Almost Symplectic Structures on $Spin(7)-$Manifolds

Sema Salur and Eyup Yalcinkaya

Abstract. A non-degenerate differential 2-form on an even dimensional manifold M^{2n} is called an almost-symplectic structure. A necessary condition for the existence of an almost-symplectic structure is that all odd-dimensional Stiefel-Whitney classes of M should vanish. In this paper, we prove that all odd-dimensional Stiefel-Whitney classes of a smooth, closed, connected, orientable 8-manifold with spin structure vanish. We also study the almost-symplectic structures on certain classes of $Spin(7)$-manifolds.

Mathematics Subject Classification (2010). Primary 53D15; Secondary 53C29.

Keywords. Holonomy, almost symplectic structure, $Spin(7)$-manifold.

1. Introduction

In this paper we study geometric structures on $Spin(7)$ manifolds. A $Spin(7)$-manifold is an 8-dimensional Riemannian manifold with the holonomy group inside the exceptional Lie group $Spin(7)$. Manifolds with special holonomy are spaces whose infinitesimal symmetries allow them to play a crucial role in M-theory compactifications. They represent the tiny curled up dimensions hiding at every point of spacetime. Examples of manifolds with special holonomy are 6-dimensional Calabi-Yau manifolds, 7-dimensional G_2 manifolds and 8-dimensional $Spin(7)$ manifolds. Despite extensive research on Calabi-Yau manifolds, the geometric properties of G_2 and $Spin(7)$ manifolds are not well understood. In this paper we initiate a program to study almost symplectic structures on Riemannian 8-manifolds with spin structure.

In particular we prove

Theorem: *All the odd-dimensional Stiefel-Whitney classes of a smooth, closed, connected, orientable 8-manifold with spin structure vanish.*

Note that a manifold M with $Spin(7)$-structure is orientable and spin. The theorem above implies that the obstructions for the existence of almost symplectic (and hence almost complex) structures on a manifold with full $Spin(7)$ holonomy vanish as well. There are inclusions between the groups

$$SU(2) \longrightarrow SU(3) \longrightarrow G_2 \longrightarrow Spin(7),$$

and

$$SU(2) \times SU(2) \longrightarrow Sp(2) \longrightarrow SU(4) \longrightarrow Spin(7).$$

These are the only connected Lie subgroups of $Spin(7)$ which can be holonomy groups of Riemannian metrics on 8-manifolds. Hence the theorem above also holds for 8-manifolds with reduced holonomy groups.

2. $Spin(7)$-structures

In this section we review the basics of $Spin(7)$ geometry. More on the subject can be found in [4], [8], [6] and [12].

Let $(x^1, ..., x^8)$ be coordinates on $\mathbb{R}^8$. The standard Cayley 4-form on $\mathbb{R}^8$ can be written as

$$\begin{aligned}\Phi_0 &= dx^{1234} + dx^{1256} + dx^{1278} + dx^{1357} - dx^{1368} - dx^{1458} - dx^{1467} \\ &\quad - dx^{2358} - dx^{2367} - dx^{2457} + dx^{2468} + dx^{3456} + dx^{3478} + dx^{5678}\end{aligned}$$

where $dx^{ijkl} = dx^i \wedge dx^j \wedge dx^k \wedge dx^l$.

The subgroup of $GL(8, \mathbb{R})$ that preserves Φ_0 is the group $Spin(7)$. It is a 21-dimensional compact, connected and simply-connected Lie group which preserves the orientation on $\mathbb{R}^8$ and the Euclidean metric g_0.

Definition 2.1. A differential 4-form Φ on an oriented 8-manifold M is called admissible if it can be identified with Φ_0 through an oriented isomorphism between T_pM and $\mathbb{R}^8$ for each point $p \in M$.

Definition 2.2. Let $\mathcal{A}(M)$ denotes the space of admissible 4-forms on M. A $Spin(7)$-structure on an 8-dimensional manifold M is an admissible 4-form $\Phi \in \mathcal{A}(M)$. If M admits such structure, (M, Φ) is called a manifold with $Spin(7)$-structure.

Each 8-manifold with a $Spin(7)$-structure Φ is canonically equipped with a metric g. Hence, we can think of a $Spin(7)$-structure on M as a pair (Φ, g) such that for all $p \in M$ there is an isomorphism between T_pM and $\mathbb{R}^8$ which identifies (Φ_p, g_p) with (Φ_0, g_0).

Existence of a $Spin(7)$-structure on an 8-dimensional manifold M is equivalent to a reduction of the structure group of the tangent bundle of M from $SO(8)$ to its subgroup $Spin(7)$. The following result gives the necessary and sufficient conditions so that the 8-manifold admits $Spin(7)$ structure.

Theorem 2.3. *([6], [4]) Let M be a differentiable 8-manifold. M admits a $Spin(7)$-structure if and only if $w_1(M) = w_2(M) = 0$ and for appropriate choice of orientation on M we have that*

$$p_1(M)^2 - 4p_2(M) \pm 8\chi(M) = 0.$$

Furthermore, if $\nabla\Phi = 0$, where ∇ is the Riemannian connection of g, then $\mathrm{Hol}(M) \subseteq Spin(7)$, and M is called a $Spin(7)$-manifold. All $Spin(7)$ manifolds are Ricci flat.

Let (M, g, Φ) be a $Spin(7)$ manifold. The action of $Spin(7)$ on the tangent space gives an action of $Spin(7)$ on the spaces of differential forms, $\Lambda^k(M)$, and so the exterior algebra splits orthogonally into components, where Λ^k_l corresponds to an irreducible representation of $Spin(7)$ of dimension l:

$$\begin{gathered}\Lambda^1(M) = \Lambda^1_8, \quad \Lambda^2(M) = \Lambda^2_7 \oplus \Lambda^2_{21}, \quad \Lambda^3(M) = \Lambda^3_8 \oplus \Lambda^3_{48}, \\ \Lambda^4(M) = \Lambda^4_+(M) \oplus \Lambda^4_-(M), \quad \Lambda^4_+(M) = \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27}, \quad \Lambda^4_- = \Lambda^4_{35} \\ \Lambda^5(M) = \Lambda^5_8 \oplus \Lambda^5_{48} \quad \Lambda^6(M) = \Lambda^6_7 \oplus \Lambda^6_{21}, \quad \Lambda^7(M) = \Lambda^7_8;\end{gathered}$$

where $\Lambda^4_\pm(M)$ are the $\pm$-eigenspaces of $*$ on $\Lambda^4(M)$ and

$$\begin{gathered}\Lambda^2_7 = \{\alpha \in \Lambda^2(M) | *(\alpha \wedge \Phi) = 3\alpha\}, \quad \Lambda^2_{21} = \{\alpha \in \Lambda^2(M) | *(\alpha \wedge \Phi) = -\alpha\}, \\ \Lambda^3_8 = \{*(\beta \wedge \Phi) | \beta \in \Lambda^1(M)\}, \quad \Lambda^3_{48} = \{\gamma \in \Lambda^3(M) | \gamma \wedge \Phi = 0\}, \\ \Lambda^4_1 = \{f\Phi | f \in \mathcal{F}(M)\}\end{gathered}$$

The Hodge star $*$ gives an isometry between Λ^k_l and Λ^{8-k}_l.

3. Almost symplectic structures and $Spin(7)$-structures

In this section we show that all the odd-dimensional Stiefel-Whitney classes on a closed, connected orientable 8-manifold with spin structure vanish.

An almost symplectic manifold M is a n-dimensional manifold ($n = 2m$) with a non degenerate 2-form ω. If in addition, ω is closed then M is called a symplectic manifold. An almost-symplectic structure defines an $Sp(m, \mathbb{R})$ structure. A necessary and sufficient condition for the existence of an almost-symplectic structure on M is the reduction of the structure group of the tangent bundle to the unitary group $U(m)$. It is therefore necessary that all odd-dimensional Stiefel-Whitney classes of M to vanish [9].

For any manifold M and integer $k \geq 0$, one can construct a graded linear map $Sq^k : H^*(M, \mathbb{Z}_2) \to H^*(M, \mathbb{Z}_2)$ of degree k. This is called the k^{th} Steenrod square. One can define Stiefel-Whitney classes using both Steenrod squares and the Thom isomorphism.

There is also a unique class $\nu_k \in H^k(M, \mathbb{Z}_2)$ such that for any $x \in H^{n-k}(M, \mathbb{Z}_2)$, $Sq^k(x) = \nu_k \cup x$. We call this class ν_k, the k^{th} Wu class.

Now suppose M is a smooth, closed, connected n-dimensional manifold. Wu's theorem states that the total Stiefel-Whitney class of the tangent bundle of M, denoted by w, Steenrod squares and Wu classes are all related by the equation $w = Sq(\nu)$, for more on the subject see [11]. This gives the following formula:

$$w_k = \sum_{i+j=k} Sq^i(\nu_j)$$

One can also compute the action of the Steenrod squares on the Stiefel-Whitney classes. This is called the Wu formula:

$$Sq^i(w_j) = \sum_{t=0}^{i} \binom{j+t-i-1}{t} w_{i-t} w_{j+t}$$

for $0 \leq i \leq j$. Thus we obtain

$$
\begin{aligned}
w_1 &= Sq^0(\nu_1) = \nu_1, \\
w_2 &= Sq^0(\nu_2) + Sq^1(\nu_1) = \nu_2 + \nu_1 \cup \nu_1, \\
w_3 &= Sq^0(\nu_3) + Sq^1(\nu_2) = \nu_3 + Sq^1(\nu_2) = \nu_3 + Sq^1(w_2) + Sq^1(w_1 \cup w_1), \\
w_4 &= Sq^0(\nu_4) + Sq^1(\nu_3) + Sq^2(\nu_2) = \nu_4 + Sq^1(\nu_3) + \nu_2 \cup \nu_2 \\
w_5 &= Sq^0(\nu_5) + Sq^1(\nu_4) + Sq^2(\nu_3) = \nu_5 + Sq^1(\nu_4) + Sq^2(w_1 \cup w_2)
\end{aligned}
$$

And one can write the corresponding Wu classes as polynomials in the Stiefel-Whitney classes as follows: For simplicity, we replace the cup product symbol by multiplication sign.

$$
\begin{aligned}
\nu_1 &= w_1, \\
\nu_2 &= w_2 + w_1^2, \\
\nu_3 &= w_1 w_2, \\
\nu_4 &= w_4 + w_3 w_1 + w_2^2 + w_1^4, \\
\nu_5 &= w_4 w_1 + w_3 w_1^2 + w_2^2 w_1 + w_2 w_1^3,
\end{aligned}
$$

In a spin manifold, $w_1 = w_2 = 0$ which imply $v_1 = v_2 = 0$ which then gives $w_3 = v_3$. One can also see that $w_3 = 0$ as follows: Note that by definition of Wu classes, $v_3 \cup x = Sq^3(x)$ for all $x \in H^{(n-3)}(M, \mathbb{Z}_2)$. Then one can see that Sq^3 is a linear combination of $Sq^1 \circ Sq^2$ and $Sq^2 \circ Sq^1$ and $Sq^1 \circ Sq^1 \circ Sq^1$ so that we get

$$v_3 \cup x = (aSq^1Sq^2 + bSq^2Sq^1 + cSq^1Sq^1Sq^1)(x) = Sq^1(y) + Sq^2(z)$$

for some y, z. This term is equal to $v_1 \cup y + v_2 \cup z = 0$. As $v_3 \cup x = 0$ for all x, Poincare duality then gives $v_3 = 0$ and hence $w_3 = 0$.

The Wu relations also imply that $w_4 = \nu_4$. Since $w_1 = 0$ (as M is orientable) this gives us $w_5 = Sq^1(w_4)$. Equivalently, w_5 is the image of w_4 under the Bockstein map induced by

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

This implies that w_5 is the mod-2 reduction of the integral Stiefel-Whitney class W_5, which is the element of $H^5(M, \mathbb{Z})$, that is the image of w_4 under the Bockstein map induced by

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

Note also that v_4 is by definition the Poincare dual to the $\mathbb{Z}_2$ linear map

$$Sq^4 : H^4(M, \mathbb{Z}_2) \to H^8(M, \mathbb{Z}_2) = \mathbb{Z}_2$$

which implies

$$w_4.x = v_4.x = x.x$$

for any element x of $H^4(M, \mathbb{Z}_2)$. In other words, w_4 just represents the mod-2 intersection form.

One can then use the Hirzebruch-Hopf theorem [7] to show that w_4 has an integer lift and therefore w_4 is in the image of $H^4(M, \mathbb{Z}) \longrightarrow H^4(M, \mathbb{Z}_2)$ and so $W_5 = 0$.

The commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_4 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0 \end{array}$$

induces a commutative diagram of the corresponding long exact sequences and hence implies that $w_5 = 0$.

And finally, in order to show that $w_7 = 0$ we use a result of W. Massey. In ([10], Thm II) it was shown that if M^{2m} is orientable (where $2m \equiv 0 \pmod 4$), then $w_{2m-1} = 0$. Hence for an 8-manifold we obtain $w_7 = 0$. One can read more about the proof in ([10], Section 5).

This completes the proof of the main theorem:

Theorem 3.1. All the odd-dimensional Stiefel-Whitney classes of a smooth, closed, connected, orientable 8-manifold with spin structure vanish.

4. A Motivating Example

Next, we discuss a special class of $Spin(7)$ manifolds that admits an almost complex structure and show how it is related to the $Spin(7)$-structure.

Let (M, Φ) be a $Spin(7)$ manifold (or more generally manifold with $Spin(7)$-structure) admitting a non-vanishing 2-plane field $\Lambda = \{u, v\} \in TM$. In [13], E. Thomas shows that the Euler characteristic $\chi(M) = 0$ and the signature $\sigma(M) \equiv 0 \pmod 4$ provides the necessary and sufficient conditions for the existence of a 2-plane field on an 8-manifold. Now define, $[u, v]^{\perp} = \{w \in TM | < u, w >=< v, w >= 0\}$. One can show that $[u, v]^{\perp}$ carries a non-degenerate 2-form $\omega_{u,v}$ which is compatible with the almost complex structure $J_{u,v} : [u, v]^{\perp} \to [u, v]^{\perp}$ and given by

$$\omega_{u,v}(w, z) = \Phi(w, u, v, z) \text{ and } J_{u,v}(z) = u \times v \times z.$$

Definition 4.1. Let (M, Φ) be a $Spin(7)$ manifold. Then $J_{u,v}(z) = u \times v \times z$ is the triple cross product defined by the identity:

$$< J_{u,v}(z), w >= \Phi(u, v, z, w). \tag{4.1}$$

Theorem 4.2. *Let (M, Φ) be a $Spin(7)$ manifold with a non-vanishing oriented 2-plane field. Then, $J_{u,v}(z) = u \times v \times z$ defines an almost complex structure on M compatible with the $Spin(7)$ structure.*

Proof. Let $\{u, v\} \in TM$ be two vectors generating the non-vanishing oriented 2-plane field. $J(z)$ is well defined since by Equation (1), $< J(z), w >= \Phi(u, v, z, w)$.

Next, we show $J^2(z) = -id$. This can be done using the properties of the $Spin(7)$-structure on M. Let $z_i, z_j \in TM$,

Then

$$\begin{aligned} < u \times v \times (u \times v \times z_i), z_j > &= \Phi(u, v, (u \times v \times z_i), z_j) \\ &= -\Phi(u, v, z_j, (u \times v \times z_i)) \\ &= - < u \times v \times z_j, u \times v \times z_i > \\ &= -\delta_{ij} \end{aligned}$$

The last equality holds since the map J is orthogonal. Note that the map J only depends on the oriented 2-plane $\Lambda = \{u, v\}$.

□

5. Interesting Questions

One major problem in the field of manifolds with special holonomy is a lack of an existence theorem that gives necessary and sufficient conditions for a 7-dimensional manifold to admit a G_2 metric. In an earlier paper, [3], Arikan, Cho and Salur proposed a program to study the relations between (almost) contact structures and G_2 structures. The main goal is to understand the topological obstructions for the existence of a G_2 metric on a Riemannian 7-manifold with spin structure. In that paper, they proved the following theorem:

Theorem 5.1. Every 7-manifold with a spin structure admits an almost contact structure.

Since every 7-manifold with spin structure admits a G_2 structure this implies:

Corollary 5.2. Every manifold with G_2-structure admits an almost contact structure.

As one might expect, a promising direction for future investigation is to obtain similar results for almost complex (and hence almost symplectic) 8-manifolds with $Spin(7)$ structures. Understanding almost complex structures on a Spin(7) manifold might help us to understand the properties of the Spin(7) metric. We plan to investigate these relations in a future paper.

Also in the papers, [1], [2] and [5], it is shown that the rich geometric structures of a G_2 manifold N with 2-plane fields provide complex and symplectic structures to certain 6-dimensional subbundles of $T(N)$. Using the 2-plane fields, one can introduce a mathematical definition of "mirror symmetry" for Calabi-Yau and G_2 manifolds. More specifically, one can assign a G_2 manifold (N, φ, Λ), with the calibration 3-form φ and an oriented 2-plane field Λ, a pair of parametrized tangent bundle valued 2 and 3-forms of N. These forms can then be used to define different complex and symplectic structures on certain 6-dimensional subbundles of $T(N)$. When these bundles are integrated they give mirror CY manifolds. This is one way of explaining duality between the symplectic and complex structures on the CY 3-folds inside of a G_2 manifold. Similarly, one can construct these structures and define mirror dual Calabi Yau manifolds inside a $Spin(7)$ manifold which admits an almost complex structure. These topics will be also studied in a future paper.

Acknowledgment

The authors would like to thank Claude Le Brun and Jonathan Pakianathan for their helpful advice on various technical issues discussed in this paper. The first named author is partially supported by NSF grant DMS–1711178.

References

[1] S. Akbulut and S. Salur, *Mirror Duality via G_2 and Spin(7) Manifolds*, Special Volume For the Arithmetic & Geometry Around Quantization, Prog. in Mathematics Series, Birkhauser, (2009), Edited by Yuri Manin & Matilde Marcolli.

[2] S. Akbulut and S. Salur, *Harvey Lawson Manifolds and Dualities*, Real and Complex Submanifolds, Daejeon, Korea, Springer Proceedings in Math and Stat. (2015), 505–514.

[3] M. F. Arikan, H. Cho, S. Salur, *Existence of Compatible Contact Structures on G_2 Manifolds*, Asian Journal of Math., Volume 17, No. 2, (2013), 321–334.

[4] M. Fernandez, *A Classification of Riemannian Manifolds with Structure Group Spin(7)*, Annali di Mat. Pura ed App., vol (143), (1986), 101–122.

[5] R. Glover and S. Salur, *Lagrangian-type Submanifolds of G_2 and Spin(7) manifolds and their deformations* (2019), Differential Geometry and its Applications, Volume 65, August 2019, 241–255.

[6] F.R. Harvey and H.B. Lawson, *Calibrated Geometries*, Acta. Math. **148** (1982), 47–157.

[7] F. Hirzebruch and H. Hopf, *Felder von Flachenelementen in $4-$dimensionalen Mannigfaltigkeiten*, Math. Ann. 136 (1958), 156–172.

[8] D. Joyce, *Riemannian Holonomy Groups and Calibrated Geometry*, Oxford Graduate Texts in Mathematics 12, OUP.

[9] P. Liberman, *Sur les structures presque complexe et autres structures infinitesimales regulieres*, Bull. Soc. Math. France, 83 (1955), 195224.

[10] W. Massey, *On the Stiefel-Whitney Classes of a Manifold*, American Journal of Mathematics, Vol. 82, No. 1 (Jan., 1960), pp. 92–102.

[11] J. Milnor and J. Stasheff, *Characteristic Classes*, No. 76 of Annals of Mathematics Studies, Princeton University. Press, 1974.

[12] I. Unal and E. Yalcinkaya, *On Symplectic 8-manifolds Admitting Spin(7)-Structure*, Preprint.

[13] E. Thomas, *Vector fields on manifolds*, Bull. of the American Math. Soc., vol 75, (1969), 643–683.

Sema Salur
Department of Mathematics
University of Rochester
Rochester NY, USA
e-mail: `salur@math.rochester.edu`

Eyup Yalcinkaya
The Scientific and Technological Research Council of Turkey
Ankara, Turkey
e-mail: `eyup.yalcinkaya@tubitak.gov.tr`