Ann. Comb. 24 (2020) 739–765

Published online October 4, 2020
https://doi.org/10.1007/s00026-020-00512-5

Annals of Combinatorics

Chain Decompositions of q, t-Catalan Numbers via Local Chains

Seongjune Han, Kyungyong Lee, Li Li and Nicholas A. Loehr

Abstract. The q, t-Catalan number $\mathrm{Cat}_n(q, t)$ enumerates integer partitions contained in an $n \times n$ triangle by their dinv and external area statistics. The paper by Lee et al. (SIAM J Discr Math 32:191–232, 2018) proposed a new approach to understanding the symmetry property $\mathrm{Cat}_n(q, t) = \mathrm{Cat}_n(t, q)$ based on decomposing the set of all integer partitions into infinite chains. Each such global chain $\mathcal{C}_\mu$ has an opposite chain $\mathcal{C}_{\mu^*}$; these combine to give a new small slice of $\mathrm{Cat}_n(q, t)$ that is symmetric in q and t. Here, we advance the agenda of Lee et al. (SIAM J Discr Math 32:191–232, 2018) by developing a new general method for building the global chains $\mathcal{C}_\mu$ from smaller elements called *local chains*. We define a *local opposite property* for local chains that implies the needed opposite property of the global chains. This local property is much easier to verify in specific cases compared to the corresponding global property. We apply this machinery to construct all global chains for partitions with deficit at most 11. This proves that for all n, the terms in $\mathrm{Cat}_n(q, t)$ of degree at least $\binom{n}{2} - 11$ are symmetric in q and t.

Mathematics Subject Classification. 05A19, 05A17, 05E05.

Keywords. q, t-Catalan numbers, Dyck paths, Dinv statistic, Joint symmetry, Integer partitions, Chain decompositions.

1. Introduction

The q, t-*Catalan numbers* $\mathrm{Cat}_n(q, t)$ are polynomials in q and t that reduce to the ordinary Catalan numbers when $q = t = 1$. These polynomials play a prominent role in modern algebraic combinatorics, with connections to representation theory, algebraic geometry, symmetric functions, knot theory, and

Kyungyong Lee was supported by NSF Grant DMS 1800207, the Korea Institute for Advanced Study (KIAS), and the University of Alabama. This work was supported by a grant from the Simons Foundation/SFARI (Grant #633564 to Nicholas A. Loehr).

other areas. Garsia and Haiman [5] originally defined these polynomials as sums of complicated rational functions indexed by integer partitions. Haglund [6] and Haiman independently discovered elegant combinatorial interpretations of the q, t-Catalan numbers as weighted sums of Dyck paths. Garsia and Haglund [4] proved that Haglund's combinatorial formula was equivalent to the original definition. More background on q, t-Catalan numbers may be found in Haglund's book [7] and in [11, Sec. 1].

One version of the combinatorial formula for $\mathrm{Cat}_n(q,t)$ is a sum over Dyck paths weighted by statistics called *area* and *dinv*. We can regard a Dyck path as the southeast border of a partition diagram contained in the triangle Δ_n with vertices $(0,0)$, $(0,n)$, and (n,n). This lets us rewrite the formula for $\mathrm{Cat}_n(q,t)$ as a weighted sum over all integer partitions that fit in this triangle:

$$\mathrm{Cat}_n(q,t) = \sum_{\gamma \subseteq \Delta_n} q^{|\Delta_n|-|\gamma|} t^{\mathrm{dinv}(\gamma)}. \tag{1.1}$$

(See Section 2.1 for the definition of $\mathrm{dinv}(\gamma)$ and other notation used in this formula.)

It is known [3,4] (see also [2,9,15]) that $\mathrm{Cat}_n(q,t) = \mathrm{Cat}_n(t,q)$ for every n, but it is a notoriously difficult open problem to give a combinatorial proof of this fact based on (1.1) or related formulas. In [13], the last three authors proposed an approach to this problem based on the following ideas. Instead of focusing only on integer partitions contained in a particular triangle Δ_n, we consider the infinite set Par of all integer partitions. We seek to decompose this set into a disjoint union of *chains* denoted $\mathcal{C}_\mu$, where each chain is indexed by an integer partition μ called a *deficit partition*. Each chain is an infinite sequence of partitions such that dinv increases by 1 as we move along the chain. Moreover, for each γ in the chain $\mathcal{C}_\mu$, the *deficit* statistic $\mathrm{defc}(\gamma) = |\gamma| - \mathrm{dinv}(\gamma)$ has the constant value $|\mu|$. Among other technical conditions, the chains $\mathcal{C}_\mu$ must satisfy the following crucial *opposite property*. For each $n \geq 0$ and collection $\mathcal{S}$ of partitions, define

$$\mathrm{Cat}_{n,\mathcal{S}}(q,t) = \sum_{\gamma \in \mathcal{S}:\, \gamma \subseteq \Delta_n} q^{|\Delta_n|-|\gamma|} t^{\mathrm{dinv}(\gamma)}. \tag{1.2}$$

The opposite property asserts that for each k, there is an involution $\mu \mapsto \mu^*$ on the set of partitions of k such that for every $n \geq 0$,

$$\mathrm{Cat}_{n,\mathcal{C}_\mu}(q,t) = \mathrm{Cat}_{n,\mathcal{C}_{\mu^*}}(t,q).$$

If such chains $\mathcal{C}_\mu$ can be constructed for all partitions μ of a fixed k, then we can deduce the joint symmetry of the terms in $\mathrm{Cat}_n(q,t)$ of degree $\binom{n}{2} - k$. At a finer level, every pair $\mathcal{C}_\mu$ and $\mathcal{C}_{\mu^*}$ that we build reveals a new "small slice" of the Catalan objects that is symmetric in q and t. A remarkable feature of this setup is that the infinite chains $\mathcal{C}_\mu$ and $\mathcal{C}_{\mu^*}$ (which do not depend on n) induce joint symmetry for all n simultaneously.

Here is a brief summary of the main results in [13] most relevant to our current work. Conjecture 6.9 of [13] gives a complete technical statement of the decomposition of Par into the chains $\mathcal{C}_\mu$ outlined above. A version of this conjecture appears as Conjecture 2.2 below. Section 2 of [13] explicitly constructs the chains $\mathcal{C}_\mu$ for one-row partitions $\mu = (k)$. In this case, $\mu^* = \mu$, and the self-opposite property $\mathrm{Cat}_{n,\mathcal{C}_{(k)}}(q,t) = \mathrm{Cat}_{n,\mathcal{C}_{(k)}}(t,q)$ is proved in Sect. 3 of [13]. Section sec:specificspschains of [13] constructs the chains $\mathcal{C}_\mu$ for two-row partitions of the form $\mu = (ab-b-1, b-1)$ and $\mu^* = (ab-a-1, a-1)$. The opposite property for these chains is proved in Sect. 5 of [13]. Finally, with the aid of results in Section 6 and extensive computer calculations, the online version of the appendix to [13] presents maps $\mu \mapsto \mu^*$ and chains $\mathcal{C}_\mu$ for all integer partitions μ of size at most 9. However, it should be emphasized that the chains in this appendix were found through exhaustive computer searches, not by any systematic construction. These searches become impractical for $|\mu| \geq 10$.

The main contribution of this paper is a new general method for building the *global* chains $\mathcal{C}_\mu$ by piecing together smaller *local* chains. The precise definition of a local chain is rather technical (see Sect. 3.5), but here is the rough idea. For each partition γ, we must keep track of the *least* integer n such that $\gamma \subseteq \Delta_n$; this integer is denoted $\min_\Delta(\gamma)$. A local chain is a sequence of partitions $\gamma(a), \gamma(a+1), \ldots, \gamma(b)$ such that $\mathrm{dinv}(\gamma(i)) = i$ for $a \leq i \leq b$, $\mathrm{defc}(\gamma(i))$ is constant for $a \leq i \leq b$, and the sequence $(\min_\Delta(\gamma(i)) : a \leq i \leq b)$ has a certain staircase structure (described later). We show how suitable local chains may be pasted together to form global chains. We introduce the idea of locally opposite local chains and use this concept to prove the needed opposite property of the global chains $\mathcal{C}_\mu$ and $\mathcal{C}_{\mu^*}$. The new local framework leads to much shorter and conceptually simpler proofs of the opposite property, compared to the very intricate computations that were given in Sects. 3 and 5 of [13]. We give a new conjecture on writing the set Par as a union of (partially overlapping) local chains, and we prove that this conjecture implies the earlier conjecture on the decomposition of Par into global chains. Finally, we construct global chains $\mathcal{C}_\mu$ satisfying the new local conjecture for all partitions μ of size at most 11. In contrast to [13], these global chains were found not through exhaustive computer searches, but rather by applying systematic operations for building local chains. The full technical details of these operations (in their general form) will be the subject of a future paper.

The rest of this article is organized as follows. Section 2 reviews the needed background material and definitions, which are included so that this paper can be read independently of [13], On the other hand, to avoid undue repetition of technical details, we do refer to [13] for the proofs of some specific results. Section 3 develops the theory of local chains, states the new structural conjecture for local chains, and proves that this conjecture implies the previous conjecture for global chains. Section 4 presents global chains $\mathcal{C}_\mu$ for $|\mu| \leq 11$ and explains how to verify that these chains satisfy the local conjecture. Section 5 contains the concluding remarks indicating intended directions of future research.

2. Background

This section reviews definitions and preliminary results on partitions, Dyck vectors, and a map ν that is useful for constructing chains.

2.1. Partition Statistics

An *integer partition* is a weakly decreasing finite sequence of positive integers. Given a partition $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_s)$, let $\gamma_i = 0$ for all $i > s$. Any of these zero parts may be appended to the sequence γ without changing the partition. The *length* of γ is $\ell(\gamma) = s$, the number of strictly positive parts of γ. The *diagram* of γ is the set

$$\mathrm{dg}(\gamma) = \{(i,j) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} : \ 1 \le i \le \ell(\gamma), \ 1 \le j \le \gamma_i\}.$$

We visualize the diagram as an array of left-justified unit squares with γ_i squares in the ith row from the top. The *conjugate partition* $\gamma' = (\gamma'_1, \gamma'_2, \ldots)$ is defined by letting γ'_j be the number of cells in the jth column of $\mathrm{dg}(\gamma)$, for $1 \le j \le \gamma_1$.

The *arm* of a cell $c = (i,j)$ in $\mathrm{dg}(\gamma)$ is $\mathrm{arm}(c) = \lambda_i - j$, which is the number of cells strictly right of c in its row. The *leg* of a cell $c = (i,j)$ in $\mathrm{dg}(\gamma)$ is $\mathrm{leg}(c) = \lambda'_j - i$, which is the number of cells strictly below c in its column. We can now define the following partition statistics:

- The *size* of γ is $|\gamma| = \sum_{i \ge 1} \gamma_i$, which is the number of cells in the diagram of γ.
- The *diagonal inversion count* $\mathrm{dinv}(\gamma)$ is the number of cells c in the diagram of γ such that $\mathrm{arm}(c) - \mathrm{leg}(c) \in \{0, 1\}$.
- The *deficit* of γ is $\mathrm{defc}(\gamma) = |\gamma| - \mathrm{dinv}(\gamma)$, which is a nonnegative integer.
- For each integer $n > 0$, the *n-triangle* Δ_n is the diagram of the partition $(n-1, n-2, \ldots, 3, 2, 1)$.
- The *minimum triangle size* of γ, denoted $\min_\Delta(\gamma)$, is the least integer n such that $\mathrm{dg}(\gamma) \subseteq \Delta_n$. Equivalently, $\min_\Delta(\gamma)$ is the least integer n such that $\gamma_i \le n - i$ for $1 \le i \le \ell(\gamma)$. (This statistic was denoted $\Delta(\gamma)$ in [13].)
- For any n such that $\mathrm{dg}(\gamma) \subseteq \Delta_n$, the *external area* of γ relative to Δ_n is $\mathrm{area}_n(\gamma) = |\Delta_n| - |\gamma| = \binom{n}{2} - |\gamma|$. This is the number of cells in the triangle Δ_n outside the diagram of γ.

Example 2.1. Let $\gamma = (5,4,1,1,1)$. This partition has length $\ell(\gamma) = 5$, size $|\gamma| = 12$, diagonal inversion count $\mathrm{dinv}(\gamma) = 8$, deficit $\mathrm{defc}(\gamma) = 4$, and minimum triangle size $\min_\Delta(\gamma) = 6$. Figure 1 shows the diagram of γ embedded in the non-minimal triangle Δ_7. Counting the shaded cells, we see that the external area of γ relative to Δ_7 is $\mathrm{area}_7(\gamma) = \binom{7}{2} - |\gamma| = 9$, whereas $\mathrm{area}_6(\gamma) = 3$. The eight cells marked with a dot contribute to $\mathrm{dinv}(\gamma)$, while the other four cells in the diagram of γ contribute to $\mathrm{defc}(\gamma)$. For example, the second cell c in row 1 contributes to $\mathrm{defc}(\gamma)$ since $\mathrm{arm}(c) = 3$ and $\mathrm{leg}(c) = 1$, while the third cell c' in row 1 contributes to $\mathrm{dinv}(\gamma)$ since $\mathrm{arm}(c') = 2$ and $\mathrm{leg}(c') = 1$.

Next, we define some special collections of integer partitions.

- Let Par be the set of all integer partitions.
- Let $\mathrm{Par}(n)$ be the set of all integer partitions of size n.

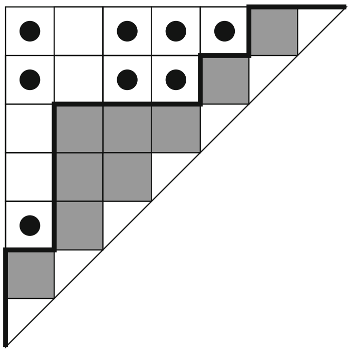

FIGURE 1. A partition contained in the triangle Δ_7

- Let $\mathcal{DP}(n) = \{\gamma \in \text{Par} : \text{dg}(\gamma) \subseteq \Delta_n\}$ be the set of all partitions whose diagrams fit in the n-triangle. We call such partitions *Dyck partitions of order n* since these partitions correspond bijectively to Dyck paths of order n by taking the southeast border of γ in Δ_n (see the thick shaded line in Fig. 1). Observe that
$$\mathcal{DP}(n) = \{\gamma \in \text{Par} : \min_\Delta(\gamma) \leq n\}. \tag{2.1}$$
- Let $\text{Def}(k) = \{\gamma \in \text{Par} : \text{defc}(\gamma) = k\}$ be the set of all partitions having deficit k. (This set was denoted $\mathcal{DP}_{*,k}$ in [13].)

Continuing Example 2.1, note that the partition $\gamma = (5, 4, 1, 1, 1)$ is a member of the sets Par(12), Def(4), and $\mathcal{DP}(n)$ for all $n \geq 6$, since $\min_\Delta(\gamma) = 6$.

We can now rewrite the definition (1.1) of q, t-Catalan numbers as follows:
$$\text{Cat}_n(q, t) = \sum_{\gamma \in \mathcal{DP}(n)} q^{\text{area}_n(\gamma)} t^{\text{dinv}(\gamma)}.$$
For example, $\mathcal{DP}(3) = \{(0), (1), (2), (1, 1), (2, 1)\}$, and
$$\text{Cat}_3(q, t) = q^3 + q^2 t + qt + qt^2 + t^3.$$

2.2. Dyck Vectors

For many calculations involving dinv, it is convenient to use Dyck vectors instead of Dyck partitions. A *Dyck vector* of order n is a list $v = (v_1, v_2, \ldots, v_n)$ of nonnegative integers such that $v_1 = 0$ and $v_{i+1} \leq v_i + 1$ for $1 \leq i < n$. Let $\mathcal{DV}(n)$ be the set of Dyck vectors of order n. There is a bijective correspondence between the sets $\mathcal{DP}(n)$ and $\mathcal{DV}(n)$, which can be defined pictorially by drawing $\gamma \in \mathcal{DP}(n)$ inside Δ_n and letting v_i be the number of external area cells in the ith row from the bottom. For example, letting $n = 7$, the partition $\gamma = (5, 4, 1, 1, 1, 0, 0)$ shown in Fig. 1 maps to the Dyck vector $v = (0, 1, 1, 2, 3, 1, 1)$. Formally, the bijection $\text{DV}_n : \mathcal{DP}(n) \to \mathcal{DV}(n)$ and its inverse $\mathrm{DP}_n : \mathcal{DV}(n) \to \mathcal{DP}(n)$ are given by these formulas:

$$\mathcal{DV}_n(\gamma_1, \gamma_2, \ldots, \gamma_n) = (0 - \gamma_n, 1 - \gamma_{n-1}, \ldots, i - \gamma_{n-i}, \ldots, n-1-\gamma_1); \tag{2.2}$$

$$\mathcal{DP}_n(v_1, v_2, \ldots, v_n) = (n-1-v_n, n-2-v_{n-1}, \ldots, n-i-v_{n-i+1}, \ldots, 1-v_2, 0-v_1). \tag{2.3}$$

We can compute the statistics area_n, dinv, and defc directly from the Dyck vector associated with a Dyck partition. In more detail, for any Dyck vector $v \in \mathcal{DV}(n)$, define

$$\begin{aligned} \mathrm{area}_n(v) &= v_1 + v_2 + \cdots + v_n; \\ \mathrm{dinv}(v) &= \text{the number of } i < j \text{ with } v_i - v_j \in \{0, 1\}; \\ \mathrm{defc}(v) &= \binom{n}{2} - \mathrm{area}_n(v) - \mathrm{dinv}(v). \end{aligned}$$

The bijections defined above preserve all three statistics, so that if $v = \mathrm{DV}_n(\gamma)$, then $\mathrm{area}_n(v) = \mathrm{area}_n(\gamma)$, $\mathrm{dinv}(v) = \mathrm{dinv}(\gamma)$, and $\mathrm{defc}(v) = \mathrm{defc}(\gamma)$. The verification of this assertion for dinv is not completely routine—see [8, Lemma 4.4.1] for details.

2.3. The Successor Map ν

This section recalls the definition and properties of the *successor map* ν, which is a function (defined on a subset of Par) that suffices to construct "almost all" of the links in the global chains $\mathcal{C}_\mu$. Intuitively, if $\gamma \in \mathcal{C}_\mu$ is a partition in the domain of ν, then $\nu(\gamma)$ is the next partition in the chain $\mathcal{C}_\mu$.

The domain of ν is $\{\gamma \in \mathrm{Par} : \gamma_1 \leq \ell(\gamma) + 2\}$. For $\gamma = (\gamma_1, \ldots, \gamma_\ell)$ in this domain, we define

$$\nu(\gamma) = (\ell(\gamma) + 1, \gamma_1 - 1, \gamma_2 - 1, \ldots, \gamma_\ell - 1).$$

Pictorially, we obtain the diagram of $\nu(\gamma)$ from the diagram of γ by removing the leftmost column, then inserting a new top row that is one cell longer than the removed column. For example, $\nu(5, 4, 1, 1, 1) = (6, 4, 3, 0, 0, 0) = (6, 4, 3)$, whereas $\nu(6, 4, 3)$ is undefined. The key property of ν (proved in Lemma 2.3 of [13]) is that for all γ in the domain of ν,

$$\mathrm{dinv}(\nu(\gamma)) = \mathrm{dinv}(\gamma) + 1 \quad \text{and} \quad \mathrm{defc}(\nu(\gamma)) = \mathrm{defc}(\gamma).$$

We may also conclude that $\mathrm{area}_n(\nu(\gamma)) = \mathrm{area}_n(\gamma) - 1$ if γ and $\nu(\gamma)$ are both in $\mathcal{DP}(n)$.

It is readily checked that the image of ν is the set $\{\delta \in \mathrm{Par} : \delta_1 \geq \ell(\delta)\}$. For a partition $\delta = (\delta_1, \ldots, \delta_s)$ in this set, we have

$$\nu^{-1}(\delta) = (\delta_2 + 1, \delta_3 + 1, \ldots, \delta_s + 1, \underline{1}^{\delta_1 - \ell(\delta)}),$$

where the notation $\underline{1}^{\delta_1 - \ell(\delta)}$ denotes $\delta_1 - \ell(\delta)$ copies of 1. We say $\delta \in \mathrm{Par}$ is an *initial* partition if $\nu^{-1}(\delta)$ is undefined, i.e., $\delta_1 < \ell(\delta)$. We say $\gamma \in \mathrm{Par}$ is a *final* partition if $\nu(\gamma)$ is undefined, i.e., $\gamma_1 > \ell(\gamma) + 2$.

Remarkably, for every partition μ, we can build the whole infinite tail of the chain $\mathcal{C}_\mu$ by starting with a particular partition $\mathrm{TI}(\mu)$ and applying ν

repeatedly. Suppose μ has n_1 parts equal to 1, n_2 parts equal to 2, and so on. Taking $N = \mu_1 + \ell(\mu) + 1$, the *tail initiator partition* of type μ is

$$\mathrm{TI}(\mu) = \textsc{dp}_N(0, 0, \underline{1}^{n_1}, 0, \underline{1}^{n_2}, 0, \ldots, 0, \underline{1}^{n_{\mu_1}}).$$

(This partition was denoted γ_μ in [13].) For example, $\mu = (4, 3, 1, 1, 1)$ has $n_1 = 3$, $n_2 = 0$, $n_3 = n_4 = 1$, and $N = 10$, so

$$\mathrm{TI}(\mu) = \textsc{dp}_{10}(0, 0, 1, 1, 1, 0, 0, 1, 0, 1) = (8, 8, 6, 6, 5, 3, 2, 1, 1, 0).$$

For any μ, the Dyck vector associated with $\mathrm{TI}(\mu)$ starts with two 0s and ends with a 1, which implies that the length of $\mathrm{TI}(\mu)$ is 1 more than the longest part of $\mathrm{TI}(\mu)$. Thus, every $\mathrm{TI}(\mu)$ is an initial partition. Moreover, it is shown in [13, Lemmas 6.7 and 6.8] that $\nu^m(\mathrm{TI}(\mu))$ is defined for every integer $m \geq 0$, $\mathrm{defc}(\mathrm{TI}(\mu)) = |\mu|$, and $\mathrm{dinv}(\mathrm{TI}(\mu)) = \binom{\mu_1+\ell(\mu)+1}{2} - \ell(\mu) - |\mu|$. We call the set

$$\textsc{tail}(\mu) = \{\nu^m(\mathrm{TI}(\mu)) : m \geq 0\}$$

the *ν-tail of the chain $\mathcal{C}_\mu$*. Note that μ is uniquely determined by the sequence $\textsc{tail}(\mu)$, as follows. First, $\mathrm{TI}(\mu)$ is the unique object with minimum dinv in $\textsc{tail}(\mu)$. Second, we can find the multiplicities of the parts of μ by counting consecutive 1s in $\textsc{dv}_N(\mathrm{TI}(\mu))$, where $N = \min_\Delta(\mathrm{TI}(\mu))$.

Now, suppose γ is any partition. The *ν-segment generated by* γ is the set of partitions obtained by applying ν and ν^{-1} to γ as many times as possible. Formally, the ν-segment $\nu^*(\gamma)$ is the set of all partitions $\nu^m(\gamma)$ for those integers m such that $\nu^m(\gamma)$ is defined. All ν-tails are ν-segments. An example of a finite ν-segment is

$$\nu^*(5, 2, 2, 2) = \{(3, 3, 3, 1), (5, 2, 2, 2), (5, 4, 1, 1, 1), (6, 4, 3)\}.$$

Since ν and ν^{-1} are one-to-one on their domains, the set Par and all of its subsets $\mathrm{Def}(k)$ are disjoint unions of ν-segments. We hope to express each chain $\mathcal{C}_\mu$ as the union of certain (suitably chosen) ν-segments, one of which is the ν-tail of $\mathcal{C}_\mu$. The hard part of the construction is figuring out which ν-segments can be combined to make the needed opposite property hold.

2.4. The Global Chain Decomposition Conjecture

We now have all the ingredients needed for the main structural conjecture on global chains. The conjecture stated here consists of parts (a), (b), (c), (d), (f), (g), and (j) of Conjecture 6.9 in [13]. Our conjecture on local chains (given in §3.7) implies this version of the global conjecture. Recall from (1.2) the notation

$$\mathrm{Cat}_{n,\mathcal{S}}(q, t) = \sum_{\gamma \in \mathcal{S} \cap \mathcal{DP}(n)} q^{\mathrm{area}_n(\gamma)} t^{\mathrm{dinv}(\gamma)}. \tag{2.4}$$

Conjecture 2.2. There exist collections of partitions $\mathcal{C}_\mu$, indexed by deficit partitions μ, and a size-preserving involution $\mu \mapsto \mu^*$ on Par, satisfying the following conditions:

(a) The collections $\mathcal{C}_\mu$ are pairwise disjoint.
(b) For all $\gamma \in \mathcal{C}_\mu$, $\mathrm{defc}(\gamma) = |\mu|$.

(c) Each $\mathcal{C}_\mu$ has the form $\{C_\mu(a), C_\mu(a+1), C_\mu(a+2), \ldots\}$, where $a = \ell(\mu^*)$ and $\operatorname{dinv}(C_\mu(i)) = i$ for all $i \geq a$.
(d) Every $\gamma \in \operatorname{Def}(k)$ belongs to $\mathcal{C}_\mu$ for some $\mu \in \operatorname{Par}(k)$.
(e) For all μ, $\textsc{tail}(\mu) \subseteq \mathcal{C}_\mu$.
(f) For all $\gamma \in \mathcal{C}_\mu$, if $\nu(\gamma)$ is defined then $\nu(\gamma) \in \mathcal{C}_\mu$.
(g) For all $n \geq 0$ and all μ, $\operatorname{Cat}_{n,\mathcal{C}_\mu}(q,t) = \operatorname{Cat}_{n,\mathcal{C}_{\mu^*}}(t,q)$.

Parts (a), (b), and (d) say that each set $\operatorname{Def}(k)$ is the disjoint union of the chains $\mathcal{C}_\mu$ with $\mu \in \operatorname{Par}(k)$. Parts (b) and (c) say that each chain $\mathcal{C}_\mu$ is an infinite string of objects of deficit $|\mu|$, where dinv increases by 1 as we move along the string, and the first object in the string has dinv equal to $\ell(\mu^*)$. Part (e) says that the "right end" of $\mathcal{C}_\mu$ is the ν-tail $\textsc{tail}(\mu)$. (More generally, Conjecture 6.9(g) of [13] asserts that each $\mathcal{C}_\mu$ is closed under ν and is therefore a union of ν-segments, but this cannot be deduced from our local conjecture.) Part (f) says that each chain $\mathcal{C}_\mu$ is closed under ν (and hence is closed under ν^{-1}). Part (f) is equivalent to requiring that for all $\gamma \in \mathcal{C}_\mu$, the whole ν-segment $\nu^*(\gamma)$ is contained in $\mathcal{C}_\mu$. Part (g) is the crucial *global opposite property* of the global chains. We note that μ^* is usually not the transpose of μ, and $\mu^* = \mu$ can occur. For instance, $(k)^* = (k)$ as seen in [13].

Remark 2.3. By invoking a deep result from [14], we can prove that there is a way to satisfy conditions (a) through (d) of Conjecture 2.2. Specifically, the main result of [14] yields an explicit (but extremely intricate) bijection Φ on Par that preserves size and sends dinv to the length of the first part:

$$|\Phi(\gamma)| = |\gamma| \text{ and } \Phi(\gamma)_1 = \operatorname{dinv}(\gamma) \text{ for all } \gamma \in \operatorname{Par}.$$

For each partition μ, let $\mathcal{B}_\mu$ be the set of partitions obtained by adding a new longest part to μ of any size $i \geq a$, where $a = \mu_1 = \ell(\mu')$. (So μ^* is μ' in this remark.) Let $\mathcal{C}_\mu = \{\Phi^{-1}(\gamma) : \gamma \in \mathcal{B}_\mu\}$. By the properties of Φ cited above, each object (i,μ) in $\mathcal{B}_\mu$ maps to an object $C_\mu(i)$ in $\mathcal{C}_\mu$ having dinv equal to i and deficit equal to $|\mu|$. Since Par is clearly the disjoint union of the sets $\mathcal{B}_\mu$, Par is also the disjoint union of the sets $\mathcal{C}_\mu$. But, we have checked that the chains $\mathcal{C}_\mu$ in this remark do *not* satisfy the opposite condition 2.2(g).

We can use Φ to compute the number of integer partitions of n having a given deficit k. For integers $a, b \geq 0$, let $p(a)$ be the number of integer partitions of a, and let $p(a,b)$ be the number of integer partitions of a with largest part at most b. These numbers satisfy the recursion $p(a,b) = p(a-1,b-1) + p(a-b,b)$ with appropriate initial conditions. To build an integer partition of n with deficit k, first choose a partition λ with $\lambda_1 = n-k$ and $(\lambda_2, \lambda_3, \ldots)$ any partition of k with largest part at most $n-k$. Then $\Phi^{-1}(\lambda)$ is a partition of n with dinv $n-k$ and deficit k. Thus the number of partitions of n with deficit k is $p(k, n-k)$. As n increases while k is fixed, this number ranges from 1 (when $n = k+1$) to $p(k)$ (when $n \geq 2k$).

3. Local Chains

This section develops the theory of local chains. Section 3.1 begins by relating the opposite property of global chains to the sequence of $\min_\Delta$ values of objects in the chains. The next three subsections study staircase sequences and sequences built from these by a pointwise minimum operation. It turns out that the needed opposite property has a remarkably simple proof in this abstract setting. The formal definition of local chains appears in Sects. 3.5 and 3.6. We state our main conjecture on local chains in Sect. 3.7, and we show how Conjecture 2.2 follows from this local conjecture.

3.1. The Opposite Property for $\min_\Delta$-Sequences

Suppose $\mathcal{S} = (\gamma(i) : i \geq a)$ is a sequence of partitions in $\mathrm{Def}(k)$ such that $\mathrm{dinv}(\gamma(i)) = i$ for all $i \geq a$. We have $\mathrm{defc}(\gamma(i)) = k$ and $|\gamma(i)| = k + i$ for all $i \geq a$. To compute the polynomials $\mathrm{Cat}_{n,\mathcal{S}}(q,t)$, we consider the function (sequence) $F : \mathbb{Z}_{\geq a} \to \mathbb{Z}$ defined by $F(i) = \min_\Delta(\gamma(i))$ for all $i \geq a$. We call F the $\min_\Delta$-*sequence associated with* $\mathcal{S}$. By (2.1) and (2.4), for any $n \geq 0$ we have

$$\mathrm{Cat}_{n,\mathcal{S}}(q,t) = \sum_{i:\ F(i)\leq n} q^{\binom{n}{2}-k-i} t^i. \tag{3.1}$$

Next, suppose $\mathcal{S}^* = (\delta(j) : j \geq b)$ is another sequence of partitions in $\mathrm{Def}(k)$ with $\mathrm{dinv}(\delta(j)) = j$ for all $j \geq b$. Let $G : \mathbb{Z}_{\geq b} \to \mathbb{Z}$ be the associated $\min_\Delta$-sequence given by $G(j) = \min_\Delta(\delta(j))$ for all $j \geq b$. For all $n \geq 0$, $\mathrm{Cat}_{n,\mathcal{S}^*}(t,q) = \sum_{j:\ G(j)\leq n} t^{\binom{n}{2}-k-j} q^j$. Making the change of variable $i = \binom{n}{2} - k - j$, we have

$$\mathrm{Cat}_{n,\mathcal{S}^*}(t,q) = \sum_{i:\ G(\binom{n}{2}-k-i)\leq n} q^{\binom{n}{2}-k-i} t^i.$$

Compared to (3.1), we see that the sequences $\mathcal{S}$ and $\mathcal{S}^*$ have the opposite property (namely, $\mathrm{Cat}_{n,\mathcal{S}^*}(t,q) = \mathrm{Cat}_{n,\mathcal{S}}(q,t)$ for all n) iff F and G are related by the following condition:

$$\text{for all integers } n \text{ and } i, F(i) \leq n \Leftrightarrow G\left(\binom{n}{2} - k - i\right) \leq n. \tag{3.2}$$

Here and below, a statement such as "$G(\binom{n}{2} - k - i) \leq n$" is an abbreviation for "$\binom{n}{2} - k - i$ is in the domain of G and $G(\binom{n}{2} - k - i) \leq n$." By definition, we say that any two functions F and G satisfying (3.2) have the *opposite property for deficit k*.

3.2. Staircase Sequences

We intend to build functions ($\min_\Delta$-sequences) having the opposite property (3.2) by taking the pointwise minimum of certain other sequences with special structure. These latter sequences are defined as follows.

Definition 3.1. Given integers a, m, h, the *infinite* (a, m, h)-*staircase* is the function (sequence) $F : \mathbb{Z}_{\geq a} \to \mathbb{Z}$ such that the values $(F(a), F(a+1), F(a+2), \ldots)$ consist of $m + 1$ copies of h, followed by h copies of $h + 1$, followed by $h + 1$ copies of $h + 2$, followed by $h + 2$ copies of $h + 3$, and so on. A *finite* (a, m, h)-*staircase* is any finite prefix of the sequence F, that is, a function obtained by restricting F to a domain of the form $\{a, a + 1, \ldots, b\}$.

Here is an explicit formula for the values of the infinite (a, m, h)-staircase F:

$$
\begin{array}{ll}
F(i) = h & \text{if } a \le i \le a+m; \\
F(i) = h+1 & \text{if } a+m+1 \le i \le a+m+h; \\
F(i) = h+2 & \text{if } a+m+h+1 \le i \le a+m+h+(h+1); \\
F(i) = h+3 & \text{if } a+m+h+(h+1)+1 \le i \le a+m+h+(h+1)+(h+2); \\
\dots & \dots \\
F(i) = h+p & \text{if } a+m+h+(h+1)+\cdots+(h+p-2)+1 \le i \\
 & \qquad \le a+m+h+(h+1)+\cdots+(h+p-1).
\end{array}
$$

So (taking $h+p=n$ above),

$$
\text{for all } n > h,\ F(i) = n \ \Leftrightarrow\ a+m+\binom{n-1}{2}-\binom{h}{2} < i \le a+m+\binom{n}{2}-\binom{h}{2}. \tag{3.3}
$$

The following result shows that staircase sequences arise naturally by taking the $\min_\Delta$-sequence associated with a ν-segment or ν-tail. The special case of a ν-tail was proved in [13, Lemma 6.8(2)].

Proposition 3.2. *Suppose γ is a partition with $\mathrm{dinv}(\gamma) = a$, $\min_\Delta(\gamma) = n$, and $m \ge 0$ is the least integer such that $\gamma'_{m+1} = n-m-1$. Let I be the set of $i \ge 0$ such that $\nu^i(\gamma)$ is defined. Then the sequence F with domain $\{a+i : i \in I\}$ given by $F(a+i) = \min_\Delta(\nu^i(\gamma))$ is an (a, m, n)-staircase.*

Proof. We first note that the value m has the following geometric interpretation. Draw the diagram of γ inside the minimal triangle Δ_n and look for the lowest point where a cell in the diagram touches the diagonal boundary $y = x$ of Δ_n. This point has coordinates $(m+1, m+1)$, as is readily checked. We call this point the *first-return point* for γ (relative to Δ_n).

Suppose $m \ge 1$ and $\nu(\gamma)$ is defined. As noted earlier, we obtain the diagram of $\nu(\gamma)$ by removing the leftmost column of $\mathrm{dg}(\gamma)$ and making a new top row that is 1 cell longer. Since the diagram of γ does not touch $(1,1)$, the new diagram still fits in Δ_n with first-return point (m, m). Thus, $\min_\Delta(\nu(\gamma)) = n$. Similarly, if $m \ge 2$ and $\nu^2(\gamma)$ is defined, then the diagram of $\nu^2(\gamma)$ still fits in Δ_n with first-return point $(m-1, m-1)$. For any $m \ge 0$, we can apply this reasoning for m steps (always assuming the relevant powers $\nu^i(\gamma)$ are defined), until we eventually obtain the diagram of $\nu^m(\gamma)$ inside Δ_n with first-return point $(1,1)$. Thus the first $m+1$ values of F are n, as needed. See Fig. 2 for an example where $\gamma = (5,5,3,3,1)$, $a = 14$, $n = 7$, $m = 2$, and the F-sequence starts $(\underline{7}^3, \underline{8}^7, \underline{9}^8, \underline{10}^9, \ldots)$.

Now consider $\nu^{m+1}(\gamma)$ (if defined). Here, we remove the first column of size $n-1$ and add a new first row of size n. This new row no longer fits inside Δ_n, but it does fit inside Δ_{n+1}. We conclude that $\min_\Delta(\nu^{m+1}(\gamma)) = n+1$, and the first-return point is now (n, n). Repeating the reasoning in the previous paragraph, we see that the next n partitions $\nu^{m+1}(\gamma), \ldots, \nu^{m+n}(\gamma)$ (if defined) will all have minimum triangle size $n+1$, as the first-return point moves from (n, n) to $(1,1)$ one step at a time. After that, the next $n+1$ partitions will have

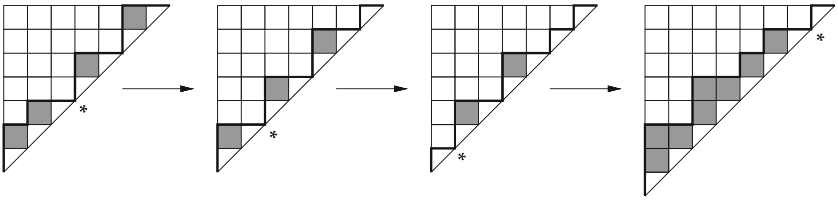

FIGURE 2. Finding the minimum triangle sizes for the sequence $(\nu^i(\gamma) : i \geq 0)$. The first-return point for each object is starred

minimum triangle size $n+2$, with first-return point moving from $(n+1, n+1)$ to $(1,1)$. This reasoning can be continued forever, unless we eventually reach a final partition where ν is undefined. In either case, we have proved that the sequence F is an (a, m, n)-staircase. $\square$

3.3. The Pointwise Minimum of Staircase Sequences

Suppose $F_1, \ldots, F_s$ are integer-valued sequences with respective domains $D_1, \ldots, D_s \subseteq \mathbb{Z}$. The *pointwise minimum* of $F_1, \ldots, F_s$ is the sequence F with domain $D = D_1 \cup \cdots \cup D_s$ such that for each $j \in D$, $F(j)$ is the least integer in the set $\{F_i(j) : j \in D_i\}$. We write $F = \min(F_1, \ldots, F_s)$. The next lemma gives an explicit description of the sets $\{i : F(i) \leq n\}$ in the case where F is the pointwise minimum of staircase sequences. For any $a, b \in \mathbb{Z}$, we use the notation $[a, b] = \{a, a+1, \ldots, b\}$ for an interval of consecutive integers. This interval is the empty set if $a > b$.

Lemma 3.3. *Let F_j be the infinite (a_j, m_j, h_j)-staircase for $1 \leq j \leq s$, and let $F = \min(F_1, \ldots, F_s)$. For every integer $n \geq 0$,*

$$\{i : F(i) \leq n\} = \bigcup_{j:\ n \geq h_j} \left[a_j, a_j + m_j + \binom{n}{2} - \binom{h_j}{2}\right]. \tag{3.4}$$

Proof. By (3.3), for $1 \leq j \leq s$ and any $n \geq h_j$,

$$\{i : F_j(i) \leq n\} = \left[a_j, a_j + m_j + \binom{n}{2} - \binom{h_j}{2}\right].$$

On the other hand, if $n < h_j$, then $\{i : F_j(i) \leq n\} = \emptyset$. Fix $n \geq 0$ and i in the domain of F. By definition of pointwise minimum and since F_j has minimum value h_j,

$$\begin{aligned} F(i) \leq n &\Leftrightarrow \text{for some } j, F_j(i) \leq n \\ &\Leftrightarrow \text{for some } j, n \geq h_j \text{ and } F_j(i) \leq n \\ &\Leftrightarrow i \in \bigcup_{j:\ n \geq h_j} \left[a_j, a_j + m_j + \binom{n}{2} - \binom{h_j}{2}\right]. \end{aligned}$$

$\square$

3.4. The Opposite Property for Minima of Staircases

Lemma 3.4. *Fix a deficit value $k \geq 0$, an integer $N \geq 2$, and nonnegative integers a_j, b_j, m_j, h_j for $0 < j < N$. For $0 < j < N$, let F_j be the infinite (a_j, m_j, h_j)-staircase, let G_j be the infinite $(b_{N-j}, m_{N-j}, h_{N-j})$-staircase, let $F = \min(F_j)$, and let $G = \min(G_j)$. Assume that*

$$a_j + b_j + m_j + k = \binom{h_j}{2} \text{ for } 0 < j < N. \tag{3.5}$$

Then, F and G have the opposite property (3.2) for deficit k.

Proof. Fix integers n and i. By Lemma 3.3 and the assumption $a_j + m_j - \binom{h_j}{2} = -b_j - k$,

$$\begin{aligned} F(i) \leq n \Leftrightarrow i \in \bigcup_{j:\ n \geq h_j} \left[a_j, \binom{n}{2} - b_j - k\right] \\ \Leftrightarrow \exists j, n \geq h_j \text{ and } a_j \leq i \leq \binom{n}{2} - b_j - k. \end{aligned}$$

Similarly, applying Lemma 3.3 to G and replacing j by $N - j$, we get:

$$\begin{aligned} & G\left(\binom{n}{2} - k - i\right) \leq n \\ & \Leftrightarrow \binom{n}{2} - k - i \in \bigcup_{j:\ n \geq h_{N-j}} \left[b_{N-j}, b_{N-j} + m_{N-j} + \binom{n}{2} - \binom{h_{N-j}}{2}\right] \\ & \Leftrightarrow \binom{n}{2} - k - i \in \bigcup_{j:\ n \geq h_j} \left[b_j, b_j + m_j + \binom{n}{2} - \binom{h_j}{2}\right] \\ & \Leftrightarrow \exists j, n \geq h_j \text{ and } b_j \leq \binom{n}{2} - k - i \leq \binom{n}{2} - k - a_j \\ & \Leftrightarrow \exists j, n \geq h_j \text{ and } a_j \leq i \leq \binom{n}{2} - b_j - k \Leftrightarrow F(i) \leq n. \end{aligned}$$

□

The next two lemmas consider a special situation where we can conclude $F_1 \geq F_2 \geq \cdots \geq F_{N-1}$; this situation will arise in our study of local chains.

Lemma 3.5. *Suppose F is the infinite (a, m, h)-staircase, G is the infinite (a', m', h')-staircase, $a + m < a'$, and $G(a') < F(a')$. Then for all $i \geq a'$, $G(i) \leq F(i)$.*

Proof. We know F and G are weakly increasing sequences whose values increase by 0 or 1 at each step. Let $y = F(a')$, let i_1 be the least integer with $F(i_1) = y$, and let i_2 be the least integer with $G(i_2) = y$. Since $a + m < a'$, we know $F(a) = \cdots = F(a + m) < F(a + m + 1) \leq y$, so $a + m < i_1 \leq a'$. Since $G(a') < y$, the definition of a staircase sequence shows that i_2 exists and $a' < i_2$. The values of the sequence F, from input i_1 onward, are $y - 1$ copies of y, then y copies of $y + 1$, and so on. The values of the sequence G, from input i_2 onward, are $y-1$ copies of y, then y copies of $y+1$, and so on. If $i_2 \le i$, then these remarks show that

$$G(i) = F(i_1 + (i - i_2)) = F(i + (i_1 - i_2)) \le F(i),$$

since $i_1 - i_2 < 0$. If $a' \le i < i_2$, then $G(i) \le G(i_2) = y = F(i_1) \le F(a') \le F(i)$. □

Lemma 3.6. *Suppose F_j is the infinite (a_j, m_j, h_j)-staircase for $1 \le j \le c$, $a_{j-1}+m_{j-1} < a_j$ and $F_j(a_j) < F_{j-1}(a_j)$ for $1 < j \le c$, and $F = \min_{1 \le j \le c}(F_j)$. Let $a_{c+1} = \infty$. If $a_j \le i < a_{j+1}$, then*

$$F(i) = \min\{F_1(i), F_2(i), \ldots, F_j(i)\} = F_j(i). \tag{3.6}$$

Proof. The domain of F_j is $\mathbb{Z}_{\ge a_j}$. If $a_j \le i < a_{j+1}$, then i is in the domain of $F_1, F_2, \ldots, F_j$, but not in the domain of $F_{j+1}, \ldots, F_c$. So the first equality in (3.6) follows from the definition of pointwise minimum. To get the second equality, we prove the following stronger statement by induction on j: for all $p < j$ and all $i \ge a_j$, $F_j(i) \le F_p(i)$. Fix $j > 1$, and assume that for all $p < j-1$ and all $i \ge a_{j-1}$, $F_{j-1}(i) \le F_p(i)$. Fix $i \ge a_j$. Since $a_{j-1} + m_{j-1} < a_j$ and $F_j(a_j) < F_{j-1}(a_j)$, Lemma 3.5 shows that $F_j(i) \le F_{j-1}(i)$. Combining this with the induction hypothesis, we see that $F_j(i) \le F_p(i)$ for all $p < j$, as needed. □

3.5. Ordinary Local Chains

We are now ready to define local chains. A sequence of partitions $\mathcal{S}$ is called an *ordinary local chain of deficit k* iff there exist nonnegative integers a, a', m, m', h, h' satisfying the following conditions. First, $\mathcal{S} = (\gamma(i) : a \le i \le a' + m')$ where $\operatorname{defc}(\gamma(i)) = k$ and $\operatorname{dinv}(\gamma(i)) = i$ for $a \le i \le a' + m'$. Second, the $\min_\Delta$-sequence $F = (\min_\Delta(\gamma(i)) : a \le i \le a' + m')$ associated with $\mathcal{S}$ satisfies

$$F(a'-1) > F(a') = F(a'+1) = \cdots = F(a'+m') = h'.$$

Third, $a + m + 1 < a'$ and the restriction of F to $\{a, a+1, \ldots, a'-1\}$ is an (a,m,h)-staircase, so in particular

$$h = F(a) = F(a+1) = \cdots = F(a+m) < F(a+m+1).$$

Since $F(a+m) < F(a+m+1)$ and $F(a'-1) > F(a')$, the integers a, a', m, m', h, h' are uniquely determined by $\mathcal{S}$, and we denote them $a_{\mathcal{S}}$, $a'_{\mathcal{S}}$, $m_{\mathcal{S}}$, $m'_{\mathcal{S}}$, $h_{\mathcal{S}}$, $h'_{\mathcal{S}}$ (respectively). We also define the *left part*, *middle part*, and *right part* of $\mathcal{S}$ to be

$$\begin{aligned} \text{LEFT}(\mathcal{S}) &= \{\gamma(j) : a \le j \le a+m\}; \\ \text{MID}(\mathcal{S}) &= \{\gamma(j) : a+m < j < a'\}; \\ \text{RIGHT}(\mathcal{S}) &= \{\gamma(j) : a' \le j \le a'+m'\}. \end{aligned}$$

It is helpful to visualize the conditions on the $\min_\Delta$-sequence of a local chain $\mathcal{S}$ by graphing the set of ordered pairs $(\operatorname{dinv}(\gamma), \min_\Delta(\gamma) : \gamma \in \mathcal{S})$ in the xy-plane. See Fig. 3 for an illustration of the structure of an ordinary local chain.

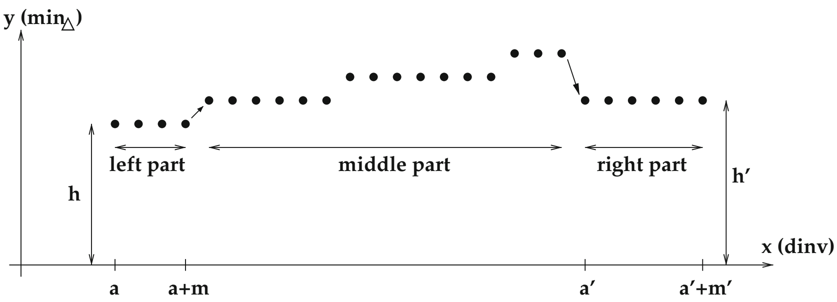

FIGURE 3. Structure of an ordinary local chain

Example 3.7. Here are two ordinary local chains of deficit 7:

$$\mathcal{S} = ((3333), (52222), (641111), (753), (44422), (633311));$$
$$\mathcal{T} = ((32222), (621111), (751), (43331), (63222), (652111)).$$

The following two-line arrays show the values of dinv and $\min_\Delta$ for objects in the sequences $\mathcal{S}$ and $\mathcal{T}$:

$$F_{\mathcal{S}} = \begin{bmatrix} \text{dinv} : 5\ 6\ 7\ 8\ 9\ 10 \\ \min_\Delta : 7\ 7\ 7\ 8\ 7\ \ \ 7 \end{bmatrix}; \qquad F_{\mathcal{T}} = \begin{bmatrix} \text{dinv} : 4\ 5\ 6\ 7\ 8\ 9 \\ \min_\Delta : 7\ 7\ 8\ 7\ 7\ 7 \end{bmatrix}.$$

We have

$$a_{\mathcal{S}} = 5,\ m_{\mathcal{S}} = 2,\ h_{\mathcal{S}} = 7,\ a'_{\mathcal{S}} = 9,\ m'_{\mathcal{S}} = 1,\ h'_{\mathcal{S}} = 7;$$
$$a_{\mathcal{T}} = 4,\ m_{\mathcal{T}} = 1,\ h_{\mathcal{T}} = 7,\ a'_{\mathcal{T}} = 7,\ m'_{\mathcal{T}} = 2,\ h'_{\mathcal{T}} = 7.$$

The left, middle, and right parts of $\mathcal{S}$ have size $m_{\mathcal{S}}+1 = 3$, 1, and $m'_{\mathcal{S}}+1 = 2$, respectively.

Suppose $\mathcal{S}$ and $\mathcal{T}$ are ordinary local chains of deficit k. We say that $\mathcal{S}$ is *locally opposite* to $\mathcal{T}$ iff $m_{\mathcal{T}} = m'_{\mathcal{S}}$, $m'_{\mathcal{T}} = m_{\mathcal{S}}$, $h_{\mathcal{T}} = h'_{\mathcal{S}}$, $h'_{\mathcal{T}} = h_{\mathcal{S}}$,

$$a_{\mathcal{S}} + m_{\mathcal{S}} + k + a'_{\mathcal{T}} = \binom{h_{\mathcal{S}}}{2}, \quad \text{and} \quad a'_{\mathcal{S}} + m'_{\mathcal{S}} + k + a_{\mathcal{T}} = \binom{h'_{\mathcal{S}}}{2}. \tag{3.7}$$

For instance, the local chains $\mathcal{S}$ and $\mathcal{T}$ in Example 3.7 are locally opposite because $m_{\mathcal{T}} = 1 = m'_{\mathcal{S}}$, $m'_{\mathcal{T}} = 2 = m_{\mathcal{S}}$, $h_{\mathcal{T}} = 7 = h'_{\mathcal{S}}$, $h'_{\mathcal{T}} = 7 = h_{\mathcal{S}}$, and (recalling $k = 7$)

$$5 + 2 + 7 + 7 = 21 = \binom{7}{2}, \quad 9 + 1 + 7 + 4 = 21 = \binom{7}{2}.$$

3.6. Exceptional Local Chains

We also need two types of *exceptional local chains of deficit k*. First, any one-element set $\mathcal{T} = \{\gamma\}$ with $\text{defc}(\gamma) = k$ is an exceptional local chain, and we define

$$a'_{\mathcal{T}} = \text{dinv}(\gamma),\ m'_{\mathcal{T}} = 0,\ h'_{\mathcal{T}} = \min_\Delta(\gamma),$$
$$\text{LEFT}(\mathcal{T}) = \emptyset,\ \text{MID}(\mathcal{T}) = \emptyset,\ \text{and}\ \text{RIGHT}(\mathcal{T}) = \{\gamma\}.$$

Second, for any $\mu \in \mathrm{Par}(k)$, the ν-tail $\mathcal{S} = \textsc{tail}(\mu) = \nu^*(\mathrm{TI}(\mu))$ is an exceptional local chain, and we define

$$a_{\mathcal{S}} = \mathrm{dinv}(\mathrm{TI}(\mu)),\ m_{\mathcal{S}} = 0,\ h_{\mathcal{S}} = \min_{\Delta}(\mathrm{TI}(\mu)),$$
$$\textsc{left}(\mathcal{S}) = \{\mathrm{TI}(\mu)\},\ \textsc{mid}(\mathcal{S}) = \mathcal{S} \setminus \{\mathrm{TI}(\mu)\},\ \text{and}\ \textsc{right}(\mathcal{S}) = \emptyset.$$

By Proposition 3.2 applied to $\gamma = \mathrm{TI}(\mu)$, the $\min_{\Delta}$ sequence associated with $\mathcal{S} = \textsc{tail}(\mu)$ is the infinite $(a_{\mathcal{S}}, m_{\mathcal{S}}, h_{\mathcal{S}})$-staircase.

Two exceptional local chains of deficit k are *locally opposite* iff one chain is $\mathcal{S} = \textsc{tail}(\mu)$ and the other chain is $\mathcal{T} = \{\gamma\}$ where $\mathrm{dinv}(\gamma) = \ell(\mu)$ and $\min_{\Delta}(\gamma) = \min_{\Delta}(\mathrm{TI}(\mu))$. Since we know $|\mu| = k$, $\min_{\Delta}(\mathrm{TI}(\mu)) = \mu_1 + \ell(\mu) + 1$, and $\mathrm{dinv}(\mathrm{TI}(\mu)) = \binom{\mu_1+\ell(\mu)+1}{2} - \ell(\mu) - |\mu|$ (see Sect. 2.3), it follows that $m'_{\mathcal{T}} = 0 = m_{\mathcal{S}}$, $h'_{\mathcal{T}} = \min_{\Delta}(\gamma) = h_{\mathcal{S}}$, and the first equation in (3.7) holds. It follows from these definitions that the relation "$\mathcal{S}$ is locally opposite to $\mathcal{T}$" is a symmetric relation on the set of all (ordinary and exceptional) local chains of deficit k.

Example 3.8. Here are two exceptional local chains of deficit 6:

$$\mathcal{S} = \textsc{tail}(411) = \nu^*(6654211) = ((6654211), (855431), (774432), \ldots);$$
$$\mathcal{T} = \{(3111111)\}.$$

Writing $\mu = (411)$, $\mathrm{TI}(\mu) = (6654211)$, and $\gamma = (3111111)$, we compute $\mathrm{dinv}(\gamma) = 3 = \ell(\mu)$ and $\min_{\Delta}(\gamma) = 8 = \min_{\Delta}(\mathrm{TI}(\mu))$. So $\mathcal{S}$ and $\mathcal{T}$ are locally opposite chains. Note that

$$a_{\mathcal{S}} = \mathrm{dinv}(\mathrm{TI}(\mu)) = 19,\ m_{\mathcal{S}} = 0,\ h_{\mathcal{S}} = 8,\ a'_{\mathcal{T}} = 3, m'_{\mathcal{T}} = 0, h'_{\mathcal{T}} = 8,$$
$$\text{and } a_{\mathcal{S}} + m_{\mathcal{S}} + k + a'_{\mathcal{T}} = 19 + 0 + 6 + 3 = 28 = \binom{h_{\mathcal{S}}}{2}.$$

3.7. The Local Chain Conjecture

We can now state our main structural conjecture on local chains.

Conjecture 3.9. For every $k \geq 0$, there is a set $\mathcal{L}$ of local chains of deficit k, and there is an involution $\mathcal{S} \mapsto \mathcal{S}^*$ on $\mathcal{L}$, satisfying the following conditions:

(a) For every $\mu \in \mathrm{Par}(k)$, $\textsc{tail}(\mu)$ belongs to $\mathcal{L}$.
(b) For any two distinct chains in $\mathcal{L}$, either the two chains are disjoint or the right part of one chain equals the left part of the other chain.
(c) Every γ in $\mathrm{Def}(k)$ belongs to exactly one or two local chains in $\mathcal{L}$. In the former case, γ belongs to the middle part of the chain. In the latter case, γ belongs to the right part of one chain and the left part of the other.
(d) For all $\mathcal{S}$ in $\mathcal{L}$, the local chains $\mathcal{S}$ and $\mathcal{S}^*$ are locally opposite.
(e) For all $\mathcal{S}, \mathcal{T} \in \mathcal{L}$, if $\mathcal{S}$ and $\mathcal{T}$ have nonempty intersection, then $\mathcal{S}^*$ and $\mathcal{T}^*$ have nonempty intersection.
(f) For all $\mathcal{S} \in \mathcal{L}$, $\textsc{left}(\mathcal{S}) \cup \textsc{mid}(\mathcal{S})$ is a union of ν-segments.

Aided by computer searches, we can explicitly construct local chains proving this conjecture for all $k \leq 11$. The details appear in Section 4. First, we prove that this conjecture for local chains implies the corresponding conjecture for global chains.

Theorem 3.10. *Conjecture 3.9 implies Conjecture 2.2.*

Proof. Assume Conjecture 3.9 holds for a fixed deficit value $k \geq 0$. We prove the conclusions of Conjecture 2.2 for partitions μ of k.

Step 1. We construct the global chains $\mathcal{C}_\mu$ for $\mu \in \mathrm{Par}(k)$. Start with the local chain $\mathcal{S} = \textsc{tail}(\mu)$, which belongs to $\mathcal{L}$. By 3.9(b) and (c), there is a unique $\mathcal{S}' \in \mathcal{L}$ with $\textsc{right}(\mathcal{S}') = \textsc{left}(\mathcal{S}) = \{\mathrm{TI}(\mu)\}$. If this chain $\mathcal{S}'$ is not a singleton, then there is a unique $\mathcal{S}'' \in \mathcal{L}$ with $\textsc{right}(\mathcal{S}'') = \textsc{left}(\mathcal{S}')$. We continue to paste together overlapping local chains in this way until eventually terminating at an exceptional (singleton) chain. Such a chain must be reached in finitely many steps, since the minimum dinv value for each local chain strictly decreases as we proceed. At the end, we have

$$\mathcal{C}_\mu = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_c, \tag{3.8}$$

where $\mathcal{S}_0 = \{\gamma\}$, $\mathcal{S}_c = \textsc{tail}(\mu)$, and $\textsc{right}(\mathcal{S}_j) = \textsc{left}(\mathcal{S}_{j+1})$ for $0 \leq j < c$. The last condition shows that $\mathcal{S}_{j+1}$ is uniquely determined by $\mathcal{S}_j$. Iterating this, we see that $\textsc{tail}(\mu)$ and hence μ are uniquely determined by any $\mathcal{S}_i$ in (3.8). So, two chains constructed in this way from two different partitions μ must be disjoint. So far, we have built chains $\mathcal{C}_\mu$ (for $\mu \in \mathrm{Par}(k)$) satisfying conditions 2.2(a), (b), (c), and (e), except for the claim $a = \ell(\mu^*)$ that will be proved later. Since a is the minimum value of dinv among all objects in $\mathcal{C}_\mu$, we see from the construction that $a = \mathrm{dinv}(\gamma)$ for the unique γ in $\mathcal{S}_0$.

Step 2. We prove two formulas expressing the $\min_\Delta$-sequence F_μ of $\mathcal{C}_\mu$ as the pointwise minimum of staircase sequences determined by the chains $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_c$ in (3.8). For brevity, write $a_j = a_{\mathcal{S}_j}$ for $0 \leq j \leq c$, and define m_j, h_j, a'_j, m'_j, and h'_j similarly. Let F_j denote the infinite (a_j, m_j, h_j)-staircase, and let F'_j denote the infinite (a'_j, m'_j, h'_j)-staircase. We claim

$$\min_{0<j\leq c} F_j = F_\mu = \min_{0\leq j<c} F'_j. \tag{3.9}$$

Since $\textsc{right}(S_{j-1}) = \textsc{left}(S_j)$ for $0 < j \leq c$, we must have $a'_{j-1} = a_j$, $m'_{j-1} = m_j$, $h'_{j-1} = h_j$, and hence $F'_{j-1} = F_j$ for $0 < j \leq c$ (compare to Figure 3). So the second equality in (3.9) follows from the first one. We prove the first equality with the help of Lemma 3.6. Let $F = \min(F_1, \ldots, F_c)$. By definition of local chains, $a_{j-1} + m_{j-1} < a'_{j-1} = a_j$ and

$$F_j(a_j) = h_j = h'_{j-1} < F_{j-1}(a'_{j-1} - 1) \leq F_{j-1}(a'_{j-1}) = F_{j-1}(a_j)$$

for $1 < j \leq c$. So for i in the range $a_j \leq i < a_{j+1}$ (taking $a_{c+1} = \infty$), the lemma tells us that $F(i) = F_j(i)$. Thus, it suffices to show that $F_\mu(i) = F_j(i)$ for all i in this range.

If $j < c$ and i satisfy $a_j \leq i < a_{j+1} = a'_j$, then the unique object $C_{\mu,i}$ in $\mathcal{C}_\mu$ with $\mathrm{dinv}(C_{\mu,i}) = i$ belongs to the left part or middle part of $\mathcal{S}_j$. Then, $F_\mu(i) = \min_\Delta(C_{\mu,i}) = F_j(i)$ by the third condition in the definition of an ordinary local chain. On the other hand, if $a_c \leq i$, then $F_\mu(i) = F_c(i)$ because the $\min_\Delta$-sequence of the ν-tail $\mathcal{S}_c$ is known to be the infinite (a_c, m_c, h_c)-staircase F_c.

Step 3. We construct the involution $\mu \mapsto \mu^*$ on $\mathrm{Par}(k)$ and verify the opposite property 2.2(g). Fix $\mu \in \mathrm{Par}(k)$ and consider the local chains $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_c$

in (3.8). Using 3.9(d), let $\mathcal{S}_0^*, \mathcal{S}_1^*, \ldots, \mathcal{S}_c^*$ be the corresponding chains in $\mathcal{L}$ such that $\mathcal{S}_j$ and $\mathcal{S}_j^*$ are locally opposite for $0 \leq j \leq c$. The exceptional chain $\mathcal{S}_0 = \{\gamma\}$ must be locally opposite to some ν-tail, so that $\mathcal{S}_0^* = \textsc{tail}(\mu^*)$ for some partition μ^* of k. Using 3.9(e), since $\mathcal{S}_0$ and $\mathcal{S}_1$ overlap, $\mathcal{S}_0^*$ and $\mathcal{S}_1^*$ must overlap as well, and in fact $\textsc{right}(\mathcal{S}_1^*) = \textsc{left}(\mathcal{S}_0^*)$. Similarly, since $\mathcal{S}_1$ and $\mathcal{S}_2$ overlap, $\mathcal{S}_1^*$ and $\mathcal{S}_2^*$ must also overlap, with $\textsc{right}(\mathcal{S}_2^*) = \textsc{left}(\mathcal{S}_1^*)$. We can continue this reasoning until reaching $\mathcal{S}_c^*$, which is locally opposite to $\mathcal{S}_c = \textsc{tail}(\mu)$ and must therefore be a singleton chain. We conclude that the decomposition (3.8) for the chain $\mathcal{C}_{\mu^*}$ looks like

$$\mathcal{C}_{\mu^*} = \mathcal{S}_c^* \cup \mathcal{S}_{c-1}^* \cup \cdots \cup \mathcal{S}_1^* \cup \mathcal{S}_0^*. \tag{3.10}$$

We now apply Lemma 3.4 with $N = c+1$, $a_j = a_{\mathcal{S}_j}$, $b_j = a'_{\mathcal{S}_j^*}$, $m_j = m_{\mathcal{S}_j} = m'_{\mathcal{S}_j^*}$, and $h_j = h_{\mathcal{S}_j} = h'_{\mathcal{S}_j^*}$ for $0 < j < N$. Define F_j, G_j, F, and G as in the lemma. The first equality in (3.9) shows that $F = F_\mu$. The second equality in (3.9) (applied to μ^*, and keeping in mind the reversal of the index order in (3.10)) shows that $G = F_{\mu^*}$. Since $\mathcal{S}_j$ and $\mathcal{S}_j^*$ are locally opposite chains, the lemma hypothesis (3.5) follows from the first equation in (3.7), which holds even if $j = c$. Therefore, the lemma applies to show that F_μ and F_{μ^*} have the opposite property for deficit k. As seen in §3.1, this property implies $\mathrm{Cat}_{n,\mathcal{C}_{\mu^*}}(t,q) = \mathrm{Cat}_{n,\mathcal{C}_\mu}(q,t)$.

Step 4. We prove $a = \ell(\mu^*)$ in 2.2(c) and deduce 2.2(d). With the above notation, we know $\mathcal{S}_0 = \{\gamma\}$ is locally opposite to $\mathcal{S}_0^* = \textsc{tail}(\mu^*)$, so $\mathrm{dinv}(\gamma) = \ell(\mu^*)$ by definition. We saw that $a = \mathrm{dinv}(\gamma)$ at the end of Step 1. Finally, Lemma 6.11 of [13] proves that part (d) of Conjecture 2.2 follows automatically from parts (a), (b), and (c), which are already known.

Step 5. We use 3.9(f) to prove 2.2(f). By construction, each global chain $\mathcal{C}_\mu$ is an overlapping union of certain local chains $\mathcal{S} \in \mathcal{L}$. By 3.9(b) and (c), we can also regard each global chain $\mathcal{C}_\mu$ as the disjoint union of the left and middle parts of these same local chains. Since each set $\textsc{left}(\mathcal{S}) \cup \textsc{mid}(\mathcal{S})$ is a union of ν-segments by hypothesis, so is the global chain $\mathcal{C}_\mu$. This proves that $\mathcal{C}_\mu$ is closed under ν (and ν^{-1}). □

4. Global and Local Chains for $|\mu| \leq 11$

This section presents specific chains $\mathcal{C}_\mu$ satisfying Conjectures 3.9 and 2.2 for all deficit partitions μ of size at most 11. First, we show that any putative global chain $\mathcal{C}_\mu$ can be decomposed into an overlapping union of local chains in at most one way. We will see that the local opposite property of the local chains comprising $\mathcal{C}_\mu$ and $\mathcal{C}_{\mu^*}$ can be checked quite easily, in contrast to the global opposite property from [13]. We give an example of this process by presenting the complete verification for deficit partitions μ of size 4. The appendix to [13] lists specific global chains $\mathcal{C}_\mu$ that happen to satisfy the new local conjecture for all μ with $0 \leq |\mu| \leq 6$. So, we do not repeat that data here. However, for deficit values larger than 6, some new chains are needed. We list these chains (and the data needed to verify the local opposite property) in the appendix at the end of this section. This proves the joint symmetry of the terms in $\mathrm{Cat}_n(q,t)$ of degree $\binom{n}{2} - k$ for all $n \geq 0$ and all $k \leq 11$.

4.1. Decomposing Global Chains into Local Chains

A given global chain $\mathcal{C}_\mu$ is an infinite sequence $(\gamma(i) : i \geq d)$ of Dyck partitions of deficit k. A convenient way to present such a chain is by specifying the initial partitions of the ν-segments comprising $\mathcal{C}_\mu$. This is a finite list that ends with the partition $\mathrm{TI}(\mu)$, which generates the ν-segment $\textsc{tail}(\mu)$. Now, it is a simple matter to compute the finite ν-segments starting at these initial partitions and tabulate the values of dinv and $\min_\Delta$ for the resulting objects. We thereby obtain a two-line array, which needs to be of the form

$$F_\mu = \begin{bmatrix} \mathrm{dinv}: & d & d+1 & d+2 & \cdots & e-1 & e & \cdots \\ \min_\Delta: & w_d & w_{d+1} & w_{d+2} & \cdots & w_{e-1} & w_e & \cdots \end{bmatrix},$$

where $w_i = \min_\Delta(\gamma(i))$ and $i = \mathrm{dinv}(\gamma(i))$ for all $i \geq d$. We can terminate the display at $e = \mathrm{dinv}(\mathrm{TI}(\mu))$, since the right end of the array (starting with the values for $\mathrm{TI}(\mu)$) is known to be an infinite $(e, 0, w_e)$-staircase by Proposition 3.2.

We now show that the global chain $\mathcal{C}_\mu$ can be decomposed into an overlapping union of local chains $\mathcal{S}_i$ in at most one way. This decomposition is readily deduced from the word $w = w_d w_{d+1} \cdots w_e$. On one hand, we know the local chain decomposition must begin with the exceptional local chain $\mathcal{S}_0 = \{\gamma(d)\}$ and end with the exceptional local chain $\mathcal{S}_c = \textsc{tail}(\mu) = \nu^*(\mathrm{TI}(\mu))$. On the other hand, we can uniquely build the local chains $\mathcal{S}_i$ for $i = 0, 1, 2, \ldots, c$ as follows. (Keep in mind Fig. 3, especially the arrows showing ascents and descents forced by the definition of local chains.) Scan w from left to right, looking for *descent positions* i where $w_i > w_{i+1}$. Each such descent marks a place where the middle part of the current local chain ends and the left part of the next local chain begins. The length of this new left part is the unique m such that $w_{i+1} = w_{i+2} = \cdots = w_{i+m} < w_{i+m+1}$. Also, the right part of the old local chain equals the left part of the new local chain. This process determines the values of a, a', m, m', h, and h' for each local chain $\mathcal{S}_i$. We must also check that the restriction of F_μ to each subinterval $\{a, a+1, \ldots, a'-1\}$ is an (a, m, h)-staircase, as required by the definition of local chains.

Example 4.1. Let $k = 6$ and $\mu = (42)$, so $\mathrm{TI}(\mu) = (554221)$. Given the global chain

$$\mathcal{C}_{(42)} = \nu^*(3111111) \cup \nu^*(42221) \cup \nu^*(44411) \cup \nu^*(554221),$$

let us find the constituent local chains for $\mathcal{C}_\mu$. The array of $(\mathrm{dinv}, \min_\Delta)$ values for the beginning of this chain is:

$$F_{(42)} = \left[\begin{array}{r|cc|ccc|ccccc|ccc} \mathrm{dinv}: & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \cdots \\ \min_\Delta: & 8 & 9 & 6 & 7 & 7 & 7 & 7 & 7 & 7 & 8 & 7 & 8 & \cdots \end{array}\right], \tag{4.1}$$

where the bars show where ν-segments begin and end. The word of $\min_\Delta$-values (with descents marked) is $w = 8, 9 > 6, \underline{7}^6, 8 > 7, \underline{8}^7, \underline{9}^8, \cdots$. Following the procedure above, we find the local chains

$$\mathcal{S}_0 = \{(3111111)\},$$

$$\mathcal{S}_1 = \nu^*(3111111) \cup \{(42221)\},$$
$$\mathcal{S}_2 = \nu^*(42221) \cup \nu^*(44411) \cup \{(554221)\},$$
$$\mathcal{S}_3 = \nu^*(554221) = \text{TAIL}(\mu).$$

Note that $\text{LEFT}(\mathcal{S}_2) \cup \text{MID}(\mathcal{S}_2)$ is the union of two ν-segments, and the necessary staircase property does hold. Because of the overlapping left and right parts of consecutive local chains, we can conveniently present the values of a, a', m, m', h, h' for all the local chains in a table such as the following:

		$\mathcal{S}_0$	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$	
a	a'	–	3	5	13	–
m	m'	–	0	0	0	–
h	h'	–	8	6	7	–

The entries in columns 1 and 2 mean that a, m, h are undefined for $\mathcal{S}_0$, while $a' = 3$, $m' = 0$, $h' = 8$ for $\mathcal{S}_0$. Reading columns 2 and 3, we see that $(a, m, h) = (3, 0, 8)$ for $\mathcal{S}_1$, whereas $(a', m', h') = (5, 0, 6)$ for $\mathcal{S}_1$. Next, we find $(a, m, h) = (5, 0, 6)$ and $(a', m', h') = (13, 0, 7)$ for $\mathcal{S}_2$. Finally, $(a, m, h) = (13, 0, 7)$, while (a', m', h') are undefined for $\mathcal{S}_3$. All of these values were found from inspection of (4.1) (compare to Fig. 3). For brevity, we use a shorter version of the table in the appendix. In this example, the abbreviated version consists of the three vectors $a = (3, 5, 13)$, $m = (0, 0, 0)$, and $h = (8, 6, 7)$.

4.2. Verifying the Local Opposite Property

Our next example shows how to check the local opposite property for the local chains comprising given global chains $\mathcal{C}_\mu$ and $\mathcal{C}_{\mu^*}$.

Example 4.2. For $\mu = (42)$, we have $\mu^* = (411)$, TI$(411) = (6654211)$, and

$$\mathcal{C}_{(411)} = \nu^*(221111) \cup \nu^*(33211) \cup \nu^*(6654211).$$

Proceeding as we did above, we find

$$F_{(411)} = \left[\begin{array}{r|cc|cccccccc|ccc} \text{dinv}: & 2 & 3 & 4 & 5 & \cdots & 10 & 11 & \cdots & 17 & 18 & 19 & 20 & \cdots \\ \min_\Delta: & 7 & 8 & 6 & 7 & \cdots & 7 & 8 & \cdots & 8 & 9 & 8 & 9 & \cdots \end{array} \right], \tag{4.2}$$

leading to local chains $\mathcal{T}_i$ with parameters shown here:

		$\mathcal{T}_0$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	
a	a'	–	2	4	19	–
m	m'	–	0	0	0	–
h	h'	–	7	6	8	–

To see that $\mathcal{C}_{(411)}$ is globally opposite to $\mathcal{C}_{(42)}$, we check that $\mathcal{T}_{3-i}$ is locally opposite to $\mathcal{S}_i$ for $i = 0, 1, 2, 3$, as follows. First, note that the m-vector for (411) is the reverse of the m-vector for (42), and the h-vector for (411) is the reverse of the h-vector for (42). Second, note that the first object in $\mathcal{C}_{(42)}$

has dinv $3 = \ell(411)$, while the first object in $\mathcal{C}_{(411)}$ has dinv $2 = \ell(42)$. Third, we directly verify Eq. (3.7) by computing

$$3+0+6+19 = 28 = \binom{8}{2}; \quad 5+0+6+4 = 15 = \binom{6}{2};$$
$$13+0+6+2 = 21 = \binom{7}{2}.$$

Example 4.3. For μ of size 4, we present the global chains $\mathcal{C}_\mu$ from the appendix to [13] and confirm the local opposite property. We list the initial objects of each ν-segment of $\mathcal{C}_\mu$, followed by the local chain parameters in abbreviated form. We can check by inspection the reversal properties of the m-vectors and h-vectors, as well as the equality of the least dinv value in $\mathcal{C}_\mu$ and the length of μ^*. We also show the verification of (3.7) in each case.

Involution on partitions of 4: $(4)^* = (4)$; $(31)^* = (22)$; $(211)^* = (1111)$.
Global chain $\mathcal{C}_{(4)}$: (11111), (2221), (3331), (44321).

Local chain parameters: $a = (1, 3, 10)$, $m = (0, 0, 0)$, $h = (6, 5, 6)$.

This chain is self-opposite, so the m-vector and h-vector are palindromes. We verify $1 + 0 + 4 + 10 = \binom{6}{2}$, $3 + 0 + 4 + 3 = \binom{5}{2}$, and $10 + 0 + 4 + 1 = \binom{6}{2}$ (this last check is redundant).

Global chain $\mathcal{C}_{(31)}$: (2211), (44311).
Local chain parameters: $a = (2, 9)$, $m = (0, 0)$, $h = (5, 6)$.

Global chain $\mathcal{C}_{(22)}$: (21111), (3221).
Local chain parameters: $a = (2, 4)$, $m = (0, 0)$, $h = (6, 5)$.
We verify $2 + 0 + 4 + 4 = \binom{5}{2}$ and $9 + 0 + 4 + 2 = \binom{6}{2}$.

Global chain $\mathcal{C}_{(211)}$: (32111), (43111), (44211).
Local chain parameters: $a = (4, 6, 8)$, $m = (0, 0, 0)$, $h = (6, 6, 6)$.

Global chain $\mathcal{C}_{(1111)}$: (31111), (42111), (43211).
Local chain parameters: $a = (3, 5, 7)$, $m = (0, 0, 0)$, $h = (6, 6, 6)$.
We verify $4 + 0 + 4 + 7 = 6 + 0 + 4 + 5 = 8 + 0 + 4 + 3 = \binom{6}{2}$.

4.3. Appendix: Chain Data

This appendix lists the global chains and values of a, m, h for all deficit partitions μ with $7 \le |\mu| \le 9$. The online extended appendix which presents this information for $|\mu| = 10$ and $|\mu| = 11$ is available in [10]. The SageMath code for checking the correctness of the global chains for $1 \le |\mu| \le 11$ is also available at the webpage in the above reference. In the data below, initial objects that do *not* start new local chains are marked N.

Involution on partitions of 7: $(7)^* = (7)$; $(31111)^* = (3211)$; $(211111)^* = (1111111)$; $(61)^* = (331)$; $(52)^* = (52)$; $(511)^* = (4111)$; $(43)^* = (322)$; $(421)^* = (421)$; $(2221)^* = (2221)$; $(22111)^* = (22111)$.
$\mathcal{C}_{(7)}$: (11111111), (22222), (33331), (44432), $(555421)^N$, $(6664321)^N$, (77654321).
$a = (1, 3, 6, 10, 28)$, $m = (0, 1, 2, 1, 0)$, $h = (9, 7, 7, 7, 9)$.

$\mathcal{C}_{(31111)}$: (3311111), (5311111), (5331111), (6431111), (6442111), (6542211), (77643211).
$a = (4, 6, 8, 10, 12, 14, 24)$, $m = (0, 0, 0, 0, 0, 0, 0)$, $h = (8, 8, 8, 8, 8, 8, 9)$.
$\mathcal{C}_{(3211)}$: (42111111), (4421111), (5422111), (5532111), (6533111), (6643111), (6644211).
$a = (5, 7, 9, 11, 13, 15, 17)$, $m = (0, 0, 0, 0, 0, 0, 0)$, $h = (9, 8, 8, 8, 8, 8, 8)$.
$\mathcal{C}_{(211111)}$: (43211111), (54211111), (54321111), (65321111), (65431111), (75432111), (76532111), (76543111), (77543211).
$a = (7, 9, 11, 13, 15, 17, 19, 21, 23)$, $m = (0, 0, 0, 0, 0, 0, 0, 0, 0)$,
$h = (9, 9, 9, 9, 9, 9, 9, 9, 9)$.
$\mathcal{C}_{(1111111)}$: (43111111), (53211111), (54311111), (64321111), (65421111), (65432111), (76432111), (76542111), (76543211).
$a = (6, 8, 10, 12, 14, 16, 18, 20, 22)$, $m = (0, 0, 0, 0, 0, 0, 0, 0, 0)$,
$h = (9, 9, 9, 9, 9, 9, 9, 9, 9)$.
$\mathcal{C}_{(61)}$: (511111), (3333), (44422), $(554421)^N$, (77654311).
$a = (3, 5, 9, 27)$, $m = (0, 2, 1, 0)$, $h = (7, 7, 7, 9)$.
$\mathcal{C}_{(331)}$: (21111111), (32222), (43331), (544311).
$a = (2, 4, 7, 11)$, $m = (0, 1, 2, 0)$, $h = (9, 7, 7, 7)$. $\mathcal{C}_{(52)}$: (2211111), (33221), $(555311)^N$, (6654221).
$a = (2, 4, 19)$, $m = (0, 0, 0)$, $h = (8, 6, 8)$.
$\mathcal{C}_{(511)}$: (32111111), (541111), (44322), (77654211).
$a = (4, 6, 8, 26)$, $m = (0, 0, 1, 0)$, $h = (9, 7, 7, 9)$.
$\mathcal{C}_{(4111)}$: (31111111), (42222), (533211), (77653211).
$a = (3, 5, 8, 25)$, $m = (0, 1, 0, 0)$, $h = (9, 7, 7, 9)$.
$\mathcal{C}_{(43)}$: (322111), (441111), (44222), (554111), (553321).
$a = (3, 5, 7, 10, 12)$, $m = (0, 0, 1, 0, 0)$, $h = (7, 7, 7, 7, 7)$.
$\mathcal{C}_{(322)}$: (222111), (332111), (43222), (544111), (553221).
$a = (2, 4, 6, 9, 11)$, $m = (0, 0, 1, 0, 0)$, $h = (7, 7, 7, 7, 7)$.
$\mathcal{C}_{(421)}$: (4111111), (522111), (443111), (552211), (6653311).
$a = (3, 5, 7, 9, 18)$, $m = (0, 0, 0, 0, 0)$, $h = (8, 7, 7, 7, 8)$.
$\mathcal{C}_{(2221)}$: (521111), (433111), (552111), (543311). $a = (4, 6, 8, 10)$, $m = (0, 0, 0, 0)$, $h = (7, 7, 7, 7)$.
$\mathcal{C}_{(22111)}$: (4221111), (532211), (6553211).
$a = (5, 7, 16)$, $m = (0, 0, 0)$, $h = (8, 7, 8)$.

Involution on partitions of 8: $(8)^* = (8)$; $(4211)^* = (4211)$; $(41111)^* = (41111)$; $(32111)^* = (32111)$; $(311111)^* = (311111)$; $(2111111)^* = (11111111)$; $(71)^* = (44)$; $(62)^* = (5111)$; $(611)^* = (521)$; $(53)^* = (2222)$; $(3221)^* = (422)$; $(431)^* = (332)$; $(3311)^* = (3311)$; $(22211)^* = (22211)$; $(221111)^* = (221111)$.
$\mathcal{C}_{(8)}$: (111111111), (222221), (33332), (444321), $(555431)^N$, $(6665321)^N$, $(77754321)^N$, (887654321).
$a = (1, 3, 6, 10, 36)$, $m = (0, 0, 1, 0, 0)$, $h = (10, 7, 7, 7, 10)$.
$\mathcal{C}_{(4211)}$: (33111111), (5411111), (5332111), (6531111), (6443111), (6642211), (77644211).
$a = (4, 6, 8, 10, 12, 14, 24)$, $m = (0, 0, 0, 0, 0, 0, 0)$, $h = (9, 8, 8, 8, 8, 8, 9)$.
$\mathcal{C}_{(41111)}$: (421111111), (4431111), (6422111), (5542111), (6533211), (887643211).

$a = (5, 7, 9, 11, 13, 32)$, $m = (0, 0, 0, 0, 0, 0)$, $h = (10, 8, 8, 8, 8, 10)$.
$\mathcal{C}_{(32111)}$: (42211111), (44211111), (54221111), (55321111), (65331111), (75431111), (75532111), (76533111), (77543111), (77553211).
$a = (5, 7, 9, 11, 13, 15, 17, 19, 21, 23)$, $m = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$, $h = (9, 9, 9, 9, 9, 9, 9, 9, 9, 9)$.
$\mathcal{C}_{(311111)}$: (431111111), (53311111), (64311111), (64421111), (65422111), (66432111), (76442111), (76542211), (887543211).
$a = (6, 8, 10, 12, 14, 16, 18, 20, 31)$, $m = (0, 0, 0, 0, 0, 0, 0, 0, 0)$, $h = (10, 9, 9, 9, 9, 9, 9, 9, 10)$.
$\mathcal{C}_{(2111111)}$: (532111111), (543111111), (643211111), (654211111), (654321111), (764321111), (765421111), (765432111), (875432111), (876532111), (876543111), (886543211).
$a = (8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30)$, $m = (\underline{0}^{12})$, $h = (\underline{10}^{12})$.
$\mathcal{C}_{(11111111)}$: (432111111), (542111111), (543211111), (653211111), (654311111), (754321111), (765321111), (765431111), (865432111), (876432111), (876542111), (876543211).
$a = (7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29)$, $m = (\underline{0}^{12})$, $h = (\underline{10}^{12})$.
$\mathcal{C}_{(71)}$: (222211), (33322), (444311), $(555331)^N$, $(6655321)^N$, (887654311).
$a = (2, 5, 9, 35)$, $m = (0, 1, 0, 0)$, $h = (7, 7, 7, 10)$.
$\mathcal{C}_{(44)}$: (211111111), (322221), (43332), (544321).
$a = (2, 4, 7, 11)$, $m = (0, 0, 1, 0)$, $h = (10, 7, 7, 7)$.
$\mathcal{C}_{(62)}$: (321111111), (551111), (44332), $(6664311)^N$, (77654221).
$a = (4, 6, 8, 26)$, $m = (0, 0, 1, 0)$, $h = (10, 7, 7, 9)$.
$\mathcal{C}_{(5111)}$: (22111111), (33222), (443211), (887653211).
$a = (2, 4, 7, 33)$, $m = (0, 1, 0, 0)$, $h = (9, 7, 7, 10)$.
$\mathcal{C}_{(611)}$: (41111111), (522211), (444211), $(554331)^N$, (887654211).
$a = (3, 5, 8, 34)$, $m = (0, 0, 0, 0)$, $h = (9, 7, 7, 10)$.
$\mathcal{C}_{(521)}$: (311111111), (422221), (533311), $(554411)^N$, (77653311).
$a = (3, 5, 8, 25)$, $m = (0, 0, 0, 0)$, $h = (10, 7, 7, 9)$.
$\mathcal{C}_{(53)}$: (422211), (444111), (552221), $(555221)^N$, (6653321).
$a = (4, 7, 9, 18)$, $m = (0, 0, 0, 0)$, $h = (7, 7, 7, 8)$.
$\mathcal{C}_{(2222)}$: (2221111), (333111), (432221), (543221).
$a = (2, 4, 6, 9)$, $m = (0, 0, 0, 0)$, $h = (8, 7, 7, 7)$.
$\mathcal{C}_{(3221)}$: (3221111), (432211), (6553111), (6643311).
$a = (3, 5, 14, 16)$, $m = (0, 0, 0, 0)$, $h = (8, 7, 8, 8)$.
$\mathcal{C}_{(422)}$: (5211111), (4331111), (542221), $(555211)^N$, (6653221).
$a = (4, 6, 8, 17)$, $m = (0, 0, 0, 0)$, $h = (8, 8, 7, 8)$.
$\mathcal{C}_{(431)}$: (322211), (442211), (6653111), (6644311).
$a = (3, 6, 15, 17)$, $m = (0, 0, 0, 0)$, $h = (7, 7, 8, 8)$.
$\mathcal{C}_{(332)}$: (5111111), (5221111), (532221), (544221).
$a = (3, 5, 7, 10)$, $m = (0, 0, 0, 0)$, $h = (8, 8, 7, 7)$.
$\mathcal{C}_{(3311)}$: (3321111), (43322), (6554211). $a = (4, 6, 16)$, $m = (0, 1, 0)$, $h = (8, 7, 8)$.
$\mathcal{C}_{(22211)}$: (4411111), (5322111), (5531111), (6433111), (6642111), (6544211).
$a = (5, 7, 9, 11, 13, 15)$, $m = (0, 0, 0, 0, 0, 0)$, $h = (8, 8, 8, 8, 8, 8)$.
$\mathcal{C}_{(221111)}$: (53111111), (6421111), (5442111), (6532211), (76643211).

$a = (6, 8, 10, 12, 22)$, $m = (0, 0, 0, 0, 0)$, $h = (9, 8, 8, 8, 9)$.

Involution on partitions of 9: $(9)^* = (9)$, $(54)^* = (54)$, $(531)^* = (432)$, $(522)^* = (33111)$, $(5211)^* = (51111)$, $(4311)^* = (4221)$, $(42111)^* = (411111)$, $(3321)^* = (3321)$, $(32211)^* = (3222)$, $(321111)^* = (321111)$, $(3111111)^* = (3111111)$, $(22221)^* = (22221)$, $(2211111)^* = (222111)$, $(21111111)^* = (21111111)$, $(111111111)^* = (111111111)$, $(81)^* = (441)$, $(72)^* = (621)$, $(711)^* = (711)$, $(63)^* = (63)$, $(6111)^* = (333)$.

$\mathcal{C}_{(9)}$: (1111111111), (222222), (333321), $(444421)^N$, (555432), $(6665421)^N$, $(77764321)^N$, $(888654321)^N$, (9987654321).
$a = (1, 3, 6, 15, 45)$, $m = (0, 1, 0, 1, 0)$, $h = (11, 8, 7, 8, 11)$.
$\mathcal{C}_{(54)}$: (2222111), (3331111), (442221), $(555111)^N$, $(553331)^N$, (6654111), (6644321).
$a = (2, 4, 6, 15, 17)$, $m = (0, 0, 0, 0, 0)$, $h = (8, 8, 7, 8, 8)$.
$\mathcal{C}_{(531)}$: (3222111), (6311111), (6331111), (6441111), (554222), (6652211), (77644311).
$a = (3, 5, 7, 9, 11, 14, 24)$, $m = (0, 0, 0, 0, 1, 0, 0)$, $h = (8, 8, 8, 8, 8, 8, 9)$.
$\mathcal{C}_{(432)}$: (51111111), (5222111), (532222), (5533111), (6633111), (6644111), (6644221).
$a = (3, 5, 7, 10, 12, 14, 16)$, $m = (0, 0, 1, 0, 0, 0, 0)$, $h = (9, 8, 8, 8, 8, 8, 8)$.
$\mathcal{C}_{(522)}$: (52211111), (533221), $(6664211)^N$, (77653221).
$a = (5, 7, 24)$, $m = (0, 0, 0)$, $h = (9, 7, 9)$.
$\mathcal{C}_{(33111)}$: (32211111), (433211), (76653211).
$a = (3, 5, 22)$, $m = (0, 0, 0)$, $h = (9, 7, 9)$.
$\mathcal{C}_{(5211)}$: (4211111111), (4432111), (6522111), (5543111), (6633211), (887644211).
$a = (5, 7, 9, 11, 13, 32)$, $m = (0, 0, 0, 0, 0, 0)$, $h = (11, 8, 8, 8, 8, 10)$.
$\mathcal{C}_{(51111)}$: (331111111), (6411111), (6332111), (6541111), (6443211), (9987643211).
$a = (4, 6, 8, 10, 12, 41)$, $m = (0, 0, 0, 0, 0, 0)$, $h = (10, 8, 8, 8, 8, 11)$.
$\mathcal{C}_{(4311)}$: (52111111), (4422111), (542222), (6442211), (77643111), (77554211).
$a = (4, 6, 8, 11, 21, 23)$, $m = (0, 0, 1, 0, 0, 0)$, $h = (9, 8, 8, 8, 9, 9)$.
$\mathcal{C}_{(4221)}$: (33211111), (54111111), (5522111), (553222), (6552211), (77643311).
$a = (4, 6, 8, 10, 13, 23)$, $m = (0, 0, 0, 1, 0, 0)$, $h = (9, 9, 8, 8, 8, 9)$.
$\mathcal{C}_{(42111)}$: (4311111111), (53321111), (65311111), (64431111), (75422111), (66532111), (76443111), (77542211), (887553211).
$a = (6, 8, 10, 12, 14, 16, 18, 20, 31)$, $m = (0, 0, 0, 0, 0, 0, 0, 0, 0)$, $h = (11, 9, 9, 9, 9, 9, 9, 9, 10)$.
$\mathcal{C}_{(411111)}$: (422111111), (44311111), (64221111), (55421111), (65332111), (76431111), (75542111), (76533211), (9987543211).
$a = (5, 7, 9, 11, 13, 15, 17, 19, 40)$, $m = (0, 0, 0, 0, 0, 0, 0, 0, 0)$, $h = (10, 9, 9, 9, 9, 9, 9, 9, 11)$.
$\mathcal{C}_{(3321)}$: (6211111), (433311), $(544411)^N$, (6553311).
$a = (4, 6, 15)$, $m = (0, 0, 0)$, $h = (8, 7, 8)$.
$\mathcal{C}_{(32211)}$: (4222111), (6321111), (5441111), (6432211), (76643111), (77544211).
$a = (4, 6, 8, 10, 20, 22)$, $m = (0, 0, 0, 0, 0, 0)$, $h = (8, 8, 8, 8, 9, 9)$.
$\mathcal{C}_{(3222)}$: (44111111), (53221111), (5433111), (6632111), (6544111), (6643221).
$a = (5, 7, 9, 11, 13, 15)$, $m = (0, 0, 0, 0, 0, 0)$, $h = (9, 9, 8, 8, 8, 8)$.

$\mathcal{C}_{(321111)}$: (531111111), (533111111), (643111111), (644211111), (654221111), (664321111), (764421111), (765422111), (775432111), (875532111), (876533111), (886543111), (886643211).

$a = (6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30)$, $m = (\underline{0}^{13})$, $h = (\underline{10}^{13})$.

$\mathcal{C}_{(3111111)}$: (4321111111), (542211111), (553211111), (653311111), (754311111), (755321111), (765331111), (865431111), (866432111), (876442111), (876542211), (9986543211).

$a = (7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 39)$, $m = (\underline{0}^{12})$, $h = (11, \underline{10}^{10}, 11)$.

$\mathcal{C}_{(22221)}$: (4322111), (5521111), (543222), (6552111), (6543311).

$a = (5, 7, 9, 12, 14)$, $m = (0, 0, 1, 0, 0)$, $h = (8, 8, 8, 8, 8)$.

$\mathcal{C}_{(2211111)}$: (43311111), (64211111), (54421111), (65322111), (66431111), (75442111), (76532211), (877543211).

$a = (6, 8, 10, 12, 14, 16, 18, 29)$, $m = (0, 0, 0, 0, 0, 0, 0, 0)$, $h = (9, 9, 9, 9, 9, 9, 9, 10)$.

$\mathcal{C}_{(222111)}$: (442111111), (55311111), (64331111), (75421111), (65532111), (76433111), (77542111), (76553211).

$a = (7, 9, 11, 13, 15, 17, 19, 21)$, $m = (0, 0, 0, 0, 0, 0, 0, 0)$, $h = (10, 9, 9, 9, 9, 9, 9, 9)$.

$\mathcal{C}_{(2\underline{1}^7)}$: $(532\underline{1}^7)$, $(543\underline{1}^7)$, $(6432\underline{1}^6)$, $(6542\underline{1}^6)$, $(65432\underline{1}^5)$, $(76432\underline{1}^5)$, (7654211111), (7654321111), (8754321111), (8765321111), (8765431111), (9765432111), (9875432111), (9876532111), (9876543111), (9976543211).

$a = (8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38)$, $m = (\underline{0}^{16})$, $h = (\underline{11}^{16})$.

$\mathcal{C}_{(\underline{1}^9)}$: $(542\underline{1}^7)$, $(5432\underline{1}^6)$, $(6532\underline{1}^6)$, $(6543\underline{1}^6)$, $(75432\underline{1}^5)$, $(76532\underline{1}^5)$, (7654311111), (8654321111), (8764321111), (8765421111), (8765432111), (9865432111), (9876432111), (9876542111), (9876543211).

$a = (9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37)$, $m = (\underline{0}^{15})$, $h = (\underline{11}^{15})$.

$\mathcal{C}_{(81)}$: (6111111), (333311), $(44442)^N$, (555422), $(6664421)^N$, $(77664321)^N$, (9987654311).

$a = (3, 5, 14, 44)$, $m = (0, 0, 1, 0)$, $h = (8, 7, 8, 11)$.

$\mathcal{C}_{(441)}$: (2111111111), (322222), (433321), $(544421)^N$, (6554311).

$a = (2, 4, 7, 16)$, $m = (0, 1, 0, 0)$, $h = (11, 8, 7, 8)$.

$\mathcal{C}_{(72)}$: (411111111), (522221), (444221), $(554431)^N$, $(77754311)^N$, (887654221).

$a = (3, 5, 8, 34)$, $m = (0, 0, 0, 0)$, $h = (10, 7, 7, 10)$.

$\mathcal{C}_{(621)}$: (221111111), (332221), (443311), $(6655311)^N$, (887653311).

$a = (2, 4, 7, 33)$, $m = (0, 0, 0, 0)$, $h = (10, 7, 7, 10)$.

$\mathcal{C}_{(711)}$: (3111111111), (422222), (44441), (555322), $(6654421)^N$, (9987654211).

$a = (3, 5, 8, 13, 43)$, $m = (0, 1, 3, 1, 0)$, $h = (11, 8, 8, 8, 11)$.

$\mathcal{C}_{(63)}$: (22211111), (333211), (533321), $(555411)^N$, $(6664221)^N$, (77653321).

$a = (2, 4, 8, 25)$, $m = (0, 0, 0, 0)$, $h = (9, 7, 7, 9)$.

$\mathcal{C}_{(6111)}$: (332211), (554322), (9987653211).

$a = (3, 12, 42)$, $m = (0, 1, 0)$, $h = (7, 8, 11)$.

$\mathcal{C}_{(333)}$: (3211111111), (432222), (543321).

$a = (4, 6, 9)$, $m = (0, 1, 0)$, $h = (11, 8, 7)$.

5. Future Work

This paper has focused on the *ordinary* q, t-Catalan numbers $\mathrm{Cat}_n(q, t)$, which enumerate integer partitions contained in the $n \times n$ triangles bounded by the diagonal line $y = x$. Recently, many researchers have studied the more general *rational* q, t-Catalan numbers $\mathrm{Cat}_{a,b}(q, t)$, which enumerate integer partitions contained in triangles with vertices $(0, 0)$, $(0, a)$, and (b, a). See [1, §6] for a combinatorial definition of rational q, t-Catalan numbers. We call a/b the *slope parameter* for $\mathrm{Cat}_{a,b}(q, t)$.

It is likely that the machinery developed here (global chains, local chains, joint symmetry proofs, etc.) can be extended to rational q, t-Catalan numbers, although many technical issues still need to be resolved. The first (straightforward) step is to generalize the partition statistics from Sect. 2.1—size, dinv, deficit, external area, and minimum triangle size—to containing triangles of a fixed slope $s = a/b$. For example, $\min_{\Delta,s}(\gamma)$ would be the least $a \geq 0$ such that the diagram of γ fits inside the triangle with vertices $(0, 0)$, $(0, a)$, and $(a/s, a)$. The next (nontrivial) step would be to develop an analog of the successor map ν for slope s and prove a version of Proposition 3.2 explaining how $\min_{\Delta,s}$ changes upon iteration of this map. For slope $s = 1/M$ where M is a positive integer, we already have a candidate successor map, namely the map f_0 in Definition 8 of [12] (see also Lemma 9 of that reference). But the interaction of f_0 and $\min_{\Delta,1/M}$ is not yet understood.

We also hope to extend the local chain technology to q, t-parking functions and their generalizations (such as rational-slope parking functions [1] and word parking functions). Here the situation is even more complicated because, in general, there must be infinitely many chains of objects for each deficit $k > 0$. We intend to explore these directions more fully in future papers.

Acknowledgements

The authors are grateful to the anonymous referees for their valuable comments.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] D. Armstrong, N. Loehr, and G. Warrington, "Rational parking functions and Catalan numbers," *Ann. Comb.* **20** (2016), 21–58.

[2] E. Carlsson and A. Mellit, "A proof of the shuffle conjecture," *J. Amer. Math. Soc.* **31** (2018), 661–697.

[3] A. Garsia and J. Haglund, "A positivity result in the theory of Macdonald polynomials," *Proc. Natl. Acad. Sci. USA* **98** (2001), 4313–4316.

[4] A. Garsia and J. Haglund, "A proof of the q, t-Catalan positivity conjecture," *Discrete Math.* **256** (2002), 677–717.

[5] A. Garsia and M. Haiman, "A remarkable q, t-Catalan sequence and q-Lagrange inversion," *J. Algebraic Combin.* **5** (1996), 191–244.

[6] J. Haglund, "Conjectured statistics for the q, t-Catalan numbers," *Adv. in Math.* **175** (2003), 319–334.

[7] J. Haglund, The q, t-Catalan Numbers and the Space of Diagonal Harmonics, with an Appendix on the Combinatorics of Macdonald Polynomials, AMS University Lecture Series (2008).

[8] J. Haglund, M. Haiman, N. Loehr, J. Remmel, and A. Ulyanov, "A combinatorial formula for the character of the diagonal coinvariants," *Duke Math. J.* **126** (2005), 195–232.

[9] M. Haiman, "Vanishing theorems and character formulas for the Hilbert scheme of points in the plane," *Invent. Math.* **149** (2002), 371–407.

[10] S. Han, K. Lee, L. Li, and N. Loehr, Extended Appendix for "Chain Decompositions of q, t-Catalan Numbers via Local Chains," available online at `sites.google.com/oakland.edu/li2345/code-and-data`

[11] K. Lee, L. Li, and N. Loehr, "Limits of modified higher q, t-Catalan numbers, *Electron. J. Combin.* **20(3)** (2013), research paper P4, 23 pages (electronic).

[12] K. Lee, L. Li, and N. Loehr, "Combinatorics of certain higher q, t-Catalan polynomials: chains, joint symmetry, and the Garsia-Haiman formula," *J. Algebraic Combin.* **39** (2014), 749–781.

[13] K. Lee, L. Li, and N. Loehr, "A combinatorial approach to the symmetry of q, t-Catalan numbers," *SIAM J. Discrete Math.* **32** (2018), 191–232.

[14] N. Loehr and G. Warrington, "A continuous family of partition statistics equidistributed with length," *J. Combin. Theory Ser. A* **116** (2009), 379–403.

[15] Anton Mellit, "Toric braids and (m, n)-parking functions," arXiv:1604.07456 (2016).

Seongjune Han and Kyungyong Lee
Department of Mathematics
University of Alabama
Tuscaloosa AL 35401
USA
e-mail: shan25@crimson.ua.edu

Kyungyong Lee
e-mail: kyungyong.lee@ua.edu;
klee1@kias.re.kr

Kyungyong Lee
Korea Institute for Advanced Study
Seoul 02455
Republic of Korea

Li Li
Department of Mathematics and Statistics
Oakland University
Rochester MI 48309
USA
e-mail: li2345@oakland.edu

Nicholas A. Loehr
Department of Mathematics
Virginia Tech
Blacksburg VA 24061-0123
USA
e-mail: nloehr@vt.edu

Received: 25 March 2020.
Accepted: 23 September 2020.