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Note on generalized symmetries, gapless excitations, generalized symmetry protected topological states, and anomaly

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Abstract. We consider quantum many body systems with generalized symmetries, such as the higher form symmetries introduced recently, and the 'tensor symmetry'. We consider a general form of lattice Hamiltonians which allow a certain level of nonlocality. Based on the assumption of dual generalized symmetries, we explicitly construct low energy excited states. We also derive the 't Hooft anomaly for the general Hamiltonians after 'gauging' the dual generalized symmetries. A 3d system with dual anomalous 1-form symmetries can be viewed as the boundary of a 4d generalized symmetry protected topological (SPT) state with 1-form symmetries. We also present a prototype example of a 4d SPT state with mixed 1-form and 0-form symmetry topological response theory as well as its physical construction. The boundary of this SPT state can be a 3d anomalous QED state, or an anomalous 1-form symmetry-enriched topological order. Insights are gained by dimensional compatification/reduction. After dimensional compactification, the 3d system with N pairs of dual 1-form symmetries reduces to a 1d system with $2N$ pairs of dual U(1) global symmetries, which is the

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boundary of an ordinary 2d SPT state, while the 3d system with tensor symmetry reduces to a 1d Lifshitz theory, which is protected by the center of mass conservation of the system.

Keywords: topological phases of matter, gauge symmetry and gauge fields

Contents

1. Introduction

Various lattice models with different emergent gauge invariance were constructed in the context of quantum many-body condensed matter systems, including models with emergent U(1) gauge invariance [1–3], and models with more exotic tensor-like gauge transformations [4–9]. The most well-known example is the quantum spin ice system with emergent electromagnetism and photon-like excitations at low energy, as well as Dirac monopoles [10]. The analysis of these lattice models usually relies on the 'spin-wave' expansion, meaning to expand the theory at a certain presumed semiclassical mean field minimum of the Hamiltonian, or the saddle point of the action in the path integral. A low energy field theory is derived from this procedure (for example, Maxwell theory), then it is expected that this field theory captures the infrared physics of the lattice model at long scale. The stability of the state of interests described by low energy field theory usually needs to be studied case-by-case for each particular example. The general procedure of such analysis is that one treats the deviation from field theory as perturbations, and demonstrates that these perturbations are irrelevant under renormalization group flow at the desired state described by the field theory. But for a general form of lattice Hamiltonian, it is unclear whether such a mean-field minimum (and its corresponding field theory) really exists, or whether the perturbative renormalization

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group argument is reliable because the deviation from the desired state can be too strong to be treated perturbatively. For example, it is known that the lattice model for the emergent photon phase can be tuned into states very different from ordinary Maxwell theory, such as a confined phase with various spin or valence bond solid orders, and the Rokhsar–Kivelson point with nonrelativistic dispersion [2, 11, 12].

Sometimes, the argument for the stability of the desired low energy state can also be translated to certain physical pictures, for example, the behavior of topological defects such as Dirac monopoles; depending on whether the Dirac monopoles are gapped or condensed, the lattice gauge theory is in its deconfined or confined phases. But this argument relies heavily on the specific form of the theory, since the physical picture and the theory describing the condensation of topological defects can vary significantly between lattice theories with different generalized gauge transformations [9].

Recently, new tools and languages, such as generalized higher-form symmetries, have been introduced to analyze gauge fields [13–21] and various features of gauge fields, for example, the physical consequence of a topological term can be clearly studied following this language [22]. In the current manuscript, the most fundamental assumption we make about the systems under study is that, though our system is defined on a lattice, at least at the long scale there exists a $\mathrm{U}(1)^{\mathrm{g}}$ symmetry. $\mathrm{U}(1)^{\mathrm{g}}$ is a generalized U(1) symmetry such as higher-form symmetry or a 'tensor symmetry', whose definition will be spelled out later in the paper. The U(1) nature of the symmetry means that the charges of the generalized symmetry take arbitrary discrete integer eigenvalues, and charges with different supports in space all commute with each other. $\mathrm{U}(1)^{\mathrm{g}}$ can be an actual symmetry on the lattice scale (UV scale), and it can also be of an emergent nature, meaning it only exists at long scale.

Depending on the dimensionality, a topological soliton exists that is associated with this presumed $\mathrm{U}(1)^{\mathrm{g}}$ symmetry. The topological soliton is defined in space but not space-time, and it has a smooth spatial energy distribution without singularity (for example, a Dirac monopole is considered as a defect, instead of a soliton). We then further assume that at long scale, the topologically quantized soliton number is conserved, which means that the system also has an emergent $\mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ symmetry. Hence, at long scale, a dual structure exists with an enlarged $\mathrm{U}(1)^{\mathrm{g}} \times \mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ symmetry where the two $\mathrm{U}(1)^{\mathrm{g}}$ and $\mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ symmetries act on two sets of degrees of freedom that are related to each other in a non-local way. In this work, we will discuss the physical implications of the presumed infrared $\mathrm{U}(1)^{\mathrm{g}} \times \mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ symmetry of general lattice Hamiltonians, without relying on any expansions around classical saddle points.

This manuscript is arranged as follows: in section 2, we start with the assumption of dual generalized symmetries in infrared, then demonstrate the 't Hooft anomaly and a stable gapless phase on general grounds, without assuming any space-time symmetry or semiclassical treatment of the lattice model; in section 3, we identify the gapless phases discussed in section 2 as the boundary of higher dimensional symmetry-protected topological (SPT) states with 1-form symmetries, and make a connection with ordinary SPT states after dimensional compactification/reduction; section 4 introduces a prototype example of a 4d SPT state with both 1-form and 0-form symmetries, whose boundary is a prototype example of anomalous 1-form symmetry-enriched topological states (1-form

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SET) with fractionalization of 1-form symmetry; in section 5 we generalize the discussions in the previous section for the case with 'tensor gauge transformation', meaning we derive the 't Hooft anomaly of a pair of dual tensor symmetries and demonstrate the gaplessness of the spectrum based on the assumption of dual tensor symmetries. We also show that after dimensional compactification, the system with a pair of dual tensor symmetries reduces to 1d Lifshitz theory.

2. 3d systems with U(1) 1-form symmetry

2.1. Consequences of 1-form symmetries

For our purpose, we do not take a specific example of state of matter and show that this example has a 1-form symmetry. Instead, we start with the assumption that at least at the long scale, our 3d system has a $\mathrm{U}(1)^{\mathrm{g}}$ symmetry, where $\mathrm{U}(1)^{\mathrm{g}}$ is a 1-form symmetry [13–21]. We will explore what consequences this general assumption can lead to. Here, 3d means 3 spatial dimensions.

There is a 1-form charge density associated with this presumed $\mathrm{U}(1)^{\mathrm{g}}$ symmetry: $Q_A = \int^A \mathrm{d}\vec{S}\cdot\vec{\rho}$. The integral is over a two-dimensional surface A. The conservation of the charge density means that the 1-form charge cannot be created or annihilated, but it can 'leak' through the boundary of A through a 1-form symmetry current. But if A is a closed surface without any boundary, Q_A must be a constant, namely

$$Q_A = \int_{\partial A=\emptyset}^{A} \mathrm{d}\vec{S}\cdot\vec{\rho} = \int_{\partial V=A}^{V} \mathrm{d}^3x\ \vec{\nabla}\cdot\vec{\rho} = \text{const.} \tag{1}$$

Since this must be valid for any closed surface, it implies that $\vec{\nabla}\cdot\vec{\rho}$ is a time-independent constant everywhere in the entire space at long scale. Hence $\vec{\rho}$ can be viewed as an electric field $\vec{e}$ which satisfies the Gauss law constraint. The equation of motion of the ordinary electromagnetic field, i.e., Maxwell equations, can be viewed as the continuity equation of the 1-form symmetries:

$$\begin{aligned}
\partial_\mu J^{(\mathrm{e})}_\mu &= \frac{\partial e_i}{\partial t} - \partial_j \epsilon_{ijk} b_k = 0,\\
\partial_\mu J^{(\mathrm{m})}_\mu &= \frac{\partial b_i}{\partial t} + \partial_j \epsilon_{ijk} e_k = 0.
\end{aligned} \tag{2}$$

This means that for ordinary Maxwell theory, the currents of the two 1-form symmetries are:

$$\begin{aligned}
J^{(\mathrm{e})} &= (\rho^{(\mathrm{e})}_i,\ J^{(\mathrm{e})}_{ij}) = (e_i,\ \epsilon_{ijk} b_k),\\
J^{(\mathrm{m})} &= (\rho^{(\mathrm{m})}_i,\ J^{(\mathrm{m})}_{ij}) = (b_i,\ -\epsilon_{ijk} e_k).
\end{aligned} \tag{3}$$

This is analogous to the more familiar fact that the equation of motion of a superfluid is also the continuity equation of its super-current. Note that the conserved current $J^{(\mathrm{e})}$ is associated with the aforementioned 1-form $\mathrm{U}(1)^{\mathrm{g}}$ symmetry. The conserved current $J^{(\mathrm{m})}$

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will be associated with a different 1-form symmetry, denoted as $\mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$, whose physical meaning and definition will be explained later in this section.

Let us denote the operator of the electric field as $\hat{\vec{e}}$. When a quantized electric field is realized in condensed matter systems, it usually only takes discrete integer eigenvalues because the physical meaning of the electric field operator is usually the number operator of certain quantum bosons (for example, the dimer number operator [2, 23]), or spin component S^z [1, 3]. We consider a lattice model for these electric field operators in the same way as previous literature on quantum spin ices. If the Gauss law constraint is imposed strictly on the lattice, the 1-form symmetry is a microscopic symmetry of the system. However, in condensed matter realizations, the Gauss law constraint is usually not imposed strictly on the lattice; instead, there is a large energy penalty for creating defects that violate the Gauss law constraint (for example, in quantum spin ice, the ice-rule of the spin configuration was enforced energetically). The Gauss law constraint and hence the 1-form symmetry (now we refer to it as the electric 1-form symmetry) is only an emergent symmetry at long scale.

Using the (emergent) Gauss law constraint, one can prove that the Hamiltonian of the system *must* have a gauge invariance: $\hat{\vec{a}}(\mathbf{x}) \to \hat{\vec{a}}(\mathbf{x}) + \vec{\nabla} f$, where $\hat{a}_i = \hat{a}_i + 2\pi$ is the canonical conjugate operator of $-\hat{e}_i$, i.e., $[\hat{e}_i(\mathbf{x}), \hat{a}_{i'}(\mathbf{x}')] = \mathrm{i}\delta_{ii'}\delta(\mathbf{x}-\mathbf{x}')$. Here, we have chosen the convention where $-\hat{e}_i$ is the canonical conjugate momentum of $\hat{a}_i$ to match the convention of ordinary Maxwell theory. We will defer the proof to the example with 'tensor symmetry' (section 5) which we will discuss later. Here, we state that by assuming there is a $\mathrm{U}(1)^{\mathrm{g}}$ 1-form symmetry, the Hamiltonian of the system must have a U(1) gauge invariance. Hence, a local Hamiltonian of the system will only involve gauge invariant operators such as $\hat{\vec{e}}$ and $\hat{\vec{b}} = \vec{\nabla} \times \hat{\vec{a}}$. Generally, a local Hamiltonian of the system that respects the 1-form $\mathrm{U}(1)^{\mathrm{g}}$ symmetry takes the form:

$$H = \sum_{\mathbf{x}} \mathcal{H}[\hat{\vec{e}}(\mathbf{x}),\ \hat{\vec{b}}(\mathbf{x})]. \tag{4}$$

$\mathcal{H}[X, Y]$ must be a periodic function of Y because $\hat{\vec{b}}(\mathbf{x})$ and $\hat{\vec{a}}(\mathbf{x})$ are both periodically defined at any spatial location $\mathbf{x}$. This means that a 2π flux has no physical effect if it is only inserted through a single plaquette of the lattice. The flux only affects physics when it is spread out in space, hence there are nontrivial fluxes through plaquettes which are not multiples of 2π. We do not assume any space-time symmetry in H, hence $\mathcal{H}$ can involve mixture terms such as $\hat{e}(\mathbf{x})_i^n \sin(\hat{b}(\mathbf{x})_j)^m + \mathrm{H.c.}$. $\mathcal{H}$ also does not need to be translationally invariant, i.e., it can have disorder. Here, we mainly focus on the local Hamiltonian of the form in equation (4). But our analysis on the local Hamiltonian equation (4) can be extended to systems with a certain degree of non-locality.

Now we are ready to define the dual $\mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ symmetry. Since $\hat{\vec{b}} = \vec{\nabla} \times \hat{\vec{a}}$, it appears that the magnetic charge density vanishes $\vec{\nabla} \cdot \hat{\vec{b}} = 0$. But just like the existence of vortices in a superfluid, singular defects like Dirac monopoles exist which complicate the scenario. We assume that $\vec{\nabla} \cdot \hat{\vec{b}} = 0$ holds at low energy or long scale, hence $\int_{\partial A=\emptyset}^{A} \mathrm{d}\vec{S} \cdot \hat{\vec{b}} = 0$ for a large enough closed surface A (unless A has nontrivial winding over the entire space), i.e., there is a $\mathrm{U}(1)^{\mathrm{g}} \times \mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ 1-form symmetry at long scale. This is similar to the

physical picture that the Dirac monopole defect has a large energy gap, hence positive and negative monopole pairs must be tightly bound at low energy. For ordinary Maxwell theory, the current associated with the $\mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ symmetry is given by the second line of equation (3). In the following, we will discuss the consequence of the $\mathrm{U}(1)^{\mathrm{g}} \times \mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ symmetry in the general Hamiltonian equation (4) of which ordinary Maxwell theory is only a special case.

For a general Hamiltonian given in equation (4), using the Heisenberg equation, we can derive the 1-form currents for both the electric and magnetic 1-form symmetries:

$$\begin{aligned}
\frac{\partial \hat{e}_i(\mathbf{x})}{\partial t} &= \mathrm{i}[H,\ \hat{e}_i(\mathbf{x})] = \int \mathrm{d}\mathbf{y}\ \mathrm{i}\frac{\partial \mathcal{H}}{\partial \hat{b}_k(\mathbf{y})}\epsilon_{ji'k}\partial_{y_j}[\hat{a}_{i'}(\mathbf{y}),\ \hat{e}_i(\mathbf{x})] = \epsilon_{ijk}\partial_{x_j}\frac{\partial \mathcal{H}}{\partial \hat{b}_k(\mathbf{x})}, \\
\frac{\partial \hat{b}_i(\mathbf{x})}{\partial t} &= \mathrm{i}[H,\ \hat{b}_i(\mathbf{x})] = \int \mathrm{d}\mathbf{y}\ \mathrm{i}\frac{\partial \mathcal{H}}{\partial \hat{e}_{k'}(\mathbf{y})}\epsilon_{ijk}\partial_{x_j}[\hat{e}_{k'}(\mathbf{y}),\ \hat{a}_k(\mathbf{x})] = -\epsilon_{ijk}\partial_{x_j}\frac{\partial \mathcal{H}}{\partial \hat{e}_k(\mathbf{x})},
\end{aligned} \tag{5}$$

which can be viewed as the generalized 1-form electric and 1-form magnetic current conservation equations. The charges associated with 1-form electric and 1-form magnetic symmetries are still identified as $\hat{\vec{e}}$ and $\hat{\vec{b}}$. The 1-form symmetry currents for a general Hamiltonian are

$$J^{(\mathrm{e})}_{ij}(\mathbf{x}) = \epsilon_{ijk}\frac{\partial \mathcal{H}}{\partial \hat{b}^k(\mathbf{x})}, \qquad J^{(\mathrm{m})}_{ij}(\mathbf{x}) = -\epsilon_{ijk}\frac{\partial \mathcal{H}}{\partial \hat{e}^k(\mathbf{x})} \tag{6}$$

respectively.

The $\mathrm{U}(1)^{\mathrm{g}} \times \mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ 1-form symmetries have the 't Hooft anomaly. For ordinary Maxwell theory, this anomaly can be seen by the form of the 1-form currents equation (3): the current of $\mathrm{U}(1)^{\mathrm{g}}$ symmetry is the charge density of the $\mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ symmetry, and vice versa. This means that the process of generating a current associated with one symmetry necessarily violates the conservation of the charge of the other symmetry. Hence, there must be a mixed anomaly between these two symmetries. The mixed $\mathrm{U}(1)^{\mathrm{g}} \times \mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ anomaly of ordinary $(3+1)$d Maxwell theory was derived in previous literature, such as in [24].

In the following, we derive the 't Hooft anomaly for systems described by the general Hamiltonian equation (4), which has the $\mathrm{U}(1)^{\mathrm{g}} \times \mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ 1-form symmetries. To demonstrate the anomaly, we start by gauging the 1-form symmetries, i.e., by coupling $J^{(\mathrm{e})}$ and $J^{(\mathrm{m})}$ to external gauge fields $A^{(\mathrm{e})}$ and $A^{(\mathrm{m})}$, both of which are rank-2 tensor (2-form) gauge fields. $A^{(\mathrm{e})}$ and $A^{(\mathrm{m})}$ carry with them the following gauge transformations:

$$\begin{aligned}
&A^{(\mathrm{e,m})}_{i,0} \to A^{(\mathrm{e,m})}_{i,0} + \partial_t f^{(\mathrm{e,m})}_i, \\
&A^{(\mathrm{e,m})}_{ij} \to A^{(\mathrm{e,m})}_{ij} + \partial_j f^{(\mathrm{e,m})}_i - \partial_i f^{(\mathrm{e,m})}_j.
\end{aligned} \tag{7}$$

These tensor gauge fields are antisymmetric: $A^{(\mathrm{e,m})}_{ij} = -A^{(\mathrm{e,m})}_{ji}$.

To explain how the rank-2 tensor gauge fields $A^{(\mathrm{e,m})}$ couple to the system described in equation (4), we need to switch to a Lagrangian formalism of the problem. Before

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considering the coupling to the gauge fields $A^{(\mathrm{e,m})}$, the Lagrangian of the system is given by

$$\mathcal{L} = \sum_{\mathbf{x}} e_i(\mathbf{x}) \frac{\delta H}{\delta e_i(\mathbf{x})} - \mathcal{H}[\vec{e}(\mathbf{x}),\ \vec{b}(\mathbf{x})], \tag{8}$$

where $\vec{e}(\mathbf{x})$ and $\vec{b}(\mathbf{x})$ should be viewed as fields (instead of as operators). In the Legendre transformation, $\dot{a}_i(\mathbf{x}) = -\delta H/\delta e_i(\mathbf{x})$, which allows us to express $\vec{e}(\mathbf{x})$ as a function of $\dot{\vec{a}}(\mathbf{x})$ and $\vec{b}(\mathbf{x})$, and further, to write the Lagrangian as a function of $\dot{\vec{a}}(\mathbf{x})$ and $\vec{b}(\mathbf{x})$, namely $\mathcal{L}[\dot{\vec{a}}(\mathbf{x}), \vec{b}(\mathbf{x})]$. Under the electric 2-form gauge transformation (whose action on $A^{(\mathrm{e})}$ is given in equation (7)), the degrees of freedom in the Lagrangian $\mathcal{L}[\dot{\vec{a}}(\mathbf{x}), \vec{b}(\mathbf{x})]$ transform as

$$\begin{aligned} a_i &\to a_i - f_i^{(\mathrm{e})}, \\ \dot{a}_i &\to \dot{a}_i - \partial_t f_i^{(\mathrm{e})}, \\ b_i &\to b_i - \epsilon_{ijk}\partial_j f_k^{(\mathrm{e})}. \end{aligned} \tag{9}$$

When the system is coupled to the background two-form gauge fields $A^{(\mathrm{e,m})}$, it can be described by the Lagrangian

$$\begin{aligned} \mathcal{L}_{\mathrm{g}} &= \mathcal{L}\left[\dot{a}_i + A^{(\mathrm{e})}_{i,0}, b_i - \frac{1}{2}\epsilon_{ijk}A^{(\mathrm{e})}_{jk}\right] \\ &\quad + \sum_{\mathbf{x}} \frac{1}{2\pi}\left(A^{(\mathrm{m})}_{ij}(\mathbf{x}) J^{(\mathrm{m})}_{ij}(\mathbf{x}) + A^{(\mathrm{m})}_{i,0}(\mathbf{x}) b_i(\mathbf{x})\right) \\ &= \mathcal{L}\left[\dot{a}_i + A^{(\mathrm{e})}_{i,0}, b_i - \frac{1}{2}\epsilon_{ijk}A^{(\mathrm{e})}_{jk}\right] \\ &\quad + \sum_{\mathbf{x}} \frac{1}{2\pi}\left(-A^{(\mathrm{m})}_{ij}(\mathbf{x})\epsilon_{ijk}\dot{a}_k(\mathbf{x}) + A^{(\mathrm{m})}_{i,0}(\mathbf{x}) b_i(\mathbf{x})\right). \end{aligned} \tag{10}$$

One can easily check that, when $A^{(\mathrm{m})} = 0$, the Lagrangian $\mathcal{L}_{\mathrm{g}}$ is invariant under the electric 2-form gauge transformations given by equations (7) and (9). The coupling to the magnetic 2-form gauge field $A^{(\mathrm{m})}$ is introduced in $\mathcal{L}_{\mathrm{g}}$ in the form of minimal coupling. Here, we have made use of the general definition of $J^{(\mathrm{m})}_{ij}$ given in equation (6) as well as the fact that $\dot{a}_i(\mathbf{x}) = -\delta H/\delta e_i(\mathbf{x})$.

It turns out that, when $A^{(\mathrm{m})} \neq 0$, the Lagrangian $\mathcal{L}_{\mathrm{g}}$ is no longer invariant under the electric 2-form gauge transformation:

$$\mathcal{L}_{\mathrm{g}} \to \mathcal{L}_{\mathrm{g}} + \sum_{\mathbf{x}} \frac{1}{2\pi}\left(A^{(\mathrm{m})}_{ij}\epsilon_{ijk}\partial_t f^{(\mathrm{e})}_k - A^{(\mathrm{m})}_{i,0}\epsilon_{ijk}\partial_j f^{(\mathrm{e})}_k\right), \tag{11}$$

which indicates a mixed 't Hooft anomaly of the $\mathrm{U}(1)^{\mathrm{g}} \times \mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ symmetry in the system. In fact, this anomaly matches that of the boundary theory of a $(4+1)$d SPT

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state that has the 1-form $\mathrm{U}(1)^{\mathrm{g}} \times \mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ symmetry and a topological response given by [19, 24, 25]

$$\mathcal{S}_{\mathrm{CS}} = \int \mathrm{d}\tau \mathrm{d}^4 x \, \frac{1}{2\pi} A^{(\mathrm{e})} \wedge \mathrm{d}A^{(\mathrm{m})}. \tag{12}$$

Hence, if the $\mathrm{U}(1)^{\mathrm{g}} \times \mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ symmetries are microscopic symmetries, the 3d state described by the Hamiltonian equation (4) must be a boundary state of a 4d SPT state with 1-form symmetries. Here, $A^{(\mathrm{e})}$ and $A^{(\mathrm{m})}$ are treated as two-form fields in $(4+1)$d.

2.2. Gapless excitations of systems with dual 1-form symmetries

Common wisdom says that a mixed 't Hooft anomaly of the dual $\mathrm{U}(1)^{\mathrm{g}} \times \mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ symmetry implies that the spectrum of the 3d system cannot be trivially gapped, namely the Hamiltonian H cannot have a unique ground state and gapped spectrum in the thermodynamics limit. Here, we explicitly construct an excited state of the general Hamiltonian equation (4) with vanishing energy in the thermodynamics limit. We define our system on a three-dimensional cubic lattice which forms a torus with size L^3, and we assume there is a unique ground state of H denoted by $|\Omega\rangle$. We consider the following state $|\Psi\rangle$:

$$|\Psi\rangle = \hat{O}_q |\Omega\rangle, \qquad \hat{O}_q = \exp\left(\mathrm{i} q \sum_{\mathbf{x}} x \frac{2\pi \hat{e}_y(\mathbf{x})}{L^2} \right), \tag{13}$$

where $\hat{O}_q$ is a function of $\hat{\vec{e}}$ only, and it creates a magnetic flux quantum $2\pi q$ with size L^2 along the $\hat{z}$ direction. $\hat{O}_q$ shifts $\hat{a}_y$ by $\hat{a}_y \to \hat{a}_y + 2\pi x/L^2$. Hence, the gauge invariant Wilson loop $W_y = \exp(\mathrm{i}\int_0^L \mathrm{d}y \hat{a}_y)$ still has a periodic boundary condition after the shift, i.e., $W_y(x=0) = W_y(x=L)$ for integer q. Notice that since $\hat{O}_q$ is a function of $\hat{\vec{e}}$, $\hat{O}_q$ must commute with any composite operator of $\hat{\vec{e}}$. This operator inserts flux $2\pi q/L^2$ on every plaquette in the XY plane. Using the language in [3], the state $|\Psi\rangle$ carries a nontrivial topological charge, but using more recently developed language, $|\Psi\rangle$ carries a different 1-form $\mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ symmetry charge compared with the ground state. To be more precise, this symmetry charge here is $\int \mathrm{d}x\mathrm{d}y \, \hat{b}_z$ (with the integration over the XY-plane).

Since we made a powerful assumption that there is an emergent magnetic 1-form symmetry $\mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ at long scale, the assumption of $|\Omega\rangle$ being the unique ground state implies that it is also an eigenstate of the 1-form $\mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ charges. $|\Psi\rangle$ must be orthogonal to $|\Omega\rangle$ when the size of the created soliton is large compared with the lattice constant, because these two states carry different charges under $\mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$. Though $|\Psi\rangle$ is not necessarily the eigenstate of the Hamiltonian, the energy of $|\Psi\rangle$ is evaluated as

$$\begin{aligned} E_\Psi &= \langle\Psi|H|\Psi\rangle = \langle\Omega|\hat{O}_q^\dagger H \hat{O}_q|\Omega\rangle \\ &= \sum_{\mathbf{x}} \langle\Omega|\mathcal{H}\left[\hat{\vec{e}}(\mathbf{x}),\ \hat{\vec{b}}(\mathbf{x}) + \frac{2\pi q}{L^2}\hat{z}\right]|\Omega\rangle \end{aligned}$$

$$= E_\Omega + \sum_{\mathbf{x}} \sum_{m=1}^{\infty} \frac{1}{m!} \langle \Omega | \partial_{\hat{b}_z}^m \mathcal{H}[\hat{\vec{e}}(\mathbf{x}),\ \hat{\vec{b}}(\mathbf{x})] | \Omega \rangle \left(\frac{2\pi q}{L^2} \right)^m, \tag{14}$$

where $\hat{z}$ is the unit vector along the z direction. We have expanded the energy as a polynomial of $1/L^2$. For our purpose, we only need to worry about the leading order expansion of $1/L^2$, because all the other terms will vanish under the limit $L \to \infty$.

The leading order expansion of E_Ψ involves the following terms:

$$\sum_{\mathbf{x}} \langle \Omega | \partial_{\hat{b}_z} \mathcal{H}[\hat{\vec{e}}(\mathbf{x}),\ \hat{\vec{b}}(\mathbf{x})] | \Omega \rangle \frac{2\pi q}{L^2}. \tag{15}$$

For a general state, this expectation value does not vanish. However, since $|\Omega\rangle$ is the ground state, $\langle \Omega | \sum_{\mathbf{x}} \partial_{\hat{b}_z} \mathcal{H}[\hat{\vec{e}}(\mathbf{x}),\ \hat{\vec{b}}(\mathbf{x})] | \Omega \rangle$ must vanish because otherwise one can always choose the sign of q to make the energy of $|\Psi\rangle$ lower than $|\Omega\rangle$ for large enough L, which violates the assumption that $|\Omega\rangle$ is the ground state.

Let us review our logic here: we do not first take ordinary Maxwell theory and demonstrate that there is a 1-form symmetry; instead we start with the assumption that one 1-form symmetry $\mathrm{U}(1)^{\mathrm{g}}$ exists at long scale, then demonstrate that there must be a gauge invariance as a consequence of the 1-form symmetry. And the gauge invariance allows us to define the dual 1-form symmetry $\mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$. Then, by further assuming $\mathrm{U}(1)^{\mathrm{g}} \times \mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ at long scale, we construct a state that is orthogonal to the ground state, with energy approaching the ground state in the thermodynamics limit. The construction also does not rely on the semiclassical 'spin-wave' expansion used often in the literature of lattice quantum spin or boson models. A similar 'soliton insertion' argument was used in the original Lieb–Shultz–Matthis theorem [26], and the Luttinger theorem [27].

The argument above can hold even with a certain degree of non-locality in the Hamiltonian. For example, if there is a term in the Hamiltonian

$$H' = \sum_{\mathbf{x},\mathbf{x}'} f(|\mathbf{x} - \mathbf{x}'|) F[\hat{\vec{b}}(\mathbf{x})] F[\hat{\vec{b}}(\mathbf{x}')], \tag{16}$$

one can show that as long as $f(|\mathbf{x}|)$ falls off faster than $1/|\mathbf{x}|^2$ at the long distance, the state $|\Psi\rangle$ constructed above still has vanishing energy with $L \to \infty$.

3. Generalized SPT states and dimensional reduction

Helpful further insights can be gained through compactifying the system discussed in the previous section to one dimension. The mixed 't Hooft anomaly between the two dual 1-form symmetries will reduce to a mixed anomaly of two ordinary (0-form) U(1) symmetries. The 4d bulk will reduce to a 2d bosonic SPT state with ordinary (0-form) symmetries.

We compactify the YZ plane to a 2d torus with a small size. Since the 1d system is along the $\hat{x}$ direction, a 2d surface A wrapping around the 1d line could be either in the XY plane, or the XZ plane. In 3d systems with the $\mathrm{U}(1)^{\mathrm{g}} \times \mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ 1-form symmetry,

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there is a 1-form charge associated with the compactified XZ plane:

$$\int_{\mathbf{x}\in XZ} \mathrm{d}^2x\ \hat{e}_y(\mathbf{x}) \sim \int \mathrm{d}x\ \hat{n}(x). \tag{17}$$

Since the system is highly compact in the Y and Z directions, we ignore the modes with finite discrete momenta in these directions. In other words, all the fields are constants in these two directions. Then, we can define a 1d particle density $\hat{n}(x) \sim \hat{e}_y(x)$ in this compactified system. After proper normalization, we can also define the canonical conjugate variable of $\hat{n}(x)$, i.e., the phase angle operator $\hat{\theta}(x)$ as

$$\int_{\mathbf{x}\in XY} \mathrm{d}^2x\ \hat{b}_z(\mathbf{x}) \sim \int \mathrm{d}x\ \nabla_x\hat{\theta}(x), \tag{18}$$

$\hat{\theta}(x) \sim \hat{a}_y(x)$. $\hat{\theta}(x)$ and $\hat{n}(x)$ obey the standard commutation relation: $[\hat{\theta}(x),\ \hat{n}(x')] = -\mathrm{i}\delta_{x,x'}$. The 1-form symmetries discussed in previous examples become the ordinary global symmetries (0-form symmetries) in 1d.

The $\mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ charge now becomes the topological soliton number in this 1d system:

$$N_{\mathrm{T}} = \frac{1}{2\pi}\int_0^L \mathrm{d}x\ \nabla_x\hat{\theta}(x). \tag{19}$$

The general Hamiltonian we considered in equation (4) becomes a 1d Hamiltonian with an ordinary U(1) symmetry

$$H = \sum_x \mathcal{H}[\hat{n}(x), \nabla_x\hat{\theta}(x)]. \tag{20}$$

All the analysis in section 2 has counterparts in the compactified system. We assume that at long scales, both the particle number $\int \mathrm{d}x\ \hat{n}(x)$ and the topological soliton number N_{T} are conserved, namely there is a $\mathrm{U}(1)\times\mathrm{U}(1)_{\mathrm{dual}}$ symmetry at long scale. We denote the ground state of the Hamiltonian described above as $|\Omega\rangle$, and then consider the following state $|\Psi\rangle$:

$$|\Psi\rangle = \hat{O}_q|\Omega\rangle = \exp\left(\mathrm{i}q\sum_x \frac{2\pi\hat{n}(x)}{L}x\right)|\Omega\rangle. \tag{21}$$

The operator $\hat{O}_q$ is the analogue of the operator $\hat{O}_q$ in equation (13) compactified to 1d. With $q = 1$, $|\Psi\rangle$ contains one extra soliton $\hat{N}_T$ compared with the ground state $|\Omega\rangle$: $\hat{O}_1$ creates one extra winding of $\hat{\theta}$ in the 1d system. Since we have assumed that the $\mathrm{U}(1)_{\mathrm{dual}}$ is an emergent symmetry at long scale, $|\Psi\rangle$ must be orthogonal to the ground state. The evaluation of the energy of $|\Psi\rangle$ is similar to the discussion in section 2. We can show that the energy of $|\Psi\rangle$ approaches the energy of $|\Omega\rangle$ as $L\to\infty$.

When the system is reduced to 1d, its $\mathrm{U}(1)\times\mathrm{U}(1)_{\mathrm{dual}}$ symmetry has an ordinary 't Hooft anomaly. In fact, the action of the $\mathrm{U}(1)\times\mathrm{U}(1)_{\mathrm{dual}}$ in the reduced 1d system mimics the spin and charge U(1) symmetry action on the boundary of a 2d quantum spin Hall insulator. It is known that the boundary of the quantum spin Hall insulator with both charge and spin U(1) symmetries has a mixed perturbative 't Hooft anomaly. To show

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this anomaly formally, one can couple the charge U(1) current to a $\mathrm{U}(1)^{(\mathrm{e})}$ background gauge field $A^{(\mathrm{e})}$, and couple the spin U(1) (or the $\mathrm{U}(1)_{\mathrm{dual}}$) current to another background $\mathrm{U}(1)^{(\mathrm{m})}$ gauge field $A^{(\mathrm{m})}$. This mixed anomaly is identical to the boundary of (2 + 1)d bulk Chern–Simons theory

$$\mathcal{S} = \int \mathrm{d}\tau \mathrm{d}^2x \, \frac{1}{2\pi} A^{(\mathrm{e})} \wedge \mathrm{d}A^{(\mathrm{m})}. \tag{22}$$

Physically, this anomaly simply means that the current of one U(1) symmetry is the charge density of the other U(1) symmetry, hence a process of creating the current of one U(1) symmetry would necessarily violate the conservation of the charge of the other U(1) symmetry.

There is another pair of dual U(1) symmetries in the 1d system after compactification, which originates from the 3d dual U(1) 1-form symmetries: the U(1) symmetry generated by $\int_{\mathbf{x}\in XY} \mathrm{d}^2x \, \hat{e}_z(\mathbf{x})$, and the $\mathrm{U}(1)_{\mathrm{dual}}$ symmetry associated with the conservation of $\int_{\mathbf{x}\in XZ} \mathrm{d}^2x \, \hat{b}_y(\mathbf{x})$. There is also a mixed 't Hooft anomaly between these two dual U(1) symmetries. Hence, one pair of dual 1-form symmetries in 3d will reduce to two pairs of ordinary dual symmetries in 1d. In general, if we start with N pairs of dual $\mathrm{U}(1)^{\mathrm{g}} \times \mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ 1-form symmetries in 3d, after compactification to 1d, there will be $2N$ pairs of dual $\mathrm{U}(1) \times \mathrm{U}(1)_{\mathrm{dual}}$ symmetries in 1d. The 4d bulk system for the 3d system with a series of 1-form symmetries can have a Chern–Simons response theory

$$\mathcal{S} = \int \mathrm{d}\tau \mathrm{d}^4x \, \frac{1}{4\pi} K_{IJ} C^I \wedge \mathrm{d}C^J, \tag{23}$$

where C^I is a two-form gauge field, and K^{IJ} is an antisymmetric matrix. Then, after dimensional reduction as discussed in this section, the corresponding 2d bulk theory for the 1d system should have a CS response theory

$$\mathcal{S} = \int \mathrm{d}\tau \mathrm{d}^2x \, \frac{1}{4\pi} K'_{IJ} C^I \wedge \mathrm{d}C^J, \qquad K' = K \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{24}$$

In the (2 + 1)d system, C^I is a 1-form gauge field, and K' is a symmetric matrix. Hence, the 4d generalized SPT state can be studied and understood as its 2d counterpart with ordinary symmetries after dimensional reduction.

4. A prototype SPT state with mixed 0-form and 1-form symmetries, 1-form symmetry enriched topological order

Equation (12) is a $(4+1)$d topological response theory involving only 1-form symmetries. In general, if there is an extra ordinary (0-form) symmetry G in the system, one can also consider the mixed topological response theory between the 0-form symmetry G and the 1-form symmetries. For example, we can consider a $(4+1)$d bulk system which has a topological response

$$\mathcal{S}_{\mathrm{topo}} = \pi \int \mathrm{d}\tau \mathrm{d}^4x \, w_2[A^{\mathrm{SO}(3)}] \cup \frac{\mathrm{d}A^{(\mathrm{e})}}{2\pi}. \tag{25}$$

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Here, $A^{\mathrm{SO}(3)}$ is the background (1-form) gauge field associated with the 0-form symmetry $G = \mathrm{SO}(3)$ and w_2 is the second Stiefel–Whitney class.

A candidate system with this response theory can be constructed as follows: we start with a $(4+1)$d QED with a microscopic electric $\mathrm{U}(1)^{\mathrm{g}}$ 1-form symmetry. We will denote the $(4+1)$d bulk dynamical gauge field as a_μ. There is no microscopic magnetic higher-form symmetry, hence there are defects with their own dynamics analogous to the Dirac monopole. The Dirac monopole defect in $(4+1)$d is a one-dimensional line/loop. It was shown that the 'decorated defect' construction is a very powerful physical picture of constructing SPT states with 0-form symmetries [28, 29], i.e., higher dimensional SPT states can be constructed by decorating the topological defects of order parameters with lower dimensional SPT states. Here, we also follow the method of decorated defects: we attach the Dirac monopole line in $(4+1)$d bulk with a one-dimensional ordinary SPT phase with $G = \mathrm{SO}(3)$ symmetry, i.e., the Haldane phase, and proliferate the Dirac monopole line. The $(4+1)$d bulk will be driven into a gapped and confined phase, while the most natural $(3+1)$d boundary state of the system will be a QED whose Dirac monopole carries a spin-1/2 under the 0-form SO(3) symmetry, while there is no electric charge. A theory which describes this boundary state is the $(3+1)\mathrm{dCP}^1$ model:

$$\mathcal{S}_{\mathrm{boundary}} = \int \mathrm{d}\tau \mathrm{d}^3x \sum_{\alpha=1}^{2} |(\partial - \mathrm{i}\tilde{a})z_\alpha|^2 + \cdots \tag{26}$$

where z_α represents a spin-1/2 representation of the SO(3) 0-form symmetry carried by the boundary termination of the Dirac monopole line in the $(4+1)$d bulk, while $\tilde{a}_\mu$ is the 'dual' gauge field of a_μ at the $(3+1)$d boundary whose gauge charge is the Dirac monopole of a_μ. As we can see from its topological response $\mathcal{S}_{\mathrm{topo}}$, this $(4+1)$d bulk is an SPT state protected by the electric 1-form U(1) symmetry and the 0-form symmetry G. Its boundary state cannot be gapped with a unique ground state without breaking the symmetries. One way to understand it is to consider the compactification of three spatial dimensions to a three-dimensional sphere S^3 with a non-trivial flux $\int_{S^3} \mathrm{d}A^{(\mathrm{e})} = 2\pi$. The effective $(1+1)$d system after the dimensional compactification/reduction has a topological response identical to the SO(3) symmetric Haldane phase in $(1+1)$d which is a $(1+1)$d SPT whose boundary does not admit a unique fully symmetric ground state.

This 'decorated monopole line' construction can be generalized to many other SPT states with mixed 1-form and 0-form symmetries. One just needs to decorate the Dirac monopole line in 4d space with a nontrivial 1d bosonic SPT state with ordinary 0-form symmetry, for example the 1d Haldane phase with $\mathrm{PSU}(N)$ symmetry with a $\mathrm{SU}(N)$ fundamental at the boundary. Then the 3d boundary could be described by a CP^{N-1} model with N flavors of the bosonic field z_α coupled with the dual U(1) gauge field in equation (26).

The system with a 0-form SO(3) symmetry and a $\mathrm{U}(1)^{\mathrm{g}}$ 1-form symmetry can also support other 3d boundary states. For example, one can condense the bound state of a pair of Dirac monopoles, which can be a singlet of SO(3) 0-form symmetry. Then the system enters a 'monopole superconductor', which is a Z_2 topological order with

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both point and loop excitations. The point excitation is a spin-1/2 of the SO(3) 0-form symmetry (z_α in equation (26)), while the loop excitation carries a half charge of the $\mathrm{U}(1)^{\mathrm{g}}$ 1-form symmetry. This fractionalization of 1-form symmetry is identical to the simple fact that in an ordinary superconductor, the vortex line carries half the magnetic flux quantum. Due to the fractionalization of the 1-form symmetry, the loop excitation must couple to a gauge field, which is precisely the 2-form gauge field dual to the condensed Dirac monopole pair.

The 3d topological order constructed here is a 1-form SET state. Both the point excitation, and the line excitation of the topological order carry nontrivial quantum numbers, and the line excitations carry 1-form symmetry charge. Moreover, this SET state is anomalous in the sense that it cannot be driven to a trivially gapped phase without breaking either the U(1) 1-form symmetry, or the SO(3) 0-form symmetry. The reason is that, in order to drive the 3d topological order to a trivial phase, we need to either condense the point particle or the line excitation. However, condensing the line excitation would lead to spontaneous breaking of the U(1) 1-form symmetry, while condensation of the point particle would lead to spontaneous breaking of the SO(3) symmetry.

This Z_2 topological order with fractionalized 1-form symmetry is the 3d analogue of a 2dZ_2 topological order whose mutual semionic Anyon excitations (the so-called e and m excitations) carry half charge and spin-1/2 representation of 0-form U(1) and SO(3) symmetries respectively. This 2dZ_2 topological order is the boundary of a 3d ordinary SPT state [28].

The example of a SET state discussed here is a prototype, and many 1-form SET states can be constructed in a similar way. For example, if the 1d SPT state decorated in the Dirac monopole line in the 4d bulk is the PSU(N) Haldane phase, the 3d boundary can be driven into a Z_N topological order by condensing an N-body bound state of the Dirac monopole, which can be a SU(N) singlet. The point particle of the Z_N topological order carries fractionalized quantum numbers of PSU(N), and the line excitation carries fractionalized 1-form symmetry. We leave the full discussion of higher symmetry-enriched SET states to future exploration.

5. 3d system with tensor symmetries

Now we turn to a system with a generalized tensor 1-form symmetry, whose lattice realization was discussed in [4–6]. A connection between this system as well as similar tensor gauge theories [7–9] and the fracton states was pointed out in recent literature (for instance, [30–48]). The fracton states are a series of novel gapped states of matter which can be obtained by partially breaking the gauge invariance in generalized tensor gauge theories. In our current paper, we will still focus on the gapless phase with tensor symmetry instead of the gapped phase. This tensor 1-form symmetry is, to a certain extent, similar to the three 1-form $\mathrm{U}(1)^{\mathrm{g}}$ symmetries discussed in the previous sections, meaning that with a given closed surface A, there are three U(1) charges: $Q^a_A = \int^A_{\partial A=\emptyset} \mathrm{d}\vec{S} \cdot \vec{\rho}^a = \int^V_{\partial V=A} \mathrm{d}^3x\ \vec{\nabla} \cdot \vec{\rho}^a$. These charges are individual constants. We further demand that ρ^{ia} is a symmetric tensor: $\rho^{ij} = \rho^{ji}$. Then ρ^{ij} can be viewed as the generalized symmetric

tensor electric field introduced in [4–6]: $\mathcal{E}^{ij}$, which is subjected to the constraints: $\partial_i \mathcal{E}^{ij} = \partial_j \mathcal{E}^{ij} = 0$.

Now we promote $\mathcal{E}^{ij}$ to an operator $\hat{\mathcal{E}}^{ij}$, whose eigenvalues are again integers. We can define the following operator $\hat{G}(f^i(\mathbf{x}))$ parameterized by an arbitrary vector function $f^i(\mathbf{x})$:

$$\begin{aligned}\hat{G}(f^i) &= \exp\left(\int \mathrm{d}^3x\ \mathrm{i}2f^i\partial_j\hat{\mathcal{E}}^{ij}\right)\\ &= \exp\left(\int \mathrm{d}^3x\ \mathrm{i}f^i\partial_j\hat{\mathcal{E}}^{ij} + \mathrm{i}f^j\partial_i\hat{\mathcal{E}}^{ij}\right)\\ &= \exp\left(-\int \mathrm{d}^3x\ \mathrm{i}\left(\partial_j f^i + \partial_i f^j\right)\hat{\mathcal{E}}^{ij}\right).\end{aligned} \tag{27}$$

Let us denote $\hat{\mathcal{A}}^{ij}$ as the canonical conjugate operator of $\hat{\mathcal{E}}^{ij}$ ($\hat{\mathcal{A}}^{ij}$ is again periodically defined). More precisely, we impose the commutation relations $[\hat{\mathcal{E}}^{ij}(\mathbf{x}), \hat{\mathcal{A}}^{i'j'}(\mathbf{x}')] = \mathrm{i}(\delta_{ii'}\delta_{jj'} + \delta_{ij'}\delta_{ji'})\delta(\mathbf{x}-\mathbf{x}')$. The $\hat{G}(f^i)$ operator will generate a gauge transformation on $\hat{\mathcal{A}}^{ij}$:

$$\hat{G}^{-1}(f^i)\hat{\mathcal{A}}^{ij}(\mathbf{x})\hat{G}(f^i) = \hat{\mathcal{A}}^{ij}(\mathbf{x}) + 2\partial_i f^j + 2\partial_j f^i. \tag{28}$$

However, because of the constraint on $\hat{\mathcal{E}}^{ij}$, $\hat{G}(f^i)$ is actually an identity operator, which must commute with any Hamiltonian of $\hat{\mathcal{E}}^{ij}$ and $\hat{\mathcal{A}}^{ij}$. It means that the Hamiltonian of the system *must* be invariant under the gauge transformation equation (28). The derivation of gauge invariance in this paragraph applies to other systems with local constraints, such as systems with generalized gauge transformations [9].

Then the Hamiltonian must be a function of $\hat{\mathcal{E}}^{ij}$ and the gauge invariant operator $\hat{\mathcal{B}}^{ij} = \epsilon_{iab}\epsilon_{jcd}\partial_a\partial_c\hat{\mathcal{A}}^{bd}$. A general local Hamiltonian should take the form:

$$H = \sum_{\mathbf{x}} \mathcal{H}[\hat{\mathcal{E}}^{ij}(\mathbf{x}),\ \hat{\mathcal{B}}^{ij}(\mathbf{x})], \tag{29}$$

and again $\mathcal{H}$ is a periodic function of $\hat{\mathcal{B}}^{ij}$. $\hat{\mathcal{B}}$ is completely dual to $\hat{\mathcal{E}}$. In addition to the more exotic gauge invariance, these Hamiltonians all have an extra center of mass conservation: H is invariant under transformation

$$\hat{\mathcal{A}}^{ij} \to \hat{\mathcal{A}}^{ij} + F^{ij}[\mathbf{x}], \tag{30}$$

where $F^{ij}[\mathbf{x}]$ is a linear function of the space coordinate. This extra conservation law in the series of tensor models [4, 6, 9] was noticed in [30], and it was realized that this center of mass conservation is a key feature of the fracton states of matter.

We can define a dual tensor 1-form symmetry $\mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$, whose charge corresponds to the generalized tensor magnetic flux through a surface A: $Q^a_A = \int \mathrm{d}S^i \cdot \hat{\mathcal{B}}^{ia}$. We assume that the generalized tensor magnetic 1-form charge density $\partial_i\hat{\mathcal{B}}^{ij} = \partial_j\hat{\mathcal{B}}^{ij} = 0$ remains zero at low energy, meaning there is an emergent dual tensor symmetry $\mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ at long scale. Then again, one can insert magnetic flux through the system through (for

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example) the following operator:

$$\hat{O}_q = \exp\left(\mathrm{i}q\sum_{\mathbf{x}}\frac{2\pi x^2}{L^2}\hat{\mathcal{E}}^{zz}(\mathbf{x})\right). \tag{31}$$

This operator is still compatible with the periodic boundary condition, and it will shift $\mathcal{A}^{zz}$ by

$$\hat{O}_q^{-1}\hat{\mathcal{A}}^{zz}\hat{O}_q = \hat{\mathcal{A}}^{zz}(\mathbf{x}) + \frac{4\pi q x^2}{L^2}. \tag{32}$$

If we denote the ground state of the system as $|\Omega\rangle$, then $|\Psi\rangle$ has nonzero extra quantized flux of $\mathcal{B}^{yy}$ through any XZ plain compared with the ground state, and the extra flux density is $\mathcal{B}^{yy} \sim 1/L^2$. Or we can create a configuration of $\mathcal{A}^{xy}(\mathbf{x})$ as $\mathcal{A}^{xy}(\mathbf{x}) = 2\pi z^2/L^2$. Then there is a nonzero flux of $\mathcal{B}^{xy}$, again with flux density $\sim 1/L^2$.

Again, we will demonstrate that the ground state of the system cannot be trivially gapped if we assume the emergent $\mathrm{U}(1)^{\mathrm{g}} \times \mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ symmetry is at the long scale. Suppose there is a unique ground state $|\Omega\rangle$ of the system, then $|\Psi\rangle = \hat{O}_q|\Omega\rangle$ must be orthogonal to $|\Omega\rangle$ for large enough L because $|\Omega\rangle$ must be an eigenstate of the tensor 1-form charge, and $|\Psi\rangle$ carries a different tensor 1-form charge from $|\Omega\rangle$. And by going through the same argument as section 2, we can demonstrate that when $L \to \infty$, the energy of $|\Psi\rangle$ must also approach the energy of $|\Omega\rangle$. This statement still holds with disorder, and also when there is a long-range interaction that falls off more rapidly than $1/|\mathbf{x}|^2$.

We have argued that an emergent $\mathrm{U}(1)^{\mathrm{g}} \times \mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ tensor symmetry rules out a trivial gapped ground state. This result can be equivalently stated as that the $\mathrm{U}(1)^{\mathrm{g}} \times \mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ tensor symmetry is anomalous. Again, the equation of motion of $\mathcal{E}^{ij}$ and $\mathcal{B}^{ij}$ can be viewed as the continuity equation of the currents of the tensor symmetries. For the simplest semiclassical limit of the theory [4, 6], the Hamiltonian of the system is approximately

$$\mathcal{H} \sim \frac{1}{4}\sum_{\mathbf{x}}\sum_{ij}\left[\left(\mathcal{E}^{ij}(\mathbf{x})\right)^2 + \left(\mathcal{B}^{ij}(\mathbf{x})\right)^2\right], \tag{33}$$

then the equation of motion reads

$$\begin{aligned}
&\frac{\partial \mathcal{E}^{ij}}{\partial t} - \partial_a\left(\epsilon_{iab}\partial_c\epsilon_{jcd}\mathcal{B}^{bd}\right) = 0,\\
&\frac{\partial \mathcal{B}^{ij}}{\partial t} - \partial_a\left(\epsilon_{iab}\partial_c\epsilon_{jcd}\mathcal{E}^{bd}\right) = 0.
\end{aligned} \tag{34}$$

This means that the currents of the tensor symmetries are:

$$\begin{aligned}
J^{(\mathrm{e})} &= (\rho^{(\mathrm{e})}_{ij},\ J^{(\mathrm{e})}_{ij,k}) = \left(\mathcal{E}^{ij},\ \frac{1}{2}\epsilon_{ikb}\epsilon_{jcd}\partial_c\mathcal{B}^{bd} + i \leftrightarrow j\right),\\
J^{(\mathrm{m})} &= (\rho^{(\mathrm{m})}_{ij},\ J^{(\mathrm{m})}_{ij,k}) = \left(\mathcal{B}^{ij},\ \frac{1}{2}\epsilon_{ikb}\epsilon_{jcd}\partial_c\mathcal{E}^{bd} + i \leftrightarrow j\right).
\end{aligned} \tag{35}$$

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Again, in a process that creates a nonzero current of one of the U(1) tensor symmetries, the charge conservation of the other U(1) tensor symmetry must be violated, hence there is a 't Hooft anomaly of the two U(1) tensor symmetries.

Formally we can still discuss the anomalies in a Lagrangian formalism. The Lagrangian is given by

$$\mathcal{L}[\dot{\mathcal{A}}^{ij}, \mathcal{B}^{ij}] = \frac{1}{4}\sum_{\mathbf{x}}\sum_{ij}\left[\left(\dot{\mathcal{A}}^{ij}(\mathbf{x})\right)^2 - \left(\mathcal{B}^{ij}(\mathbf{x})\right)^2\right], \tag{36}$$

where $\dot{\mathcal{A}}^{ij} \equiv \delta H/\delta \mathcal{E}^{ij} = \mathcal{E}^{ij}$ is introduced through the Legendre transformation $\mathcal{L} = \left(\sum_{\mathbf{x}}\sum_{ij}\mathcal{E}^{ij}\frac{\delta H}{\delta \mathcal{E}^{ij}}\right) - H$. The electric $\mathrm{U}(1)^{\mathrm{g}}$ tensor symmetry is defined by the symmetry transformation

$$\mathcal{A}^{ij} \to \mathcal{A}^{ij} + \Lambda^{(\mathrm{e})}_{ij}, \tag{37}$$

where $\Lambda^{(\mathrm{e})}_{ij}$ is a constant symmetric tensor, namely $\Lambda^{(\mathrm{e})}_{ij} = \Lambda^{(\mathrm{e})}_{ji}$. In the following, we will use the terms $\mathrm{U}(1)^{\mathrm{g}}$ tensor symmetry and electric tensor symmetry interchangeably. We can gauge the electric tensor symmetry by promoting $\Lambda^{(\mathrm{e})}_{ij}$ to a space-time function, and introducing the electric tensor gauge fields $G^{(\mathrm{e})}_{ij,0}$ and $G^{(\mathrm{e})}_{ij,k}$ which are symmetric under the exchange of the first two indices, namely $G^{(\mathrm{e})}_{ij,0} = G^{(\mathrm{e})}_{ji,0}$ and $G^{(\mathrm{e})}_{ij,k} = G^{(\mathrm{e})}_{ji,k}$. Under the electric tensor gauge transformation, we have

$$\begin{aligned}
&\mathcal{A}^{ij} \to \mathcal{A}^{ij} + \Lambda^{(\mathrm{e})}_{ij}\\
&\dot{\mathcal{A}}^{ij} \to \dot{\mathcal{A}}^{ij} + \partial_t \Lambda^{(\mathrm{e})}_{ij}\\
&\mathcal{B}^{ij} \to \mathcal{B}^{ij} + \epsilon_{iab}\epsilon_{jcd}\partial_a\partial_c\Lambda^{(\mathrm{e})}_{bd}\\
&G^{(\mathrm{e})}_{ij,0} \to G^{(\mathrm{e})}_{ij,0} + \partial_t \Lambda^{(\mathrm{e})}_{ij}\\
&G^{(\mathrm{e})}_{ij,k} \to G^{(\mathrm{e})}_{ij,k} + \partial_k \Lambda^{(\mathrm{e})}_{ij}.
\end{aligned} \tag{38}$$

When the electric tensor background gauge field is turned on, the system is described by the Lagrangian

$$\begin{aligned}
&\mathcal{L}\left[\dot{\mathcal{A}}^{ij} - G^{(\mathrm{e})}_{ij,0}, \mathcal{B}^{ij} - \frac{1}{2}\left(\epsilon_{iab}\epsilon_{jcd} + \epsilon_{icb}\epsilon_{jad}\right)\partial_a G^{(\mathrm{e})}_{bd,c}\right]\\
&\quad = \mathcal{L}[\dot{\mathcal{A}}^{ij}, \mathcal{B}^{ij}] - \sum_{ijk}\sum_{\mathbf{x}}\left(G^{(\mathrm{e})}_{ij,0}\rho^{(\mathrm{e})}_{ij} + G^{(\mathrm{e})}_{ij,k}J^{(\mathrm{e})}_{ij,k}\right) + \cdots,
\end{aligned} \tag{39}$$

which is explicitly gauge invariant under the gauge transformation given by equation (38). The '...' part contains higher order terms in $\dot{\mathcal{A}}^{ij}$ and $\mathcal{B}^{ij}$. As a sanity check, we notice that the Lagrangian above effectively introduces the minimal coupling between the electric tensor gauge fields $\left(G^{(\mathrm{e})}_{ij,0},\ G^{(\mathrm{e})}_{ij,k}\right)$ and the current $J^{(\mathrm{e})}$ introduced in equation (35).

Similarly, we can introduce the magnetic tensor gauge fields $G^{(\mathrm{m})}_{ij,0}$ and $G^{(\mathrm{m})}_{ij,k}$ which are also symmetric under the exchange of the first two indices, namely $G^{(\mathrm{m})}_{ij,0} = G^{(\mathrm{m})}_{ji,0}$ and

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$G^{(\mathrm{m})}_{ij,k} = G^{(\mathrm{m})}_{ji,k}$. The magnetic tensor gauge fields are associated with the emergent $\mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ symmetry. They transform under the magnetic tensor gauge transformation as

$$\begin{aligned} G^{(\mathrm{m})}_{ij,0} &\to G^{(\mathrm{m})}_{ij,0} + \partial_t \Lambda^{(\mathrm{m})}_{ij}, \\ G^{(\mathrm{m})}_{ij,k} &\to G^{(\mathrm{m})}_{ij,k} + \partial_k \Lambda^{(\mathrm{m})}_{ij}. \end{aligned} \tag{40}$$

We can introduce the minimal coupling between the magnetic tensor gauge fields and the current $J^{(\mathrm{m})}$ introduced in equation (35), which yields

$$\begin{aligned} \mathcal{L}_{\mathrm{g}} &= \mathcal{L}\left[\dot{\mathcal{A}}^{ij} - G^{(\mathrm{e})}_{ij,0},\ \mathcal{B}^{ij} - \frac{1}{2}\left(\epsilon_{iab}\epsilon_{jcd} + \epsilon_{icb}\epsilon_{jad}\right)\partial_a G^{(\mathrm{e})}_{bd,c}\right] \\ &\quad - \sum_{ijk}\sum_{\mathbf{x}}\left(G^{(\mathrm{m})}_{ij,0}\rho^{(\mathrm{m})}_{ij} + G^{(\mathrm{m})}_{ij,k}J^{(\mathrm{m})}_{ij,k}\right) \\ &= \mathcal{L}\left[\dot{\mathcal{A}}^{ij} - G^{(\mathrm{e})}_{ij,0},\ \mathcal{B}^{ij} - \frac{1}{2}\left(\epsilon_{iab}\epsilon_{jcd} + \epsilon_{icb}\epsilon_{jad}\right)\partial_a G^{(\mathrm{e})}_{bd,c}\right] \\ &\quad - \sum_{ijk}\sum_{\mathbf{x}}\left(G^{(\mathrm{m})}_{ij,0}\mathcal{B}^{ij} + \frac{1}{2}G^{(\mathrm{m})}_{ij,k}\left(\epsilon_{ikb}\epsilon_{jcd} + \epsilon_{jkb}\epsilon_{icd}\right)\partial_c \dot{\mathcal{A}}^{bd}\right). \end{aligned} \tag{41}$$

When the magnetic tensor gauge field $\left(G^{(\mathrm{m})}_{ij,0},\ G^{(\mathrm{m})}_{ij,k}\right)$ is turned on, the Lagrangian $\mathcal{L}_{\mathrm{g}}$ is no longer invariant under the electric tensor gauge transformation equation (38):

$$\begin{aligned} \mathcal{L}_{\mathrm{g}} \to \mathcal{L}_{\mathrm{g}} - \sum_{ijk}\sum_{\mathbf{x}}\Big(&G^{(\mathrm{m})}_{ij,0}\epsilon_{iab}\epsilon_{jcd}\partial_a\partial_c\Lambda^{(\mathrm{e})}_{bd} \\ &+ \frac{1}{2}G^{(\mathrm{m})}_{ij,k}\left(\epsilon_{ikb}\epsilon_{jcd} + \epsilon_{jkb}\epsilon_{icd}\right)\partial_c\partial_t\Lambda^{(\mathrm{e})}_{bd}\Big). \end{aligned} \tag{42}$$

The fact that $\mathcal{L}_{\mathrm{g}}$ is no longer gauge invariant once the magnetic tensor gauge field $\left(G^{(\mathrm{m})}_{ij,0},\ G^{(\mathrm{m})}_{ij,k}\right)$ is turned on indicates a 't Hooft anomaly of the emergent $\mathrm{U}(1)^{\mathrm{g}} \times \mathrm{U}(1)^{\mathrm{g}}_{\mathrm{dual}}$ tensor symmetry.

The 3d system with tensor symmetry can also be compactified to 1d. After compactification, one can still define several ordinary 1d global U(1) symmetries. One of the U(1) symmetries has the following charge:

$$\int_{\mathbf{x}\in XY} \mathrm{d}^2x\ \hat{\mathcal{E}}_{zz}(\mathbf{x}) \sim \int \mathrm{d}x\ \hat{n}(x). \tag{43}$$

The conjugate variable of $\hat{n}(x)$, i.e., the phase angle $\hat{\theta}(x)$ is defined as

$$\int_{\mathbf{x}\in XZ} \mathrm{d}^2x\ \hat{\mathcal{B}}_{yy}(\mathbf{x}) \sim \int \mathrm{d}x\ \nabla_x^2\hat{\theta}(x), \tag{44}$$

and $\hat{\theta}(x) \sim \hat{\mathcal{A}}_{zz}(x)$. The 3d Hamiltonian then reduces to 1d Lifshitz theory: $H = \sum_x \mathcal{H}[\hat{n}(x),\ \nabla_x^2\hat{\theta}(x)]$.

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The 1d Hamiltonian H inherits the center of mass conservation equation (30), which in 1d becomes $\hat{\theta} \to \hat{\theta} + Bx$ with constant B. This center of mass conservation prohibits terms like $\cos(\nabla_x \hat{\theta})$ after compactification. Hence, after compactification, the 2d bulk of the system should be an exotic SPT state with a special center of mass conservation, whose nature deserves further studies.

6. Discussion

In this note, we explored the consequences of the assumption of a pair of dual generalized symmetries. We discussed the implication of the dual symmetries on low energy excitations, 't Hooft anomaly, their bulk description, and the corresponding state after dimensional compactification. Then, we extended all these discussions to tensor gauge theories.

Further studies can be pursued following the questions raised in this work. We have shown that for N pairs of dual 1-form symmetries in 3d, there will be $2N$ pairs of dual 0-form symmetries after compactification to 1d. If we break the dual 1-form symmetries to a certain combination of these two 1-form symmetries, a bound state of electric and magnetic charges (a dyon) is allowed and has its own dynamics. The 3d system can be driven to a gapped phase by condensing these dyons, and the gapped 3d system may have a topological order which depends on the condensed object. There should be a systematic formalism describing the relation between the gapped 3d systems and the corresponding gapped 1d systems after dimensional compactification. The problem would be further enriched if there is a topological Θ-term in the 3d system [49, 50].

The 1d system after compactification is described by ordinary boson operators $\hat{n}$ and $\hat{\theta}$, and these bosons do not fractionalize. Hence, it is sufficient to view the 1d system as the boundary of a 2d SPT state instead of a 2d topological order with fractionalization. The 4d bulk of the 3d system is also a generalized SPT state with 1-form symmetries rather than a topological order. But a topological order with fractionalized 1-form symmetries would be an interesting direction to explore. In section 4, we presented one prototype of such topological order. A more general and systematic discussion of the fractionalization of higher form symmetry is worth studying in the future.

Besides the higher-form symmetries and tensor like symmetries, many other generalized concepts of symmetries have been discussed in the past (for early examples, please see [51–54]). Much of the topics discussed in this paper, such as the SPT states and anomalies involving these generalized symmetries, also provide interesting future directions.

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