^* + \tilde{C} C_t),$$

without further information about the dependence on t of the constant C_t.

Regular diamond fractals

Since $n_i = n$ and $j_i = j$ for all $i \geq 1$, Assumption 1 can be checked in this case by a direct computation (here we use mathematica). For any $k \geq 0$,

$$\sum_{\ell=k}^{\infty} n^{\ell-k+1} j^{\ell-k} e^{-j^{2(\ell-k)}} \leq \frac{n}{e} + \int_0^{\infty} n^{\xi} j^{\xi} e^{-j^{2\xi}}\, d\xi = \frac{n}{e} + \frac{1}{2\log j}\Gamma\Big(\frac{1}{2}\Big(1+\frac{\log n}{\log j}\Big)\Big),$$

where for the latter integral one does the change of variables $\eta = j^{\xi}$. Similarly, we can compute explicitly the bound in (12) to get

$$\sum_{\ell=k}^{\infty} n^{\ell-k+1} e^{-j^{2(\ell-k)}} \leq \frac{n}{e} + \int_0^{\infty} n^{\xi} e^{-j^{2\xi}}\, d\xi = \frac{n}{e} + \frac{1}{2\log j}\Gamma\Big(\frac{\log n}{2\log j}\Big).$$

Since $\frac{\log N_\ell}{\log J_\ell} = \frac{\log n}{\log j}$ for all $\ell \geq 1$, Theorem 4.1 provides the following global estimate of the sup-norm.

Corollary 4.2. *On a regular diamond fractal with parameters $n, j \geq 2$ there exists $C_{n,j} > 0$ such that*

$$\frac{1}{\sqrt{4\pi}} t^{-1/2} \leq \|p_t^{F_\infty}\|_\infty \leq \frac{1}{2\pi} + \frac{1}{\sqrt{4\pi}} t^{-1/2} + C_{n,j}\, t^{-\frac{1}{2}\left(1+\frac{\log n}{\log j}\right)}. \tag{13}$$

Note that $d_S = 1 + \frac{\log n}{\log j}$ is the spectral dimension of F_∞ [24, Theorem A.2] and coincides with the Hausdorff dimension d_H, in agreement with the observation that the walk dimension of a diamond fractal is $d_w = \frac{2d_H}{d_S} = 2$.

4.2. Continuity estimates

The recursive nature of the underlying space is also reflected in the proof of the continuity of the heat kernel. In particular, the case $i = 1$ serves both as guideline and as first induction step. The different pair-point configurations for that level, summarized in Fig. 3, will be analyzed by means of standard estimates recorded in Appendix B.

Theorem 4.3. *For any $t > 0$, the heat kernel $p_t^{F_\infty} : F_\infty \times F_\infty \to [0, \infty)$ is Lipschitz continuous in (F_∞, d_∞) and satisfies for any $x, y_1, y_2 \in F_\infty$*

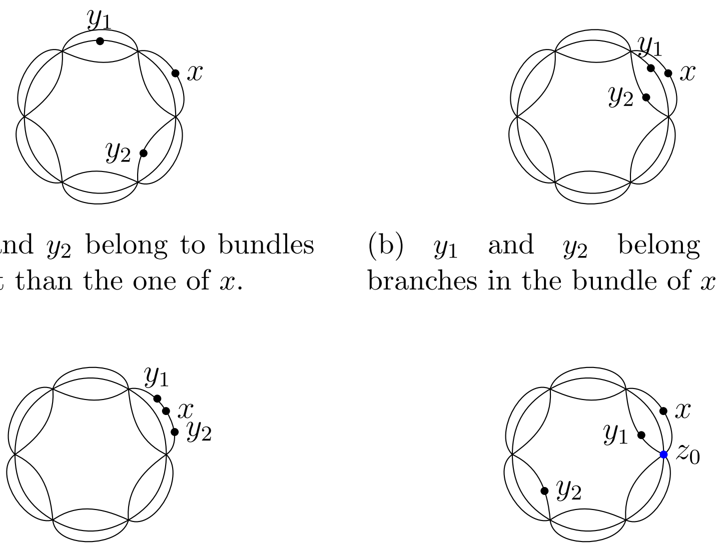
$$|p_t^{F_\infty}(x, y_1) - p_t^{F_\infty}(x, y_2)| \leq C t^{-1-\frac{1}{2}d(\ell_t^*)} d_\infty(y_1, y_2), \tag{14}$$

where $d(\ell_t^*) = \frac{\log N_{\ell_t^*-1}}{\log J_{\ell_t^*-1}}$, $\ell_t^* := \inf\{\ell \geq 1 : J_\ell^{-2} \leq t\}$ and some constant $C > 0$ depending on the parameter sequences $\mathcal{N}$, $\mathcal{J}$.

Note that, for short times, the estimate (14) is better than what could be obtained from the continuity of the semigroup proved in Theorem 5.1, and the uniform bound from Theorem 2.4.

Proof. Since $p_t^{F_i}(x, y)$ converges uniformly to $p_t^{F_\infty}(x, y)$, see [2, Remark 8], the latter is continuous and its Lipschitz constant $C_L(t)$ may be bounded by taking the limit $i \to \infty$ in Proposition 4.4, which leads to

$$C_L(t) \leq \frac{2}{\pi} \sum_{\ell=0}^{\infty} N_\ell \Big(J_\ell^2 + \frac{1}{2t}\Big) e^{-J_\ell^2 t}. \tag{15}$$



(a) y_1 and y_2 belong to bundles different than the one of x.

(b) y_1 and y_2 belong different branches in the bundle of x.

(c) Both y_1 and y_2 belong to the same branch as x.

(d) y_1 and y_2 belong to different bundles, and y_2 to the bundle of x.

Fig. 3. Pair-point configurations.

To estimate this series, we decompose (15) into

$$\sum_{\ell=0}^{\ell_t^*-1} N_\ell\Big(J_\ell^2+\frac{1}{2t}\Big)e^{-J_\ell^2 t}+\sum_{\ell=\ell_t^*}^{\infty} N_\ell\Big(J_\ell^2+\frac{1}{2t}\Big)e^{-J_\ell^2 t} =: S_1+S_2.$$

Analogous computations to those in (11) allow us to bound the first term by

$$\frac{3}{2t}\sum_{\ell=0}^{\ell_t^*-1} N_\ell e^{-J^2_{\ell_t^*}t} \leq \frac{3}{2t}N_{\ell_t^*-1}\sum_{\ell=0}^{\ell_t^*-1} N^{-1}_{\ell+1,\ell_t^*-1} \leq \frac{3}{2t}N_{\ell_t^*-1}\sum_{\ell=0}^{\ell_t^*-1} 2^{-(\ell_t^*-\ell-1)} \leq \frac{3}{t}N_{\ell_t^*-1}.$$

For the second, using $J^{-2}_{\ell_t^*}\leq t<J^{-2}_{\ell_t^*-1}$ we get

$$S_2 \leq \frac{3}{2t}\sum_{\ell=\ell_t^*}^{\infty} N_\ell J_\ell^2 t e^{-J_\ell^2 t} \leq \frac{3}{2t}N_{\ell_t^*-1}\sum_{\ell=\ell_t^*}^{\infty} N_{\ell_t^*,\ell}J^2_{\ell_t^*,\ell}e^{-J^2_{\ell_t^*+1,\ell}}. \tag{16}$$

The latter series is bounded independently of t by Assumption 1. Setting $d(\ell_t^*) := \frac{\log N_{\ell_t*-1}}{\log J_{\ell_t*-1}}$, the two previous estimates yield (14). □

Proposition 4.4. *For any $t>0$, the heat kernel $p_t^{F_i}$ is Lipschitz continuous in (F_i, d_i) and*

$$|p_t^{F_i}(x,y_1)-p_t^{F_i}(x,y_2)| \leq \frac{2}{\pi}\sum_{\ell=0}^{i} N_\ell\Big(J_\ell^2+\frac{1}{2t}\Big)e^{-J_\ell^2 t}d_i(y_1,y_2). \tag{17}$$

This result is proved by induction. To ease the notation and to remain consistent with [2], we set $\theta_x := \phi_{i0}(x)$ for $x \in F_i$ and $\theta_x := \Phi_0(x)$ for $x \in F_\infty$.

Proposition 4.5. *For each $t > 0$, the heat kernel $p_t^{F_1}$ is Lipschitz continuous in (F_1, d_1) and*

$$|p_t^{F_1}(x,y_1) - p_t^{F_1}(x,y_2)| \leq \frac{2}{\pi}\Big[\Big(1+\frac{1}{2t}\Big)e^{-t} + n_1\Big(j_1^2 + \frac{1}{2t}\Big)e^{-j_1^2 t}\Big] d_1(y_1,y_2).$$

Proof. By virtue of Lemma 2.1(iii) and the triangle inequality, it is enough to analyze the basic cases shown in Figs. 3(a) through 3(c).

(a) In view of the expression of $p_t^{F_1}(x,y)$, Lemmas B.1 and 2.1 we have (recall $\theta_x := \phi_{i0}(x)$)

$$\begin{aligned}|p_t^{F_1}(x,y_1) - p_t^{F_1}(x,y_2)| = |p_t^{F_0}(\theta_x,\theta_{y_1}) - p_t^{F_0}(\theta_x,\theta_{y_2})| &\leq \frac{1}{\pi}\sum_{k=1}^\infty e^{-k^2 t} k|\theta_{y_1} - \theta_{y_2}|\\
&\leq \frac{1}{\pi}\Big(e^{-t} + \int_1^\infty \xi e^{-\xi^2 t} d\xi\Big) d_1(y_1,y_2)\\
&= \frac{1}{\pi}\Big(1+\frac{1}{2t}\Big)e^{-t} d_1(y_1,y_2).\end{aligned}$$

(b) In this case, writing

$$\begin{aligned}|p_t^{F_1}(x,y_1) - p_t^{F_1}(x,y_2)| \leq\, &|p_t^{F_0}(\theta_x,\theta_{y_1}) - p_t^{F_0}(\theta_x,\theta_{y_2})|\\
&+ |p_t^{[0,L_1]_D}(\theta_x,\theta_{y_1}) - p_t^{[0,L_1]_D}(\theta_x,\theta_{y_2})|\end{aligned}$$

we can estimate the first term as in (i), and the second by

$$\begin{aligned}&\frac{2}{L_1}\sum_{k=1}^\infty e^{-\frac{k^2\pi^2}{L_1^2}t}\Big|\sin\Big(\frac{k\pi\theta_{y_1}}{L_1}\Big) - \sin\Big(\frac{k\pi\theta_{y_2}}{L_1}\Big)\Big| \leq \Big(\frac{2j_1^2}{\pi}\sum_{k=1}^\infty k e^{-k^2 j_1^2 t}\Big) d_0(\phi_1(y_1),\phi_1(y_2))\\
&\leq \frac{2}{\pi}\Big(j_1^2 e^{-j_1^2 t} + \int_1^\infty j_1^2 \xi e^{-\xi^2 j_1^2 t} d\xi\Big) d_1(y_1,y_2) = \frac{2}{\pi}\Big(j_1^2 e^{-j_1^2 t} + \frac{1}{2t}e^{-j_1^2 t}\Big) d_1(y_1,y_2).\end{aligned}$$

(c) Fig. 3(c) reduces to the previous case with an extra factor $(n_1 - 1)$ in the second summand. □

Proposition 4.5 serves both as proof schema and first step to show the corresponding estimate for the Lipschitz constant in a generic finite level.

Proof of Proposition 4.4. By induction, the case $i = 1$ is Proposition 4.5. By virtue of the triangle inequality it suffices to prove (17) for any fixed $t > 0$, $x \in F_i$ and y_1, y_2 such that (x, y_1) and (x, y_2) have the same pair-point configuration. Otherwise, decompose the path y_1 to y_2 as in Fig. 3(d). Let us thus assume that the Lipschitz constant for the $(i-1)$approximation, $C_L^{(i-1)}(t)$, admits the bound (17).

(i) If (x, y_1) and (x, y_2) are both as in Fig. 3(a), the expression of the heat kernel and Lemma 2.1 yield

$$\begin{aligned}|p_t^{F_i}(x,y_1) - p_t^{F_i}(x,y_2)| &= |p_t^{F_{i-1}}(\phi_i(x),\phi_i(y_1)) - p_t^{F_{i-1}}(\phi_i(x),\phi_i(y_2))|\\
&\leq C_L^{(i-1)}(t) d_{i-1}(\phi_i(y_1),\phi_i(y_2)) \leq C_L^{(i-1)}(t) d_i(y_1,y_2).\end{aligned}$$

(ii) If (x, y_1) and (x, y_2) are as in Fig. 3(b), then

$$\begin{aligned}|p_t^{F_i}(x,y_1) - p_t^{F_i}(x,y_2)| \leq\, &|p_t^{F_{i-1}}(\phi_i(x),\phi_i(y_1)) - p_t^{F_{i-1}}(\phi_i(x),\phi_i(y_2))|\\
&+ N_{i-1}|p_t^{[0,L_i]_D}(\theta_x,\theta_{y_1}) - p_t^{[0,L_i]_D}(\theta_x,\theta_{y_2})|.\end{aligned}$$

The first term can be estimated as in (i). For the second term, following the proof of Proposition 4.5 (substituting L_1 by L_i) and Lemma 2.1 yield

$$|p_t^{[0,L_i]_D}(\theta_x,\theta_{y_1})-p_t^{[0,L_i]_D}(\theta_x,\theta_{y_2})|\leq\frac{2}{\pi}\Big(J_i^2e^{-J_i^2t}+\frac{1}{2t}e^{-J_i^2t}\Big)d_i(y_1,y_2).$$

(iii) The case (x,y_1) and (x,y_2) both as in Fig. 3(c) reduces to the previous one with an extra factor (n_i-1).

Putting all estimates together and using the induction hypothesis we obtain

$$\begin{aligned}|p_t^{F_i}(x,y_1)-p_t^{F_i}(x,y_2)|&\leq\Big(C_L^{(i-1)}(t)+\frac{2}{\pi}N_i\Big(J_i^2e^{-J_i^2t}+\frac{1}{2t}e^{-J_i^2t}\Big)\Big)d_i(y_1,y_2)\\&=\Big(\frac{2}{\pi}\sum_{\ell=0}^{i}N_\ell\Big(J_\ell^2+\frac{1}{2t}\Big)e^{-J_\ell^2t}\Big)d_i(y_1,y_2).\quad\square\end{aligned}$$

Remark 4.2. The Lipschitz constant of Theorem 4.3 is bounded by the series (15), which converges under the weaker condition (2). However, at this level of generality little can be said about its behavior as a function of t.

5. Continuity estimates of the heat semigroup

The aim of this section is to study the regularity of the heat semigroup $\{P_t^{F_\infty}\}_{t\geq0}$ as a means to describe the geometry of F_∞ along the lines of the *weak Barky–Émery curvature condition* from [3]. This condition reads

$$|P_t^{F_\infty}f(x)-P_t^{F_\infty}f(y)|\leq C\frac{d_\infty(x,y)^\kappa}{t^{\kappa/d_w}}\|f\|_\infty \tag{18}$$

for all $f\in L^\infty(F_\infty,\mu_\infty)$, where $\kappa>0$ denotes a (curvature) parameter and $d_w>0$ the *walk dimension* of the space. We refer to [3] for further details and functional analytic consequences in the context of Dirichlet spaces with sub-Gaussian heat kernel estimates. Proposition 5.5 shows that the condition (18) is satisfied in each approximation level with $\kappa=1$ and $d_w=2$. While the latter is expected because each F_i is a one-dimensional object, the situation in the limit is less clear. The estimate in Theorem 5.1 reveals in concrete computations, cf. Theorem 5.6, a logarithmic correction that is also observed in diffusion processes with multifractal structures, see e.g. [12]. Whether this time dependence is optimal remains an open question.

Theorem 5.1. *For any $t>0$, there exists a constant $C>0$ such that*

$$|P_t^{F_\infty}f(x)-P_t^{F_\infty}f(y)|\leq\frac{C}{\sqrt{t}}(1+\ell_t^*)\,d_\infty(x,y)\,\|f\|_\infty, \tag{19}$$

for any $f\in L^\infty(F_\infty,\mu_\infty)$ and $x,y\in F_\infty$, where $\ell_t^*:=\inf\{\ell\geq1:\ J_\ell^{-2}\leq t\}$.

The proof of Theorem 5.1 is presented in detail at the end of the section. Here, the "chain property" from Lemma 2.1(iii) turns out crucial to reduce the analysis of pair-point configurations to the case of pairs that belong to the same branch. We continue using the notation $\theta_x:=\phi_{i0}(x)$ for any $x\in F_i$, $i\geq1$ and $\theta_x:=\Phi_0(x)$ for $x\in F_\infty$.

Remark 5.1. In view of the estimate (28), the constant C seems to be independent of the parameter sequence $\mathcal{N}$ that gives the number of copies ("parallel universes", as named in [10]) of a level that give rise to the next.

5.1. Key lemma

The following estimate is applied several times throughout the steps that yield the main result. To be consistent with the notation in [2], we write $L_i := \pi/J_i$.

Lemma 5.2. *Let $i \geq 0$. For any $x, y \in F_i$ and $t > 0$,*

$$\int_0^{L_i} J_i |p^{F_0}_{J_i^2 t}(J_i\rho, J_i\theta_x) - p^{F_0}_{J_i^2 t}(J_i\rho, J_i\theta_y)|\, d\rho \leq \min\Big\{\frac{2}{\sqrt{\pi t}}, \Big(J_i + \frac{1}{2J_i t}\Big)e^{-J_i^2 t}\Big\}|\theta_x - \theta_y|. \tag{20}$$

In view of (B.3), Lemma 5.2 readily implies another useful inequality.

Corollary 5.3. *Let $i \geq 1$. For any $f \in L^\infty(F_i)$,*

$$\int_0^{L_i} |(p_t^{[0,L_i]_D}(\rho, \theta_x) - p_t^{[0,L_i]_D}(\rho, \theta_y))f(\rho)|\, d\rho$$
$$\leq 2\min\Big\{\frac{2}{\sqrt{\pi t}}, \Big(J_i + \frac{1}{2J_i t}\Big)e^{-J_i^2 t}\Big\}\|f\|_\infty |\theta_x - \theta_y|.$$

Proof of Lemma 5.2. The strategy consists in estimating the left hand side of (20) using both representations of the heat kernel $p_t^{F_0}(x, y)$ given in (B.1).

(a) Using the first representation in (B.1), the triangle inequality yields

$$\begin{aligned}
\sqrt{4\pi t}\, I &:= \int_0^{L_i} \Big|\sum_{k\in\mathbb{Z}}\Big(e^{-\frac{(J_i\rho - J_i\theta_x - 2\pi k)^2}{4J_i^2 t}} - e^{-\frac{(J_i\rho - J_i\theta_y - 2\pi k)^2}{4J_i^2 t}}\Big)\Big|\, d\rho \\
&\leq \int_0^{L_i} \sum_{k\in\mathbb{Z}}\Big|e^{-\frac{(J_i\rho - J_i\theta_x - 2\pi k)^2}{4J_i^2 t}} - e^{-\frac{(J_i\rho - J_i\theta_y - 2\pi k)^2}{4J_i^2 t}}\Big|\, d\rho \\
&= \int_0^{L_i} \Big|e^{-\frac{(J_i\rho - J_i\theta_x)^2}{4J_i^2 t}} - e^{-\frac{(J_i\rho - J_i\theta_y)^2}{4J_i^2 t}}\Big|\, d\rho \\
&\quad + \int_0^{L_i} \sum_{k\geq 1}\Big|e^{-\frac{(J_i\rho - J_i\theta_x - 2\pi k)^2}{4J_i^2 t}} - e^{-\frac{(J_i\rho - J_i\theta_y - 2\pi k)^2}{4J_i^2 t}}\Big|\, d\rho \\
&\quad + \int_0^{L_i} \sum_{k\geq 1}\Big|e^{-\frac{(J_i\rho - J_i\theta_x + 2\pi k)^2}{4J_i^2 t}} - e^{-\frac{(J_i\rho - J_i\theta_y + 2\pi k)^2}{4J_i^2 t}}\Big|\, d\rho =: I_1 + I_2 + I_3.
\end{aligned}$$

Without loss of generality, let us assume that $\theta_x \leq \theta_y$. Then,

$$\begin{aligned}
I_1 &\leq \int_0^{L_i}\Big|\int_{\theta_x}^{\theta_y} \frac{J_i\rho - J_i\tilde\rho}{2J_i t} e^{-\frac{(J_i\rho - J_i\tilde\rho)^2}{4J_i^2 t}}\, d\tilde\rho\Big|\, d\rho \leq \int_0^{L_i}\int_{\theta_x}^{\theta_y} \frac{|\rho - \tilde\rho|}{2t} e^{-\frac{(\rho-\tilde\rho)^2}{4t}}\, d\tilde\rho\, d\rho \\
&= \int_{\theta_x}^{\theta_y}\int_0^{\tilde\rho} -\frac{\rho - \tilde\rho}{2t} e^{-\frac{(\rho-\tilde\rho)^2}{4t}}\, d\rho\, d\tilde\rho + \int_{\theta_x}^{\theta_y}\int_{\tilde\rho}^{L_i} \frac{\rho - \tilde\rho}{2t} e^{-\frac{(\rho-\tilde\rho)^2}{4t}}\, d\rho\, d\tilde\rho \\
&= \int_{\theta_x}^{\theta_y}\Big(1 - e^{-\frac{\tilde\rho^2}{4t}}\Big)d\tilde\rho + \int_{\theta_x}^{\theta_y}\Big(1 - e^{-\frac{(L_i-\tilde\rho)^2}{4t}}\Big)d\tilde\rho \leq 2|\theta_y - \theta_x|.
\end{aligned}$$

Moreover, recall that $\theta_x, \theta_y \in [0, L_i)$ for any $x, y \in F_i$. Hence, for any $k \geq 1$, $\rho \in [\theta_x, \theta_y]$ and $\tilde{\rho} \in [0, L_i)$, the quantity $\rho - \tilde{\rho} - 2kL_i \leq L_i - 2kL_i$ is nonpositive. Thus,

$$\begin{aligned} I_2 &\leq \int_0^{L_i} \sum_{k\geq 1} \Big| \int_{\theta_x}^{\theta_y} \frac{2(J_i\rho - J_i\tilde{\rho} - 2\pi k)}{4J_i t} e^{-\frac{(J_i\rho - J_i\tilde{\rho} - 2\pi k)^2}{4J_i^2 t}} d\tilde{\rho} \Big| \, d\rho \\ &\leq \int_{\theta_x}^{\theta_y} \sum_{k\geq 1} \int_0^{L_i} \frac{|\rho - \tilde{\rho} - 2\pi k/J_i|}{2t} e^{-\frac{(\rho - \tilde{\rho} - 2\pi k/J_i)^2}{4t}} d\tilde{\rho}\, d\rho \\ &\leq \int_{\theta_x}^{\theta_y} \int_0^{L_i} \int_0^{\infty} -\frac{\rho - \tilde{\rho} - \xi}{2t} e^{-\frac{(\rho - \tilde{\rho} - \xi)^2}{4t}} d\tilde{\rho}\, d\rho = \frac{1}{L_i} \int_{\theta_x}^{\theta_y} \int_0^{L_i} e^{-\frac{(\rho-\tilde{\rho})^2}{4t}} d\tilde{\rho}\, d\rho \\ &\leq |\theta_x - \theta_y| \end{aligned}$$

Analogously, because $\rho - \tilde{\rho} + 2kL_i \geq L_i + 2kL_i > 0$ for any $k \geq 1$, we obtain

$$I_3 \leq \int_{\theta_x}^{\theta_y} \int_0^{L_i} \sum_{k\geq 1} \frac{\rho - \tilde{\rho} + 2kL_i}{2t} e^{-\frac{(\rho - \tilde{\rho} + 2kL_i)^2}{4t}} d\tilde{\rho}\, d\rho \leq |\theta_x - \theta_y|.$$

Adding up these estimates leads to $I \leq \frac{1}{\sqrt{4\pi t}}(I_1 + I_2 + I_3) = \frac{2}{\sqrt{\pi t}}|\theta_x - \theta_y|$.

(b) Using the second representation of $p_t^{F_0}(x, y)$ in (B.1) and $L_i = \pi/J_i$,

$$\begin{aligned} I &:= \frac{J_i}{\pi} \int_0^{L_i} \Big| \sum_{k\geq 1} e^{-k^2 J_i^2 t} \big(\cos(kJ_i(\theta_x - \rho)) - \cos(kJ_i(\theta_y - \rho))\big) \Big| \, d\rho \\ &\leq \frac{J_i}{\pi} \int_0^{L_i} \sum_{k\geq 1} e^{-k^2 J_i^2 t} |\cos(kJ_i(\theta_x - \rho)) - \cos(kJ_i(\theta_y - \rho))| \, d\rho \\ &\leq \frac{J_i}{\pi} \int_0^{L_i} \sum_{k\geq 1} e^{-k^2 J_i^2 t} kJ_i |\theta_x - \theta_y| \, d\rho = \frac{J_i L_i}{\pi} |\theta_x - \theta_y| \sum_{k\geq 1} e^{-k^2 J_i^2 t} kJ_i \\ &\leq |\theta_x - \theta_y| \Big(J_i e^{-J_i^2 t} + \int_1^{\infty} J_i \xi e^{-\xi^2 J_i^2 t} d\xi \Big) = e^{-J_i^2 t} \Big(J_i + \frac{1}{2J_i t} \Big) |\theta_x - \theta_y|. \end{aligned}$$

The assertion now follows from (a) and (b). □

5.2. First approximation level

The weak Bakry–Émery condition (18) with $\kappa = 1$ and $d_w = 2$ is obtained on each finite approximation F_i by an inductive argument and this paragraph is devoted to the first induction step. Any notation appearing in the proof for the first time follows [2] and is briefly recalled in Appendix A.

Proposition 5.4. *For any $t > 0$, $f \in L^\infty(F_1)$ and $x, y \in F_1$,*

$$|P_t^{F_1} f(x) - P_t^{F_1} f(y)| \leq C_1(t) d_1(x, y) \|f\|_\infty, \tag{21}$$

where

$$C_1(t) \leq 2\Big(\min\Big\{ \frac{2}{\sqrt{\pi t}}, \Big(1 + \frac{1}{2t}\Big) e^{-t} \Big\} + \min\Big\{ \frac{2}{\sqrt{\pi t}}, \Big(j_1 + \frac{1}{2j_1 t}\Big) e^{-j_1^2 t} \Big\} \Big). \tag{22}$$

Proof. By virtue of the triangle inequality and [2, Proposition 3], see also (A.3), we have (recall $\theta_x := \phi_{i0}(x)$)

$$\begin{aligned} |P_t^{F_1} f(x) - P_t^{F_1} f(y)| &\leq |P_t^{F_0}(\mathcal{I}_1 f)(\theta_x) - P_t^{F_0}(\mathcal{I}_1 f)(\theta_y)| \\ &\quad + |P_t^{[0,L_1]_D}(\mathrm{P}_1^{\perp} f)|_{I_{\alpha_x}}(\theta_x) - P_t^{[0,L_1]_D}(\mathrm{P}_1^{\perp} f)|_{I_{\alpha_y}}(\theta_y)| \\ &= D_1 + D_2. \end{aligned} \tag{23}$$

Let us assume first that $x, y \in F_1$ belong to the same branch , in particular $\alpha_x = \alpha_y$. Applying Lemma 5.2 with $L_0 = 2\pi$ and $J_0 = 1$,

$$\begin{aligned} D_1 &\leq \|\mathcal{I}_1 f\|_\infty \int_0^{L_0} |p_t^{F_0}(\theta,\theta_x) - p_t^{F_0}(\theta,\theta_y)|\, d\theta \\ &\leq \min\Big\{\frac{2}{\sqrt{\pi t}}, \Big(1+\frac{1}{2t}\Big)e^{-t}\Big\}\|f\|_\infty|\theta_x-\theta_y|. \end{aligned}$$

By virtue of Corollary 5.3,

$$\begin{aligned} D_2 &= |P_t^{[0,L_1]_D}(\mathrm{P}_1^{\perp} f)_{\alpha_x}(\theta_x) - P_t^{[0,L_1]_D}(\mathrm{P}_1^{\perp} f)\,\|_{I_{\alpha_x}}\,(\theta_y)| \\ &\leq \int_0^{L_1} |(\mathrm{P}_1^{\perp} f)_{\alpha_x}(\theta)||p_t^{[0,L_1]_D}(\theta,\theta_x) - p_t^{[0,L_1]_D}(\theta,\theta_y)|\, d\theta \\ &\leq 2\min\Big\{\frac{2}{\sqrt{\pi t}}, \Big(J_1+\frac{1}{2J_1 t}\Big)e^{-J_1^2 t}\Big\}\|\,\mathrm{P}_1^{\perp} f\,\|_\infty|\theta_x-\theta_y| \\ &\leq 2\min\Big\{\frac{2}{\sqrt{\pi t}}, \Big(J_1+\frac{1}{2J_1 t}\Big)e^{-J_1^2 t}\Big\}\|f\|_\infty d_1(x,y). \end{aligned}$$

Putting both estimates together and noticing that $d_1(x,y) = d_0(\phi_1(x),\phi_1(y)) = |\theta_x - \theta_y|$ because x and y belong to the same branch yields (21) with $C_1(t)$ as in (21). If x and y belong to different branches, there exists by construction a sequence $x_1,\ldots,z_N \in B_1$ with $1 \leq N \leq J_1$ such that $d_1(x,y) = d_1(x,z_1) + \sum_{\ell=1}^{N-1} d_1(x_\ell,x_{\ell+1}) + d_1(z_N,y)$. Each pair of points in the summands belongs to the same branch, hence applying the triangle inequality and estimating each term as in the previous case proves again (21). □

5.3. Generic approximation level

The recursive nature of the construction of F_i also underlies the proof of the weak Bakry–Émery condition for an arbitrary level.

Proposition 5.5. *Let $i \geq 1$. For any $t > 0$, $f \in L^\infty(F_i)$ and $x, y \in F_i$,*

$$|P_t^{F_i} f(x) - P_t^{F_i} f(y)| \leq 2\sum_{\ell=0}^{i} \min\Big\{\frac{2}{\sqrt{\pi t}}, \Big(J_\ell+\frac{1}{2J_\ell t}\Big)e^{-J_\ell^2 t}\Big\} d_i(x,y)\|f\|_\infty. \tag{24}$$

Proof. Let $i \geq 2$. With the notation from (A.2), applying [2, Proposition 3], see also (A.2), and the triangle inequality we have

$$\begin{aligned} |P_t^{F_i} f(x) - P_t^{F_i} f(y)| &\leq |P_t^{F_{i-1}}(\mathcal{I}_i f)(\phi_i(x)) - P_t^{F_{i-1}}(\mathcal{I}_i f)(\phi_i(y))| \\ &\quad + |P_t^{[0,L_i]_D}(\mathrm{P}_i^{\perp} f)|_{I_{\alpha_x}}(\theta_x) - P_t^{[0,L_i]_D}(\mathrm{P}_i^{\perp} f)_{I_{\alpha_y}}(\theta_y)| \\ &= D_{i,1} + D_{i,2}. \end{aligned} \tag{25}$$

To estimate these terms, assume first that $x, y \in F_i$ belong to the same branch. By hypothesis of induction, there is $C_{i-1}(t) > 0$ such that

$$D_{i,1} \leq C_{i-1}(t)\|f\|_\infty\, d_{i-1}(\phi_i(x), \phi_i(y)) = C_{i-1}(t)\, d_i(x, y)\|f\|_\infty,$$

where the last equality is due to the fact that for points in the same branch

$$d_i(x, y) = d_{i-1}(\phi_i(x), \phi_i(y)) = |\theta_x - \theta_y|. \tag{26}$$

In addition, also $\alpha_x = \alpha_y$ hence Corollary 5.3 and (26) yield

$$\begin{aligned} D_{i,2} &= |P_t^{[0,L_i]_D}(\mathrm{P}_i^\perp f)|_{I_{\alpha_x}}(\theta_x) - P_t^{[0,L_i]_D}(\mathrm{P}_i^\perp f)|_{I_{\alpha_x}}(\theta_y)| \\ &\leq \int_0^{L_1} |(\mathrm{P}_i^\perp f)_{\alpha_x}(\theta)||p_t^{[0,L_i]_D}(\theta, \theta_x) - p_t^{[0,L_i]_D}(\theta, \theta_y)|\, d\theta \\ &\leq 2\min\Big\{\frac{2}{\sqrt{\pi t}}, \Big(J_i + \frac{1}{2J_i t}\Big)e^{-J_i^2 t}\Big\}\|f\|_\infty d_i(x, y). \end{aligned}$$

Putting both estimates together we obtain for x and y in the same branch

$$|P_t^{F_i} f(x) - P_t^{F_i} f(y)| \leq \Big(C_{i-1}(t) + 2\min\Big\{\frac{2}{\sqrt{\pi t}}, \Big(J_i + \frac{1}{2J_i t}\Big)e^{-J_i^2 t}\Big\}\Big)\|f\|_\infty d_i(x, y). \tag{27}$$

If $x, y \in F_i$ belong to different branches, we find $z_1, \ldots, z_{N_{xy}} \in B_i$ that connect both branches so that $d_i(x, y) = d_i(x, z_1) + d_i(z_1, z_2) + \cdots + d_i(z_{N_{xy}}, y)$, cf. Lemma 2.1(iii). Regarding points in each pair as belonging to the same branch, the triangle inequality and the previous computations for each of the terms yield (27). Finally, (24) is obtained by solving the recursive inequality $C_i(t) \leq C_{i-1}(t) + 2\min\Big\{\frac{2}{\sqrt{\pi t}}, \Big(J_i + \frac{1}{2J_i t}\Big)e^{-J_i^2 t}\Big\}$ with $C_1(t)$ as in (22). □

5.4. Continuity estimates in the limit. proof of Theorem 5.1

We are now ready to apply Proposition 5.5 to obtain the estimate (19). Once more we see the important role that the intertwining property (4) plays in order to "pass to the limit". By virtue of Theorem 3.6, it suffices to prove the statement for $f \in \mathcal{C}_0 = \bigcup_{i\geq 0} \Phi_i^* C(F_i)$, i.e. $f = h \circ \Phi_i$ for some $i \geq 1$ and $h \in C(F_i)$. Let $x, y \in F_\infty$. By virtue of Lemma 2.3 and Proposition 5.5,

$$\begin{aligned} |P_t^{F_\infty} f(x) - P_t^{F_\infty} f(y)| &= |P_t^{F_\infty} \Phi_i^* h(x) - P_t^{F_\infty} \Phi_i^* h(y)| = |\Phi_i^* P_t^{F_i} h(x) - \Phi_i^* P_t^{F_i} h(y)| \\ &= |P_t^{F_i} h(\Phi_i(x)) - P_t^{F_i} h(\Phi_i(y))| \leq C_i(t)\, d_i(\Phi_i(x), \Phi_i(y))\, \|h\|_\infty. \end{aligned}$$

Letting $i \to \infty$, cf. (1), yields $|P_t^{F_\infty} f(x) - P_t^{F_\infty} f(y)| \leq C(t)\, d_\infty(x, y)\, \|f\|_\infty$ with

$$C(t) \leq 2\sum_{\ell=0}^{\infty} \min\Big\{\frac{2}{\sqrt{\pi t}}, \Big(J_\ell + \frac{1}{2J_\ell t}\Big)e^{-J_\ell^2 t}\Big\}. \tag{28}$$

To estimate the series on the right hand side, we notice that $J_{\ell_t^*}^{-2} \leq t < J_{\ell_t^*-1}^{-2}$ and split the series into the three terms

$$\sum_{\ell=0}^{\ell_t^*-1} \frac{2}{\sqrt{\pi t}} + \frac{1}{\sqrt{t}}\sum_{\ell=\ell_t^*}^{\infty} J_\ell \sqrt{t} e^{-J_\ell^2 t} + \frac{1}{\sqrt{t}}\sum_{\ell=\ell_t^*}^{\infty} \frac{1}{J_\ell \sqrt{t}} e^{-J_\ell^2 t} =: \frac{2\ell_t^*}{\sqrt{\pi t}} + \frac{1}{\sqrt{t}} S_1 + \frac{1}{\sqrt{t}} S_2. \tag{29}$$

For the first series, analogous arguments as (16) give

$$S_1 = J_{\ell_{t^*-1}}\sqrt{t} e^{-J_{\ell_t^*}^2 t} \sum_{\ell=\ell_t^*}^{\infty} J_{\ell_t^*,\ell}^2 e^{-J_\ell^2 t} \leq \sum_{\ell=\ell_t^*}^{\infty} J_{\ell_t^*,\ell}^2 e^{-J_{\ell_t^*+1,\ell}^2} \leq \sum_{\ell=\ell_t^*}^{\infty} N_{\ell_t^*} J_{\ell_t^*,\ell}^2 e^{-J_{\ell_t^*+1,\ell}^2}$$

which is uniformly bounded and independent of t by Assumption 1. The second series is bounded by $\sum_{\ell=\ell_t^*}^{\infty} e^{-J^2_{\ell_t^*+1,\ell}}$, which is finite by definition of J_ℓ and in particular independent of t. The claim now follows from (28) and (29).

5.5. Regular case

As far as computations allow for regular diamond fractals, the estimates obtained in Proposition 5.5 and Theorem 5.1 provide local continuity estimates with a logarithmic correction.

Theorem 5.6. *For a regular diamond fractal F_∞ with parameters $n, j \geq 2$, there exists $C_j > 0$ such that*

$$|P_t^{F_\infty} f(x) - P_t^{F_\infty} f(y)| \leq \frac{C_j}{\sqrt{t}}(1 + |\log t|) d_\infty(x, y) \|f\|_\infty \tag{30}$$

for any $f \in L^\infty(F_\infty)$, $x, y \in F_\infty$ and $0 < t < 1$.

Proof. To simplify constants which do not depend on J_ℓ, N_ℓ or t, we will estimate the quantity appearing in (28) by $2\sum_{\ell=0}^{i} \min\left\{\frac{1}{\sqrt{t}}, \left(J_\ell + \frac{1}{J_\ell t}\right) e^{-J_\ell^2 t}\right\}$. Since $J_\ell = j^\ell$, we have $j^{-2\ell_t^*} \leq t < j^{-2(\ell_t^*-1)}$ and $\ell_t^* - 1 \leq \left|\frac{\log\sqrt{t}}{\log j}\right| < \ell_t^*$. Thus,

$$S_1 \leq \sum_{\ell=\ell_t^*}^{\infty} j^{\ell-\ell_t^*} e^{-j^{2(\ell-\ell_t^*-1)}} \leq 1 + j + \int_0^\infty j^\xi e^{-j^{2\xi}} = 1 + j + \frac{\Gamma(1/2)}{2\log j}$$

and

$$S_2 \leq \sum_{\ell=\ell_t^*}^{\infty} e^{-j^{2(\ell-\ell_t^*-1)}} = e^{-\frac{1}{j^2}} + 1 + e^{-j^2} + \sum_{k=2}^{\infty} e^{-\frac{1}{j^2}} \leq 3 + \int_1^\infty e^{-j^{2\xi}} d\xi \leq 3 + \frac{\sqrt{\pi}}{j\log j}.$$

From (29) we conclude the bound $C(t) \leq \frac{C_j}{\sqrt{t}}(1 + |\log t|)$. □

6. Applications in functional inequalities. overview

The estimates obtained in previous sections allow to analyze other functional inequalities to further investigate the properties of the diffusion process on a generalized diamond fractal. In this section we formulate some of these and outline the main ideas to prove them.

6.1. Ultracontractivity

Among the many formulations of this property that can be found in the literature we consider here that of [18, Chapter 2]: the semigroup $\{P_t^{F_\infty}\}_{t\geq 0}$ is contractive if it is a bounded operator from $L^2(F_\infty, \mu_\infty)$ to $L^\infty(F_\infty)$ for all $t > 0$. A direct application of the estimate from Theorem 4.1 for short times leads to the desired statement.

Theorem 6.1. *There exists $C_{\mathcal{N},\mathcal{J}} > 0$ such that*

$$\|P_t^{F_\infty}\|_{2\to\infty} \leq C_{\mathcal{N},\mathcal{J}}\, t^{-\frac{1}{4}(1+d(\ell_t^*))} \tag{31}$$

for any $0 < t < 1$.

Similarly, Corollary 4.2 can be applied to deduce the result in the regular case, that reads

$$\|P_t^{F_\infty}\|_{2\to\infty} \leq C_{j,n}\, t^{-\frac{1}{4}(1+\frac{\log n}{\log j})}$$

with an explicit constant. Again, the spectral dimension $d_S = 1 + \frac{\log n}{\log j}$ appears in the exponent. Since by symmetry $\|P_t^{F_\infty}\|_{1\to\infty} \leq \|P_t^{F_\infty}\|_{2\to\infty}^2$, we recover [23, Proposition 4.9] without using Poincaré inequality.

6.2. Poincaré inequality

The ultracontractivity proved in Theorem 6.1 can be applied to adapt the argument from [23, Proposition 4.8] and prove a global Poincaré inequality in the present general (non self-similar) framework. Further inequalities of this type that require a notion of gradient, as for instance the weak (1–1) Poincaré inequality studied in [27], are left to be the subject of future investigations.

Theorem 6.2. *A diamond fractal F_∞ with parameters $\mathcal{J}$ and $\mathcal{N}$ satisfies the uniform global Poincaré inequality*

$$\int_{F_\infty} |f - \overline{f}|^2\, d\mu_\infty \leq \mathcal{E}^{F_\infty}(f, f) \tag{32}$$

for any $f \in \mathcal{F}^{F_\infty}$, where $\overline{f} = \frac{1}{2\pi}\int_{F_\infty} f\, d\mu_\infty$.

Since the space (F_∞, d_∞) is compact and has finite measure, ultracontractivity implies compactness of the semigroup $P_t^{F_\infty}$ on $L^p(F_\infty, \mu_\infty)$ for any $1 \leq p \leq \infty$ and $t > 0$ (see e.g. [18, Theorem 2.1.5]). This can be used to deduce the existence of spectral gap [7, Theorem A.6.4] and follow [23, Proposition 4.8] to obtain (32), where the constant one is the inverse of the lowest non-zero eigenvalue. The latter eigenvalue coincides with the lowest eigenvalue of the infinitesimal operator L_{F_0}, that is the Laplacian on the circle F_0; see e.g. [26, Proposition 2.5].

6.3. Logarithmic Sobolev inequality

This inequality provides information about the (exponential) convergence to the equilibrium of the diffusion process in terms of the *entropy*, given by the expression on the left hand side of (33).

Theorem 6.3. *For any non-negative $f \in \mathcal{F}^{F_\infty} \cap L^1(F_\infty, \mu_\infty) \cap L^\infty(F_\infty)$ it holds that $f^2 \log f \in L^1(F_\infty, \mu_\infty)$ and there exists $M_{\mathcal{N},\mathcal{J}} > 0$ such that*

$$\int_{F_\infty} f^2 \log f^2\, d\mu_\infty - \int_{F_\infty} f^2\, d\mu_\infty \log\Big(\int_{F_\infty} f\, d\mu_\infty\Big) \leq M_{\mathcal{N},\mathcal{J}}\, \mathcal{E}^{F_\infty}(f, f). \tag{33}$$

The estimate in Theorem 4.1 and classical arguments [18, Theorem 2.2.3] provide a defective Sobolev inequality as e.g. [7, Proposition 5.1.3], which by virtue of Theorem 6.2 implies the logarithmic Sobolev inequality.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A

For completeness, this section briefly summarizes several facts from [2] that are mentioned in some of the proofs, especially in that of Theorem 5.1. For each $i \geq 0$, the space of square integrable functions on F_i is decomposed into

$$L^2(F_i,\mu_i) = L^2_{\rm sym}(F_i,\mu_i) \oplus L^2_{{\rm sym}^\perp}(F_i,\mu_i),$$

where $L^2_{\rm sym}(F_i,\mu_i)$ denotes the invariant subspace of $L^2(F_i,\mu_i)$ under the action of the symmetric group $S(n_i)^{2j_i}$. The projection operator $\mathrm{P}_i\colon C(F_i) \to L^2_{\rm sym}(F_i,\mu_i) \cap C(F_i)$ is defined as

$$\mathrm{P}_i f(x) = \begin{cases} \frac{1}{n_i}\sum_{w=1}^{n_i} f(\phi_i(x)w) & \text{if } x \in F_i \setminus B_i, \\ f(x) & \text{if } x \in B_i \end{cases} \tag{A.1}$$

and its orthogonal complement operator, $\mathrm{P}_i^\perp\colon C(F_i) \to L^2_{{\rm sym}^\perp}(F_i,\mu_i) \cap C(F_i)$ by $\mathrm{P}_i^\perp f(x) = f(x) - \mathrm{P}_i f(x)$. Analogous formal definitions of these operators apply to bounded Borel functions. The projection P_i is related to the so-called integration over fibers in [16], $\mathcal{I}_{\mathcal{D}_i}\colon C(F_i) \to C(F_{i-1})$ which in this case has the expression

$$\mathcal{I}_i f(x) := \mathcal{I}_{\mathcal{D}_i} f(x) = \frac{1}{n_i}\sum_{w=1}^{n_i} f(xw). \tag{A.2}$$

Thus, for any $f \in C(F_i)$, $\mathrm{P}_i f(x) = \phi_i^* \mathcal{I}_i f(x)$. With this notation, the semigroups $\{P_t^{F_i}\}_{t\geq 0}$ admit the decomposition

$$P_t^{F_i} f(x) = P_t^{F_{i-1}}(\mathcal{I}_i f)(\phi_i(x)) + P_t^{[0,L_i]_D}(\mathrm{P}_i^\perp f)|_{I_{\alpha_x}}(\phi_{i0}(x)), \tag{A.3}$$

where I_{α_x} denotes the branch in F_i where x belongs to.

Appendix B. Useful equalities and inequalities

We record the following identities relating the heat kernel on an interval and on the circle. Explicit computations can be fairly reproduced with a mathematical computing software.

Lemma B.1. *The heat kernel on the unit circle admits the representations*

$$p_t^{F_0}(\theta,\tilde\theta) = \frac{1}{\sqrt{4\pi t}}\sum_{k\in\mathbb{Z}} e^{-\frac{(\theta-\tilde\theta-2\pi k)^2}{4t}} = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k\geq 1} e^{-k^2 t}\cos(k(\tilde\theta-\theta)). \tag{B.1}$$

For any $L > 0$, the heat kernel on the interval $[0, L]$ with Dirichlet boundary conditions admits the representation

$$p_t^{[0,L]_D}(\theta,\tilde\theta) = \frac{2}{L}\sum_{k=1}^{\infty} e^{-\frac{k^2\pi^2 t}{L^2}} \sin\Big(\frac{k\pi\theta}{L}\Big)\sin\Big(\frac{k\pi\tilde\theta}{L}\Big). \tag{B.2}$$

Both heat kernels are related through the identity

$$p_t^{[0,L]_D}(\theta,\tilde\theta) = \frac{\pi}{L}\Big(p^{F_0}_{\pi^2 t/L^2}\Big(\frac{\pi\theta}{L},\frac{\pi\tilde\theta}{L}\Big) - p^{F_0}_{\pi^2 t/L^2}\Big(\frac{\pi\theta}{L},-\frac{\pi\tilde\theta}{L}\Big)\Big). \tag{B.3}$$

Lemma B.2. *For any $a > 0$,*

$$\sum_{k=1}^{\infty} e^{-ak^2} \leq \min\Big\{\frac{\sqrt{\pi}}{2\sqrt{a}}, \frac{1}{a}e^{-a}\Big\}.$$

Proof. The series can be estimated in two different ways. On the one hand,

$$\sum_{k=1}^{\infty} e^{-ak^2} \leq \int_0^{\infty} e^{-a\xi^2}\,d\xi = \frac{\sqrt{\pi}}{2\sqrt{a}}.$$

On the other hand, since $ak^2 \geq 2ak \geq a + ak$ for any $k \geq 1$,

$$\sum_{k=1}^{\infty} e^{-ak^2} \leq e^{-a}\sum_{k=1}^{\infty} e^{-ak} \leq e^{-a}\int_0^{\infty} e^{-a\xi}\,d\xi = e^{-a}\frac{1}{a}. \qquad \square$$

Lemma B.3. *For any $\theta, \tilde{\theta} \in [0, 2\pi)$ and $t > 0$,*

$$|p_t^{F_0}(\theta, \tilde{\theta})| \leq \frac{1}{2\pi} + \frac{1}{\sqrt{4\pi t}}.$$

In particular, $|p_t^{F_0}(\theta, \tilde{\theta})| \leq \frac{1}{\sqrt{\pi t}}$ for $t \in (0, 1)$.

Proof. In view of the second expression in (B.1),

$$|p_t^{F_0}(\theta, \tilde{\theta})| \leq \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k\geq 1} e^{-k^2 t} \leq \frac{1}{2\pi} + \frac{1}{\pi}\int_0^{\infty} e^{-\xi^2 t}\,d\xi = \frac{1}{2\pi} + \frac{1}{\sqrt{4\pi t}}. \qquad \square$$

Lemma B.4. *For any $a > 0$,*

$$\int_a^{\infty} \frac{1}{\xi^2} e^{-\xi^2 t}\,d\xi = \frac{1}{a}e^{-a^2} - \sqrt{\pi}\,\mathrm{Erfc}(a).$$

Moreover,

$$\int_a^{\infty} \frac{1}{\xi^2} e^{-\xi^2 t}\,d\xi = \frac{1}{a} - \sqrt{\pi t} + a \cdot t - \frac{a^3 t^2}{6} + \frac{a^5 t^3}{30} - \frac{a^7 t^4}{168} + O(t^5) \qquad \textit{as } t \to 0.$$

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