

Physics of symmetry protected topological phases involving higher symmetries and its applications

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We discuss physical constructions and the boundary properties of various symmetry-protected topological phases that involve 1-form symmetries from one spatial dimension to four spatial dimensions ($4d$). For example, the prototype three-dimensional ($3d$) boundary state of $4d$ SPT states involving 1-form symmetries can be either a gapless photon phase (quantum electrodynamics) or gapped topological order enriched by 1-form symmetries; that is, the loop excitations of these topological orders carry nontrivial 1-form symmetry charges. This study also serves the purpose of diagnosing anomalies of $3d$ states of matter. Connection between SPT states with 1-form symmetries and condensed-matter systems such as quantum dimer models at one lower dimension will also be discussed. Whether a quantum dimer model can have a trivial gapped phase depends on the nature of its corresponding bulk state in one higher dimension.

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I. INTRODUCTION

Symmetry-protected topological (SPT) phases [1,2] have greatly enriched our understanding of quantum states of matter. With certain symmetries, the boundary of these SPT states cannot be trivially gapped without degeneracy. In particular, many exotic states of matter can be realized at the two-dimensional ($2d$) boundary of three-dimensional ($3d$) bosonic SPT states. For example, exotic quantum critical points (QCPs) in $2d$ with spatial symmetries (both on the square or triangular lattice) can be realized at the boundary of certain $3d$ SPT states [3,4], and the conjectured emergent symmetry of the deconfined QCP matches well with the bulk symmetry of the SPT state, sometimes these emergent symmetries are only revealed through certain dualities [5,6] between $(2+1)$ -dimensional $[(2+1)D]$ quantum field theories.¹ The analysis of the SPT state in the $(d+1)D$ bulk can also be used as a diagnostic of the “Lieb-Schultz-Mattis theorem” in d -dimensional systems with spatial symmetries, i.e., whether the d -dimensional system can be gapped without degeneracy [7–12] is related to the nature of the corresponding bulk state in one higher dimension.

In recent years it was realized that the very concept of symmetry can be generalized to higher-dimensional objects rather than just point-like operators [13–21]. Examples of SPT states that involve these generalized symmetries were discussed in previous literature [18,21–30]. For example, a classification of SPT states based on generalized cobordism theory was given in Refs. [27,28], and exactly soluble lattice models for a class of SPT states were constructed in Refs. [29,30]. In the

current paper we focus on physical construction and boundary properties of a series of SPT states with generalized concepts of symmetries, from $(1+1)D$ to $(4+1)D$. We do not seek exactly soluble models, instead we focus on general physical pictures of these states. For example, the prototype $4d$ [or $(4+1)D$] SPT state we will discuss can be constructed by “decorated Dirac monopole loop” picture, which is analogous to the flux attachment construction in the $2d$ SPT state. And the prototype $3d$ boundary state of the four-dimensional ($4d$) SPT state is a photon phase with various constraints of dynamics, quantum numbers, and statistics on the electric and magnetic charges. We assume that the gauge-invariant objects or excitations, i.e., objects that do not couple to a dynamical gauge field, are always bosonic. These include point particles and higher-dimensional excitations such as loops.

The 1-form symmetry transformation acts on loop-like operators such as the Wilson loop or the ’t Hooft loop of a dynamical gauge field. The existence of an electric 1-form symmetry demands that the electric charge of the gauge field is infinitely heavy. In condensed-matter systems the quantum dimer model [31] naturally fits this criterion. It is well known that the quantum dimer model can be mapped to a lattice gauge field [32]. In a quantum dimer model, every site of the lattice is connected to a fixed number of dimers, which implies that there is a background electric charge distribution, but no dynamical charge in the system. Hence the quantum dimer model naturally has a 1-form symmetry. The quantum dimer model on certain d -dimensional lattice may be mapped to the boundary of a $(d+1)D$ SPT state with 1-form symmetry in a certain limit, and the spatial symmetries of the quantum dimer model is mapped to the onsite symmetry of the bulk SPT state. The analysis of the SPT state in the bulk has strong indications on the allowed phenomena of the quantum dimer model at d dimensions.

¹In this work the symbol “ d ” labels spatial dimensions, and “ D ” labels space-time dimensions. For example, the $3d$ space is identical to the $(3+1)D$ space-time.

Due to the inevitable complexity of notations used in this paper, we keep a self-consistent conventions of notations:

The N -form symmetry G will be labeled as $G^{(N)}$, such as $U(1)^{(1)}$, $Z_n^{(1)}$, etc. Ordinary 0-form symmetry will be labeled without superscript.

Gauge symmetries associated with a *dynamical* gauge field will be labeled as $u(1)^{(1)}$, $z_n^{(2)}$, etc. depending on the nature of the gauge fields. A topological order which corresponds to a dynamical discrete gauge field will also be labeled as, for example, a z_n topological order.

Gauge symmetries associated with *background* gauge fields will be labeled as $\mathcal{U}(1)^{(1)}$, $\mathcal{Z}_n^{(2)}$, etc.

Classifications of SPT states will be labeled as $\mathbb{Z}$, $\mathbb{Z}_n$, etc.

For space and space-time dimensions, for example, $3d$ space refers to three spatial dimensions; $(3+1)D$ refers to the space-time dimension, which is the same as $4D$ Euclidean space-time. Also, QED_4 refers to quantum electrodynamics in $(3+1)D$ or $4D$ space-time dimensions.

For QED_4 , there are point-like particles such as electric charge, and Dirac monopole. We label bosonic (fermionic) electric charges as $e_b(e_f)$, and bosonic (fermionic) Dirac monopoles as $m_b(m_f)$. Some of these point excitations have no dynamics (infinitely heavy) due to the 1-form symmetries; we label these immobile point particles as e_{0b} , e_{0f} , etc. A QED_4 with bosonic electric charge and fermionic Dirac monopole is labeled as “ $\text{QED}_4\{e_b, m_f\}$.”

II. BUILDING BRICKS: $1d$ SYMMETRY-PROTECTED TOPOLOGICAL STATE WITH 1-FORM SYMMETRIES

The simplest SPT state that involves a 1-form symmetry exists in one-dimensional ($1d$) space or $(1+1)D$ space-time. $1d$ SPT state with a 1-form symmetry is analogous to an ordinary SPT state in zero-dimensional ($0d$) space. For a $U(1)^{(1)}$ -form symmetry, a SPT state in $1d$ simply corresponds to a state with integer electric flux through the system. Let us take a $1d$ chain with electric-field operators defined on the links. Due to the Gauss law constraint, $\nabla_x \hat{e}(x) = 0$, the electric field $\hat{e}(x)$ takes a uniform integer eigenvalue on the entire chain [in a compact $u(1)$ lattice gauge theory, the electric-field operator $\hat{e}(x)$ takes discrete integer value, while its conjugate operator $\hat{a}(x)$ is periodically defined], hence for a $U(1)^{(1)}$ -form symmetry, the classification of $1d$ SPT states is $\mathbb{Z}$, which corresponds to different integer eigenvalues of $\hat{e}(x)$. It is analogous to the $\mathbb{Z}$ classification of a zero-dimensional ordinary SPT state with $U(1)$ symmetry [1,2].

The Hamiltonian of a $1d$ lattice $U(1)$ gauge field is also very simple, for example:

$$H = \sum_x g[\hat{e}(x) - k]^2. \quad (1)$$

Due to the Gauss law constraint, a Hamiltonian must be invariant under gauge transformation $\hat{a} \rightarrow \hat{a} + \nabla_x f(x)$, where $\hat{a}$ is the conjugate operator of $\hat{e}$. A local $1d$ Hamiltonian that involves $\hat{a}$ cannot be gauge invariant, hence a local gauge-invariant Hamiltonian is only a function of $\hat{e}$. In Eq. (1) k can take continuous values. When k is half integer, the system is at the transition between two SPT states, and the ground state of the Hamiltonian is twofold degenerate with $\hat{e}(x) = k \pm 1/2$, namely, the transition is a level crossing between two eigen-

values of $\hat{e}(x)$. This transition should be viewed as a first-order transition.

One can also couple the electric field to a background 2-form $\mathcal{U}(1)^{(2)}$ gauge field:

$$S = \int d\tau dx i f_{\mu\nu} B_{\mu\nu} \quad (2)$$

In $(1+1)D$ the stress tensor of the $u(1)$ gauge field is just the electric field: $f_{10} - f_{01} = e(x)$, and $B_{01} = -B_{10}$ is a Lagrange multiplier. Hence the $(1+1)D$ topological response theory for the SPT state is

$$S_{1d\text{-topo}} = \int_{(1+1)D} ikB, \quad (3)$$

which is a $(1+1)D$ Chern-Simons action of the 2-form gauge field B , and its level k takes only integer values. For each integer level k , the electric field (the 1-form symmetry charge)

$$e(x) = \frac{\delta S_{1d\text{-topo}}}{i\delta B(x)} = k. \quad (4)$$

The $1d$ SPT state with 1-form symmetries will be the building bricks for SPT states in higher dimensions. Suppose we break the $U(1)^{(1)}$ down to $Z_n^{(1)}$ symmetry, the topological response theory Eq. (3) still applies, but B is now a 2-form $\mathcal{Z}_n^{(2)}$ background gauge field. The classification of the SPT state will reduce to $\mathbb{Z}_n$, which means that in Eq. (3) the integer $k+n=k$.

III. $4d$ SYMMETRY-PROTECTED TOPOLOGICAL STATES WITH $G_1^{(1)} \times G_2^{(1)}$ SYMMETRY

A. Parent $4d$ symmetry-protected topological state with $U(1)^{(1)} \times U(1)^{(1)}$ symmetry

We now discuss SPT states in $4d$ space that involves 1-form symmetries. This discussion is useful for diagnosing anomalies of $3d$ states of matter, namely, some $3d$ states of matter can only be realized at the boundary of a $4d$ SPT state. The parent SPT state that we start with is the $(4+1)D$ state with the $U(1)^{(1)} \times U(1)^{(1)}$ -form symmetry. With two $U(1)^{(1)}$ -form symmetries, the system can couple to two background $\mathcal{U}(1)^{(2)}$ -form gauge fields B^1 and B^2 , and the response theory in $(4+1)D$ reads

$$S_{4d\text{-topo}} = \int_{(4+1)D} \frac{ik}{4\pi} \epsilon_{IJ} B^I \wedge dB^J, \quad (5)$$

where $\epsilon_{IJ} = i\sigma^y$. For each integer k , Eq. (5) is a different Chern-Simons theory, and the system should correspond to a different SPT state, hence these SPT states described by Eq. (5) have a $\mathbb{Z}$ classification. The $(3+1)D$ boundary of this state is a QED_4 without dynamical electric or magnetic charge (Dirac monopole). This QED_4 has a $U(1)^{(1)} \times U(1)^{(1)}$ mixed 't Hooft anomaly as was derived in previous literature [18,21,22].

To construct this $4d$ SPT state, we can start with two $(4+1)Du(1)$ gauge fields $\vec{a}^1$ and $\vec{a}^2$. These two gauge fields both have *electric* 1-form $U(1)^{(1)}$ symmetry, namely, both gauge fields have no dynamical electric charges, i.e., the Gauss law constraint on the electric field is strictly enforced. This is equivalent to tuning the electric charges in the $4d$ bulk to be infinitely heavy. Both $u(1)$ gauge fields allow dynamical

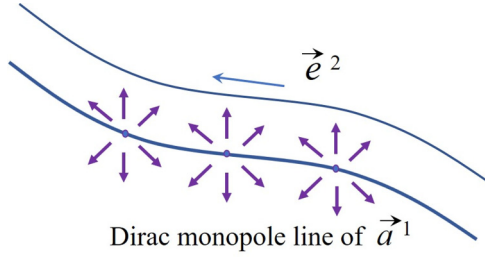


FIG. 1. The decorated Dirac monopole loop construction of the parent SPT state in $4d$ space. The Dirac monopole loop of gauge field $\vec{a}^1$ is decorated with the $1d$ SPT state of the $U(1)^{(1)}$ -form symmetry associated with gauge field $\vec{a}^2$. After the condensation of the decorated Dirac monopole loops, the $4d$ system is driven into a SPT state described by response theory (5).

Dirac monopole loop or line defects in $4d$ space. We first discuss the cases where the charges of $\vec{a}^1$ and $\vec{a}^2$ are both bosons, otherwise $\vec{a}^1$ and $\vec{a}^2$ would be Spin^C connections. Situations with fermionic gauge charges of $\vec{a}^1$ and $\vec{a}^2$ will be discussed later.

We use the analog of the “flux attachment” (or “decorated defect”) construction of the SPT state which was used to construct a $2d$ bosonic SPT state [33]. In $2d$ space, a $U(1) \times U(1)$ SPT state (the parent state of many $2d$ SPT states) can be constructed by binding the vortex defect of one $U(1)$ symmetry with the charge of the other $U(1)$ symmetry, and condense the bound state, which drives the system into a gapped SPT phase. In $4d$ space, the analog of the vortex defect of an ordinary $U(1)$ 0-form symmetry, is the Dirac monopole loop or line of a $u(1)$ gauge field. We decorate the Dirac monopole loop of $\vec{a}^1$ with the $1d$ SPT state defined with the 1-form symmetry associated with $\vec{a}^2$ with level $(+k)$ in Eq. (3), and condense or proliferate the decorated loops (Fig. 1). Once the bound state between the monopole loop of $\vec{a}^1$ and the $(+k)$ unit of electric flux of $\vec{a}^2$ is condensed, the monopole loop of $\vec{a}^2$ will be automatically bound with $(-k)$ unit of electric flux of $\vec{a}^1$.

Condensation of Dirac monopole loops would normally drive a $(4+1)Du(1)$ gauge field to the gapped confined phase (the loop excitation is coupled to a dual dynamical 2-form gauge field, and the condensate is gapped due to the Higgs mechanism). But because the Dirac monopole loop is decorated with another SPT state with 1-form symmetry in our case, after the condensation of the decorated monopole loops, the phase in the $4d$ bulk is not an ordinary confined phase, it is actually a SPT phase described by Eq. (5). In fact, Eq. (5) directly implies that the 1-form symmetry charge (electric field) $\vec{e}^2(x)$, which is the variation $\delta S_{4d-\text{topo}}/(i\delta B_{01}^2)$, equals to the flux of B^1 , which is attached to the monopole of $\vec{a}^1$.

The $3d$ boundary of the $4d$ SPT state is most naturally a $(3+1)D$ QED₄ with both magnetic and electric 1-form symmetries. The electric 1-form symmetry of the boundary, QED₄ is inherited from the 1-form symmetry of $\vec{a}^1$ in the bulk, while the magnetic 1-form symmetry of QED₄ corresponds to the electric 1-form symmetry of $\vec{a}^2$ in the bulk, because the Dirac monopole line of $\vec{a}^1$ in the $4d$ bulk is bound decorated with the electric 1-form symmetry charge of $\vec{a}^2$. As we mentioned previously, we first discuss the situation with bosonic point particles, hence in this QED₄ the infinitely heavy electric charge and Dirac monopoles are both bosons.

We label this QED₄ as QED₄{ e_{0b} , m_{0b} }. Even though these point particles have infinite mass, their statistics still matter, because their Wilson loops (or ’t Hooft loops) still exist. If these point particles are fermions, the Wilson loop will need a framing structure, and the Wilson loop or ’t Hooft loop with a twist will acquire a minus sign.

B. Descendant $4d$ symmetry-protected topological state with $U(1)^{(1)} \times Z_n^{(1)}$ symmetry

Now we break one of the $U(1)^{(1)}$ -form symmetry down to the $Z_n^{(1)}$ symmetry. The topological response theory remains unchanged from Eq. (5), although one of the background 2-form gauge fields will become a $Z_n^{(2)}$ background 2-form gauge field. The decorated monopole line construction discussed in the previous section still applies here. One key difference is that, because the $1d$ SPT phase with $Z_n^{(1)}$ -form symmetry has a $\mathbb{Z}_n$ classification itself, the flux attachment or decorated defect construction mentioned in the previous section will naturally lead to a $\mathbb{Z}_n$ classification of the $4d$ SPT state also. Namely, when $k = n$ in Eq. (5), this bulk SPT state will be trivialized, because the $1d$ SPT state decorated on the Dirac monopole line is trivial.

We can always start with QED₄ as a candidate boundary state. Now since the magnetic 1-form symmetry is only $Z_n^{(1)}$, it means that there are dynamical Dirac monopoles with n magnetic charges (Dirac monopole with $2\pi n$ flux quanta). As we mentioned before we first focus on the cases where the point excitations are bosons, then we can condense the n magnetic charge at the $3d$ boundary without breaking any symmetry. The condensate of the $2\pi n$ Dirac monopole will drive the boundary into a $3dz_n$ topological order.

An ordinary $3dz_n$ topological order is the deconfined phase of a dynamical $z_n^{(1)}$ gauge field. In an ordinary $3dz_n$ topological order, normally there are two types of excitations: a point particle which is the remnant of the 2π Dirac monopole; and also another line or loop excitation which is coupled to a $z_n^{(2)}$ -form gauge field. If the loop excitation is condensed (proliferated in $4D$ Euclidean space), the z_n topological order is trivialized, and the system becomes gapped and nondegenerate.

The dynamics of the loop excitation can be schematically described by the following Hamiltonian:

$$H_{\text{loop}} = \sum_{\mathcal{C}} -t_{\mathcal{C}} \cos \left(\sum_{\vec{l} \in \mathcal{C}} \hat{c}_{\vec{l}} - \sum_{\vec{p} \in \mathcal{A}_{\mathcal{C}}} \hat{b}_{\vec{p}} \right) + \dots \quad (6)$$

In this equation, $\mathcal{C}$ represents certain loop configuration; $\vec{l}$ is a link which is part of this loop, and $\mathcal{A}_{\mathcal{C}}$ is a membrane whose boundary is the loop $\mathcal{C}$ ($\partial \mathcal{A}_{\mathcal{C}} = \mathcal{C}$); $\vec{p}$ is a plaquette that belongs to $\mathcal{A}_{\mathcal{C}}$. $\Psi_{\vec{l}}^{\dagger} \sim \exp(i\hat{c}_{\vec{l}})$ is the creation operator of the loop segment on link $\vec{l}$, and $\hat{b}_{\vec{p}}$ is a 2-form gauge field defined on plaquette $\vec{p}$. The direction of the link and the unit plaquette can be absorbed into the definition of $\hat{c}$ and $\hat{b}$ and render them a 1-form and 2-form fields.

For an ordinary z_n topological order, both $\hat{c}_{\vec{l}}$ and $\hat{b}_{\vec{p}}$ take eigenvalues $2\pi N/n$ with integer N . Hence the “condensation” of the loop excitation will not lead to degeneracy because of the existence of the $z_n^{(2)}$ -form gauge field $\hat{b}$. Or in other words, the condensation of the loop excitation will be fully

“Higgsed” due to the coupling to the $z_n^{(2)}$ dynamical gauge field $\hat{b}$, and this Higgs phase is the confined phase of the $z_n^{(1)}$ gauge theory.

However, if the loop excitation carries a $U(1)^{(1)}$ -form charge, the situation would be very different. Now $\hat{c}_l$ can take continuous values between 0 and 2π . Condensing the loop would just drive the system back into a gapless photon phase. Physically because the loop excitation carries a $U(1)^{(1)}$ -form charge, condensing the loop excitations would lead to spontaneous $U(1)^{(1)}$ -form symmetry breaking, whose “Goldstone mode” is precisely the photon.

With the bulk response action Eq. (5), the loop excitation of 3d boundary carries charge quantum k/n of the $U(1)^{(1)}$ -form symmetry. However, when $k = n$, the quantum number of the loop excitation can be screened by binding with unfractionalized integer 1-form symmetry charge, hence the loop excitations become completely neutralized. Then when $k = n$ the neutralized loop excitation can proliferate and drive the boundary to a fully gapped and nondegenerate state, just like the case of an ordinary $z_n^{(1)}$ gauge theory. This argument again leads to a $\mathbb{Z}_n$ classification.

C. Descendant 4d symmetry-protected topological state with $Z_q^{(1)} \times Z_n^{(1)}$ symmetry

We can further break the left $U(1)^{(1)}$ -form symmetry down to $Z_q^{(1)}$ from the previous example. Now in the condensate of the $2\pi n$ Dirac monopole, the loop excitation will carry k/n units of the $Z_q^{(1)}$ -form symmetry charge, and the loop excitation is coupled to a dual $z_n^{(2)}$ gauge field. Our interest is to ask when this 3d boundary can be fully gapped without degeneracy.

Let us start with the simple example with $k = 1$, $q = 3$, and $n = 2$. Following the discussion in the previous section, we consider the z_2 topological order after condensing the 4π Dirac monopole at the boundary QED_4 (the $2\pi n$ monopole has dynamics and can condense). There is a loop excitation of this z_2 topological order, which couples to a dual $z_2^{(2)}$ gauge field, and carries half charge of the $Z_3^{(1)}$ -form symmetry. Now consider a loop excitation whose creation operator is $P_C^\dagger$:

$$P_C^\dagger \sim \prod_{\vec{l} \in \mathcal{C}} \Psi_{\vec{l}}^\dagger \sim \exp \left(i \sum_{\vec{l} \in \mathcal{C}} \hat{c}_{\vec{l}} \right). \quad (7)$$

$P_C^\dagger$ carries half charge under $Z_3^{(1)}$, and it also couples to a dual $z_2^{(2)}$ gauge field. Under both the $Z_3^{(1)}$ symmetry and the $z_2^{(2)}$ gauge symmetry, $\mathcal{C}$ transforms as

$$\begin{aligned} Z_3^{(1)} : P_C^\dagger &\rightarrow e^{i\frac{1}{2} \frac{2\pi N}{3}} P_C^\dagger, \\ z_2^{(2)} - \text{gauge} : P_C^\dagger &\rightarrow -P_C^\dagger, \end{aligned} \quad (8)$$

with integer N . One can check that, by combining the loop operator P_C with unfractionalized integer 1-form charges, the $Z_3^{(1)}$ transformation can be completely cancelled by a $z_2^{(2)}$ gauge transformation. In other words the fractional $Z_3^{(1)}$ charge carried by the $P_C^\dagger$ can be “neutralized” by binding a gauge invariant $Z_3^{(1)}$ charge, and the 3d boundary system can

be driven into a trivial gapped phase by condensing this $Z_3^{(1)}$ neutral loop excitation.

The discussions above can be generalized to other q and n . With $k = 1$ in Eq. (5), after condensing the $2\pi n$ monopole, the 3d boundary system is driven into a z_n topological order whose loop excitation carries $1/n$ fractional $Z_q^{(1)}$ -form symmetry charge. Our interest is to check when this fractional 1-form symmetry charge can be “neutralized” by integer 1-form symmetry charge, namely, by binding integer 1-form symmetry charge the $Z_q^{(1)}$ transformation can be completely absorbed or cancelled by the dual $z_n^{(2)}$ gauge transformation.

Under a $Z_q^{(1)}$ transformation, the loop creation operator P_C acquires phase angle $2\pi/(nq)$; after binding with Q units of integer $Z_q^{(1)}$ charge, the loop would acquire phase angle $2\pi/(nq) + 2\pi Q/q$. Now we seek for a pair of integers (Q, N) which satisfies the following equation:

$$\frac{1}{nq} + \frac{Q}{q} = \frac{N}{n}. \quad (9)$$

This would mean that the $Z_q^{(1)}$ transformation can be totally absorbed or cancelled by a gauge transformation. For $(q, n) = (3, 2)$ one can choose $(Q, N) = (1, 1)$. In general the question is equivalent to finding a pair of integers (Q, N) that satisfies $Nq - Qn = 1$, which is only possible when q and n are coprime. When q and n are not coprime, the loop quantum number can be fully neutralized when $k = \text{gcd}(q, n)$. This implies a $\mathbb{Z}_{\text{gcd}(q, n)}$ classification.

More states

All the SPT states discussed so far have bosonic electric charge and Dirac monopoles at its boundary QED_4 , namely, the boundary of all the SPT states are $\text{QED}_4\{e_{0b}, m_{0b}\}$ states. Let us revisit the starting point of our bulk construction of Eq. (5). The two $u(1)$ gauge fields $\vec{a}^1$ and $\vec{a}^2$ can have either bosonic or fermionic electric charges with infinite mass in the bulk, which become the static electric charges and Dirac monopoles of the boundary QED_4 . Hence logically there will also be $\text{QED}_4\{e_{0b}, m_{0f}\}$, $\text{QED}_4\{e_{0f}, m_{0b}\}$, $\text{QED}_4\{e_{0f}, m_{0f}\}$ states that we need to discuss. As we pointed out before, the statistics of static particles still affect the Wilson and 't Hooft loops. We defer discussions of these states to Sec. V.

IV. 4d SYMMETRY-PROTECTED TOPOLOGICAL STATE WITH $U(1)^{(1)} \times G$ SYMMETRY AND 3d QUANTUM DIMER MODEL

Here we consider 4d SPT states with both a $U(1)^{(1)}$ symmetry and an ordinary 0-form symmetry G . The decorated defect construction in the previous section can be generalized here: we start with one $(4+1)D$ $u(1)$ gauge field $\vec{a}$ with a 1-form electric symmetry, and decorate its Dirac monopole line with the 1d SPT state with symmetry G , then condense the monopole line in the bulk. A prototype 4d SPT state with such construction was discussed previously, whose G symmetry is $\text{SO}(3)$, and its topological response theory is [34]

$$\mathcal{S}_{4d-\text{topo}} = i\pi \int_{(4+1)D} w_2[A^{\text{SO}(3)}] \cup \frac{dB}{2\pi}, \quad (10)$$

where $A^{\text{SO}(3)}$ is the external 1-form $\text{SO}(3)$ gauge field.

Generally speaking the discussion of $4d$ SPT state with 1-form symmetry has implications on properties of $3d$ systems with loop-like excitations. If in certain limit a $3d$ system with spatial symmetries can be mapped to the boundary of a $4d$ state with onsite symmetries, then whether the $4d$ bulk is a nontrivial SPT state has strong implication on whether the $3d$ system can be trivially gapped, i.e., the nature of the $4d$ bulk helps us prove a Lieb-Schultz-Mattis (LSM) theorem [35,36] of the $3d$ system. In recent years much progress has been made in understanding the LSM theorems for quantum spin systems using the anomaly analysis of its corresponding higher-dimensional bulk states [7–12,37]. In condensed-matter theories the quantum dimer model is an example of systems with loop-like excitations. Dimers are defined on the links of the lattice, and each site of the lattice is connected to a fixed number of dimers. Previous literature has shown that the $3d$ quantum dimer model can be mapped to a QED₄ without dynamical electric charge [38], but its monopole can carry a nontrivial quantum number under spatial group due to the Berry phase, and in particular, for the quantum dimer model on the cubic lattice, the monopole of the QED₄ carries a “spin-1/2” representation (projective representation) of an emergent SO(3) symmetry [39,40]. Hence this quantum dimer model is analogous to the boundary of a $4d$ SPT state with symmetry $U(1)^{(1)} \times SO(3)$, and there should be a LSM theorem for this quantum dimer model.

This LSM theorem for the quantum dimer model is consistent with the LSM theorem for spin-1/2 systems on the cubic lattice. In Ref. [7], various quantum spin systems on the cubic lattice were considered. For example, a $SU(N)$ spin system on the cubic lattice with fundamental and antifundamental representations on the two sublattices of the cubic lattice has a LSM theorem for even integer N , but there is no LSM theorem for odd integer N , i.e., the quantum spin system described above with odd integer N can have a featureless gapped ground state on the cubic lattice. However, a quantum dimer model on the cubic lattice could be the low-energy effective description of all these systems, since two nearest-neighbor AB sites can always form a dimer (spin singlet), regardless of even or odd integer N .

One simple extension of Eq. (10) is that, when we break SO(3) down to its subgroup $U(1) \rtimes Z_2$, Eq. (10) reduces to

$$S_{4d-\text{topo}} = i \frac{\Theta}{(2\pi)^2} \int_{(4+1)D} dB \wedge dA, \quad (11)$$

where A is the background $U(1)$ gauge field. The integral in Eq. (11) is quantized, hence Θ is periodically defined: $\Theta = \Theta + 2\pi$. Under the Z_2 subgroup of SO(3), A changes sign, hence a symmetric response theory demands $\Theta = k\pi$ with integer k . Equation (11) with $k = 1$ corresponds to the nontrivial $4d$ SPT phase.

Equation (11) also describes the corresponding $4d$ bulk state if instead we consider a quantum dimer model defined on a $3d$ tetragonal lattice, here the $U(1)$ symmetry is further reduced to a Z_4 symmetry, and the Z_4 corresponds to the rotation of the square lattice in each layer. In this case, in the topological response theory (11), A is a background Z_4 gauge field. Equation (11) still describes a nontrivial $4d$ SPT state with 1-form symmetry.

The situation will be very different if we consider a quantum dimer model on a $3d$ bipartite lattice with an effective $Z_3 \rtimes Z_2 = S^3$ symmetry. The Z_3 should correspond to a three-fold rotation C_3 in the XY plane, and Z_2 is a π rotation about the x axis. Such quantum dimer models can potentially be mapped to the boundary of a $4d$ system with $U(1)^{(1)} \times S^3$ symmetry. But there is no $1d$ SPT state with the S^3 symmetry, hence the $4d$ bulk with the $U(1)^{(1)} \times S^3$ symmetry is also trivial as a descendant state of the SPT state described by Eq. (11). Hence there should be no LSM theorem for these quantum dimer models, i.e., these quantum dimer models can in general have a gapped ground state without degeneracy, unless this model has higher symmetries than the lattice itself.

V. OTHER $4d$ SYMMETRY-PROTECTED TOPOLOGICAL STATES

With just a $U(1)^{(1)}$ symmetry, there is already a nontrivial $4d$ SPT phase, whose boundary is a QED₄ with a 1-form electric symmetry, and the Dirac monopole is a fermion (labeled as m_f). The unit electric charge (labeled as e_{ob}) is infinitely heavy at the boundary QED₄ due to the $U(1)^{(1)}$ symmetry. We label this boundary QED₄ as state QED₄ $\{e_{ob}, m_f\}$. The bulk is a nontrivial SPT state, namely, its boundary QED₄ cannot be trivially gapped. One can condense a Cooper pair of the fermionic Dirac monopole m_f , and drive the QED₄ to a “monopole superconductor,” which is also a z_2 topological order. The loop excitation of the z_2 topological order will carry a fractional half charge of the $U(1)^{(1)}$ 1-form symmetry and hence cannot lead to a fully gapped and nondegenerate state after condensation for the reasons explained previously in this paper. Although the electric charges are infinitely heavy due to the 1-form symmetry, its statistics still matters to physical observables such as the Wilson loops of the QED₄. And in this QED₄ the infinitely heavy electric charge is a boson.

This state remains a nontrivial SPT after breaking the $U(1)^{(1)}$ down to $Z_n^{(1)}$ with even integer n , the cases with $n = 2, 4$ were discussed in Refs. [27,28]. But this state will be trivialized if n is an odd integer. For odd integer n , in the monopole superconductor constructed above, the loop excitation carries half charge of the $Z_n^{(1)}$ 1-form symmetry, and it can be “neutralized” by binding unfractionalized 1-form symmetry charge, i.e., the $Z_n^{(1)}$ transformation on the loop excitation can be completely canceled by the $z_2^{(2)}$ gauge transformation on the loop excitation, then the condensation of the neutralized loop can lead to a trivially gapped phase.

There is even a nontrivial bosonic SPT state in $4d$ space without any symmetry; its boundary is a QED₄ whose both electric charge and Dirac monopole (including their bound state dyon) are fermions [41,42]. We label this QED as the QED₄ $\{e'_f, m'_f\}$ state. We view QED₄ $\{e_{ob}, m_f\}$ and QED₄ $\{e'_f, m'_f\}$ as two root states, and by “gluing” these two QED₄ states together, another new state can be constructed. One can condense the bound state of the Dirac monopoles [labeled as (m_f, m'_f)] of both QED₄ systems, then the gauge fields from both QED₄ will be identified due to the Higgs mechanism, and e_{ob} and e_f are both confined since they both have nontrivial statistics with the condensed bound state of monopoles. Although e_{ob} is infinitely heavy, its confinement can still be defined by the behavior of the Wilson loop

of its gauge field. In the condensed phase of bound state (m_f, m'_f) , the Wilson loop of each individual gauge field obeys the area law. But the bound state $(e_{0b}, -e'_f)$, which has trivial mutual statistics with (m_f, m'_f) , remains deconfined, although it is still infinitely heavy. This new QED state has infinitely heavy fermionic electric charge, and dynamical fermionic Dirac monopole. This new state is labeled as $\text{QED}_4\{e_{0f}, m_f\}$. One can also exchange e and m and label the state as $\text{QED}_4\{e_f, m_{0f}\}$, i.e., a state with dynamical fermionic gauge charge, but infinitely heavy fermionic Dirac monopole.

Summary of 4d symmetry-protected topological states with 1-form symmetries

Let us reinvestigate the states discussed in the end of Sec. III. As we briefly discussed there, besides the states $\text{QED}_4\{e_{0b}, m_{0b}\}$, logically there should also be $\text{QED}_4\{e_{0b}, m_{0f}\}$, $\text{QED}_4\{e_{0f}, m_{0b}\}$, $\text{QED}_4\{e_{0f}, m_{0f}\}$, which can all be boundary states of $(4+1)D$ SPT bulk. It turns out that these states can be constructed by gluing states in Secs. III and V. For example, starting with the state $\text{QED}_4\{e_{0b}, m_{0b}\}$ discussed in Sec. III (we label its gauge field as $\vec{a}$), one can combine it with the state $\text{QED}_4\{e'_{0b}, m'_f\}$ (with gauge field $\vec{a}'$) discussed in Sec. V, and consider the charge bound state $(e_{0b}, -e'_{0b})$. This bound state carries zero total gauge charge of $\vec{a}$ and $\vec{a}'$. We assume that there is only one $U(1)^{(1)}$ 1-form symmetry, hence the charge bound state $(e_{0b}, -e'_{0b})$, which carries zero total gauge charge, is no longer necessarily infinitely heavy and can acquire dynamics and condense. Its condensate would render $\vec{a} = \vec{a}'$ through the Higgs mechanism, and in the condensate the monopole bound state (m_{0b}, m'_f) remains deconfined, because it has trivial mutual statistics with $(e_{0b}, -e'_{0b})$. The final state is identical to state $\text{QED}_4\{e_{0b}, m_{0f}\}$ discussed in Sec. III. Following the same argument, through gluing $\text{QED}_4\{e_{0b}, m_{0f}\}$ and state $\text{QED}_4\{e'_f, m'_{0f}\}$ discussed in Sec. V [by condensing the bound state $(m_{0f}, -m'_{0f})$], one can obtain another state $\text{QED}_4\{e_{0f}, m_{0f}\}$ discussed in Sec. III.

The construction of all these states discussed so far can be summarized mathematically in a single unified topological response theory in the $(4+1)D$ bulk:

$$S_{4d-\text{topo}} = \int_{(4+1)D} \frac{ik_0}{2\pi} B^1 \wedge dB^2 + \frac{ik_1}{2} dB^1 \cup w_2 + \frac{ik_2}{2} dB^2 \cup w_2 + i\pi k_3 w_2 \cup w_3. \quad (12)$$

w_2 and w_3 are the second and third Stiefel-Whitney class of the space-time manifold. k_0 takes arbitrary integer values, while k_1, k_2 , and k_3 only take value 0 and 1, since the Stiefel-Whitney class is defined mod 2. This topological response theory is equivalent to the discussion based on the cobordism theory in Refs. [27,28]. We also note that the state with only $k_3 \neq 0$ corresponds to the state that does not need any symmetry protection discussed earlier in this section.

The classification of 4d SPT states discussed so far is summarized as follows:

$$\begin{aligned} U(1)^{(1)} : \mathbb{Z}_2 \times \mathbb{Z}_2, \\ \mathbb{Z}_n^{(1)} : \mathbb{Z}_2 \times \mathbb{Z}_{\text{gcd}(2,n)}, \\ U(1)^{(1)} \times U(1)^{(1)} : \mathbb{Z} \times \mathbb{Z}_2^3, \end{aligned}$$

$$\begin{aligned} U(1)^{(1)} \times \mathbb{Z}_n^{(1)} : \mathbb{Z}_n \times \mathbb{Z}_2^2 \times \mathbb{Z}_{\text{gcd}(2,n)}, \\ \mathbb{Z}_q^{(1)} \times \mathbb{Z}_n^{(1)} : \mathbb{Z}_{\text{gcd}(q,n)} \times \mathbb{Z}_{\text{gcd}(2,q)} \\ \times \mathbb{Z}_{\text{gcd}(2,n)} \times \mathbb{Z}_2. \end{aligned} \quad (13)$$

VI. 3d SYMMETRY-PROTECTED TOPOLOGICAL STATE WITH $G_1^{(1)} \times G_2$ SYMMETRY

A. Parent 3d symmetry-protected topological state with $U(1)^{(1)} \times U(1)$ symmetry

The parent 3d SPT state we consider is a state with $U(1)^{(1)} \times U(1)$ symmetry. We can couple its symmetry currents to a background 2-form gauge field B , and a 1-form gauge field A . The response theory for this SPT state is

$$S_{3d-\text{topo}} = \int \frac{ik}{2\pi} B \wedge dA = \int \frac{ik}{2\pi} A \wedge dB. \quad (14)$$

To construct such a state, again one can rely on the decorated defect picture. We can start with a photon phase with an electric $U(1)^{(1)}$ 1-form symmetry, namely, there is no dynamical electric charge, or equivalently the electric charge is infinitely heavy, but there are dynamical Dirac monopoles. Then we decorate the Dirac monopole with a zero-dimensional bosonic SPT state with $U(1)$ symmetry, which is a bosonic charge with $U(1)$ symmetry. This zero-dimensional bosonic SPT state has $\mathbb{Z}$ classification, which corresponds to states with integer charges of a boson with $U(1)$ symmetry. These states can also be equivalently constructed by decorating the vortex line of the $U(1)$ order parameter with a 1d SPT state with $U(1)^{(1)}$ 1-form symmetry, i.e., the building bricks discussed in Sec. II.

After condensing the decorated Dirac monopole, the 3d bulk of the system is driven into a fully gapped state without degeneracy. The 2d boundary of the system would most naturally be a QED_3 whose dynamical $u(1)$ gauge field $\vec{a}$ has no dynamical gauge charge, but its magnetic flux carries conserved $U(1)$ quantum number that couples to A . The QED_3 is a dual of the superfluid phase with spontaneous breaking of the $U(1)$ symmetry. And the assumption that there is no dynamical electric charge of gauge field $\vec{a}$ is equivalent to the statement that there is no dynamical vortex of the dual superfluid, hence the superfluid cannot be disordered by condensing the vortices.

B. Descendant 3d symmetry-protected topological state with $U(1)^{(1)} \times \mathbb{Z}_n$ symmetry

We can break the $U(1)$ 0-form symmetry coupled to A in Eq. (14) down to a $\mathbb{Z}_n$ symmetry—now the entire symmetry becomes $U(1)^{(1)} \times \mathbb{Z}_n$. The topological response theory (14) still applies, but now A becomes a $\mathbb{Z}_n^{(1)}$ background gauge field. The decorated defect construction in the previous case would lead to a $\mathbb{Z}_n$ classification, because the zero-dimensional SPT state with $\mathbb{Z}_n$ symmetry decorated at the Dirac monopole has a $\mathbb{Z}_n$ classification.

This classification can be understood at the boundary as well. The $(2+1)D$ boundary is a QED_3 whose flux carries k units of the $\mathbb{Z}_n$ quantum number, where k is given in Eq. (14). With $k = n$, the flux of the QED_3 basically carries a trivial quantum number, and the QED_3 can be driven into a trivial

confined phase. This boundary state is similar to the quantum dimer model on a $2d$ bipartite lattice, such as the square lattice. The quantum dimer model can be mapped to a compact QED₃ with no electric charge (the quantum dimer constraint, i.e., every site is connected to precisely one dimer, is strictly enforced), but the flux of the compact QED₃ carries nontrivial lattice quantum number. The description of the quantum dimer model in terms of QED₃ is analogous to the boundary of the $3d$ SPT state with $U(1)^{(1)} \times Z_4$ symmetry at $k = 1$. It is well known that the confined phase of the quantum dimer model on the square lattice cannot be a trivial gapped phase, instead it must have ground degeneracy due to spontaneous breaking of lattice symmetry. But in the quantum dimer model because the Z_4 symmetry is a non-onsite lattice symmetry, the quantum dimer model exists as a well-defined system in $2d$.

This effect is inherited from the LSM theorem for spin-1/2 systems on the square lattice. There is no LSM theorem for a spin-2 system on the square lattice, and a spin-2 system can be viewed as four copies of spin-1/2 systems glued together, or a system with four spin-1/2s in each unit cell. All these observations are consistent with the Z_4 classification of the $3d$ SPT state with $U(1)^{(1)} \times Z_4$ symmetry discussed in this section.

C. Descendant $3d$ symmetry-protected topological state with $Z_q^{(1)} \times U(1)$ symmetry

Next we consider the $3d$ SPT states as descendant states of Eq. (14) with $Z_q^{(1)} \times U(1)$ symmetry. Again we first consider the cases where all the point particles in the bulk are bosons. When we break the $U(1)^{(1)}$ symmetry down to $Z_q^{(1)}$, the $2d$ boundary is a QED₃ whose flux carries $U(1)$ quantum number, and there are dynamical q -fold electric charges. The boundary can only be driven to a z_q topological order by condensing the q -fold electric charge. One of the point-like anyons of this topological order is the remnant of the $2\pi/q$ flux of the QED₃, which carries k/q charges of the $U(1)$ symmetry quantum number. When $k = q$ this anyon carries unfractioalized quantum number, and hence can be neutralized by binding with gauge-invariant integer charge of the $U(1)$ symmetry. This neutralized anyon is a self-boson, and after condensation it drives the boundary into a trivial gapped state. Hence this $3d$ SPT state should have a Z_q classification.

To facilitate further discussions let us also consider a different $3d$ bulk state with $U(1)$ global symmetry only. This is a QED₄ whose electric charge is fermion, and Dirac monopole is a boson (using the notations introduced before, this bulk state is QED₄ $\{e_f, m_b\}$). Again one can bind the Dirac monopole with another boson that carries $U(1)$ quantum number, and condense the bound state in the $3d$ bulk. Then the bulk is gapped and nondegenerate, while the $2d$ boundary is a QED₃ whose electric charge is a fermion, while the gauge flux carries $U(1)$ quantum number. However, this $3d$ bulk is not a SPT state, since one can put the electric charge at the boundary in a $2d$ Chern insulator with Hall conductivity 1, then the $2d$ boundary is gapped without breaking any symmetry. This is consistent with the classification of ordinary SPT states without higher form symmetries. With only $U(1)$ symmetry, there is no nontrivial SPT state in $3d$. One needs another time-reversal symmetry to construct a $3d$ bosonic SPT state, since the boundary Chern insulator of the fermionic gauge

charge as we constructed above necessarily breaks the time reversal.

One can again glue the $2d$ boundary states in the previous two paragraphs together. Let us recall that the boundary of a nontrivial $3d$ SPT state with $Z_q^{(1)} \times U(1)$ symmetry is a QED₃ whose flux carries a $U(1)$ quantum number, and its bosonic electric charges are infinitely heavy; the boundary of the trivial state discussed in the last paragraph is a QED₃ whose flux also carries $U(1)$ quantum number, and its electric charge is a fermion with nonzero dynamics. Once we couple the two $2d$ systems together, the tunneling between the gauge fluxes between the two QED₃ will be turned on, which identifies the two gauge fields. Now the $2d$ boundary state is a QED₃ whose gauge flux still carries $U(1)$ quantum number, but its static electric charge is a fermion. This state is not a new SPT state since it can be constructed by gluing the $2d$ boundaries of the two systems discussed above.

D. Descendant $3d$ symmetry-protected topological state with $Z_q^{(1)} \times Z_n$ symmetry

Finally, we can break the $U(1)^{(1)}$ -form symmetry in Eq. (14) to $Z_q^{(1)}$. Again we can start with the QED₃ state at the $(2+1)D$ boundary. In this case there are dynamical q -fold electric charge of the $u(1)$ gauge field, and the magnetic flux of the $u(1)$ gauge field still carries Z_n quantum number. One can condense the charge- q bound state, and drive the $2d$ boundary into a $2dz_q$ topological order. In an ordinary $2dz_q$ topological order, there are two sets of anyons. The e anyon is a remnant of the unit charge excitation of the QED₃ before the condensation of the q -fold electric charge, and the m anyon is a $2\pi/q$ flux quantum of the $u(1)$ gauge flux. Both e and m anyons are self-bosons but have a mutual $2\pi/q$ statistical angle. In our current case, due to the $Z_q^{(1)}$ -form symmetry, the e anyons are not dynamical, and a m anyon carries a fractional quantum number $1/q$ of the Z_n symmetry [assuming $k = 1$ in Eq. (14)]. Both e and m anyons are coupled to z_q gauge fields. Following the arguments in Sec. III, we can demonstrate that when q and n are coprime, the fractional quantum number of the m anyon can always be “neutralized” by binding with integer charges of the Z_n symmetry, in the sense that the Z_n transformation on the decorated m anyon can always be canceled by a z_q gauge transformation. When q and n are not coprime, the quantum number of the m anyon can be neutralized when $k = \gcd(q, n)$. The neutralized m anyon can condense and drive the $2d$ boundary to a trivial gapped state without degeneracy. Hence as a descendant state of Eq. (14), the classification of the $3d$ SPT state with $Z_q^{(1)} \times Z_n$ symmetry is $Z_{\gcd(q,n)}$.

Summary of $3d$ SPT states with 1-form symmetries

Here we summarize the classification of $3d$ SPT states that are descendants of Eq. (14). If there are special SPT states that cannot be described by Eq. (14), such as some of the states discussed in Refs. [29,30], they are not included in this list:

$$\begin{aligned} U(1)^{(1)} \times U(1) &: Z, \\ Z_q^{(1)} \times U(1) &: Z_q, \\ U(1)^{(1)} \times Z_n &: Z_n, \\ Z_q^{(1)} \times Z_n &: Z_{\gcd(q,n)}. \end{aligned} \quad (15)$$

Using the SPT states discussed in this section, one can also construct 1-form symmetry-enriched topological states. For example, for the SPT state with $U(1)^{(1)} \times Z_n$ symmetry, one can “gauge” the Z_n symmetry by coupling the system to a $Z_n^{(1)}$ dynamical 1-form gauge field. This will generally drive the system into a $3dZ_n$ topological order. There are line excitation and point excitation in the Z_n topological order. The line excitation will carry the $U(1)^{(1)}$ 1-form symmetry charges. Hence this topological order is a 1-form symmetry enriched topological order. The same construction applies to the $Z_q^{(1)} \times Z_n$ SPT. By coupling the Z_n symmetry to a dynamical $Z_n^{(1)}$ 1-form gauge field, the system is driven into a $3dZ_n$ topological order enriched by $Z_q^{(1)}$ 1-form symmetry, in the sense that the line excitation of the Z_n topological order carries $Z_q^{(1)}$ 1-form charge. But when q and n are coprime, the fusion of Z_n line excitation is incompatible with the $Z_q^{(1)}$ 1-form symmetry charges, hence in this case the 1-form symmetry-enriched Z_n topological order is the same as Z_n topological order without symmetries, which is consistent with the classification in Eq. (15).

VII. 2d SYMMETRY-PROTECTED TOPOLOGICAL STATE WITH $G_1^{(1)} \times Z_2^T$ SYMMETRY

Several different $(2+1)D$ SPT states that involve 1-form symmetries can be described by the following topological response term:

$$S_{2d\text{-topo}} = \int_{(2+1)D} \frac{i\Theta}{2\pi} dB. \quad (16)$$

In principle Θ can take arbitrary value, because dB is gauge invariant. But some extra symmetry can pin Θ to a specific value, like the Θ term of the ordinary topological insulator [43] and the bosonic SPT state [3].

As an example of such states, we assume that the 2-form background gauge field B is unchanged under time-reversal transformation, this means that the 1-form symmetry charge will change sign under time reversal. This implies that the total symmetry of the system is a direct product between the 1-form symmetry and time reversal. Θ is clearly defined periodically, namely $\Theta + 2\pi = \Theta$, hence the time-reversal-invariant states correspond to $\Theta = \pi k$ with arbitrary integer k .

For even integer k , the $(2+1)D$ topological response theory (16) reduces to a boundary topological term that is identical to the topological response theory with 1d SPT state with a 1-form symmetry (Sec. II). This means that, for even integer k , the boundary corresponds to a well-defined 1d state, hence an even integer k would correspond to a trivial state in $(2+1)D$. On the other hand, for odd integer k , the boundary is a “half” 1d SPT state with 1-form symmetry $G^{(1)}$. Then the $(2+1)D$ bulk could be a SPT state.

As we mentioned before, due to the strict constraint $\nabla_x \hat{e}(x) = 0$ for 1-form charge in one dimension, a 1d system with 1-form symmetry is analogous to a 0d system with ordinary 0-form symmetry. Then whether there is a $(2+1)D$ SPT state with $G^{(1)} \times Z_2^T$ symmetry can also be determined by the existence of a projective representation of $G \times Z_2^T$. And there is a two-dimensional projective representation of $U(1) \times Z_2^T$, but not for $U(1) \rtimes Z_2^T$. Indeed, if the symmetry of the system is $G^{(1)} \rtimes Z_2^T$, namely, B is odd under time reversal,

the Θ coefficient is unchanged under time reversal, hence time reversal will not pin Θ to any specific value.

To summarize our result in two spatial dimensions, there is a nontrivial 2d SPT state with $U(1)^{(1)} \times Z_2^T$ symmetry, and this state remains nontrivial when $U(1)^{(1)}$ is broken down to $Z_q^{(1)}$ with even integer q .

The decorated defect construction also applies in this scenario, which is analogous to what was discussed in Ref. [44] for ordinary SPT states. We can construct the SPT state with $k=1$ in Eq. (16) by first creating a domain wall of time-reversal symmetry, then embedding each domain wall with a 1d SPT state described by Eq. (3), and finally proliferating the domain walls. Besides construction from 1d SPT state, we can also obtain this 2d SPT state by reduction from higher dimensions. For example, starting with the 3d SPT state with $U(1)^{(1)} \times U(1)$ symmetry described by the response theory (14), one can compactify one of the three spatial dimensions (the 3d space R^3 becomes $R^2 \otimes S^1$), and insert a π flux of the 1-form gauge field A through S^1 . Then the response theory Eq. (14) reduces to Eq. (16) with $k=1$. This is the same procedure of dimensional reduction introduced in Ref. [43].

VIII. DISCUSSION

In this work we discussed the classification, construction, and boundary properties of SPT states involving higher symmetries, from one to four spatial dimensions. Our discussion is mostly based on physical arguments. As an application of our discussion, we make connection between the SPT states with 1-form symmetry to a quantum dimer model at one lower dimension. The quantum dimer model with spatial symmetries can be mapped to the boundary of a bulk state with onsite symmetries. Some of the universal features of the quantum dimer model is dictated by the nature of the corresponding bulk state.

In this work we only discussed quantum dimer models on bipartite lattices, which can be mapped to a QED with $U(1)^{(1)}$ 1-form symmetry. It is well known that some other dimer models can be naturally mapped to a z_2 gauge field, such as quantum dimer model on the triangular lattice [45]. Then these models would be examples of systems with $Z_2^{(1)}$ 1-form symmetry, and they can also be potentially mapped to the boundary of one higher dimensions. Insights for these systems gained from higher dimensions will be studied in later works.

In this work we have also briefly discussed the construction of 1-form symmetry-enriched topological states. These 1-form SET states can appear at the boundary of a higher dimensional SPT states (these would be anomalous 1-form SET states), or can be constructed by gauging the symmetries of the SPT states in the bulk. The 0-form SET states have been studied very actively in the past. We believe a full systematic study of 1-form SET states, either anomalous or not, is worth exploration in the future.

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