Scalable Vehicle Team Continuum Deformation Coordination with Eigen Decomposition

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Abstract—The continuum deformation leader-follower cooperative control strategy models vehicles in a multi-agent system as particles of a deformable body. A desired continuum deformation is defined based on leaders' trajectories and acquired by followers in real-time through local communication. The existing continuum deformation theory requires followers to be placed inside the convex simplex defined by leaders. This constraint is relaxed in this paper. We prove that, under suitable assumptions, any n+1 (n=1,2,3) vehicles forming an n-D simplex can be selected as leaders while followers, arbitrarily positioned inside or outside the leading simplex, can acquire a desired continuum deformation in a decentralized fashion. The paper's second contribution is to assign a one-to-one mapping between leaders' smooth trajectories and homogeneous deformation features obtained by continuum deformation eigen-decomposition. Therefore, a safe and smooth continuum deformation coordination can be planned either by shaping homogeneous transformation features or by choosing appropriate leader trajectories. This is beneficial to efficiently plan and guarantee collision avoidance in a large-scale group. A simulation case study is reported in which a virtual convex simplex contains a quadcopter vehicle team at any time t; A* search is applied to optimize quadcopter team continuum deformation in an obstacle-laden environment.

Keywords—Path Planning, Collision Avoidance, Multi-vehicle System (MVS), Eigen Decomposition, Local Communication

I. INTRODUCTION

Formation and cooperative control algorithms [1] have been applied to problems in biology [2], computer science [3], aerospace engineering [4], and elsewhere. Virtual structure [5], [6], consensus [7]–[9], containment [10]–[12], and continuum deformation [13] are some of the existing multi-agent system (MAS) coordination methods. While the virtual structure (VS) method is commonly exploited for centralized coordination, the other three methods provide decentralized solutions. The VS method treats MAS as particles of a virtual rigid body; rigid body translation and rotation prescribe agents' trajectories in a 3-D motion space. Consensus algorithm stability has been analyzed under fixed and switching communication topologies [14], [15] and in the presence of fixed and time-varying delays [16]–[18]. Finite-time consensus under fixed and switching communication topologies is developed in Refs. [19], [20], while leader-follower consensus is investigated in Refs. [8],

In containment control, leaders independently guide collective motion, and followers acquire the desired coordination via local communication. Containment control stability and

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convergence [21] with fixed [10] and switching [22] communication topologies have been analyzed in the existing literature. Retarded containment with fixed [10] and timevarying [11] time-delays and finite-time containment control and coordination [23] have also been investigated. Similar to containment control, continuum deformation is a decentralized leader-follower approach [13], [24], where followers' communications are weighted and consistent with agents' reference positions. Furthermore, continuum deformation formally specifies and rigorously ensures safety of the formation.

Our manuscript offers new contributions compared to the existing literature as well as the authors' previous work. One of the key contributions of our paper is that it relaxes the containment requirement condition needed in the previous work on continuum deformation based coordination. In particular, we prove that any n+1 vehicles forming an n-D simplex can be selected as the leader vehicles while followers can be either inside or outside the leading simplex. This greatly enhances scalability and flexibility of our MVS continuum deformation approach which in turn implies that a large number of agents with an arbitrary distribution can be safely coordinated in a geometrically-constrained environment. Furthermore, this paper uses eigen-decomposition to provide a significantly less conservative condition for ensuring safety and inter-agent collision avoidance in a large-scale continuum deformation coordination. While the existing continuum deformation coordination method ensures inter-agent collision avoidance by assigning a single lower-limit for all deformation eigenvalues, this paper guarantees inter-agent collision avoidance by assigning a lower-limit on one of the eigenvalues of the pure deformation matrix that is characterized based on the minimum separation distance in the reference configuration of the vehicles.

This paper is organized as follows. Preliminaries presented in Section II are followed by inter-agent communication topology and graph theory definitions in Section III. Section IV presents the formulations and statements of the problems considered in this paper. MVS collective dynamics is obtained in Section V. Safety requirements of MVS continuum deformation are obtained in Section VI. Continuum deformation planning is formulated in Section VII. Case study results in Section VIII are followed by a conclusion in Section IX.

II. PRELIMINARIES

A. Position Notations

Agent positions are expressed with respect to a Cartesian frame with unit basis vectors $\hat{\mathbf{e}}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $\hat{\mathbf{e}}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, and $\hat{\mathbf{e}}_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. For agent i, $\mathbf{r}_i = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$

 $\begin{bmatrix} x_i \ y_i \ z_i \end{bmatrix}^T$, $\mathbf{r}_{i,s} = \begin{bmatrix} x_{i,s} \ y_{i,s} \ z_{i,s} \end{bmatrix}^T$, $\mathbf{r}_{i,0} = \begin{bmatrix} x_{i,0} \ y_{i,0} \ z_{i,0} \end{bmatrix}^T$, $\mathbf{r}_{i,HT} = \begin{bmatrix} x_{i,HT} \ y_{i,HT} \ z_{i,HT} \end{bmatrix}^T$, and $\mathbf{r}_{d,i} = \begin{bmatrix} x_{d,i} \ y_{d,i} \ z_{d,i} \end{bmatrix}^T$ denote actual, initial, reference, global desired, and local desired positions, respectively. For continuum deformation coordination, $\mathbf{r}_i = \begin{bmatrix} x_i \ y_i \ z_i \end{bmatrix}^T$ is considered as the output of the control system of every vehicle $i \in \mathcal{V}$. Global and local desired positions of agent i are defined in Sections II-C and III, respectively. For n=1, reference position components $y_{i,0} = z_{i,0} = 0$ for every agent $i \in \mathcal{V}$ and agents are all distributed along the $\hat{\mathbf{e}}_1$ axis. For n=2, $z_{i,0}=0$ for every agent $i \in \mathcal{V}$ and agents lie in the plane defined by $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$.

B. Motion Space Discretization

Let $\mathbf{p}_1 \in \mathbb{R}^{n+1}$, ..., $\mathbf{p}_{n+1} \in \mathbb{R}^{n+1}$, and \mathbf{c} be position vectors of n+2 points on an n-D hyperplane. Defining a scalar function

$$\Psi_n(\mathbf{p}_1,\cdots,\mathbf{p}_{n+1}) = \operatorname{rank}([\mathbf{p}_2 - \mathbf{p}_1 \quad \cdots \quad \mathbf{p}_{n+1} - \mathbf{p}_1]), \quad (1)$$

vectors $\mathbf{p}_1 \in \mathbb{R}^{n+1}$, \cdots , $\mathbf{p}_{n+1} \in \mathbb{R}^{n+1}$ assign positions of vertices of an n-D simplex, if $\Psi_n(\mathbf{p}_1, \cdots, \mathbf{p}_{n+1}) = n$. If $\Psi_n(\mathbf{p}_1, \cdots, \mathbf{p}_{n+1}) = n$, we can define a vector function

$$\mathbf{\Theta}_{n}\left(\mathbf{p}_{1},\cdots,\mathbf{p}_{n+1},\mathbf{c}\right) = \begin{bmatrix} \mathbf{p}_{1} & \cdots & \mathbf{p}_{n+1} \\ 1 & \cdots & 1 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{c} \\ 1 \end{bmatrix}. \tag{2}$$

Per Eq. (2), the sum of the entries of Θ_n is 1, i.e. $\mathbf{1}_{1\times(n+1)}\Theta_n = 1$, where $\mathbf{1}_{1\times(n+1)} \in \mathbb{R}^{1\times(n+1)}$ is a vector with all components equal to 1. If $\Theta_n(\mathbf{p}_1, \dots, \mathbf{p}_{n+1}, \mathbf{c}) > \mathbf{0}$, then, the point \mathbf{c} is inside the simplex defined by $\mathbf{p}_1, \dots, \mathbf{p}_{n+1}$. Otherwise \mathbf{c} is outside this simplex.

C. MVS Homogeneous Deformation Coordination

We consider collective motion of an MVS consisting of N vehicles and treat them as particles of a deformable body. The global desired position of vehicle i is defined by

$$t \ge t_s$$
, $\mathbf{r}_{i,HT}(t) = \mathbf{Q}(t)\mathbf{r}_{i,0} + \mathbf{d}(t)$, (3)

where t is the current time, t_s is the initial time, $\mathbf{Q} \in \mathbb{R}^{3\times 3}$ is the Jacobian matrix, and $\mathbf{d} = [d_1 \ d_2 \ d_3]^T \in \mathbb{R}^{3\times 1}$ is a *rigid body displacement vector*. The matrix $\mathbf{Q}(t)$ is nonsingular for all t > t.

Leader-Follower Homogeneous Deformation: Because (3) is a linear transformation, global desired position of vehicle $i \in \mathcal{V}_R$ can be expressed as follows [13]:

$$i \in \mathcal{V}_R, \ t \ge t_s, \qquad \mathbf{r}_{i,HT}(t) = \sum_{i=1}^{n+1} \alpha_{i,j} \mathbf{r}_{j,HT}(t), \qquad (4)$$

where leaders form an *n*-D simplex in the reference configuration and

$$\begin{bmatrix} \alpha_{i,1} & \cdots & \alpha_{i,n+1} \end{bmatrix}^T = \mathbf{\Theta}_n \left(\mathbf{r}_{1,0}, \cdots, \mathbf{r}_{n+1,0}, \mathbf{r}_{i,0} \right). \tag{5}$$

Homogeneous Deformation Decomposition: Let angles β_1 , β_2 , and β_3 define a rotation matrix

$$\mathbf{R}(\beta_{1},\beta_{2},\beta_{3}) = \begin{bmatrix} C_{\beta_{2}}C_{\beta_{3}} & C_{\beta_{2}}S_{\beta_{3}} & -S_{\beta_{2}} \\ S_{\beta_{1}}S_{\beta_{2}}C_{\beta_{3}} - C_{\beta_{1}}S_{\beta_{3}} & S_{\beta_{1}}S_{\beta_{2}}S_{\beta_{3}} + C_{\beta_{1}}C_{\beta_{3}} & S_{\beta_{1}}C_{\beta_{2}} \\ C_{\beta_{1}}S_{\beta_{2}}C_{\beta_{3}} + S_{\beta_{1}}S_{\beta_{3}} & C_{\beta_{1}}S_{\beta_{2}}S_{\beta_{3}} - S_{\beta_{1}}C_{\beta_{3}} & C_{\beta_{1}}C_{\beta_{2}} \end{bmatrix},$$
(6)

where $C_{(\cdot)}$ and $S_{(\cdot)}$ abbreviate $\cos(\cdot)$ and $\sin(\cdot)$, respectively. The rotation matrix $\mathbf{R}(\beta_1, \beta_2, \beta_3)$ is orthogonal. Using the form of rotation matrix given in (6), matrix $\mathbf{Q}(t)$ in (3) can be decomposed as

$$\mathbf{Q}(t) = \mathbf{R}_D(t)\mathbf{U}_D(t),\tag{7}$$

where $\mathbf{R}_D(t) = \mathbf{R} (\phi_r(t), \theta_r(t), \psi_r(t)),$

$$\mathbf{U}_{D}(t) = \sum_{i=1}^{3} \lambda_{i} \hat{\mathbf{u}}_{i} \left(\phi_{u}(t), \theta_{u}(t), \psi_{u}(t) \right) \hat{\mathbf{u}}_{i}^{T} \left(\phi_{u}(t), \theta_{u}(t), \psi_{u}(t) \right),$$
(8a)

$$i = 1, 2, 3,$$
 $\hat{\mathbf{u}}_i = \mathbf{R}^T \left(\phi_u(t), \theta_u(t), \psi_u(t) \right) \hat{\mathbf{e}}_i.$ (8b)

Note that $\hat{\mathbf{u}}_1$, $\hat{\mathbf{u}}_2$, and $\hat{\mathbf{u}}_3$ are the eigenvectors of \mathbf{U}_D while $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_3$ are the base vectors of the inertial coordinate system defined in Section II-A. A desired homogeneous transformation (3) can thus be uniquely expressed by the following features: (i) Rotation angles $\phi_r(t)$, $\theta_r(t)$, $\psi_r(t)$, (ii) Deformation eigenvalues $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$, (iii) Deformation angles $\phi_u(t)$, $\theta_u(t)$, and $\psi_u(t)$, and (iv) Rigid body displacement components $d_1(t)$, $d_2(t)$, and $d_3(t)$. For n=1, $\phi_u(t)=\theta_u(t)=\psi_u(t)=\phi_r(t)=0$ and $\lambda_2(t)=\lambda_3(t)=1$ but $\lambda_1(t)$, $\theta_r(t)$, $\psi_r(t)$, $d_1(t)$, $d_2(t)$, and d_3 can be designed to shape a desired homogeneous deformation coordination. For n=2, $\lambda_3(t)=1$, $\phi_u(t)=\theta_u(t)=0$ at any time t and the remaining features can be used to specify a desired homogeneous transformation coordination. For n=3, all 12 features can be used to plan a continuum deformation coordination.

We include the following result for n = 2:

Theorem 1. Consider a 2-D continuum deformation coordination in a 3-D motion space where vehicles are distributed on the plane normal to $\hat{\mathbf{u}}_3(t)$ at any time t. Given reference positions of three leaders $(\mathbf{r}_{1,0}, \mathbf{r}_{2,0}, \text{ and } \mathbf{r}_{3,0})$ and leaders' global desired positions at a time t $(\mathbf{r}_{1,HT}(t), \mathbf{r}_{2,HT}(t),$ and $\mathbf{r}_{3,HT}(t))$, eigenvalues $\lambda_1(t)$ and $\lambda_2(t)$ of $\mathbf{U}_D(t)$ and deformation angle $\psi_u(t)$ are obtained by

$$\begin{cases} \lambda_{1} = \sqrt{\frac{a+c}{2} + \sqrt{\left[\frac{1}{2}(a-c)\right]^{2} + b^{2}}} \\ \lambda_{2} = \sqrt{\frac{a+c}{2} - \sqrt{\left[\frac{1}{2}(a-c)\right]^{2} + b^{2}}} \\ \psi_{u} = \frac{1}{2} \tan^{-1} \left(\frac{2b}{a-c}\right) \end{cases} , \tag{9}$$

where the dependence on t is omitted to simplify the notation

and

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} (x_{2,0} - x_{1,0})^2 & 2(x_{2,0} - x_{1,0}) & (y_{2,0} - y_{1,0}) & (y_{2,0} - y_{1,0})^2 \\ (x_{3,0} - x_{2,0})^2 & 2(x_{3,0} - x_{2,0}) & (y_{3,0} - y_{2,0}) & (y_{3,0} - y_{2,0})^2 \\ (x_{1,0} - x_{3,0})^2 & 2(x_{1,0} - x_{3,0}) & (y_{1,0} - y_{3,0}) & (y_{1,0} - y_{3,0})^2 \end{bmatrix}^{-1} \\ \begin{bmatrix} (x_{2,HT} - x_{1,HT})^2 + (y_{2,HT} - y_{1,HT})^2 + (z_{2,HT} - z_{1,HT})^2 \\ (x_{3,HT} - x_{2,HT})^2 + (y_{3,HT} - y_{2,HT})^2 + (z_{3,HT} - z_{2,HT})^2 \\ (x_{1,HT} - x_{3,HT})^2 + (y_{1,HT} - y_{3,HT})^2 + (z_{1,HT} - z_{3,HT})^2 \end{bmatrix}.$$

$$(10)$$

Proof: Substituting $\lambda_3 = 1$ and $\phi_u = \theta_u = 0$ into Eq. (8a), it follows that

$$\mathbf{U}_D^2 = \begin{bmatrix} a & b & 0 \\ b & c & 0 \\ 0 & 0 & 1 \end{bmatrix},\tag{11}$$

where

$$\begin{cases} \lambda_1^2 \cos^2 \psi_u + \lambda_2^2 \sin^2 \psi_u = a, \\ (\lambda_1^2 - \lambda_2^2) \sin \psi_u \cos \psi_u = b, \\ \lambda_1^2 \sin^2 \psi_u + \lambda_2^2 \cos^2 \psi_u = c. \end{cases}$$
 (12)

By solving (12), λ_1 , λ_2 , and ψ_u are related to a, b, c by Eq. (9). For every two different leaders i and j with global desired trajectories defined by (3), the following relation holds:

$$\begin{aligned} \left(\mathbf{r}_{i,HT} - \mathbf{r}_{j,HT}\right)^T \left(\mathbf{r}_{i,HT} - \mathbf{r}_{j,HT}\right) &= \left(\mathbf{r}_{i,HT} - \mathbf{r}_{j,HT}\right)^T \mathbf{Q}^T \mathbf{Q} \left(\mathbf{r}_{i,0} - \mathbf{r}_{j,0}\right) \\ \text{where } \mathbf{Q}^T \mathbf{Q} &= \mathbf{U}_D^2. \end{aligned}$$
 This implies that

$$(\mathbf{r}_{2,HT} - \mathbf{r}_{1,HT})^{T} (\mathbf{r}_{2,HT} - \mathbf{r}_{1,HT}) = (\mathbf{r}_{2,0} - \mathbf{r}_{1,0})^{T} \mathbf{U}_{D}^{2} (\mathbf{r}_{2,HT} - \mathbf{r}_{1,HT}),$$

$$(13a)$$

$$(\mathbf{r}_{3,HT} - \mathbf{r}_{2,HT})^{T} (\mathbf{r}_{3,HT} - \mathbf{r}_{2,HT}) = (\mathbf{r}_{3,0} - \mathbf{r}_{2,0})^{T} \mathbf{U}_{D}^{2} (\mathbf{r}_{3,HT} - \mathbf{r}_{2,HT}),$$

$$(13b)$$

$$(\mathbf{r}_{1,HT} - \mathbf{r}_{3,HT})^{T} (\mathbf{r}_{1,HT} - \mathbf{r}_{3,HT}) = (\mathbf{r}_{1,0} - \mathbf{r}_{3,0})^{T} \mathbf{U}_{D}^{2} (\mathbf{r}_{1,HT} - \mathbf{r}_{3,HT}).$$

$$(13c)$$

Now, \mathbf{U}_D^2 can be replaced by Eq. (11) and Eq. (13) reduces to a set of three linear algebraic equations with three unknowns a, b, and c. By solving Eq. (13), a, b, and c are obtained as given in (10).

III. INTER-AGENT COMMUNICATION

Suppose an MVS consists of N vehicles moving in a 3-D motion space. The set $\mathcal{V}_R = \{1, 2, \cdots, N\}$ defining identification (index) numbers of the vehicles is expressed as $\mathcal{V}_R = \mathcal{V}_L \bigcup \mathcal{V}_F$ where \mathcal{V}_L and \mathcal{V}_F define index numbers of leaders and followers, respectively. The paper considers cases in which vehicles are distributed in an n-D (n = 1,2,3) Euclidean space in \mathbb{R}^3 . The MVS is guided by n+1 leaders with index numbers $\mathcal{V}_L = \{1, 2, \cdots, n+1\}$. Followers' index numbers are defined by the set $\mathcal{V}_F = \{n+2, \cdots, N\}$.

Let Ω_c be an arbitrary closed domain enclosing all real vehicles at a reference configuration, where N_a auxiliary nodes are arbitrarily distributed on the boundary $\partial \Omega_c$ and identified by the set $\mathcal{V}_{aux} = \{N+1, N+2, \cdots, N+N_a\}$.

Remark 1. Auxiliary nodes do not represent real agents and they are introduced only to ensure MVS coordination stability. Note that followers only communicate with real in-neighbors at any time *t* during continuum deformation coordination. This is further described in Section III-B below.

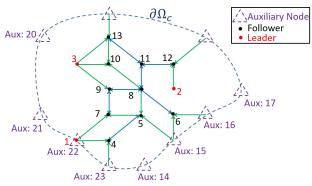


Fig. 1: Schematic of a communication graph with real and auxiliary nodes used in a 2-D continuum deformation.

A. Reference Communication Weights and Weight Matrix

Inter-agent communication is defined by the graph $\mathcal{G}_{w} = \mathcal{G}_{w}\left(\mathcal{V},\mathcal{E}_{w}\right)$ with node set \mathcal{V} and edge set $\mathcal{E}_{w} \in \mathcal{V} \times \mathcal{V}$. \mathcal{V} defines real and auxiliary (virtual) agents, e.g. $\mathcal{V} = \mathcal{V}_{R} \bigcup \mathcal{V}_{aux}$ where \mathcal{V}_{R} and \mathcal{V}_{aux} correspond to real and auxiliary vehicle index numbers, respectively. For every node $i \in \mathcal{V}$, reference in-neighbor set $\mathcal{N}_{i} = \{j \in \mathcal{V} | (j,i) \in \mathcal{E}_{w}\}$ defines the inneighbor nodes in the reference configuration. Every follower communicates with n+1 in-neighbors and is inside of an n-D communication simplex with n+1 vertices occupied by the in-neighbors.

An example communication graph \mathcal{G}_w for a 2-D MVS coordination is shown in Fig. 1. Real nodes are defined by $\mathcal{V}_R = \{1, \cdots, 13\}$, where $\mathcal{V}_L = \{1, 2, 3\}$ and $\mathcal{V}_F = \{4, \cdots, 13\}$ define leaders and followers, respectively. $\mathcal{V}_{aux} = \{14, \cdots, 23\}$ defines auxiliary nodes. The set of all nodes is given by $\mathcal{V} = \{1, \cdots, 23\}$. Each auxiliary node communicates with all three leaders, and communication between auxiliary nodes and leaders is not shown in Fig. 1. An auxiliary node may or may not be coincident with a real node positioned at boundary $\partial \Omega_c$ at reference time t_0 .

Defining reference in-neighbor set of follower $i \in (\mathcal{V}_F \bigcup \mathcal{V}_{aux})$ as $\mathcal{N}_i = \{i_1, \cdots, i_{n+1}\}$, the communication weight between $i \in (\mathcal{V}_F \bigcup \mathcal{V}_{aux})$ and in-neighbor vehicle $i_k \in \mathcal{V}$ $(k = 1, 2, \cdots, n+1)$ is denoted by w_{i,i_k} and obtained as follows:

$$\begin{bmatrix} w_{i,i_1} & \cdots & w_{i,i_{n+1}} \end{bmatrix}^T = \mathbf{\Theta}_n \left(\mathbf{r}_{i_1,0}, \cdots, \mathbf{r}_{i_{n+1},0}, \mathbf{r}_{i,0} \right), \quad (14)$$

where n = 1, 2, 3 is the dimension of the homogeneous deformation coordination and Θ_n is defined by (2). Because every follower is inside the communication simplex, followers' communication weights are all positive. We define the weight matrix $\mathbf{W} = [W_{ij}] \in \mathbb{R}^{(N+N_a)\times (N+N_a)}$ as follows:

$$\mathbf{W}_{ij} = \begin{cases} -1 & i = j \\ w_{i,j} > 0 & i \in (\mathcal{V}_F \bigcup \mathcal{V}_{aux}), \ j \in \mathcal{N}_i \\ 0 & \text{otherwise.} \end{cases}$$
 (15)

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The matrix **W** can be partitioned as follows:

$$\mathbf{W} = \begin{bmatrix} -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{W}_{f,l} & \mathbf{A} & \mathbf{W}_{f,a} \\ \mathbf{W}_{a,l} & \mathbf{0} & -\mathbf{I} \end{bmatrix}, \tag{16}$$

where $\mathbf{W}_{f,l} \in \mathbb{R}^{(N-n-1)\times(n+1)}$ and $\mathbf{W}_{f,a} \in \mathbb{R}^{(N-n-1)\times N_a}$ are non-negative matrices, $\mathbf{A} \in \mathbb{R}^{(N-n-1)\times(N-n-1)}$, and $\mathbf{W}_{a,l} \in \mathbb{R}^{N_a \times 3}$. Let $\mathbf{z}_{q,l,0} = [q_{1,0} \cdots q_{n+1,0}]^T$, $\mathbf{z}_{q,f,0} = [q_{n+2,0} \cdots q_{N,0}]^T$, and $\mathbf{z}_{q,aux,0} = [q_{N+1,0} \cdots q_{N+N_a,0}]^T$ define component $q \in \{x,y,z\}$ of leaders, followers, and auxiliary nodes, respectively. If followers' communication weights are consistent with agents' reference positions and defined by (14), $\mathbf{z}_{q,f,0}$ is related to $\mathbf{z}_{q,l,0}$ by

$$\begin{bmatrix} -\mathbf{I}_{n+1} & \mathbf{0} \\ \mathbf{W}_{f,l} + \mathbf{W}_{f,a} \mathbf{W}_{a,l} & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{z}_{q,l,0} \\ \mathbf{z}_{q,f,0} \end{bmatrix} = \begin{bmatrix} -\mathbf{z}_{q,l,0} \\ \mathbf{0} \end{bmatrix}.$$
(17)

Theorem 2. ([24]) Assume leaders form an n-D simplex in the reference configuration, in-neighbors of follower i form an n-D simplex enclosing follower i in the reference configurations, followers' communication weights are consistent with vehicles' reference positions, and graph \mathcal{G}_w is defined such there exists at least one directed path from every node $i \in \mathcal{V}_L$ to every node $j \in \mathcal{V}_F$. Then the matrix $\mathbf{A} \in \mathbb{R}^{(N-n-1)\times (N-n-1)}$ is Hurwitz and

$$\mathbf{W}_{L} = \mathbf{A}^{-1} \left(\mathbf{W}_{f,l} + \mathbf{W}_{f,a} \mathbf{W}_{a,l} \right) \in \mathbb{R}^{(N-n-1) \times (n+1)}$$
 (18)

is one-sum row. for an arbitrary placement of the auxiliary agents on $\partial\Omega_c$, i.e. the sum of the row-elements is one for every row of matrix \mathbf{W}_L , where Ω_c is an arbitrary closed domain enclosing MVS reference configuration ($\Omega_0 \subset \Omega_c$).

Remark 2. Let $\mathbf{z}_{q,l,HT}(t) = [q_{1,HT} \cdots q_{n+1,HT}]^T \in \mathbb{R}^{n+1}$ and $\mathbf{z}_{q,f,HT}(t)(t) = [q_{n+2,HT} \cdots q_{N,HT}]^T \in \mathbb{R}^{N-n-1}$ denote the vector form component q $(q \in \{x,y,z\})$ of the global desired positions of leaders and real vehicles. Then,

$$j = 0, 1, \dots, \rho_a,$$
 $\mathbf{z}_{a,f,HT} = \mathbf{W}_L \mathbf{z}_{a,l,HT}.$

B. Coordination Graph and Real Communication Weights

When the MVS is moving, follower vehicles only communicate with real in-neighbor where inter-agent communication is defined by coordination graph \mathcal{G}_c ($\mathcal{V}_R, \mathcal{E}_c$) with edge set $\mathcal{E}_c \subset \mathcal{V}_R \times \mathcal{V}_R$. Real in-neighbors of follower $i \in \mathcal{V}_F$ are defined by time-invariant set $\mathcal{I}_i = \{j \mid (j,i) \in \mathcal{E}_c\}$ which is called real in-neighbor set. We define the local desired position of vehicle $i \in \mathcal{V}_R$ by

$$\mathbf{r}_{d,i} = \begin{cases} \mathbf{r}_{i,HT} & i \in \mathcal{V}_L \\ \sum_{j \in I_i} \varpi_{i,j} \mathbf{r}_j = i \in \mathcal{V}_F \end{cases}, \tag{19}$$

where $\varpi_{i,j} > 0$ is the <u>real communication weight</u> between follower $i \in \mathcal{V}_F$ and vehicle $j \in \mathcal{I}_i \subset \mathcal{V}_R$ and $\sum_{j \in I_i} \varpi_{i,j} = 1$. If $\mathcal{N}_i \cap \mathcal{V}_{aux} = \emptyset$, then, $I_i = \mathcal{N}_i$ and $\varpi_{i,j} = w_{i,j}$ for every vehicle $i \in \mathcal{V}_F$ with every in-neighbor $j \in \mathcal{I}_i = \mathcal{N}_i$. If $\mathcal{N}_i \cap \mathcal{V}_{aux} \neq \emptyset$, then, $I_i \neq \mathcal{N}_i$ and $\varpi_{i,j}$ is defined as follows:

$$\varpi_{i,j} = \begin{cases} w_{i,j} & j \in (I_i \cap \mathcal{N}_i) \\ \sum_{h \in (\mathcal{N}_i \cap \mathcal{V}_{aux})} w_{i,h} \alpha_{h,j} & \text{otherwise} \end{cases}.$$

IV. PROBLEM STATEMENT

This paper studies the properties of the homogeneous deformation approach for an N-vehicle MVS where vehicles are treated as particles of an n-D deformable body (n = 1, 2, 3). The desired MVS vehicle positions are guided by n+1 leaders and acquired by the remaining followers through local communication, where followers are arbitrarily distributed in a 3D motion space. Dynamics of the vehicle $i \in \mathcal{V}_R$ is represented by a nonlinear model

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i) + \mathbf{g}_i(\mathbf{x}_i) \, \mathbf{u}_i, \\ \mathbf{r}_i = \mathbf{r}_i(\mathbf{x}_i), \end{cases}$$
 (20)

where $\mathbf{x}_i \in \mathbb{R}^{n_x \times 1}$ and $\mathbf{u}_i \in \mathbb{R}^{n_u \times 1}$ are state and control input, respectively, the actual position $\mathbf{r}_i \in \mathbb{R}^{3 \times 1}$ is the output of vehicle dynamics (20), and $\mathbf{f}_i : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ and $\mathbf{g}_i : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x \times n_u}$ are smooth. We assume the dynamics of every vehicle $i \in \mathcal{V}_L \cup \mathcal{V}_F$ is input-output linearizable and study the following problems:

Problem 1 (MVS Continuum Deformation Coordinated Control): Determine $U_i(t)$ such that

$$\forall t \in [t_s, t_f], \ \forall i \in \mathcal{V}, \qquad \|\mathbf{r}_i(t) - \mathbf{r}_{i,HT}(t)\|_2 \le \delta,$$
 (21)

where $\delta > 0$ is constant. Every agent only accesses the local desired trajectory $\mathbf{r}_{d,i}(t)$ defined in Section II-A as the reference input for every vehicle $i \in \mathcal{V}_L \cup \mathcal{V}_F$. Follower $i \in \mathcal{V}_F$ does not know $\mathbf{r}_{i,HT}$ but acquires it through local communication. We investigate stability and convergence of the MVS collective dynamics in Sections V and VI.

Problem 2 (Guaranteeing Continuum Deformation Safety Specification): Ensure inter-agent collision avoidance, defined by

$$\forall t \in [t_s, t_f], \ \forall i, j \in \mathcal{V}, \ i \neq j, \qquad \|\mathbf{r}_i(t) - \mathbf{r}_i(t)\|_2 \ge 2\epsilon, \ (22)$$

and a follower containment condition defined by

$$t \in [t_s, t_f], \ \forall i \in \mathcal{V}, \qquad \mathbf{\Theta}_n(\mathbf{h}_1(t), \cdots, \mathbf{h}_{n+1}(t), \mathbf{r}_i(t)) > \mathbf{0},$$
(23)

with low computation cost, where ϵ is the radius of the smallest ball enclosing every vehicle i and Θ_n was previously defined by Eq. (2). Specifically, we show how inter-agent collisions can be avoided only by constraining one of the eigenvalues of the deformation matrix if the shear-deformation angles remain constant at all times t. This can significantly improve the flexibility and maneuverability of the large-scale continuum deformation coordination.

Problem 3 (MVS Continuum Deformation Coordination Planning): To plan a continuum deformation coordination, eigen-decomposition and A* search methods are applied to determine the global desired trajectories of team leaders in obstacle-free and obstacle-laden environments, respectively. The third problem is how to enforce continuum deformation safety conditions for all instants, $t \in [t_s, t_f]$.

V. PROBLEM 1: MVS CONTINUUM DEFORMATION COORDINATED CONTROL

5

We define state transformation $\mathbf{x}_i \to \left(\mathbf{r}_i, \cdots, \mathbf{r}_i^{\rho-1}, \varpi_i\right)$ such that (20) is decomposed into external dynamics

$$q \in \{x, y, z\}, \qquad \frac{d^{\rho_q} q_i}{dt^{\rho_q}} = v_{q,i} \tag{24}$$

and internal dynamics

$$\dot{\omega}_{i} = \mathbf{f}_{\text{int},i} \left(\omega_{i}, x_{i}, \cdots, L_{\mathbf{f}_{i}}^{\rho_{x}-1} x_{i}, y_{i}, \cdots, L_{\mathbf{f}_{i}}^{\rho_{y}-1} y_{i}, z_{i}, \cdots, L_{\mathbf{f}_{i}}^{\rho_{z}-1} z_{i} \right)$$
(25)

where $v_{q,i}$ is assigned as

$$v_{q,i} = -\sum_{k=1}^{\rho_q - \Xi - 1} \gamma_{i,k} L_{\mathbf{f}_i}{}^{\rho_q - k} q_i + \sum_{k=\rho_q - \Xi}^{\rho_q} \gamma_{i,k} L_{\mathbf{f}_i}{}^{\rho_q - k} \left(q_{d,i} - q_i \right), \tag{26}$$

and where $L_{\mathbf{f}_i}q_i = (\nabla_{\mathbf{x}_i}q_i)^T \mathbf{f}_i$ is the Lie derivative of a smooth function q_i with respect to a vector field \mathbf{f}_i , $q_{d,i}$ is the component $q \in \{x, y, z\}$ of local desired position $\mathbf{r}_{d,i}$ defined by Eq. (19). Control gains $\gamma_{i,1}$ through γ_{i,ρ_q} are assigned such that the MVS collective dynamics is stable. In Eq. (24), ρ_q $(q \in \{x, y, z\})$ is the relative degree, $\rho = \rho_x + \rho_y + \rho_z \le n_x$ is the total relative degree, $\Xi < \rho_q$ is constant, and

$$\begin{bmatrix} v_{x,i} \\ v_{y,i} \\ v_{z,i} \end{bmatrix} = \begin{bmatrix} L_{\mathbf{g}i} L_{\mathbf{f}i}^{\rho_x - 1} x_i \\ L_{\mathbf{g}i} L_{\mathbf{f}i}^{\rho_y - 1} y_i \\ L_{\mathbf{g}i} L_{\mathbf{f}i}^{\rho_z - 1} z_i \end{bmatrix} \mathbf{u}_i + \begin{bmatrix} L_{\mathbf{f}i}^{\rho_x} x_i \\ L_{\mathbf{f}i}^{\rho_y} y_i \\ L_{\mathbf{f}i}^{\rho_z} z_i \end{bmatrix}.$$

Define $\mathbf{X}_{\mathrm{SYS},q} = \begin{bmatrix} q_1 & \cdots & q_N & \cdots & L_{\mathbf{f}_1}^{\rho_q} q_1 & \cdots & L_{\mathbf{f}_N}^{\rho_q} q_N \end{bmatrix}^T$

$$\mathbf{U}_{\text{SYS},q} = \sum_{j=\rho_q-\Xi}^{\rho_q} \mathbf{\Gamma}_{j,l} \begin{bmatrix} L_{\mathbf{f}_i}^{\rho_q-j} q_{1,HT} \\ \vdots \\ L_{\mathbf{f}_i}^{\rho_q-j} q_{n+1,HT} \end{bmatrix}, \tag{27}$$

where q_i and $q_{i,HT}$ denote component $q \in \{x, y, z\}$ of actual and global desired positions of vehicle i. The MVS collective dynamics can be expressed by the following normal form:

$$\begin{bmatrix} \dot{\mathbf{X}}_{SYS,x} \\ \dot{\mathbf{X}}_{SYS,y} \\ \dot{\mathbf{X}}_{SYS,z} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{SYS,x} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{SYS,y} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{SYS,z} \end{bmatrix} \begin{bmatrix} \mathbf{X}_{SYS,x} \\ \mathbf{X}_{SYS,z} \\ \mathbf{X}_{SYS,z} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_{SYS,x} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{SYS,y} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{SYS,z} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{SYS,x} \\ \mathbf{U}_{SYS,z} \\ \mathbf{U}_{SYS,z} \end{bmatrix}$$

$$\frac{d\omega}{dt} = \mathbf{F}_{\text{INT}}(\omega, \mathbf{X}_{\text{SYS}}), \tag{28b}$$

where
$$\omega = \begin{bmatrix} \omega_1^T & \cdots & \omega_N^T \end{bmatrix}^T$$
, $\mathbf{F}_{\text{INT}} = \begin{bmatrix} \mathbf{f}_{\text{int},1}^T & \cdots & \mathbf{f}_{\text{int},N}^T \end{bmatrix}^T$,

$$j = \rho_q - \Xi + 1, \dots, \rho_q,$$
 $\Gamma_{j,l} = \operatorname{diag}(\gamma_{j,1}, \dots, \gamma_{j,n+1}) \in \mathbb{R}^{(n+1) \times (n+1)}$

$$j = 1, \dots, \rho_q$$
 $\Gamma_j = \operatorname{diag}(\gamma_{j,1}, \dots, \gamma_{j,N}) \in \mathbb{R}^{N \times N},$

$$q \in \left\{x, y, z\right\}, \qquad \mathbf{B}_{\mathrm{SYS}, q} = \begin{bmatrix} \mathbf{0}_{(n+1) \times \left(\rho_q - 1\right)N} & \mathbf{I}_{n+1} & \mathbf{0}_{(n+1) \times (N-n-1)} \end{bmatrix}^T,$$

$$q \in \{x,y,z\}, \qquad \mathbf{A}_{\mathrm{SYS},q} = \begin{bmatrix} \mathbf{0}_N & \mathbf{I}_N & \cdots & \mathbf{0}_N \\ \vdots & \vdots & \ddots & \vdots \ddots \\ \mathbf{0}_N & \mathbf{0}_N & \cdots & \mathbf{I}_N \\ \mathbf{\Gamma}_{\rho_q} \mathbf{H}_{\rho_q,q} & \mathbf{\Gamma}_{\rho_q-1} \mathbf{H}_{\rho_q-1,q} & \cdots & \mathbf{\Gamma}_1 \mathbf{H}_{1,q} \end{bmatrix}.$$

Note that $\mathbf{I}_N \in \mathbb{R}^{N \times N}$ and $\mathbf{0}_N \in \mathbb{R}^{N \times N}$ are the identity and zero-entry matrices. Matrix $\mathbf{H}_{i,q} \in \mathbb{R}^{N \times N}$ $(i = 1, \cdots, \rho_q)$ is defined as follows:

$$q \in \{x, y, z\}, \qquad \mathbf{H}_{i, q} = \begin{cases} -\mathbf{I}_N & 1 \leq i \leq \rho_q - \Xi - 1 \\ \mathbf{L} & \rho_q - \Xi \leq i \leq \rho_q \end{cases}.$$

 $\dot{\omega}_{i} = \mathbf{f}_{\text{int},i} \left(\omega_{i}, x_{i}, \cdots, L_{\mathbf{f}_{i}}^{\rho_{x}-1} x_{i}, y_{i}, \cdots, L_{\mathbf{f}_{i}}^{\rho_{y}-1} y_{i}, z_{i}, \cdots, L_{\mathbf{f}_{i}}^{\rho_{z}-1} z_{i} \right), \text{ eigenvalues of the characteristic equation of external dynamics}$ (25)(28a), $|s\mathbf{I} - \mathbf{A}_{SYS,q}| = 0$ $(q \in \{x, y, z\})$, are all located in the open left-half s-plane. Also, zero dynamics of the MVS internal dynamics (28b) is locally asymptotically stable and $\mathbf{F}_{\mathrm{I},\omega} = \frac{\partial \mathbf{F}_{\mathrm{INT}}}{\partial \omega}$ is Hurwitz.

VI. PROBLEM 2: CONTINUUM DEFORMATION SAFETY SPECIFICATION

For continuum deformation coordination in an obstacleladen environment, the MVS needs to be contained by a virtual n-D simplex, called virtual containment simplex (VCS), at any time t. Let $\mathbf{h}_{i,0} = h_{x,i,0}\hat{\mathbf{e}}_1 + h_{y,i,0}\hat{\mathbf{e}}_2 + h_{z,i,0}\hat{\mathbf{e}}_3$ and $\mathbf{h}_i(t) =$ $h_{x,i}\hat{\mathbf{e}}_1 + h_{y,i}\hat{\mathbf{e}}_2 + h_{z,i}\hat{\mathbf{e}}_3$ $(i = 1, \dots, n+1)$ denote positions of vertex *i* of the VCS in the reference and current configurations, respectively. VCS evolution is defined by a homogeneous transformation, therefore, $\mathbf{h}_{i,0}$ and $\mathbf{h}_{i}(t)$ are related by

$$i = 1, \dots, n+1,$$

$$\mathbf{h}_i(t) = \mathbf{Q}(t)\mathbf{h}_{i,0} + \mathbf{d}(t), \quad (29)$$

where **Q** and **d** are computed based on leaders' positions in the reference configuration and the current configuration at current time t per Section II-C. The MVS containment condition is mathematically specified by

$$t \in [t_s, t_f], \ \forall i \in \mathcal{V}, \qquad \mathbf{\Theta}_n(\mathbf{h}_1(t), \cdots, \mathbf{h}_{n+1}(t), \mathbf{r}_i(t)) > \mathbf{0},$$
(30)

where Θ_n was previously defined by Eq. (2).

Theorem 3. ([24]) Assume each vehicle is enclosed by a ball with radius ϵ . Given deviation upper-bound δ (Eq. (21)), we define $\delta_{\text{max}} = \min \left\{ (d_b - \epsilon), \frac{1}{2} (d_s - 2\epsilon) \right\}$, where d_s is the minimum separation distance and d_b is the minimum distance from the boundary of the leading simplex in the reference configuration (See Fig. 2 (a)). The vehicle containment and inter-agent collision avoidance are guaranteed at time instant t if eigenvalues λ_1 , λ_2 , λ_3 of $\mathbf{U}_D(t)$ satisfy the following inequality constraint [25]:

$$j = 1, 2, 3, \qquad \lambda_j \ge \frac{\delta + \epsilon}{\delta_{\text{max}} + \epsilon}.$$
 (31)

Relaxation of the Collision Avoidance Condition Theorem 3 provides a conservative collision avoidance guarantee independent of the total number of agents (N). However, the conditions (31) can be overly restrictive when agents are not uniformly distributed in the reference configuration (See Fig. 2 (b)). This issue can be dealt with, if we only constrain the eigenvalue λ_1 of matrix $\mathbf{U}_D(t)$ and assign the direction of

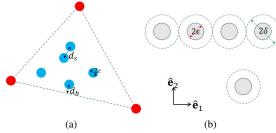


Fig. 2: (a) Schematic for MVS reference configuration and the VCS containing all vehicles. (b) Schematic of a non-uniform MVS reference configuration where $\delta = \delta_{max}$.

eigenvector $\hat{\mathbf{u}}_1$ by solving the following max-min optimization problem:

$$(\psi_{u,0}, \theta_{u,0}) = \underset{\psi_{u}, \theta_{u}}{\operatorname{argmax}} \left\{ \underset{i,j \in \mathcal{V}_{R}, i \neq j}{\min} \left(\left(\mathbf{r}_{i,0} - \mathbf{r}_{j,0} \right) \cdot \hat{\mathbf{u}}_{1} \left(0, \theta_{u}, \psi_{u} \right) \right) \right\}.$$
(32)

Since we use the 3-2-1 Euler angles to define a rigid-body rotation, the eigenvector $\hat{\mathbf{u}}_1$ of $\mathbf{U}_D(t)$ only depends on θ_u and ψ_u at any time t (see Eq. (8b)). Thus, the reference deformation angles $\theta_{u,0}$ and $\psi_{u,0}$ are obtained by

$$\theta_{u,0} = -\sin^{-1}\left(\hat{\mathbf{u}}_{1,0} \cdot \hat{\mathbf{e}}_{3}\right),\tag{33a}$$

$$\psi_{u,0} = \tan^{-1} \left(\frac{\hat{\mathbf{u}}_{1,0} \cdot \hat{\mathbf{e}}_2}{\hat{\mathbf{u}}_{1,0} \cdot \hat{\mathbf{e}}_1} \right). \tag{33b}$$

Suppose now that the desired continuum deformation is designed such that deformation angles $\phi_u(t) = \phi_{u,0}$, $\theta_u(t) = \theta_{u,0}$ and $\psi_u(t) = \psi_{u,0}$ remain constant for $t \in [t_s, t_f]$, where $\phi_{u,0}$ takes an arbitrary value between 0 and 2π .

Lemma 1. If $\phi_u(t) = \phi_{u,0}$ $\theta_u(t) = \theta_{u,0}$ and $\psi_u(t) = \psi_{u,0}$ are constant for $t \in [t_s, t_f]$ and $\mathbf{r}_{i,HT}$ and $\mathbf{r}_{j,HT}$ are defined by the homogeneous transformation given in (3) $(\forall i, j \in V_R)$, then, the following relation holds for all $i, j \in V_R$:

$$\left(\mathbf{r}_{i,HT}(t) - \mathbf{r}_{j,HT}(t)\right)^{T} \left(\mathbf{R}_{D}(t)\hat{\mathbf{u}}_{l,0}\right) = \lambda_{l}(t) \left(\mathbf{r}_{i,0} - \mathbf{r}_{j,0}\right)^{T} \hat{\mathbf{u}}_{l,0} \tag{34}$$

Proof: Given global desired positions of vehicles $i, j \in V_R$ defined by Eq. (3), we can express $\mathbf{r}_{i,HT} - \mathbf{r}_{i,HT}$ as

$$\mathbf{r}_{i,HT}(t) - \mathbf{r}_{j,HT}(t) = \mathbf{Q}(t) \left(\mathbf{r}_{i,0} - \mathbf{r}_{j,0} \right) = \mathbf{R}_{D}(t) \sum_{h=1}^{3} \lambda_{h}(t) \hat{\mathbf{u}}_{h,0} \hat{\mathbf{u}}_{h,0}^{T} \left(\mathbf{r}_{i,0} - \mathbf{r}_{j,0} \right) = \sum_{h=1}^{3} \lambda_{h}(t) \left[\left(\mathbf{r}_{i,0} - \mathbf{r}_{j,0} \right)^{T} \hat{\mathbf{u}}_{h,0} \right] \mathbf{R}_{D}(t) \hat{\mathbf{u}}_{h,0}.$$
(35)

Note that $\lambda_h(t) \left[\left(\mathbf{r}_{i,0} - \mathbf{r}_{j,0} \right)^T \hat{\mathbf{u}}_{h,0} \right] \in \mathbb{R}$ at any time t (h = 1,2,3). Thus, pre-multiplying both sides of Eq. (35) by $\hat{\mathbf{u}}_{l,0}^T \mathbf{R}_D^T(t)$ and noting that $\hat{\mathbf{u}}_{l,0}^T \mathbf{R}_D^T(t) \mathbf{R}_D(t) \hat{\mathbf{u}}_{h,0}$ is 1 if h = l and 0 otherwise, Eq. (34) follows.

Theorem 4. Assume every vehicle is enclosed by a ball of radius ϵ . MVS inter-agent collision avoidance is guaranteed if

$$\forall t \in [t_s, t_f], \qquad \lambda_1(t) \ge 2\frac{\delta + \epsilon}{d_s},$$
 (36)

where $\lambda_1(t)$ is the first eigenvalue of $\mathbf{U}_D(t)$ and d_s is the minimum separation distance in the reference configuration.

Proof: Vectors $\mathbf{R}_D(t)\hat{\mathbf{u}}_{1,0}$, $\mathbf{R}_D(t)\hat{\mathbf{u}}_{2,0}$, $\mathbf{R}_D(t)\hat{\mathbf{u}}_{3,0}$ are parallel to $\hat{\mathbf{u}}_1(t)$, $\hat{\mathbf{u}}_2(t)$, $\hat{\mathbf{u}}_3(t)$, respectively. Because the lowest minimum separation distance is along the unit vector $\hat{\mathbf{u}}_1(t) = \mathbf{R}_D(t)\hat{\mathbf{u}}_{1,0}$, inter-agent collision is avoided if

$$\forall i, j \in \mathcal{V}_R, \qquad \left(\mathbf{r}_{i,HT}(t) - \mathbf{r}_{j,HT}(t)\right)^T \left(\mathbf{R}_D(t)\mathbf{u}_{1,0}\right) \ge 2\left(\delta + \epsilon\right).$$
(37)

By Lemma 1, it follows that

$$\min_{i,j \in \mathcal{V}_R} \left\{ \left(\mathbf{r}_{i,HT}(t) - \mathbf{r}_{j,HT}(t) \right)^T \left(\mathbf{R}_D(t) \mathbf{u}_{1,0} \right) \right\} = \lambda_1(t) \min_{i,j \in \mathcal{V}_R} \left\{ \left(\mathbf{r}_{i,0} - \mathbf{r}_{j,0} \right)^T \hat{\mathbf{u}}_{1,0} \right\} = \lambda_1(t) d_s,$$

Thus the inter-agent collision is avoided if $\lambda_1(t)d_s \ge 2(\delta + \epsilon)$ at any time $t \in [t_s, t_f]$. This is ensured if (36) is satisfied. \blacksquare Remark 3. Eigenvalues $\lambda_2(t)$ and $\lambda_3(t)$ of matrix \mathbf{U}_D must be positive at any time t as is required for continuum deformation coordination. Furthermore, shear deformation angle $\phi_{u,0}^*$ can be arbitrarily selected. Without loss of generality, this paper chooses $\phi_u(t) = \phi_{u,0}^* = 0$ at any time t.

VII. PROBLEM 3: CONTINUUM DEFORMATION PLANNING

We define the desired trajectory of every leader $i \in \mathcal{V}_L$ by

$$\mathbf{r}_{i,HT}(t) = \mathbf{r}_{i,HT} \left(\mathbf{s}_{\varrho}^{n}(t) \right), \tag{38}$$

where $\mathbf{s}_{\varrho}^{n}(t)$ is the generalized coordinate vector and $n \in \{1,2,3\}$ is the dimension of the homogeneous deformation coordination.

The generalized coordination vector $\mathbf{s}_{o}^{n}(t)$ is given by

$$t \in [t_k, t_{k+1}], \quad \mathbf{s}_{\varrho}^n(t) = \bar{\mathbf{s}}_{k,\varrho}^n(1 - \beta(t, T_k)) + \beta(t, T_k)\bar{\mathbf{s}}_{k+1,\varrho}^n,$$
(39)

where $k=1,\cdots,n_{\tau},\ \varrho\in\{\text{OL},\text{OF}\}$. OF and OL denote obstacle-free and obstacle-laden, respectively. Conditions $t_1=t_s,\ t_{n_{\tau}+1}=t_f,\ \bar{\mathbf{s}}^n_{1,\varrho},\ \cdots,\ \bar{\mathbf{s}}^n_{n_{\tau}+1,\varrho}$ determine intermediate configurations for the generalized coordinate, $T_k=t_{k+1}-t_k$, and $\beta(t,T_k)$ is defined by the following fifth order polynomial:

$$t \in [t_k, t_{k+1}], \qquad \beta(t, T_k) = \sum_{i=0}^{5} \zeta_{j,k} \left(\frac{t - t_k}{T_k}\right)^5,$$
 (40)

where $\beta(t_k, T_k) \in [0, 1]$ is an increasing function of t over $[t_k, t_{k+1}]$, $\beta(t_k, T_k) = 0$, $\beta(t_{k+1}, T_k) = 1$, $\zeta_{0,k}$ through $\zeta_{5,k}$ are constant and determined based on boundary conditions at times t_k and t_{k+1} . Note that T_1 through T_n are the design parameters determined such that the continuum deformation safety conditions are all satisfied.

Path Planning: For continuum deformation coordination in an obstacle-free environment, $n_{\tau} = 1$ generalized coordinate vector \mathbf{s}_{OF}^{n} is defined as follows:

where \mathbf{s}_{OF}^n is defined by quintic polynomial (39) with $t_1 = t_s$ and $t_2 = t_f$.

For motion planning in obstacle-laden environments, $\mathbf{r}_{i,HT}\left(\mathbf{s}_{o}^{n}(t)\right) = \mathbf{s}_{o}^{n}(t)$, where

$$\mathbf{s}_{\varrho}^{n}(t) = \text{vec}\left(\begin{bmatrix}\mathbf{r}_{1,HT}(t) & \cdots & \mathbf{r}_{n+1,HT}(t)\end{bmatrix}^{T}\right).$$
 (42)

This paper uses A* search such that the leaders' travel distance is minimized.

Trajectory Planning and Planning of Travel Time: Assume consecutive generalized coordinate vectors $\bar{\mathbf{s}}_{k,\varrho}^n$ and $\bar{\mathbf{s}}_{k+1,\varrho}^n$ are known for $\varrho \in \{\text{OF}, \text{OL}\}$ and $k=1,\cdots,n_\tau$. Then, the travel time planning problem is defined as follows. Choose $T_k \geq T_k^*$, where minimum travel time T_k^* is assigned by solving the following optimization problem:

$$T_k^* = \min \ T_k \tag{43}$$

subject to Eqs. (21), (40), and

$$\mathbf{s}_{\varrho}^{n}(t) = \bar{\mathbf{s}}_{k,\varrho}^{n}(1 - \beta(t, T_{k})) + \beta(t, T_{k})\bar{\mathbf{s}}_{k+1,\varrho}^{n}, \qquad t \in [t_{k}, t_{k+1}],$$

$$\mathbf{u}_{i}(t) \in \mathcal{U}, \qquad \forall t \in [t_{k}, t_{k+1}], \forall i \in \mathcal{V},$$

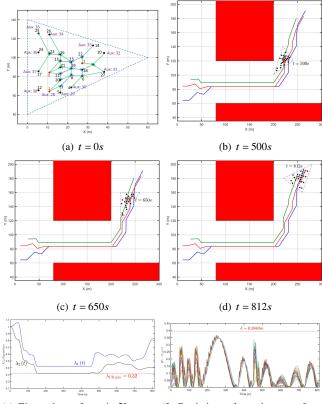
(44)

where $t_s = t_1$, $t_f = t_{n_\tau + 1}$, $t_{k+1} = t_k + T_k$, and $\zeta_{0,k}$ through $\zeta_{5,k}$ are specified coefficients satisfying assumptions discussed after (40). Because $\beta(t, T_k)$ is assigned by Eq. (40) for $t \in [t_k, t_{k+1}]$, the term $\left| \frac{d^j \beta(t, T_k)}{dt^j} \right|$ is a decreasing function with respect to T_k at any time $t \in [t_k, t_{k+1}]$ for $j \in \{\rho_q - \Xi, \cdots \rho_q\}$ $(q \in \{x, y, z\})$. Consequently, there exists a minimum travel time T_k^* such that the safety conditions (21), (40), and (44) are all satisfied.

VIII. SIMULATION RESULTS

Case studies of 2-D, and 3-D continuum deformation are presented below with and without obstacles. We assume each vehicle is a quadcopter with the input-output linearizable dynamics presented in [24].

For the 2-D example, we consider continuum deformation coordination of N=27 quadcopters with reference (initial) configuration shown in Fig. 3 (a), where $V_L=\{1,2,3\}$, $V_F=\{4,\cdots,27\}$ and $V_{aux}=\{28,29,\cdots,38\}$. Leaders move independently and every follower communicates with three in-neighbor nodes where followers' reference communication weights are determined by Eq. (14). As shown in Fig. 3 (a), reference VCS is a containing triangle with vertices positioned at $\mathbf{h}_{1,0}=(0,40)$, $\mathbf{h}_{2,0}=(60,100)$, and $\mathbf{h}_{3,0}=(0,140)$. Given the reference configuration and final formation shown in Fig. 3 (d), the optimal leaders' paths are determined using A* search and shown in Fig. 3 (b-d). Given leaders' optimal paths, eigenvalues λ_1 and λ_2 are plotted versus time in Fig. 3 (e).



(e) Eigenvalues of matrix \mathbf{U}_D ver- (f) Deviation of quadcopters from sus time.

Fig. 3: (a-d) MVS at sample times 0s, 500s, and 812s. (e) \mathbf{U}_D eigenvalues λ_1 and λ_2 versus time. (f) Deviation of each follower $i \in \mathcal{V}_F$ versus time. Note that $\sup_t \|\mathbf{r}_i - \mathbf{r}_{i,HT}\| \le \delta = 0.3940m$, $\forall i \in \mathcal{V}_F$.

Assuming every quadcopter is enclosed by a ball with radius $\epsilon = 0.5m$ and $\delta = 0.3940m$, leaders' travel times are assigned such that safety requirements are all satisfied. Fig. 3 (f) shows that $\|\mathbf{r}_i(t) - \mathbf{r}_{i,HT}(t)\| \le \delta$ at any time $t \in [0,812]$.

For the 3-D example, we simulate collective takeoff with a N=16 quadcopter MVS where $\mathcal{V}_L=\{1,2,3,4\}$), $\mathcal{V}_F=\{5,\cdots,16\}$, and $\mathcal{V}_{aux}=\{17,18,19\}$ with $\mathbf{r}_{14,0}=\mathbf{r}_{17,0}$, $\mathbf{r}_{15,0}=\mathbf{r}_{18,0}$, and $\mathbf{r}_{16,0}=\mathbf{r}_{19,0}$. The MVS initial formation is shown in Fig. 4 (a) where leaders are illustrated by red, followers 5 through 13 are shown by black. In addition, followers 14, 15, and 16 are shown in green in Fig. 4 (b).

Given quadcopter initial reference positions, the shear deformation angles $\theta_{u,0} = 0.3770$ rad and $\psi_{u,0} = 1.0053$ rad are obtained using Eq. (33). Furthermore, we choose $\phi_{u,0} = 0$ at any time t, per Remark 3). We consider MVS collective motion over the time interval [0,600s] ($t_s = 0s$ and $t_f = 600$) where $\phi_r(t_f) = 0$ rad, $\theta_r(t_f) = 0.0713$ rad, $\psi_r(t_f) = \frac{\pi}{2}$ rad, $d_1(t_f) = 100m$, $d_2(t_f) = 165m$, and $d_3(t_f) = 200m$. Given $\mathbf{s}_{OF}^3(0)$ and $\mathbf{s}_{OF}^3(600)$, a homogeneous transformation is defined by Eq. (3) and acquired by followers through local communication. MVS formations at sample times $t_s = 0,250,400,600s$ are shown in

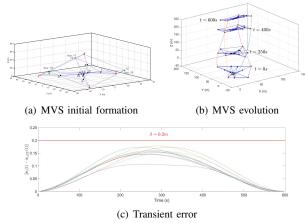


Fig. 4: (a) MVS initial formation and inter-agent communication. (b) MVS at times 0s, 250s, 400s, 250s. Leader 1, 2, 3, 4 paths are shown by black, green, red, and pink curves. (c) Deviation of each follower $i \in V_F$ versus time.

Fig. 4 (b). Assuming $\epsilon = 0.18m$ and obtaining $\delta = 0.6652m$, deviation of every follower i from $\mathbf{r}_{i,HT}$ is plotted versus time in Fig. 4 (c). Because $\sup \|\mathbf{r}_i(t) - \mathbf{r}_{i,HT}(t)\| \le 0.6652 \ (\forall i \in \mathcal{V}_F)$ collision avoidance is gauranteed.

IX. CONCLUSION

This paper advanced continuum deformation coordination by relaxing existing containment constraints. We showed that any n+1 agents forming an n-D simplex can be considered as leaders; followers can be placed inside or outside the leading simplex in an n-D homogeneous transformation (n = 1, 2, 3). This paper also formulated continuum deformation coordination eigen-decomposition to determine a nonsingular mapping between leader position components and homogeneous transformation features assigned by continuum deformation eigendecomposition. With this approach, leader trajectories ensuring collision avoidance and quadcopter containment can be safely planned. Furthermore, this paper advances the existing condition for inter-agent collision avoidance in a large-scale continuum deformation. This new safety condition is much less restrictive and significantly advances the maneuverability and flexibility of the continuum deformation coordination.

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