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KUMMER COVERINGS AND SPECIALISATION

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Abstract We prove versions of various classical results on specialisation of fundamental groups in the context of log schemes in the sense of Fontaine and Illusie, generalising earlier results of Hoshi, Lepage and Orgogozo. The key technical result relates the category of finite Kummer étale covers of an fs log scheme over a complete Noetherian local ring to the Kummer étale coverings of its reduction.

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1. Introduction

1.1. Let $\widehat{A}$ be a complete Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k, and let $f : X_{\widehat{A}} \to \operatorname{Spec}(\widehat{A})$ be a proper morphism with closed fibre X_k. Then it follows from the Grothendieck existence theorem that the pullback functor

$$\text{Fét}(X_{\widehat{A}}) \to \text{Fét}(X_k)$$

is an equivalence of categories (see, e.g., [9, Exposé X, Théorème 2.1]), where for a scheme Y we write $\text{Fét}(Y)$ for the category of finite étale Y-schemes.

This was generalised to the logarithmic setting by Hoshi [10, Corollary 1, p. 83] under assumptions and by Orgogozo [24]. In this article we give a stack-theoretic proof of the logarithmic version of [9, Exposé X, Théorème 2.1] and deduce various consequences with an eye towards future applications to fundamental groups.

Let (S, M_S) be an fs log scheme with $S = \operatorname{Spec}(\widehat{A})$ a complete Noetherian local ring as before, and let $(f, f^b) : (X, M_X) \to (S, M_S)$ be a morphism of fs log schemes with underlying morphism of schemes $X \to S$ proper. The main purpose of this article is to explain how to deduce the following result – originally due to Orgogozo [24], who references ideas of Gabber – from stack-theoretic considerations:

Theorem 1.2. *The restriction functor*

$$\text{Fét}(X, M_X) \to \text{Fét}(X_k, M_{X_k}) \tag{1.2.1}$$

is an equivalence of categories, where for an fs log scheme (Y, M_Y) *we write* $\text{Fét}(Y, M_Y)$ *for the category of fs log schemes over* (Y, M_Y) *which are finite and Kummer étale.*

Remark 1.3. The log structure M_S on S plays no role in the statement of Theorem 1.2, and there is no loss of generality in assuming that $M_S = \mathscr{O}_S^*$.

Remark 1.4. Using Artin approximation, we also prove a variant of Theorem 1.2 with $\widehat{A}$ replaced by a henselian local ring; see 5.1. This variant over a Henselian local ring has also been obtained by Lepage [16, Theorem 1.8].

Using Theorem 1.2, we generalise two classical results on fundamental groups to log schemes.

1.5. For a Noetherian fs log scheme (Y, M_Y), the category $\text{Fét}(Y, M_Y)$ is a Galois category by [10, Theorem B.1]. We can therefore talk about an object $(U, M_U) \in \text{Fét}(Y, M_Y)$ being Galois. For a prime p, let $\text{Fét}^{(p)}(Y, M_Y) \subset \text{Fét}(Y, M_Y)$ be the full subcategory of objects which can be written as quotients $(U, M_U)/H$ for some Galois object (U, M_U) of degree (a locally constant function on Y) prime to p and H a finite group of automorphisms of (U, M_U) over (Y, M_Y). More generally, for a set of primes $\mathbf{L}$ we can consider the category $\text{Fét}^{\mathbf{L}}(Y, M_Y)$, defined to be the intersection of the categories $\text{Fét}^{(p)}(Y, M_Y)$ for $p \in \mathbf{L}$.

1.6. Let (B, M_B) be an fs log scheme and let $(f, f^b) : (X, M_X) \to (B, M_B)$ be a morphism of fs log schemes with underlying morphism $f : X \to B$ proper. For a log geometric point $\bar{b}^{\log} = (\bar{b}, M_{\bar{b}^{\log}}) \to (B, M_B)$, we consider the following category, introduced by Hoshi in [10]:

$$\text{Fét}((X, M_X)_{(\bar{b}^{\log})}) := \text{colim}_\lambda \ \text{Fét}((X, M_X) \times_{(B, M_B)} \bar{b}_\lambda^{\log}), \tag{1.6.1}$$

where the colimit is taken over fs log structures $M_\lambda \subset M_{\bar{b}^{\log}}$ containing the image of $M_{B, \bar{b}}$ and we write $\bar{b}_\lambda^{\log}$ for the log scheme $(\bar{b}, M_\lambda)$ (here and throughout this article, fibre products are taken in the category of fs log schemes). This category should be viewed as the category of covers of the fibre of (X, M_X) over $\bar{b}^{\log}$, though this does not make literal sense, because $M_{\bar{b}^{\log}}$ is not fine.

Theorem 1.7. *Let* $h : \bar{b}'^{\log} \to \bar{b}^{\log}$ *be a morphism of log geometric points over* (B, M_B)*. Then the pullback functor*

$$\text{Fét}^{(p)}((X, M_X)_{(\bar{b}^{\log})}) \to \text{Fét}^{(p)}((X, M_X)_{(\bar{b}'^{\log})}) \tag{1.7.1}$$

is an equivalence of categories, where p *is the residue characteristic of* $\bar{b}^{\log}$*.*

Remark 1.8. Lepage [16, Theorem 2.15] earlier obtained this result for saturated morphisms $(X, M_X) \to (B, M_B)$ and morphisms h which are isomorphisms on underlying geometric points using a different argument, which holds also for non-proper morphisms and without the prime-to-p assumption.

A second application concerns the variation of the category of covers of the fibre in log smooth proper families:

Theorem 1.9. *Let (B, M_B) be an fs log scheme with B connected and of finite type over a field or excellent Dedekind ring. Let (X, M_X) be an fs log scheme and $f : (X, M_X) \to (B, M_B)$ a log smooth morphism with underlying morphism of schemes proper. Then for any two log geometric points*

$$\bar{b}_i^{\log} \to (B, M_B), \quad i = 1, 2,$$

the categories $\mathrm{Fét}^{\mathbf{L}}((X, M_X)_{(\bar{b}_1^{\log})})$ *and* $\mathrm{Fét}^{\mathbf{L}}((X, M_X)_{(\bar{b}_2^{\log})})$ *are equivalent, where* $\mathbf{L}$ *is the set of residue characteristics of* $\bar{b}_1^{\log}$ *and* $\bar{b}_2^{\log}$*.*

Remark 1.10. This result was previously obtained by Lepage for "proper polystable log fibrations" [17, Theorem 3.3].

Remark 1.11. In the proof of Theorem 1.9 we will also make explicit how to relate the two categories using specialisation and cospecialisation functors.

Remark 1.12. As in the classical case [9, Exposé X, Corollaire 3.9], the prime-to-p assumptions in Theorem 1.9 are necessary and arise in the proofs with the applications of the purity theorem, which in the logarithmic context is [18, Theorem 3.3].

Remark 1.13. In the analytic context, the analogue of Theorem 1.9 for exact morphisms follows from the stronger topological results proven by Nakayama and Ogus in [20, Theorem 5.1].

Remark 1.14. Mattia Talpo suggested an alternate proof of Theorem 1.2 based on his result with Vistoli [27, Theorem 6.22]. The basic idea is to show that the category $\mathrm{Fét}(X, M_X)$ is equivalent to the category of finite étale covers of the infinite root stack associated to (X, M_X), and this latter category is then equivalent to the colimit of the categories of finite étale covers of the finite-level root stacks. The technology of infinite root stacks may well be the "correct" language for proving Theorem 1.2, but in this article we choose to develop the stack-theoretic tools needed directly. However, the reader familiar with [27] may in places find more direct proofs of some of the technical results using that theory.

Example 1.15. An interesting example to consider with regard to Theorem 1.9 is the case when (B, M_B) is log smooth over a field k, and (X, M_X) is a log blowup of (B, M_B) with respect to a coherent sheaf of ideals. In this case the morphism $(X, M_X) \to (B, M_B)$ is an isomorphism over a dense open subset of B, and therefore Theorem 1.9 includes the statement that the geometric fundamental group of the fibre of a log blowup is trivial. This result is already known by work of Fujiwara and Kato [11, Theorem 6.10], and is in fact crucial for the argument in this article. Since a proof of this result is not published, we provide one in Section 9.

1.16. Conventions

We assume that the reader is familiar with the basics of log geometry as developed in [13] and algebraic stacks as developed in [15].

Throughout this article, we consider only fine saturated (abbreviated “fs”) log schemes, and fibre products of log schemes are always considered in this category.

We use the notion of the *log geometric point* of an fs log scheme (X, M_X), introduced in [19, Definition 2.5]. Recall from that reference that this a morphism of log schemes $\bar{b}^{\log} = (\bar{b}, M_{\bar{b}^{\log}}) \to (X, M_X)$, where $\bar{b}$ is the spectrum of a separably closed field k and $M_{\bar{b}^{\log}}$ is an integral log structure such that for every integer $n > 0$ prime to $\mathrm{char}(k)$, the multiplication-by-n map on $\overline{M}_{\bar{b}^{\log}}$ is bijective. A morphism of log geometric points of (X, M_X) is defined to be a morphism of log schemes over (X, M_X).

If k is a separably closed field of characteristic p (possibly 0) and P is a sharp fs monoid, then we can consider the monoid $P_{\mathbb{Z}_p}$, defined to be the saturation of P inside $P^{\mathrm{gp}} \otimes \mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)}$ is the localisation of $\mathbb{Z}$ away from p. Writing simply $k^* \oplus P$ for the log structure on $\mathrm{Spec}(k)$ given by the map $k^* \oplus P \to k$ sending all nonzero elements of P to 0, we get for any morphism of log schemes $(\mathrm{Spec}(k), k^* \oplus P) \to (X, M_X)$ a log geometric point of (X, M_X) by considering the induced morphism

$$(\mathrm{Spec}(k), k^* \oplus P_{\mathbb{Z}_{(p)}}) \to (X, M_X).$$

Every log geometric point of (X, M_X) can be written as a limit of log geometric points of this form. Indeed, for any log geometric point $\bar{b}^{\log} = (\bar{b}, M_{\bar{b}^{\log}}) \to (X, M_X)$, write $\overline{M}_{\bar{b}^{\log}} = \mathrm{colim}_\lambda P_\lambda$, where P_λ is a sharp fs submonoid of $\overline{M}_{\bar{b}^{\log}}$ containing the image of $\overline{M}_X$. Then for each λ, the inclusion $P_\lambda \hookrightarrow \overline{M}_{\bar{b}^{\log}}$ extends uniquely to an inclusion $P_{\lambda, \mathbb{Z}_{(p)}} \hookrightarrow \overline{M}_{\bar{b}^{\log}}$. Let M_λ denote the preimage in $M_{\bar{b}^{\log}}$ of $P_{\lambda, \mathbb{Z}_{(p)}}$. Then $(\mathrm{Spec}(k), M_\lambda)$ is noncanonically isomorphic to $(\mathrm{Spec}(k), k^* \oplus P_{\mathbb{Z}_{(p)}})$ and

$$\bar{b}^{\log} = \lim_{\lambda} (\mathrm{Spec}(k), M_\lambda)$$

in the category of log geometric points over (X, M_X) as well as the category of log schemes.

Similarly, for any morphism of log geometric points $f : \bar{b}'^{\log} \to \bar{b}^{\log}$ of (X, M_X) with underlying morphism of schemes an isomorphism, we can present the log geometric points as limits

$$\bar{b}'^{\log} = \lim_{\lambda} (\mathrm{Spec}(k), M'_\lambda), \quad \bar{b}^{\log} = \lim_{\lambda} (\mathrm{Spec}(k), M_\lambda),$$

where $M'_\lambda \simeq k^* \oplus Q_{\lambda, \mathbb{Z}_{(p)}}$ and $M_\lambda \simeq k^* \oplus P_{\lambda, \mathbb{Z}_{(p)}}$ for fs monoids P_λ and Q_λ, and f is induced by morphisms

$$f_\lambda : (\mathrm{Spec}(k), k^* \oplus Q_\lambda) \to (\mathrm{Spec}(k), k^* \oplus P_\lambda)$$

defined by maps of monoids $h_\lambda : P_\lambda \to Q_\lambda$.

We will use the Kummer étale site and topos of a fs log scheme. We refer to the survey article [11] and references therein for basics on the Kummer étale topology.

2. Kummer coverings and root stacks

2.1. Recall from [19, Definition 2.1.2] that a morphism of fs log schemes $f : (Y, M_Y) \to (X, M_X)$ is *of Kummer type* if for every geometric point $\bar{y} \to Y$ the induced map

$$h : \overline{M}_{X, f(\bar{y})} \to \overline{M}_{Y, \bar{y}}$$

is injective and for every element $\bar{m} \in \overline{M}_{Y,\bar{y}}$ there exists an integer $N > 0$ such that $N\bar{m}$ is in the image of h. Note that such a morphism is exact.

If Y is quasi-compact then, since $\overline{M}_X$ is constructible, this implies that there exists an integer $N > 0$ such that the map of sheaves

$$\cdot N : f^{-1}\overline{M}_X \to f^{-1}\overline{M}_X$$

given by multiplication by N factors as

$$f^{-1}\overline{M}_X \xrightarrow{f^b} \overline{M}_Y \xrightarrow{f^\sharp_N} f^{-1}\overline{M}_X \tag{2.1.1}$$

for a morphism of sheaves of monoids $f^\sharp_N$, as indicated. Note also that we have a commutative diagram

$$\begin{array}{ccccc} f^{-1}\overline{M}_X & \xrightarrow{f^b} & \overline{M}_Y & \xrightarrow{f^\sharp_N} & f^{-1}\overline{M}_X \\ \downarrow & & \downarrow & & \downarrow \\ f^{-1}\overline{M}^{\mathrm{gp}}_X \otimes \mathbb{Q} & \xrightarrow{f^{b,\mathrm{gp}}} & \overline{M}^{\mathrm{gp}}_Y \otimes \mathbb{Q} & \xrightarrow{f^{\sharp,\mathrm{gp}}_N} & f^{-1}\overline{M}^{\mathrm{gp}}_X \otimes \mathbb{Q}, \end{array}$$

where the vertical morphisms are injective, since the sheaves of monoids are saturated and the morphisms $f^{b,\mathrm{gp}}$ and $f^{\sharp,\mathrm{gp}}_N$ are isomorphisms. From this it follows that f^b and $f^\sharp_N$ are both injective, and we can view $\overline{M}_Y$ as being contained in $\frac{1}{N}f^{-1}\overline{M}_X$ inside $(f^{-1}\overline{M}^{\mathrm{gp}}_X) \otimes \mathbb{Q}$.

2.2. Fix a morphism $f : (Y, M_Y) \to (X, M_X)$ of Kummer type with Y quasi-compact, and let N be a positive integer such that we have a factorisation (2.1.1). We can then describe the (X, M_X)-log scheme (Y, M_Y) as follows, using just (X, M_X) and certain morphisms of stacks.

Let $\mathscr{Y}$ denote the stack over Y which to any Y-scheme $g : T \to Y$ associates the groupoid of morphisms of fs log structures

$$u : g^*M_Y \to M_T$$

such that there exists an isomorphism $\eta : g^{-1}f^{-1}\overline{M}_X \to \overline{M}_T$ such that the diagram

$$\begin{array}{ccc} g^{-1}\overline{M}_Y & \xrightarrow{f^\sharp_N} & g^{-1}f^{-1}\overline{M}_X \\ & {\scriptstyle u} \searrow & \downarrow {\scriptstyle \eta} \\ & & \overline{M}_T \end{array}$$

commutes. Note that the isomorphism η is unique if it exists, since $\overline{M}_T$ is torsion free.

Taking $(Y, M_Y) = (X, M_X)$, we also get a stack $\mathscr{X}_N$ classifying morphisms of log structures $M_X \to M$ such that the induced map $\overline{M}_X \to \overline{M}$ identifies $\overline{M}$ with $\frac{1}{N}\overline{M}_X$ inside $\overline{M}^{\mathrm{gp}}_X \otimes \mathbb{Q}$.

The morphism f induces a functor

$$q : \mathscr{Y} \to \mathscr{X}_N.$$

Given a Y-scheme $g: T \to Y$ and an object $u: g^*M_Y \to M_T$, the X-scheme $f \circ g: T \to X$ and the morphism of log structures

$$g^*f^*M_X \xrightarrow{g^*f^b} g^*M_Y \xrightarrow{u} M_T$$

is an object of $\mathscr{X}_N(T)$ and this defines q.

Remark 2.3. This construction is a special case of the general construction of root stacks discussed in [4, §4.2, especially Proposition 4.13].

2.4. For later use, let us explicate the local structure of these stacks, which implies in particular that they are algebraic stacks, and even tame stacks in the sense of [1, Definition 3.1].

Let $\bar{y} \to Y$ be a geometric point with image $\bar{x} \to X$. Let P (resp. Q) denote the monoid $\overline{M}_{X,\bar{x}}$ (resp. $\overline{M}_{Y,\bar{y}}$), so we have a morphism of monoids $\theta: P \to Q$. By our assumptions we also have a morphism of monoids $\theta^\sharp: Q \to P$ such that the composition

$$P \xrightarrow{\theta} Q \xrightarrow{\theta^\sharp} P$$

is multiplication by N. Let $\boldsymbol{\mu}_{P/Q}$ denote the diagonalisable group scheme associated to the quotient $P^{\text{gp}}/Q^{\text{gp}}$, where Q^{gp} is included in P^{gp} by $\theta^\sharp$. Similarly, define $\boldsymbol{\mu}_{Q/P}$ to be the diagonalisable group scheme associated to $Q^{\text{gp}}/\theta(P^{\text{gp}})$.

Lemma 2.5. *After replacing X by an étale neighbourhood of $\bar{x}$ and Y by an fppf neighbourhood of $\bar{y}$, we can find a commutative diagram*

$$\begin{array}{ccc} Q & \xrightarrow{\alpha_Y} & M_Y(Y) \\ \uparrow{\scriptstyle\theta} & & \uparrow{\scriptstyle f^b} \\ P & \xrightarrow{\alpha_X} & M_X(X), \end{array} \tag{2.5.1}$$

where α_X and α_Y are charts inducing the given identifications $Q \simeq \overline{M}_{Y,\bar{y}}$ and $P \simeq \overline{M}_{X,\bar{x}}$. If N is invertible in $k(\bar{y})$, we can find such charts étale locally on Y.

Proof. This is very similar to [23, Proposition 2.1]. First, after replacing X and Y by étale neighbourhoods, we can find charts α_X and α_Y inducing the isomorphisms $Q \simeq \overline{M}_{Y,\bar{y}}$ and $P \simeq \overline{M}_{X,\bar{x}}$. Diagram (2.5.1) may not commute: the failure is measured by a homomorphism $h: P^{\text{gp}} \to \mathscr{O}_Y^*$. The obstruction to extending this to a homomorphism $\tilde{h}: Q^{\text{gp}} \to \mathscr{O}_Y^*$ is a class in $\text{Ext}^1(Q^{\text{gp}}/P^{\text{gp}}, \mathscr{O}_Y^*)$. Therefore, after replacing Y by an fppf neighbourhood of $\bar{y}$, we can lift h to a homomorphism $\tilde{h}: Q^{\text{gp}} \to \mathscr{O}_Y^*$, and if N is invertible in Y we can do so étale locally. Modifying our chart α_Y by this homomorphism, we then have that (2.5.1) commutes. □

2.6. Localising, we now assume as chosen such charts α_X and α_Y. In this case, for any Y-scheme $g: T \to Y$ and object $u: g^*M_Y \to M_T$ there exists a unique extension $\beta_T: P \to \overline{M}_T$ of the composition

$$Q \xrightarrow{\bar{\alpha}_Y} g^{-1}\overline{M}_Y \xrightarrow{\bar{u}} \overline{M}_T,$$

and the map β_T lifts étale locally on T to a chart. This extension is defined to be the composition

$$P \xrightarrow{\bar{\alpha}_X} g^{-1}f^{-1}\overline{M}_X \xrightarrow{\bar{\eta}} \overline{M}_T.$$

From this and [23, Proposition 5.20], we deduce that the stack $\mathscr{Y}$ in this local situation is described as the stack quotient

$$[\underline{\mathrm{Spec}}_Y(\mathscr{O}_Y \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P])/\boldsymbol{\mu}_{P/Q}],$$

where the action is induced by the natural action of $\boldsymbol{\mu}_{P/Q}$ on $\mathrm{Spec}(\mathbb{Z}[P])$ over $\mathrm{Spec}(\mathbb{Z}[Q])$.

Similarly, we have a description of $\mathscr{X}_N$ as the quotient

$$[\underline{\mathrm{Spec}}_X(\mathscr{O}_X \otimes_{\mathbb{Z}[P],\cdot N} \mathbb{Z}[P])/\boldsymbol{\mu}_{P,N}],$$

where $\boldsymbol{\mu}_{P,N}$ denotes the diagonalisable group scheme associated to $P^{\mathrm{gp}} \otimes \mathbb{Z}/(N)$.

The morphism $q : \mathscr{Y} \to \mathscr{X}_N$ is the morphism of stacks induced by the natural map

$$\underline{\mathrm{Spec}}_Y(\mathscr{O}_Y \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]) \to \underline{\mathrm{Spec}}_X(\mathscr{O}_X \otimes_{\mathbb{Z}[P],\cdot N} \mathbb{Z}[P])$$

by taking quotients. Note here that there is a natural inclusion $\boldsymbol{\mu}_{P/Q} \hookrightarrow \boldsymbol{\mu}_{P,N}$.

2.7. From this we can read off a number of properties of the stacks $\mathscr{Y}$ and $\mathscr{X}_N$:

(i) The stack $\mathscr{Y}$ (resp. $\mathscr{X}_N$) is a tame algebraic stack with finite diagonal over Y (resp. X). Furthermore, if N is invertible on Y (resp. X), then $\mathscr{Y}$ (resp. $\mathscr{X}_N$) is Deligne–Mumford.

(ii) The coarse space of $\mathscr{Y}$ (resp. $\mathscr{X}_N$) is Y (resp. X).

(iii) The morphism $q : \mathscr{Y} \to \mathscr{X}_N$ is representable and finite if the morphism $Y \to X$ is finite.

Remark 2.8. Note that because $\mathscr{Y}$ and $\mathscr{X}_N$ are tame stacks, the formation of their coarse spaces commutes with arbitrary base change by [1, Theorem 3.2].

2.9. Let $p_X : \mathscr{X}_N \to X$ and $p_Y : \mathscr{Y} \to Y$ be the projections. We have tautological morphisms of log structures $p_X^* M_X \to M_{\mathscr{X}_N}$ and $p_Y^* M_Y \to M_{\mathscr{Y}}$ and an isomorphism $q^* M_{\mathscr{X}_N} \simeq M_{\mathscr{Y}}$. So we have a commutative square of log stacks

$$\begin{array}{ccc} (\mathscr{Y}, M_{\mathscr{Y}}) & \xrightarrow{q} & (\mathscr{X}_N, M_{\mathscr{X}_N}) \\ \downarrow{\scriptstyle p_Y} & & \downarrow{\scriptstyle p_X} \\ (Y, M_Y) & \xrightarrow{f} & (X, M_X), \end{array} \tag{2.9.1}$$

where the morphism q is strict.

Remark 2.10. The vertical morphisms in (2.9.1) are log étale. This follows from [23, Corollary 5.24].

Lemma 2.11. *The map $M_Y \to p_{Y*}^{\log} M_{\mathscr{Y}}$ is an isomorphism, where $p_{Y*}^{\log}$ denotes the pushforward in the category of log structures.*

Proof. Notice that the sheaf $\overline{M}_{\mathscr{X}_N}$ is isomorphic to $p_X^{-1}\overline{M}_X$, and therefore $\overline{M}_{\mathscr{Y}}$ descends to Y. In fact, $\overline{M}_{\mathscr{Y}} \simeq p_Y^{-1}f^{-1}\overline{M}_X$.

Since Y is the coarse moduli space of $\mathscr{Y}$, we have

$$p_{Y*}^{\log}M_{\mathscr{Y}} = p_{Y*}M_{\mathscr{Y}},$$

where the right side is the pushforward in the category of sheaves. Indeed, by definition, $p_{Y*}^{\log}M_{\mathscr{Y}}$ is the fibre product of the diagram

$$\begin{array}{ccc} & & \mathscr{O}_Y \\ & & \downarrow \simeq \\ p_{Y*}M_Y & \longrightarrow & p_{Y*}\mathscr{O}_{\mathscr{Y}}, \end{array}$$

where the vertical morphism is an isomorphism. For ease of notation, we write M'_Y for the log structure $p_{Y*}^{\log}M_{\mathscr{Y}}$. Note that by the left exactness of p_{Y*}, the natural map

$$\overline{M}'_Y \simeq (p_{Y*}M_{\mathscr{Y}})/(p_{Y*}\mathscr{O}^*_{\mathscr{Y}}) \to p_{Y*}\overline{M}_{\mathscr{Y}}$$

is injective. We therefore have an inclusion

$$\overline{M}'_Y \hookrightarrow p_{Y*}p_Y^{-1}f^{-1}\overline{M}_X = f^{-1}\overline{M}_X,$$

and it suffices to show that this map identifies $\overline{M}'_Y$ with $\overline{M}_Y$. Here we use the fact that the map $\overline{M}_X \to p_{Y*}p_Y^{-1}\overline{M}_X$ is an isomorphism, which follows for example from the proper base change theorem [3, Expose XII, Theorem 5.1].

Now to prove the lemma, we can work locally in the fppf topology on Y. Let $\bar{y} \to Y$ be a geometric point with image $\bar{x} \to X$, and choose charts as in Lemma 2.5, after possibly shrinking on X and Y in the fppf topology. Write $\mathscr{Y}_{(\bar{y})} := \mathscr{Y} \times_Y \operatorname{Spec}(\mathscr{O}_{Y,\bar{y}})$ and let $m \in \overline{M}_{X,\bar{x}} = P$ be a section. Then m lifts to $(p_{Y*}M_{\mathscr{Y}})_{\bar{y}}$ if and only if the $\mathscr{O}^*_{\mathscr{Y}_{(\bar{y})}}$-torsor $\mathscr{L}_m$ on $\mathscr{Y}_{(\bar{y})}$ of liftings of m to $M_{\mathscr{Y}_{(\bar{y})}}$ is trivial. Indeed, we have $(p_{Y*}M_{\mathscr{Y}})_{\bar{y}} = H^0(\mathscr{Y}_{(\bar{y})}, M_{\mathscr{Y}_{(\bar{y})}})$, and the (possibly empty) set of trivialisations of $\mathscr{L}_m$ is precisely the set of global sections of $M_{\mathscr{Y}_{(\bar{y})}}$ mapping to m. Now observe that with the description of $\mathscr{Y}$ in Paragraph 2.6, the closed point defines a closed immersion

$$B\boldsymbol{\mu}_{P/Q} \hookrightarrow \mathscr{Y}_{(\bar{y})}.$$

The pullback of $\mathscr{L}_m$ to $B\boldsymbol{\mu}_{P/Q}$ is the torsor corresponding to the character of $\boldsymbol{\mu}_{P/Q}$ defined by m. It follows that a necessary condition for $\mathscr{L}_m$ to be trivial is that m lie in $Q = Q^{\mathrm{gp}} \cap P \subset P^{\mathrm{gp}}$ (using the fact that Kummer morphisms are exact). It follows that $M_Y \to p_{Y*}M_{\mathscr{Y}}$ is surjective and therefore an isomorphism. □

2.12. Adding to the list in Paragraph 2.7:

(iv) The morphism of log schemes $f : (Y, M_Y) \to (X, M_X)$ is obtained from the data of the representable morphism of stacks $q : \mathscr{Y} \to \mathscr{X}_N$ and the morphism of log stacks $(\mathscr{X}_N, M_{\mathscr{X}_N}) \to (X, M_X)$ by taking Y the coarse moduli space of $\mathscr{Y}$ and M_Y the pushforward $p_{Y*}q^*M_{\mathscr{X}_N}$.

2.13. Conversely, we can try to reverse the preceding constructions to get a Kummer morphism from a representable morphism of stacks. Fix a representable morphism of stacks $\mathscr{Y} \to \mathscr{X}_N$ separated over X and let $Y \to X$ be the coarse moduli space and M_Y the pushforward log structure $p^{\log}_{Y*} q^* M_{\mathscr{X}_N}$. Note that a priori it is not clear what good properties (e.g., fine and/or saturated) the log structure $p^{\log}_{Y*} q^* M_{\mathscr{X}_N}$ posseses.

Let $\bar{y} \to Y$ be a geometric point with image $\bar{x} \to X$. Then, following the foregoing arguments, the stalk $\overline{M}_{Y,\bar{y}}$ can be described as follows. Let $\mathscr{Y}_{(\bar{y})}$ be the fibre product $\mathscr{Y} \times_Y \mathrm{Spec}(\mathscr{O}_{Y,\bar{y}})$. Let P denote the stalk $\overline{M}_{X,\bar{x}}$. Let $\boldsymbol{\mu}$ denote the stabiliser group scheme of $\mathscr{Y}$ over $\bar{y}$, and let $\boldsymbol{\mu}_{P,N}$ denote the diagonalisable group scheme

$$\mathrm{Hom}(P^{\mathrm{gp}}, \boldsymbol{\mu}_N).$$

Then $\boldsymbol{\mu}_{P,N}$ is the stabiliser group scheme of $\mathscr{X}_N$ at $\bar{x}$, and since $\mathscr{Y} \to \mathscr{X}_N$ is representable we have a closed immersion $\boldsymbol{\mu} \hookrightarrow \boldsymbol{\mu}_{P,N}$. Since a closed subgroup scheme of a diagonalisable group scheme is again diagonalisable, the group scheme $\boldsymbol{\mu}$ is also diagonalisable, and the inclusion $\boldsymbol{\mu} \hookrightarrow \boldsymbol{\mu}_{P,N}$ corresponds to a quotient

$$P^{\mathrm{gp}}/NP^{\mathrm{gp}} \to A,$$

or equivalently a subgroup $Q^{\mathrm{gp}} \subset P^{\mathrm{gp}}$ containing NP^{gp}. Let $Q \subset P$ denote $P \cap Q^{\mathrm{gp}}$.

Lemma 2.14. *We have $\overline{M}_{Y,\bar{y}} = Q$ inside $\overline{M}_{X,\bar{x}} = P$.*

Proof. With notation as in the proof of Lemma 2.11, let $m \in P$ be a section and let $\mathscr{L}_m$ denote the $\mathscr{O}^*_{\mathscr{Y}}$-torsor over $\mathscr{Y}_{(\bar{y})}$ of liftings of m to a section of $M_{\mathscr{Y}} := q^* M_{\mathscr{X}_N}$. Then $\mathscr{L}_m^{\otimes N}$ is trivial, and fixing a trivialisation we get a $\boldsymbol{\mu}_N$-torsor $\widetilde{\mathscr{L}}_m$ whose pushout along $\boldsymbol{\mu}_N \subset \mathscr{O}^*_{\mathscr{Y}}$ is the torsor $\mathscr{L}_m$. The result then follows from the following lemma, where we use the natural identifications of $H^1(B\boldsymbol{\mu}, \boldsymbol{\mu}_N)$ and $\mathrm{Hom}(\boldsymbol{\mu}, \boldsymbol{\mu}_N) \simeq A$. □

Lemma 2.15. *Pullback to $i : B\boldsymbol{\mu} \subset \mathscr{Y}_{(\bar{y})}$ defines a bijection between $H^1(\mathscr{Y}_{(\bar{y})}, \boldsymbol{\mu}_N)$ and A.*

Proof. By the general theory of tame stacks [1, Proposition 3.6], the stack $\mathscr{Y}_{(\bar{y})}$ can be written as a quotient $[V/\boldsymbol{\mu}]$, where V is a finite $\mathrm{Spec}(\mathscr{O}_{Y,\bar{y}})$-scheme. In particular, there is a retraction $r : \mathscr{Y}_{(\bar{y})} \to B\boldsymbol{\mu}$ of i. It follows that the restriction map

$$H^1(\mathscr{Y}_{(\bar{y})}, \boldsymbol{\mu}_N) \to H^1(B\boldsymbol{\mu}, \boldsymbol{\mu}_N) = A$$

is surjective. To prove injectivity it suffices to show that a $\boldsymbol{\mu}_N$-torsor $\mathscr{L}$ which pulls back to the trivial torsor over $B\boldsymbol{\mu}$ is trivial. Using the Artin approximation theorem, it suffices to prove that such a torsor is trivial after base change to the completion $\mathrm{Spec}(\widehat{\mathscr{O}}_{Y,\bar{y}})$, and then by the Grothendieck existence theorem it suffices to show that each of the reductions of $\mathscr{L}$ to infinitesimal neighbourhoods of $B\boldsymbol{\mu}$ is trivial. Now the deformation theory of $\boldsymbol{\mu}_N$-torsors is governed by the groups $H^i(B\boldsymbol{\mu}, \mathrm{Lie}(\boldsymbol{\mu}_N))$. These groups vanish for $i > 0$, since $\boldsymbol{\mu}$ is linearly reductive, which proves the lemma. □

2.16. To further understand the log structure M_Y, we analyse the situation locally. Replacing Y by some étale neighbourhood of $\bar{y}$, we can find a morphism $\boldsymbol{\alpha}_Y : Q \to \overline{M}_Y$

extending the given isomorphism $Q \simeq \overline{M}_{Y,\bar{y}}$. Furthermore, as in the proof of Lemma 2.5, we can arrange it so that we have a chart $\alpha_X : P \to M_X$ inducing the given isomorphism $P \simeq \overline{M}_{X,\bar{x}}$ and that (2.5.1) commutes. Let M'_Y be the log structure associated to $Q \to M_Y \to \mathscr{O}_Y$, so we have an induced morphism of log structures $M'_Y \to M_Y$. We can then apply the construction of Paragraph 2.2 to get another stack $\mathscr{Y}' \to Y$, and the natural map $M'_Y \to q^* M_{\mathscr{X}_N}$ induces a morphism of stacks

$$g : \mathscr{Y} \to \mathscr{Y}'.$$

This is a morphism of algebraic stacks proper and quasi-finite over Y with finite diagonal equipped with representable morphisms to $\mathscr{X}_N$. This implies that g is a representable morphism and therefore a finite morphism.

Proposition 2.17. *Suppose that for every integer m, the base change of g to* $\mathrm{Spec}(\mathscr{O}_{Y,\bar{y}}/\mathfrak{m}^m)$ *is an isomorphism. Then there exists an étale neighbourhood of $\bar{y}$ over which g is an isomorphism and $M'_Y \to M_Y$ is an isomorphism.*

Proof. Since g is finite, we have $\mathscr{Y} = \underline{\mathrm{Spec}}_{\mathscr{Y}'}(g_* \mathscr{O}_{\mathscr{Y}})$. Under the assumptions in the proposition, we have that the map of coherent $\mathscr{O}_{\mathscr{Y}'}$-modules

$$\mathscr{O}_{\mathscr{Y}'} \to g_* \mathscr{O}_{\mathscr{Y}} \tag{2.17.1}$$

pulls back to an isomorphism over each of the $\mathrm{Spec}(\mathscr{O}_{Y,\bar{y}}/\mathfrak{m}^m)$. By the Grothendieck existence theorem, it follows that (2.17.1) is an isomorphism in a neighbourhood of $\bar{y}$, which implies the first statement in the proposition. The statement that $M'_Y \to M_Y$ is an isomorphism over this étale neighbourhood follows from Lemma 2.11. □

Remark 2.18. The formation of the stack $\mathscr{Y}$ is functorial in the log scheme (Y, M_Y). If $g : (Y, M_Y) \to (Y', M_{Y'})$ is a morphism over (X, M_X) between Kummer (X, M_X)-log schemes, then there is an induced morphism of stacks

$$\tilde{g} : \mathscr{Y} \to \mathscr{Y}'$$

over $\mathscr{X}_N$, where N is chosen appropriately. The fibre of this morphism over $h : T \to Y$ is given by sending a morphism of log structures $h^* M_Y \to M_T$ defining an object of $\mathscr{Y}(T)$ to the composition

$$h^* g^* M_{Y'} \to h^* M_Y \to M_T,$$

which is an object of $\mathscr{Y}'(T)$.

2.19. We are particularly interested in the case when $(Y, M_Y) \to (X, M_X)$ is Kummer étale. In this case, consideration of (2.9.1) shows that the morphism of log stacks $(\mathscr{Y}, M_{\mathscr{Y}}) \to (\mathscr{X}_N, M_{\mathscr{X}_N})$ is strict and log étale, and therefore the underlying morphism of stacks $\mathscr{Y} \to \mathscr{X}_N$ is representable and étale.

If, furthermore, the underlying morphism $Y \to X$ is finite, then the representable morphism $\mathscr{Y} \to \mathscr{X}_N$ is also proper and quasi-finite, and therefore a finite étale morphism.

2.20. We can use this to understand morphisms of log étale schemes better. Let

$$(Y_i, M_i) \to (X, M_X), \quad i = 1, 2,$$

be two finite Kummer étale morphisms with associated finite étale morphisms $\mathscr{Y}_i \to \mathscr{X}_N$.

Let $\mathscr{H}om_{\mathscr{X}_N}(\mathscr{Y}_1, \mathscr{Y}_2)$ be the functor over $\mathscr{X}_N$ which associates to any morphism $T \to \mathscr{X}_N$ the set of morphisms

$$\mathscr{Y}_1 \times_{\mathscr{X}_N} T \to \mathscr{Y}_2 \times_{\mathscr{X}_N} T.$$

Since $\mathscr{Y}_1$ and $\mathscr{Y}_2$ are finite étale over $\mathscr{X}_N$, this functor is representable by a finite étale morphism $h : \mathscr{H} \to \mathscr{X}_N$. Likewise, define

$$\mathscr{H}om_{(X, M_X)}((Y_1, M_{Y_1}), (Y_2, M_{Y_2}))$$

to be the functor on the category of X-schemes which to any $f : T \to X$ associates the set of morphisms of log schemes

$$r : (Y_1, M_{Y_1}) \times_{(X, M_X)} (T, f^*M_X) \to (Y_1, M_{Y_1}) \times_{(X, M_X)} (T, f^*M_X) \tag{2.20.1}$$

over (T, f^*M_X).

Proposition 2.21. *The functor $\mathscr{H}om_{(X, M_X)}((Y_1, M_{Y_1}), (Y_2, M_{Y_2}))$ is representable by a scheme finite and étale over X.*

Proof. By the functoriality discussed in Remark 2.18, any morphism (2.20.1) defines a morphism of stacks

$$t_r : \mathscr{Y}_1 \times_{\mathscr{X}_N} T \to \mathscr{Y}_2 \times_{\mathscr{X}_N} T,$$

and conversely such a morphism of stacks defines a morphism of log schemes (2.20.1) by passing to coarse moduli spaces. In this way, the functor $\mathscr{H}om_{(X, M_X)}((Y_1, M_{Y_1}), (Y_2, M_{Y_2}))$ is identified with the functor H which to any X-scheme T associates the set of sections $s : \mathscr{X}_{N, T} \to \mathscr{H}_T$ of the base change of $\mathscr{H} \to \mathscr{X}_N$ to T. The result therefore follows from the following stack-theoretic lemma. □

Lemma 2.22. *Let $\mathscr{X}$ be a tame Artin stack with coarse moduli space $\pi : \mathscr{X} \to X$ and let $h : \mathscr{H} \to \mathscr{X}$ be a finite étale morphism. Let H be the functor on X-schemes sending a scheme T to the set of sections $s : \mathscr{X}_T \to \mathscr{H}_T$ of the base change $h_T : \mathscr{H}_T \to \mathscr{X}_T$ of h to T. Then H is representable by a scheme H finite and étale over X.*

Proof. The assertion is étale local on X, so by [1, Theorem 3.2] we may assume that $\mathscr{X} = [U/G]$, where U is a finite X-scheme and G is a finite linearly reductive group scheme acting on U over X. Let $h_U : V \to U$ be the fibre product $\mathscr{H} \times_{\mathscr{X}} U$, so h is finite and étale and there is an action of G on V such that $\mathscr{H} = [V/G]$. Then giving a section of $\mathscr{H} \to \mathscr{X}$ is equivalent to giving a G-equivariant closed subscheme $\Gamma \subset U \times_X V$ such that the projection $\Gamma \to U$ is an isomorphism.

Since U is finite over X, we can, after further localising on X, arrange it so that V is a trivial étale cover of U. Indeed, by a standard limit argument it suffices to note that if $\bar{x} \to X$ is a geometric point and $\mathscr{O}_{X,\bar{x}}$ the strictly Henselian local ring associated to $\bar{x}$, then

$$V \times_X \mathrm{Spec}(\mathscr{O}_{X,\bar{x}}) \to U \times_X \mathrm{Spec}(\mathscr{O}_{X,\bar{x}})$$

is a trivial étale covering, since $U \times_X \mathrm{Spec}(\mathscr{O}_{X,\bar{x}})$ is the spectrum of a product of strictly Henselian local rings by [26, Tag 04GH].

In this case, H is representable by the set of G-invariant sections of the projection $\pi_0(V) \to \pi_0(U)$. □

Remark 2.23. The various functors already considered can also be studied using the general methods in [22]. However, in my cases we need slightly stronger results, under stronger hypotheses, than what we get directly from the results in [22].

3. An example

To illustrate the constructions and results of the preceding section, we make them explicit in this section in the case of a log point. This will also make clear the connection with infinite root stacks in the sense of [27], and the result [27, Theorem 6.22] can be viewed as a vast generalisation of (3.2.1) constructed later.

3.1. Let (b, M_b) be a log point with $b = \mathrm{Spec}(k)$ the spectrum of a separably closed field and M_b an fs log structure. Let Q denote the monoid $\overline{M}_b$. Choosing a section of the projection $M_b \to \overline{M}_b$, we get a decomposition $M_b = k^* \oplus Q$ with the map to k given by sending all nonzero elements of Q to 0. For $N > 0$, let $\mathcal{B}_N$ denote the associated Nth root stack over $\mathrm{Spec}(k)$. The stack $\mathcal{B}_N$ can be described as the stack quotient

$$\mathcal{B}_N = [\mathrm{Spec}(k \otimes_{k[Q], N} k[Q])/\boldsymbol{\mu}_{Q,N}].$$

In particular, there is a closed immersion defined by a nilpotent ideal

$$j_N : B\boldsymbol{\mu}_{Q,N} \hookrightarrow \mathcal{B}_N.$$

This enables us to completely describe the category of finite étale covers of $\mathcal{B}_N$.

Let $U \to \mathcal{B}_N$ be a finite étale morphism with U connected. Then $U_0 := j_N^* U$ is connected and finite étale over $B\boldsymbol{\mu}_{Q,N}$ and therefore isomorphic to

$$B\boldsymbol{\mu}_{U_0} \to B\boldsymbol{\mu}_{Q,N}$$

for some closed subgroup $\boldsymbol{\mu}_{U_0} \subset \boldsymbol{\mu}_{Q,N}$. Such a subgroup is given by a quotient z: $Q^{\mathrm{gp}}/NQ^{\mathrm{gp}} \to A$. By the invariance of the étale site under infinitesimal thickenings (see, e.g., [7, Lemma 3.41]), it follows that $U \to \mathcal{B}_N$ is isomorphic to the quotient

$$[\mathrm{Spec}(k \otimes_{k[Q], N} k[Q])/\boldsymbol{\mu}_{U_0}]$$

with its natural map to $\mathcal{B}_N$. Let $Q' \subset Q$ be the set of elements $q' \in Q$ for which $z(q') = 0$. Then the log scheme obtained from U by the construction in Paragraph 2.13 is the scheme

$$\mathrm{Spec}(k \otimes_{k[Q} k[Q'])$$

with the natural log structure $M_{Q'}$ induced by Q'.

Observe also that the projection $Q^{\mathrm{gp}} \to A$ induces an isomorphism $Q^{\mathrm{gp}}/Q'^{\mathrm{gp}} \simeq A$. Indeed, it is clear that Q'^{gp} is in the kernel of the map to A, and if $q \in Q^{\mathrm{gp}}$ is an element in the kernel, then there exists $y \in Q$ such that $q' := q + Ny \in Q \cap \mathrm{Ker}(Q^{\mathrm{gp}} \to A) = Q'$. Therefore, $q = q' - Ny$ lies in Q'^{gp}.

It follows that the finite étale cover $U \to \mathcal{B}_N$ is the stack obtained by the construction of Paragraph 2.2 from the morphism of log schemes

$$(\mathrm{Spec}(k \otimes_{k[Q} k[Q']), M_{Q'}) \to (b, M_b).$$

3.2. If $N|M$, then there is a natural morphism of stacks

$$\pi_{M,N} : \mathcal{B}_M \to \mathcal{B}_N.$$

In fact, if $q_N : \mathcal{B}_N \to b$ is the structure morphism and $\alpha : q_N^* M_b \to M_{\mathcal{B}_N}$ the tautological morphism of log structures over $\mathcal{B}_N$, then we can consider the stack $\mathcal{B}_{N,M/N}$ over $\mathcal{B}_N$ classifying M/Nth roots of $M_{\mathcal{B}_N}$, and it follows immediately from the construction that composition with α defines an isomorphism of stacks

$$\mathcal{B}_{N,M/N} \to \mathcal{B}_M.$$

In particular, we obtain the projection $\pi_{M,N}$ by taking the inverse of this isomorphism followed by the projection $\mathcal{B}_{N,M/N} \to \mathcal{B}_N$. Furthermore, it follows from the preceding discussion that if $U \to \mathcal{B}_N$ is a finite étale morphism with associated Kummer morphism $(c, M_c) \to (b, M_b)$, then the base change $U \times_{\mathcal{B}_N} \mathcal{B}_M$ is the finite étale morphism over $\mathcal{B}_M$ also corresponding to (c, M_c). Since every object of $\mathrm{Fét}((b, M_b))$ is obtained by this construction for some N prime to the characteristic of k, we obtain an equivalence of categories

$$\mathrm{Fét}((b, M_b)) \simeq \mathrm{colim}_{N \in \mathbb{N}'} \mathrm{Fét}(\mathcal{B}_N), \tag{3.2.1}$$

where the colimit on the right is taken with respect to the morphisms $\pi_{M,N}$ for $N|M$ for N prime to p.

4. Proof of Theorem 1.2

4.1. We can without loss of generality assume that the log structure M_S is trivial. For an $\widehat{A}$-algebra B, write (X_B, M_{X_B}) for the base change of (X, M_X) to $\mathrm{Spec}(B)$, so $(X_{\widehat{A}}, M_{X_{\widehat{A}}}) = (X, M_X)$. Let $\mathfrak{m} \subset \widehat{A}$ be the maximal ideal, and write A_n for the quotient $\widehat{A}/\mathfrak{m}^n$. First we show the full faithfulness of (1.2.1):

Proposition 4.2. *For any two objects $(U, M_U), (V, M_V) \in \mathrm{Fét}(X_{\widehat{A}}, M_{X_{\widehat{A}}})$, the map*

$$\mathrm{Hom}_{(X_{\widehat{A}}, M_{X_{\widehat{A}}})}((U, M_U), (V, M_V)) \to \mathrm{Hom}_{(X_k, M_{X_k})}((U_k, M_{U_k}), (V_k, M_{V_k})) \tag{4.2.1}$$

is bijective.

Proof. Let $H_{\widehat{A}} \to X_{\widehat{A}}$ be the finite étale scheme classifying morphisms $(U, M_U) \to (V, M_V)$ over $(X_{\widehat{A}}, M_{X_{\widehat{A}}})$ as in Proposition 2.21. We then want to show that a section of $H_k \to X_k$ lifts uniquely to a section of $H_{\widehat{A}} \to X_{\widehat{A}}$. This follows from the fact that the reduction functor

$$\mathrm{Fét}(X_{\widehat{A}}) \to \mathrm{Fét}(X_k)$$

is an equivalence of categories [9, Exposé X, Théorème 2.1]. □

4.3. To complete the proof of Theorem 1.2, it suffices to show that every object $(U_k, M_{U_k}) \in \mathrm{Fét}(X_k, M_{X_k})$ is in the essential image of (1.2.1). To see this, recall the general fact that if $i : (T_0, M_{T_0}) \hookrightarrow (T, M_T)$ is an exact closed immersion of fine log schemes defined by a nilpotent ideal, then the reduction functor

$$\mathrm{Fét}(T, M_T) \to \mathrm{Fét}(T_0, M_{T_0})$$

is an equivalence of categories. This follows from the same argument as in the classical case [9, Exposé I, Théorème 8.3], combined with [13, Proposition 3.14]. Therefore, for each $n \geq 1$ there exists a unique lifting $(U_n, M_{U_n}) \in \mathrm{Fét}(X_{A_n}, M_{X_{A_n}})$ of (U_k, M_{U_k}), so we have a compatible system $\{(U_n, M_{U_n})\}_n$ of log schemes. Let $N > 0$ be an integer as in Paragraph 2.1. For each n we then obtain, as in Paragraph 2.9, a commutative square of log stacks

$$\begin{array}{ccc} (\mathscr{U}_n, M_{\mathscr{U}_n}) & \xrightarrow{q_n} & (\mathscr{X}_{A_n,N}, M_{\mathscr{X}_{A_n,N}}) \\ \downarrow{\scriptstyle p_{U_n}} & & \downarrow{\scriptstyle p_{X_{A_n}}} \\ (U_n, M_{U_n}) & \xrightarrow{f} & (X_{A_n}, M_{X_{A_n}}). \end{array}$$

The morphisms $q_n : \mathscr{U}_n \to \mathscr{X}_{A_n,N}$ are strict and étale, as noted in Paragraph 2.19.

By the Grothendieck existence theorem for stacks [22, Theorem A.1], the system $\{q_n : \mathscr{U}_n \to \mathscr{X}_{A_n,N}\}$ of finite étale morphisms is uniquely algebraisable to a finite étale morphism $q : \mathscr{U} \to \mathscr{X}_{\widehat{A},N}$. Let $U \to X_{\widehat{A}}$ be the coarse moduli space of $\mathscr{U}$ and let M_U be the log structure on U given by $p^{\log}_{U*} q^* M_{\mathscr{X}_{\widehat{A},N}}$, where $p_U : \mathscr{U} \to U$ is the projection. We claim that (U, M_U) is an object of $\mathrm{Fét}(X_{\widehat{A}}, M_{X_{\widehat{A}}})$ reducing to (U_k, M_{U_k}).

Note first of all that since $\mathscr{U}$ is a tame Deligne–Mumford stack, the formation of its coarse moduli space commutes with arbitrary base change [1, Corollary 3.3]. This implies that U reduces to the system $\{U_n\}$ over A_n.

The log structure M_U is an fs log structure. Indeed, since U is proper over $\widehat{A}$, it suffices to show that every geometric point $\bar{u} \to U$ of the closed fibre admits an étale neighbourhood over which M_U is fine and saturated. This follows from Proposition 2.17 and the fact that $\mathscr{U}$ reduces to $\mathscr{U}_n$ by definition.

Furthermore, the log structure M_U reduces to M_{U_n} over U_n. Indeed, there is a natural map $M_U|_{U_n} \to M_{U_n}$; to verify that this is an isomorphism, it suffices to show that for every geometric point $\bar{u} \to U$ of the closed fibre, this map induces an isomorphism $\overline{M}_{U,\bar{u}} \to \overline{M}_{U_n,\bar{u}}$. This follows from Lemma 2.14.

To complete the proof of Theorem 1.2, it now suffices to observe that the morphism $(U, M_U) \to (X_{\widehat{A}}, M_{X_{\widehat{A}}})$ is log étale and Kummer. Indeed, the locus in U where this morphism is log étale and Kummer is open (this statement for log étale morphisms follows from Kato's structure theorem [13, Theorem 3.5] and the corresponding statement for schemes [9, Exposé I, Proposition 4.5]; the statement that the Kummer type condition is open follows from the constructibility of the sheaves $\overline{M}_U$ and $\overline{M}_{X_{\widehat{A}}}$), and since $X_{\widehat{A}}$ – and therefore also U – is proper over $\mathrm{Spec}(\widehat{A})$, this open set, which contains the closed fibre, must be all of U. □

5. Artin approximation and étale covers

As in [2], Theorem 1.2 can be generalised to the case when A is not necessarily complete but Henselian. This result in the classical case is [2, Theorem 3.1], and in the log setting it is due to Lepage. For the convenience of the reader we provide a proof.

Proposition 5.1 ([16, Theorem 2.7]). *Let A be a Henselian local ring with residue field k, and let $f : (X_A, M_{X_A}) \to \mathrm{Spec}(A)$ be an fs log scheme over A with underlying morphism $X_A \to \mathrm{Spec}(A)$ proper and locally of finite presentation. Then the pullback functor*

$$\mathrm{Fét}(X_A, M_{X_A}) \to \mathrm{Fét}(X_k, M_{X_k}) \tag{5.1.1}$$

is an equivalence of categories.

Proof. By a standard reduction, as in the proof of [2, Theorem 3.1], it suffices to consider the case when A is the strict Henselisation of a finite type affine $\mathbb{Z}$-scheme $\mathrm{Spec}(S)$, and (X_A, M_{X_A}) is obtained by base change from a morphism of fs log schemes $f : (X, M_X) \to \mathrm{Spec}(S)$ with underlying morphism of schemes proper. Let k denote the residue field and let $\widehat{A}$ denote the completion of A. Let $\mathfrak{m} \subset \widehat{A}$ denote the maximal ideal, and for an integer $n \geq 1$ let A_n denote the quotient $\widehat{A}/\mathfrak{m}^n$, so $A_1 = k$.

For an S-algebra B, write (X_B, M_{X_B}) for the base change of (X, M_X) to $\mathrm{Spec}(B)$. By Theorem 1.2, it then suffices to show that the functor

$$\mathrm{Fét}(X_A, M_{X_A}) \to \mathrm{Fét}(X_{\widehat{A}}, M_{X_{\widehat{A}}}) \tag{5.1.2}$$

is an equivalence of categories.

To prove this statement we first show that every object of $\mathrm{Fét}(X_{\widehat{A}}, M_{X_{\widehat{A}}})$ is in the essential image of (5.1.2). For this, consider the functor

$$F : \mathrm{Alg}_S \to \mathrm{Set}$$

sending an S-algebra B to the set of isomorphism classes of objects in $\mathrm{Fét}(X_B, M_{X_B})$. This functor is limit preserving; that is, for any filtering inductive system of S-algebras $\{B_i\}$ with $B = \mathrm{colim}_i B_i$, the natural map

$$\mathrm{colim}_i F(B_i) \to F(B)$$

is bijective. Indeed, the functor sending an S-algebra R to isomorphism classes of finite X_R-schemes is limit preserving by [6, IV, Théorème 8.5.2 and Proposition 8.5.5], and since the stacks $\mathscr{L}og_{(X, M_X)}$ introduced in [23] are locally of finite type, we further have that the functor sending an S-algebra to isomorphism classes of morphism of log schemes $(Y, M_Y) \to (X_R, M_{X_R})$ with $Y \to X_R$ finite is limit preserving. It therefore suffices to observe that the property of being Kummer étale is a condition locally of finite presentation which is immediate.

By the Artin approximation theorem [2, Theorem 1.12], it follows that given $(\widehat{U}, M_{\widehat{U}}) \in \mathrm{Fét}(X_{\widehat{A}}, M_{X_{\widehat{A}}})$, there exists an object $(U, M_U) \in \mathrm{Fét}(X_A, M_{X_A})$ such that $(\widehat{U}, M_{\widehat{U}})$ and (U, M_U) map to isomorphic objects in $\mathrm{Fét}(X_k, M_{X_k})$. By the bijectivity of (4.2.1), it follows that in fact $(\widehat{U}, M_{\widehat{U}})$ is isomorphic to the image of (U, M_U) under (5.1.2).

It remains to show that given two objects $(U, M_U), (V, M_V) \in \text{Fét}(X_A, M_{X_A})$ with induced objects $(\widehat{U}, M_{\widehat{U}}), (\widehat{V}, M_{\widehat{V}}) \in \text{Fét}(X_{\widehat{A}}, M_{X_{\widehat{A}}})$, the map

$$\text{Hom}_{(X_A, M_{X_A})}((U, M_U), (V, M_V)) \to \text{Hom}_{(X_{\widehat{A}}, M_{X_{\widehat{A}}})}((\widehat{U}, M_{\widehat{U}}), (\widehat{V}, M_{\widehat{V}})) \qquad (5.1.3)$$

is bijective. Let $H_A \to X_A$ be the finite étale X_A-scheme classifying morphisms $(U, M_U) \to (V, M_V)$, as in Proposition 2.21. Then the bijectivity of (5.1.3) is equivalent to the statement that base change gives a bijection between the set of sections of $H_A \to X_A$ and the set of sections of $H_{\widehat{A}} \to X_{\widehat{A}}$. This follows from [2, Theorem 3.1]. This completes the proof. □

6. Proof of Theorem 1.7 for integral morphisms

In this section we prove Theorem 1.7 in the case when (f, f^b) is assumed in addition to be integral [21, III, Definition 2.5.1].

6.1. Let $h : \bar{b}'^{\log} \to \bar{b}^{\log}$ be a morphism of log geometric points over (B, M_B). Then h factors as

$$\bar{b}'^{\log} \xrightarrow{a} \bar{b}''^{\log} \xrightarrow{b} \bar{b}^{\log},$$

where b is strict and given by an extension of separably closed fields and a is an isomorphism on underlying fields.

6.2. By [12, Theorem A.4.2], any integral morphism of fs log schemes becomes saturated after a finite Kummer étale base change. Theorem 1.7 for a follows from this and [16, Theorem 2.15].

6.3. This therefore reduces the proof for integral morphisms to the case when h is strict, given by an inclusion of separably closed fields $k(\bar{b}) \subset k(\bar{b}')$.

The category of Kummer étale coverings forms a stack for the fppf topology. From this it follows immediately that (1.7.1) is an equivalence when $k(\bar{b}) \subset k(\bar{b}')$ is a purely inseparable algebraic extension. Indeed, in this case every object of $\text{Fét}((X, M_X)_{(\bar{b}'^{\log})})$ has unique descent data, since the kernel of the surjection

$$k(\bar{b}') \otimes_{k(\bar{b})} k(\bar{b}') \otimes_{k(\bar{b})} \cdots \otimes_{k(\bar{b})} k(\bar{b}') \to k(\bar{b}')$$

is nilpotent. It follows that in order to show that (1.7.1) is an equivalence in the case when h is strict, we may assume that $k(\bar{b})$ is algebraically closed.

6.4. Next we show that (1.7.1) is fully faithful. Consider two objects $(U, M_U), (V, M_V) \in \text{Fét}((X, M_X)_{(\bar{b}^{\log})})$, and let $(U', M_{U'}), (V', M_{V'}) \in \text{Fét}((X, M_X)_{(\bar{b}'^{\log})})$ be their base changes. We show that the natural map

$$\text{Hom}_{\text{Fét}((X, M_X)_{(\bar{b}^{\log})})}((U, M_U), (V, M_V)) \to \text{Hom}_{\text{Fét}((X, M_X)_{(\bar{b}'^{\log})})}((U', M_{U'}), (V', M_{V'})) \qquad (6.4.1)$$

is bijective. The injectivity is clear by faithfully flat descent. For surjectivity, let $f' : (U', M_{U'}) \to (V', M_{V'})$ be a morphism. By spreading out, we can find an integral finite type $k(\bar{b})$-scheme T with function field $k(\bar{b}')$ and an extension of the morphism f' to a morphism $f_T : (U_T, M_{U_T}) \to (V_T, M_{V_T})$ between the base changes of (U, M_U) and (V, M_V) to T. Restricting f_T to the fibre over a $k(\bar{b})$-point of T, using the fact that $k(\bar{b})$ is algebraically closed, we get a morphism $f : (U, M_U) \to (V, M_V)$ whose base change to T agrees with f_T at a point. By Proposition 4.2 and the injectivity of (6.4.1) already shown, it follows that f_T agrees everywhere with the map obtained from f by base change to T. This completes the proof of full faithfulness.

6.5. To show essential surjectivity, let $(U', M_{U'}) \in \mathrm{F\acute{e}t}((X, M_X)_{(\bar{b}'^{\log})})$ be an object which we show is obtained by base change from an object of $\mathrm{F\acute{e}t}((X, M_X)_{(\bar{b}^{\log})})$. Spreading out and looking at a $k(\bar{b})$-point as in the proof of full faithfulness, we find a finite type $k(\bar{b})$-scheme T, an extension $(U'_T, M_{U'_T})$ of $(U', M_{U'})$ to T and an object $(U, M_U) \in \mathrm{F\acute{e}t}((X, M_X)_{(\bar{b}^{\log})})$ whose base change to T is isomorphic to $(U'_T, M_{U'_T})$ at a point. Then by Proposition 4.2 there exists an extension of $k(\bar{b}')$ over which $(U', M_{U'})$ becomes isomorphic to the base change of (U, M_U). By the full faithfulness already shown, this implies that $(U', M_{U'})$ is isomorphic to the base change of (U, M_U).

This completes the proof of Theorem 1.7 for integral morphisms. □

7. Variations on the log purity theorem

7.1. Local version

In this subsection we consider variants (which follow from the original case) of Kato's log purity theorem [18, Theorem 3.3].

We will consider two setups.

7.2. Setup 1. Let k be a field of characteristic p (possibly 0) and let M be an fs monoid with the torsion subgroup of M^{gp} of order invertible in k. Let $F \subset M$ be a face and let (A_F, M_{A_F}) denote the log scheme whose underlying scheme is $\mathrm{Spec}(k[F])$ and whose log structure is induced by the map

$$M \to k[F], \quad m \mapsto \begin{cases} m & \text{if } m \in F \\ 0 & \text{otherwise.} \end{cases}$$

Let $A_F^\circ \subset A_F$ denote the open subset

$$\mathrm{Spec}(k[F^{\mathrm{gp}}]) \subset \mathrm{Spec}(k[F]).$$

7.3. Setup 2. Let V be a discrete valuation ring with uniformiser π and residue field of characteristic p (possibly 0). Let M be an fs monoid, let $F \subset M$ be a face of M and let $f \in F$ be an element. Let (A_F, M_{A_F}) denote the log scheme whose underlying scheme is

$$\mathrm{Spec}(V[F]/(\pi - f)),$$

where we abusively also write $f \in V[F]$ for the "monomial" corresponding to f, and log structure M_{A_F} induced by the map of monoids

$$M \to V[F]/(\pi - f), \quad m \mapsto \begin{cases} m & \text{if } m \in F \\ 0 & \text{otherwise.} \end{cases}$$

Let $A_F^\circ \subset A_F$ denote the open subset

$$\mathrm{Spec}(V[F^{\mathrm{gp}}]/(\pi - f)) \subset \mathrm{Spec}(V[F]/(\pi - f)).$$

Note that A_F° is a scheme over $K := \mathrm{Frac}(V)$.

7.4. Considering either setup, let

$$(X, M_X) \to (A_F, M_{A_F})$$

be a strict étale morphism and let

$$(X^\circ, M_{X^\circ}) \subset (X, M_X)$$

denote the preimage of A_F°.

Theorem 7.5. *The restriction functor*

$$\mathrm{F\acute{e}t}^{(p)}(X, M_X) \to \mathrm{F\acute{e}t}^{(p)}(X^\circ, M_{X^\circ}) \tag{7.5.1}$$

is an equivalence of categories.

Proof. Recall from [21, Theorem 2.1.17 (3)] that a face of a fine monoid is finitely generated. From this and the definition of a face, it follows that in either setup 1 or setup 2 the face F, viewed as a monoid in its own right, is an fs monoid. From this and [14, Theorem 4.1], it follows that the underlying scheme X is normal. This implies that if $H \to X$ is a finite étale morphism then, any section $X^\circ \to H$ over X° extends uniquely to a section over X. From this and Proposition 2.21, the full faithfulness of the restriction functor follows.

With the full faithfulness of the restriction functor already shown, to prove that (7.5.1) is an equivalence it suffices to show that for every Kummer étale morphism

$$(U^\circ, M_{U^\circ}) \to (X^\circ, M_{X^\circ}) \tag{7.5.2}$$

in $\mathrm{F\acute{e}t}^{(p)}(X^\circ, M_{X^\circ})$, there exists a Kummer étale covering

$$(W, M_W) \to (X, M_X)$$

such that the base change

$$(T^\circ, M_{T^\circ}) := (U^\circ, M_{U^\circ}) \times_{(X, M_X)} (W, M_W) \tag{7.5.3}$$

extends to an object (T, M_T) of

$$\mathrm{F\acute{e}t}^{(p)}(W, M_W).$$

Indeed, by the full faithfulness, the canonical descent data on (T°, M_{T°) extends uniquely to descent data for (T, M_T) relative to the morphism $(W, M_W) \to (X, M_X)$.

By [11, Proposition 3.13] these descent data are effective, and the resulting object of $\mathrm{Fét}^{(p)}(X, M_X)$ is then an extension of (U°, M_{U°).

From this argument it also follows that it suffices to consider the case when X is quasi-compact (and even affine, though this is not necessary).

To construct the local extension of (7.5.3), define for an integer $n \geq 1$ prime to p a morphism

$$\times n : (A_{F,n}, M_{A_{F,n}}) \to (A_F, M_{A_F}) \tag{7.5.4}$$

as follows.

Setup 1. Take $(A_{F,n}, M_{A_{F,n}})$ to be equal to (A_F, M_{A_F}) and the map $\times n$ to be the morphism induced by multiplication by n on M and F.

Setup 2. Let V_n be the extension of V obtained by adjoining an nth root π_n of π, setting

$$A_{F,n} := \mathrm{Spec}(V_n[F]/(\pi_n - f))$$

with log structure induced by the map $M \to V_n[F]$ as before and $\times n$ induced by the natural map $V \to V_n$, and multiplication by n on F and M.

We then have a commutative diagram

$$\begin{array}{ccc} (A_{F,n}, M_{A_{F,n}}) & \hookrightarrow & (A_F, M_{A_F}) \times_{\mathrm{Spec}(M \to \mathbf{Z}[M]), \times n} \mathrm{Spec}(M \to \mathbf{Z}[M]) \\ & \searrow^{\times n} & \downarrow \\ & & (A_F, M_{A_F}), \end{array}$$

where the right vertical morphism is the Kummer étale morphism given by multiplication by n on M, and the top inclusion is a strict closed immersion defined by a nilpotent ideal.

We claim that there exists an integer n prime to p such that the base change of (7.5.2) extends to

$$(X, M_X) \times_{\mathrm{Spec}(M \to \mathbf{Z}[M]), \times n} \mathrm{Spec}(M \to \mathbf{Z}[M]). \tag{7.5.5}$$

By the invariance of the log étale site under nilpotent thickenings and the fact that

$$(X, M_X) \times_{(A_F, M_{A_F}), \times n} (A_{F,n}, M_{A_{F,n}}) \tag{7.5.6}$$

is defined by a nilpotent ideal inside (7.5.5), to construct an extension to (7.5.5) it suffices to construct an extension to (7.5.6).

For this, note that by the definition of a Kummer étale morphism, we can find an integer n prime to p such that the base change

$$(U^\circ, M_{U^\circ}) \times_{(A_F, M_{A_F}), \times n} (A_{F,n}, M_{A_{F,n}}) \tag{7.5.7}$$

is a strict étale cover of the base change

$$(X^\circ, M_{X^\circ}) \times_{(A_F, M_{A_F}), \times n} (A_{F,n}, M_{A_{F,n}}). \tag{7.5.8}$$

Indeed, by the assumed quasi-compactness of X, there exists an integer n prime to p such that for every geometric point $\bar{u} \to U^\circ$ with image $\bar{x} \to X$ we have

$$\overline{M}_{U^\circ,\bar{u}} \subset \frac{1}{n}\overline{M}_{X,\bar{x}} \subset \overline{M}^{\mathrm{gp}}_{X,\bar{x}} \otimes \mathbf{Q},$$

and for such an integer n a direct calculation of pushouts in the category of fs monoids shows that the base change (7.5.7) is strict over (7.5.8).

Making such a base change, we are reduced to the case when (7.5.2) is strict and étale.

Let (Y, M_Y) be the log scheme with underlying scheme $Y = X$ and log structure M_Y induced by the map $F \to \mathscr{O}_X$, so we have a morphism of fs log schemes

$$(X, M_X) \to (Y, M_Y).$$

Since we are now in the case when the morphism (7.5.2) is strict, which implies that $U^\circ \to X^\circ$ is an étale cover and M_{U° is the pullback of M_X, we can write

$$(U^\circ, M_{U^\circ}) = (X, M_X) \times_{(Y, M_Y)} (U^\circ, N_{U^\circ}),$$

where N_{U° is the pullback of M_Y to U°. By the log purity theorem [18, Theorem 3.3], the cover U° of X° extends to a Kummer étale cover $(U, N_U) \to (Y, M_Y)$, and applying further base change to (X, M_X), we get the desired extension. □

Remark 7.6. In the foregoing proof we could also have considered X étale over the power series ring $k[[F]]$ or $V[[F]]/(\pi - f)$. The same result holds with the same proof (simply take $A_F = \mathrm{Spec}(k[[F]])$ or $\mathrm{Spec}(V[[F]]/(\pi - f))$ in the proof).

7.7. Global version

Let (A_F, M_{A_F}) be as in the previous section (either setup 1 or setup 2), and let

$$(f, f^b) : (X, M_X) \to (A_F, M_{A_F})$$

be a log smooth integral morphism of fs log schemes. As before, let (X°, M_{X°) denote the preimage of $(A_F^\circ, M_{A_F^\circ})$. Then we have the same result:

Theorem 7.8. *The restriction functor*

$$\mathrm{Fét}^{(p)}(X, M_X) \to \mathrm{Fét}^{(p)}(X^\circ, M_{X^\circ})$$

is an equivalence of categories.

Proof. We will reduce the proof to the previous case. We can work étale locally on X, so by [13, Proposition 4.1] (see also [25, Theorem 2.2.8], where more refined properties that emerge from the proof of this result are discussed, as well as [21, Chapter, IV, Theorem 3.3.1]) we can assume that we have an integral morphism of fs monoids

$$\theta : M \to N,$$

with

$$\theta^{\mathrm{gp}} : M^{\mathrm{gp}} \to N^{\mathrm{gp}}$$

injective with torsion of $M^{\mathrm{gp}}/N^{\mathrm{gp}}$ of order invertible in k, and an étale morphism

$$X \to \mathrm{Spec}(R),$$

where

$$R = \begin{cases} k[F] \otimes_{k[M]} k[N] & \text{Setup 1,} \\ (V[F]/(\pi - f)) \otimes_{V[M]} V[N] & \text{Setup 2.} \end{cases}$$

We can further assume that N^{gp} is torsion free.

Let $Z \subset \mathrm{Spec}(R)$ be an irreducible component, viewed as a closed subscheme with the reduced structure. We can describe the coordinate ring $\mathscr{O}_Z$ as follows.

Setup 1. By definition we have surjections

$$k[N] \twoheadrightarrow R \twoheadrightarrow \mathscr{O}_Z.$$

The ideal of $\mathrm{Spec}(R)$ in $\mathrm{Spec}(k[N])$ is generated by the monomials corresponding to the elements $\theta(M - F) \subset N$. In particular, the $D(N^{\mathrm{gp}})$-action on $\mathrm{Spec}(k[N])$ restricts to an action on $\mathrm{Spec}(R)$. Since Z is an irreducible component and $D(N^{\mathrm{gp}})$ is connected, this action further restricts to an action on Z. For $s \in N^{\mathrm{gp}}$, let $\mathscr{O}_{Z,s}$ denote the s-eigenspace for the action (so an element $g \in \mathscr{O}_Z$ lies in $\mathscr{O}_{Z,s}$ if and only if for all scheme-valued points $u \in D(N^{\mathrm{gp}})$ we have $u * g = u(s) \cdot g$). Let $S \subset N^{\mathrm{gp}}$ denote those s for which $\mathscr{O}_{Z,s} \neq 0$. Since $D(N^{\mathrm{gp}})$ is diagonalisable we have

$$\mathscr{O}_Z = \oplus_{s \in S} \mathscr{O}_{Z,s}.$$

Furthermore, since $\mathscr{O}_Z$ admits an equivariant surjection from $k[N]$, we must have $S \subset N$ and each $\mathscr{O}_{Z,s}$ has dimension 1 as a k-vector space whose image is the monomial corresponding to s. Furthermore, since Z is integral for $s, s' \in S$, the multiplication map

$$\mathscr{O}_{Z,s} \otimes \mathscr{O}_{Z,s'} \to \mathscr{O}_{Z,s+s'} \subset \mathscr{O}_Z$$

is nonzero, which implies that S is a submonoid of N. Finally, if $n, n' \in N$ are two elements with corresponding monomials $m_n, m_{n'} \in k[N]$, then the image of $m_{n+n'}$ in $\mathscr{O}_Z$ is the product of the images of m_n and $m_{n'}$. It follows that if $n + n'$ is in S, then both n and n' are in S. That is, $S \subset N$ is a face and

$$Z = Z_S := \mathrm{Spec}(k[S]).$$

Furthermore, we must have $S \cap M \subset F$, since the map $Z \to \mathrm{Spec}(k[M])$ factors through $\mathrm{Spec}(k[F])$. In fact, $S \cap M$ is a face of M and the map factors through the closed subscheme of $\mathrm{Spec}(k[F])$ defined by this face. Since

$$\mathrm{Spec}(k[F] \otimes_{k[M]} k[N]) \to \mathrm{Spec}(k[F])$$

is flat (recall from [13, Proposition 4.1] that this follows from the fact that $M \to N$ is integral), Z dominates $\mathrm{Spec}(k[F])$ and therefore we must have $S \cap M = F$.

Setup 2. Let $S \subset N$ be the face of elements whose image in $\mathscr{O}_Z$ is nonzero. Then $S \cap M = F$. Indeed, since θ is an integral morphism, the map $Z \to A_F$ is dominant, which implies that all elements of F map to nonzero elements in $\mathscr{O}_Z$ and therefore $F \subset$

$S\cap M$, and the reverse inclusion $S\cap M\subset F$ is immediate. Let $Z_S\subset \mathrm{Spec}(R)$ be the closed subscheme given by the surjection

$$R\to V[S]/(\pi - f)$$

induced by the map $V[N]\to V[S]$ sending $n\in N$ to n for $n\in S$, and 0 otherwise. By the definition of S, the surjection $R\to \mathscr{O}_Z$ factors through $V[S]/(\pi - f)$, and therefore we get an inclusion $i: Z\hookrightarrow Z_S$. This inclusion i is in fact an isomorphism. To see this, write K for the field of fractions of V and let M_F denote the localisation of M at F. The coordinate ring of $\mathrm{Spec}(R)\times_{A_F} A_F^\circ$ is given by the ring

$$R^\circ := K[F^{\mathrm{gp}}]/(f-\pi)\otimes_{K[M_F]} K[N_F],$$

which comes equipped with an action of the diagonalisable group scheme $D(N^{\mathrm{gp}}/(f))$ induced by the standard action on $K[N_F]$. Proceeding as in setup 1, let $S'\subset N_F$ denote those elements whose associated monomial in $k[N_F]$ maps to a nonzero element in $\mathscr{O}_{Z^\circ} = \mathscr{O}_Z\otimes_R R^\circ$. Since Z° is an integral domain, S' is a face of N_F with $S'\cap M_F = F^{\mathrm{gp}}$. Furthermore, since Z is flat over A_F, we have $S = S'\cap N$, which in turn implies that $S' = S_F$. We conclude that i restricts to an isomorphism over A_F°, and since Z is flat over A_F this implies that i is an isomorphism.

Returning to the proof of the theorem in either setup 1 or 2, let $\mathscr{S}$ denote the set of faces $S\subset N$ such that $S\cap M = F$, and for $S\in\mathscr{S}$ let $X_S\subset X$ be the preimage of

$$Z_S\subset A_F$$

and let M_{X_S} be the restriction of M_X to X_S. Note that for $S_1, S_2\in\mathscr{S}$ we have

$$X_{S_1}\cap X_{S_2} = X_{S_1\cap S_2},$$

and $S_1\cap S_2\in\mathscr{S}$. Then the functor induced by restriction

$$\mathrm{F\acute{e}t}(X, M_X)\to \lim_{S\in\mathscr{S}}\mathrm{F\acute{e}t}(X_S, M_{X_S}) \tag{7.8.1}$$

is an equivalence, and similarly for $\mathrm{F\acute{e}t}(X^\circ, M_{X^\circ})$. In the classical case without log structures this follows from [3, Exposé VIII, Théorème 9.4]. The logarithmic version can be deduced from this as follows. Consider the fibred category $\mathscr{F}$ (resp. $\mathscr{F}_S$ for $S\in\mathscr{S}$) over the Kummer étale site of (X, M_X) which to any (U, M_U) associates the category of finite étale U-schemes (resp. U_S-schemes, where $(U_S, M_{U_S}) := (U, M_U)\times_{(X,M_X)}(X_S, M_{X_S})$). Let $\mathscr{F}^a$ (resp. $\mathscr{F}_S^a$) be the stack over the Kummer étale site of (X, M_X) associated to $\mathscr{F}$ (resp. $\mathscr{F}_S$). Then it follows from [11, Proposition 3.13] that we have

$$\mathrm{F\acute{e}t}(X, M_X)\simeq \mathscr{F}^a(X, M_X),\quad \mathrm{F\acute{e}t}(X_S, M_{X_S}) = \mathscr{F}_S^a(X, M_X).$$

Furthermore, the natural functor

$$(\lim_{S\in\mathscr{S}}\mathscr{F}_S)^a\to \lim_{S\in\mathscr{S}}\mathscr{F}_S^a$$

is an equivalence (using the fact that there are only finitely many categories in this limit). To prove that (7.8.1) is an equivalence, it therefore suffices to show that the map of prestacks

$$\mathscr{F} \to \lim_{S \in \mathscr{S}} \mathscr{F}_S$$

is an equivalence, which follows from the case of ordinary schemes.

This reduces the proof to showing that each of the functors

$$\mathrm{Fét}(X_S, M_{X_S}) \to \mathrm{Fét}(X_S^\circ, M_{X_S^\circ})$$

is an equivalence, which follows from Theorem 7.5. □

8. Proof of Theorem 1.9 in the case of an integral morphism

We proceed with the notation of Theorem 1.9. Since B is connected, in order to prove Theorem 1.9 it suffices to consider the case when B is integral. Furthermore, it suffices to consider the case when (B, M_B) is defined over a field or a complete discrete valuation ring.

8.1. We say that a log geometric point $\bar{b}^{\log} \to (B, M_B)$ is *quasi-strict* if the map

$$\overline{M}_{B,\bar{b}} \to \overline{M}_{\bar{b}^{\log}}$$

induces an isomorphism $\overline{M}_{B,\bar{b},\mathbb{Z}_{(p)}} \to \overline{M}_{\bar{b}^{\log}}$, where p is the residue charcteristic of $\bar{b}$.

For any log geometric point $\bar{b}^{\log} \to (B, M_B)$ there exists a morphism of log geometric points of (B, M_B)

$$\begin{array}{ccc} \bar{b}^{\log} & \longrightarrow & \bar{b}'^{\log} \\ & \searrow & \downarrow \\ & & (B, M_B), \end{array} \tag{8.1.1}$$

where $\bar{b}'^{\log} \to (B, M_B)$ is quasi-strict. Indeed, by the definition of a log geometric point we get an induced map $\overline{M}_{B,\bar{b},\mathbb{Z}_{(p)}} \to \overline{M}_{\bar{b}^{\log}}$. Choose a lifting $\overline{M}_{B,\bar{b},\mathbb{Z}_{(p)}} \to M_{\bar{b}^{\log}}$ of this map, and let $M_{\bar{b}'^{\log}}$ denote the associated log structure on $\bar{b}$. Setting $\bar{b}'^{\log} := (\bar{b}, M_{\bar{b}'^{\log}})$ gives the desired factorisation (8.1.1).

In particular, by this discussion and Theorem 1.7, it suffices to prove Theorem 1.9 in the case when $\bar{b}^{\log} \to (B, M_B)$ is quasi-strict.

8.2. Let $\bar{s}^{\log} \to (B, M_B)$ and $\bar{b}^{\log} \to (B, M_B)$ be two log geometric points of (B, M_B). A *specialisation* $\bar{s}^{\log} \rightsquigarrow \bar{b}^{\log}$ is a commutative diagram

$$\begin{array}{ccccc} \bar{b}^{\log} & \xrightarrow{\alpha} & (E, M_E) & \xleftarrow{\beta} & \bar{s}^{\log} \\ & \searrow & \downarrow & \swarrow & \\ & & (B, M_B), & & \end{array}$$

where α and β are strict and (E, M_E) is log strictly local in the sense of [19, Paragraph 2.8 (6)], with residue field given by $\bar{b}$.

Given such a specialisation, we get an induced cospecialisation functor

$$\mathrm{cosp}: \mathrm{Fét}((X, M_X)_{(\bar b^{\log})}) \to \mathrm{Fét}((X, M_X)_{(\bar s^{\log})}) \tag{8.2.1}$$

defined as follows.

Write $\bar b^{\log} = (\bar b, M_{\bar b^{\log}})$, and $M_{\bar b^{\log}} = \mathrm{colim}_\lambda\ M_{\bar b, \lambda}$, where $M_{\bar b,\lambda}$ is an fs log structure contained in $M_{\bar b^{\log}}$ and containing the image of $M_{B, \bar b}$. Let $\bar b_\lambda^{\log}$ denote $(\bar b, M_{\bar b, \lambda})$, so $\bar b^{\log} = \lim_\lambda\ \bar b_\lambda^{\log}$. Let $M_{E,\lambda} \subset M_E$ be the sublog structure induced by the submonoid $M_{E, \bar b} \times_{\overline{M}_{E, \bar b}} \overline{M}_{\bar b, \lambda}$, so $M_E = \mathrm{colim}_\lambda\ M_{E,\lambda}$, and let $M_{\bar s, \lambda} \subset M_{\bar s^{\log}}$ denote the log structure defined by the image of $M_{E,\lambda}$. Setting $\bar s_\lambda^{\log} := (\bar s, M_{\bar s, \lambda})$, we then have a commutative diagram for all λ:

$$\begin{array}{ccccc} \bar b^{\log} & \hookrightarrow & (E, M_E) & \hookleftarrow & \bar s^{\log} \\ \downarrow & & \downarrow & & \downarrow \\ \bar b_\lambda & \hookrightarrow & (E, M_{E,\lambda}) & \hookleftarrow & \bar s_\lambda^{\log}. \end{array}$$

We then get restriction functors

$$\mathrm{Fét}((X, M_X)_{\bar b_\lambda^{\log}}) \xleftarrow{u} \mathrm{Fét}((X, M_X)_{(E, M_{E,\lambda})}) \xrightarrow{v} \mathrm{Fét}((X, M_X)_{\bar s_\lambda^{\log}}),$$

where u is an equivalence by Proposition 5.1 and Theorem 1.2. The functor $v \circ u^{-1}$ then is a functor

$$\mathrm{Fét}((X, M_X)_{\bar b_\lambda^{\log}}) \to \mathrm{Fét}((X, M_X)_{\bar s_\lambda^{\log}}).$$

Passing to the limit over λ, we get (8.2.1).

Lemma 8.3. *Let $\bar x \to B$ be a geometric point. After (B, M_B) is replaced by a strict étale neighbourhood of $\bar x$, there exists a morphism of log schemes $(\widetilde B, M_{\widetilde B}) \to (B, M_B)$ with the following properties:*

(i) *$\bar x$ lifts to a geometric point of $\widetilde B$.*

(ii) *$\widetilde B$ is integral and the morphism $\widetilde B \to B$ is dominant.*

(iii) *There exists an fs monoid M with a face $F \subset M$ and a strict étale morphism*

$$(\widetilde B, M_{\widetilde B}) \to (A_F, M_{A_F}),$$

where (A_F, M_{A_F}) is as in setup 1 or setup 2 in Paragraphs 7.2 and 7.3 (for some discrete valuation ring V).

Proof. We can assume that we have a chart $\beta_B : P \to \mathscr{O}_B$ for M_B, with P an fs sharp monoid. Let $I \subset P$ be the face of elements whose image in $\mathscr{O}_B$ is nonzero, and let $Z \subset B$ be the closed subscheme defined by I.

Note that the quotient $P^{\mathrm{gp}}/I^{\mathrm{gp}}$ is a torsion-free group. Indeed, if $a, b \in P$, $c, d \in I$ and $n > 0$ is an integer such that

$$n(a - b) = c - d,$$

then

$$n(a+d-b) = c + (n-1)d \in I,$$

which – since P is saturated and I is a face – implies that $a+d-b \in I$ and so $a-b \in I^{\mathrm{gp}}$.

We now apply de Jong's alteration results.

In the case when the base ring is a field k, we apply [5, Theorem 4.1] to find a dominant morphism $Y \to B$ such that Y is smooth over k and the preimage of Z is a divisor with simply normal crossings. Let B_1 denote an open neighbourhood of a point of Y lying over the image of $\bar{x}$ such that the restriction of the divisor to B_1 is given by a morphism $\mathbf{N}^r \to \mathscr{O}_{B_1}$.

By this we mean that the restriction to B_1 of the divisor over Z has r irreducible components cut out by the images in $\mathscr{O}_{B_1}$ of the standard generators of $\mathbf{N}^r$. There is a morphism of monoids

$$a : I \to \mathbf{N}^r$$

sending an element $p \in I$ to the element of $\mathbf{N}^r$ whose jth entry is the order of zero of p along the jth divisor in B_1. Let M_{int} (resp. M) denote the pushout in the category of fine (resp. fs) monoids of the diagram

$$\begin{array}{ccc} I & \longrightarrow & P \\ \big\downarrow{\scriptstyle a} & & \\ \mathbf{N}^r. & & \end{array}$$

By [21, Chapter I, Proposition 2.2.1] there exists a morphism $h : P \to \mathbf{N}$ such that $h^{-1}(0) = I$. Let $\widetilde{h} : M \to \mathbf{N}$ be the unique extension which is 0 on $\mathbf{N}^r$, and let $F \subset M$ denote $\tilde{h}^{-1}(0)$. If $m = p + f \in M_{\mathrm{int}} \cap F$, where $p \in P$ and $f \in \mathbf{N}^r$, then we must have $p \in I$ and therefore $m \in \mathbf{N}^r$. It follows that F is the saturation of $\mathbf{N}^r$ in M and that $\mathbf{N}^r$ is a face in M_{int}. The given map $\mathbf{N}^r \to \mathscr{O}_{B_1}$ then extends to a map $M_{\mathrm{int}} \to \mathscr{O}_{B_1}$. Let $\widetilde{B}$ denote the fibre product

$$B_1 \times_{\mathrm{Spec}(k[\mathbf{N}^r])} \mathrm{Spec}(k[F]),$$

and let $M_{\widetilde{B}}$ be the log structure induced by the map $M \to k[F]$, so we have a chart $\beta_{\widetilde{B}} : M \to \mathscr{O}_{\widetilde{B}}$.

If $\alpha : P \to M$ is the map arising from the construction of M as a pushout, then the induced diagram

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & M \\ \big\downarrow{\scriptstyle \beta_B} & & \big\downarrow{\scriptstyle \beta_{\widetilde{B}}} \\ \mathscr{O}_B & \longrightarrow & \mathscr{O}_{\widetilde{B}} \end{array}$$

may not commute. By the definition of the map $a : I \to \mathbf{N}^r$, however, after possibly shrinking on $\widetilde{B}$ there exists a homomorphism $\gamma : I \to \mathscr{O}^*_{\widetilde{B}}$ such that for $s \in I$ we have

$$\beta_B(s) = \gamma(s)\beta_{\widetilde{B}}(\alpha(s))$$

in $\mathscr{O}_{\widetilde{B}}$. Now, as noted earlier, the quotient $P^{\text{gp}}/I^{\text{gp}}$ is torsion free, so we can extend γ to a homomorphism $\tilde{\gamma} : P \to \mathscr{O}^*_{\widetilde{B}}$, and then the resulting diagram

$$\begin{array}{ccc} P & \xrightarrow{\tilde{\gamma}\oplus\alpha} & \mathscr{O}^*_{\widetilde{B}} \oplus M \\ \downarrow{\scriptstyle \beta_B} & & \downarrow{\scriptstyle \beta_{\widetilde{B}}} \\ \mathscr{O}_B & \longrightarrow & \mathscr{O}_{\widetilde{B}} \end{array}$$

does commute. In this way we get the desired morphism of log schemes

$$(\widetilde{B}, M_{\widetilde{B}}) \to (B, M_B).$$

In the case when the base is a complete discrete valuation ring (recall that at the beginning of the section we reduced to the cases of the base being a field or a complete discrete valuation ring), we proceed in a very similar manner. By [5, Theorem 6.5] we can find $B_1 \to B$ and an étale map

$$B_1 \to \text{Spec}(V[\mathbf{N}^r]/(\pi - f))$$

for some $f \in \mathbf{N}^r$ and a discrete valuation ring V. Now proceeding with the pushout of monoids as in the previous case, we obtain the desired $(\widetilde{B}, M_{\widetilde{B}})$. □

8.4. Using Lemma 8.3, we are then further reduced to the case when (B, M_B) admits a strict étale morphism

$$(B, M_B) \to (A_F, M_{A_F}),$$

and to showing that for a specialisation of quasi-strict log geometric points $\bar{\eta}^{\log} \rightsquigarrow \bar{b}^{\log}$, with $\bar{\eta}^{\log}$ lying over the generic point of B, the cospecialisation functor is an equivalence of categories (recall that we already reduced to the case when B is integral).

8.5. Let S denote the strict Henselisation of B at $\bar{b}$, and for an integer $n \geq 1$ prime to p let $\text{Spec}(\widetilde{S}_n)$ denote the fibre product of the diagram

$$\begin{array}{ccc} & & A_{F,n} \\ & & \downarrow{\scriptstyle \times n} \\ \text{Spec}(S) & \longrightarrow & A_F, \end{array}$$

where $\times n : A_{F,n} \to A_F$ is as in the proof of Theorem 7.5, and let $M_{\widetilde{S}_n}$ denote the log structure on $\text{Spec}(\widetilde{S}_n)$ induced by that on $A_{F,n}$. The ring $\widetilde{S}_n$ is finite over S, since $\times n$ is a finite morphism, and therefore isomorphic to a product of strictly Henselian local rings. Choosing a compatible system of lifts of the morphism

$$\bar{b}^{\log} \to (B, M_B) \to (A_F, M_{A_F})$$

to the $(A_{F,n}, M_{A_{F,n}})$, we obtain a lifting of $\bar{b}$ to each $\text{Spec}(\widetilde{S}_n)$. These lifts and the given specialisation from $\eta^{\log}$ to $\bar{b}^{\log}$ then also define compatible liftings of $\eta^{\log}$ to each $(\text{Spec}(\widetilde{S}_n), M_{\widetilde{S}_n})$, which specialise to the liftings of $\bar{b}^{\log}$. Let S_n denote the local ring of

$\mathrm{Spec}(\widetilde{S}_n)$ at $\bar{b}$, so S_n is a finite S-algebra. Denote by M_{S_n} the log structure on $\mathrm{Spec}(S_n)$ obtained by restriction from $M_{\widetilde{S}_n}$.

Let (X_n, M_{X_n}) denote the base change of (X, M_X) to $(\mathrm{Spec}(S_n), M_{S_n})$. Let

$$(b_n, M_{b_n}) \hookrightarrow (\mathrm{Spec}(S_n), M_{S_n})$$

be the closed point with the induced log structure, and let $(\mathrm{Spec}(K_n), M_{K_n})$ be the generic point of $(\mathrm{Spec}(S_n), M_{S_n})$.

Finally, we can assume that

$$\bar{b}^{\log} = \lim_n (b_n, M_{b_n})$$

and that $\bar{\eta}^{\log}$ admits a strict morphism

$$\bar{\eta}^{\log} \to \lim_n (\mathrm{Spec}(K_n), M_{K_n})$$

whose underlying morphism of schemes is given by a separable closure of $\mathrm{colim}_n K_n$.

8.6. We then get a diagram

$$\begin{array}{ccccc} \mathrm{colim}_n \mathrm{F\acute{e}t}^{(p)}((X, M_X) \times_{(b, M_b)} (b_n, M_{b_n})) & \xleftarrow{u} & \mathrm{colim}_n \mathrm{F\acute{e}t}^{(p)}(X_n, M_{X_n}) & \xrightarrow{v} & \mathrm{F\acute{e}t}^{(p)}((X, M_X)_{(\bar{\eta}^{\log})}) \\ \downarrow \simeq & & & & \\ \mathrm{F\acute{e}t}^{(p)}((X, M_X)_{(\bar{b}^{\log})}). & & & & \end{array}$$

By Theorem 1.2 the functor u is an equivalence.

To show that v is an equivalence it suffices to show that for any morphism

$$\rho_{\bar{\eta}^{\log}} : (U_{\bar{\eta}}, M_{U_{\bar{\eta}}}) \to (V_{\bar{\eta}}, M_{V_{\bar{\eta}}})$$

in $\mathrm{F\acute{e}t}^{(p)}((X, M_X)_{\bar{\eta}^{\log}})$, there exists an integer n prime to p and a morphism

$$\rho_n : (U_n, M_{U_n}) \to (V_n, M_{V_n})$$

in $\mathrm{F\acute{e}t}^{(p)}(X_n, M_{X_n})$ inducing $\rho_{\bar{\eta}^{\log}}$. Note that such a morphism is necessarily unique by Theorem 7.8.

By the definition of $\mathrm{F\acute{e}t}^{(p)}((X, M_X)_{\bar{\eta}^{\log}})$ as a direct limit, there exists an integer m and a finite extension L/K_m such that if M_L is the log structure on $\mathrm{Spec}(L)$ obtained by pullback from M_{K_m}, then there exists a morphism

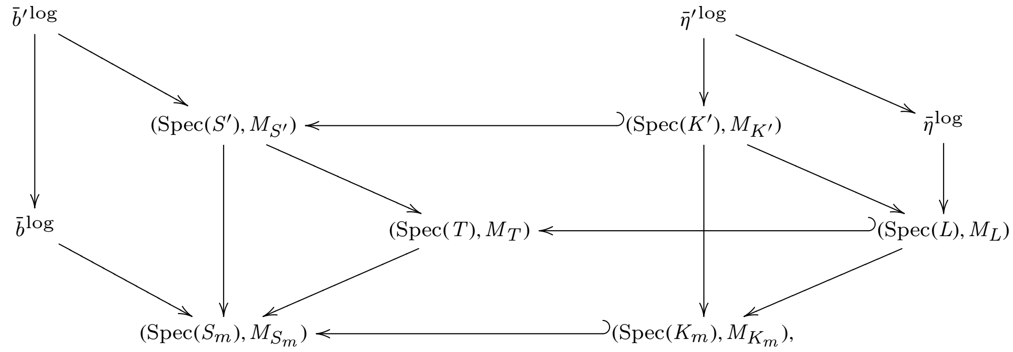
$$\rho_L : (U_L, M_{U_L}) \to (V_L, M_{V_L})$$

in $\mathrm{F\acute{e}t}^{(p)}((X, M_X) \times_{(B, M_B)} (\mathrm{Spec}(L), M_L))$ inducing $\rho_{\bar{\eta}^{\log}}$.

Let T denote the normalisation of S_m in L, so we get a finite strict morphism

$$(\mathrm{Spec}(T), M_T) \to (\mathrm{Spec}(S_m), M_{S_m})$$

inducing the map $(\mathrm{Spec}(L), M_L) \to (\mathrm{Spec}(K_m), M_{K_m})$. Applying Lemma 8.3 to a finite type approximation of $(\mathrm{Spec}(T), M_T)$, we then obtain a commutative diagram



where $(\mathrm{Spec}(S'), M_{S'})$ admits a strict étale morphism to $(A_{F'}, M_{A_{F'}})$ for suitable fs monoid M' with face F' (and in setup 2 a discrete valuation ring V'). By the purity theorem we then get an extension of ρ_L to a morphism in $\mathrm{Fét}^{(p)}((X, M_X) \times_{(B,M_B)} (\mathrm{Spec}(S'), M_{S'}))$. By Theorem 1.7 the image of this morphism in $\mathrm{Fét}^{(p)}((X, M_X)_{(\bar{b}'^{\log})})$ is induced by a morphism in $\mathrm{Fét}^{(p)}((X, M_X)_{(\bar{b}^{\log})})$. Using the fact that the reduction functor u is an equivalence, we therefore obtain a morphism ρ_n in $\mathrm{Fét}^{(p)}(X_n, M_{X_n})$, for some n, whose image in $\mathrm{Fét}^{(p)}((X, M_X)_{(\bar{\eta}'^{\log})})$ agrees with the image of $\rho_{\bar{\eta}^{\log}}$. By Theorem 1.7 we then conclude that $v(\rho_n) = \rho_{\bar{\eta}^{\log}}$ as desired.

This completes the proof of Theorem 1.9 in the case of an integral morphism. □

Remark 8.7. It is in the application of the log purity theorem [18, Theorem 3.3] that the prime-to-p assumption is crucial.

9. Invariance under log blowups

To prove Theorems 1.7 and 1.9 in general we will use results of Fujiwara and Kato on invariance under log blowups of the category of finite Kummer étale covers. Since these results are not published, we provide a proof.

9.1. Let (X, M_X) be an fs log scheme and let $\mathscr{I} \subset M_X$ be a coherent sheaf of ideals (see [21, Chapter II, Proposition 2.6.1]), all of whose stalks are nonempty. We can then consider the log blowup

$$\pi : (X', M_{X'}) \to (X, M_X)$$

defined as in [21, Chapter III, Definition 2.6.2].

Theorem 9.2 (Fujiwara-Kato [11, Theorem 6.10 and references therein])**.** *Assume that X is locally of finite type over a field or an excellent Dedekind ring A, and let $\mathbf{L}$ denote the set of residue characteristics of X. Then the pullback functor*

$$\mathrm{Fét}^{\mathbf{L}}(X, M_X) \to \mathrm{Fét}^{\mathbf{L}}(X', M_{X'}) \tag{9.2.1}$$

is an equivalence of categories.

Proof. By [11, Proposition 3.13], the categories of Kummer étale covers of (X, M_X) and $(X', M_{X'})$ form stacks for the Kummer étale topology. It therefore suffices to prove the theorem after replacing X by an étale cover. We can therefore assume that $X = \mathrm{Spec}(R)$ is affine and that we have a chart $P \to R$ such that $(X', M_{X'})$ is defined by blowing up a finitely generated ideal $I \subset P$, where P is an fs sharp monoid.

Observe that a log blowup has connected fibres. Indeed, for this it suffices to consider the case when $X = \mathrm{Spec}(\mathbf{Z}[P])$. In this case, $\pi_* \mathscr{O}_{X'}$ is a coherent sheaf of algebras corresponding to a finite $\mathbf{Z}[P]$-algebra C which is an integral domain and with $\mathbf{Z}[P] \to C$ an isomorphism over $\mathbf{Z}[P^{\mathrm{gp}}]$. Since P is saturated, which implies that $\mathbf{Z}[P]$ is normal, this implies that $C = \mathbf{Z}[P]$; that is, $\mathscr{O}_X = \pi_* \mathscr{O}_{X'}$. From this and [6, III, Théorème 4.3.2] we conclude that the fibres are connected. This, in turn, implies that (9.2.1) is fully faithful.

To prove that (9.2.1) is an equivalence, it therefore suffices to show that any object

$$(U', M_{U'}) \in \mathrm{F\acute{e}t}^{\mathbf{L}}(X', M_{X'}) \tag{9.2.2}$$

descends to (X, M_X), and using the full faithfulness it suffices to show that it descends locally in the Kummer étale topology on (X, M_X).

For an integer $n \geq 1$ not divisible by the primes in $\mathbf{L}$, let $(X_n, M_{X_n}) \to (X, M_X)$ be the Kummer étale cover given by the fibre product

$$X_n := X \times_{\mathrm{Spec}(\mathbf{Z}[P]), \times n} \mathrm{Spec}(\mathbf{Z}[P]), \tag{9.2.3}$$

where $\times n : \mathrm{Spec}(\mathbf{Z}[P]) \to \mathrm{Spec}(\mathbf{Z}[P])$ is induced by multiplication by n on P and M_{X_n} is induced by the natural log structure on the second factor. Let $\mathscr{I}_n \subset M_{X_n}$ be the coherent sheaf of ideals given by $I \subset P$ (again on the second factor in (9.2.3)). Let $(X'_n, M_{X'_n})$ denote the log blowup of (X_n, M_{X_n}) along $\mathscr{I}_n$. Note that the ideal $\mathscr{I}$ generates an invertible sheaf of ideals in $M_{X'_n}$, so we obtain a commutative diagram

$$\begin{array}{ccc} (X'_n, M_{X'_n}) & \xrightarrow{q_n} & (X', M_{X'}) \\ \downarrow & & \downarrow \\ (X_n, M_{X_n}) & \longrightarrow & (X, M_X). \end{array} \tag{9.2.4}$$

We can describe this diagram explicitly locally on X'. Let $a_1, \dots, a_r \in I$ be generators for I, and for $j = 1, \dots, r$ define $P_j \subset P^{\mathrm{gp}}$ to denote the saturated submonoid generated by P and the elements

$$z_i := a_i - a_j.$$

Then X' is covered by the affine schemes

$$\mathrm{Spec}(R \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P_j]).$$

Now observe that the diagram of monoids

$$\begin{array}{ccc} P & \xrightarrow{\times n} & P \\ \downarrow & & \downarrow \\ P_j & \xrightarrow{\times n} & P_j \end{array} \tag{9.2.5}$$

is a pushout diagram in the category of fs monoids. From this it follows that (9.2.4) is cartesian, in the category of fs log schemes, and that for any object $(U', M_{U'}) \in \mathrm{Fét}^{\mathbf{L}}(X', M_{X'})$ there exists an integer n not divisible by primes in $\mathbf{L}$ such that the pullback in the fs category of $(U', M_{U'})$ to $(X'_n, M_{X'_n})$ is given by a finite étale morphism $U'_n \to X'_n$ with the pullback log structure (this follows from a similar argument to the one given in the proof of Theorem 7.5).

Making a base change $(X_n, M_{X_n}) \to (X, M_X)$, to prove that (9.2.2) descends to (X, M_X) we are therefore reduced to the case when $(U', M_{U'}) \to (X', M_{X'})$ is strict – that is, given by a finite étale covering of schemes $U' \to X'$.

By a standard limit argument and an application of Artin approximation, we are then reduced to the variant statement that if R (the coordinate ring of X) is a complete local ring with separably closed residue field, which is the completion of the strict Henselisation of a finite type algebra over a field or discrete valuation ring A at a point, then any object of $\mathrm{Fét}^{\mathbf{L}}(X')$ descends to a Kummer étale covering of (X, M_X). Let R_0 be a finite type algebra over A such that R is the completion of the local ring $\mathscr{O}_{\mathrm{Spec}(R_0),\bar{x}}$ at a geometric point $\bar{x} \to \mathrm{Spec}(R_0)$, and choose an epimorphism $S_0 := A[X_1, \ldots, X_t] \to R_0$ for some t. Combining this with the chart $P \to R$, we obtain a morphism

$$S_0[P] \to R.$$

The closed point of $\mathrm{Spec}(R)$ maps to a geometric point of $\mathrm{Spec}(S_0[P])$, and we write S for the completion of the local ring of $\mathrm{Spec}(S_0[P])$ at this geometric point. Since R is complete, we then get an epimorphism

$$S \twoheadrightarrow R.$$

The natural map $P \to S$ defines a log structure M_S on $\mathrm{Spec}(S)$, and we define $(X'_S, M_{X'_S})$ to be the log blowup of $(\mathrm{Spec}(S), M_S)$ with respect to the ideal $I \subset P$. We then have a cartesian diagram

$$\begin{array}{ccc} (X'_S, M_{X'_S}) & \hookleftarrow & (X', M_{X'}) \\ \downarrow & & \downarrow \\ (\mathrm{Spec}(S), M_S) & \hookleftarrow & (\mathrm{Spec}(R), M_R). \end{array}$$

Here we write M_R for the log structure M_X on $\mathrm{Spec}(R) = X$.

Now observe that the restriction functor

$$\mathrm{Fét}^{\mathbf{L}}(X'_S, M_{X'_S}) \to \mathrm{Fét}^{\mathbf{L}}(X', M_{X'})$$

is an equivalence of categories, since the further pullback from either of these categories to the corresponding category of Kummer étale covers of the fibre of $(X', M_{X'})$ over the closed point of $\mathrm{Spec}(R)$ is an equivalence by Theorem 1.2. In this way we are further reduced to the case when the complete local base $(X, M_X) = (\mathrm{Spec}(R), M_R)$ is log regular. Let $X_{\mathrm{triv}} \subset X$ be the maximal open subset over which M_X is trivial. By the log purity theorem [18, Theorem 3.3], both the restriction functors

$$\mathrm{Fét}^{\mathbf{L}}(X, M_X) \to \mathrm{Fét}^{\mathbf{L}}(X_{\mathrm{triv}}), \quad \mathrm{Fét}^{\mathbf{L}}(X', M_{X'}) \to \mathrm{Fét}^{\mathbf{L}}(X_{\mathrm{triv}})$$

are equivalences of categories, from which it follows that U' is obtained from a Kummer étale covering $u : (U, M_U) \to (X, M_X)$. This completes the proof. □

Remark 9.3. The Kummer étale covering $u : (U, M_U) \to (X, M_X)$ obtained at the end of the proof must in fact be strict, since the map $u^{-1}\overline{M}_X^{\mathrm{gp}} \to \overline{M}_U^{\mathrm{gp}}$ is an isomorphism (since this can be verified over X'). Therefore $U \to X$ is a finite étale covering and consequently trivial, since R is a complete local ring with separably closed residue field.

10. Lifting étale covers of fibres

10.1. In order to prove Theorems 1.7 and 1.9, it will be useful to have some results about lifting étale covers from a geometric fibre to the total space of a fibration. In the classical setting of topology, the problem of finding such liftings can be understood in terms of the higher homotopy groups of the base, using the long exact sequence of homotopy groups of a fibration. We use this idea to obtain results in the logarithmic setting.

For a log scheme (T, M_T), write $(T, M_T)_{\mathrm{két}}$ for the Kummer étale topos of (T, M_T).

10.2. Let

$$(f, f^b) : (X, M_X) \to (B, M_B)$$

be a log smooth integral morphism of fs log schemes, with B connected. Let $\mathbf{L}$ be a set of primes which includes all the residue characteristics of B.

Theorem 10.3. *Assume that for any locally constant sheaf of finite abelian groups A on $(B, M_B)_{\mathrm{két}}$ of order not divisible by any prime in $\mathbf{L}$ and class $\alpha \in H^2((B, M_B)_{\mathrm{két}}, A)$, there exists a finite Kummer étale covering $(B_\alpha, M_{B_\alpha}) \to (B, M_B)$ such that the class α maps to 0 in $H^2((B_\alpha, M_{B_\alpha})_{\mathrm{két}}, A)$.*

Let $\bar{b}^{\log} \to (B, M_B)$ be a log geometric point and let $(U_{\bar{b}^{\log}}, M_{U_{\bar{b}^{\log}}}) \in \mathrm{Fét}^{\mathbf{L}}((X, M_X)_{(\bar{b}^{\log})})$ be an object. Then, after possibly replacing (B, M_B) by a covering in $\mathrm{Fét}(B, M_B)$, there exists an object $(U, M_U) \in \mathrm{Fét}^{\mathbf{L}}(X, M_X)$ inducing $(U_{\bar{b}^{\log}}, M_{U_{\bar{b}^{\log}}})$. Moreover, any two such objects become isomorphic after a finite Kummer étale extension of (B, M_B).

Proof. By [11, Theorem 9.9], for any prime ℓ not in $\mathbf{L}$ the sheaf (pushforward for Kummer étale topos)

$$f_{\mathrm{két}*}(\mathbf{Z}/(\ell))$$

is locally constant constructible on (B, M_B) and its formation commutes with base change. Replacing (B, M_B) by a Kummer étale covering over which this locally constant sheaf is trivial, we are then reduced to the case when the geometric fibres of f are connected, which we assume for the rest of the proof.

Let G be a finite group of order not divisible by any prime of $\mathbf{L}$. To prove the theorem it suffices to show the variant statement that if $(U_{\bar{b}^{\log}}, M_{U_{\bar{b}^{\log}}})$ is a covering of the geometric fibre which is Galois with group G, then after replacing (B, M_B) by a covering as in the theorem we can find a G-covering (U, M_U) of (X, M_X) inducing the given G-cover in the fibre.

Let $R^1 f_{\text{két}*} G$ be the sheaf associated to the presheaf on the Kummer étale site $\text{Ket}(B, M_B)$ which to any $(U, M_U) \to (B, M_B)$ associates the pointed set of isomorphism classes of G-torsors on $(X, M_X) \times_{(B, M_B)} (U, M_U)$.

Lemma 10.4. *The natural map*

$$(R^1 f_{\text{két}*} G)_{\bar{b}^{\log}} \to |\text{Fét}^G((X, M_X)_{(\bar{b}^{\log})})| \tag{10.4.1}$$

is an isomorphism, where $|\text{Fét}^G((X, M_X)_{(\bar{b}^{\log})})|$ *denotes the set of isomorphism classes in the category of* G*-coverings* $\text{Fét}^G((X, M_X)_{(\bar{b}^{\log})})$*, defined as in [11, Definition 3.1].*

Proof. This is an immediate consequence of Proposition 5.1. □

Let $\mathscr{F}_{\bar{b}^{\log}} \subset (R^1 f_{\text{két}*} G)_{\bar{b}^{\log}}$ be the subset obtained from the $\pi_1((b, M_b), \bar{b}^{\log})$-orbit of the class of $(U_{\bar{b}^{\log}}, M_{U_{\bar{b}^{\log}}})$ and (10.4.1).

Lemma 10.5. *There exists a unique locally constant subsheaf* $\mathscr{F} \subset R^1 f_{\text{két}*} G$ *whose stalk at* $\bar{b}^{\log}$ *is* $\mathscr{F}_{\bar{b}^{\log}}$*. Furthermore, for any other log geometric point* $\bar{b}'^{\log} \to (B, M_B)$ *with image point* $(b', M_{b'})$*, the stalk* $\mathscr{F}_{\bar{b}'^{\log}} \subset (R^1 f_{\text{két}*} G)_{\bar{b}'^{\log}}$ *is an orbit for the* $\pi_1((b', M_{b'}), \bar{b}'^{\log})$*-action.*

Proof. By uniqueness, it suffices to consider the case when B is integral and $\bar{b}^{\log}$ maps to the generic point of B.

Consider the sheaf $\mathscr{I}$ on $(B, M_B)_{\text{két}}$, which to any $(V, M_V) \to (B, M_B)$ associates

$$\prod_{\rho: \bar{b}^{\log} \to (V, M_V)} |\text{Fét}^G((X, M_X)_{(\bar{b}^{\log})}|,$$

where the product is taken over lifts to (V, M_V) of the morphism $\bar{b}^{\log} \to (B, M_B)$. Let $T \subset |\text{Fét}^G((X, M_X)_{(\bar{b}^{\log})}|$ denote the orbit of the class of $(U_{\bar{b}^{\log}}, M_{U_{\bar{b}^{\log}}})$ under the action of the automorphism group $\pi_1((b, M_b), \bar{b}^{\log})$ of $\bar{b}^{\log}$ over (b, M_b). Since $(U_{\bar{b}^{\log}}, M_{U_{\bar{b}^{\log}}})$ is defined over some finite extension of (b, M_b), the set T is finite. Let $\mathscr{T}$ denote the subsheaf of $\mathscr{I}$ which to any $(V, M_V) \to (B, M_B)$ associates

$$\prod_{\rho: \bar{b}^{\log} \to (V, M_V)} T,$$

and let $\mathscr{F} \subset R^1 f_{\text{két}*} G$ denote the preimage of $\mathscr{T}$ under the natural inclusion

$$R^1 f_{\text{két}*} G \hookrightarrow \mathscr{I}.$$

It then follows from inspection of the stalks that $\mathscr{F}$ is a locally constant constructible subsheaf of $R^1 f_{\text{két}*}G$ whose restriction to $\bar{b}^{\log}$ is the set T. □

Returning to the proof of Theorem 10.3, after replacing (B, M_B) by a finite Kummer étale cover we can assume that $\mathscr{F}$ is a constant sheaf. We therefore have a distinguished section $\gamma \in H^0((B, M_B)_{\text{két}}, R^1 f_{\text{két}*}G)$ whose image in $|\text{Fét}^G((X, M_X)_{(\bar{b}^{\log})})|$ is the class of $(U_{\bar{b}^{\log}}, M_{U_{\bar{b}^{\log}}})$.

Let $\mathscr{G}_\gamma$ be the fibred catgory on the Kummer étale site of (B, M_B) which to any $(V, M_V) \to (B, M_B)$ associates the category of G-torsors $(P, M_P) \to (X, M_X) \times_{(B, M_B)} (V, M_V)$ whose induced class in $H^0((V, M_V), R^1 f_{\text{két}*}G)$ is the image of the class of γ. Then $\mathscr{G}_\gamma$ is a gerbe with associated band, in the sense of [8, Chapitre IV, §1 and 2], equal to the band L_G associated to G. Let $Z \subset G$ be the centre. Then by [8, Chapitre IV, 3.3.3], the natural action of $H^2((B, M_B)_{\text{két}}, Z)$ on $H^2((B, M_B)_{\text{két}}, L_G)$ is simply transitive. This, combined with our assumption that every class in $H^2((B, M_B)_{\text{két}}, Z)$ can be killed by a finite Kummer étale cover, implies that there exists such a cover of (B, M_B) over which $\mathscr{G}_\gamma$ is trivial. Making such a cover, we obtain the desired global torsor.

Finally, to see that any two such torsors

$$(U_i, M_{U_i}) \to (X, M_X), \quad i = 1, 2,$$

become isomorphic after a Kummer étale finite extension of (B, M_B), note that it follows from the foregoing discussion that the sheaf on the Kummer étale site of (B, M_B) of isomorphisms between these two covers is a locally constant nonempty sheaf.

This completes the proof of Theorem 10.3. □

11. Proofs of Theorems 1.7 and 1.9 in general

11.1. Let $f : (X, M_X) \to (B, M_B)$ be a log smooth proper morphism of finite type. Combining [21, Chapter III, Theorem 2.6.7] and [12, Proposition A.3.4], we know that étale locally on B there exists a sequence of morphisms

$$(B_1, M_{B_1}) \xrightarrow{q} (B_2, M_{B_2}) \xrightarrow{b} (B, M_B),$$

where b is a log blowup and q is a Kummer étale covering, such that the base change in the category of fs log schemes

$$(X_1, M_{X_1}) \to (B_1, M_{B_1})$$

is integral.

In fact, we can describe this sequence more precisely. Working étale locally on B, we can assume that $B = \text{Spec}(R)$ is affine and that there exists a chart $P \to R$ for M_B and a finitely generated ideal

$$K = \langle f_1, \ldots, f_r \rangle \subset P$$

such that if

$$b : (B_2, M_{B_2}) \to (B, M_B)$$

is the log blowup with respect to K, then the base change

$$(X_2, M_{X_2}) \to (B_2, M_{B_2})$$

is $\mathbf{Q}$-integral in the sense of [21, Chapter I, Definition 4.7.4]. If we describe the blowup using charts as in the proof of Theorem 9.2, so that B_2 is covered by open sets $U_j := \operatorname{Spec}(R \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P_j])$ and the restriction M_{U_j} of M_{B_2} to U_j is given by the chart $P_j \to R \otimes_{\mathbf{Z}[P]} \mathbf{Z}[P_j]$, then [12, Proposition A.3.4] implies that we can take q to be the Kummer étale covering obtained from the finite Kummer étale covers of the (U_j, M_{U_j}) induced by multiplication by n on P_j for a suitable integer n invertible in B. In other words, we can take q to be the Kummer étale covering

$$(B_2^{1/n}, M_{B_2^{1/n}}) \to (B_2, M_{B_2})$$

induced by multiplication by n on P (see also the observations around (9.2.5)).

This implies in particular that in this local setting, when $X = \operatorname{Spec}(R)$ with a chart $P \to R$ and b given by the blowup of a finitely generated ideal K, we can reverse the order of the Kummer étale cover and the log blowup. Let

$$(B^{1/n}, M_{B^{1/n}}) \to (B, M_B)$$

be the Kummer étale coverings induced by multiplication by n on P. Then $M_{B^{1/n}}$ again admits a chart $\beta_n : P \to M_{B^{1/n}}$, and the natural map

$$(B_2^{1/n}, M_{B_2^{1/n}}) \to (B^{1/n}, M_{B^{1/n}})$$

is the log blowup of the target with respect to $\beta_n(K)$.

Since the assertions of Theorems 1.7 and 1.9 are local for the Kummer étale topology on (B, M_B), we see that it suffices to prove both results under the further assumption that there exists a log blowup

$$(B', M_{B'}) \to (B, M_B)$$

such that the base change

$$(X', M_{X'}) \to (B', M_{B'})$$

is integral.

Note furthermore that $B' \to B$ has geometrically connected fibres, as discussed in the proof of Theorem 9.2. In particular, if B is connected, then so is B'.

By the case of an integral morphism discussed in Section 8, we know that Theorem 1.9 holds for $(X', M_{X'}) \to (B', M_{B'})$. Using the case of an integral morphism, we reduce the proof of Theorem 1.9 to Theorem 1.7 as follows. Choose log geometric points

$$\bar{b}_i^{\prime\log} \to (B', M_{B'})$$

lying over $\bar{b}_i^{\log}$, and let $\bar{c}_i^{\log} \to (B, M_B)$ denote the composition

$$\bar{b}_i^{\prime\log} \longrightarrow (B', M_{B'}) \longrightarrow (B, M_B).$$

Then by definition we have

$$\mathrm{F\acute{e}t}^{\mathbf{L}}((X', M_{X'})_{(\bar{b}_i'^{\log})}) \simeq \mathrm{F\acute{e}t}^{\mathbf{L}}((X, M_X)_{(\bar{c}_i^{\log})}).$$

Therefore, if we prove Theorem 1.7 in general, we get equivalences

$$\begin{array}{l} \mathrm{F\acute{e}t}^{\mathbf{L}}((X, M_X)_{(\bar{b}_1^{\log})}) \xrightarrow{1.7} \mathrm{F\acute{e}t}^{\mathbf{L}}((X, M_X)_{(\bar{c}_1^{\log})}) \xrightarrow{\simeq} \mathrm{F\acute{e}t}^{\mathbf{L}}((X', M_{X'})_{(\bar{b}_1'^{\log})}) \\ \qquad \xrightarrow{\simeq} \mathrm{F\acute{e}t}^{\mathbf{L}}((X', M_{X'})_{(\bar{b}_2'^{\log})}) \xrightarrow{\simeq} \mathrm{F\acute{e}t}^{\mathbf{L}}((X', M_{X'})_{(\bar{c}_2^{\log})}) \xrightarrow{1.7} \mathrm{F\acute{e}t}^{\mathbf{L}}((X', M_{X'})_{(\bar{b}_2^{\log})}), \end{array}$$

where the middle diagonal equivalence is using Theorem 1.9 in the case of an integral morphism.

To complete the proofs of Theorems 1.7 and 1.9, we are thus reduced to proving Theorem 1.7 in the case when $(B, M_B) = (\mathrm{Spec}(k), M_k)$ is a log point and $\bar{b}'^{\log} \to (B, M_B)$ lifts to $(B', M_{B'})$. We can further assume that we have a chart $P \to M_B$ inducing an isomorphism $k^* \oplus P \simeq M_B$, that $(B', M_{B'})$ is obtained by blowing up a finitely generated ideal $K \subset P$ and that k is separably closed. We can further assume that $\bar{b}^{\log}$ is given by the geometric point

$$(\mathrm{Spec}(k), k^* \oplus P_{\mathbf{Z}_{(p)}}) \to (\mathrm{Spec}(k), k^* \oplus P).$$

11.2. For an integer $n \geq 1$ invertible in k, let

$$\times n : (B, M_B) \to (B, M_B), \quad \times n : (B', M_{B'}) \to (B', M_{B'})$$

be the maps induced by multiplication by n on P, and denote the base changes by

$$(X_n, M_{X_n}) := (X, M_X) \times_{(B, M_B), \times n} (B, M_B), \ \ (X_n', M_{X_n'}) := (X', M_{X'}) \times_{(B', M_{B'}), \times n} (B', M_{B'}).$$

As before, the map $\times n : (B', M_{B'}) \to (B', M_{B'})$ can be described explicitly in terms of the open cover $\mathrm{Spec}(k \otimes_{k[P]} k[P_j])$ obtained by choosing generators $f_1, \ldots, f_r \in K$, and the multiplication by n maps on P_j. By the previous discussion, each morphism

$$(X_n', M_{X_n'}) \to (X_n, M_{X_n})$$

is a log blowup. Since the system of log structures

$$k^* \oplus \frac{1}{n} P \subset k^* \oplus P_{\mathbf{Z}_{(p)}}$$

is cofinal among fine log structures in $k^* \oplus P_{\mathbf{Z}_{(p)}}$ containing M_B, it follows from this and Theorem 9.2 that the pullback functor

$$\mathrm{F\acute{e}t}^{(p)}((X, M_X)_{(\bar{b}^{\log})}) \simeq \mathrm{colim}_n \mathrm{F\acute{e}t}^{(p)}(X_n, M_{X_n}) \to \mathrm{colim}_n \mathrm{F\acute{e}t}^{(p)}(X_n', M_{X_n'})$$

is an equivalence of categories.

This reduces to showing that the functor

$$\mathrm{colim}_n \mathrm{F\acute{e}t}^{(p)}(X_n', M_{X_n'}) \to \mathrm{F\acute{e}t}^{(p)}((X', M_{X'})_{(\bar{b}'^{\log})}) \qquad (11.2.1)$$

is an equivalence of categories for any log geometric point $\bar{b}'^{\log} \to (B', M_{B'})$. For this we will apply Theorem 10.3 to the morphism

$$(X', M_{X'}) \to (B', M_{B'}).$$

In order to do so, note that by Theorem 10.3 the pullback functor

$$\mathrm{Fét}^{(p)}(B, M_B) \to \mathrm{Fét}^{(p)}(B', M_{B'}) \tag{11.2.2}$$

is an equivalence of categories, so any locally constant sheaf of finite abelian groups A' on $(B', M_{B'})_{\text{két}}$ of order prime to p is pulled back from a locally constant sheaf A on $(B, M_B)_{\text{két}}$, and by [11, Theorem 6.2] the pullback functor

$$H^i((B, M_B)_{\text{két}}, A) \to H^i((B', M_{B'}), A')$$

is an isomorphism. Since any class in $H^2((B, M_B)_{\text{két}}, A)$ is killed by a finite Kummer étale covering of (B, M_B), it follows that the same is true for a class in $H^2((B', M_{B'}), A')$, and the assumptions of Theorem 10.3 hold.

Now since (11.2.2) is an equivalence, the collection of finite Kummer étale covers

$$(X'_n, M_{X'_n}) \to (X', M_{X'})$$

are cofinal among finite Kummer étale coverings of $(X', M_{X'})$, and therefore we conclude from Theorem 10.3 that (11.2.1) is an equivalence.

This completes the proofs of Theorems 1.7 and 1.9. □

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