



Divisibility results for zero-cycles

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Abstract

Let X be a product of smooth projective curves over a finite unramified extension k of \mathbb{Q}_p . Suppose that the Albanese variety of X has good reduction and that X has a k -rational point. We propose the following conjecture. The kernel of the Albanese map $CH_0(X)^0 \rightarrow \text{Alb}_X(k)$ is p -divisible. When p is an odd prime, we prove this conjecture for a large family of products of elliptic curves and certain principal homogeneous spaces of abelian varieties. Using this, we provide some evidence for a local-to-global conjecture for zero-cycles of Colliot-Thélène and Sansuc (Duke Math J 48(2):421–447, 1981), and Kato and Saito (Contemporary Mathematics, vol. 55:255–331, 1986).

Keywords Elliptic curves · Chow groups · Local fields

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1 Introduction

Let X be a smooth, projective, and geometrically connected variety over a field k . We consider the group $CH_0(X)$ of zero cycles on X modulo rational equivalence. This group is a direct generalization of the Picard group $\text{Pic}(C)$ of a curve C , and as such it inherits many of its properties. Namely, there is a degree map, $CH_0(X) \xrightarrow{\deg} \mathbb{Z}$, whose kernel will be denoted by $F^1(X)$. Moreover, there is a generalization of the

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Abel–Jacobi map,

$$F^1(X) \xrightarrow{\text{alb}_X} \text{Alb}_X(k),$$

called the *Albanese map* of X , where Alb_X is the dual abelian variety to the Picard variety of X . When X has a k -rational point, the degree map is surjective. When k is algebraically closed it follows by Roitman’s theorem that the Albanese map is also surjective [37], but this is not always true over arbitrary fields, except in some special cases. Some examples when surjectivity holds include $K3$ surfaces and geometrically rationally connected varieties (in these cases surjectivity holds trivially since $\text{Alb}_X = 0$), and products of curves all having a k -rational point [20, 36]. Coming to the question of injectivity, unlike the case of curves when the map alb_X is always injective, in higher dimensions the situation is rather chaotic and the map alb_X has often a very significant kernel, which we will denote by $F^2(X)$. Mumford [33] was the first to find examples of surfaces over \mathbb{C} with enormous F^2 , in particular not finitely generated. The key feature of these examples was the positive geometric genus, $p_g(X) > 0$.

When k is a finite extension over its prime field, the expectations for the structure of $F^2(X)$ are on the other extreme, predicting that $F^2(X)$ is rather small. When k is a finite field, $F^2(X)$ is indeed finite and its structure can be understood by geometric class field theory [22, 26]. When k is a number field, that is, a finite extension of \mathbb{Q} , we have fascinating conjectures due to Beilinson and Bloch. Namely, Beilinson predicts [1] that $F^2(X \otimes_k \overline{\mathbb{Q}}) = 0$, which would imply that $F^2(X)$ is a torsion group, while Bloch [2] expects that the group $CH_0(X)$ is a finitely generated abelian group. The two conjectures combined suggest that $F^2(X)$ should be finite. Apart from curves for which the above conjectures follow by the Mordell–Weil theorem, there is some evidence for surfaces with $p_g(X) = 0$ by the work of Colliot-Thélène and Raskind [6] and Salberger [39], and for the self-product $E \times E$ of an elliptic curve E over \mathbb{Q} by the work of Langer and Saito [29], and Langer and Raskind [28]. Namely, for all these classes of surfaces it has been shown that the torsion subgroup of $F^2(X)$ is finite.

The intermediate case of a p -adic field k is rather interesting, as it features similarities with both \mathbb{C} and \mathbb{Q} . In this case the group $F^2(X)$ is conjectured by Raskind and Spiess to have the following structure.

Conjecture 1.1 (Raskind, Spiess [36, Conjecture 3.5.4]) *Let X be a smooth projective variety over a finite extension of the p -adic field \mathbb{Q}_p . The Albanese kernel $F^2(X)$ is the direct sum of a divisible group and a finite group.*

This conjecture was inspired by earlier considerations of Colliot-Thélène [4, Conjecture 1.4(d, e, f)]. A celebrated result in this direction is due to Saito and Sato [38], who proved a weaker form of Conjecture 1.1, namely that the group $F^1(X)$ is the direct sum of a finite group and a group divisible by any integer m coprime to p . The full conjecture has been verified in very limited cases including rationally connected varieties with good reduction [27, Theorem 5] (those in fact satisfy $F^2(X) = 0$) and certain products of curves, [36, Theorem 1.1], [14, Theorem 1.2]. Moreover, for rationally connected varieties with semistable reduction and abelian varieties with good ordinary reduction it has been established that the group $F^2(X)$ is the direct sum of a divisible group and a torsion group (cf. [27, Corollary 9] and [13, Theorem 1.1])

respectively). We also refer to [34] for a list of results on CH_0 for quadric fibrations over various types of fields, including number fields.

When X has good reduction, it follows by [38, Theorem 0.3, Corollary 0.10] and [22, Theorem 1] that the group $F^1(X)$ is l -divisible for every prime $l \neq p$. For the classes of varieties for which Conjecture 1.1 has been verified, this is also true for the group $F^2(X)$. It is natural therefore to ask if the same holds for the “ p ”-part, a question which only makes sense for the group $F^2(X)$.

Question 1.2 Suppose that the variety X has good reduction and that Conjecture 1.1 holds for X . Is the group $F^2(X)$ p -divisible?

The answer is no in general. In fact all the known results [14, 16, 36, 52] indicate that the group $F^2(X)/p$ is nontrivial when k is *ramified enough*. The purpose of this article is to investigate what happens when k is unramified over \mathbb{Q}_p . We expect that in this case Question 1.2 should have an affirmative answer, at least for certain classes of varieties, including some cases of bad reduction. This expectation is strongly motivated by certain local-to-global expectations for zero-cycles, which will be discussed in more detail in Sect. 1.1. We suggest the following conjecture.

Conjecture 1.3 Suppose that k is a finite unramified extension of \mathbb{Q}_p . Let $X = C_1 \times \cdots \times C_r$ be a product of smooth projective curves over k such that for $i = 1, \dots, r$, $C_i(k) \neq \emptyset$. Suppose we are in one of the following two situations:

- (a) The Jacobian variety J_i of C_i has good reduction, for $i = 1, \dots, r$.
- (b) The Jacobian variety J_i of C_i has split multiplicative reduction, for $i = 1, \dots, r$, that is, C_i is a Mumford curve over k .

Then, the kernel of the Albanese map $F^2(X)$ is p -divisible.

Our first significant evidence for the situation (a) of the above conjecture is the following theorem, which constitutes the main result of this article.

Theorem 1.4 (cf. Theorem 4.5) Let $X = E_1 \times E_2$ be a product of two elliptic curves over a finite unramified extension k of \mathbb{Q}_p with good reduction, where p is an odd prime. Suppose that one of the curves has good ordinary reduction. Then, the Albanese kernel $F^2(X)$ is p -divisible.

We note that Conjecture 1.1 has already been established for such products by joint work of the first author with Isabel Leal [14, Theorem 1.2]. Moreover, using easy descent arguments, and related work of Takao Yamazaki [52], we verify Conjecture 1.3 in the following additional cases, including the situation (b) of Conjecture 1.3.

Corollary 1.5 (cf. Corollaries 4.10, 4.11, and 4.13) Let k be a finite unramified extension of \mathbb{Q}_p , with p is odd. Then, Conjecture 1.3 is true for each of the following classes of varieties:

- (a) An abelian variety A such that there is an isogeny $A \xrightarrow{\phi} E_1 \times \cdots \times E_r$ of degree coprime to p , where E_i are elliptic curves over k with good reduction with at most one having good supersingular reduction.
- (b) A principal homogeneous space X of an abelian variety A , such that $X \otimes_k L \simeq A \otimes_k L$ for some finite extension L/k of degree coprime to p and with A as in (a).

(c) A product $X = C_1 \times \cdots \times C_r$ of Mumford curves over k .

Our current techniques work well only for odd primes p . However, we have no significant reason to exclude $p = 2$ from Conjecture 1.3. When $X = E_1 \times E_2$ with both curves having good supersingular reduction we only obtain a partial result, which we will state in Sect. 1.2. Before doing that, we would like to discuss some connections of Conjecture 1.3 with certain local-to-global expectations.

1.1 Local-to-global approximations for zero-cycles

For a smooth projective geometrically connected variety X over a number field F , it is customary to consider the diagonal embedding $X(F) \hookrightarrow X(\mathbf{A}_F)$ to the set of adelic points, $X(\mathbf{A}_F) := \prod_{v \in \Omega} X(F_v)$, where Ω is the set of places in F . When $X(F) \neq \emptyset$, the question that arises is whether X satisfies weak approximation, that is, whether $X(F)$ is dense in $X(\mathbf{A}_F)$. The Brauer group $\text{Br}(X)$ of X is known to often obstruct Weak Approximation (cf. [35, 44]). Namely, it gives rise to an intermediate closed subset, $X(F) \subset X(\mathbf{A}_F)^{\text{Br}(X)} \subset X(\mathbf{A}_F)$, which is often properly contained in $X(\mathbf{A}_F)$. This obstruction is called *Brauer–Manin obstruction to Weak Approximation*.

Although this obstruction cannot always explain the failure of Weak Approximation for points, its zero-cycles counterparts are conjectured to explain all phenomena. We are particularly interested in the following conjecture.

Conjecture 1.6 ([7, Section 4], [23, Section 7], see also [4, Conjecture 1.5(c)] and [50, Conjecture (E_0)]) *Let X be a smooth projective geometrically connected variety over a number field F . The following complex is exact:*

$$\varprojlim_n F^1(X)/n \xrightarrow{\Delta} \varprojlim_n F_{\mathbf{A}}^1(X)/n \rightarrow \text{Hom}(\text{Br}(X)/\text{Br}(F), \mathbb{Q}/\mathbb{Z}).$$

Here, the adelic Chow group $F_{\mathbf{A}}^1(X)$ is essentially $\prod_{v \in \Omega_f} F^1(X \otimes_F F_v)$ with a small contribution from the infinite real places, where Ω_f is the set of all finite places in F . For a precise definition see Sect. 5.

Conjecture 1.6 was originally suggested by Colliot-Thélène and Sansuc [7] for geometrically rational varieties built upon some evidence. It was later extended to general varieties by Kato and Saito [22]. However, to this day the only evidence we have for this conjecture is for several classes of rationally connected varieties, starting with the work of Colliot-Thélène, Sansuc and Swinnerton-Dyer on Châtelet surfaces [8, 9], and continued by multiple authors (cf. [51] for a great survey article). There is some recent partial evidence for $K3$ surfaces by work of Ieronymou [18], which is the only known result for varieties with positive geometric genus.

We are interested to see whether the above conjecture has any chance of being true for a product $X = C_1 \times C_2$ of two curves over F having an F -rational point. In what follows we do a quantitative analysis of this problem. In this case the Albanese variety Alb_X is just the product $J_1 \times J_2$ of the Jacobian varieties of C_1, C_2 , and it follows by a result of Raskind and Spiess [36, Corollary 2.4.1] that we have a decomposition

$$CH_0(X) \simeq \mathbb{Z} \oplus \text{Alb}_X(F) \oplus F^2(X) \simeq \mathbb{Z} \oplus J_1(F) \oplus J_2(F) \oplus F^2(X).$$

If we assume the finiteness of the Tate–Shafarevich group of $J_1 \times J_2$, then verifying Conjecture 1.6 is reduced to proving that the following complex is exact (cf. Proposition 5.4):

$$\varprojlim_n F^2(X)/n \xrightarrow{\Delta} \varprojlim_n F_{\mathbf{A}}^2(X)/n \rightarrow \mathrm{Hom}(\mathrm{Br}(X)/\mathrm{Br}_1(X), \mathbb{Q}/\mathbb{Z}), \quad (1.1)$$

where $\mathrm{Br}_1(X) := \ker(\mathrm{Br}(X) \rightarrow \mathrm{Br}(X \otimes_F \overline{F}))$ is the *algebraic Brauer group*, and the quotient $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is the *transcendental Brauer group* of X . By a result of Skorobogatov and Zarhin [46] the quotient $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is finite for such a product of curves X defined over a number field. At the same time, the Beilinson–Bloch conjectures predict that the group $F^2(X)$ is finite, and should therefore coincide with $\varprojlim_n F^2(X)/n$. As a conclusion, for Conjecture 1.6 to be compatible with the global expectations, the adelic Albanese kernel $\varprojlim_n F_{\mathbf{A}}^2(X)/n$ must also be finite. Conjecture 1.3 when combined with Conjecture 1.1 and [36, Theorem 3.5] imply this finiteness, suggesting that the group $\varprojlim_n F_{\mathbf{A}}^2(X)/l^n$ vanishes, for all rational primes l lying below unramified places of F of good reduction. Theorem 1.4 yields the following corollary, which is precisely of that flavor.

Corollary 1.7 (cf. Corollary 5.7) *Let $X = E_1 \times E_2$ be the product of two elliptic curves over a number field F . Assume that for $i = 1, 2$ the elliptic curve E_i has potentially good reduction at all finite places of F . There is an infinite set T of rational primes such that $\prod_{v \in \Omega} \prod_{l \in T} \varprojlim_n F^2(X \otimes_F F_v)/l^n = 0$. In particular, the result holds when for $i = 1, 2$ the elliptic curve $E_i \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ has complex multiplication by the ring of integers of a quadratic imaginary field K_i .*

Unfortunately, the complement of T may be infinite, because in order to use Theorem 1.4, we need to exclude the rational primes below all places of bad reduction, all ramified places, and all places where both curves have good supersingular reduction, and the latter subset is infinite. In the present article we can only treat the case of potentially good reduction. We hope that in a future paper we will explore higher ramification cases and primes of bad multiplicative reduction, where it is very likely that global zero-cycles need to be constructed.

Remark 1.8 One could suggest extending Conjecture 1.3 to $K3$ surfaces, as most of the above analysis carries over to that case. Namely, in this case the groups $F^1(X)$ and $F^2(X)$ coincide, which allows once again to reduce Conjecture 1.6 to proving exactness of (1.1). Additionally, the quotient $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is finite [45, Theorem 1.2]. Our methods do not provide any information for those at the moment. However, if one could verify Conjecture 1.3 for $K3$ surfaces, this would strengthen very significantly the recent result of Ieronymou [18, Theorem 1.2], which could potentially lead to a full proof of Conjecture 1.6 for $K3$ surfaces. Another interesting case to consider is surfaces with $p_g(X) = 0$, which are not rationally connected. These are known to have finite Albanese kernel ([6], see also [4, Théorème 2.2]).

1.2 Outline of the method and additional results

The key tool to prove Theorem 1.4 is the use of the Somekawa K -group $K(k; E_1, E_2)$ attached to E_1, E_2 . This group is a quotient of the group $\bigoplus_{L/k \text{ finite}} E_1(L) \otimes E_2(L)$, and it is a generalization of the Milnor K -group $K_2^M(k)$ of k . For a finite extension L/k and points $a_i \in E_i(L)$, the image of a tensor $a_1 \otimes a_2$ inside $K(k; E_1, E_2)$ is denoted as a symbol $\{a_1, a_2\}_{L/k}$. Raskind and Spiess [36] proved an isomorphism, $\rho: F^2(X) \xrightarrow{\sim} K(k; E_1, E_2)$. As an example, if $(x, y) \in X(k)$ is a k -rational point, then ρ sends the zero-cycle $[x, y] - [x, 0] - [0, y] + [0, 0] \in F^2(X)$ to the symbol $\{x, y\}_{k/k}$.

Some limited cases of Theorem 1.4 were obtained in [14] for good ordinary reduction, but the arguments were very ad hoc. In the current article, we develop a uniform method to prove p -divisibility of $K(k; E_1, E_2)$. Our method roughly involves the following main steps:

Step 1: (cf. Theorem 4.4) We show that the K -group $K(k; E_1, E_2)/p$ is generated by symbols of the form $\{x, y\}_{k/k}$ for $(x, y) \in X(k)$.

Step 2: (cf. Theorem 4.5) We prove that all symbols of the form $\{x, y\}_{k/k}$ are p -divisible.

The key to prove both steps is to consider the extension $L = k(\widehat{E}_1[p], \widehat{E}_2[p])$, where \widehat{E}_i is the formal group of E_i , and study the restriction map

$$K(k; E_1, E_2)/p \xrightarrow{\text{res}_{L/k}} K(L; E_1, E_2)/p.$$

We may reduce to the case when the extension L/k is of degree coprime to p , in which case $\text{res}_{L/k}$ is injective. The advantage of looking at the restriction is that under the reduction assumptions of Theorem 1.4, we have a complete understanding of the group $K(L; E_1, E_2)/p$; namely it is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^r$ for some $0 \leq r \leq 2$.

When both curves have good supersingular reduction, Step 2 still holds. It is much harder to establish Step 1 however. In this case we obtain the following partial result.

Theorem 1.9 (cf. Theorems 4.6, 4.9) *Let k be a finite unramified extension of \mathbb{Q}_p . Let $X = E \times E$ be the self product of an elliptic curve over k with good supersingular reduction. Let $L = k(E[p])$. Then all symbols of the form $\{a, b\}_{L/k}$ and $\{a, b\}_{k/k}$ vanish in $K(k; E, E)/p$.*

In order to give some content to Theorem 1.9, we note that in all other cases it has been established by previous work of Isabel Leal and the authors [14, 16] that the group $K(L; E_1, E_2)/p$ is generated by symbols $\{x, y\}_{L/L}$ defined over L . We expect the same to be true in the case of two elliptic curves with good supersingular reduction, and then Theorem 1.9 would imply p -divisibility. However, this case appears to be much harder, and at the moment we do not have a good way to control all finite extensions.

1.3 Notation

For a variety X over a field k and an extension L/k , we will denote by $X_L := X \otimes_k L$ the base change to L . For an abelian group A and an integer $n \geq 1$, we will denote by $A[n]$ and A/n the n -torsion and n -cotorsion respectively. For a field k we will denote by $G_k := \text{Gal}(\bar{k}/k)$ the absolute Galois group of k . Moreover, for a G_k -module M we will denote by $H^i(k, M)$ the Galois cohomology groups of M , for $i \geq 0$.

2 Background

In this section we review some necessary background. We start with the definitions of Mackey functors and Somekawa K -groups. Throughout this section, k will be a field with characteristic 0.

2.1 Mackey functors

Following [36, (3.2)], we introduce Mackey functors and their product. For properties of Mackey functors, see also [19, 20].

Definition 2.1 A Mackey functor \mathcal{M} (over k) is a presheaf of abelian groups in the category of étale k -schemes equipped with push-forward maps $f_*: \mathcal{M}(X) \rightarrow \mathcal{M}(Y)$ for finite morphisms $X \xrightarrow{f} Y$, satisfying the following properties:

- (i) $\mathcal{M}(X_1 \sqcup X_2) = \mathcal{M}(X_1) \oplus \mathcal{M}(X_2)$, for étale k -schemes X_1, X_2 .
- (ii) If $Y' \xrightarrow{g} Y$ is a finite morphism and

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a Cartesian diagram, then the induced diagram

$$\begin{array}{ccc} \mathcal{M}(X') & \xrightarrow{g'_*} & \mathcal{M}(X) \\ f'^* \uparrow & & \uparrow f_* \\ \mathcal{M}(Y') & \xrightarrow{g_*} & \mathcal{M}(Y) \end{array}$$

commutes.

Property (i) implies that a Mackey functor \mathcal{M} is fully determined by its value on $\text{Spec}(K)$ where K is a finite field extension of k . We will denote by $\mathcal{M}(K) := \mathcal{M}(\text{Spec } K)$.

Notation 2.2 For finite field extensions $k \subset K \subset L$, the map $j_\star: \mathcal{M}(L) \rightarrow \mathcal{M}(K)$ induced by the projection $\text{Spec}(L) \xrightarrow{j} \text{Spec}(K)$ will be denoted by $N_{L/K}: \mathcal{M}(L) \rightarrow \mathcal{M}(K)$ and will be referred to as the *norm*. Similarly, the induced pull-back map $j^\star: \mathcal{M}(K) \rightarrow \mathcal{M}(L)$ will be denoted by $\text{res}_{L/K}: \mathcal{M}(K) \rightarrow \mathcal{M}(L)$ and will be referred to as the *restriction*.

The category of Mackey functors over k is abelian ([21, p. 5], [36, p. 14]) with a tensor product defined by Kahn in [19], whose definition we review below.

Definition 2.3 For Mackey functors \mathcal{M}, \mathcal{N} , their *Mackey product* $\mathcal{M} \otimes \mathcal{N}$ is defined as follows. For a finite field extension k'/k ,

$$(\mathcal{M} \otimes \mathcal{N})(k') = \left(\bigoplus_{K/k': \text{finite}} \mathcal{M}(K) \otimes_{\mathbb{Z}} \mathcal{N}(K) \right) / (\mathbf{PF}),$$

where (\mathbf{PF}) is the subgroup generated by elements of the following form. For a tower of finite extensions $k' \subset K \subset L$,

(PF1) $N_{L/K}(x) \otimes y - x \otimes \text{res}_{L/K}(y) \in (\mathbf{PF})$, for elements $x \in \mathcal{M}(L)$, $y \in \mathcal{N}(K)$.

(PF2) $x \otimes N_{L/K}(y) - \text{res}_{L/K}(x) \otimes y \in (\mathbf{PF})$, for elements $x \in \mathcal{M}(K)$, $y \in \mathcal{N}(L)$.

These relations are referred in the literature as *projection formula*.

For $x \in \mathcal{M}(K)$, $y \in \mathcal{N}(K)$ the image of $x \otimes y$ in $(\mathcal{M} \otimes \mathcal{N})(k')$ is traditionally denoted as a symbol $\{x, y\}_{K/k'}$. Moreover, for a finite extension F/k' the norm map $N_{F/k'}$ is given by

$$\begin{aligned} N_{F/k'}: (\mathcal{M} \otimes \mathcal{N})(F) &\rightarrow (\mathcal{M} \otimes \mathcal{N})(k') \\ \{x, y\}_{K/F} &\mapsto \{x, y\}_{K/k'}. \end{aligned}$$

In other words, $N_{F/k'}(\{x, y\}_{K/F}) = \{x, y\}_{K/k'}$, for $x \in \mathcal{M}(K)$, $y \in \mathcal{N}(K)$.

Remark 2.4 Using the symbolic notation, the projection formula (\mathbf{PF}) , can be rewritten as

$$\begin{aligned} \{N_{L/K}(x), y\}_{K/k'} &= \{x, \text{res}_{L/K}(y)\}_{L/k'}, \\ \{x, N_{L/K}(y)\}_{K/k'} &= \{\text{res}_{L/K}(x), y\}_{L/k'}. \end{aligned} \quad (2.1)$$

Example 2.5 (1) Let G be a commutative algebraic group over k . Then G induces a Mackey functor by defining $G(K) := G(\text{Spec } K)$ for K/k finite.

(2) Let \mathcal{M} be a Mackey functor and $n \in \mathbb{N}$ be a positive integer. We define a Mackey functor \mathcal{M}/n as follows: $(\mathcal{M}/n)(K) := \mathcal{M}(K)/n$.

(3) For every integer $n \geq 1$ and any finite extension K/k , in an unpublished work due to Kahn, we have an isomorphism

$$\left(\frac{\mathbb{G}_m \otimes \mathbb{G}_m}{n} \right)(K) \simeq \frac{K_2^{\mathcal{M}}(K)}{n},$$

where $K_2^M(K)$ is the Milnor K -group of K (cf. [36, Remark 4.2.5 (b)]). When k is a finite extension of the p -adic field \mathbb{Q}_p , this follows also from [36, Lemma 4.2.1].

The restriction map

Suppose $k \subset K \subset L$ is a tower of finite extensions of k . In the following sections we are going to use extensively the restriction map

$$\text{res}_{L/K} : (\mathcal{M} \otimes \mathcal{N})(K) \rightarrow (\mathcal{M} \otimes \mathcal{N})(L).$$

We review its definition here. Let F/K be a finite extension. There is an isomorphism of L -algebras, $F \otimes_K L \simeq \prod_{i=1}^n A_i$, where for each $i \in \{1, \dots, n\}$, A_i is an Artin local ring of length e_i over L with residue field L_i . Let $x \in \mathcal{M}(F)$, $y \in \mathcal{N}(F)$. Then, we define

$$\text{res}_{L/K}(\{x, y\}_{F/K}) = \sum_{i=1}^n e_i \{ \text{res}_{L_i/F}(x), \text{res}_{L_i/F}(y) \}_{L_i/L}.$$

The following lemma gives a more concrete description of $\text{res}_{L/K}$ in some special cases.

- Lemma 2.6** (1) Suppose L/k is a finite extension and $x \in \mathcal{M}(k)$, $y \in \mathcal{N}(k)$. Then $\text{res}_{L/k}(\{x, y\}_{k/k}) = \{x, y\}_{L/L}$.
 (2) Suppose L/k is a finite Galois extension and $x \in \mathcal{M}(L)$, $y \in \mathcal{N}(L)$. Let $G = \text{Gal}(L/k)$. Then,

$$\text{res}_{L/k}(\{x, y\}_{L/k}) = \text{res}_{L/k}(N_{L/k}(\{x, y\}_{L/L})) = \sum_{g \in G} g\{x, y\}_{L/L}.$$

Proof The assertion (1) follows immediately, since we have an isomorphism $k \otimes_k L \simeq L$. Hence, in this case we have $n = 1$, $L_i = L$ and $e_i = 1$.

To prove (2), note that since L/k is Galois, it is the splitting field of some polynomial $f(x) \in k[x]$ so that $L \simeq k[x]/\langle f(x) \rangle$. By taking a root $\alpha \in L$ of $f(x)$, we have

$$L \otimes_k L = \frac{k[x]}{\langle f(x) \rangle} \otimes_k L = \frac{L[x]}{\langle f(x) \rangle} \simeq \prod_{g \in G} \frac{L[x]}{\langle x - g(\alpha) \rangle} \simeq \prod_{i=1}^n L_i,$$

where for each $i \in \{1, \dots, n\}$ there is an isomorphism $g : L_i \xrightarrow{\sim} L$ given by some $g \in G$. \square

Remark 2.7 Let \mathcal{M} be a Mackey functor over k . Suppose that we have an equality $N_{L/K} \circ \text{res}_{L/K} = [L : K]$, for every finite extension L/K , e.g. \mathcal{M} is given by a commutative algebraic group (cf. Example 2.5 (1)). The equality implies that for every

prime number p coprime to $[L : K]$, the restriction map $(\mathcal{M}/p)(K) \xrightarrow{\text{res}_{L/K}} (\mathcal{M}/p)(L)$ is injective, while the norm $(\mathcal{M}/p)(L) \xrightarrow{N_{L/K}} (\mathcal{M}/p)(K)$ is surjective.

Remark 2.8 (*The vanishing trick*) In the following sections we are going to prove the vanishing of certain Mackey products of the form $(\mathcal{M}/p \otimes \mathcal{N}/p)(k)$, where p is a prime number. A usual strategy to show that a symbol $\{a, b\}_{k/k} \in (\mathcal{M}/p \otimes \mathcal{N}/p)(k)$ vanishes is to consider a finite extension k'/k over which there exists an element $a' \in \mathcal{M}(k')$ with $pa' = \text{res}_{k'/k}(a)$. If $b = N_{k'/k}(b')$, for some $b' \in \mathcal{N}(k')$, then the projection formula (2.1) yields an equality

$$\{a, b\}_{k/k} = \{a, N_{k'/k}(b')\}_{k/k} = \{\text{res}_{k'/k}(a), b'\}_{k'/k} = N_{k'/k}(\{pa', b'\}_{k'/k}) = 0.$$

This trick was used in [49, Proposition 4.3] and refined in the context of Mackey functors by Kahn in [20].

Notation 2.9 Let G be a commutative algebraic group over k and $a \in G(k)$. For every $n \geq 1$ the multiplication-by- n map $G(\bar{k}) \xrightarrow{n} G(\bar{k})$ is surjective. Suppose that $G[n] := G(\bar{k})[n] \subset G(k)$. We will denote by $k(\frac{1}{n}a)$ the smallest finite extension k' of k over which there exists an element $a' \in G(k')$ such that $na = a'$. The assumption $G[n] \subset G(k)$ implies that this is a Galois extension.

2.2 Somekawa K -group

For semi-abelian varieties G_1, \dots, G_r over k the *Somekawa K -group* $K(k; G_1, \dots, G_r)$ attached to G_1, \dots, G_r is a quotient of the Mackey product $(G_1 \otimes \cdots \otimes G_r)(k)$ (see [47] for the precise definition). Although our statements will often concern the K -group $K(k; G_1, \dots, G_r)$, all our computations will be at the level of Mackey products, hence we omit the definition of $K(k; G_1, \dots, G_r)$. We only highlight the following facts:

- For every K/k finite there is a surjection, $(G_1 \otimes \cdots \otimes G_r)(K) \twoheadrightarrow K(K; G_1, \dots, G_r)$.
- When $G_1 = \cdots = G_r = \mathbb{G}_m$, there is an isomorphism $K(k; G_1, \dots, G_r) \simeq K_r^M(k)$ with the Milnor K -group of k (cf. [47]).
- The elements of $K(k; G_1, \dots, G_r)$ will also be denoted as linear combinations of symbols of the form $\{x_1, \dots, x_r\}_{K/k}$, where K/k is some finite extension and $x_i \in G_i(K)$ for $i = 1, \dots, r$.
- The Somekawa K -group $K(k; G_1, \dots, G_r)$ inherits all the properties of the Mackey product $(G_1 \otimes \cdots \otimes G_r)(k)$ discussed in the previous subsection.

2.3 Galois symbol map

Let G be a semi-abelian variety over k and p be a prime number. Since we assumed that the field k has characteristic zero, for any finite extension K/k the Kummer sequence

$$0 \rightarrow G[p] \rightarrow G(\bar{K}) \xrightarrow{p} G(\bar{K}) \rightarrow 0$$

is a short exact sequence of G_K -modules and hence it induces a connecting homomorphism

$$\delta_G: G(K)/p \hookrightarrow H^1(K, G[p]), \quad (2.2)$$

which is often called the *Kummer map*.

Definition 2.10 (cf. [47, Proposition 1.5]) Let G_1, G_2 be semi-abelian varieties over k and p be a prime. By considering G_1, G_2 as Mackey functors (as in Example 2.5 (1)), the *Galois symbol map*

$$s_p: (G_1 \otimes G_2)(k)/p \rightarrow H^2(k, G_1[p] \otimes G_2[p])$$

is defined by the cup product and the corestriction as follows:

$$s_p(\{x, y\}_{K/k}) = \text{Cor}_{K/k}(\delta_{G_1}(x) \cup \delta_{G_2}(y)).$$

For semi-abelian varieties G_1, G_2 over k , it is known that the Galois symbol s_p defined above factors through the Somekawa K -group $K(k; G_1, G_2)/p$ [47, Proposition 1.5]. The induced homomorphism $K(k; G_1, G_2)/p \rightarrow H^2(k, G_1[p] \otimes G_2[p])$ will also be denoted by s_p . In particular, when $G_1 = G_2 = \mathbb{G}_m$, we have the following commutative diagram:

$$\begin{array}{ccc} (\mathbb{G}_m \otimes \mathbb{G}_m)(k)/p & \xrightarrow{\simeq} & K_2^M(k)/p \\ s_p \downarrow & \swarrow g_p & \\ H^2(k, \mu_p^{\otimes 2}), & & \end{array} \quad (2.3)$$

where the map $g_p: K_2^M(k)/p \rightarrow H^2(k, \mu_p^{\otimes 2})$ is the classical Galois symbol, which is an isomorphism by the Merkurjev/Suslin theorem [31] (for the isomorphism $(\mathbb{G}_m \otimes \mathbb{G}_m)(k)/p \simeq K_2^M(k)/p$ see Example 2.5 (3)). Now, we suppose $\mu_p \subset k^\times$. In this case the map g_p sends the symbol $\{x, y\}_{k/k}$ to the central simple algebra $(x, y)_p$. From [49, Proposition 4.3], we have the following equivalences:

$$\{x, y\}_{k/k} = 0 \text{ in } (\mathbb{G}_m \otimes \mathbb{G}_m)(k)/p \iff (x, y)_p = 0 \iff x \in N_{k_1/k}(k_1^\times), \quad (2.4)$$

where $k_1 = k(\sqrt[p]{y})$. Note that the last implication \Leftarrow follows from the same argument as in Remark 2.8.

2.4 Relation to zero-cycles

Let X be a smooth projective variety over k . We consider the Chow group of zero-cycles, $CH_0(X)$. Recall that this group has a filtration

$$CH_0(X) \supset F^1(X) \supset F^2(X) \supset 0,$$

where $F^1(X) := \ker(\deg: CH_0(X) \rightarrow \mathbb{Z})$ is the kernel of the degree map, and $F^2(X) := \ker(\text{alb}_X: F^1(X) \rightarrow \text{Alb}_X(k))$ is the kernel of the Albanese map. When $X = C_1 \times \cdots \times C_r$ is a product of smooth projective, geometrically connected curves over k such that $C_i(k) \neq \emptyset$ for $i = 1, \dots, r$, the Albanese kernel $F^2(X)$ has been related to the Somekawa K -group attached to the Jacobian varieties J_1, \dots, J_r of C_1, \dots, C_r by Raskind and Spiess. Namely, we have the following theorem.

Theorem 2.11 ([36, Theorem 2.2, Corollary 2.4.1]) *For $X = C_1 \times \cdots \times C_r$ as above there is a canonical isomorphism,*

$$CH_0(X) \xrightarrow{\cong} \mathbb{Z} \oplus \bigoplus_{1 \leq v \leq r} \bigoplus_{1 \leq i_1 < i_2 < \cdots < i_v \leq r} K(k; J_{i_1}, \dots, J_{i_v}).$$

Since each curve C_i has a k -rational point, we have $CH_0(X) \simeq \mathbb{Z} \oplus F^1(X)$. Hence, we have

$$F^1(X) \xrightarrow{\cong} \bigoplus_{1 \leq v \leq r} \bigoplus_{1 \leq i_1 < i_2 < \cdots < i_v \leq r} K(k; J_{i_1}, \dots, J_{i_v}).$$

Since we have $K(k; J_i) \simeq J_i(k)$ and the Albanese map $\text{alb}_X: F^1(X) \rightarrow \text{Alb}_X(k) = J_1(k) \oplus \cdots \oplus J_r(k)$ is surjective, we obtain

$$F^2(X) \xrightarrow{\cong} \bigoplus_{2 \leq v \leq d} \bigoplus_{1 \leq i_1 < i_2 < \cdots < i_v \leq d} K(k; J_{i_1}, \dots, J_{i_v}) \quad (2.5)$$

when $r \geq 2$. In particular, when $X = C_1 \times C_2$, we have an isomorphism $F^2(X) \xrightarrow{\cong} K(k; J_1, J_2)$, which has the following explicit description. Fix base points $x_0 \in C_1(k)$, $y_0 \in C_2(k)$. Let $x \in C_1(L)$, $y \in C_2(L)$ where L/k is some finite extension. Let $\pi_{L/k}: X_L \rightarrow X$ be the projection. Then

$$\pi_{L/k*}([x, y] - [x, y_0] - [x_0, y] + [x_0, y_0]) \mapsto \{x - x_0, y - y_0\}_{L/k}.$$

3 Preliminary computations

Convention 3.1 From now on we assume that k is a finite extension of the p -adic field \mathbb{Q}_p with absolute ramification index e_k , where p is an odd prime. We will denote by \mathcal{O}_k the ring of integers of k , \mathfrak{m}_k the maximal ideal of \mathcal{O}_k and \mathbb{F} its residue field. For a finite extension k'/k , we will denote by $v_{k'}$ the discrete valuation of k' that extends the one of k , and by $\mathbb{F}_{k'}$ the residue field of k' . We also denote by $\mathcal{O}_{k'}^\times = U_{k'}^0$ the group of units in $\mathcal{O}_{k'}$ and $U_{k'}^i := 1 + \mathfrak{m}_{k'}^i$ for $i \geq 1$ the higher unit groups.

In this section we will obtain some essential information about the Mackey functor E/p (cf. Example 2.5(2)) for an elliptic curve E over k , and we will discuss some necessary ramification theory.

3.1 Formal groups

Let F be a formal group law (which is commutative of dimension one) over \mathcal{O}_k . The group $F(\mathfrak{m}_k)$ associated to F is denoted by $F(k)$. The group $F(k)$ has a natural filtration given by $F^i(k) := F(\mathfrak{m}_k^i)$, for $i \geq 1$. Let $\phi: F \rightarrow F'$ be an isogeny of formal groups over \mathcal{O}_k which is given by a formal power series $\phi(T) = a_1T + a_2T^2 + \dots$ in $\mathcal{O}_k[[T]]$. We define a filtration on $F'(k)/\phi(F(k))$ by

$$\mathcal{J}_k^i := \text{Im}(F'^i(k) \rightarrow F'(k)/\phi(F(k))) = \frac{F'^i(k)}{\phi(F(k)) \cap F'^i(k)}, \quad (3.1)$$

for $i \geq 1$. Recall that the *height* of ϕ is defined to be the positive integer n such that $\phi(T) \equiv \psi(T^{p^n}) \pmod{\mathfrak{m}_k}$ for some $\psi \in \mathcal{O}_k[[T]]$ whose leading coefficient is a unit in \mathcal{O}_k . The structure of these graded pieces are known as follows.

Proposition 3.2 ([25, Lemma 2.1.4]) *Assume that the isogeny $\phi: F \rightarrow F'$ which is given by the power series $\phi(T) = a_1T + a_2T^2 + \dots$ has height 1 and $F[\phi] := \ker(\phi) \subset F(k)$. Putting $t(\phi) = v_k(a_1)$ and $t_0(\phi) = t(\phi)/(p-1)$, we have*

- (a) *If $1 \leq i < pt_0(\phi)$ and i is coprime to p , then $\mathcal{J}_k^i/\mathcal{J}_k^{i+1} \simeq F'^i(k)/F'^{i+1}(k) \simeq \mathbb{F}$.*
- (b) *If $1 \leq i < pt_0(\phi)$ and i is divisible by p , then $\mathcal{J}_k^i/\mathcal{J}_k^{i+1} = 0$.*
- (c) *If $i = pt_0(\phi)$, then $\mathcal{J}_k^i/\mathcal{J}_k^{i+1} \simeq \mathbb{Z}/p$.*
- (d) *If $i > pt_0(\phi)$, then $\mathcal{J}_k^i = 0$.*

The above isomorphisms are induced by the standard isomorphism $F'^i(k)/F'^{i+1}(k) \xrightarrow{\simeq} \mathbb{F}$ as in [41, IV.2, Proposition 6] and they depend on the choice of a uniformizer π_k of k .

A typical example of a height 1 isogeny is the multiplication $[p]: \widehat{\mathbb{G}}_m \rightarrow \widehat{\mathbb{G}}_m$ by p on the multiplicative group $\widehat{\mathbb{G}}_m$. This isogeny is given by the power series $[p](T) = pT + \dots$ and $\widehat{\mathbb{G}}_m[p] = \mu_p$, the group of p -th roots of unity. In particular, $t([p]) = e_k$ is the absolute ramification index of k . The filtration $\widehat{\mathbb{G}}_m^i(k) = U_k^i$ defines a filtration on $k^\times/p = k^\times/(k^\times)^p$ by

$$\overline{U}_k^i := \text{Im}(U_k^i \rightarrow k^\times/p) \quad \text{for } i \geq 0.$$

Applying Proposition 3.2 to $[p]: \widehat{\mathbb{G}}_m \rightarrow \widehat{\mathbb{G}}_m$, the structure of the graded quotients $\overline{U}_k^i/\overline{U}_k^{i+1}$ is summarized as follows.

Lemma 3.3 *Assume $\mu_p \subset k$. Set $e_0(k) = e_k/(p-1)$ (which is an integer).*

- (a) *If $0 \leq i < pe_0(k)$ and i is coprime to p , then $\overline{U}_k^i/\overline{U}_k^{i+1} \simeq \mathbb{F}_k$.*
- (b) *If $0 \leq i < pe_0(k)$ and i is divisible by p , then $\overline{U}_k^i/\overline{U}_k^{i+1} = 1$.*
- (c) *If $i = pe_0(k)$, then $\overline{U}_k^i/\overline{U}_k^{i+1} \simeq \mathbb{Z}/p$.*
- (d) *If $i > pe_0(k)$, then $\overline{U}_k^i = 1$.*

For the remaining of this subsection we assume that $\phi: F \rightarrow F'$ is a height 1 isogeny with $F[\phi] \subset F(k)$. For a point $x \in F'(k)$, let $k' := k(\phi^{-1}(x))$ be the smallest extension of k over which there exists a point $y \in F(k')$ such that $\phi(y) = x$. Under

our assumptions this is a finite Galois extension of k of degree p (cf. Lemma 3.4 below). Let $G = \text{Gal}(k'/k)$ be the Galois group of k'/k . Consider the ramification filtration $(G^\lambda)_{\lambda \geq -1}$ in the upper numbering of G [41, IV.3]. Then there is an integer s , called the *jump* of G , such that $G^\lambda = G$, for every $\lambda \leq s$ and $G^\lambda = 1$, for every $\lambda > s$ [41, IV.3, Theorem]. The ramification of the extension k'/k is described in the following lemma due to Kawachi.

Lemma 3.4 ([25, Lemma 2.1.5]) *For a height 1 isogeny $\phi: F \rightarrow F'$ with $F[\phi] \subset F(k)$, and $x \in \mathcal{J}_k^i \setminus \mathcal{J}_k^{i+1}$, the ramification of the extension $k' = k(\phi^{-1}(x))/k$ is known as follows:*

- (a) *If $1 \leq i < pt_0(\phi)$, then k'/k is a totally ramified extension of degree p with the jump in the ramification filtration of $G := \text{Gal}(k'/k)$ occurring at $s = pt_0(\phi) - i$.*
- (b) *If $i = pt_0(\phi)$, then k'/k is an unramified extension of degree p .*

Note that Proposition 3.2 implies that Lemma 3.4 covers all possible cases. Next we further assume that $\mu_p \subset k^\times$. The short exact sequences of G_k -modules $1 \rightarrow \mu_p \rightarrow k^\times \xrightarrow{p} k^\times \rightarrow 1$ and $0 \rightarrow F[\phi] \rightarrow F \xrightarrow{\phi} F' \rightarrow 0$ give rise to connecting homomorphisms (Kummer maps, cf. (2.2))

$$k^\times/p \xrightarrow{\simeq} H^1(k, \mu_p), \quad \text{and} \quad F'(k)/\phi(F(k)) \hookrightarrow H^1(k, F[\phi]).$$

After fixing an isomorphism $F[\phi] \simeq \mu_p$ of (trivial) G_k -modules, we get noncanonical homomorphisms

$$f: F'(k)/\phi(F(k)) \hookrightarrow H^1(k, F[\phi]) \simeq H^1(k, \mu_p) \xleftarrow{\simeq} k^\times/p.$$

In fact, the induced map is compatible with filtrations and the following theorem holds.

Theorem 3.5 ([25, Theorem 2.1.6]) *The map $f: F'(k)/\phi(F(k)) \rightarrow k^\times/p$ satisfies*

$$f(\mathcal{J}_k^i) = \overline{U}_k^{p(e_0(k) - t_0(\phi)) + i}$$

for any $i \geq 1$, where $e_0(k) = e_k/(p-1)$ and $t_0(\phi)$ is defined in Proposition 3.2.

We close this subsection by defining a Mackey functor by the higher unit groups \overline{U}_k^i .

Definition 3.6 Let $i \geq 0$. We define the sub-Mackey functor \overline{U}^i , of \mathbb{G}_m/p as follows. If L is a finite extension of k , then $\overline{U}^i(L) := \overline{U}_L^{ie(L/k)}$. For a finite extension F/L , the norm $N_{F/L}$ and restriction maps $\text{res}_{F/L}$ are induced by the ones on \mathbb{G}_m .

3.2 Elliptic curves

Let E be an elliptic curve over k with good reduction. Let \mathcal{E} be the Néron model of E , which is an abelian scheme over $\text{Spec}(\mathcal{O}_k)$. Let $\overline{E} := \mathcal{E} \otimes_{\mathcal{O}_k} \mathbb{F}$ be the reduction of E , and \widehat{E} be the formal group of E . We have a short exact sequence of abelian groups,

$$0 \rightarrow \widehat{E}(\mathfrak{m}_k) \xrightarrow{j} E(k) \xrightarrow{r} \overline{E}(\mathbb{F}) \rightarrow 0, \quad (3.2)$$

where $r: E(k) \rightarrow \overline{E}(\mathbb{F})$ is the reduction map. This induces a *short exact sequence of Mackey functors*,

$$0 \rightarrow \widehat{E} \rightarrow E \rightarrow [E/\widehat{E}] \rightarrow 0, \quad (3.3)$$

where \widehat{E} is the Mackey functor given by $K \mapsto \widehat{E}(K) := \widehat{E}(\mathfrak{m}_K)$, and $[E/\widehat{E}]$ is a Mackey functor defined by $L \mapsto \overline{E}(\mathbb{F}_L)$. For a finite extension L/K , the restriction $\text{res}_{L/K}: [E/\widehat{E}](K) \rightarrow [E/\widehat{E}](L)$ is the usual restriction, $\overline{E}(\mathbb{F}_K) \xrightarrow{\text{res}_{L/K}} \overline{E}(\mathbb{F}_L)$, while the norm $N_{L/K}: [E/\widehat{E}](L) \rightarrow [E/\widehat{E}](K)$ is the map $\overline{E}(\mathbb{F}_L) \xrightarrow{e(L/K) \cdot N_{L/K}} \overline{E}(\mathbb{F}_K)$. The fact that $[E/\widehat{E}]$ is a Mackey functor has been shown by Raskind and Spiess [36, p. 15].

From now on we will drop the notation $[p]: \widehat{E}(k) \rightarrow \widehat{E}(k)$ and will denote it simply by $p: \widehat{E}(k) \rightarrow \widehat{E}(k)$. Recalling from Sect. 3.1, the group $\widehat{E}(k) = \widehat{E}(\mathfrak{m}_k)$ has a natural filtration given by $\widehat{E}^i(k) := \widehat{E}(\mathfrak{m}_k^i)$, for $i \geq 1$.

Definition 3.7 Following (3.1), we define a filtration on $\widehat{E}(k)/p = \widehat{E}(k)/p\widehat{E}(k)$ by

$$\mathcal{D}_k^i := \frac{\widehat{E}^i(k)}{p\widehat{E}(k) \cap \widehat{E}^i(k)}, \quad i \geq 1.$$

In the case when $\widehat{E}[p] \subset E(k)$, we can decompose E/p using the Mackey functors \mathbb{G}_m/p and \overline{U}^i (cf. Definition 3.6). This decomposition depends on the reduction type of E , which we review in the following two subsections.

3.3 Elliptic curves with good ordinary reduction

We first consider the case when E has *good ordinary reduction*; that is, \overline{E} is an ordinary elliptic curve over \mathbb{F} . In this case the G_k -module $E[p]$ has a one-dimensional G_k -invariant submodule. Namely, we have the connected-étale short exact sequence of G_k -modules

$$0 \rightarrow E[p]^\circ \rightarrow E[p] \rightarrow E[p]^{\text{ét}} \rightarrow 0, \quad (3.4)$$

where $E[p]^\circ := \widehat{E}[p]$ are the $[p]$ -torsion points of the formal group \widehat{E} of E . For more details on the connected-étale exact sequence we refer to [48, Section 8]. After a finite unramified extension k'/k the short exact sequence (3.4) becomes

$$0 \rightarrow \mu_p \rightarrow E[p] \rightarrow \mathbb{Z}/p \rightarrow 0. \quad (3.5)$$

This follows because after base change to the maximal unramified extension k^{nr} of k the formal group \widehat{E}^{nr} becomes isomorphic to the multiplicative group \mathbb{G}_m (cf. [30, Lemma 4.27]). As the multiplication $p: \widehat{E} \rightarrow \widehat{E}$ has height 1, Theorem 3.5 now leads to the following proposition.

Proposition 3.8 ([25, Theorem 2.1.6], see also [14, Proposition 3.12]) *Let E be an elliptic curve over k with good ordinary reduction, and \widehat{E} be its formal group. Assume that $\widehat{E}[p] \subset \widehat{E}(k)$ and $\mu_p \subset k^\times$. Then, we have an isomorphism $f: \widehat{E}(k)/p \xrightarrow{\sim} \overline{U}_k^1 \simeq \overline{U}_k^0$, which satisfies $f(\mathcal{D}_k^i) = \overline{U}_k^i$ for $i \geq 1$.*

Assume we are in the situation of Proposition 3.8. In particular, $\mu_p \subset k^\times$ and after a finite unramified extension (3.5) holds. Next we consider the short exact sequence (3.2) and apply the right exact functor $\otimes_{\mathbb{Z}} \mathbb{Z}/p$. Using Proposition 3.8 we get an exact sequence of \mathbb{F}_p -vector spaces,

$$\overline{U}_k^0 \xrightarrow{j} E(k)/p \xrightarrow{r} \overline{E}(\mathbb{F})/p \rightarrow 0.$$

The map j is not always injective. Namely, there is a unit $u \in \mathcal{O}_k^\times / \mathcal{O}_k^{\times p}$, known as the *Serre–Tate* parameter of E , that generates the kernel. For more details see [14, Section 3.1]. All we need in this article is that there is an exact sequence of Mackey functors,

$$\overline{U}^0 \rightarrow E/p \rightarrow [E/\widehat{E}]/p \rightarrow 0,$$

where $\overline{U}^0, [E/\widehat{E}]$ are the Mackey functors defined previously.

3.4 Elliptic curves with good supersingular reduction

Lastly, we consider the case when E has *good supersingular reduction*. This means that the elliptic curve \overline{E} has no p -torsion, and hence we get a surjection of Mackey functors,

$$\widehat{E}/p \xrightarrow{j} E/p \rightarrow 0. \quad (3.6)$$

For the rest of this subsection we assume that $E[p] \subset E(k)$. Let $e_0(k) = \frac{e_k}{p-1}$. Suppose y_0, y_1 are two generators of $\widehat{E}[p] \simeq \mathbb{Z}/p \oplus \mathbb{Z}/p$ such that

$$v_k(y_0) = \max \{v_k(y) : y \in \widehat{E}[p], y \neq 0\}.$$

From now on we will denote $t_0(k) := v_k(y_0)$. We have a decomposition of Mackey functors (see [16, Proof of Theorem 4.1]),

$$E/p \simeq \overline{U}^{p(e_0(k)-t_0(k))} \oplus \overline{U}^{pt_0(k)}, \quad (3.7)$$

where $\overline{U}^{p(e_0(k)-t_0(k))}, \overline{U}^{pt_0(k)}$ are Mackey functors defined as in Definition 3.6. We recall how this decomposition is obtained. The main reference for what follows is [25]. Consider the isogenous elliptic curve $E' := E/\langle y_0 \rangle$ and the isogeny $E \xrightarrow{\phi} E'$, and its dual, $E' \xrightarrow{\check{\phi}} E$, which are both defined over k . As $[p] = \check{\phi} \circ \phi : E \rightarrow E$ has height 2, these isogenies correspond to isogenies of formal groups, $\widehat{E} \xrightarrow{\phi} \widehat{E}', \widehat{E}' \xrightarrow{\check{\phi}} \widehat{E}$, which are both of height 1. We have a splitting short exact sequence of \mathbb{F}_p -vector spaces,

$$0 \rightarrow \frac{E'(k)}{\phi(E(k))} \xrightarrow{\check{\phi}} \frac{E(k)}{p} \xrightarrow{\varepsilon} \frac{E(k)}{\check{\phi}(E'(k))} \rightarrow 0, \quad (3.8)$$

where ε is just the projection. By [17, Corollary 2.3], we have $t_0(\phi) = t_0(k)$. From [25, Theorem 3.2.6(2)], we have isomorphisms

$$\frac{E'(k)}{\phi(E(k))} \simeq \frac{\widehat{E}'(k)}{\phi(\widehat{E}(k))} \xrightarrow{f} \overline{U}_k^{p(e_0(k)-t_0(k))} \simeq \overline{U}_k^{p(e_0(k)-t_0(k))+1},$$

where the middle isomorphism f follows from Theorem 3.5, and the last one follows from Lemma 3.3(b). It is clear that the dual isogeny $\check{\phi}$ identifies E with the quotient $E'/\langle\phi(y_1)\rangle$. By [17, Corollary 2.3] again, we have $t_0(\check{\phi}) = v_k(\phi(y_1))$. The leading coefficient of $[p] = \check{\phi} \circ \phi$ equals to p so that $t(\phi) + t(\check{\phi}) = e_k$ and hence $t_0(k) + t_0(\check{\phi}) = e_0(k)$. Then the same theorem [25, Theorem 3.2.6(2)] gives an isomorphism,

$$\frac{E(k)}{\check{\phi}(E'(k))} \simeq \frac{\widehat{E}(k)}{\check{\phi}(\widehat{E}'(k))} \xrightarrow{f} \overline{U}_k^{p(e_0(k)-t_0(\check{\phi}))+1} = \overline{U}_k^{pt_0(k)+1}.$$

Behavior with respect to the filtration of the formal group

Notation 3.9 Consider the filtration $\mathcal{D}_k^i = \frac{\widehat{E}^i(k)}{p\widehat{E}(k) \cap \widehat{E}^i(k)}$, $i \geq 1$ (cf. Definition 3.7). We define additionally $\mathcal{F}_k^i := \frac{\widehat{E}^i(k)}{\check{\phi}(\widehat{E}'(k)) \cap \widehat{E}^i(k)}$, and $\mathcal{G}_k^i := \frac{\widehat{E}^i(k)}{\phi(\widehat{E}(k)) \cap \widehat{E}^i(k)}$, for $i \geq 1$.

Note that since the isogenies $\phi, \check{\phi}$ are of height 1, Proposition 3.2 applies for the quotients $\mathcal{F}_k^i/\mathcal{F}_k^{i+1}$, $\mathcal{G}_k^i/\mathcal{G}_k^{i+1}$.

Remark 3.10 It is clear that the map $\varepsilon: \mathcal{D}_k^1 \rightarrow \mathcal{F}_k^1$ preserves the filtration. The following lemma gives a complete description of the quotients $\mathcal{D}_k^i/\mathcal{D}_k^{i+1}$ for $i \geq 1$.

Lemma 3.11 *Let $i \geq 1$ be an integer. Suppose that we have $e_0(k) - t_0(k) = pt_0(k)$. Then the following are true for the quotient $\mathcal{D}_k^i/\mathcal{D}_k^{i+1}$:*

(a) *If $i < p(e_0(k) - t_0(k))$ and $(i, p) = 1$, then ε induces an isomorphism,*

$$\varepsilon: \mathcal{D}_k^i/\mathcal{D}_k^{i+1} \xrightarrow{\simeq} \mathcal{F}_k^i/\mathcal{F}_k^{i+1} \simeq \mathbb{F}.$$

Moreover, both quotients are isomorphic to $\widehat{E}^i(k)/\widehat{E}^{i+1}(k)$.

(b) *If $i < p(e_0(k) - t_0(k))$ and $i = pj$, for some $j \geq 1$, then $\check{\phi}$ induces an isomorphism,*

$$\check{\phi}: \mathcal{G}_k^j/\mathcal{G}_k^{j+1} \xrightarrow{\simeq} \mathcal{D}_k^i/\mathcal{D}_k^{i+1}.$$

Moreover, we have an isomorphism $\mathbb{F} \simeq \widehat{E}^j(k)/\widehat{E}^{j+1}(k) \simeq \mathcal{D}_k^i/\mathcal{D}_k^{i+1}$ if $(j, p) = 1$.

(c) *If $i = p(e_0(k) - t_0(k))$, then we have a short exact sequence of \mathbb{F}_p -vector spaces,*

$$0 \rightarrow \frac{\mathcal{G}_k^{e_0(k)-t_0(k)}}{\mathcal{G}_k^{p(e_0(k)-t_0(k))+1}} \xrightarrow{\check{\phi}} \frac{\mathcal{D}_k^{p(e_0(k)-t_0(k))}}{\mathcal{D}_k^{p(e_0(k)-t_0(k))+1}} \xrightarrow{\varepsilon} \frac{\mathcal{F}_k^{p(e_0(k)-t_0(k))}}{\mathcal{F}_k^{p(e_0(k)-t_0(k))+1}} \rightarrow 0.$$

Moreover, we have isomorphisms, $\frac{\mathcal{G}_k^{e_0(k)-t_0(k)}}{\mathcal{G}_k^{e_0(k)-t_0(k)+1}} \simeq \frac{\mathcal{F}_k^{p(e_0(k)-t_0(k))}}{\mathcal{F}_k^{p(e_0(k)-t_0(k))+1}} \simeq \mathbb{Z}/p$.

(d) If $i > p(e_0(k) - t_0(k))$, then $\mathcal{D}_k^i = 0$.

Proof All the parts will follow from Proposition 3.2 and [17, Lemma 2.6]. For the first claim Proposition 3.2(a) gives an isomorphism, $\mathcal{F}_k^i/\mathcal{F}_k^{i+1} \simeq \mathbb{F}$. At the same time, we have a surjective map of finite dimensional \mathbb{F}_p -vector spaces, $\mathcal{D}_k^i/\mathcal{D}_k^{i+1} \xrightarrow{\varepsilon} \mathcal{F}_k^i/\mathcal{F}_k^{i+1} \rightarrow 0$. Moreover, the projection $\widehat{E}^i(k) \rightarrow \mathcal{D}_k^i/\mathcal{D}_k^{i+1}$ induces a surjection $\widehat{E}^i(k)/\widehat{E}^{i+1}(k) \rightarrow \mathcal{D}_k^i/\mathcal{D}_k^{i+1} \rightarrow 0$. Since the quotient $\widehat{E}^i(k)/\widehat{E}^{i+1}(k)$ is isomorphic to \mathbb{F} [43, Chapter IV, Proposition 3.2], the claim follows.

For the second claim, the proof of [25, Lemma 2.1.4(1)] shows an isomorphism $\mathcal{G}_k^j/\mathcal{G}_k^{j+1} \xrightarrow{\simeq} \mathcal{D}_k^j/\mathcal{D}_k^{j+1}$ induced by $\check{\phi}$. If $p \nmid j$, then $\mathbb{F} \simeq \widehat{E}'^j(k)/\widehat{E}'^{j+1}(k) \simeq \mathcal{G}_k^j/\mathcal{G}_k^{j+1}$ by Proposition 3.2(a).

Next, suppose $i = p(e_0(k) - t_0(k))$. The isomorphisms

$$\frac{\mathcal{G}_k^{e_0(k)-t_0(k)}}{\mathcal{G}_k^{e_0(k)-t_0(k)+1}} \simeq \frac{\mathcal{F}_k^{p(e_0(k)-t_0(k))}}{\mathcal{F}_k^{p(e_0(k)-t_0(k))+1}} \simeq \mathbb{Z}/p$$

follow by Proposition 3.2(c). The short exact sequence follows easily by the short exact sequence $0 \rightarrow \mathcal{G}_k^1 \xrightarrow{\check{\phi}} \mathcal{D}_k^1 \xrightarrow{\varepsilon} \mathcal{F}_k^1 \rightarrow 0$, after restricting to $\mathcal{D}_k^{p(e_0(k)-t_0(k))}$.

Finally, claim (d) follows by Proposition 3.2(d). \square

3.5 Computing ramification jumps

In the remaining two subsections we focus on the following special case. We consider an elliptic curve E with *good supersingular* reduction over a finite *unramified* extension k/\mathbb{Q}_p , that is, $e_k = 1$. In this situation, for the mod p Galois representation

$$\rho: \text{Gal}(\bar{k}/k) \rightarrow \text{Aut}(E[p]) \simeq \text{GL}_2(\mathbb{F}_p),$$

the image of the inertia subgroup by ρ is known to be cyclic of order $p^2 - 1$ [40, Proposition 12]. This implies that

- the extension $L := k(E[p])/k$ corresponding to the kernel of ρ has ramification index $e_{L/k} = p^2 - 1$, and
- the inertia subgroup of $\text{Gal}(L/k)$ is cyclic.

Lemma 3.12 *For every non-zero $x \in \widehat{E}[p]$, we have $v_L(x) = 1$.*

Proof Since the valuation of the p -th coefficient, a_p , of multiplication by p in the formal group \widehat{E} satisfies $v_k(a_p) \geq pe/(p+1) = p/(p+1)$, we have $v_L(x) = e_{L/k} \frac{1}{p^2-1} = 1$ (cf. [24, Theorem 3.10.7]). \square

From (3.7), we obtain a decomposition,

$$E(L)/p \simeq \overline{U}_L^{p(e_0(L)-t_0(L))} \oplus \overline{U}_L^{pt_0(L)} = \overline{U}_L^{p^2+1} \oplus \overline{U}_L^{p+1}, \quad (3.9)$$

since $t_0(L) = \max \{v_L(x) : 0 \neq x \in \widehat{E}[p]\} = 1$ and $e_0(L) = p + 1$.

Next we make a simplifying assumption that the extension $L = k(E[p])/k$ is *totally ramified* of degree $p^2 - 1$. We consider the restriction

$$\text{res}_{L/k} : \mathcal{D}_k^1 \rightarrow \mathcal{D}_L^{p^2-1},$$

which is an injective map of \mathbb{F}_p -vector spaces, since $[L : k]$ is coprime to p (cf. Remark 2.7). Let $x \in \mathcal{D}_k^1$. We will identify x with its image, $\text{res}_{L/k}(x)$. We are interested in obtaining concrete information about the ramification of the finite Galois extension $L(\frac{1}{p}x)$. This will be achieved in Lemma 3.15, but we first need some reminders from Kummer theory.

Let F/k be an arbitrary finite extension such that $\mu_p \subset F^\times$. Let $a \in F^\times/p = F^\times/(F^\times)^p$. Consider the Kummer extension $F_1 = F(\sqrt[p]{a})$. Let $G = \text{Gal}(F_1/F)$ be the Galois group of F_1/F . In this setting, Lemma 3.4 now reads as follows:

Lemma 3.13 *Let $a \in F^\times/p$. Consider the Galois extension $F_1 = F(\sqrt[p]{a})$.*

- (a) *If $a \in \overline{U}_F^i \setminus \overline{U}_F^{i+1}$, for some $0 \leq i < pe_0(F)$, then F_1/F is a totally ramified extension of degree p with the jump at $s = pe_0(F) - i$.*
- (b) *If $a \in \overline{U}_F^{pe_0(F)} \setminus \overline{U}_F^{pe_0(F)+1}$, then the extension F_1/F is unramified extension of degree p .*

Next, consider the Herbrand $\psi_{F_1/F}$ function defined as follows:

$$\psi_{F_1/F}(x) = \begin{cases} x, & x \leq s \\ s + p(x - s), & x > s. \end{cases}$$

We will need the following facts about the norm map $N_{F_1/F}$.

Proposition 3.14 (cf. [41, V.3, Corollaries 2 & 3])

- (1) *For every integer $1 \leq i < s$, the norm map $N_{F_1/F} : F_1^\times \rightarrow F^\times$ induces a surjection*

$$\frac{U_{F_1}^i}{U_{F_1}^{i+1}} \xrightarrow{N_{F_1/F}} \frac{U_F^i}{U_F^{i+1}} \rightarrow 0.$$

- (2) *For every integer $i > s$, the norm map induces a surjection*

$$U_{F_1}^{\psi_{F_1/F}(i)} \xrightarrow{N_{F_1/F}} U_F^i \rightarrow 0.$$

We now come back to the element $x \in \mathcal{D}_k^1$ considered before the Kummer theory aside. We think of the element x as lying in $\mathcal{D}_L^{p^2-1}$ under the restriction map. The following lemma gives us how x decomposes under the decomposition (3.9). We note that this lemma is a key computation that will be used several times in the next section.

Lemma 3.15 *Let $x \in \mathcal{D}_k^1$. Consider the finite extension $L_0 = L(\frac{1}{p}x)$.*

- (1) *Suppose $x \in \mathcal{D}_k^1 \setminus \mathcal{D}_k^2$. Under the decomposition (3.9) the image of x in $E(L)/p = \mathcal{D}_L^1$ can be written as (x_0, x_1) with $x_0 \in \overline{U}_L^{p^2+p}$ and $x_1 \in \overline{U}_L^{p+p^2-1} \setminus \overline{U}_L^{p+p^2}$. In particular, we have a tower of finite extensions $L_0 \supset L_1 \supset L$, with $L_1 \supset L$ unramified and $L_0 \supset L_1$ totally ramified of degree p . The jump in the ramification filtration of $\text{Gal}(L_0/L_1)$ happens at $s = 1$.*
- (2) *Suppose $x \in \mathcal{D}_k^2$. Then, $L_0 = L$.*

Proof In what follows we will denote by \oplus the addition law given by the formal group \widehat{E}_L . Moreover, we will fix π, π_L uniformizer elements of k, L respectively.

We first prove (2). If $x \in \mathcal{D}_k^2$, then $v_L(x) \geq 2(p^2 - 1)$, and hence its image in \mathcal{D}_L^1 lies in $\mathcal{D}_L^{2(p^2-1)}$. Since $2(p^2 - 1) > p^2$, the claim (2) follows by Lemma 3.11 (d), since $\mathcal{D}_L^{2(p^2-1)} = 0$.

We next prove (1). Following the notation from (3.8), for every finite extension L'/k we have a splitting short exact sequence of \mathbb{F}_p -vector spaces,

$$0 \rightarrow \frac{\widehat{E}'^1(L')}{\phi(\widehat{E}^1(L'))} \xrightarrow{\check{\phi}} \frac{\widehat{E}^1(L')}{p} \xrightarrow{\varepsilon} \frac{\widehat{E}^1(L')}{\check{\phi}(\widehat{E}'^1(L'))} \rightarrow 0.$$

We can therefore decompose the image of x in $\widehat{E}^1(L)p$ as $x = \check{\phi}(w_0) \oplus w_1$, where w_0 is the class of some $\tilde{w}_0 \in \widehat{E}'^1(L) \bmod \phi(\widehat{E}^1(L))$ and w_1 is the class of some $\tilde{w}_1 \in \widehat{E}^1(L) \bmod \check{\phi}(\widehat{E}'^1(L))$. Note that the same decomposition holds also over any finite extension L'/L . This implies that there exists an element $y \in \widehat{E}^1(L')$ such that $x = py$ if and only if there exist elements $\tilde{z}_0 \in \widehat{E}^1(L')$ and $\tilde{z}_1 \in \widehat{E}'^1(L')$ such that $\tilde{w}_0 = \phi(\tilde{z}_0)$ and $\tilde{w}_1 = \check{\phi}(\tilde{z}_1)$. This means that the extension $L(\frac{1}{p}x)$ is precisely the compositum of the extensions $L_1 = L(\phi^{-1}(\tilde{w}_0))$ and $L_2 = L(\check{\phi}^{-1}(\tilde{w}_1))$ (for this notation see paragraph preceding Lemma 3.4).

Next we analyze the extensions L_1, L_2 . We start with $L_1 = L(\phi^{-1}(\tilde{w}_0))$ and we claim that the extension L_1/L is unramified. It is enough to show that we are in the set-up of Lemma 3.4 (b), which we apply for the height 1 isogeny $\phi: \widehat{E} \rightarrow \widehat{E}'$. Recall from the discussion in Sect. 3.4 and Lemma 3.12 that we have an equality $t_0(\phi) = t_0(L) = 1$. Thus, it suffices to show that $v_L(\tilde{w}_0) \geq p$. Since x is the restriction of an element from $\mathcal{D}_k^1 \setminus \mathcal{D}_k^2$, $v_L(x) = p^2 - 1$. This yields $v_L(\check{\phi}(\tilde{w}_0)) \geq p^2 - 1$. Let i be such that $\tilde{w}_0 \in \widehat{E}^{i^*}(L) \setminus \widehat{E}^{i+1}(L)$. Since $\check{\phi}$ is a height 1 isogeny, it follows that

$$v_L(\check{\phi}(\tilde{w}_0)) = pi$$

(cf. [17, Lemma 2.2]). In particular, it is divisible by p . Combining the two relations, we conclude that $i = v_L(\tilde{w}_0) \geq p$ as required. Next consider the isomorphism

$$\frac{\widehat{E}'^1(L)}{\phi(\widehat{E}^1(L))} \xrightarrow{\cong} \overline{U}_L^{p^2+1}$$

and let x_0 be the image of w_0 under this isomorphism. The above computation together with Theorem 3.5 give us that $x_0 \in \overline{U}_L^{p^2+p}$.

Next we consider the extension $L_2 = L(\check{\phi}^{-1}(\tilde{w}_1))$, and let $\tilde{z}_1 \in \widehat{E}'(L_2)$ be such that $\check{\phi}(\tilde{z}_1) = \tilde{w}_1$. We apply Lemma 3.4(a) for the height 1 isogeny $\check{\phi}: \widehat{E}' \rightarrow \widehat{E}$. By the discussion in Sect. 3.4 it follows that

$$t_0(\check{\phi}) = e_0(L) - t_0(L) = p + 1 - 1 = p.$$

Thus, in order to show that we are in the set-up of Lemma 3.4(a), it is enough to verify that $v_L(w_1) < p^2$, which follows directly by the previous paragraph and the equality $v_L(x) = p^2 - 1$. We conclude that the extension L_2/L is totally ramified of degree p and Lemma 3.4(a) gives that the jump of the ramification filtration of $\text{Gal}(L_2/L)$ occurs at $s = pt_0(\check{\phi}) - (p^2 - 1)$. Moreover, if x_1 is the image of w_1 under the isomorphism $\widehat{E}(L)/\check{\phi}(\widehat{E}'(L)) \simeq \overline{U}_L^{p+1}$, then Theorem 3.5 implies that $x_1 \in \overline{U}_L^{p+p^2-1} \setminus \overline{U}_L^{p^2+p}$.

We conclude the proof by noticing that the extensions L_1/L , L_2/L are totally disjoint, one being unramified and the other totally ramified of degree p . Thus, for the compositum $L(\frac{1}{p}x) = L_1 \cdot L_2$ we have an isomorphism $\text{Gal}(L_1 L_2/L_1) \simeq \text{Gal}(L_2/L)$. \square

3.6 Galois action on graded quotients

We close this section with a technical computation that will be used in the proofs of Theorems 4.4 and 4.9. We continue working in the set-up of Sect. 3.5. Namely, E is an elliptic curve over a finite unramified extension k of \mathbb{Q}_p and the extension $L = k(E[p])/k$ is totally ramified of degree $[L:k] = e(L/k) = p^2 - 1$. We want to understand how the Galois group $\text{Gal}(L/k)$ acts on the graded quotients $\mathcal{D}_L^i/\mathcal{D}_L^{i+1}$. This will be achieved in Lemma 3.17. Before that we start with a preliminary discussion about tame extensions and formal groups.

Suppose k/\mathbb{Q}_p is finite and L/k is a totally ramified extension of degree coprime to p (that is, L/k is a tame extension). Fix a uniformizer π of L . Let $G_0 = \text{Gal}(L/k)$ be the Galois group. Because L/k is tame, G_0 is cyclic [41, IV.2, Corollary 1]. Let σ be a generator of G_0 . We have an injection, $f: G_0 \hookrightarrow U_L^0/U_L^1$ given by $\sigma^m \mapsto \frac{\sigma^m(\pi)}{\pi}$. Since there is an isomorphism $U_L^0/U_L^1 \simeq \mathbb{F}_L^\times$, f identifies G_0 with a subgroup of \mathbb{F}_L^\times . Set $\bar{u} = \sigma(\pi)/\pi \in \mathbb{F}_L^\times$. The following lemma shows how σ acts on the quotient $U_L^i/U_L^{i+1} \simeq \mathbb{F}_L$, when $i \geq 1$.

Lemma 3.16 *Let $i \geq 1$. Then σ induces an automorphism, $\bar{\sigma}: U_L^i/U_L^{i+1} \rightarrow U_L^i/U_L^{i+1}$, which is given by multiplication by \bar{u}^i .*

Proof Since $U_L^i/U_L^{i+1} \simeq \mathbb{F}_L$, and L/k is totally ramified, the map $\bar{\sigma}$ is an \mathbb{F}_L -linear map, $\bar{\sigma}: \mathbb{F}_L \rightarrow \mathbb{F}_L$, and hence it is given by multiplication by a non-zero scalar $c \in \mathbb{F}_L^\times$. We will show that $c = \bar{u}^i$. For every $i \geq 1$, we have an isomorphism of

groups,

$$\frac{U_L^i}{U_L^{i+1}} = \frac{\widehat{\mathbb{G}}_m(\mathfrak{m}_L^i)}{\widehat{\mathbb{G}}_m(\mathfrak{m}_L^{i+1})} \simeq \frac{\widehat{\mathbb{G}}_a(\mathfrak{m}_L^i)}{\widehat{\mathbb{G}}_a(\mathfrak{m}_L^{i+1})} = \frac{(\pi^i)}{(\pi^{i+1})},$$

(cf. [43, Chapter IV, Proposition 3.2]). It is immediate that the isomorphism is G_0 -equivariant. Thus, it is enough to see how $\bar{\sigma}$ acts on the quotient $\frac{(\pi^i)}{(\pi^{i+1})}$. But this is immediate, since

$$\pi^i \mapsto \sigma(\pi^i) = \left(\frac{\sigma(\pi)}{\pi} \right)^i \pi^i = \bar{u}^i \pi^i,$$

and the class of π^i corresponds to 1 under the isomorphism $\frac{(\pi^i)}{(\pi^{i+1})} \simeq \mathbb{F}_L$. \square

Note that there was nothing special about the multiplicative group in Lemma 3.16. Namely, according to [43, Chapter IV, Proposition 3.2], if \mathcal{F} is any formal group over \mathcal{O}_k , then for each $i \geq 1$ the map $\mathcal{F}(\mathfrak{m}_k^i)/\mathcal{F}(\mathfrak{m}_k^{i+1}) \rightarrow \widehat{\mathbb{G}}_a(\mathfrak{m}_k^i)/\widehat{\mathbb{G}}_a(\mathfrak{m}_k^{i+1})$ induced by the identity map on sets is an isomorphism of groups. Thus, if L/k is a tame extension, then the induced map $\bar{\sigma}: \mathcal{F}(\mathfrak{m}_L^i)/\mathcal{F}(\mathfrak{m}_L^{i+1}) \rightarrow \mathcal{F}(\mathfrak{m}_L^i)/\mathcal{F}(\mathfrak{m}_L^{i+1})$, which is obviously G_0 -equivariant, is given by scalar multiplication by $\bar{u}^i = (\sigma(\pi)/\pi)^i$.

We are now going to apply this to our elliptic curve E satisfying the assumptions of the beginning of this subsection. If $G_0 = \text{Gal}(L/k) = \langle \sigma \rangle$, then σ induces a \mathbb{F}_L -linear automorphism

$$\bar{\sigma}: \widehat{E}^1(L)/\widehat{E}^2(L) \xrightarrow{\cdot \bar{u}} \widehat{E}^1(L)/\widehat{E}^2(L)$$

of exact order $p^2 - 1$. In fact, the Galois group G_0 permutes the p -torsion points. Next we see how the G_0 -action behaves under the decomposition (3.9). Following the notation of Sect. 3.4 we may take $E' = E_L/\langle y_0 \rangle$, where y_0 is some fixed torsion point. Consider the isogenous elliptic curve $E'^\sigma = E_L/\langle \sigma(y_0) \rangle$ and the isogenies $E_L \xrightarrow{\phi^\sigma} E'^\sigma$, $E'^\sigma \xrightarrow{\check{\phi}^\sigma} E_L$. It is clear that we have an equality $\phi^\sigma = \sigma \circ \phi \circ \sigma^{-1}$. We therefore get a commutative diagram,

$$\begin{array}{ccccc} \widehat{E}'(L)/\phi(\widehat{E}_L(L)) & \xrightarrow{\check{\phi}} & \widehat{E}_L(L)/p & \xrightarrow{\varepsilon} & \widehat{E}_L(L)/\check{\phi}(\widehat{E}'(L)) \\ \sigma \downarrow & & \downarrow \sigma & & \downarrow \sigma \\ \widehat{E}'^\sigma(L)/\phi^\sigma(\widehat{E}_L(L)) & \xrightarrow{\check{\phi}^\sigma} & \widehat{E}_L(L)/p & \xrightarrow{\varepsilon^\sigma} & \widehat{E}_L(L)/\check{\phi}^\sigma(\widehat{E}'^\sigma(L)). \end{array}$$

Combining this diagram with the information in Lemma 3.11 we obtain the following lemma.

Lemma 3.17 *Consider the decomposition $E_L(L)/p \simeq \overline{U}_L^{p^2+1} \oplus \overline{U}_L^{p+1}$ given in (3.9). Let σ be the generator of $G_0 = \text{Gal}(L/k)$. Then*

- (1) For each $j \in \{1, \dots, p-1\}$, σ induces an automorphism $\frac{\overline{U}_L^{p^2+j}}{\overline{U}_L^{p^2+j+1}} \xrightarrow{\bar{\sigma}} \frac{\overline{U}_L^{p^2+j}}{\overline{U}_L^{p^2+j+1}}$ given by multiplication by \bar{u}^{pj} .
- (2) For each $i \in \{1, \dots, p^2-1\}$ which is coprime to p , σ induces an automorphism $\frac{\overline{U}_L^{p+i}}{\overline{U}_L^{p+i+1}} \xrightarrow{\bar{\sigma}} \frac{\overline{U}_L^{p+i}}{\overline{U}_L^{p+i+1}}$ given by multiplication by \bar{u}^i .

Proof This follows directly by Lemma 3.11. Namely, to prove (1) note that, for each $j \in \{1, \dots, p-1\}$ which is coprime to p , we have a commutative diagram of isomorphisms,

$$\begin{array}{ccccc} \overline{U}_L^{p^2+j}/\overline{U}_L^{p^2+j+1} & \xrightarrow{\simeq} & \widehat{E}'^j(L)/\widehat{E}'^{j+1}(L) & \xrightarrow{\check{\sigma}} & \widehat{E}^{pj}(L)/\widehat{E}^{pj+1}(L) \\ \bar{\sigma} \downarrow & & \downarrow \bar{\sigma} & & \downarrow \bar{\sigma} \\ \overline{U}_L^{p^2+j}/\overline{U}_L^{p^2+j+1} & \xrightarrow{\simeq} & (\widehat{E}^\sigma)^j(L)/(\widehat{E}^{\sigma})^{j+1}(L) & \xrightarrow{\check{\sigma}^\sigma} & \widehat{E}^{pj}(L)/\widehat{E}^{pj+1}(L), \end{array}$$

where the leftmost vertical map is obtained by the diagram. Since the rightmost vertical map is given by multiplication by \bar{u}^{pj} , the claim follows.

Similarly, to prove (2) note that for each $i \in \{1, \dots, p^2-1\}$ we have a commutative diagram of isomorphisms,

$$\begin{array}{ccccc} \widehat{E}_L^i(L)/\widehat{E}_L^{i+1}(L) & \xrightarrow{\varepsilon} & \mathcal{F}_L^i/\mathcal{F}_L^{i+1} & \xrightarrow{\simeq} & \overline{U}_L^{p+i}/\overline{U}_L^{p+i+1} \\ \bar{\sigma} \downarrow & & \downarrow \bar{\sigma} & & \downarrow \bar{\sigma} \\ \widehat{E}_L^i(L)/\widehat{E}_L^{i+1}(L) & \xrightarrow{\varepsilon^\sigma} & (\mathcal{F}^\sigma)_L^i/(\mathcal{F}^\sigma)_L^{i+1} & \xrightarrow{\simeq} & \overline{U}_L^{p+i}/\overline{U}_L^{p+i+1}, \end{array}$$

where $\mathcal{F}_L^i = \widehat{E}_L^i(L)/(\widehat{E}_L^i(L) \cap \check{\sigma}(\widehat{E}'(L)))$, $(\mathcal{F}^\sigma)_L^i = \widehat{E}_L^i(L)/(\widehat{E}_L^i(L) \cap \check{\sigma}^\sigma(\widehat{E}'^\sigma(L)))$, and the rightmost vertical map is obtained by the diagram. Since the leftmost vertical map is given by multiplication by \bar{u}^i , the claim follows. \square

Remark 3.18 We note that the various isomorphisms in Lemmas 3.16 and 3.17 are not canonical; namely they depend on the choice of a uniformizer element π of L .

4 Main results

In this section we give proofs for Theorems 1.4 and 1.9. All our statements will be in terms of the Mackey product $(E_1 \otimes E_2)(k)$, and the Somekawa K -group $K(k; E_1, E_2)$. Using the identification of the latter with the Albanese kernel $F^2(E_1 \times E_2)$ (cf. (2.11)), the corresponding statements for zero-cycles will follow. We start with some preliminary lemmas.

Lemma 4.1 *Let E_1, E_2 be elliptic curves with good reduction over a finite extension k of \mathbb{Q}_p . Then, we have a surjection of Mackey functors,*

$$\frac{\widehat{E}_1 \otimes \widehat{E}_2}{p} \rightarrow \frac{E_1 \otimes E_2}{p} \rightarrow 0.$$

Proof We have an exact sequence of Mackey functors,

$$\widehat{E}_1/p \rightarrow E_1/p \rightarrow [E_1/\widehat{E}_1]/p \rightarrow 0.$$

Applying the right exact functor $\otimes E_2/p$, we obtain an exact sequence of Mackey functors,

$$\widehat{E}_1/p \otimes E_2/p \rightarrow E_1/p \otimes E_2/p \rightarrow [E_1/\widehat{E}_1]/p \otimes E_2/p \rightarrow 0.$$

We claim that $[E_1/\widehat{E}_1]/p \otimes E_2/p = 0$, that is $([E_1/\widehat{E}_1]/p \otimes E_2/p)(K) = 0$, for any finite extension K/k . To see this, let $\{x, y\}_{F/K} \in ([E_1/\widehat{E}_1]/p \otimes E_2/p)(K)$, where F/K is some finite extension, $x \in [E_1/\widehat{E}_1](F)/p$ and $y \in E_2(F)/p$. Recall that $[E_1/\widehat{E}_1](F) = \overline{E}_1(\mathbb{F}_F)$. Let $x' \in \overline{E}_1(\mathbb{F}_F)$ be a point such that $x = px'$. Then x' is defined over some finite extension \mathbb{F}'/\mathbb{F}_F . Let F'/F be a finite unramified extension with $\mathbb{F}_{F'} = \mathbb{F}'$ (cf. [41, I.6]). Since the elliptic curve E_2 has good reduction, [30, Corollary 4.4] gives that the norm map $E_2(F') \xrightarrow{N_{F'/F}} E_2(F)$ is surjective. Thus, we can find $y' \in E_2(F')$ such that $y = N_{F'/F}(y')$. The projection formula (2.1) yields (cf. Remark 2.8),

$$\begin{aligned} \{x, y\}_{F/K} &= \{x, N_{F'/F}(y')\}_{F/K} \\ &= N_{F'/F}(\{x, y'\}_{F'/F'}) = N_{F'/F}(\{px', y'\}_{F'/F'}) = 0. \end{aligned}$$

We conclude that there is a surjection of Mackey functors,

$$\widehat{E}_1/p \otimes E_2/p \rightarrow E_1/p \otimes E_2/p \rightarrow 0.$$

With a similar argument, by applying the right exact functor $\widehat{E}_1/p \otimes$ to the exact sequence (3.3) for E_2 we obtain an exact sequence of Mackey functors,

$$\widehat{E}_1/p \otimes \widehat{E}_2/p \rightarrow \widehat{E}_1/p \otimes E_2/p \rightarrow \widehat{E}_1/p \otimes [E_2/\widehat{E}_2]/p \rightarrow 0.$$

We claim that $\widehat{E}_1/p \otimes [E_2/\widehat{E}_2]/p = 0$. The argument is exactly the same as before, noting that if F'/F is a finite unramified extension, then the norm map on formal groups, $\widehat{E}_1(\mathfrak{m}_{F'}) \xrightarrow{N_{F'/F}} \widehat{E}_1(\mathfrak{m}_F)$ is surjective (cf. [15, Proposition 3.1]). \square

Notation 4.2 For $X = E_1 \times E_2$ we will denote by $\text{Symb}_X(k)$ the subgroup of $(E_1 \otimes E_2)(k)$ generated by symbols of the form $\{x, y\}_{k/k}$. Moreover, let $\widetilde{\text{Symb}}_X(k)$ be the image of $\text{Symb}_X(k)$ in $(E_1 \otimes E_2)(k)/p$.

Lemma 4.3 *Let E_1, E_2 be elliptic curves over a finite extension k of \mathbb{Q}_p . Assume that at least one of the curves has good reduction. Let k'/k be a finite unramified extension. Then,*

- (1) *We have an inclusion $\overline{\text{Sym}}_X(k) \subseteq N_{k'/k}(\overline{\text{Sym}}_X(k'))$ in $(E_1 \otimes E_2)(k)/p$.*
- (2) *If the group $(E_1 \otimes E_2)(k')$ is p -divisible, then so is $(E_1 \otimes E_2)(k)$.*

Proof Without loss of generality, assume that E_1 has good reduction. To prove (1), let $\{a, b\}_{k/k} \in \overline{\text{Sym}}_X(k)$. Because k'/k is unramified, and E_1 has good reduction, the norm map $E_1(k') \xrightarrow{N_{k'/k}} E_1(k)$ is surjective [30, Corollary 4.4]. Thus, we may find $a' \in E_1(k')$ such that $a = N_{k'/k}(a')$. This yields equalities,

$$\{a, b\}_{k/k} = \{N_{k'/k}(a'), b\}_{k/k} = \{a', b\}_{k'/k} = N_{k'/k}(\{a', b\}_{k'/k}).$$

To prove (2), assume that $(E_1 \otimes E_2)(k')$ is p -divisible. Let $\{a, b\}_{F/k} \in (E_1 \otimes E_2)(k)/p$, where F/k is some finite extension. Consider the extension $F' = F \cdot k'$. Then F'/F is unramified, and hence the norm $E_1(F') \xrightarrow{N_{F'/F}} E_1(F)$ is surjective. Thus there exists some $a' \in E_1(F')$ such that $a = N_{F'/F}(a')$. The projection formula yields (cf. Remark 2.8),

$$\begin{aligned} \{a, b\}_{F/k} &= \{N_{F'/F}(a'), b\}_{F/k} \\ &= \{a', b\}_{F'/k} = N_{k'/k}(\{a', b\}_{F'/k'}) = 0 \in (E_1 \otimes E_2)(k)/p, \end{aligned}$$

and hence $(E_1 \otimes E_2)(k)/p = 0$. □

4.1 Proof of Theorem 1.4

As already mentioned in the introduction, the proof of this theorem will be split up into two steps. Let k be a finite unramified extension of \mathbb{Q}_p and let E_1, E_2 be elliptic curves over k with good reduction. Put $X = E_1 \times E_2$. Suppose that E_1 has good ordinary reduction. In the following, we consider the following two cases:

(ord) E_2 has good ordinary reduction,

(ssing) E_2 has good supersingular reduction.

We can consider the smallest extension L/k such that the following are true:

- $L \supset k(\widehat{E}_1[p], \widehat{E}_2[p])$.
- If the curve E_i has good ordinary reduction, then the G_L -module $E_{iL}[p]$ fits into a short exact sequence of the form (3.5), and hence $\overline{E}_i[p] \subset \overline{E}_i(\mathbb{F}_L)$.

The above assumptions imply that $L \supset k(\mu_p)$ and that L fits into a tower of finite extensions

$$k \subset k' \subset L \tag{4.1}$$

with k'/k unramified and L/k' totally ramified of degree coprime to p . More precisely, Sects. 3.3–3.5 imply the following:

- In case (ord), then L/k has ramification index $p - 1$, and hence $e_0(L) = 1$.

- In case **(ssing)**, then L/k has ramification index $p^2 - 1$, and hence $e_0(L) = p + 1$.

Theorem 4.4 *Let $L/k'/k$ be the tower of extensions as in (4.1). Then, we have an equality $\overline{\text{Symb}}_{X_{k'}}(k') = (E_1 \otimes E_2)(k')/p$.*

Proof For simplicity we assume that the extension L/k is totally ramified, and hence $k = k'$. We need to show that the group $(E_1 \otimes E_2)(k)/p$ is generated by symbols of the form $\{a, b\}_{k/k}$. Since $[L : k]$ is coprime to p , the norm map $(E_1 \otimes E_2)(L)/p \xrightarrow{N_{L/k}} (E_1 \otimes E_2)(k)/p$ is surjective. It is enough therefore to show that the image of the norm lies in $\text{Symb}_X(k)$.

Claim A $\overline{\text{Symb}}_{X_L}(L) = (E_1 \otimes E_2)(L)/p$. That is, the group $(E_1 \otimes E_2)(L)/p$ is generated by symbols of the form $\{x, y\}_{L/L}$, with $x \in E_1(L)$, $y \in E_2(L)$.

This claim follows easily by the computations in [16] as follows: Using Lemma 4.1, the problem is reduced to showing that the Mackey product $(\widehat{E}_1/p \otimes \widehat{E}_2/p)(L)$ is generated by symbols of the form $\{x, y\}_{L/L}$, with $x \in \widehat{E}_1(L)$, $y \in \widehat{E}_2(L)$. We have the following subcases:

- In case **(ord)**, Proposition 3.8 gives for $i = 1, 2$ an isomorphism $\widehat{E}_i/p \simeq \overline{U}^0$ of Mackey functors over L . This implies an isomorphism of abelian groups,

$$(\widehat{E}_1/p \otimes \widehat{E}_2/p)(L) \simeq (\overline{U}^0 \otimes \overline{U}^0)(L).$$

- In case **(ssing)**, (3.9) gives an isomorphism $\widehat{E}_2/p \simeq \overline{U}^{p^2} \oplus \overline{U}^p$ of Mackey functors over L , and [16, Lemma 3.3] gives,

$$(\widehat{E}_1/p \otimes \widehat{E}_2/p)(L) \simeq (\overline{U}^0 \otimes \overline{U}^{p^2})(L) \oplus (\overline{U}^0 \otimes \overline{U}^p)(L).$$

The argument discussed in [16, Proposition 3.11] gives that these Mackey products $(\overline{U}^0 \otimes \overline{U}^j)(L)$ are generated by symbols defined over L . Thus, Claim A holds.

By [16, Theorem 3.6], for each $j < pe_0(L)$, the composition

$$(\overline{U}^0 \otimes \overline{U}^j)(L) \xrightarrow{\iota} (\mathbb{G}_m \otimes \mathbb{G}_m)(L)/p \xrightarrow{s_p} H^2(L, \mu_p^{\otimes 2}) \simeq \mathbb{Z}/p,$$

is bijective, where ι is induced by the maps $\overline{U}^i \hookrightarrow \mathbb{G}_m/p$, and s_p is the Galois symbol map (cf. (2.3)). By (2.4), for a symbol $\{x, y\}_{L/L}$ in $(\overline{U}^0 \otimes \overline{U}^j)(L)$

$$\{x, y\}_{L/L} \text{ generates } (\overline{U}^0 \otimes \overline{U}^j)(L) \iff x \notin N_{L_1/L}(L_1^\times), \quad (4.2)$$

where $L_1 = L(\sqrt[p]{y})$.

Claim B *Suppose we are in case **(ord)**. Then, the Mackey product $(E_1 \otimes E_2)(L)/p$ is generated by symbols of the form $\{x, y\}_{L/L}$ with either $x \in E_1(k)$, or $y \in E_2(k)$.*

It is enough to show that the group $(\widehat{E}_1/p \otimes \widehat{E}_2/p)(L) \simeq (\overline{U}^0 \otimes \overline{U}^0)(L)$ is generated by a symbol $\{x, y\}_{L/L}$ with $x \in \widehat{E}_1(k)$. Take any $x \in \widehat{E}_1^1(k) \setminus \widehat{E}_1^2(k)$. Then its restriction in $\widehat{E}_1(L)$ lies in $\widehat{E}_1^{p-1}(L) \setminus \widehat{E}_1^p(L)$. In particular, we can view x as an element of $\overline{U}_L^{p-1} \setminus \overline{U}_L^p$. The extension $L_1 = L(\sqrt[p]{x})/L$ is totally ramified extension of degree p with the jump at $s = pe_0(L) - (p-1) = 1$ (cf. Lemma 3.13 (a)). Then the isomorphism $U_L/N_{L_1/L}(U_{L_1}) \simeq L^\times/N_{L_1}(L_1^\times)$ of order p (cf. [41, V.3, Corollary 7]) gives us that there exists some $y \in U_L$ such that $y \notin N_{L_1/L}(U_{L_1})$. By (4.2), the symbol $\{x, y\}_{L/L}$ generates the group $(\overline{U}^0 \otimes \overline{U}^0)(L) \simeq (\widehat{E}_1/p \otimes \widehat{E}_2/p)(L)$.

The above computation guarantees that in case **(ord)**, $(E_1 \otimes E_2)(k)/p$ coincides with $\overline{\text{Symb}}_X(k)$. For, find elements $\{x_i, y_i\}_{L/L} \in (E_1 \otimes E_2)(L)/p$ that generate this group and are such that $x_i \in E_1(k)$. Then $N_{L/k}(\{x_i, y_i\}_{L/L})$ generate $(E_1 \otimes E_2)(L)/p$. But the projection formula yields,

$$N_{L/k}(\{x_i, y_i\}_{L/L}) = \{x_i, N_{L/k}(y_i)\}_{k/k} \in \overline{\text{Symb}}_X(k).$$

This completes the proof in case **(ord)**.

Lastly, we consider case **(ssing)**, namely, E_1 has good ordinary and E_2 has good supersingular reduction (hence $e_0(L) = p+1$). In this case we have a commutative diagram with exact rows and surjective vertical maps,

$$\begin{array}{ccccc} (\overline{U}^0 \otimes \overline{U}^{p^2})(L) \oplus (\overline{U}^0 \otimes \overline{U}^p)(L) & \longrightarrow & (E_1 \otimes E_2)(L)/p & \longrightarrow & 0 \\ \downarrow N_{L/k} & & \downarrow N_{L/k} & & \\ (\widehat{E}_1/p \otimes \widehat{E}_2/p)(k) & \longrightarrow & (E_1 \otimes E_2)(k)/p & \longrightarrow & 0. \end{array}$$

With a similar argument as in case **(ord)** we can show that the subgroup $(\overline{U}^0 \otimes \overline{U}^p)(L)$ of $(\widehat{E}_1/p \otimes \widehat{E}_2/p)(L)$ is generated by a symbol $\{x, y\}_{L/L}$ with $y \in \widehat{E}_2^1(k)$. Namely, let $y \in \widehat{E}_2^1(k) \setminus \widehat{E}_2^2(k)$. Then by Lemma 3.15 (1) the image of y in $E_2(L)/p$ decomposes as (y_0, y_1) , with $y_0 \in \overline{U}_L^{p^2+p}$, and $y_1 \in \overline{U}_L^{p+p^2-1} \setminus \overline{U}_L^{p+p^2}$. Since the extension $L(\sqrt[p]{y_1})/L$ is totally ramified of degree p (cf. Lemma 3.13 (a)), by [41, V.3, Corollary 7] we can find some $x \in \overline{U}_L^0$ such that $\{x, y_1\}_{L/L} \neq 0 \in (\overline{U}^0 \otimes \overline{U}^p)(L)$. Viewing x as an element of $E_1(L)/p$, we have that $\{x, y\}_{L/L}$ generates the subgroup $(\overline{U}^0 \otimes \overline{U}^p)(L) \subset (\widehat{E}_1/p \otimes \widehat{E}_2/p)(L)$. Notice that such an argument will not work for the other subgroup of $(\widehat{E}_1/p \otimes \widehat{E}_2/p)(L)$. Namely, no matter which symbol $\{x, y\}_{L/L}$ we take with either $x \in \widehat{E}_1(k)/p$ or $y \in \widehat{E}_2(k)/p$, the coordinate $\{x, (y_0, 1)\}_{L/L}$ will always vanish. For, if $y \in \widehat{E}_2(k)/p$ this follows immediately from Lemma 3.15. On the other hand, if $x \in \widehat{E}_1(k)/p$, then $x \in \widehat{E}_1^{p^2-1}(L)/p$. An easy computation shows that for $y = (y_0, y_1) \in \widehat{E}_2(L)/p$, and $L' = L(\sqrt[p]{y_0})$, it follows that $x \in N_{L'/L}(L'^\times)$. We suggest the following claim instead.

Claim C *The norm map $(\overline{U}^0 \otimes \overline{U}^{p^2})(L) \xrightarrow{N_{L/k}} (\widehat{E}_1/p \otimes \widehat{E}_2/p)(k)$ is zero.*

To prove this claim let $\{a, b\}_{L/L}$ be a generator of $(\overline{U}^0 \otimes \overline{U}^{p^2})(L) \simeq \mathbb{Z}/p$. For notational simplicity we will identify this symbol with its image, $\{a, (b, 1)\}_{L/L} \in$

$(\widehat{E}_1/p \otimes \widehat{E}_2/p)(L)$. We will show $N_{L/k}(\{a, b\}_{L/L}) = 0$. Since the restriction $\text{res}_{L/k}$ is injective, it suffices to show

$$\text{res}_{L/k}(N_{L/k}(\{a, b\}_{L/L})) = 0 \in (\overline{U}^0 \otimes \overline{U}^{p^2})(L).$$

We first obtain some information about the generator $\{a, b\}_{L/L}$.

As noted in (4.2), the symbol $\{a, b\}_{L/L}$ is a generator of $(\overline{U}^0 \otimes \overline{U}^{p^2})(L)$ if and only if $a \notin N_{L_1/L}(L_1^\times)$, where $L_1 = L(\sqrt[p]{b})$. Suppose that $a \in \overline{U}_L^i \setminus \overline{U}_L^{i+1}$, and $b \in \overline{U}_L^{p^2+j} \setminus \overline{U}_L^{p^2+j+1}$. By [16, Lemma 3.4(ii)] we may assume that i is coprime to p , $i < p^2 + p$ and $1 \leq j < p$, otherwise $\{a, b\}_{L/L} \neq 0$, and hence $\{a, b\}_{L/L}$ is not a generator of $(\overline{U}^0 \otimes \overline{U}^{p^2})(L)$. Moreover, by [41, V.3, Corollary 7] we obtain the following equivalence:

$$\{a, b\}_p \neq 0 \iff i + p^2 + j = pe_0(L) = p^2 + p \iff i + j = p.$$

Let $G_0 = \text{Gal}(L/k)$. Note that $L = k(\widehat{E}_2[p])$, and hence G_0 is a cyclic group of order $p^2 - 1$ (see Sect. 3.5 for details). Let σ be a generator of G_0 . Then Lemma 2.6 yields,

$$\text{res}_{L/k}(N_{L/k}(\{a, b\}_{L/L})) = \sum_{r=0}^{p^2-2} \sigma^r(\{a, b\}_{L/L}).$$

Set $V = (\overline{U}^0 \otimes \overline{U}^{p^2})(L)$. Using the previous remarks, the symbol $\{a, b\}_{L/L}$ is in the image of the symbol map $\overline{U}_L^i \otimes \overline{U}_L^{p^2+j} \rightarrow V$ defined by $x \otimes y \mapsto \{x, y\}_{L/L}$. This map factors through $\overline{U}_L^i / \overline{U}_L^{i+1} \otimes \overline{U}_L^{p^2+j} / \overline{U}_L^{p^2+j+1}$, inducing a symbol map

$$\overline{U}_L^i / \overline{U}_L^{i+1} \otimes \overline{U}_L^{p^2+j} / \overline{U}_L^{p^2+j+1} \rightarrow V. \quad (4.3)$$

In fact, for any $x \in \overline{U}_L^l$, $y \in \overline{U}_L^{p^2+l'}$ with $l + l' = p + 1$, $s_p(\{x, y\}_{L/L}) = (x, y)_p = 0$ by [17, Lemma 3.4] as $l + p^2 + l' = p^2 + p + 1 > pe_0(L)$. This implies $\{x, y\}_{L/L} = 0$. Let $\bar{\sigma}$ be the endomorphism of $\overline{U}_L^i / \overline{U}_L^{i+1} \otimes \overline{U}_L^{p^2+j} / \overline{U}_L^{p^2+j+1}$ induced by σ . Fix a uniformizer π_L of L . Then Lemma 3.17 yields the following equality in $\overline{U}_L^i / \overline{U}_L^{i+1} \otimes \overline{U}_L^{p^2+j} / \overline{U}_L^{p^2+j+1}$:

$$\bar{\sigma}(\bar{a} \otimes \bar{b}) = \bar{\sigma}(\bar{a}) \otimes \bar{\sigma}(\bar{b}) = \left(\frac{\sigma(\pi_L)}{\pi_L} \right)^i \bar{a} \otimes \left(\frac{\sigma(\pi_L)}{\pi_L} \right)^{pj} \bar{b} = \bar{u}^i \bar{a} \otimes \bar{u}^{pj} \bar{b},$$

where $\bar{u} = \sigma(\pi_L)/\pi_L \in \mathbb{F}_L^\times$. Note that since σ has exact order $p^2 - 1$, \bar{u} lies in the (possibly smaller) field \mathbb{F}_{p^2} . The next claim is that we have an equality, $\bar{u}^i \bar{a} \otimes \bar{u}^{pj} \bar{b} = \bar{u}^{i+pj} \bar{a} \otimes \bar{b}$. This follows by Lemma 3.3. Namely, we have isomorphisms $\overline{U}_L^i / \overline{U}_L^{i+1} \simeq \mathbb{F}_L$, and $\overline{U}_L^{p^2+j} / \overline{U}_L^{p^2+j+1} \simeq \mathbb{F}_L$. Thus, the group $\overline{U}_L^i / \overline{U}_L^{i+1} \otimes \overline{U}_L^{p^2+j} / \overline{U}_L^{p^2+j+1}$

becomes an \mathbb{F}_L -vector space by defining for $c \in \mathbb{F}_L$ and $\bar{x} \otimes \bar{y} \in \bar{U}_L^i / \bar{U}_L^{i+1} \otimes \bar{U}_L^{p^2+j} / \bar{U}_L^{p^2+j+1}$,

$$c \cdot \bar{x} \otimes \bar{y} = (c\bar{x}) \otimes \bar{y} = \bar{x} \otimes (c\bar{y}).$$

We conclude that $\bar{\sigma}$ acts as follows:

$$\bar{\sigma}(\bar{a}) \otimes \bar{\sigma}(\bar{b}) = \bar{u}^{i+pj} \bar{a} \otimes \bar{b}.$$

Since $i + j = p$, and both i, j are coprime to p , we have $1 \leq j \leq p - 1$, and hence $pj \leq p^2 - p$. This gives

$$i + pj \leq p^2 - p + p - 1 = p^2 - 1.$$

In order for the inequality to become an equality we need both $j = p - 1$ and $i = p - 1$. But this is not true, since $i + j = p$. Thus, $i + pj < p^2 - 1$. Set $\bar{v} = \bar{u}^{i+pj} \in \mathbb{F}_{p^2}$. The above inequality implies that $\bar{v} \neq 1$. At the same time $\bar{v}^{p^2-1} = 1$, which means that \bar{v} is a root of the polynomial $f(x) = x^{p^2-2} + x^{p^2-3} + \cdots + x + 1 \in \mathbb{F}_{p^2}[x]$. We then can compute,

$$\sum_{r=0}^{p^2-2} \bar{\sigma}^r(\bar{a} \otimes \bar{b}) = \left(\sum_{r=0}^{p^2-2} \bar{v}^r \right) \bar{a} \otimes \bar{b} = 0 \quad \text{in } \bar{U}_L^i / \bar{U}_L^{i+1} \otimes \bar{U}_L^{p^2+j} / \bar{U}_L^{p^2+j+1}.$$

Then (4.3) yields a vanishing $\text{res}_{L/k}(N_{L/k}(\{a, b\}_{L/L})) = 0$, which completes the proof. \square

The next theorem completes the proof of Theorem 1.4.

Theorem 4.5 *Let k be a finite unramified extension of \mathbb{Q}_p . Let E_1, E_2 be elliptic curves over k with good reduction. Assume that E_1 has good ordinary reduction. Then, the Mackey product $(E_1 \otimes E_2)(k)$ is p -divisible. In particular, the same holds for $K(k; E_1, E_2)$.*

Proof Using Lemma 4.3 (2), we may assume $k = k'$, where k'/k is as in (4.1). By Theorem 4.4, it is enough to show the vanishing of $\overline{\text{Symb}}_X(k)$. Notice that throughout the proof of Theorem 4.4, we used the Mackey product $(\widehat{E}_1 \otimes \widehat{E}_2)(k)/p$. It is therefore enough to show that $\{a, b\}_{k/k} = 0$, for every $a \in \widehat{E}_1(k)/p$ and $b \in \widehat{E}_2(k)/p$. Consider the finite extension L/k as in (4.1), which is totally ramified of degree $p^2 - 1$ or $p - 1$, depending on whether the curve E_2 has good supersingular reduction or not.

Case (ord): When both elliptic curves have good ordinary reduction. In this case we have $e_0(L) = 1$. We consider the restriction map, $(\widehat{E}_1 \otimes \widehat{E}_2)(k)/p \xrightarrow{\text{res}_{L/k}} (\widehat{E}_1 \otimes \widehat{E}_2)(L)/p$. Since the extension L/k is of degree coprime to p , $\text{res}_{L/k}$ is injective. It is therefore enough to show that $\text{res}_{L/k}(\{a, b\}_{k/k}) = \{a, b\}_{L/L} = 0$, for every $a \in \widehat{E}_1(k)/p$ and $b \in \widehat{E}_2(k)/p$. Using the identification $\widehat{E}_i(L)/p \simeq \bar{U}_L^0$ for $i = 1, 2$ (Proposition 3.8), we will show $\{a, b\}_{L/L} = 0$ in $(\bar{U}_L^0 \otimes \bar{U}_L^0)(L)$. Using the vanishing trick (2.8), it is enough to show that $b \in \text{Im}(\bar{U}_{L_1}^0 \xrightarrow{N_{L_1/L}} \bar{U}_L^0)$ for $L_1 = L(\sqrt[p]{a})$. Since

$a \in \widehat{E}_1^1(k)$, its restriction in $\widehat{E}_1(L)$ lies in $\widehat{E}_1^{p-1}(L)$, and the same holds for b . Thus in order to calculate the symbol $\{a, b\}_{L/L}$, we may view a, b as units in $\overline{U}_L^{p-1} \subset \overline{U}_L^0$. We distinguish the following two cases. First, if $v_k(a) = 1$, that is, $a \in \widehat{E}_1^1(k) \setminus \widehat{E}_1^2(k)$, then its restriction over L lies in $\overline{U}_L^{p-1} \setminus \overline{U}_L^p$. In this case Lemma 3.13 (a) gives us that L_1/L is a totally ramified degree p extension and its Galois group $\text{Gal}(L_1/L)$ has jump in its ramification filtration at $s = pe_0(L) - p + 1 = 1$. Since $b \in \overline{U}_L^{p-1}$, and $p - 1 > s$, Proposition 3.14 (2) gives a surjection $\overline{U}_{L_1}^{\psi(p-1)} \twoheadrightarrow \overline{U}_L^{p-1}$, where $\psi := \psi_{L_1/L}$. Since $\overline{U}_{L_1}^{\psi(p-1)} \subset \overline{U}_{L_1}^0$ and $\widehat{E}_2(L_1)/p \simeq \overline{U}_{L_1}^0$, we conclude that b is in the image of the norm map, and hence $\{a, b\}_{L/L} = 0$. Second, if $v_k(a) > 1$, then its restriction over L lies in $\overline{U}_L^p \subset \overline{U}_L^0$. In this case the extension L_1/L is unramified (cf. Lemma 3.13 (b)), and hence the norm map $U_{L_1} \xrightarrow{N_{L_1/L}} U_L$ is surjective. This completes the proof for the case of two elliptic curves with good ordinary reduction.

Case (ssing): When E_2 has good supersingular reduction. In this case we have $[L:k] = p^2 - 1$, and $e_0(L) = p + 1$. The argument is analogous to the previous case. Namely, let $\{a, b\}_{k/k} \in (\widehat{E}_1 \otimes \widehat{E}_2)(k)/p$. Since the extension L/k is of degree coprime to p , in order to show that $\{a, b\}_{k/k} = 0$, it is enough to show that $\{a, b\}_{L/L} = 0$. Since $b \in \widehat{E}_2(k)$, $v_L(b) \geq p^2 - 1$. Let $F = L(\sqrt[p]{b})$. We will show that $a \in N_{F/L}(\widehat{E}_1(F)/p)$. Lemma 3.15 allows us to reduce to the case when $b \in \widehat{E}_2^{p^2-1}(L) \setminus \widehat{E}_2^{p^2}(L)$. Then Lemma 3.15 (1) gives us that b decomposes as (b_0, b_1) under the decomposition (3.9), the extension $F_0 = L(\sqrt[p]{b_0})$ is unramified over L , while the extension $F_1 = L(\sqrt[p]{b_1})/L$ is totally ramified of degree p and the jump in the ramification filtration of $\text{Gal}(L(\sqrt[p]{b_1})/L)$ happens at $s = 1$. We may write,

$$\{a, b\}_{L/L} = \{a, (b_0, 1)\}_{L/L} + \{a, (1, b_1)\}_{L/L}.$$

We claim that both these symbols are zero. The first one vanishes, since the norm map $E_1(F_0) \xrightarrow{N_{F_0/L}} E_1(L)$ is surjective. For the second symbol, we can identify the restriction of a over L with a unit $a \in \overline{U}_L^{p^2-1}$. Since $p^2 - 1 > s = 1$, Proposition 3.14 gives us that $a \in N_{F_1/L}(F_1^\times)$, and hence $a \in N_{F_1/L}(\widehat{E}_1(F_1)/p)$. \square

4.2 The case of two elliptic curves with good supersingular reduction

In this subsection we focus on the only case which is not covered in Theorem 1.4; namely when $X = E_1 \times E_2$ is a product of two elliptic curves with good supersingular reduction. This case appears to be much harder than all others, and we can only obtain a partial result. Our first computation shows that a weaker form of Theorem 4.5 is still true in this case.

Theorem 4.6 *Let $X = E_1 \times E_2$ be a product of two elliptic curves with good supersingular reduction over an unramified extension k of \mathbb{Q}_p . Then, we have $\text{Symb}_X(k) = 0$.*

Proof The proof is very analogous to Theorem 4.5. Consider the finite extension $L = k(E_1[p], E_2[p])$. Using Lemma 4.3 we may assume that L/k is totally ramified.

Since for $i = 1, 2$ the extension $k_i := k(E_i[p])/k$ is totally ramified of degree $p^2 - 1$, L/k is a tamely ramified extension of degree $e(p^2 - 1)$, where $e = [L : k_1] = [L : k_2]$.

It follows by (3.6), that once again it suffices to show the vanishing of every symbol $\{a, b\}_{k/k} \in (\widehat{E}_1 \otimes \widehat{E}_2)(k)/p$. Let $\{a, b\}_{k/k} \in (\widehat{E}_1 \otimes \widehat{E}_2)(k)/p$. We will show that $\{a, b\}_{L/L} = 0$. Decompose $a = (a_0, a_1) \in \widehat{E}_1(k_1)/p$, and $b = (b_0, b_1) \in \widehat{E}_2(k_2)/p$ by identifying

$$\widehat{E}_i(k_i)/p \simeq \overline{U}_{k_i}^{p(e_0(k_i) - t_0(k_i))} \oplus \overline{U}_{k_i}^{pt_0(k_i)} = \overline{U}_{k_i}^{p^2} \oplus \overline{U}_{k_i}^p.$$

From Lemma 3.15, we may assume that $v_k(a) = v_k(b) = 1$ and we have

$$\begin{aligned} a_0 &\in \overline{U}_{k_1}^{p^2+p} \quad \text{and} \quad a_1 \in \overline{U}_{k_1}^{p^2+p-1} \setminus \overline{U}_{k_1}^{p^2+p}, \\ b_0 &\in \overline{U}_{k_2}^{p^2+p} \quad \text{and} \quad b_1 \in \overline{U}_{k_2}^{p^2+p-1} \setminus \overline{U}_{k_2}^{p^2+p}. \end{aligned}$$

By restricting them to L , we obtain

$$a = (a_0, a_1), \quad b = (b_0, b_1)$$

with $a_0, b_0 \in \overline{U}_L^{e(p^2+p)}$ and $a_1, b_1 \in \overline{U}_L^{e(p^2+p-1)} \setminus \overline{U}_L^{e(p^2+p)}$ in the decomposition $\widehat{E}_i(L)/p \simeq \overline{U}_L^{p(e_0(L) - t_0(L))} \oplus \overline{U}_L^{pt_0(L)} = \overline{U}_L^{ep^2} \oplus \overline{U}_L^{ep}$. This in turn gives,

$$\begin{aligned} \{a, b\}_{L/L} &= \{(a_0, 1), (b_0, 1)\}_{L/L} + \{(1, a_1), (b_0, 1)\}_{L/L} \\ &\quad + \{(a_0, 1), (1, b_1)\}_{L/L} + \{(1, a_1), (1, b_1)\}_{L/L}. \end{aligned}$$

Since the extensions $L(\sqrt[p]{a_0})$, and $L(\sqrt[p]{b_0})$ are unramified over L , the first three symbols vanish. It remains to show $\{(1, a_1), (1, b_1)\}_{L/L} = 0$. Consider the finite extension $L_1 = L(\sqrt[p]{b_1})$. We need to show that $(1, a_1) \in N_{L_1/L}(\widehat{E}_1(L_1)/p)$. By the same reason, we may assume that L_1/L is ramified. Lemma 3.13 gives us that L_1/L is totally ramified of degree p with jump at $s = pe_0(L) - i = e(p^2 + p) - i$ for some $e(p^2 + p - 1) \leq i < e(p^2 + p)$. In particular, we have $0 < s \leq e$. Considering the decomposition

$$\widehat{E}_1(L_1)/p \simeq \overline{U}_{L_1}^{ep^3} \oplus \overline{U}_{L_1}^{ep^2},$$

it is enough to show that $a_1 \in \text{Im}(\overline{U}_{L_1}^{ep^2} \xrightarrow{N_{L_1/L}} \overline{U}_L^{ep})$. By $e(p^2 + p - 1) > e \geq s$, Proposition 3.14 gives us a surjection

$$\overline{U}_{L_1}^{\psi(e(p^2+p-1))} \xrightarrow{N_{L_1/L}} \overline{U}_L^{e(p^2+p-1)} \rightarrow 0.$$

In order for $(1, a_1)$ to be a norm, we need to verify that $\overline{U}_{L_1}^{\psi(e(p^2+p-1))} \subset \overline{U}_{L_1}^{ep^2}$. This follows from $\psi(e(p^2 + p - 1)) \geq e(p^2 + p - 1) > ep^2$. Here, the first inequality follows from the definition of the Herbrand function. \square

In order to extend Theorem 1.4 to this case, we would need an analog of Theorem 4.4. The problem is that in this case we do not know how large the K -group $K(L; E_1, E_2)/p$ is. Let us focus for simplicity on the case of a self-product $X = E \times E$. It follows by [17, Proposition 3.6] that the image of the Galois symbol $(E/p \otimes E/p)(L) \xrightarrow{s_p} H^2(L, E[p] \otimes E[p])$ is isomorphic to \mathbb{Z}/p . In all other reduction cases this map is an isomorphism [16, 36, 52].

Question 4.7 Is the map $(E/p \otimes E/p)(L) \xrightarrow{s_p} H^2(L, E[p] \otimes E[p])$ injective when E has good supersingular reduction?

It is very likely that Question 4.7 has a negative answer. At least the Mackey product $(E/p \otimes E/p)(L) \simeq (\overline{U}^p \otimes \overline{U}^{p^2})(L)$ does not seem to give enough relations that guarantee injectivity. A weaker question is whether the analog of Claim A in the proof of Theorem 4.4 is true in this case.

Question 4.8 Is the K -group $K(L; E_1, E_2)/p$ generated by symbols of the form $\{a, b\}_{L/L}$, with $a \in E_1(L)$, $b \in E_2(L)$?

The next theorem provides some indication that if Question 4.8 has an affirmative answer, then Theorem 4.4 can indeed be extended to this case.

Theorem 4.9 Let k be a finite unramified extension of \mathbb{Q}_p . Suppose $X = E \times E$ is the self-product of an elliptic curve over k with good supersingular reduction. Let $L = k(E[p])$. Then, for every $a, b \in E(L)/p$, we have

$$N_{L/k}(\{a, b\}_{L/L}) = \{a, b\}_{L/k} = 0$$

in the Mackey product $(E \otimes E)(k)/p$.

Proof The proof will be along the lines of Claim C in the proof of Theorem 4.4. With the usual argument we may assume that the extension $L = k(\widehat{E}[p])/k$ is totally ramified of degree $p^2 - 1$ with cyclic Galois group. By Lemma 4.1 and (3.6) it is enough to prove that for every $a, b \in \widehat{E}(L)/p$, it holds $N_{L/k}(\{a, b\}_{L/L}) = 0$, and since $\text{res}_{L/k}$ is injective, this is equivalent to proving that $\text{res}_{L/k}(N_{L/k}(\{a, b\}_{L/L})) = 0$.

Set $W = (\widehat{E}/p \otimes \widehat{E}/p)(L)$. Recall that the group $\widehat{E}(L)/p$ has a filtration $\{\mathcal{D}_L^i\}$ given by $\mathcal{D}_L^i = \widehat{E}^i(L)/(p\widehat{E}(L) \cap \widehat{E}^i(L))$, for $i \geq 1$. This induces a filtration on W as follows. For every $t \geq 1$ the symbol maps define

$$\text{Fil}^t(W) := \sum_{n+m=t} \text{Im}(\mathcal{D}_L^n \otimes \mathcal{D}_L^m \rightarrow W).$$

Claim A Proving $\text{res}_{L/k} \circ N_{L/k} = 0$ is equivalent to proving that if $\{a, b\}_{L/L} \in \text{Fil}^t(W)$ for some $t \geq 1$, then $\text{res}_{L/k}(N_{L/k}(\{a, b\}_{L/L})) \in \text{Fil}^{t+1}(W)$.

The direction (\Rightarrow) is clear. To prove (\Leftarrow) , fix $a, b \in \widehat{E}(L)/p$. Assume that $\{a, b\}_{L/L} \in \text{Fil}^t(W)$ for some $t \geq 1$. Since by assumption $\text{res}_{L/k}(N_{L/k}(\{a, b\}_{L/L})) \in \text{Fil}^{t+1}(W)$, we may write $\text{res}_{L/k}(N_{L/k}(\{a, b\}_{L/L}))$ in the form, $\sum_{l=1}^N \{a_l, b_l\}_{L/L}$, for some $N \geq 1$ and some $a_l \in \mathcal{D}_L^n$, $b_l \in \mathcal{D}_L^m$ with $n + m \geq t + 1$. We claim that $\{a, b\}_{L/k} = 0$ if and

only if $\sum_{l=1}^N \{a_l, b_l\}_{L/k} = 0$. To see this note that the element $\sum_{l=1}^N \{a_l, b_l\}_{L/k}$ can be rewritten as,

$$\begin{aligned} \sum_{l=1}^N \{a_l, b_l\}_{L/k} &= N_{L/k} \left(\sum_{l=1}^N \{a_l, b_l\}_{L/L} \right) \\ &= N_{L/k}(\text{res}_{L/k}(\{a, b\}_{L/L})) = (p^2 - 1)\{a, b\}_{L/k}. \end{aligned}$$

Since $p^2 - 1 \equiv -1 \pmod{p}$, the claim follows. Using the above argument allows us to reduce to proving that $\{a, b\}_{L/k} = 0$, whenever $\{a, b\}_{L/L} \in \text{Fil}^{t+1}(W)$. At this point we can continue inductively. The process is guaranteed to terminate in finitely many steps, since Lemma 3.11 (d) implies that $\text{Fil}^{2p^2}(W) = 0$.

From now we focus on proving the implication,

$$\{a, b\}_{L/L} \in \text{Fil}^t(W) \implies \text{res}_{L/k}(N_{L/k}(\{a, b\}_{L/L})) \in \text{Fil}^{t+1}(W).$$

Let $G_0 = \text{Gal}(L/k) = \langle \sigma \rangle$ be the Galois group of L/k . Then for all $a, b \in \widehat{E}(L)/p$ we have,

$$\text{res}_{L/k}(N_{L/k}(\{a, b\}_{L/L})) = \sum_{r=0}^{p^2-2} \sigma^r(\{a, b\}_{L/L}).$$

Suppose that $a \in \mathcal{D}_L^i \setminus \mathcal{D}_L^{i+1}$, and $b \in \mathcal{D}_L^j \setminus \mathcal{D}_L^{j+1}$, for some $i, j \in \{1, \dots, p^2\}$ with $i + j = t$. Note that if either i or j is equal to p^2 , then Lemmas 3.11 and 3.15 imply that the extension $L_0 = L(\frac{1}{p}b)$ is unramified over L . Using the surjectivity of the norm, $E_1(L_0) \xrightarrow{N_{L_0/L}} E_1(L)$, we immediately get that $\{a, b\}_{L/L} = 0$. We may therefore reduce to the case when $i, j \in \{1, \dots, p^2 - 1\}$. Consider the quotient $\text{Fil}^t(W)/\text{Fil}^{t+1}(W)$ and the well-defined map,

$$\mathcal{D}_L^i/\mathcal{D}_L^{i+1} \otimes \mathcal{D}_L^j/\mathcal{D}_L^{j+1} \rightarrow \text{Fil}^t(W)/\text{Fil}^{t+1}(W).$$

Denote by $\bar{a} \otimes \bar{b}$ the image of $a \otimes b$ in $\mathcal{D}_L^i/\mathcal{D}_L^{i+1} \otimes \mathcal{D}_L^j/\mathcal{D}_L^{j+1}$. Moreover, let $\bar{\sigma}$ be the endomorphism induced by σ ,

$$\bar{\sigma}: \mathcal{D}_L^i/\mathcal{D}_L^{i+1} \otimes \mathcal{D}_L^j/\mathcal{D}_L^{j+1} \rightarrow \mathcal{D}_L^i/\mathcal{D}_L^{i+1} \otimes \mathcal{D}_L^j/\mathcal{D}_L^{j+1}.$$

Fix a uniformizer π_L of L and set $\bar{u} = \sigma(\pi_L)/\pi_L \in \mathbb{F}_{p^2}^\times$. Then Lemmas 3.16 and 3.17 (2) yield an equality,

$$\bar{\sigma}(\bar{a} \otimes \bar{b}) = \bar{u}^i \bar{a} \otimes \bar{u}^j \bar{b}.$$

Similarly to the proof of Claim C in Theorem 4.4, the tensor product $\mathcal{D}_L^i/\mathcal{D}_L^{i+1} \otimes \mathcal{D}_L^j/\mathcal{D}_L^{j+1}$ becomes an \mathbb{F}_L -vector space, and hence we can rewrite,

$$\overline{\sigma}(\overline{a} \otimes \overline{b}) = \overline{u}^{i+j} \overline{a} \otimes \overline{b}.$$

When $i + j$ is not divisible by $p^2 - 1$, we may proceed exactly as in the proof of Claim C in Theorem 4.4 to deduce that

$$\sum_{r=0}^{p^2-2} \overline{\sigma}^r(\overline{a} \otimes \overline{b}) = \left(\sum_{r=0}^{p^2-2} (\overline{u}^{i+j})^r \right) \overline{a} \otimes \overline{b} = 0,$$

and hence $\text{res}_{L/k}(N_{L/k}(\{a, b\}_{L/L})) \in \text{Fil}^{t+1}(W)$. It remains to consider the cases when $i + j$ is a multiple of $p^2 - 1$. Since $2 \leq i + j \leq 2p^2 - 2$, the only multiples of $p^2 - 1$ in that range are $p^2 - 1$ and $2p^2 - 2$. We consider each case separately.

Case 1: Suppose that $i + j = 2p^2 - 2$. This is only possible if $i = j = p^2 - 1$. In this case we can prove that $\{a, b\}_{L/L} = 0$, imitating the proof of Theorem 4.6.

Case 2.1: Suppose that $i + j = p^2 - 1$ and both i, j are coprime to p . Using (3.9) we can decompose $a = (a_0, a_1)$, and $b = (b_0, b_1)$ with $a_0, b_0 \in \overline{U}_L^{p^2+1}$, and $a_1, b_1 \in \overline{U}_L^{p+1}$. Then Lemma 3.11 gives us that $a_1 \in \overline{U}_L^{p+i} \setminus \overline{U}_L^{p+i+1}$, $b_1 \in \overline{U}_L^{p+j} \setminus \overline{U}_L^{p+j+1}$, while $a_0 \in \overline{U}_L^{p^2+t'}$, $b_0 \in \overline{U}_L^{p^2+t'}$, for some t, t' such that $pt > i$, and $pt' > j$. Notice that the symbols $\{(1, a_1), (b_0, 1)\}_{L/k}$, $\{(a_0, 1), (1, b_1)\}_{L/k}$ and $\{(a_0, 1), (b_0, 1)\}_{L/k}$ vanish due to previous considerations in the proof. It remains to show that $\{(1, a_1), (1, b_1)\}_{L/k} = 0$. In fact, we claim that $\{(1, a_1), (1, b_1)\}_{L/L} = 0$. We consider the finite extensions $L_1 = L(\sqrt[p]{a_1})$, and $L_2 = L(\sqrt[p]{b_1})$, both of which are totally ramified of degree p over L . Using the usual vanishing trick (see Remark 2.8), it is enough to establish the following claim.

Claim B *The assumption $i + j = p^2 - 1$ forces one of the elements to be a norm. That is, either $(1, a_1) \in N_{L_2/L}(E(L_2)/p)$, or $(1, b_1) \in N_{L_1/L}(E(L_1)/p)$.*

To prove Claim B, we need to show that either a_1 lies in the image of the norm map $\overline{U}_{L_2}^{p^2} \xrightarrow{N_{L_2/L}} \overline{U}_L^p$, or b_1 lies in the image of $\overline{U}_{L_1}^{p^2} \xrightarrow{N_{L_1/L}} \overline{U}_L^p$. Let $G_1 = \text{Gal}(L_1/L)$, $G_2 = \text{Gal}(L_2/L)$ be the Galois groups of L_1, L_2 respectively. Then the jump in the ramification filtration of G_1 (resp. of G_2) occurs at $s_1 = p^2 + p - (p + i) = p^2 - i$ (resp. at $s_2 = p^2 - j$). Observe that since $i + j = p^2 - 1$, it follows that,

$$p + i = p^2 + p - 1 - j > p^2 - j \Rightarrow p + i > s_2.$$

Since the computation is symmetric, we also get $p + j > s_1$. Then Proposition 3.14 yields,

$$\begin{aligned} \psi_{L_2/L}(p + i) &= s_2 + p(p + i - s_2) = p^2 - j + p(p + i - p^2 + j) \\ &= p^2 - j + p(p - 1) = 2p^2 - p - j. \end{aligned}$$

A symmetric computation yields $\psi_{L_1/L}(p+j) = 2p^2 - p - i$. In order to prove Claim B, we must show that at least one of $\psi_{L_2/L}(p+i)$ and $\psi_{L_1/L}(p+j)$ is larger than p^2 . Assume to contradiction that $\psi_{L_1/L}(p+j) \leq p^2$, and $\psi_{L_2/L}(p+i) \leq p^2$. Adding these inequalities yields,

$$\begin{aligned} 2p^2 - p - j + 2p^2 - p - i &\leq 2p^2 \implies 2p^2 - 2p \leq i + j \\ &\implies 2p^2 - 2p \leq p^2 - 1 \implies (p-1)^2 \leq 0, \end{aligned}$$

which is a contradiction. We conclude that either $\psi_{L_2/L}(p+i) > p^2$, or $\psi_{L_1/L}(p+j) > p^2$, which respectively give that $(1, a_1) \in N_{L_2/L}(E(L_2)/p)$, or $(1, b_1) \in N_{L_1/L}(E(L_1)/p)$.

Case 2.2: Suppose that $i+j = p^2 - 1$ and one of the integers i, j is divisible by p . Without loss of generality, assume that $i = pl$, for some $1 \leq l < p$. We decompose $a = (a_0, a_1)$, $b = (b_0, b_1)$ with $a_0, b_0 \in \overline{U}_L^{p^2+1}$, and $a_1, b_1 \in \overline{U}_L^{p+1}$. Lemma 3.17 and [25, Lemma 2.1.4] yield that $a_0 \in \overline{U}_L^{p^2+l} \setminus \overline{U}_L^{p^2+l+1}$, $a_1 \in \overline{U}_L^{p+i+1}$, while $b_1 \in \overline{U}_L^{p+j} \setminus \overline{U}_L^{p+j+1}$. Finally, $b_0 \in \overline{U}_L^{p^2+t}$ for some t such that $pt > j$. The symbols $\{(1, a_1), (b_0, 1)\}_{L/k}$, $\{(1, a_1), (1, b_1)\}_{L/k}$ and $\{(a_0, 1), (b_0, 1)\}_{L/k}$ vanish due to earlier considerations in this proof. It remains to show that $\{(a_0, 1), (1, b_1)\}_{L/k} = 0$. We proceed similarly to Case 2.1, to show $\{(a_0, 1), (1, b_1)\}_{L/L} = 0$. Consider the finite extension $L_3 = L(\sqrt[p]{a_0})$, which is totally ramified of degree p over L . It is enough to establish the following claim.

Claim C *The assumption $i+j = p^2 - 1$ forces the element $(1, b_1) \in E(L)/p$ to be a norm, namely $(1, b_1) \in N_{L_3/L}(E(L)/p)$.*

To prove Claim C we need to show that b_1 lies in the image of the norm $\overline{U}_{L_3}^{p^2} \xrightarrow{N_{L_3/L}} \overline{U}_L^p$. Let $G_3 = \text{Gal}(L_3/L)$ be the Galois group of L_3/L . The jump in the ramification filtration of G_3 occurs at $s_3 = p^2 + p - (p^2 + l) = p - l$. It is then clear that $p+j > s_3$, and hence we can compute $\psi_{L_3/L}(p+j)$ as follows:

$$\begin{aligned} \psi_{L_3/L}(p+j) &= s_3 + p(p+j-s_3) = p-l + p(p+j-p+l) \\ &= p-l + p(j+l) = p-l + pj + pl. \end{aligned}$$

Using that $i+j = p^2 - 1$, and hence $pl+j = p^2 - 1$, the last equality can be rewritten as,

$$\psi_{L_3/L}(p+j) = p-l + pj + p^2 - 1 - j = p^2 + p + (p-1)j - l - 1.$$

We want to prove that $\psi_{L_3/L}(p+j) > p^2$. Equivalently, $(p-1)j + p-l > 1$. But this is clear, since $l < p$ and $j \geq 1$. We conclude that $b_1 \in \text{Im}(\overline{U}_{L_3}^{p^2} \xrightarrow{N_{L_3/L}} \overline{U}_L^p)$, as required. \square

4.3 Extensions

We close this section by giving some easy extensions of Theorem 4.5.

Corollary 4.10 *Let k be a finite unramified extension of \mathbb{Q}_p and E_1, \dots, E_r be elliptic curves over k with good reduction. Assume that at most one of the curves has good supersingular reduction. Let $X = E_1 \times \dots \times E_r$. Then, the Albanese kernel $F^2(X)$ is p -divisible.*

Proof Using the relation of $F^2(X)$ with the Somekawa K -groups given in (2.5), we are reduced to proving that the K -group $K(k; E_{i_1}, \dots, E_{i_v})$ is p -divisible, for every $2 \leq v \leq r$ and $1 \leq i_1 < i_2 < \dots < i_v \leq r$. It is enough to show that the Mackey product $(E_{i_1}/p \otimes \dots \otimes E_{i_v}/p)(k) = 0$. Note that Theorem 4.5 implies that this is true when $v = 2$. The general case follows by the fact that the product \otimes in the category of Mackey functors is associative. \square

The following corollary follows by an easy descent argument.

Corollary 4.11 *Let k be a finite unramified extension of \mathbb{Q}_p . Let X be a principal homogeneous space of an abelian variety A , such that $X_L \simeq A_L$ for some finite extension L/k of degree coprime to p . Suppose additionally that there is an isogeny $A \xrightarrow{\phi} E_1 \times \dots \times E_r$ of degree coprime to p , where E_i are elliptic curves over k satisfying the assumptions of Corollary 4.10. Then, the groups $F^2(A)$ and $F^2(X)$ are p -divisible.*

Proof We first show that the Albanese kernel $F^2(A)$ is p -divisible. Suppose that the isogeny $A \xrightarrow{\phi} E_1 \times \dots \times E_r$ has degree n . Let $\check{\phi}$ be the dual isogeny. These induce push-forward maps,

$$F^2(A)/p \xrightarrow{\phi_*} F^2(E_1 \times \dots \times E_r)/p \xrightarrow{\check{\phi}_*} F^2(A)/p,$$

with $\check{\phi}_* \circ \phi_* = n$. Since by assumption n is coprime to p , multiplication by n is injective on $F^2(A)/p$. At the same time $F^2(E_1 \times \dots \times E_r)/p = 0$ by the previous corollary. We conclude that $F^2(A)$ is p -divisible.

Next suppose that X is a principal homogeneous space of A such that $X_L \simeq A$ for some finite extension L/k of degree m which is coprime to p . Consider the projection $X_L \xrightarrow{g} X$, which induces a push-forward, $F^2(X_L)/p \xrightarrow{g_*} F^2(X)/p$, and a pull-back $F^2(X)/p \xrightarrow{g^*} F^2(X_L)/p$, satisfying $g_* \circ g^* = m$. By the previous case we have $F^2(X_L)/p \simeq F^2(A)/p = 0$, which forces g^* to vanish. Since g^* is also injective, the claim follows. \square

Proposition 4.12 *Let k be a finite unramified extension of \mathbb{Q}_p . Let A_1 and A_2 be abelian varieties over k . For each $i = 1, 2$, we assume that the connected component of the special fiber of the Néron model of A_i is a split torus. Then, $(A_1 \otimes A_2)(k)$ is p -divisible.*

Proof From the assumption on A_i , there exists a split torus $T_i \simeq \mathbb{G}_m^{\oplus g_i}$ over k with $g_i = \dim A_i$ and a free abelian subgroup $L_i \subset T_i(k)$ such that $T_i(k')/L_i \simeq A_i(k')$ for any finite extension k'/k . This gives surjections of Mackey functors $T_i/p \rightarrow A_i/p$, which induce a surjection

$$(T_1/p \otimes T_2/p)(k) \rightarrow (A_1/p \otimes A_2/p)(k) \rightarrow 0$$

(cf. [52, Remark 4.2 (2)]). As the tori T_1 and T_2 split, we have

$$(T_1/p \otimes T_2/p)(k) \simeq (\mathbb{G}_m/p \otimes \mathbb{G}_m/p)(k)^{\oplus g_1 g_2} \simeq (K_2(k)/p)^{\oplus g_1 g_2} = 0,$$

where the last equality follows from [11, Chapter IX, Proposition 4.2]. This shows that $(A_1/p \otimes A_2/p)(k) = 0$. \square

Recall that the Jacobian variety of a Mumford curve satisfies the assumption in the above proposition. Using the associativity of the Mackey products as in the proof of Corollary 4.10, we obtain the following corollary.

Corollary 4.13 *Let k be a finite unramified extension of \mathbb{Q}_p . Let $X = C_1 \times \cdots \times C_r$ be a product of Mumford curves over k . Then, the Albanese kernel $F^2(X)$ is p -divisible.*

5 Local-to-global results

In this last section we focus on the local-to-global Conjecture 1.6, which constitutes one of the main motivations of this article.

Notation 5.1 For a \mathbb{Z} -module M we will denote by $\widehat{M} := \varprojlim_n M/n$ the completion of M . Let F be a number field, that is, a finite extension of $\overline{\mathbb{Q}}$. We will denote by Ω , Ω_f and Ω_∞ the set of all places, all finite places and all infinite places of F respectively.

5.1 The Brauer group

Let X be a smooth projective and geometrically connected variety over F . The Brauer group $\mathrm{Br}(X) = H_{\text{ét}}^2(X, \mathbb{G}_m)$ of X has a filtration,

$$\mathrm{Br}(X) \supset \mathrm{Br}_1(X) \supset \mathrm{Br}_0(X),$$

induced by the Hochschild–Serre spectral sequence

$$H^i(F, H_{\text{ét}}^j(X_{\overline{F}}, \mathbb{G}_m)) \Rightarrow H_{\text{ét}}^{i+j}(X, \mathbb{G}_m).$$

The two subgroups are defined as follows; $\mathrm{Br}_0(X) := \mathrm{Im}(\mathrm{Br}(F) \rightarrow \mathrm{Br}(X))$, and $\mathrm{Br}_1(X) := \ker(\mathrm{Br}(X) \rightarrow \mathrm{Br}(X_{\overline{F}}))$. The latter is usually referred in the literature as the *algebraic Brauer group*, while the quotient $\mathrm{Br}(X)/\mathrm{Br}_1(X)$ is called the *transcendental Brauer group*. When X is an abelian variety, a $K3$ surface [45, Theorem 1.1], or a product of curves [46, Theorem B] over a number field, the transcendental Brauer group is finite.

5.2 The Brauer–Manin pairing

For each place v of F , put $X_v = X_{F_v}$. Suppose v is a finite place of F . There is a pairing,

$$\langle \cdot, \cdot \rangle_v: CH_0(X_v) \times \text{Br}(X_v) \rightarrow \text{Br}(F_v) \simeq \mathbb{Q}/\mathbb{Z},$$

known as the Brauer–Manin pairing, which is defined using evaluation at closed points, and the invariant map of local class field theory, $\text{inv}_v: \text{Br}(F_v) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$. Precisely, for each closed point $P \in X_v$ and $\alpha \in \text{Br}(X_v)$, the pull-back of α along the closed point $P \rightarrow X_v$ is denoted by $\alpha(P) \in \text{Br}(F_v(P))$, where $F_v(P)$ is the residue field of P . The pairing above is defined by $\langle P, \alpha \rangle_v := \text{Cor}_{F_v(P)/F_v}(\alpha(P))$. In a similar manner, one can define a Brauer–Manin pairing for every real place v of F ,

$$\langle \cdot, \cdot \rangle_v: CH_0(X_v) \times \text{Br}(X_v) \rightarrow \text{Br}(F_v) \simeq \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}.$$

Note that in this case the subgroup $\pi_{\overline{F}_v/F_v\star}(CH_0(X_v \otimes_{F_v} \overline{F}_v))$ of $CH_0(X_v)$ is contained in the left kernel of this pairing, where $X_{\overline{F}_v} \xrightarrow{\pi_{\overline{F}_v/F_v}} X$ is the projection.

Definition-Notation 5.2 The adelic Chow group of X is defined as

$$CH_{0,\mathbb{A}}(X) = \prod_{v \in \Omega} \overline{CH}_0(X_v),$$

where for every finite place v of F , we have an equality $\overline{CH}_0(X_v) = CH_0(X_v)$, while for every infinite place v , $\overline{CH}_0(X_v) := \frac{CH_0(X_v)}{\pi_{\overline{F}_v/F_v\star}(CH_0(X_v \otimes_{F_v} \overline{F}_v))}$. In a similar way, we define the adelic group $F_{\mathbb{A}}^1(X) := \prod_{v \in \Omega} \overline{F}^1(X_v)$ of zero-cycles of degree zero, and the adelic Albanese kernel, $F_{\mathbb{A}}^2(X) := \prod_{v \in \Omega} \overline{F}^2(X_v)$ (cf. Sect. 2.4 for the definition of the filtration).

Notice that for every infinite complex place v we have an equality, $\overline{CH}_0(X_v) = 0$. The local pairings induce a global pairing,

$$\langle \cdot, \cdot \rangle: CH_{0,\mathbb{A}}(X) \times \text{Br}(X) \rightarrow \mathbb{Q}/\mathbb{Z},$$

defined by $\langle (z_v)_v, \alpha \rangle = \sum_v \langle z_v, \iota^*(\alpha) \rangle_v$, where ι^* is the pull-back of $\iota: X_v \rightarrow X$. We note that, if v is a real place, then [4, Théorème 1.3] gives us that the group $\overline{F}^1(X_v)$ is isomorphic to a finite number of copies of $\mathbb{Z}/2\mathbb{Z}$.

The short exact sequence of global class field theory,

$$0 \rightarrow \text{Br}(F) \rightarrow \bigoplus_{v \in \Omega} \text{Br}(F_v) \xrightarrow{\sum \text{inv}_v} \mathbb{Q}/\mathbb{Z} \rightarrow 0,$$

implies that the group $CH_0(X)$ lies in the left kernel of $\langle \cdot, \cdot \rangle$, thus giving rise to a complex,

$$CH_0(X) \xrightarrow{\Delta} CH_{0,\mathbb{A}}(X) \rightarrow \text{Hom}(\text{Br}(X), \mathbb{Q}/\mathbb{Z}).$$

Behavior with respect to filtrations

Under the Brauer–Manin pairing, the filtration $CH_0(X) \supset F^1(X) \supset F^2(X) \supset 0$ is compatible with the filtration $\text{Br}(X) \supset \text{Br}_1(X) \supset \text{Br}_0(X)$ of the Brauer group. Namely, restricting the map $CH_{0,\mathbb{A}}(X) \rightarrow \text{Hom}(\text{Br}(X), \mathbb{Q}/\mathbb{Z})$ to $F_{\mathbb{A}}^1(X)$, it factors through

$$F_{\mathbb{A}}^1(X) \rightarrow \text{Hom}(\text{Br}(X)/\text{Br}_0(X), \mathbb{Q}/\mathbb{Z}) \hookrightarrow \text{Hom}(\text{Br}(X), \mathbb{Q}/\mathbb{Z}).$$

Similarly, restricting the latter to $F_{\mathbb{A}}^2(X)$, it factors through,

$$F_{\mathbb{A}}^2(X) \rightarrow \text{Hom}\left(\frac{\text{Br}(X)}{\text{Br}_1(X)}, \mathbb{Q}/\mathbb{Z}\right) \hookrightarrow \text{Hom}(\text{Br}(X), \mathbb{Q}/\mathbb{Z}).$$

This compatibility follows by considering the local pairings, $CH_0(X_v) \rightarrow \text{Hom}(\text{Br}(X_v), \mathbb{Q}/\mathbb{Z})$ and using local Tate duality. For a proof of this compatibility we refer to [52, Proof of Proposition 3.1] and [12, Proof of Theorem 6.9].

5.3 The conjecture

We are interested in the following conjecture.

Conjecture 5.3 ([7, Section 4], [23, Section 7], [4, Conjecture 1.5(c)] and [50, Conjecture (E_0)]) *Let X be a smooth projective geometrically connected variety over a number field F . The following complex is exact:*

$$\widehat{F^1(X)} \xrightarrow{\Delta} \widehat{F_{\mathbb{A}}^1(X)} \rightarrow \text{Hom}(\text{Br}(X)/\text{Br}_0(X), \mathbb{Q}/\mathbb{Z}), \quad (5.1)$$

where $F_{\mathbb{A}}^1(X)$ is the group defined in 5.2.

When E is an elliptic curve, it is a theorem of Cassels [3] that Conjecture 5.3 is true, if the Tate–Shafarevich group of E contains no nonzero divisible element; in particular, it is true if it is finite. This result has been generalized by Colliot-Thélène [5, paragraphe 3] to all curves, assuming that the Tate–Shafarevich group of their Jacobian contains no nonzero divisible element. We next consider what happens for a product $X = C_1 \times C_2$ of two curves over F .

Proposition 5.4 *Let $X = C_1 \times C_2$ be a product of smooth projective curves over a number field F . Let $\text{Alb}_X = J_1 \times J_2$ be the Albanese variety of X , where J_1, J_2 are the Jacobian varieties of C_1, C_2 respectively. Assume that X contains a k -rational point and that the Tate–Shafarevich group of Alb_X contains no nonzero divisible*

element. Then the exactness of (5.1) can be reduced to the exactness of the following complex:

$$\widehat{F^2(X)} \xrightarrow{\Delta} \widehat{F_A^2(X)} \rightarrow \operatorname{Hom}\left(\frac{\operatorname{Br}(X)}{\operatorname{Br}_1(X)}, \mathbb{Q}/\mathbb{Z}\right). \quad (5.2)$$

Proof It follows by [36, Corollary 2.4.1] that we have a direct sum decomposition,

$$F^1(X) \simeq J_1(F) \oplus J_2(F) \oplus F^2(X),$$

and the same holds over F_v , for every place v of F . This decomposition passes to the completions. Thus, proving exactness of (5.1) amounts to proving exactness of (5.2) and exactness of the following complex:

$$\widehat{\operatorname{Alb}_X(F)} \xrightarrow{\Delta} \prod_{v \in \Omega} \widehat{\operatorname{Alb}_{X_v}(F_v)} \rightarrow \operatorname{Hom}(\operatorname{Br}_1(X)/\operatorname{Br}_0(X), \mathbb{Q}/\mathbb{Z}). \quad (5.3)$$

It follows directly by the Hochschild–Serre spectral sequence,

$$E_2^{i,j} = H^i(F, H^j(X_{\overline{F}}, \mathbb{G}_m)) \Rightarrow H^{i+j}(X, \mathbb{G}_m),$$

that we have a surjection $H^1(F, \operatorname{Pic}(X_{\overline{F}})) \rightarrow \operatorname{Br}_1(X)/\operatorname{Br}_0(X) \rightarrow 0$. Namely, $H^1(F, \operatorname{Pic}(X_{\overline{F}})) = E_2^{1,1}$, while the quotient $\operatorname{Br}_1(X)/\operatorname{Br}_0(X)$ is precisely the graded piece $\operatorname{gr}_1(E^2) = \operatorname{gr}_1(\operatorname{Br}(X)) = E_{\infty}^{1,1}$. Note that the differential $d_2^{1,1}: E_2^{1,1} \rightarrow E_2^{3,0}$ vanishes, since $E_2^{3,0} = H^3(F, \overline{F}^{\times}) = 0$. This is a byproduct of global class field theory (cf. [35, Remark 6.7.10]). Thus, we get a surjection $E_2^{1,1} \rightarrow E_{\infty}^{1,1} \rightarrow 0$. This induces an injection on the dual groups, $0 \rightarrow \operatorname{Hom}(\operatorname{Br}_1(X)/\operatorname{Br}_0(X), \mathbb{Q}/\mathbb{Z}) \rightarrow \operatorname{Hom}(H^1(F, \operatorname{Pic}(X_{\overline{F}})), \mathbb{Q}/\mathbb{Z})$. Moreover, there is a map $\operatorname{Hom}(H^1(F, \operatorname{Pic}(X_{\overline{F}})), \mathbb{Q}/\mathbb{Z}) \rightarrow \operatorname{Hom}(H^1(F, \operatorname{Pic}^0(X_{\overline{F}})), \mathbb{Q}/\mathbb{Z})$ induced by the short exact sequence of G_F -modules,

$$0 \rightarrow \operatorname{Pic}^0(X_{\overline{F}}) \rightarrow \operatorname{Pic}(X_{\overline{F}}) \rightarrow NS(X_{\overline{F}}) \rightarrow 0.$$

The variety $\operatorname{Pic}^0(X)$ is the dual abelian variety to Alb_X . Since the latter is equal to the product $J_1 \times J_2$, it is self dual. We conclude that the exactness of (5.3) follows by the exactness of the following complex:

$$\widehat{\operatorname{Alb}_X(F)} \xrightarrow{\Delta} \prod_{v \in \Omega} \widehat{\operatorname{Alb}_{X_v}(F_v)} \xrightarrow{\beta} \operatorname{Hom}(H^1(F, \operatorname{Alb}_X), \mathbb{Q}/\mathbb{Z}).$$

Here the map $\beta: \prod_{v \in \Omega} \widehat{\operatorname{Alb}_{X_v}(F_v)} \rightarrow \operatorname{Hom}(H^1(F, \operatorname{Alb}_X), \mathbb{Q}/\mathbb{Z})$ is obtained by all the local isomorphisms $\widehat{\operatorname{Alb}_{X_v}(F_v)} \simeq \operatorname{Hom}(H^1(F_v, \operatorname{Alb}_{X_v}), \mathbb{Q}/\mathbb{Z})$ induced by local Tate duality (cf. [32, I. Corollary 3.4]) and composing it with the map

$$\prod_{v \in \Omega} \operatorname{Hom}(H^1(F_v, \operatorname{Alb}_{X_v}), \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sum_v} \operatorname{Hom}(H^1(F, \operatorname{Alb}_X), \mathbb{Q}/\mathbb{Z}).$$

The fact that the map \sum_v is well-defined follows by [32, I. Lemma 6.3]. This complex is known to be exact under the assumption on the Tate–Shafarevich group of Alb_X (cf. [32, II. 5.6(b)]). \square

5.4 Elliptic curves with potentially good reduction

In this section we consider a product $X = E_1 \times E_2$ of two elliptic curves over F . We assume that for $i = 1, 2$ the elliptic curve E_i has potentially good reduction at all finite places of F . It is known that an elliptic curve over F has potentially good reduction if and only if its j -invariant is integral [43, Chapter VII, Proposition 5.5]. An important class of elliptic curves with this property are elliptic curves with complex multiplication ([10, p. 225] and [42, Theorem 7], see also [43, Chapter VII, Exercise 7.10]).

Suppose that v_1, \dots, v_r are all the places of bad reduction of X . Then, there exists some finite extension L_{v_i} of F_{v_i} such that $X_{L_{v_i}}$ has good reduction. We set $n_i := [L_{v_i} : F_{v_i}]$.

Lemma 5.5 *Let $i \in \{1, \dots, r\}$. Let p be a prime number such that $v_i \nmid p$ and p is coprime to n_i . Then the Albanese kernel $F^2(X_{v_i})$ is p -divisible.*

Proof Consider the projection $X_{L_{v_i}} \xrightarrow{\pi_{L_{v_i}/F_{v_i}}} X_{v_i}$, and let $CH_0(X_{L_{v_i}}) \xrightarrow{\pi_{L_{v_i}/F_{v_i}}^*} CH_0(X_{v_i})$, and $CH_0(X_{v_i}) \xrightarrow{\pi_{L_{v_i}/F_{v_i}}^*} CH_0(X_{L_{v_i}})$ be the induced push-forward and pull-back maps respectively. Since $X_{L_{v_i}}$ has good reduction and $v_i \nmid p$, it follows by [38, Corollary 0.10] that the Albanese kernel $F^2(X_{L_{v_i}})$ is p -divisible. Moreover, the endomorphism

$$\frac{F^2(X_{v_i})}{p} \xrightarrow{n_i} \frac{F^2(X_{v_i})}{p}$$

can be factored as $n_i = \pi_{L_{v_i}/F_{v_i}}^* \circ \pi_{L_{v_i}/F_{v_i}}^*$. Since we assumed that p does not divide n_i , this forces the multiplication by n_i to be injective modulo p . At the same time the map $\pi_{L_{v_i}/F_{v_i}}^*$ is the zero map. We conclude that the group $F^2(X_{v_i})/p$ vanishes. \square

Definition 5.6 Let S be the set of rational primes consisting of:

- all the primes p such that $v_i \mid p$ for some $1 \leq i \leq r$,
- all the prime divisors of $\prod_{i=1}^r n_i$,
- all the primes p such that both E_1, E_2 have good supersingular reduction at v for some place $v \in \Omega_f$ that lies above p ,
- all the ramified primes in the extension F/\mathbb{Q} , and
- $p = 2$.

Moreover, let T_S be the set of all integers whose prime divisors do not belong to S , namely

$$T_S = \{n \geq 1 : p \nmid n, \text{ for all } p \in S\}.$$

The following corollary follows directly from Theorem 4.4.

Corollary 5.7 *Let $X = E_1 \times E_2$ be the product of two elliptic curves over F . Assume that for $i = 1, 2$ the elliptic curve E_i has potentially good reduction at all finite places of F . Then, we have*

$$\prod_{v \in \Omega} \varprojlim_{n \in T_S} \frac{\overline{F^2(X_v)}}{n} = 0.$$

In particular, the following complex is exact:

$$\varprojlim_{n \in T_S} \frac{F^2(X)}{n} \xrightarrow{\Delta} \prod_{v \in \Omega} \varprojlim_{n \in T_S} \frac{\overline{F^2(X_v)}}{n} \xrightarrow{\varepsilon} \mathrm{Hom}\left(\frac{\mathrm{Br}(X)}{\mathrm{Br}_1(X)}, \mathbb{Q}/\mathbb{Z}\right).$$

Proof Fix $v \in \Omega$ and $n \in T_S$. We will show that the group $\overline{F^2(X_v)}$ is n -divisible. It is enough to show that $\overline{F^2(X_v)}$ is p -divisible, for each prime divisor p of n . By definition of the set T_S , we have $p \notin S$.

Case 1: Suppose $v \in \Omega_\infty$ is a real place of F . By our assumption, p is odd. Since $\overline{F^2(X_v)} = (\mathbb{Z}/2\mathbb{Z})^s$, for some $s \geq 0$ [4, Théorème 1.3], we conclude that $\overline{F^2(X_v)}$ is p -divisible.

Case 2: Suppose $v \in \Omega_f$ and $v \nmid p$. If X_v has good reduction, then $F^2(X_v)$ is p -divisible [38, Corollary 0.10]. If X_v has bad reduction, then it follows by Lemma 5.5 that $F^2(X_v)$ is p -divisible.

Case 3: Suppose $v \in \Omega_f$ and $v \mid p$. By $p \notin S$, the surface X_v has good reduction. Moreover, the extension F_v/\mathbb{Q}_p is unramified, and at least one of the curves E_{1v}, E_{2v} over F_v has good ordinary reduction. It then follows by Theorem 4.5 that the group $F^2(X_v)$ is p -divisible. \square

Example 5.8 Consider the product $X = E_1 \times E_2$ of the elliptic curves given by the Weierstrass equations $y^2 = x^3 + x$ and $y^2 = x^3 + 1$ respectively. The curve $E_1 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ has complex multiplication by $\mathbb{Z}[i]$, while $E_2 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ has complex multiplication by $\mathbb{Z}[\omega]$, where ω is a primitive third of unity. Then Corollary 5.7 in this case reads as follows: for every prime $p \geq 5$ such that $p \not\equiv 11 \pmod{12}$, the group $\varprojlim_n F_{\mathbb{A}}^2(X)/p^n = 0$.

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