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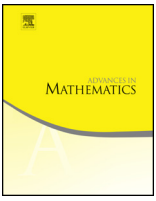

Fragments of the theory of the enumeration degrees ☆

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ABSTRACT

We prove that every finite distributive lattice can be strongly embedded into the enumeration degrees as an interval, i.e., that there is an interval $[\mathbf{a}_0, \mathbf{a}_1]$ of enumeration degrees isomorphic to the lattice, and any enumeration degree $\mathbf{b} \leq \mathbf{a}_1$ lies in this interval or below $\mathbf{a}_0$. As corollaries, we conclude that the $\exists\forall\exists$-theory of $\mathcal{D}_e$ is undecidable, while the extension of embeddings problem (a subproblem of the $\forall\exists$-theory) is decidable.

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1. Introduction

Consider a degree structure $\mathcal{D}$, a partial order induced by an arithmetically definable reducibility on sets of natural numbers. When studying such a structure, we start by investigating which first-order facts about the structure are true and which are false. Ideally, we would like to characterize the theory of the structure by giving an algorithm which decides whether a given sentence (in the language of partial orders) is true or not. Unfortunately, most degree structures have first-order theories that are far from decidable. Once undecidability is established, two natural questions follow: What is the Turing degree of the theory $\mathrm{Th}(\mathcal{D})$ of the structure $\mathcal{D}$ (viewed as a set of codes for sentences), and at what quantifier complexity does decidability break down?

An interesting phenomenon in degree theory is that when we can provide answers to the questions above, the answers seem to always follow the same pattern: For the partial order of the Turing degrees $\mathcal{D}_T$, Simpson [22] proved that $\mathrm{Th}(\mathcal{D}_T)$ is computably isomorphic to the second-order theory of true arithmetic. Shore [20] and Lerman [15] independently proved that $\forall\exists$-theory of $\mathcal{D}_T$ is decidable, while Lerman and Schmerl (see Lerman [15]) proved that the $\exists\forall\exists$-theory of $\mathcal{D}_T$ is undecidable. For the partial order of the many-one degrees $\mathcal{D}_m$, Nerode and Shore [17] proved that $\mathrm{Th}(\mathcal{D}_m)$ is computably isomorphic to the second-order theory of arithmetic; Dëgtev [5] proved that the $\forall\exists$-theory of $\mathcal{D}_m$ is decidable, while Nies [18] proved that the $\exists\forall\exists$-theory of $\mathcal{D}_m$ is undecidable. For the local structure of the Δ^0_2-Turing degrees $\mathcal{D}_T(\leq \mathbf{0}')$, Shore [21] proved that $\mathrm{Th}(\mathcal{D}_T(\leq \mathbf{0}'))$ is computably isomorphic to the first-order theory of arithmetic. Shore and Lerman [16] proved that the $\forall\exists$-theory of $\mathcal{D}_T(\leq \mathbf{0}')$ is decidable, while the same proof that is used for $\mathcal{D}_T$ by Lerman and Schmerl showed that the $\exists\forall\exists$-theory of $\mathcal{D}_T(\leq \mathbf{0}')$ is undecidable. Similar results were shown for the arithmetic and hyperarithmetic degrees.

In this paper, we will focus on the structure of the enumeration degrees, where less is known. Enumeration reducibility captures a natural relationship between sets of natural numbers in which positive information about the first set is used to produce positive information about the second set. Friedberg and Rogers [6] introduced enumeration reducibility in 1959.

Definition 1.1. $A \subseteq \omega$ is *enumeration reducible* to $B \subseteq \omega$ (denoted as $A \leq_e B$) if there is a c.e. set W such that

$$A = \{n \colon (\exists e)\ \langle n, e\rangle \in W \text{ and } D_e \subseteq B\},$$

where D_e is the eth finite set in a canonical enumeration.

An equivalent way to define this reducibility is to say that $A \leq_e B$ if there is a uniform way to compute an enumeration of A from every enumeration of B. In fact, Selman [19] proved that the uniformity condition can be dropped.

The degree structure $\mathcal{D}_e$ induced by $\leq_e$ is the partial order of the *enumeration degrees*. $\mathcal{D}_e$ is, in fact, an upper semilattice with a least element $\mathbf{0}_e$ (the degree of all c.e.

sets) and a jump operator, just like $\mathcal{D}_T$. Note that enumeration reducibility is a definable relation in second-order arithmetic. Thus it is easy to interpret the partial order $\mathcal{D}_e$ in second-order arithmetic. Slaman and Woodin [25] proved that the converse is true as well, and so $\mathrm{Th}(\mathcal{D}_e)$ is computably isomorphic to second-order arithmetic, just like the theory of the Turing degrees. Lagemann [11] proved that every countable partial order can be embedded in the enumeration degrees and so the $\exists$-theory of $\mathcal{D}_e$ is decidable. However, it is not known where decidability breaks down.

In this paper, we make several advancements towards a solution of this open question. Our main structural result on which the other results rely is the existence of a *strong interval embedding* of every finite distributive lattice—generalizing the embedding of the two element lattice as a nonzero degree and its strong minimal cover. This result and an application of the Nies Transfer lemma allow us to conclude that the $\exists\forall\exists$-theory of $\mathcal{D}_e$ is not decidable. On the other hand, we show that the *extension of embeddings problem* for $\mathcal{D}_e$ is decidable. The extension of embeddings problem captures a nontrivial fragment of the $\forall\exists$-theory of a partial order. We also prove that this is the maximal fragment on which the Turing degrees and the enumeration degrees are elementarily equivalent.

2. The $\forall\exists$-theory of an upper semilattice

We start by reviewing the algorithm that decides the $\forall\exists$-theory of the Turing degrees and the reasons why the same algorithm cannot apply to the structure $\mathcal{D}_e$. Our first step is to rephrase the problem of deciding the $\forall\exists$-theory of an upper semilattice $\mathcal{D}$ in a structural way. A decision procedure for the following problem is easily seen to be equivalent to a decision procedure for $\forall\exists$-$\mathrm{Th}(\mathcal{D})$:

Problem 2.1. Given a finite partial order P and finitely many finite extensions $Q_1, \ldots, Q_k$ of P, does every embedding of P into $\mathcal{D}$ extend to an embedding of Q_i for at least one $i \leq k$?

The special case when $k = 1$ is known as the *extension of embeddings problem for* $\mathcal{D}$.

Consider the case when $\mathcal{D} = \mathcal{D}_T$. Lerman [14] showed that every finite lattice P can be embedded as an initial segment of the Turing degrees $\mathcal{D}_T$. Suppose that P is a lattice and Q extends P as a partial order. The embedding of P as an initial segment of $\mathcal{D}_T$ can be extended to an embedding of Q only if no new element in $Q \setminus P$ is bounded by any element of P. In addition, Q must respect least upper bounds; i.e., if $x \in Q \setminus P$ is above two old elements $u, v \in P$ then x must be above $u \vee v$. If P is simply a partial order, we first extend P to a lattice P^* by adding a minimal number of new elements (this can be done in a unique way) and then ask that new elements in $Q \setminus P$ either satisfy the previous conditions or can be mapped to one of the added elements from $P^* \setminus P$. Shore [20] and Lerman [15] independently proved that these are the only obstacles, yielding an algorithm for the solution of an instance of Problem 2.1 in $\mathcal{D}_T$: Output "Yes" if one of the Q_i satisfies the conditions above, and "No" otherwise. The algorithm

does not even use the possibility of selecting different possible extensions in different situations, it is reduced to its simplest case, the extension of embeddings problem.

The algorithm described above cannot work for a dense structure, such as the partial orders of the c.e. Turing degrees $\mathcal{R}$ or that of the Σ^0_2-enumeration degrees $\mathcal{D}_e(\leq \mathbf{0}'_e)$. In both of these cases, we know that the extension of embeddings problem is decidable (by work of Slaman and Soare [23] for $\mathcal{R}$, and by Lempp, Slaman, and Sorbi [13] for $\mathcal{D}_e(\leq \mathbf{0}')$). In both cases, we also know that the $\exists\forall\exists$-theory is undecidable (by work of Lempp, Nies and Slaman [12] for $\mathcal{R}$, and by Kent [9] for $\mathcal{D}_e(\leq \mathbf{0}')$). A decision procedure for the more general Problem 2.1, i.e., for the $\forall\exists$-theory, remains out of reach in both cases.

In $\mathcal{D}_e$, the situation is very interesting for the following reasons. Gutteridge [7] showed that the enumeration degrees are downward dense. Hence, in this case as well, there can be no initial segment embeddings of finite lattices. Cooper [4] proved, however, that the enumeration degrees are not dense and Slaman and Calhoun [3] extended Coopers's result by showing that there are empty intervals in the Π^0_2-enumeration degrees. Kent, Lewis-Pye, and Sorbi [10] showed that there are strong minimal covers in the enumeration degrees:

Definition 2.2. A degree $\mathbf{b}$ is a *strong minimal* cover of a degree $\mathbf{a}$ if $\mathbf{a} < \mathbf{b}$ and every degree $\mathbf{x} < \mathbf{b}$ is $\leq \mathbf{a}$.

Consider the two-element lattice P consisting of two elements $u < v$. In $\mathcal{D}_T$, we can embed this lattice as an initial segment: u is mapped to $\mathbf{0}_T$, and v is mapped to some minimal degree. The only way that this embedding can be extended to an embedding of Q is if every element of $Q \setminus P$ is incomparable to or above v. In the enumeration degrees, the situation is slightly different: The embedding of P to enumeration degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ is a strong minimal cover of $\mathbf{a}$ extends to an embedding of Q only if new elements $x \in Q \setminus P$ that are strictly below v are also below u. The embedding of P to degrees $\mathbf{0}_e < \mathbf{b}$, on the other hand, extends to an embedding of Q only if all new elements $x \in Q \setminus P$ are above u. Slaman and Sorbi [24] show that every countable partial order can be embedded below any nonzero enumeration degree. This, along with a standard forcing argument, allows us to conclude that these are the only obstacles. Thus, for this particular lattice P, we can decide Problem 2.1: Every embedding of P extends to an embedding of $Q_1, \ldots, Q_n$, if and only if there is a Q_i that places new elements strictly below v also below u and there is a (possibly different) Q_j that places all new elements above u. The decision procedure is already slightly more complicated than that for the same lattice in $\mathcal{D}_T$.

A first step towards a possible extension of the algorithm outlined above to the general case, where P is an arbitrary finite lattice, requires the generalization of embedding the two-element lattice to a nonzero degree and a strong minimal cover of it. We introduce the notion of a *strong interval embedding*:

Definition 2.3. Let P be a finite lattice and $\mathcal{D}$ a degree structure. An embedding $f : P \to \mathcal{D}$ is a *strong interval embedding* if the range of the embedding f is an interval $[\mathbf{a}, \mathbf{b}] \subseteq \mathcal{D}$ and every degree $\mathbf{x} \leq \mathbf{b}$ that is not in the range of f is bounded by $\mathbf{a}$.

The main technical result of this paper is the following.

Theorem 2.4. *Every finite distributive lattice has a strong interval embedding into $\mathcal{D}_e$.*

We postpone the proof of this theorem until Section 7 and focus first on several applications.

3. The undecidability of the ∃∀∃-theory

Recall that a set of sentences Ω in a language L is *hereditarily undecidable* if no subset $\Phi \subseteq \Omega$ that contains all validities in Ω is decidable. For example, Nies [18] proved that the $\forall\exists\forall$-theory of finite distributive lattices is hereditarily undecidable. In the same paper, he gave a general recipe for transferring undecidability between classes of structures. The following definition is adapted from Nies [18] to our specific setting. (Here, we adopt Nies's notation of Σ^0_k-formulas for $\exists_k$-formulas, and Π^0_k-formulas for $\forall_k$-formulas; so, e.g., Σ^0_3, $\exists_3$ and $\exists\forall\exists$ all mean the same.)

Definition 3.1. Let $\mathcal{C}$ be a class of structures in a finite relational language $L = \{R_1, \ldots, R_n\}$. We say that $\mathcal{C}$ is Σ^0_k*-elementarily definable with parameters* in $\mathcal{D}_e$ if there are Σ^0_k-formulas φ_U, φ_{R_i}, and $\varphi_{\neg R_i}$ for $i \leq n$ such that for every $C \in \mathcal{C}$, there are parameters $\vec{\mathbf{p}} \in \mathcal{D}_e$ that make the structure with universe $U = \{\mathbf{x} \mid \mathcal{D}_e \models \varphi_U(\mathbf{x}, \vec{\mathbf{p}})\}$ and relations R_i defined as $\{\vec{\mathbf{x}} \mid \mathcal{D}_e \models \varphi_{R_i}(\vec{\mathbf{x}}, \vec{\mathbf{p}})\} = \{\vec{\mathbf{x}} \mid \mathcal{D}_e \models \neg\varphi_{\neg R_i}(\vec{\mathbf{x}}, \vec{\mathbf{p}})\}$ isomorphic to C.

Theorem 2.4 implies that the class of finite distributive lattices is Σ^0_1-elementarily definable in the partial order $\mathcal{D}_e$ with two parameters: $\varphi_U(\mathbf{x}, \mathbf{a}, \mathbf{b})$ is the formula $\mathbf{a} \leq \mathbf{x}$ & $\mathbf{x} \leq \mathbf{b}$, and $=$, $\neq$, $\leq$ and $\not\leq$ are interpreted by $=$, $\neq$, $\leq$ and $\not\leq$, respectively. We next apply the Nies Transfer Lemma to our setting:

Lemma 3.2 *(Nies [18]). Let $r \geq 2$ and $k \geq 1$. If a class of models $\mathcal{C}$ is Σ^0_k-elementarily definable in $\mathcal{D}_e$ with parameters and the Π^0_{r+1}-theory of $\mathcal{C}$ is hereditarily undecidable, then the Π^0_{r+k}-theory of $\mathcal{D}_e$ is hereditarily undecidable.*

We can now state, using the hereditary undecidability of the Π^0_3-theory of finite distributive lattices mentioned above, the following.

Theorem 3.3. *The $\exists\forall\exists$-theory of $\mathcal{D}_e$ is (hereditarily) undecidable.* □

This uses the fact that a $\forall\exists\forall$-sentence φ is true in $\mathcal{D}_e$ if and only if the $\exists\forall\exists$-sentence $\neg\varphi$ is false in $\mathcal{D}_e$, and so the undecidability of the $\forall\exists\forall$-theory of $\mathcal{D}_e$ implies the undecidability of the $\exists\forall\exists$-theory of $\mathcal{D}_e$.

4. The extension of embeddings problem

In this section, we give an algorithm to decide the extension of embeddings problem for $\mathcal{D}_e$: Given finite partial orders $P \subseteq Q$, we give necessary and sufficient conditions on P and Q to make the statement "every embedding of P extends to an embedding of Q" true.

In addition to Theorem 2.4, we will need to use properties of sufficiently generic sets. Recall, that a set G is n-generic relative to B if and only if for every $\Sigma^0_n(B)$-set S of finite binary strings, there is an initial segment $G \upharpoonright \ell$ of G such that $G \upharpoonright \ell$ is in S or no extension of $G \upharpoonright \ell$ belongs to S. If $\{G_i\}_{i<\omega}$ is a sequence of sets and F is a set of natural numbers, we use $\bigoplus_{i\in F} G_i$ to denote the set $\{\langle i, x\rangle \mid i \in F \ \&\ x \in G_i\}$.

Proposition 4.1. *Let G be 2-generic relative to B. Define G_i so that $G = \bigoplus_{i<\omega} G_i$. For every pair of sets $A_1, A_2 \leq_e B$, $i \in \omega$ and finite set $F \subseteq \omega$, we have that $A_1 \oplus G_i \leq_e A_2 \oplus \bigoplus_{j\in F} G_j$ if and only if $i \in F$ and $A_1 \leq_e A_2$.*

Proof. Fix $A_1, A_2 \leq_e B$, i and F and suppose that $A_1 \oplus G_i \leq_e A_2 \oplus \bigoplus_{j\in F} G_j$. Let $G_i = \Gamma(A_2 \oplus \bigoplus_{j\in F} G_j)$. Given a string $\tau \in 2^{<\omega}$, we let τ_j be the shortest string such that for all $\langle j, n\rangle < |\tau|$, we have $\tau(\langle j, n\rangle) = \tau_j(n)$. Consider the set $U = \{\tau \in 2^{<\omega} \mid (\exists x)[\tau_i(x) = 0 \ \&\ x \in \Gamma(A_2 \oplus \bigoplus_{j\in F} \tau_j)]\}$. This set is $\Sigma^0_1(B)$ (in fact, it is enumeration reducible to B) and hence, by our assumption, G must avoid it. Let $\mu \prec G$ be such that no extension of μ is in U. As G is generic, the set G_i is infinite, and so there is some $x > |\mu|$ such that $G_i(x) = 1$. It follows that $x \in \Gamma(A_2 \oplus \bigoplus_{j\in F} G_j)$ and so there is some finite extension $\tau \succeq \mu$ such that $x \in \Gamma(A_2 \oplus \bigoplus_{j\in F} \tau_j)$. If $i \notin F$ then we can modify the $\langle i, x\rangle$-th bit of τ to get a string τ^* such that $\tau^*_i(x) = 0$ and $x \in \Gamma(A_2 \oplus \bigoplus_{j\in F} \tau^*_j)$, i.e., an extension of μ in the set U. It follows that i must be in F.

Now suppose that $A_1 = \Gamma(A_2 \oplus \bigoplus_{j\in F} G_j)$. Consider the set $V = \{\tau \in 2^{<\omega} \mid (\exists x)[A_1(x) = 0 \ \&\ x \in \Gamma(A_2 \oplus \bigoplus_{j\in F} \tau_j)]\}$. The set V is $\Sigma^0_2(B)$ (in fact, it is enumeration reducible to B'). Once again, we must have some initial segment $\mu \prec G$ with no extension in V. But then $A_1 = \{x \mid \exists \tau \succeq \mu[x \in \Gamma(A_2 \oplus \bigoplus_{j\in F} \tau_j)]\}$ and so $A_1 \leq_e A_2$.

The reverse direction is clearly true. □

Note that a special case of the proposition above gives us that, in particular, the degrees of B and each G_i form a minimal pair. Furthermore, since G_i and G_j (for distinct i and j) are mutually 2-generic, their degrees also form a minimal pair.

We are now ready to present an algorithm for deciding the extension of embeddings problem.

Theorem 4.2. *The extension of embedding problem for $\mathcal{D}_e$ is decidable.*

Proof. Fix finite partial orders $P \subseteq Q$. For any set $S \subseteq Q$, let

$$A(S) = \{p \in P \mid (\forall s \in S)[p \geq s]\} \text{ and}$$
$$B(S) = \{p \in P \mid (\forall s \in S)[p \leq s]\}.$$

We will use $A(q)$ to denote $A(\{q\})$ and $B(q)$ for $B(\{q\})$.

We outline several cases in which we can construct an embedding of P that does not extend to an embedding of Q.

Case 1: *There is $q \in Q \setminus P$ such that $A(q) = \emptyset$ and $B(q) \neq B(A(B(q)))$.*

Suppose that there is $q \in Q \setminus P$ with $A(q) = \emptyset$. We will show that we can obstruct an extension with such a q if $B(q) \neq B(A(B(q)))$. Note that since we always have $B(q) \subseteq B(A(B(q)))$, these two conditions imply the existence of some $p \in B(A(B(q))) \setminus B(q)$. (In particular, it follows that $B(q)$ has no greatest element.) We will construct an embedding of P such that any degree that is above all elements in the image of $B(q)$ is also above the image of p; this embedding will therefore not be extendable to an embedding of Q.

If $B(q) = \emptyset$ then $A(B(q)) = P$ and hence the element $p \in B(A(B(q)))$ is the least element in P. Any embedding of P that maps p to $\mathbf{0}_e$ will do the job.

Suppose that $B(q) \neq \emptyset$. Let $P = \{p_0, p_1, \ldots, p_n\}$. We fix a 2-generic set G and break it up into $|P|$ many mutually generic pieces $G_0, \ldots, G_n$. We map p_i to the degree of the set $X_{p_i} = \bigoplus_{p_j \leq p_i} G_j$, which we denote as $g(p_i)$. By genericity, we have that $i \not\leq j$ implies $X_{p_i} \not\leq_e X_{p_j}$. Next we want to modify this embedding to achieve the desired result. Take $X_p = \bigoplus_{p_j \leq p} G_j$ and break it up into $|B(q)|$ many pieces $Y_0, \ldots, Y_l$ as follows: break up each G_j into $|B(q)|$ many mutually generic pieces for $p_j \leq p$, and let Y_i consist of the join of the i-th pieces in the sets G_j for $p_j \leq p$. Let $B(q) = \{r_0, \ldots, r_l\}$. We modify our embedding g to an embedding f on the elements $s \in \bigcup_{r \in B(q)} A(r)$ by setting $f(s) = g(s) \oplus \deg_e(\bigoplus_{r_j \leq s} Y_j)$. In this way, we have that the least upper bound of the elements in $B(q)$ enumerates all the pieces that make up X_p, preventing an extension of f which maps q not above p.

All we need to do is prove that this modification does not change the order. If $p_i \leq p_j$ then $f(p_i) \leq f(p_j)$: To see this, first consider the case when $p_j \notin \bigcup_{r \in B(q)} A(r)$. Then $p_i \notin \bigcup_{r \in B(q)} A(r)$ as well, and so $f(p_i) = g(p_i) \leq g(p_j) = f(p_j)$. If, on the other hand, $p_j \in \bigcup_{r \in B(q)} A(r)$, then $f(p_i)$ is either $g(p_i)$ or $g(p_i) \oplus \deg_e(\bigoplus_{r_k \leq p_i} Y_j)$ and so

$$f(p_i) \leq g(p_i) \oplus \deg_e(\bigoplus_{r_k \leq p_i} Y_j) \leq_e g(p_j) \oplus \bigoplus_{r_k \leq p_j} \deg_e(Y_k) = f(p_j).$$

Now suppose that $p_i \not\leq p_j$. Once again, the case when $p_j \notin \bigcup_{r \in B(q)} A(r)$ is easy because $f(p_j) = g(p_j)$ and $f(p_i) \geq g(p_i)$. So suppose that $p_j \in \bigcup_{r \in B(q)} A(r)$. Recall that G_i is part of the image of p_i. If $p_i \not\leq p$, then $G_i \not\leq_e f(p_j)$ by the properties of

mutually generic sets. If $p_i \leq p$, then there is some $r_k \in B(q)$ such that $r_k \not\leq p_j$ (since otherwise $p_j \in A(B(q))$ and hence $p_j \geq p \geq p_i$). But then the ith column of Y_k, which was broken off from the set $G_i \leq_e f(p_i)$, is still mutually generic with all the other pieces into which we have broken up G, hence it is not below $f(p_j)$.

From now on, in all cases that we consider, let us assume that we have that $A(q) \neq \emptyset$. Our next group of cases examines possible obstructions when $B(q) = \emptyset$. We break this into three cases, based on whether $B(A(q))$ is empty or not, and whether $B(A(q)) \subseteq A(q)$ or not.

Case 2: *There is $q \in Q \setminus P$ such that $A(q) \neq \emptyset$, $B(q) = \emptyset$, $B(A(q)) = \emptyset$, but q is not the least element of Q.*

In this case, we know that $A(q)$ is not principal, i.e., does not have a least element. We use the columns of a 2-generic to embed P into $\mathcal{D}_e$. The only degree that is bounded by all degrees that are images of the elements of $A(q)$ is then $\mathbf{0}_e$. This embedding of P can only be extended to an embedding of Q if q is mapped to $\mathbf{0}_e$. So if q is not the least element of Q, then we have exhibited an embedding of P that does not extend to an embedding of Q.

Case 3: *There is $q \in Q \setminus P$ such that $A(q) \neq \emptyset$, $B(q) = \emptyset$, $B(A(q)) \neq \emptyset$, and $B(A(q)) \subseteq A(q)$.*

If $B(A(q)) \subseteq A(q)$ then $A(q)$ is principal above p_0 and p_0 is a minimal element in P. If p_0 is the least element in P then we can embed P by sending p_0 to $\mathbf{0}_e$ and then use the columns of a 2-generic as before to embed the rest of P. This embedding of P cannot be extended to an embedding of Q because there is no possible image for q strictly below $\mathbf{0}_e$.

If p_0 is not least, then fix a minimal element $p_1 \in P$ distinct from p_0. Let A_0 and A_1 form an *Ahmad pair*, i.e., $A_0 \not\leq_e A_1$ and $(\forall \mathbf{x})[\mathbf{x} < \deg_e(A_0) \to \mathbf{x} < \deg_e(A_1)]$. The existence of such pairs was first proved by Ahmad [1], but also follows from Theorem 2.4. Next pick a set G that is 2-generic relative to $A_0 \oplus A_1$. Split G into $|P|-2$ many mutually generic sets. If $P = \{p_0, p_1, \ldots, p_n\}$ then let $f(p_i) = \deg_e(\bigoplus_{p_j \leq p_i} X_j)$, where $X_0 = A_0$, $X_1 = A_1$ and $X_{i+2} = G_i$ for $i < |P| - 2$. Once again, it is clear from the definition of f that if $p_i \leq p_j$ then $f(p_i) \leq f(p_j)$. On the other hand, by our choice of generics and of A_0 and A_1, we have that $X_i \leq_e \bigoplus_{p_k \leq p_j} X_k$ if and only if X_i is one of the elements in $\{X_k \mid p_k \leq p_j\}$ if and only if $p_i \leq p_j$. So if $p_i \not\leq p_j$ then $f(p_i) \not\leq f(p_j)$. This embedding cannot be extended to an embedding to Q because any degree that is strictly below the image of p_0 must also be below the image of p_1, so q cannot be embedded.

We are left with the case when $B(q) = \emptyset$ and $B(A(q)) \not\subseteq A(q)$. We will be able to obstruct this case as well using a slightly more complicated embedding. In short, we extend $B(A(q))$ to a distributive lattice L only adding points if some finite set of points in $B(A(q))$ is missing a least upper bound. Since $B(q)$ is empty, we can argue that no

point will be added that will fit the type of q over P in this way. Here by the *type of q over P*, we mean the set of atomic facts that describe the position of q with respect to the elements of P. We then embed L using a strong embedding, and the rest of P using 2-generics relative to the image of the top element in L. We argue that any degree strictly below all elements in $A(q)$ must be an image of an element in L or bounded below the image of the least element in L. No such degree can be the image of q. We use the same construction in case $B(q) \neq \emptyset$ and either $B(A(q)) \neq B(q)$ or $A(B(q)) \neq A(q)$ (and so it is inconsistent to place q as the least upper bound of $B(q)$ and the greatest lower bound of $A(q)$ at the same time). We combine both of these situations in Case 4 below.

Case 4: *There is $q \in Q \setminus P$ such that $A(q) \neq \emptyset$ and either*

(a) $B(q) = \emptyset$ *and* $B(A(q)) \not\subseteq A(q)$*; or*
(b) $B(q) \neq \emptyset$*, and either* $A(B(q)) \neq A(q)$ *or* $B(A(q)) \neq B(q)$*.*

Since we will build an embedding of P which blocks any extension to $P \cup \{q\}$, we may assume in this case that $Q = P \cup \{q\}$. As we already hinted in the previous case, this is the most complicated case and the one where we will make use of the strong embedding of all finite distributive lattices from Theorem 2.4.

We first enlarge P by adding new elements in a minimal way to make $S_0 = B(A(q))$ into an upper semilattice with least element: For each nonempty subset $F \subseteq S_0$ such that F has no greatest element and $A(F)$ has no least element, add a new element $s_{A(F)}$ and specify that $B(s_{A(F)}) = B(A(F))$ and $A(s_{A(F)}) = A(F)$. Note that if F and G are distinct such subsets with $A(F) = A(G)$, then this will add only a single point $s_{A(F)} = s_{A(G)}$. We order new points $s_{A(F)} < s_{A(G)}$ if and only if $A(G) \subset A(F)$. If $B(A(q))$ has no least element then we add one additional point $s_{A(\emptyset)}$ bounded below all elements in $B(A(q))$. Denote by P' the union of P and of all the newly added elements $s_{A(F)}$, and let S be the union of S_0 and of all the newly added elements $s_{A(F)}$. We show that S is an upper semilattice. It is easy to check that any subset $F \subseteq S_0$ either has a least upper bound in P or has a new least upper bound $s_{A(F)}$. Indeed, if $A(F)$ has a least element (which would be implied by F having a greatest element), then, by definition, this is the least upper bound of F in P, and since $F \subseteq B(A(q))$, we have that $A(q) \subseteq A(F)$ and so the least element of $A(F)$ is below all elements in $A(q)$, hence in $B(A(q))$. Now, if $F' \subseteq S$ has newly added elements, we can transform it into $F \subseteq S_0$ by replacing each $s_{A(G)} \in F'$ by G. Thus the least upper bound of F is the same as the least upper bound of F'.

For future reference, we also note that every element $s \in S_0$ can be written uniquely as $s_{A(F)}$ for $F = B(s)$. Fixing such $s = s_{A(F)}$, note that $A(s) = A(F)$ and $B(s) = B(A(F))$ just like for elements in $S \setminus S_0$. In addition, note that by construction, for such $s \in S_0$, there cannot be a newly added point $s' \in S \setminus S_0$ of the form $s_{A(F)}$ for such a set F since $A(F)$ has a least element.

We next expand P' to make our upper semilattice S into a distributive lattice L in a minimal way, avoiding adding any new elements which have the same type over P as q does. Let M_S be the set of *meet-irreducible* elements of S (i.e., all elements $s \in S$ such that for no $s_0, s_1 > s$ in S, $s_0 \wedge s_1 = s$). Let L be the set of all nonempty upward closed subsets of M_S. As S has a greatest element, which is meet-irreducible by definition, L is closed under union and intersection and thus forms a lattice, in fact, a distributive lattice. Note further that S naturally embeds into L by mapping each $s \in S$ to the set $\{m \in M_S \mid s \leq m\}$. Note that an upward closed subset M of M_S does not correspond to an element $s \in S$ under this embedding only if it contains two incomparable meet-irreducible elements m_0, m_1 which are minimal in M. We complete the definition of the partial ordering on $L \cup P'$ by simply taking the transitive closure, i.e., we define $l \leq p$ for $l \in L$ and $p \in P \setminus S$ iff there is some $s \in S$ with $l \leq s$ and $s < p$. Denote by P'' the union of P' and of L.

Suppose, for the sake of a contradiction, that we add an element $l \in P'' \setminus P$ of the same type over P as q. First consider the case that $l = s_{A(F)}$ for some $F \subseteq S_0$. Since $B(q) = B(l) = B(A(F))$, we have $A(B(q)) = A(B(A(F))) = A(F) = A(l) = A(q)$ (here we use the fact that whenever $F \subseteq P$ we have that $A(B(A(F))) = A(F)$, which can be easily verified just from the definitions); but by our case assumption, we must then have $B(q) \neq B(A(q)) = B(A(l)) = B(A(F)) = B(l)$, contradicting $B(q) = B(l)$.

Next suppose that $l \in P'' \setminus P'$; then there is an upward closed subset M of M_S containing two incomparable meet-irreducible elements m_1, m_2 which are minimal in M such that $l < m_0, m_1$. But now $B(q) = B(l) \subseteq B(m_0) \cap B(m_1)$, and so, as explained above, $m_0 = s_{A(F_0)}$ and $m_1 = s_{A(F_1)}$ for some $F_0, F_1 \subseteq B(A(q))$ (even if $m_0 \in S_0$ or $m_1 \in S_0$), implying that $A(m_0) = A(F_0) \supseteq A(q)$ and $A(m_1) = A(F_1) \supseteq A(q)$. In addition, $l < m_0, m_1$ implies $A(q) = A(l) \supseteq A(m_0), A(m_1)$ and thus $A(q) = A(l) = A(m_0) = A(m_1)$. But $A(m_0) = A(F_0)$ and $A(m_1) = A(F_1)$, and so $m_0 = m_1$ by construction, a contradiction.

Now we can proceed with our embedding: We embed the distributive lattice L as a strong interval $[\mathbf{a}, \mathbf{b}]$ invoking Theorem 2.4. Note that since we started with the downward closed set $B(A(q))$, we have that elements in $P \setminus L$ are all above or incomparable with elements in L. Extend our embedding to an embedding of $P'' = P \cup L$ using the columns of a set G that is 2-generic relative to $\mathbf{b}$. In this embedding, we have that any point that is below the degrees of the images of all elements in $A(q)$ (which is nonempty by assumption) is bounded by the image $\mathbf{b}$ of the top element in L. It is therefore bounded by $\mathbf{a}$ or else equals one of the degrees which are images of L. We ruled out the possibility that q takes the place of one of the elements in L and so q must be mapped to a degree strictly below $\mathbf{a}$. But then $B(A(q)) \subseteq A(q)$, and since $B(A(q)) \neq \emptyset$ by assumption, this can only happen if $B(q) = \emptyset$, contradicting our case assumption.

Suppose that no element in Q satisfies the conditions of the previous four cases. We have one more possible obstruction related to the relative type of two elements in Q.

Case 5: *There exists $q \in Q \setminus P$ and $r \in Q$, such that $A(q) \neq \emptyset$, $B(q) \subseteq B(r)$ and $q \not\leq r$.*

Note that by the fact that none of the Cases 2, 3, or 4 applies to q, we know that $B(q)$ is nonempty. Since we assume that Case 4 fails, we have that $B(A(q)) = B(q)$ and $A(B(q)) = A(q)$. (In particular, neither $B(q)$ nor $A(q)$ is principal.) We now embed $B(q)$ in some arbitrary way and use a 2-generic relative to the least upper bound of the elements in the image of $B(q)$ to complete our embedding of P. Thus the greatest lower bound of the degrees in the image of $A(q)$ is the least upper bound of the degrees that are images of $B(q)$. The only possible degree that q can be mapped to is the image of $\bigwedge A(q) = \bigvee B(q)$. Since $B(q) \subseteq B(r)$, the image of r can only be above the image of $\bigvee B(q)$, but this conflicts with $q \not\leq r$.

We claim that in all other cases, every embedding of P can be extended to an embedding of Q. Fix such $P \subseteq Q$. To summarize, we have that:

(A) For all $q \in Q \setminus P$, if $A(q) = \emptyset$, then $B(q) = B(A(B(q)))$ by the failure of Case 1.
(B) For all $q \in Q \setminus P$, if $A(q) \neq \emptyset$ and $B(q) = \emptyset$, then q is the least element of Q and $B(A(q)) = \emptyset$ by the failure of Cases 2, 3, and 4(a).
(C) For all $q \in Q \setminus P$, if $A(q) \neq \emptyset$ and $B(q) \neq \emptyset$, then $A(B(q)) = A(q)$ and $B(A(q)) = B(q)$ by the failure of Case 4(b).
(D) For all $q \in Q \setminus P$ and $r \in Q$, if $A(q) \neq \emptyset$ and $B(q) \subseteq B(r)$, then $q \leq r$ by the failure of Case 5.

Let f be an embedding of P in $\mathcal{D}_e$. Order the elements of $Q \setminus P = \{q_0, \ldots, q_n\}$ so that

- $q_i \leq q_j$ implies $i \leq j$, and
- $A(q_i) \neq \emptyset$ and $A(q_j) = \emptyset$ implies $i < j$.

We consider q_i in turn, and for each, we build $f(q_i)$.

(1) If $A(q_i) \neq \emptyset$ and $B(q_i) = \emptyset$, then by (B), we have that q_i is the least element of Q, hence we can send q_i to $f(q_i) = \mathbf{0}_e$.
(2) If $A(q_i) \neq \emptyset$ and $B(q_i) \neq \emptyset$, then send q_i to the least upper bound of the image of $B(q_i)$, setting $f(q_i) = \bigvee_{p \in B(q_i)} f(p)$.
(3) Finally, we are left with $\{q_k, \ldots, q_n\}$ with $A(q_i) = \emptyset$ for all i with $k \leq i \leq n$. Let G be 2-generic relative to the least upper bound of all degrees in the range of our embedding so far. We break G up into columns $\{G_i\}_{i<\omega}$ and map q_i to $\deg_e(G_i) \vee \bigvee_{q<q_i} f(q)$. (If $B(q_i) = \emptyset$, then $\bigvee_{q<q_i} f(q) = \mathbf{0}_e$.)

Now we need to prove that this embedding works. Suppose first that $q \leq q'$. The case when $q, q' \in P$ is handled by the assumption that f is an embedding. So we may assume that at least one of q or q' is in $Q \setminus P$. We consider the different possibilities:

- If $q \in P$ then by our construction we clearly have $f(q) \leq f(q')$.

- If $q' \in P$ then $A(q) \neq \emptyset$ and so $f(q) = \mathbf{0}_e$ or $f(q) = \bigvee f(B(q))$. However, since f is a valid embedding of P as a partial order into $\mathcal{D}_e$, we must have that $\bigvee f(B(q)) \leq f(q')$.
- Finally, suppose that q and q' are both in $Q \setminus P$. Then by construction, we build the image of q first. Suppose first that $A(q') \neq \emptyset$. First, note that $A(q) \supseteq A(q')$, so $A(q) \neq \emptyset$ as well. If $B(q)$ is empty, then $f(q) = \mathbf{0}_e$, so $f(q) \leq f(q')$. If $B(q) \neq \emptyset$, then $B(q') \neq \emptyset$ as well since $B(q') \supseteq B(q)$, so

$$f(q) = \bigvee_{p \in B(q)} \leq \bigvee_{p \in B(q')} = f(q').$$

  On the other hand, if $A(q') = \emptyset$, then $f(q') \geq f(q)$ by construction.

Suppose now that $q \not\leq q'$. Again, we only need to consider cases when at least one of q or q' is in $Q \setminus P$.

- If $q \in P$ and $A(q') = \emptyset$, then by (A), we have that $B(q') = B(A(B(q')))$. Since $q \notin B(q')$, it must be that $q \notin B(A(B(q')))$ and so $f(q) \not\leq \bigvee f(B(q'))$. Since we use a set that is generic with respect to $f(q) \vee \bigvee f(B(q'))$, and $f(q')$ is constructed by joining $\bigvee f(B(q'))$ and several columns of that generic set, we have that $f(q) \not\leq f(q')$.
- If $q \in P$ and $A(q') \neq \emptyset$, then there are two possibilities: If $B(q') = \emptyset$ then by (B), we have that $f(q') = \mathbf{0}_e$ and $B(A(q')) = \emptyset$. In that case, P does not have a least element (or else that least element would be in $B(A(q'))$), and so $f(q) \neq \mathbf{0}_e$.
  Otherwise, $B(q') \neq \emptyset$ and so $f(q') = \bigvee f(B(q'))$. By (C), we have that $B(q') = B(A(q'))$ and so $q \notin B(A(q'))$; thus there is some $r \in A(q')$ such that $q \not\leq r$. But then $f(q) \not\leq f(r)$. On the other hand, $\bigvee f(B(q')) \leq f(r)$ and thus $f(q) \not\leq f(q')$.
- If $q' \in P$ and $A(q) = \emptyset$, then the use of a generic with respect to $f(q')$ ensures that $f(q) \not\leq f(q')$.
- If $q' \in P$ and $A(q) \neq \emptyset$, then since q cannot be least in Q as $q \not\leq q'$, by (B) we have that $B(q) \neq \emptyset$. This means that $f(q) = \bigvee f(B(q))$. But then $\bigvee f(B(q)) \leq f(q')$ would imply $B(q) \subseteq B(q')$ which is impossible by (D).
- If q and q' are both in $Q \setminus P$ and $A(q) \neq \emptyset$, then by (D), we have $B(q) \not\subseteq B(q')$, so there is some $p \in B(q) \setminus B(q')$. Since $p \not\leq q'$, we have already shown $f(p) \not\leq f(q')$, implying $f(q) \not\leq f(q')$.
- If q and q' are both in $Q \setminus P$ and $A(q) = \emptyset$, then the use of a generic guarantees that $f(q) \not\leq f(q')$.

This completes the proof. □

5. The common fragment of the theories of the Turing and the enumeration degrees

In this section, we characterize the largest "natural" common fragment of the first-order theories of the Turing degrees and the enumeration degrees. More precisely, we will

show that the $\exists\forall$-theory of the Turing degrees is a supertheory of the $\exists\forall$-theory of the enumeration degrees; in fact, it is a proper supertheory since there is an $\exists\forall$-difference $\varphi \in \mathrm{Th}(\mathcal{D}_T) \setminus \mathrm{Th}(\mathcal{D}_e)$, namely, the existence of a minimal degree:

$$\exists x \forall y (x > 0 \ \& \ (y < x \to y = 0)),$$

or equivalently, in the language of partial orderings only, without a constant symbol for 0:

$$\varphi : \exists x \exists z (z < x \ \& \ \forall y (z \leq y \ \& \ \neg (z < y < x))).$$

Recall (e.g., from Lerman [15, proof of Theorem VII.4.4]) that any $\exists\forall$-sentence is a disjunction of $\exists\forall$-sentences ψ of the format in Problem 2.1: "For some finite partial order P and finite extensions $Q_1, \ldots, Q_k$ of P, the sentence ψ states that there is an embedding of P that cannot be extended to an embedding of any of the $Q_1, \ldots, Q_k$." For example, the above sentence φ can be expressed as a statement of this format with $k = 2$, setting $P = \{a < b\}$, $Q_1 = \{c < a < b\}$ and $Q_2 = \{a < d < b\}$.

We first give a model-theoretic characterization of the $\exists\forall$-theory of the Turing degrees.

Definition 5.1. Let U be an upper semilattice with least element. We say that U *exhibits end-extensions* if for every pair of a finite lattice P and a finite partial order $Q \supseteq P$ such that if $x \in Q \setminus P$ then x is not below any element of P and x respects least upper bounds from P, every embedding of P into U extends to an embedding of Q into U.

Note that both $\mathcal{D}_T$ and $\mathcal{D}_e$ are upper semilattices with least element that exhibit end-extensions. We claim that for the Turing degrees, this property characterizes its $\exists\forall$-theory:

Theorem 5.2. *Let φ be an $\exists\forall$-sentence in the language of partial orders. Then the sentence φ is true in $\mathcal{D}_T$ if and only if there is an upper semilattice U with least element that exhibits end-extensions such that φ is true in U. Thus the $\exists\forall$-theory of the Turing degrees is a supertheory of the $\exists\forall$-theory of the enumeration degrees.*

Proof. Note that this theorem is implicit in the proof of the decidability of the $\exists\forall$-theory of $\mathcal{D}_T$ by Shore [20] and Lerman [15], rephrased in our language.

Suppose that φ is true in some upper semilattice U with least element that exhibits end-extensions. By the remark above, we can fix a disjunct ψ of φ which has the format "For some finite partial order P and finite extensions $Q_1, \ldots, Q_k$ of P, there is an embedding of P that cannot be extended to an embedding of any of the $Q_1, \ldots, Q_k$" and holds in U. Fix an embedding f of P into U witnessing this. Let P^* be the upper semilattice with least element generated by the range of f in U, taking least upper bounds as in U and adding a least element into P^* if the least element of U is not already in the range of f. Then P^* is a finite lattice, and so by Lerman [14], we can embed P^* as an

initial segment of $\mathcal{D}_T$ via a mapping g. Any finite extension of the embedding g of P^* into $\mathcal{D}_T$ satisfies the end-extension requirements: No new element is below any member of the range of g since this range is an initial segment, and new elements respect least upper bounds of elements in the range since $\mathcal{D}_T$ is an upper semilattice. As U exhibits end-extensions, it follows that any extension g into $\mathcal{D}_T$ can be pulled back to an isomorphic extension of P^* (and hence of P) into U. It follows that g cannot be extended to an embedding of any of the partial orders $Q_1, \ldots, Q_k$ into $\mathcal{D}_T$. Thus $\mathcal{D}_T \models \psi$ and so $\mathcal{D}_T \models \varphi$.

The reverse direction is trivially true since $\mathcal{D}_T$ is an upper semilattice with least element that exhibits end-extensions. □

Definition 5.3. We denote by E the set of $\exists\forall$-sentences ψ from the extension of embeddings problem: "For some finite partial order P and finite extension Q of P, the sentence ψ states that there is an embedding of P that cannot be extended to an embedding of Q."

We can now state precisely what the "natural" common fragment of the first-order theories of the Turing degrees and the enumeration degrees is in the following.

Theorem 5.4. *For the above set E of $\exists\forall$-sentences,*

$$E \cap \mathrm{Th}(\mathcal{D}_e) = E \cap \mathrm{Th}(\mathcal{D}_T).$$

Recall from the minimal degree example above that even loosening the restriction in the extension of embeddings problem from $k = 1$ to $k = 2$ results in an $\exists\forall$-difference.

Proof. Suppose first that $\varphi \in E \cap \mathrm{Th}(\mathcal{D}_e)$. The structure $\mathcal{D}_e$ is an upper semilattice with least element that exhibits end-extensions. Note that φ is an $\exists\forall$-sentence and so, by Theorem 5.2, if φ is true in $\mathcal{D}_e$, then it must be true in $\mathcal{D}_T$ as well. It follows that φ is true in $\mathcal{D}_T$ and hence $\varphi \in E \cap \mathrm{Th}(\mathcal{D}_T)$.

Now suppose that $\varphi \in E \setminus \mathrm{Th}(\mathcal{D}_e)$. Suppose that φ is the statement that expresses that some embedding of the finite partial order P does not extend to an embedding of the partial order Q. If $\neg\varphi$ is true in $\mathcal{D}_e$ (and so every embedding of P extends to an embedding of Q), then the properties (A), (B), (C), and (D) from the proof of Theorem 4.2 apply to the pair P, Q. To prove that φ also fails in $\mathcal{D}_T$, we essentially use the same construction as in Theorem 4.2:

Fix some embedding f of P into $\mathcal{D}_T$. Order the elements of $Q \setminus P = \{q_0, \ldots, q_n\}$ so that

- $q_i \leq q_j$ implies $i \leq j$ and
- $A(q_i) \neq \emptyset$ and $A(q_j) = \emptyset$ implies $i < j$.

We consider q_i in turn. We define $f(q_i)$ using the fact that the four properties (A), (B), (C), and (D) are true:

(1) If $B(q_i) = \emptyset$ and $A(q_i) \neq \emptyset$, then by property (B), we have that q_i is the least element of Q and $B(A(q_i)) = \emptyset$. It follows that $P = A(q_i)$ does not have a least element, hence we can send q_i to $f(q_i) = \mathbf{0}_T$.
(2) If $B(q_i) \neq \emptyset$ and $A(q_i) \neq \emptyset$, then by (C), we have that $B(q_i) = B(A(q_i))$. By (D), we know that for every $j < i$, $B(q_i) = B(q_j)$ implies that $q_i = q_j$ (since $A(q_j) \neq \emptyset$), so we can send q_i to the least upper bound of the image of $B(q_i)$: $f(q_i) = \bigvee_{p \in B(q_i)} f(p)$ without violating injectivity of f.
(3) Finally, we are left with $\{q_k, \ldots, q_n\}$ with $A(q_i) = \emptyset$ for all i with $k \leq i \leq n$. Let G be generic relative to the least upper bound of all degrees in the range of our embedding so far. In the Turing case, even 1-genericity suffices. We break up G into columns $\{G_i\}_{i<\omega}$ and map q_i to $\deg_T(G_i) \vee \bigvee_{q<q_i} f(q)$.

Mutually generic sets have similar properties with respect to Turing reducibility as to enumeration reducibility. If $A_1, A_2 \leq_T B$ and G is 1-generic with respect to B and $\bigoplus_{i<\omega} G_i = G$, then for any $i \in \omega$ and any finite set F, we have that $A_1 \oplus G_i \leq_T A_2 \oplus \bigoplus_{j \in F} G_j$ if and only if $A_1 \leq_T A_2$ and $i \in F$. Thus, the same argument as was used in Theorem 4.2 will prove that f is an embedding as required. □

6. Conjectures and open problems

The most glaring open problem is, of course, the decidability of the $\exists\forall$-theory of the enumeration degrees. Our work opens up two related problems that we would like to explicitly state.

The first question asks whether we can extend the work presented in the next section by removing the distributivity requirement from our statement. We conjecture that this is possible:

Conjecture 6.1. *Every finite lattice has a strong interval embedding in $\mathcal{D}_e$.*

Confirming the above conjecture will not lead to an algorithm for deciding the $\exists\forall$-theory of $\mathcal{D}_e$ in a straightforward way. What we would like to have is a model-theoretic characterization of the $\exists\forall$-theory of $\mathcal{D}_e$ along the lines of Theorem 5.2. One possible attempt at getting such a characterization is to incorporate the theorem of Slaman and Sorbi [24], which proves a strong form of downward density. Consider the statement:

> An $\exists\forall$-sentence φ is true in $\mathcal{D}_e$ if and only if there is an upper semilattice U with least element that exhibits end-extensions and *strong downward density* such that φ is true in U,

where U *exhibits strong downward density* if every countable partial order can be embedded below any nonzero element of U.

Consider the $\exists\forall$-sentence φ that states that there is an embedding of the diamond lattice $P = \{d < a, b < c\}$ that cannot be extended to an embedding of any of $Q_1, \ldots, Q_4$, where Q_1 puts a new element below a, b and above d, Q_2 puts a new element below c, above d but incomparable to each of a and b, Q_3 puts a new element below in the interval (a, c) and Q_4 puts a new element in the interval (b, c). We can easily imagine an upper semilattice that makes φ true: In it, a and b would be mapped to a minimal pair $\{\mathbf{a}, \mathbf{b}\}$, d to the least element, and c would be mapped to the least upper bound of $\mathbf{a}$ and $\mathbf{b}$, which has the additional property that every element of U strictly bounded by $\mathbf{a} \vee \mathbf{b}$ is either below $\mathbf{a}$ or below $\mathbf{b}$.

Unfortunately, φ is not true in $\mathcal{D}_e$. Jacobsen-Grocott and Soskova (see Jacobsen-Grocott [8]) prove that strong interval embedding cannot be combined with minimal pairs:

Theorem 6.2 *(Jacobsen-Grocott, Soskova). If* $\mathbf{a}$ *and* $\mathbf{b}$ *are enumeration degrees such that every degree* $\mathbf{x} < \mathbf{a} \vee \mathbf{b}$ *is bounded by* $\mathbf{a}$ *or bounded by* $\mathbf{b}$*, then* $\{\mathbf{a}, \mathbf{b}\}$ *is not a minimal pair.*

This leaves open the following

Question 6.3. Is there a natural class of upper semilattices $\mathcal{U}$ so that an $\exists\forall$-sentence φ is true in $\mathcal{D}_e$ if and only if it is true in some upper semilattice $U \in \mathcal{U}$?

7. Strong interval embeddings

We will devote this section to the rather technical proof of the existence of a strong interval embedding of any finite distributive lattice. Recall that a strong interval embedding of a lattice L is a bijective map f between L and some interval of enumeration degrees $[\mathbf{a}, \mathbf{b}]$ such that for any degree $\mathbf{x} \leq \mathbf{b}$ that is not in the range of f we have that $\mathbf{x} < \mathbf{a}$. We came to this definition by generalizing the notion of a strong minimal cover, which gives an example of a strong interval embedding of the two-element lattice. Kent, Lewis-Pye, and Sorbi [10] proved the existence of degrees with strong minimal covers. We start by giving an alternative proof of their result.

7.1. A strong minimal cover in the Π^0_2*-enumeration degrees*

The construction of Kent, Lewis-Pye, and Sorbi [10] yields a Δ^0_3-degree $\mathbf{b}$ with a Π^0_2-strong minimal cover $\mathbf{a}$. Our plan is to extend this theorem to a strong embedding of arbitrary finite distributive lattices, so we will need some more uniformity for the images constructed. We start by giving a slightly different construction of a strong minimal cover that extends the previous result. We will then extend the ideas in this subsection to obtain our general theorem.

Theorem 7.1. *There are Π^0_2-enumeration degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\mathbf{a}$ *is a strong minimal cover of* $\mathbf{b}$ *in the enumeration degrees.*

We will build Π^0_2-sets A and B so that $\deg_e(A \oplus B)$ is a strong minimal cover of $\deg_e(B)$. We need to satisfy the following two groups of requirements:

$$\mathcal{M}_e : \exists \Gamma[\Psi_e(A \oplus B) = \Gamma(B)] \vee \exists \Delta[A \oplus B = \Delta(\Psi_e(A \oplus B))],$$

where Γ and Δ are enumeration operators we construct and $\{\Psi_e\}_{e<\omega}$ lists all enumeration operators, and

$$\mathcal{T}_e : \Phi_e(B) \neq A,$$

where $\{\Phi_e\}_{e<\omega}$ lists all enumeration operators.

We will build A and B as Π^0_2-sets as follows: We approximate them stage by stage via finite sets $\{A_s\}_{s<\omega}$, $\{B_s\}_{s<\omega}$. We use X to denote A or B. Ultimately, X consists of the elements that are enumerated into X_s at infinitely many stages s. The construction will take place on a tree of strategies. We use lower-case Greek letters α, β, etc., to denote nodes on the tree. The nodes are ordered by setting $\alpha \prec \beta$ iff α is a strict predecessor of β on the tree (and $\alpha \preceq \beta$ iff $\alpha = \beta$ or $\alpha \prec \beta$). Each node on the tree works towards satisfying a requirement. We associate *outcomes* to each node, which represent different ways in which we may satisfy the requirement and which determine its immediate successors. The outcomes are linearly ordered by $<_L$. This ordering extends to a different partial ordering on the nodes: We say that $\alpha <_L \beta$ (α *is to the left of* β) if α and β have a common predecessor γ, say, with outcomes $o_1 <_L o_2$ such that $\gamma\hat{}o_1 \preceq \alpha$ and $\gamma\hat{}o_2 \preceq \beta$. We combine the two partial orders on nodes into a total order: We say that α has *higher priority* than β (and write $\alpha < \beta$) if $\alpha <_L \beta$ or $\alpha \prec \beta$. If we identify nodes on the tree with strings in the alphabet of outcomes, then this is just the lexicographical order on such strings. During our construction, we will visit nodes on the tree and activate their *strategy* which works to satisfy their associated requirement. (We will often identify a node with its strategy.) Which node we activate next depends on the outcome currently representing our best guess as to how the requirement will be satisfied. Nodes of higher priority may *injure* the work done by lower-priority nodes, but lower-priority nodes must respect the work done so far by higher-priority nodes. The intention is that there will be a *true path* of nodes visited at infinitely many stages and injured only finitely many times, which can therefore implement their strategies successfully. Before we give a formal construction, we first consider the two types of strategies in the context of the tree.

A node α working on an $\mathcal{M}$-requirement (an $\mathcal{M}$-strategy α) first tries to build Γ_α by associating an *axiom location* and a *promise* to every axiom enumerated into Ψ_α. (For simplicity, we use the index α to refer to the operators involved in the requirement associated with α.) If $\langle x, F\rangle$ enters Ψ_α, the strategy selects a *suitable* element b as an axiom location from a stream of numbers $\mathcal{S}^B$ handed down to α by its predecessor on

the tree at each stage at which α is active. This number b is then taken out of the stream before that stream is handed down. (We will define "suitability" in a precise way later in the construction.) The strategy α then enumerates the axiom $\langle x, \{b\}\rangle$ into Γ_α and makes the promise (b, x, F) by recording it in a list of promises $\mathcal{P}_\alpha$. The intent of the promise is that if there is ever evidence that F is a subset of $A \oplus B$, then the axiom location b will be enumerated into B. Lower-priority strategies are asked to respect the promises that higher-priority strategies make.

Now let's consider a node β working on a $\mathcal{T}$-requirement (a $\mathcal{T}$-strategy β). It starts by selecting a witness z from a stream $\mathcal{S}^A$ (which is also handed down to it by its predecessor on the tree at every stage at which the node is visited). At every stage that this strategy β is visited while $z \notin \Phi_\beta(B)$, it enumerates z into A and takes its wait outcome w. The strategy must ensure that promises made by higher-priority nodes are kept, so z entering A might set off a chain reaction of numbers being enumerated into B. As there are only finitely many promises made at any given moment, this process is finitary. Furthermore, when evaluating B, the strategy takes into account what strategies below the outcome w might enumerate into the sets A and B, along with the chain reaction that higher-priority promises require. If it ever sees that z can be realized via an axiom $\langle z, D\rangle$, then it would like to keep $D \subseteq B$ and stop enumerating z into A. The elements that are enumerated into the stream of strategies below outcome w are dumped into *dump sets* U^A and U^B. These sets are enumerated into A and B, respectively, at every future stage, so they will not cause problems. If there are no higher-priority $\mathcal{M}$-strategies, then this leads us to a successful diagonalization denoted by outcome d to the left of outcome w. An actual problem might arise if there are higher-priority $\mathcal{M}$-strategies. Suppose that there is just one higher-priority $\mathcal{M}$-strategy $\alpha \prec \beta$ for simplicity. Consider the following situation:

It is possible that α has a promise $(b, x, F_A \oplus F_B)$, where $z \in F_A$ and $b \in D$. Enumerating z into A might cause b to enter B, but if we stop enumerating z into A, then b must leave B. Thus our goal of taking z out of A while keeping $D \subseteq B$ is in conflict with a promise of the higher-priority strategy α. There might be a way around this conflict in certain situations: If $z \notin A$ does not cause x to leave $\Psi_\alpha(A \oplus B)$, then we can afford to break the promise $(b, x, F_A \oplus F_B)$, as this will not cause an error in Γ_α, i.e., we will have $\Gamma_\alpha(B)(x) = \Psi_\alpha(A \oplus B)(x)$.

If, on the other hand, z leaving A causes x to leave $\Psi_\alpha(A \oplus B)$, we should be more careful. We use this relationship instead to switch α to a backup strategy: We start building an enumeration operator Δ_α by enumerating its first axiom which relies on this relationship between z and x. This situation will be marked by a visit to an outcome ∞ between outcome d and outcome w. We restart the strategy β with a new witness z'. The stream for A that is passed on to strategies below outcome ∞ is reduced to the realized witnesses $z, z', \ldots$. That is, if $\beta\hat{}\infty$ is on the true path, then, using Δ_α, we can enumerate A from $\Psi_\alpha(A \oplus B)$. All natural numbers that are in A and that are not in the stream S^A below $\beta\hat{}\infty$ are dumped into A by strategies $\gamma \preceq \beta$ and form a c.e. set. Recall that to satisfy the requirement $\mathcal{M}$, we need to ensure that B can also be

enumerated by $\Psi_\alpha(A \oplus B)$ and so Δ_α cannot quite serve as the operator Δ required to satisfy α's requirement. We use every other element of the stream S^A below $\beta^\frown\infty$ to code B into A. We do this through a list H_β of *attachments* (z, u), which is similar to the list of promises: It requires from lower-priority strategies to ensure that if u enters B, then z is enumerated into A. So if ∞ is the true outcome, then we will have $A = \Delta_\alpha(\Psi_\alpha(A \oplus B))$ and $B \leq_e A$, and so we can produce the required operator Δ to satisfy α's requirement. If the connection between z and x is lost due to more axioms entering Ψ_α, then we say that z is *cleared* by α, and we can revert to our original plan to diagonalize to satisfy β's requirement.

The rest of the mechanics of the construction is standard. If β extends more than one $\mathcal{M}$-strategy above it, say, $\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_{k-1} \prec \beta$, then in order to diagonalize with a witness z, it must be cleared by all α_i. We try to clear it in turn, starting with α_{k-1} and ending with α_0, with the possibility of switching each α_i to its backup strategy with an outcome ∞_i if we cannot clear the witness. The outcomes of β are:

$$d <_L \infty_0 <_L \infty_1 <_L \cdots <_L \infty_{k-1} <_L w.$$

The full construction will give the precise details on how this is organized.

The tree of strategies. Our tree of strategies will be a partial function $T : \{w, d, \infty_i : i < \omega\}^{<\omega} \to \mathcal{R}$, where $\mathcal{R}$ is the set of all requirements. We will define $T(\alpha)$ along with the set $\mathcal{C}(\alpha)$ of active $\mathcal{M}$-strategies along α. Let $T(\emptyset) = \mathcal{M}_0$ and $\mathcal{C}(\emptyset) = \emptyset$. Suppose $T(\alpha) = \mathcal{M}_\alpha$, then $T(\alpha^\frown d)$ is defined and equals the least $\mathcal{T}$-requirement that has not been assigned to any node $\beta \preceq \alpha$; we also set $\mathcal{C}(\alpha^\frown d) = \mathcal{C}(\alpha) \cup \{\alpha\}$. On the other hand, suppose that $T(\alpha) = \mathcal{T}_\alpha$ and $\mathcal{C}(\alpha) = \{\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_{k-1}\}$. We set $T(\alpha^\frown w)$ and $T(\alpha^\frown d)$ to be the least $\mathcal{M}$-requirement that has not yet been assigned to any node $\beta \preceq \alpha$, and we set $\mathcal{C}(\alpha^\frown w) = \mathcal{C}(\alpha^\frown d) = \mathcal{C}(\alpha)$. For every $l < k$, we set $T(\alpha^\frown\infty_l) = T(\alpha)$, and we set $\mathcal{C}(\alpha^\frown\infty_l) = \mathcal{C}(\alpha) \setminus \{\alpha_l\}$.

Approximating X. Recall that X stands for either the set A or the set B. At stage s of the construction, we build a finite path f_s of length s in the (domain of the) tree of strategies. Strategies to the right of f_s are initialized at (the end of) stage s. The set X_s is constructed in substages X_s^t where $t \leq s$, starting with $X_s^{-1} = \emptyset$ and letting X_s^t be the set X_s^{t-1} along with all elements enumerated into X by $f_s \upharpoonright t$ at stage s. We will omit reference to specific substages when they are understood from the context. As we said before, we will have that $n \in X$ if and only if $n \in X_s$ at infinitely many stages s. At first sight, this means that for a finite set F, we might have $F \subseteq X$ but $F \nsubseteq X_s$ at any stage s. We will ensure that the leftmost path f of strategies visited at infinitely many stages is the true path, i.e., it correctly approximates the true outcome of every strategy. We will prove that $n \in X$ if and only if n is enumerated into X by a unique strategy σ along the true path at all but finitely many stages s at which σ is visited. And so for

finite sets F, we will have as well that $F \subseteq A$ if and only if $F \subseteq A_s$ for infinitely many stages s.

Let β be a $\mathcal{T}$-strategy with active $\mathcal{M}$-strategies $\mathcal{C}(\beta) = \{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\}$ and suppose that $\beta_0, \ldots, \beta_{m-1}$ are all predecessors of β so that $\beta_l \hat{} \infty_u \preceq \beta$ for some u. When we visit β, we must take into account the promises and attachments that these strategies have made whenever β enumerates a number into the set A. For finite sets Y and Z, we define $g^A_\beta(Y, Z)$ formally as $\bigcup_n Y_n$ and $g^B_\beta(Y, Z)$ as $\bigcup Z_n$, where $Y_0 = U^A \cup Y$, $Z_0 = U^B \cup Z$ and

- $Y_{n+1} = Y_n \cup \{a \mid (\exists l < m)[(a, b) \text{ is an attachment of } \beta_l \ \&\ b \in Z_n]\}$;
- $Z_{n+1} = Z_n \cup \{b \mid (\exists l < k)[(b, x, F) \text{ is a promise of } \alpha_l \ \&\ F \subseteq Y_{n+1} \oplus Z_n]\}$.

Suitability. The streams $\mathcal{S}^A_\alpha$ and $\mathcal{S}^B_\alpha$ are defined inductively during the construction. We list all strategies in order type ω and associate to each a unique number $\hat{\alpha}$ corresponding to $\alpha's$ position in this list. We will say that x is *suitable* for α if $x > |\alpha|$ and x is the $\langle \hat{\alpha}, j\rangle$-th number among all elements that are in the stream $\mathcal{S}^X(\alpha)$ and not used in attachments for some j. For every $\alpha \prec f$, we will ensure that $\mathcal{S}^X_\alpha$ is infinite and so there are infinitely many numbers suitable for α.

The construction. At stage 0, all strategies are in initial state: We set $U^A = U^B = \emptyset$; for each $\mathcal{M}$-strategy α, we set $\Gamma_\alpha = \emptyset$ and the list of promises $\mathcal{P}_\alpha = \emptyset$; for each $\mathcal{T}$-strategy β with $\mathcal{C}(\beta) = \{\alpha_0, \ldots, \alpha_{k-1}\}$, we set $\Delta^\beta_l = \emptyset$ for all $l < k$, the list of attachments $H^\beta_l = \emptyset$ for $l < k$, and let the current witness z_β be undefined. During the construction, *initializing* a strategy will mean that we restore it to its initial state.

At stage $s > 0$, we build f_s of length at most s, activating strategies along f_s. We begin by enumerating U^X into X.

We then start at the root and let $\mathcal{S}^X_{\emptyset,s} = \mathcal{S}^X_{\emptyset,s-1} \cup \{s\} = [0, s]$. Suppose we have constructed $f_s \upharpoonright n$, along with $\mathcal{S}^X_{f_s\upharpoonright n,s} = \mathcal{S}^X_{f_s\upharpoonright n,s-1} \cup \{y^X\}$ and A_s and B_s (or rather A^n_s and B^n_s, the approximation to the sets A and B at substage n of stage s). If $n = s$, then we end this stage and move on to the next stage. If $n < s$, then we activate $f_s \upharpoonright n$ and let it pick its outcome o. Then $f_s \upharpoonright n + 1 = (f_s \upharpoonright n) \hat{} o$ unless f_s ends the stage prematurely. At the end of stage s, we initialize all strategies σ such that $f_s <_L \sigma$.

Case 1. If $f_s \upharpoonright n = \alpha$ is an $\mathcal{M}$-strategy and α did not end the previous stage at which it was visited prematurely, then we scan Ψ_α for new axioms that have not yet been assigned axiom locations. If such axioms exist, then we pick the *oldest* such, say, $\langle x, F_A \oplus F_B\rangle$. (Here by oldest we mean the one that was enumerated into Ψ_α first.) If $b = y^B$ is suitable for α and $b > \max(F_A \cup F_B)$, then we assign b to the axiom and enumerate the promise $(b, x, F_A \oplus F_B)$ into $\mathcal{P}_\alpha$ as well as the axiom $\langle x, \{b\}\rangle$ into Γ_α. We end this stage prematurely (note we do not initialize strategies $\beta \succ \alpha$, only strategies $\beta >_L \alpha$). Otherwise (in particular if α did end the previous stage at which it was visited

prematurely), we enumerate y^X into $\mathcal{S}^X(\alpha\hat{}d)$. In all cases, we enumerate into X the set $g^X_\alpha(A_s, B_s)$, and if we don't end the stage, we let d be α's outcome.

Case 2. If $f_s \upharpoonright n = \beta$ is a $\mathcal{T}$-strategy with $\mathcal{C}(\beta) = \{\alpha_0 \prec \cdots \prec \alpha_{k-1}\}$, then we pick the first case which applies:

(1) *The strategy β was successful via a realized witness z and had outcome d at the previous stage at which β was active:* In that case, enumerate D_z into B and then $g^X_\beta(A_s, B_s)$ into X, the number y^X into $\mathcal{S}^X_{\beta\hat{}d}$, and let the outcome be d.
(2) *The current witness z_β is not defined:* If $y^A = a$ is defined and suitable for β, then let $z_\beta = a$ be the *current witness* and end the stage prematurely. Otherwise, if a is not suitable, then enumerate y^X into $\mathcal{S}^X_{\beta\hat{}w}$ and let the outcome be w. In both cases, we enumerate $g^X_\beta(A_s, B_s)$ into X.
(3) $z_\beta \notin \Phi_\beta(V)$*, where* $V = g^B_\beta(A_s \cup \mathcal{S}^A_{\beta\hat{}w,s} \cup \{z_\beta\}, B_s \cup \mathcal{S}^B_{\beta\hat{}w,s})$*:* We then enumerate $g^X_\beta(A_s \cup \{z_\beta\}, B_s)$ into X, the number y^X into $\mathcal{S}^X_{\beta\hat{}w}$, and let the outcome be w.
(4) *Otherwise:* Call the witness z_β is *realized*. This is the only case in which we grow the dump sets. We start by enumerating $S^X_{\beta\hat{}w,s}$ into U^X. Let D_{z_β} be the set of axiom locations in the finite subset of the axiom that puts z_β into $\Phi_\beta(B)$ which are not enumerated into B if $z \notin A$:

$$D_{z_\beta} = g^B_\beta(A_s \cup \{z_\beta\}, B_s) \setminus g^B_\beta(A_s, B_s).$$

For every $l < k$, let

$$E^{z_\beta}_l = \Psi_{\alpha_l}(g^A_\beta(A_s \cup \{z_\beta\}, B_s) \oplus g^B_\beta(A_s \cup \{z_\beta\}, B_s)) \setminus \Psi_{\alpha_l}(g^A_\beta(A_s, B_s) \oplus g^B_\beta(A_s, B_s)).$$

Make the *current witness* z_β undefined.
Now, for every realized witness z and every $l < k$, let

$$G^X_{l,z} = g^X_\beta((A_s \cup \bigcup_{l \le j < k} \mathcal{S}^A_{\beta\hat{}\infty_j}) \setminus \{z\}, B_s \cup \bigcup_{l \le j < k} \mathcal{S}^B_{\beta\hat{}\infty_j}).$$

We say that z is α_l-cleared if $E^z_l \subseteq \Psi_{\alpha_l}(G^A_{l,z} \oplus G^B_{l,z})$. We search for the least pair (l, z) (in the lexicographical order) such that z is a realized witness, $z \notin \mathcal{S}^A_{\beta\hat{}\infty_j}$ for $j < l$, $z \notin U^A$, and z is j-cleared for all $j > l$. (Note that the pair $(k-1, z_\beta)$ satisfies these conditions, so such (l, z) must exist.) We enumerate $(\bigcup_{l<j<k} \mathcal{S}^A_{\beta\hat{}\infty_j}) \setminus \{z\}$ into U^A, $\bigcup_{l<j<k} \mathcal{S}^B_{\beta\hat{}\infty_j}$ into U^B, and set $\Delta_j = H_j = \emptyset$ for all $j > l$.
(a) If $l \ge 0$, then enumerate the axiom $\langle z, E^z_l\rangle$ into Δ_l, the set $g^X_\beta(A_s, B_s)$ into X, the number y^B into $\mathcal{S}^B(\beta\hat{}\infty_l)$, and z into $\mathcal{S}^A(\beta\hat{}\infty_l)$. If z is the $2n$-th number in $\mathcal{S}^A(\beta\hat{}\infty_l)$, then we enumerate (z, n) into H_l and end this stage. Otherwise, we let the outcome be ∞_l.

(b) Otherwise, we have a witness z that is α_l-cleared for all $l < k$. We say that z is *successful* and that the numbers in D_z are *associated* with z at β. Enumerate D_z into B, as well as $g_\beta^X(A_s, B_s \cup D_z)$ into X. Let the outcome be d.

7.1.1. The verification

As mentioned before, we define the infinite *true path* f by

$$f(n) = \liminf_{s>n} f_s(n).$$

It is straightforward to see that the strategies σ along f are visited at infinitely many stages and initialized at only finitely many stages. For $\sigma \prec f$, we say that s is a *true stage* if σ is visited at stage s. Let s_σ be the least stage after which σ is not initialized. Let $\mathcal{S}^X(\sigma) = \bigcup_{s>s_\sigma} \mathcal{S}^X_{\sigma,s}$. A simple induction on the length of σ proves that $\mathcal{S}^X(\sigma)$ is infinite. Furthermore, if σ is visited at consecutive stages $s > t > s_\sigma$, then $A_s^{|\sigma|} \supseteq A_t^{|\sigma|}$, as in order for a strategy above σ to stop enumerating an element into A, it must move its outcome left of σ and hence initialize σ. We now verify the important claims about enumeration into A and B that we made earlier. Once again, X denotes either the set A or the set B.

Lemma 7.2. *If a number x is in X, then it is either eventually dumped into X at almost every stage, or there is a strategy $\sigma \prec f$ such that at all but finitely many stages at which x is enumerated into X, σ is the least strategy that enumerates x into X, and σ does so at cofinitely many stages at which σ takes its true outcome.*

Proof. Suppose $x \in X$. If x is dumped into U^X at stage s, then it is enumerated into X at the beginning of all stages $t > s$. So suppose that x is not dumped.

Consider first the case when $X = B$ and denote x by b for convenience. Note that when an $\mathcal{M}$-strategy α picks a number b as an axiom location, that number is taken out of the stream, and it is never returned to the stream, so no other strategy can use it. If b is not an axiom location for any strategy, then it is not enumerated into B at any stage unless it is dumped, so let α be the unique strategy that uses b as an axiom location for the axiom $\langle x, F\rangle$. If α is ever initialized, then b is dumped into U^B. So, by assumption, α is never initialized after b is chosen.

There are infinitely many stages at which b is enumerated into B. At stage s, this could be because a $\mathcal{T}$-strategy σ with $\alpha \in \mathcal{C}(\sigma)$ causes $F \subseteq A_s \oplus B_s$, or because b is associated with a witness z at a successful $\mathcal{T}$-strategy β that is visited at stage s.

First, note that there can be only finitely many strategies at which b is associated with a witness. This is because if $z \in A$ causes $b \in B$ and $z \notin A$ causes $b \notin B$, then there is a sequence of promises and attachments witnessing the recursive relationship between z and b that drives the definition of g_β. The sequence starts with a promise $(b_0, x_0, F_A^0 \oplus F_B^0)$ such that $z \in F_A^0$ and ends with a promise $(b, x_k, F_A^k \oplus F_B^k)$. In every promise $(b_i, x_i, F_A^i \oplus F_B^i)$, we have that $b_i > \max(F_A^i \oplus F_B^i)$, and in every attachment

(z_i, b_i), we have that $z_i > b_i$, and so $z < b$. Furthermore, once b is associated with a witness z at β, we have that z is out of A at all further stages unless β is initialized. It keeps z out of A, and so b is an element of B_s only if we visit β at stage s. In particular, this means that unless β is initialized, b will not be associated at any other strategy. If β is initialized, then the witness z is either dumped or moved to a stream to the left of β, and so β will never have z as its witness again. There are finitely many numbers $z < b$, and each is suitable for finitely many $\mathcal{T}$-strategies, namely, the ones of length smaller than z, hence there can be only finitely many associations, and never more than one at a time.

So, to sum up, b can be associated with z at β only for finitely many pairs (β, z). If at stage s, the number b is associated with z at β, then β is the only strategy that b is associated with at stage s, and $b \in B_s$ if and only if $\beta \preceq f_s$ and enumerates it into B_s. So there are two cases: Either some strategy β is associated with b at all but finitely many stages, in which case it is never initialized and as $b \in B$, it is visited infinitely often, in which case $\beta \prec f$ and satisfies the conditions. Otherwise, at all but finitely many stages s, we have that b is enumerated into B_s only if $F \subseteq A_s \oplus B_s$. It follows that $F \subset A \oplus B$, and since $b > \max F$, by induction, for every element in F, some strategy along the true path enumerates it into the corresponding set. Pick the longest such σ. It follows that $F \subseteq A_s \oplus B_s$ only if $\sigma \preceq f_s$. So, as $b \in B$, it must be that $\alpha \in \mathcal{C}(\sigma)$, and hence σ enumerates b into B_s when visited, or $\sigma \preceq \alpha$ and then $b \in B_s$ whenever we visit $\alpha^\frown d$.

Now consider the case when $X = A$ and denote x by a. We are assuming that $a \in A$. At every stage s, there is at most one $\mathcal{T}$-strategy $\beta_s{}^\frown o_s$ such that a is an unrealized witness of β_s and $o_s = w$, or such that a is a realized witness of β_s and β_s uses a as an attachment to code whether some n is in B below outcome $o_s = \infty_i$ for some $i < k$. There are only finitely many strategies β that can ever fulfill this role, as a must be suitable for β, hence β is of length less than a. If at stage $t > s$, such a strategy for a changes, then $\beta_t{}^\frown o_t <_L \beta_s{}^\frown o_s$. On the other hand, this role is filled by some strategy and outcome visited at infinitely many stages, as those are the situations in which a is enumerated into A. So fix β such that $\beta^\frown o$ is least and hence the same at all but finitely many stages. If $o = w$, then a enters A only at stages at which we visit β and β has outcome w. It follows that $\beta^\frown w \prec f$.

If $o = \infty_i$, then $(a, n) \in H_i^\beta$ and hence the only way that a can enter A_s is if some strategy compatible with $\beta^\frown \infty_i$ enumerates n into B. As $n < a$, we have by induction that there is a least strategy $\sigma \prec f$ that causes this. So either $\sigma \preceq \beta$, in which case $\beta^\frown \infty_i \prec f$ and $a \in A$ at all but finitely many $\beta^\frown \infty_i$-true stages, or else $\beta^\frown \infty_i \preceq \sigma$ and σ enumerates $a \in A$ along with n via the function g_σ. □

Lemma 7.3. *Every $\mathcal{T}$-requirement is satisfied.*

Proof. Fix a requirement $\mathcal{T}_e$. Let $\beta \prec f$ be the longest strategy such that $T(\beta) = \mathcal{T}_e$. Such a strategy exists because once $\mathcal{T}_e^A$ is assigned to a node σ with $|\mathcal{C}^A(\delta)| = k$, $\mathcal{T}_e$ can

be assigned at most k many more times along any path through σ. It follows that $\beta^\frown d \prec f$ or $\beta^\frown w \prec f$.

If $\beta^\frown w \prec f$, then there is an unrealized witness $z_\beta \in A$ such that at every β-true stage s, $z_\beta \notin \Phi_\beta(V_s)$, where $V_s = g^B_\beta(A_s \cup \mathcal{S}^A_{\beta^\frown w,s} \cup \{z_\beta\}, B_s \cup \mathcal{S}^B_{\beta^\frown w,s})$. By Lemma 7.2, $B \subseteq \bigcup_s V_s$ since the true path passes through $\beta^\frown w$. So $z_\beta \notin \Phi_\beta(B)$.

If $\beta^\frown d \prec f$, then there is a successful witness z that is α_i-cleared for all $i < k$ at all stages $t > s_z$. As the strategy β enumerates all elements of D_z into B at all true stages $t > s_z$, we have that $z \in \Phi_\beta(B)$. On the other hand, z is never enumerated into A_t for $t > s_z$ by β or any other strategy, so $z \notin A$. □

Lemma 7.4. *Every $\mathcal{M}$-requirement is satisfied.*

Proof. Fix e. There is a unique strategy $\alpha \prec f$ associated with $\mathcal{M}_e$. Suppose that α is switched to a backup strategy by some $\beta \succ \alpha$ along the true path. Then $\beta^\frown\infty_i \prec f$, $\mathcal{C}(\beta) = \{\alpha_0, \ldots, \alpha_{k-1}\}$, and $\alpha = \alpha_i$. There are three types of elements that make up the set A in this case: elements that are eventually dumped (a c.e. set), elements that belong to the stream $\mathcal{S}^A(\beta^\frown\infty_i)$, and elements that are used by higher-priority $\mathcal{T}$-strategies for coding purposes. We will show that $\Psi(A \oplus B)$ can enumerate the elements in $A \cap \mathcal{S}^A(\beta^\frown\infty_i)$. Once we have that, we will show that $\Psi(A \oplus B)$ can enumerate the set B. Knowing B will then let $\Psi(A \oplus B)$ figure out which of the elements of the third kind, the ones used for attachments by higher-priority strategies, end up in the set A.

For all elements $z \in \mathcal{S}^A(\beta^\frown\infty_i)$, we have that $z \in A$ if and only if $z \in \Delta_i(\Psi_\alpha(A \oplus B))$. This is because if we ever see an axiom stop being valid, we would move to an outcome to the left of ∞_i. First of all, we claim that $B \leq_e \Delta_i(\Psi_\alpha(A \oplus B))$. By Lemma 7.2, we have that $b \in B$ if and only if a least strategy $\sigma \prec f$ enumerates b at all but finitely many stages at which σ takes its true outcome. The strategy β forms the association $(z, b) \in H^i$ for some $z \in \mathcal{S}^A(\beta^\frown\infty_i)$, and so if $b \in B$, then at all stages s at which we visit the longer of the strategies σ and $\beta^\frown\infty_i$, we have $z \in A_s$. On the other hand, z is enumerated into A_s only if $b \in B_s$, so we have that $B = \{b \mid (z, b) \in H^i \ \&\ z \in \Delta(\Psi_\alpha(A \oplus B))\}$.

Next, we claim that

$$A = \bigcup_{s:\beta^\frown\infty_i \preceq f_s} g^A_{\beta^\frown\infty_i}(A_s^{|\beta|} \cup \Delta_i(\Psi_\alpha(A \oplus B)), B)$$

and hence is as well enumeration reducible to $\Delta_i(\Psi_\alpha(A \oplus B))$. First, suppose that $a \in A$. By Lemma 7.2, a is either dumped into A, or is enumerated into A by some least $\sigma \prec f$ at all but finitely many stages at which σ takes its true outcome. If $\sigma \preceq \beta$, then $a \in A_s^{|\beta|}$ at all but finitely stages s at which β is active. If $\beta^\frown\infty_i \preceq \sigma$, then $a \in \mathcal{S}^A(\beta^\frown\infty_i)$, or else a is the attachment at some $\delta \prec \beta$ and so a is enumerated into A at stages at which some fixed number b is enumerated into B. It follows that $b \in B$ and hence $a \in \bigcup_{s:\beta^\frown\infty_i \preceq f_s} g^A_{\beta^\frown\infty_i}(A_s^{|\beta|} \cup \Delta_i(\Psi_\alpha(A \oplus B)), B)$.

On the other hand, if

$$a \in \bigcup_{s:\beta^\frown\infty_i \preceq f_s} g^A_{\beta^\frown\infty_i}(A_s^{|\beta|} \cup \Delta_i(\Psi_\alpha(A \oplus B)), B),$$

then for some $\beta^\frown\infty_i$-true stage s, we have that $a \in g^A_{\beta^\frown\infty_i}(A_s^{|\beta|} \cup \Delta_i(\Psi_\alpha(A \oplus B)), B)$. But then, since whether a enters this set depends only on numbers smaller than a, and all numbers in $A_s^{|\beta|}$ are in A (since $A_s^{|\beta|}$ only grows at $\beta^\frown\infty_i$-true stages), it follows that $a \in A$.

Finally, suppose that α is never switched to a backup strategy. We claim that $\Gamma(B) = \Psi_\alpha(A \oplus B)$. Suppose that $x \in \Psi_\alpha(A \oplus B)$; then there is a valid axiom $\langle x, F_A \oplus F_B\rangle$ in Ψ_α. This axiom is assigned a marker b, and the axiom $\langle x, \{b\}\rangle$ is enumerated into Γ. By Lemma 7.2, we have that $F_A \oplus F_B \subseteq A_s \oplus B_s$ at all σ-true stages for some least $\alpha \preceq \sigma \prec f$. As α is active at σ and σ respects α's promises (see the definition of g_σ), it follows that $b \in B$, and so $x \in \Gamma(B)$.

On the other hand, suppose that an axiom location $b \in B$ is associated with the axiom $\langle x, F_A \oplus F_B\rangle$ in Ψ_α and this axiom is not valid. After a fixed stage in the construction, b is enumerated into B only at stages at which some strategy β such that b is associated with a witness z at β is visited. As z is α-cleared, we know that $x \in \Psi_\alpha(A_s \oplus B_s)$ (even though $z \notin A$ results in that the original axiom is invalid). As we discussed in the proof of Lemma 7.2, b is associated to at most one unique pair (β, z) at any stage, and there are only finitely many possibilities. It follows that if $b \in B$, then one of the finitely many axioms that cause $x \in \Psi_\alpha(A_s \oplus B_s)$ must be valid. □

7.2. Building up the intuition for the general case

We would like to generalize the previous construction to the general case of an arbitrary distributive lattice. To build up to that, we first consider two special cases: the three-element lattice and the diamond lattice.

7.2.1. The three-element lattice

Suppose first that we want to construct Π^0_2-enumeration degrees $\mathbf{a} > \mathbf{b} > \mathbf{c}$ such that $\mathbf{a}$ is a strong minimal cover of $\mathbf{b}$ and $\mathbf{b}$ is a strong minimal cover of $\mathbf{c}$. We can approach this by building three Π^0_2-sets A, B, C so that $\mathbf{c} = \deg_e(C)$, $\mathbf{b} = \deg_e(B \oplus C)$ and $\deg_e(A \oplus B \oplus C)$. Now we will have two groups of requirements: $\mathcal{M}^A$ and $\mathcal{M}^B$ mirroring the $\mathcal{M}$-requirements but for the pairs of sets (A, B) and (B, C), respectively, along with $\mathcal{T}^A$ and $\mathcal{T}^B$, proving that we have a strictly increasing sequence of degrees. One complication that arises immediately is that the set B now plays two roles: On the one hand, it serves as a set that supplies coding locations for the requirements of the form $\mathcal{M}^A$, and on the other hand, it supplies $\mathcal{T}^B$-requirements with witnesses. To keep things tidy, we will treat $B = B^a \oplus B^w$ as consisting of two parts: B^a will be used by $\mathcal{M}^A$-requirements, and B^w will be used by $\mathcal{T}^B$-requirements. With this idea in mind, we have the following list of requirements:

$$\mathcal{M}_e^A : \exists\Gamma[\Psi_e(A \oplus B \oplus C) = \Gamma(B^a)] \vee \exists\Delta[A \oplus B \oplus C = \Delta(\Psi(A \oplus B \oplus C))],$$
$$\mathcal{M}_e^B : \exists\Gamma[\Psi_e(B \oplus C) = \Gamma(C)] \vee \exists\Delta[B \oplus C = \Delta(\Psi(B \oplus C))],$$

where Γ and Δ are enumeration operators we construct and $\{\Psi_e\}_{e<\omega}$ lists all enumeration operators, and

$$\mathcal{T}_e^A : \Phi_e(B \oplus C) \neq A,$$
$$\mathcal{T}_e^B : \Phi_e(C) \neq B^w,$$

where $\{\Phi_e\}_{e<\omega}$ lists all enumeration operators.

The $\mathcal{T}$-requirements ensure that $C <_e B \oplus C <_e A \oplus B \oplus C$, the $\mathcal{M}^A$-requirements ensure that if $X <_e A \oplus B \oplus C$ then $X \leq_e B^a \leq_e B \oplus C$, and the $\mathcal{M}^B$-requirements ensure that if $X <_e B \oplus C$ then $X \leq_e C$.

We will have streams associated with each set that we are constructing which are handed off from strategy to strategy much like in the previous construction: Every strategy σ has streams $\mathcal{S}_\sigma^A$, $\mathcal{S}_\sigma^{B^a}$, $\mathcal{S}_\sigma^{B^w}$, and $\mathcal{S}_\sigma^C$.

The actions of $\mathcal{M}^A$- and $\mathcal{M}^B$-strategies are very similar to the actions of the $\mathcal{M}$-strategy from the previous construction. The only difference is that $\mathcal{M}^A$-strategies pick coding locations out of the stream $\mathcal{S}^{B^a}$, and $\mathcal{M}^B$-strategies pick coding locations out of the stream $\mathcal{S}^C$.

A $\mathcal{T}^A$-strategy β will pick a witness z from $\mathcal{S}^A$. This witness is enumerated into A while $z \notin \Phi_\beta(B \oplus C)$. When evaluating $B \oplus C$, the strategy takes into account which numbers strategies below the outcome w might enumerate into each of the sets A, B and C, and the reaction that higher-priority $\mathcal{M}_A$-strategies and $\mathcal{M}_B$-strategies might have. If it ever sees that z can be realized via an axiom $\langle z, D\rangle$, then it would like to keep $D \subseteq B \oplus C$ and stop enumerating z into A. This could be in conflict with higher-priority $\mathcal{M}^A$-strategies directly because of coding locations in B^a, but there is no direct conflict with higher-priority $\mathcal{M}^B$-strategies: We would like to change the approximation to A, which does not directly interfere with $\Psi(B \oplus C)$ that an $\mathcal{M}^B$-strategy is working on. There could, however, be an indirect interaction: Suppose that a higher-priority $\mathcal{M}^A$-strategy α has a promise (b, x, F_1) where $z \in F_1$, and a higher-priority $\mathcal{M}^B$-strategy γ has a promise (c, y, F_2) where $b \in F_2$ and $c \in D$. Now even though α's axiom location is not directly in the set D, the chain reaction starting with $z \notin A$ would still cause a problem as then b would need to leave B^a, and then c would need to leave C, causing $D \nsubseteq B \oplus C$. If we are able to clear b via another axiom for x entering Ψ_α, then enumerating b into B will have the effect of enumerating c into C, so we can still get the desired result. The conclusion is that the strategy β can switch higher-priority $\mathcal{M}_A$-strategies to their backup versions (and need not consider the active $\mathcal{M}_B$-strategies). However, when clearing a witness z, it needs to take into account all elements b that may leave B once we remove z, not just the ones in the finite set of a realizing axiom $\langle z, D\rangle$.

Another modification to this strategy is needed in case the strategy has one of its infinite outcomes. In the simpler case, we reserved half of the stream below an infinite

outcome to code the set B. Now we need to ensure that each of B^a, B^w and C is reducible to $\Delta(A \oplus B \oplus C)$. We take a similar approach: We split up the stream generated into four parts: one to code B^a, one for B^w, one to code C, and the last one is reserved for lower-priority $\mathcal{T}^A$-strategies to pick witnesses.

A $\mathcal{T}^B$-strategy δ selects its witness z from the stream $\mathcal{S}^{B^w}$ and tries to clear it with respect to all $\mathcal{M}_B$-strategies. Once again, not enumerating a witness z into the set B can cause a chain reaction involving all kinds of axiom locations for $\mathcal{M}^A$- or $\mathcal{M}^B$-strategies, and even coding locations for higher-priority $\mathcal{T}^A$-strategies that have an infinite outcome. Nevertheless, the only thing that is important to δ is to free up the axiom locations from the realizing axiom $\langle z, D\rangle$ that stop being enumerated into C if z is not enumerated into B^w. For this reason, δ takes into account higher-priority $\mathcal{M}^B$-strategies and tries to get their clearance to diagonalize or switches them to a backup version. Below an infinite outcome, it codes the sets B^a and C with a portion of the stream $\mathcal{S}^{B^w}$.

7.2.2. The diamond lattice

Suppose next that we want to construct Π^0_2-enumeration degrees $\mathbf{a} > \mathbf{b}, \mathbf{c} > \mathbf{d}$ such that $\mathbf{a} = \mathbf{b} \vee \mathbf{c}$ and for all $\mathbf{x} \leq \mathbf{a}$ we have that $\mathbf{x} \neq \mathbf{a}, \mathbf{c}, \mathbf{b}$ implies that $\mathbf{x} \leq \mathbf{d}$. We will build three Π^0_2-sets $B = B^a \oplus B^w$, $C = C^a \oplus C^w$ and D so that $\mathbf{a} = \deg_e(B \oplus C \oplus D)$, $\mathbf{b} = \deg_e(B \oplus D)$, $\mathbf{c} = \deg_e(C \oplus D)$, and $\mathbf{d} = \deg_e(D)$. We will need to satisfy the following list of requirements:

$$\mathcal{M}_e^{B,C} : \exists\Gamma[\Psi_e(B \oplus C \oplus D) = \Gamma(B^a)] \vee \exists\Delta[C \oplus D = \Delta(\Psi_e(B \oplus C \oplus D))],$$
$$\mathcal{M}_e^{C,B} : \exists\Gamma[\Psi_e(B \oplus C \oplus D) = \Gamma(C^a)] \vee \exists\Delta[B \oplus D = \Delta(\Psi_e(B \oplus C \oplus D))],$$
$$\mathcal{M}_e^{D,B} : \exists\Gamma[\Psi_e(B \oplus D) = \Gamma(D)] \vee \exists\Delta[B \oplus D = \Delta(\Psi_e(B \oplus D))],$$
$$\mathcal{M}_e^{D,C} : \exists\Gamma[\Psi_e(C \oplus D) = \Gamma(D)] \vee \exists\Delta[C \oplus D = \Delta(\Psi_e(C \oplus D))],$$

where Γ and Δ are enumeration operators we construct and $\{\Psi_e\}_{e<\omega}$ lists all enumeration operators, and

$$\mathcal{T}_e^C : \Phi_e(B \oplus D) \neq C^w,$$
$$\mathcal{T}_e^B : \Phi_e(C \oplus D) \neq B^w,$$

where again $\{\Phi_e\}_{e<\omega}$ lists all enumeration operators.

The $\mathcal{T}$-requirements ensure that $B \oplus D$ and $C \oplus D$ are incomparable and hence $D <_e B \oplus D, C \oplus D <_e B \oplus C \oplus D$. The $\mathcal{M}^{D,B}$-requirements ensure that if $X <_e B \oplus D$ then $X \leq_e D$, similarly the $\mathcal{M}^{D,C}$-requirements ensure that if $X <_e C \oplus D$ then $X \leq_e D$. The new idea comes from the combined use of the $\mathcal{M}^{C,B}$ and $\mathcal{M}^{D,B}$ requirement: Fix $X \leq_e B \oplus C \oplus D$. If $X \not\leq_e B \oplus D$ then by the $\mathcal{M}^{B,C}$ requirements we have that $C \oplus D \leq_e X$. On the other hand if $X \not\leq_e C \oplus D$ then $C \oplus D \leq_e X$ and so $X \equiv_e B \oplus C \oplus D$.

As usual, we attach a stream $\mathcal{S}^X$ to every set $X \in \{B^a, B^w, C^a, C^w, D\}$. The $\mathcal{M}$-strategies function in a similar way as before. The only difference is that they pick coding locations from different streams.

A $\mathcal{T}^C$-strategy selects its witnesses from $\mathcal{S}^{C^w}$ and tries to clear them with respect to two kinds of higher-priority $\mathcal{M}$-strategies: $\mathcal{M}^{B,C}$- and $\mathcal{M}^{D,C}$-strategies. It can also switch these strategies to a backup version. To see why this is reasonable, note that the goal of this strategy, once it has a realized witness z, is to keep a finite set in $B \oplus D$. Extracting z from C^w can cause axiom locations to leave B^a via an $\mathcal{M}^{B,C}$-strategy and D via an $\mathcal{M}^{D,C}$-strategy directly. The change in B^a can then cause axiom locations to leave the set D also via an $\mathcal{M}^{D,B}$-strategy. However, if we are able to re-enumerate all such axiom locations into B^a, then that will erase the indirect change in D. Thus when we ask for clearance, we consider all axiom locations that leave B^a if z leaves C^w, not just the ones involved in the realizing axiom. Below an infinite outcome, we code the sets C^a and D using a portion of the stream $\mathcal{S}^{C^w}$.

Similarly, a $\mathcal{T}^B$-strategy works primarily with the stream $\mathcal{S}^{B^w}$ and with respect to higher-priority $\mathcal{M}^{C,B}$- and $\mathcal{M}^{D,B}$-strategies. Below its infinite outcome, it codes the sets B^a and D.

7.3. Embedding finite distributive lattices

In this section, we generalize the ideas from the previous two subsections to prove our main technical result:

Theorem 2.4. *Every finite distributive lattice has a strong interval embedding into the enumeration degrees. (In fact, the range of the embedding will be inside the* Π^0_2*-enumeration degrees.)*

Fix a finite distributive lattice $\mathcal{L}$. Suppose that a_0 is the least element and $a_1, \ldots, a_n$ are the nonzero *join-irreducible* elements, i.e., the nonzero elements which cannot be represented as the join of strictly smaller elements. Then every element of the lattice has a unique representation as $a_F = \bigvee_{i \in F} a_i$, where $F \subseteq \{0, 1, \ldots, n\}$ has the property that if $a_i \leq_{\mathcal{L}} a_j$ and $j \in F$ then $i \in F$. (We will call such F *downward closed*.) This is easily seen as follows: If $F, G \subseteq \{0, 1, 2, \ldots, n\}$ are downward closed sets, then $a_F \leq_{\mathcal{L}} a_G$ if and only if $F \subseteq G$. One direction is obvious: If $F \subseteq G$, then, of course, $a_F \leq_{\mathcal{L}} a_G$. On the other hand, if $a_F \leq_{\mathcal{L}} a_G$ then fix $i \in F$. We have that $a_i \leq_{\mathcal{L}} a_F \leq_{\mathcal{L}} a_G$ and so $a_i = a_i \wedge a_G$. By distributivity, $a_i = \bigvee_{j \in G}(a_i \wedge a_j)$. Since a_i is join-irreducible, we have $a_i \wedge a_j = a_i$ for some $j \in G$, and so $a_i \leq_{\mathcal{L}} a_j$. Since G is downward closed, this implies $i \in G$ as desired.

It is worth pointing out that this property is characteristic of distributive lattices. Consider the lattice M_3 consisting of incomparable elements a, b, c, their least upper bound 1, and their meet 0. The top element has three different downward closed representations: $0 \vee a \vee b$, $0 \vee a \vee c$, and $0 \vee b \vee c$. Consider the lattice N_5 consisting of

elements $0 <_{N_5} a <_{N_5} b <_{N_5} 1$ and $0 <_{N_5} c <_{N_5} 1$, where c is incomparable with both a and b. Here the top element has two different downward closed representations: $0 \vee a \vee c$ and $0 \vee a \vee b \vee c$. It is well-known (see Birkhoff [2, Theorems I.12 and II.13]) that every non-distributive lattice embeds at least one of M_3 and N_5.

Requirements. We will build sets $X_0, X_1, \ldots, X_n$, and let the enumeration degree of $A_F = \bigoplus_{i \in F} X_i$ be the image of the element a_F in the lattice $\mathcal{L}$ under the embedding into the enumeration degrees. This automatically ensures that $a_F \leq_{\mathcal{L}} a_G$ implies ($F \subseteq G$ and hence) $A_F \leq_e A_G$. (We will sometimes abuse notation and write A_F for the set A_{F^*}, where F^* is the downward closure of F with respect to $\mathcal{L}$.)

To ensure that we have strict inequality, i.e., that if $a_F \not\leq_{\mathcal{L}} a_G$ then $A_F \not\leq_e A_G$, we will have $\mathcal{T}$-requirements. The $\mathcal{T}$-requirements will be assigned to pairs (i, F_i) where $i \in \{1, \ldots, n\}$ and $F_i = \{j \mid a_i \not\leq_{\mathcal{L}} a_j\}$. We claim that a_{F_i} is the greatest element in $\mathcal{L}$ that is not above a_i: Note that F_i is downward closed by definition. By the argument above, since $i \notin F_i$, we have $a_i \not\leq_{\mathcal{L}} a_{F_i}$. And a_{F_i} is greatest not above a_i since, if $a_G \not\leq_{\mathcal{L}} a_{F_i}$, then $G \setminus F \neq \emptyset$, but then G contains the index of some $a_j \geq_{\mathcal{L}} a_i$ and hence i.

For every $i \leq n$, the set X_i will consist of two parts, $X_i^a \oplus X_i^w$. For each such pair (i, F_i), we have the requirements

$$\mathcal{T}_{i,e} : X_i^w \neq \Phi_e(A_{F_i}).$$

Let's check that this ensures what we want. Suppose that $a_F \not\leq_{\mathcal{L}} a_G$. Then $F \not\subseteq G$, so fix $i \in F \setminus G$. Since $i \notin G$ and G is downward closed, we have that $G \subseteq F_i$ and so from $X_i \not\leq_e A_{F_i}$, $X_i \leq_e A_F$, and $A_G \leq_e A_{F_i}$ we conclude $A_F \not\leq_e A_G$.

Next, we need to ensure that a set U that is enumeration reducible to the top element $A_{\{0,1,2,\ldots,n\}}$ is either enumeration equivalent to A_F for some downward closed set F or else is enumeration reducible to the set X_0. For this reason, we need a requirement $\mathcal{M}^{F,G}$ for every pair of elements (a_F, a_G) such that a_G is minimal above a_F. It follows that $G = F \cup \{i\}$ for some i. Indeed if $i, j \in G \setminus F$ and $i \neq j$, then the downward closure of $F \cup \{i\}$ and the downward closure of $F \cup \{j\}$ represent two different elements in the interval $(a_F, a_G]$, contradicting minimality. The requirement will say that a set $U \leq_e A_G$ is either below A_F or else is above the set $A_{\{j \mid a_j \leq_{\mathcal{L}} a_i\}} = A_{G \setminus F}$ (Recall that, by our convention, $G \setminus F$ denotes the downward closure of the set $\{i\}$.) Note that $a_{G \setminus F}$ is the least element below a_G that is not below a_F. Indeed, if a_H is below a_G and not below a_F, then $H \not\subseteq F$ and $H \subseteq G$, hence $i \in H$ and, by downward closure, $G \setminus F \subseteq H$.

To see that this set of requirements ensures what we want, fix $U \leq_e A_G$. We will have one requirement for every possible $F \subset G$ representing an element a_F such that a_G is minimal above a_F. If all requirements turn out with outcome $A_{G \setminus F} \leq_e U$, then we claim that $U \equiv_e A_G$. Fix a maximal join-irreducible element $a_j \leq_{\mathcal{L}} a_G$. Then $F = G \setminus \{j\}$ is downward closed and a_G is minimal above a_F, hence $U \geq_e A_{G \setminus F} = A_{\{i \mid a_i \leq_{\mathcal{L}} a_j\}}$. As a_G is the join of all maximal join-irreducible elements $a_j \leq_{\mathcal{L}} a_G$, it follows that $U \geq_e A_G$.

On the other hand, for this G, consider a specific pair (F, G) and let $G = \{j\} \cup F$. Furthermore, recall that, for every i, the set X_i will consist of two parts, $X_i^a \oplus X_i^w$. Denote by A_F^a the set $\bigoplus_{i \in F} X_i^a$. Suppose that $A_{G \setminus F} \not\leq_e U$; then we will want $U \leq_e A_F^a \leq_e A_F$ so that we can continue with "pushing" the degree of U down. Collectively, we thus end up, for all such pairs (F, G), with the requirements

$$\mathcal{M}_e^{F,G} : (\exists \Gamma)[\Psi_e(A_G) = \Gamma(A_F^a)] \vee (\exists \Delta)[A_{G \setminus F} = \Delta(\Psi_e(A_G))],$$

which will ensure that U either has the same degree as some A_F or else is reducible to X_0, as desired.

Conflicts. A $\mathcal{T}_i$-strategy β wants to change X_i and restrain A_{F_i}. It will be in conflict with an $\mathcal{M}^{F,G}$-strategy $\alpha \prec \beta$ only if $G = F \cup \{i\}$ (and hence $F \subseteq F_i$). Indeed, if $i \notin G$, then enumerating and then extracting a witness from X_i^w does not affect $\Psi(A_G)$. On the other hand, if $F \not\subseteq F_i$, then the enumeration of a witness w into X_i^w may make an axiom in Ψ for some number x valid and so cause the $\mathcal{M}$-strategy to enumerate markers m_j into A_j for $j \in F$ to make an axiom for x in Γ valid; now later, the extraction of the witness w may mean that x leaves $\Psi(A_G)$; however, we can still restrain in A_j the markers m_j for all $j \in F_i \cap F$ because there is a marker m_j for some $j \in F \setminus F_i$ that can be used to extract x from $\Gamma(A_F^a)$. This means that all $\mathcal{M}^{F,G}$-strategies that are in conflict with $\mathcal{T}_i$ have $G = F \cup \{i\}$ and hence the same set $A_{G \setminus F} = A_{\{j | a_j \leq_{\mathcal{L}} a_i\}}$.

Streams, approximations, and parameters. For every $i \leq n$ and for every X_i, we have two streams $\mathcal{S}_\delta^{i^a}$ and $\mathcal{S}_\delta^{i^w}$ at every strategy δ. The streams at the root strategy $\emptyset$ at stage s consist of the interval $[0, s]$. For the other strategies, we will define them recursively during the construction. When an $\mathcal{M}$-strategy α chooses axiom locations for X_i, it picks them out of the stream $\mathcal{S}_\alpha^{i^a}$. When a $\mathcal{T}_i$-strategy β picks witnesses, it picks them out of the stream $\mathcal{S}_\beta^{i^w}$. An element from a stream is *suitable* for a strategy δ if it is the $\langle \hat{\delta}, x \rangle$-th element that is not used in an attachment for some $x \in \omega$, where $\delta \mapsto \hat{\delta}$ is a computable injective function that maps a strategy δ to a natural number.

We will also have dump sets U_i^a and U_i^w for all $i \leq n$.

Next, an $\mathcal{M}^{F,G}$-strategy α has as parameters the enumeration operator Γ_α that it builds, and a list of promises P_α. A promise is of the form $(\{m_j\}_{j \in F}, x, D)$, where $\langle x, D \rangle$ is an axiom in Ψ_α and m_j is an axiom location picked from the stream $S_\alpha^{j^a}$ and targeted for X_j^a.

A $\mathcal{T}_i$-strategy β has a parameter $\mathcal{C}_\beta = \{\alpha_0, \ldots, \alpha_{k-1}\}$ in which we list in order of priority all $\mathcal{M}$-strategies of higher priority that are in conflict with β and still active along β (i.e., no $\mathcal{T}$-strategy of higher priority has switched them to a backup strategy). The infinite outcomes of β are determined by the number k. Below each infinite outcome ∞_u, say, the strategy β will build an operator Δ_u^β that enumerates X_i^w from $\Psi_{\alpha_u}(A_{G_{\alpha_u}})$, and we will code into X_i^w all sets X_j^a where $a_j \leq_{\mathcal{L}} a_i$ as well as all sets X_j^w where $a_j <_{\mathcal{L}} a_i$. For each such X, we will have an attachment set at outcome ∞_u denoted by $H_{u,\beta}^X$. The

entries in this set are of the form (z, y), where z is a witness of β enumerated into the stream $\mathcal{S}^{i^w}_{\beta^\frown\infty_u}$. The strategy β will also have a current witness z_β as well as other minor parameters that we will define during the construction.

The $\mathcal{M}$-strategies make promises; the $\mathcal{T}$-strategies make attachments. Whenever a new number is enumerated into a set by a strategy α, this sets off a chain reaction in which higher-priority strategies respond by possibly enumerating more elements. At any moment, the set of their promises or attachments are all finite, so this process is finitary. We define functions g_α to explain formally how this process works. The function g_α has $2(n+1)$ arguments $\vec{D}^a = D^a_0, D^a_1, \ldots, D^a_n$ and $\vec{D}^w = D^w_0, D^w_1, \ldots, D^w_n$, where D^a_i and D^w_i are finite sets targeted for X^a_i and X^w_i, respectively. The function g_α outputs a vector of $2(n+1)$ sets in turn:

$$g_\alpha(\vec{D}^a, \vec{D}^w) = (\bigcup_l D^a_{0,l}, \ldots, \bigcup_l D^a_{n,l}, \bigcup_l D^w_{0,l}, \ldots, \bigcup_l D^w_{n,l}),$$

where $D^a_{i,0} = D^a_i \cup U^a_i$ and $D^w_{i,0} = D^w_i \cup U^w_i$, and where

$$D^w_{i,l+1} = D^w_{i,l} \cup \{z \mid (\exists \sigma, u, j)(\exists y)[\sigma \text{ assigned to } \mathcal{T}_i,\ \sigma^\frown\infty_u \preceq \alpha, \text{ and } [[(z,y) \in H^{X^a_j}_{u,\sigma} \text{ and } y \in D^a_{j,l}] \text{ or } [(z,y) \in H^{X^w_j}_{u,\sigma} \text{ and } y \in D^w_{j,l}]]]\}$$

and

$$D^a_{i,l+1} = D^a_{i,l} \cup \{m_i \mid (\exists F, G, e, \sigma)(\exists x, D)[i \in F,\ \sigma \preceq \alpha \text{ is active at } \alpha, \sigma \text{ is assigned to } \mathcal{M}^{F,G}_e,\ (\{m_j\}_{j\in F}, x, D) \in \mathcal{P}_\sigma, \text{ and } D \subseteq \bigoplus_{j\in G}(D^a_{j,l} \oplus D^w_{j,l})]\}.$$

Tree of strategies. The set of outcomes is, as before, $\{w, d, \infty_l : l < \omega\}$, ordered by:

$$d <_L \infty_0 <_L \infty_1 <_L \cdots <_L w.$$

The tree of strategies will be a partial computable function

$$T : \{w, d, \infty_l : l < \omega\}^{<\omega} \to \mathcal{R},$$

where $\mathcal{R}$ is an effective listing of all $\mathcal{M}$- and $\mathcal{T}$-requirements. We will define $T(\alpha)$ and C_α by induction: Set $T(\emptyset) = \mathcal{M}_0$. If $T(\alpha) = \mathcal{M}_\alpha$, then $T(\alpha^\frown d)$ is defined and equals the least $\mathcal{T}$-requirement that has not been assigned to any node $\beta \preceq \alpha$, and we set $\mathcal{C}_{\alpha^\frown d} = \mathcal{C}_\alpha \cup \{\alpha\}$. If $T(\alpha) = \mathcal{T}_\alpha$ and $\mathcal{C}_\alpha = \{\alpha_0 \prec \alpha_1 \prec \cdots \prec \alpha_{k-1}\}$, then we set $T(\alpha^\frown w)$ and $T(\alpha^\frown d)$ to be the least $\mathcal{M}$-requirement that has not yet been associated to any node $\beta \preceq \alpha$. For every $l < k$, we set $T(\alpha^\frown\infty_l) = T(\alpha)$. Note that in that case $\mathcal{C}(\alpha^\frown\infty_l) = \mathcal{C}(\alpha) \setminus \{\alpha_l\}$, hence a specific $\mathcal{T}$-requirement can only be assigned finitely often along each branch. It follows that every requirement is assigned along each infinite path, and furthermore, that it is assigned only a finite number of times.

Construction. At stage 0, the global dump sets U_i^a and U_i^w are empty, and all strategies are in initial state: For an $\mathcal{M}$-strategy α, we have that $\Gamma_\alpha = \emptyset$ and $\mathcal{P}_\alpha = \emptyset$; for a $\mathcal{T}_i$-strategy β with $|\mathcal{C}_\beta| = k$, we have that $\Delta_l^\beta = H_{l,\beta}^X = \emptyset$ for all $l < k$ and all $X \in \{X_j^a \mid a_j \leq_{\mathcal{L}} a_i\} \cup \{X_j^w \mid a_j <_{\mathcal{L}} a_i\}$, and that the current witness z_β is undefined.

At stage $s > 0$, we build f_s of length at most s, activating strategies along f_s. We begin by enumerating U_i^a and U_i^w into X_i^a and X_i^w for all $i \leq n$, respectively. We then start at the root and let $\mathcal{S}_{\emptyset,s}^{i^x} = \mathcal{S}_{\emptyset,s-1}^{i^x} \cup \{s\}$, for $x \in \{a, w\}$ and $i \leq n$. Suppose we have constructed $f_s \upharpoonright t$, along with $\mathcal{S}_{f_s\upharpoonright t,s}^{i^x} = \mathcal{S}_{f_s\upharpoonright t,s-1}^{i^x} \cup \{y_i^x\}$ and $X_{i,s}^a$, $X_{i,s}^w$ (or rather, the approximations to these sets at substage t of stage s). If $t = s$, then we end this stage and move on to the next. We initialize all strategies on the tree that are to the right of f_s by returning them to their initial state as defined at stage 0. If $t < s$, then we activate $f_s \upharpoonright t$ and let it pick its outcome o. Then $f_s \upharpoonright (t+1) = (f_s \upharpoonright t)^\frown o$:

Case 1: Suppose $f_s \upharpoonright t = \alpha$ is an $\mathcal{M}^{F,G}$-strategy. If α did not end the previous stage at which it was visited prematurely, and there is an axiom in Ψ_α which has not yet been assigned all axiom locations m_j for $j \in F$, then pick the oldest such axiom $\langle x, D\rangle$, i.e., the one that was first enumerated into Ψ_α. For every $j \in F$ such that m_j is not yet defined, check whether y_j^a is suitable and larger than $\max(D)$. If so, we take y_j^a out of the stream, set $m_j = y_j^a$, and end this stage of the construction prematurely. If we now have a suitable m_j for all $j \in F$, then we enumerate the promise $(\{m_j\}_{j\in F}, x, D)$ into $\mathcal{P}_\alpha$ and the axiom $\langle x, \bigoplus_{j\in F}\{m_j\}\rangle$ into Γ_α. Otherwise (if α ended the previous stage at which it was visited prematurely, if no y_j^a is suitable, or if all axioms in Ψ_α have been assigned axiom locations), then we enumerate each y_j^x into the stream $\mathcal{S}_{\alpha^\frown d}^{j^x}$ and let d be α's outcome. In either case, we enumerate $g_\alpha(\vec{X}_{j,s}^a, \vec{X}_{j,s}^w)$ for $j \leq n$ into the sets $(\vec{X}_j^a, \vec{X}_j^w)$.

Case 2: If $f_s \upharpoonright t = \beta$ is a $\mathcal{T}^i$-strategy and $\mathcal{C}_\beta = \{\alpha_0 \prec \cdots \prec \alpha_{k-1}\}$ is the list of active $\mathcal{M}$-strategies of higher priority in conflict with β, then we pick the first case which applies:

(1) *The strategy β was successful via a realized witness z and had outcome d at the previous stage at which it was active.* In that case, for all $j \in F_i$, enumerate D_z^j into X_j^a. (Here, D_z^j consists of axiom locations that belong to higher-priority $\mathcal{M}^{F,G}$-strategies still active at β such that $i \in G$, and these D_z^j were defined when the witness became realized.) For all $j \leq n$, enumerate y_j^x into $\mathcal{S}_{\beta^\frown d}^{j^x}$. We enumerate $g_\beta(\vec{X}_{j,s}^a, \vec{X}_{j,s}^w)$ into the corresponding sets $(\vec{X}_j^a, \vec{X}_j^w)$ for $j \leq n$, and let the outcome be d.

(2) *The current witness z_β is not defined.* If y_i^w is suitable for β, and larger than $|\beta|$ and larger than the last stage when β was initialized, then let the *current witness* be $z_\beta = y_i^w$ and end this stage. Otherwise, enumerate y_j^x into $\mathcal{S}_{\beta^\frown w}^{j^x}$ for all $j \leq n$. We enumerate $g_\beta(\vec{X}_{j,s}^a, \vec{X}_{j,s}^w)$ into the corresponding sets $(\vec{X}_j^a, \vec{X}_j^w)$ and let the outcome be w.

(3) *The witness z_β is not realized*, i.e., $z_\beta \notin \Phi_\beta(\bigoplus_{j\in F_i}(V_j^a \oplus V_j^w))$, where

$$(\vec{V}_j^a, \vec{V}_j^w) = g_\beta(\overrightarrow{(X_{j,s}^a \cup \mathcal{S}^{j^a}_{\beta\hat{}w,s})_{j\le n}}, \overrightarrow{(X_{j,s}^w \cup \mathcal{S}^{j^w}_{\beta\hat{}w,s})_{j<i}}, X_i^w \cup \{z_\beta\} \cup \mathcal{S}^{i^w}_{\beta\hat{}w}, \overrightarrow{(X_{j,s}^w \cup \mathcal{S}^{j^w}_{\beta\hat{}w,s})_{i<j\le n}}).$$

We then enumerate

$$g_\beta((\vec{X}_{j,s}^a)_{j\le n}, (\vec{X}_{j,s}^w)_{j<i}, X_{i,s}^w \cup \{z_\beta\}, (\vec{X}_{j,s}^w)_{j>i})$$

into $((\vec{X}_{j,s}^a)_{j\le n}, (\vec{X}_j^w)_{j\le n})$, the numbers y_j^x into the corresponding stream $\mathcal{S}^{j^x}_{\beta\hat{}w}$, and let the outcome be w.

(4) *The witness z_β is realized.* Enumerate the numbers from the stream $S^{j^x}_{\beta\hat{}w,s}$ into U^{x_j}. For every $j \in F_i$, let $D^j_{z_\beta}$ be the set of axiom locations m of higher-priority active $\mathcal{M}^{F,G}$-strategies with $i \in G$ and targeted for X_j^a such that if z_β is in X_i^w, then those axiom locations are enumerated into X_j^a via the function g_β, but if $z_\beta \notin X_i^w$, then they are not enumerated. (We will argue that axiom locations that have the same behavior but belong to other types of $\mathcal{M}^{F,G}$-strategies, i.e., with $i \notin G$ but $F_i \cap F \neq \emptyset$, are automatically enumerated into their respective set once all $D^j_{z_\beta}$ are.) For every $u < k$, let $E_u^{z_\beta}$ be the set of numbers that enter $\Psi_{\alpha_u}(A_{G_{\alpha_u}})$ when z_β is enumerated into X_i^w but leave $\Psi_{\alpha_u}(G_{\alpha_u})$ when z_β is taken out. Make the *current witness* z_β undefined.

Now, for every realized witness z, $x \in \{a, w\}$, $j \le n$, and every $u < k$, let

$$M_{u,z}^{X_i^w} = X_{i,s}^w \cup \bigcup_{u\le v\le k} \mathcal{S}^{i^w}_{\beta\hat{}\infty_v} \setminus \{z\}, \text{ and}$$

$$M_{u,z}^{X_j^x} = X_{j,s}^x \cup \bigcup_{u\le v\le k} \mathcal{S}^{j^x}_{\beta\hat{}\infty_v} \text{ for } (x,j) \neq (w,i).$$

Let

$$(\vec{L}_{u,z}^{X_j^a}, \vec{L}_{u,z}^{X_j^w}) = g_\beta(\vec{M}_{u,z}^{X_j^a}, \vec{M}_{u,z}^{X_j^w}).$$

We say that z is *α_u-cleared* if $E_u^z \subseteq \Psi_{\alpha_u}(\bigoplus_{j\in G_{\alpha_u}} L_{u,z}^{X_j^a} \oplus L_{u,z}^{X_j^w})$. We search for the (lexicographically) least pair (u,z) such that z is a realized witness, $z \notin \mathcal{S}^{X_i^w}_{\beta\hat{}\infty_v}$ for $v < u$, $z \notin U_i^w$ and z is α_v-cleared for all $v > u$. Note that the pair $(k-1, z_\beta)$ satisfies these conditions.

For all $j \le n$, we enumerate $\bigcup_{u<v<k} M_{v,z}^{X_j^a}$ into U_j^a, and $\bigcup_{u<v<k} M_{v,z}^{X_j^w}$ into U_j^w. We set $\Delta_v = H_v^Y = \emptyset$ for $v > u$ and all $Y \in \{X_j^a \mid a_j \le_{\mathcal{L}} a_i\} \cup \{X_j^w \mid a_j <_{\mathcal{L}} a_i\}$.

(a) If $u \ge 0$, then we enumerate the axiom $\langle z, E_u^z\rangle$ into Δ_u, the sets $g_\beta(\vec{X}_{j,s}^a, \vec{X}_{j,s}^w)$ into the corresponding sets $(\vec{X}_j^a, \vec{X}_j^w)$, the number z into the stream $\mathcal{S}^{i^w}(\beta\hat{}\infty_u)$, the numbers y_j^x (for $(x,j) \neq (w,i)$) into the corresponding streams $\mathcal{S}^{j^x}_{\beta\hat{}\infty_u}$, and

let the outcome be ∞_u. List the set $\{X^a_j \mid a_j \leq_{\mathcal{L}} a_i\} \cup \{X^w_j \mid a_j <_{\mathcal{L}} a_i\}$ as $\{Y_0, Y_1, \ldots, Y_{m-1}\}$. If z is the $(m \cdot q + v - 1)$-st number enumerated into $\mathcal{S}^{i^w}(\beta^\frown\infty_i)$ for $v < m$, then we enumerate the attachment $(z, q) \in H^{Y_v}_{\beta,u}$ and end this stage. If z is the $(m \cdot q + m - 1)$-st number enumerated into the stream, then we do not end the stage and let the next strategy act.

(b) Otherwise, we have a witness z that is α_v-cleared for all $v < k$. We say that z is *successful*. We enumerate D^j_z into X^a_j and then $g_\beta(\vec{X}^a_{j,s}, \vec{X}^w_{j,s})$ into $(\vec{X}^a_j, \vec{X}^w_j)$. We let the outcome be d.

Verification. We define the infinite path f by $f(t) = \liminf_{s>t} f_s(t)$. It is straightforward to see that strategies σ along f are visited at infinitely many stages and initialized at only finitely many stages. Let s_σ be the least stage after which σ is not initialized. Let $X \in \{i^a, i^w\}_{i \leq n}$. Let $\mathcal{S}^X_\sigma = \bigcup_{s > s_\sigma} \mathcal{S}^X_{\sigma,s}$. A simple induction on the length of σ proves that $\mathcal{S}^X_\sigma$ is infinite. Furthermore, if a strategy σ is visited at two consecutive σ-stages $s > t > s_\sigma$, then $X^a_{i,s} \supseteq X^a_{i,t}$ and $X^w_{i,s} \supseteq X^w_{i,t}$ (as seen at substage $|\sigma|$), as in order for a strategy above σ to stop enumerating an element, it must move its outcome left of σ and hence initialize σ. Our next lemma holds the key to the way numbers may enter the sets X^a_i and X^w_i for $i \leq n$.

Lemma 7.5. *If a number $m \in X^a_i$ or $z \in X^w_i$ where $i \leq n$, then either m or z is dumped into X^a_i or X^w_i, respectively, starting at some stage, or there is a strategy $\sigma \prec f$ such that at all but finitely many stages, σ is the least strategy that enumerates m or z into its corresponding set, and it does so at cofinitely many stages at which σ takes its true outcome.*

Proof. First note that if a number ever enters a dump set U^x_i for $x \in \{a, w\}$ and $i \leq n$ at stage s, say, then it is enumerated into X^x_i at the beginning of every stage $t > s$.

So suppose that $m \in X^a_i$, but is never dumped into U^a_i. Note that when an $\mathcal{M}^{F,G}$-strategy picks a number as an axiom location, then that number is taken out of the stream. It is never returned to any stream, so no other strategy can use it. If m is not the axiom location to any strategy then it is not enumerated into X^a_i at any stage unless it is dumped, so let α be the unique $\mathcal{M}^{F,G}$-strategy that uses m as an axiom location, say, for the axiom $\langle y, D\rangle$. It follows that $i \in F$. If α is ever initialized, then m is dumped. Indeed, if α is initialized, then a higher-priority $\mathcal{T}$-strategy σ with $\sigma^\frown o \preceq \alpha$ moves to an outcome o' that is to the left of o. Every time a $\mathcal{T}$-strategy moves its outcome to the left, it dumps all elements into the streams associated with outcomes to the right unless the element is a specific witness (but then this element is in a different stream). In particular, it dumps $\mathcal{S}^{i^a}_\alpha \subseteq \mathcal{S}^{i^a}_{\sigma^\frown o}$. So, by assumption, α is never initialized.

There are infinitely many stages at which m is enumerated into X^a_i. At stage s, this could be because a strategy $\sigma \succeq \alpha$ such that α is active at σ causes $D \subseteq A_G$ or because a successful $\mathcal{T}_j$-strategy β has $m \in D^i_z$ for some realized cleared witness z and is visited at stage s. We first claim that there are only finitely many possible strategies β that can

have $m \in D^i_z$ for a realized witness z. This is because if z entering X^w_j causes $D \subseteq A_G$, and z not entering X^w_j does not cause $D \subseteq A_G$, there is a sequence of promises and attachments that realize this. This sequence is triggered by z and ends in $D \subseteq A_G$. It follows that $z \leq \max(D) < m$. This is because every time we pick axiom locations for an axiom, we require that they are larger than all elements in the finite set associated with the axiom, and every time we make an attachment (z, n), we have that $n < z$. There are finitely many $\mathcal{T}$-strategies that can have a witness $z < m$, because the witness for β is always selected to be larger than $|\beta|$. Once a strategy β succeeds with a witness z with $m \in D^i_z$, it will keep $z \notin X^w_j$ at all further stages unless it is initialized. This means that while β is not initialized, no other strategy will have a reason to enumerate m into X^a_i, as in order for m to enter X^a_i (and that needs to happen in order for m to possibly enter another $D^i_{z'}$ for the realized witness of a different strategy), we must have that $z \in X^w_j$. This means that unless β is initialized, m will be enumerated into X^a_i only at stages at which β is visited. If β is initialized, then it will have witnesses of size greater than the stage at which it is initialized. It follows that there are only finitely many pairs (β, z) such that $m \in D^i_z$ and z is a successful witness for β. If there is such a strategy β that is never initialized, then we have argued that β must be visited infinitely often (as $m \in X^a_i$), and so β is on the true path. It enumerates m along with D^i_z every time it is visited after z's success.

If no such strategy remains uninitialized, then after some stage s_0, the axiom location m can only be enumerated into X^a_i via the promise at α and the function g_σ for some σ extending α. It follows that at infinitely many stages s, $D \subseteq A_{G,s}$, and thus $D \subseteq A_G$. Now, since $\max(D) < m$, we can use the induction hypothesis: For every element $u \in D$, there is a strategy $\sigma_u \prec f$ that enumerates u into its respective set at all stages at which it is visited after some fixed stage s_u. Pick the longest such σ_u. Then at all stages $t > \max\{s_u \mid u \in D\}$ at which σ_u is visited, it will enumerate m into X^a_i via the function g_{σ_u}.

Now suppose $z \in X^w_i$ and is never dumped. At every stage s, there is at most one strategy $\gamma_s = \beta^\frown o$ visited at stage s such that z is an unrealized witness of β and $o = w$, or such that z is a realized witness of β and β uses z in an attachment below outcome $o = \infty_u$ for some $u < k$. There are only finitely many strategies β that can ever fulfill this role, as z must be suitable for β and β has length no more than z. If between stages $s < t$, such a strategy for z changes, then $\gamma_t <_L \gamma_s$. So let β be such that $\beta^\frown o$ is least among all such γ_t, and hence equal to γ_t at all but finitely many stages t. If $o = w$, then z enters X^w_i only at stages at which we visit β and β has outcome w. It follows that $\beta^\frown w \prec f$.

If $o = \infty_u$, then $(z, y) \in H^Y_{\beta,u}$ and hence the only way that z can enter X^w_i is if some strategy compatible with $\beta^\frown\infty_u$ enumerates y into Y. As $y < z$, we have by induction that there is a unique strategy $\sigma \prec f$ that causes this. So either $\sigma \preceq \beta$, in which case $\beta^\frown\infty_u \prec f$ and $z \in X^w_i$ at all but finitely many $\beta^\frown\infty_u$-true stages, or else $\beta^\frown\infty_u \preceq \sigma$ and σ enumerates $z \in X^w_i$ along with y via the function g_σ. □

Lemma 7.6. *Every $\mathcal{T}$-requirement is satisfied.*

Proof. Fix a requirement $\mathcal{T}_{i,e}$. Let $\beta \prec f$ be the longest strategy that is assigned to requirement $\mathcal{T}_{i,e}$. Such a strategy exists because once $\mathcal{T}_{i,e}$ is assigned to a node σ with $|\mathcal{C}_\sigma| = k$, $\mathcal{T}_{i,e}$ can be assigned at most k many more times along any path through σ. It follows that $\beta\hat{}d \prec f$ or $\beta\hat{}w \prec f$.

If $\beta\hat{}w \prec f$, then there is an unrealized witness z such that at every β-true stage s, $z \notin \Phi_e(\bigoplus_{j\in F_i}(V_j^a \oplus V_j^w))$, where

$$(\vec{V}_j^a, \vec{V}_j^w) = g_\beta(\overrightarrow{(X_{j,s}^a \cup \mathcal{S}_{\beta\hat{}w,s}^{j^a})}_{j\le n}, \overrightarrow{(X_{j,s}^w \cup \mathcal{S}_{\beta\hat{}w,s}^{j^w})}_{j<i},$$
$$X_i^w \cup \{z\} \cup \mathcal{S}_{\beta\hat{}w}^{i^w}, \overrightarrow{(X_{j,s}^w \cup \mathcal{S}_{\beta\hat{}w,s}^{j^w})}_{i<j\le n}).$$

By Lemma 7.5, we have that $X_j^a \subseteq \bigcup_s V_{j,s}^a$ and $X_j^w \subseteq \bigcup_s V_{j,s}^w$, where s ranges over all $\beta\hat{}w$-true stages. This is because the true path passes through $\beta\hat{}w$ and so any element enumerated into any set Y is either enumerated by a strategy of higher priority than β and hence in the set Y_s by the time we reach β, enumerated by β and hence it is a member of the corresponding set in the sequence $(\vec{V}_j^a, \vec{V}_j^w)$, or else it is enumerated by a strategy extending $\beta\hat{}w$ and hence belongs to the stream $\mathcal{S}_{\beta\hat{}w}^Y$ at some β-true stage s. It follows that $z \notin \Phi_e(\bigoplus_{j\in F_i} X_j)$. On the other hand, z is enumerated into X_i^w at every sufficiently large β-true stage, so $z \in X_i^w$.

If $\beta\hat{}d \prec f$, then there is a successful witness z that is α_u-cleared for all $u < k$. This means that z was realized, and so at some earlier stage, we saw $z \in \Phi_e(\bigoplus_{j\in F_i}(V_j^a \oplus V_j^w))$. The strategy β then proceeded to dump into the respective stream all elements that contribute to the definition of the sets V_j^a and V_j^w apart from the witness z. This leaves out numbers that enter A_{F_i} when z is enumerated into X_i^w, but are not enumerated into A_{F_i} when z is left out of X_i^w via the function g_β. From those numbers, the axiom locations that are used by higher-priority active $\mathcal{M}^{F,G}$-strategies with $i \in G$ enter the set D_z^j (where $j \in F_i$) and are enumerated into their corresponding set when β is visited. This leaves axiom locations for $\mathcal{M}^{F,G}$-strategies where $i \notin G$, and attachments.

We reason by induction. We will show that any axiom location targeted for X_j^a or attachment targeted for X_j^w, where $a_j \not\geq_\mathcal{L} a_i$, that is enumerated when z is enumerated, and is not enumerated if z is not enumerated at the stage when z became realized is enumerated into its targeted set when we visit β. Fix an axiom location m targeted for X_j^a by $\alpha \preceq \beta$ where $j \in F_i$. The strategy α is an $\mathcal{M}^{F,G}$-strategy. If $i \in G$, then $m \in D_z^j$, so suppose that $i \notin G$. This means that m is part of a promise $(\{m_i\}_{i\in F}, x, D)$ at α and D is enumerated into the set that it is targeted for if z enters X_i^w. Since $i \notin G$, we have that the set D consists of axiom locations and attachments targeted for X_k^w, where $k \neq j$. Since G is downward closed, we even have that $a_k \not\geq_\mathcal{L} a_i$. As $m > \max(D)$, we can apply the induction hypothesis, namely, that D is enumerated into its targeted set at every stage at which we visit β, and so we enumerate m via the function g_β as well.

Now consider an attachment y targeted for X_j^w, where $a_j \not\geq_{\mathcal{L}} a_i$. Then it belongs to a $\mathcal{T}_j$-strategy $\beta' \preceq \beta$. The $\mathcal{T}_j$-strategy codes all Y_k^x, where $a_k <_{\mathcal{L}} a_j$, as well as Y_j^a, so y is a part of an attachment (y, q) where $q < y$ is targeted for one of these Y_k^x or Y_j^a. Since $a_k \leq_{\mathcal{L}} a_j$ and $a_j \not\geq_{\mathcal{L}} a_i$, it follows that $a_k \not\geq_{\mathcal{L}} a_i$. Once again, the induction hypothesis applies, and so $q \in Y_k^x$ at all sufficiently large β-true stages. The definition of the function g_β then ensures that $y \in X_j^w$ at all sufficiently large β-true stages. □

Lemma 7.7. *Every $\mathcal{M}$-requirement is satisfied.*

Proof. Fix e and let $F \subset G \subseteq \{0, 1, \ldots, n\}$ be downward closed sets in $\mathcal{L}$ such that F represents a_F, G represents a_G, and a_G is minimal above a_F. Then there is a unique strategy $\alpha \prec f$ assigned to the requirement $\mathcal{M}_e^{F,G}$. Suppose first that α is switched to a backup strategy by some $\mathcal{T}_i$-strategy $\beta \succ \alpha$ along the true path. We know that $G = F \cup \{i\}$ and hence we have that $F \subseteq F_i$ and $G \setminus F = \{a_j \mid a_j \leq_{\mathcal{L}} a_i\}$ under our convention. We have that $\beta^\frown\infty_u \prec f$, $\mathcal{C}(\beta) = \{\alpha_0, \ldots, \alpha_{k-1}\}$ and $\alpha = \alpha_u$. It follows that for all elements $z \in \mathcal{S}^{X_i^w}(\beta^\frown\infty_u)$, we have that $z \in X_i^w$ if and only if $z \in \Delta_u(\Psi_\alpha(A_G))$. This is because if we ever see an axiom stop being valid, we would move to an outcome to the left of ∞_u. First of all, we claim that $\bigoplus_{a_j <_{\mathcal{L}} a_i}(X_j^a \oplus X_j^w) \oplus X_i^a \leq_e \Delta_u(\Psi_\alpha(A_G))$. By Lemma 7.5, we have that $y \in Y$ (where $Y \in \{X_i^a\} \cup \{X_j^a, X_j^w \mid a_j <_{\mathcal{L}} a_i\}$) if and only if a there is a strategy $\sigma \prec f$ that enumerates y at all but finitely many σ-true stages. The strategy β has an attachment $(z, y) \in H_u^Y$ for some $z \in \mathcal{S}^{x_i^w}(\beta^\frown\infty_u)$, and so if $y \in Y$ then at all $\max(\sigma, \beta^\frown\infty_u)$-true stages, z would be enumerated into X_i^w. On the other hand, z is enumerated into X_i^w at stage s only if $y \in Y_s$, so we have that $Y = \{y \mid (z, y) \in H_u^Y \ \&\ z \in \Delta_u(\Psi_\alpha(A_G))\}$.

Next, we note that X_i^w consists of three types of elements: the elements that are enumerated into X_i^w by strategies of higher priority than β at β-true stages s (which is a c.e. set); the elements in $X_i^w \cap (\bigcup_s \mathcal{S}^{i^w}_{\beta^\frown\infty_u}) = \Delta_u(\Psi_\alpha(A_G))$; and the set of all numbers z such that (z, y) is an attachment at some $\mathcal{T}_i$-strategy $\sigma \preceq \beta$ coding some $Y \in \{X_i^a\} \cup \{X_j^a, X_j^w \mid a_j <_{\mathcal{L}} a_i\}$ and $y \in Y$. We have already argued that $Y \leq_e \Delta_u(\Psi_\alpha(A_G))$, and so the last part of X_i^w is also reducible to $\Delta_u(\Psi_\alpha(A_G))$. Altogether, we obtain that $A_{G\setminus F} \leq_e \Psi_\alpha(A_G))$, and so the requirement $\mathcal{M}_e^{F,G}$ is satisfied.

On the other hand, suppose that α is never switched to a backup strategy. We claim that $\Gamma_\alpha(A_F^a) = \Psi_\alpha(A_G)$. If $x \in \Psi_\alpha(A_G)$, then there is a valid axiom $\langle x, D\rangle$ in Ψ_α. This axiom is assigned a set of axiom locations $\{m_j\}_{j\in F}$, and the axiom $\langle x, \bigoplus_{j\in F}\{m_j\}\rangle$ is enumerated into Γ_α. By Lemma 7.5, we have that $D \subseteq A_{G,s}$ at cofinitely many σ-true stages s for some sufficiently long $\sigma \prec f$. As α is active at all of its successor nodes along the true path, it follows that $g_{\max(\sigma,\alpha)}$ enumerates $\bigoplus_{j\in F}\{m_j\}$ into A_F^a, and so $x \in \Gamma_\alpha(A_F^a)$.

Now suppose that $x \in \Gamma_\alpha(A_F^a)$ via an axiom $\langle x, \bigoplus_{j\in F}\{m_j\}\rangle$, associated with an axiom $\langle x, D\rangle \in \Psi_\alpha$. If the axiom $\langle x, D\rangle \in \Psi_\alpha$ is not valid, then after finitely many stages, the axiom locations can only be enumerated into A_F^a if they enter sets D_z^j for some witness z and are enumerated by a $\mathcal{T}$-strategy. By Lemma 7.5, we have that if any

axiom location m is in A^a_F, then there is a least strategy $\sigma \prec f$ that enumerates it at cofinitely many σ-true stages. Fix such an axiom location m and the corresponding σ. If σ is a $\mathcal{T}_i$-strategy, where $F \cup \{i\} = G$, then z is a realized witness that is cleared by α, meaning that σ has found evidence that all x associated with axiom locations that are enumerated if z is enumerated, and not enumerated if z is not enumerated, and that belong to strategies in conflict with α, are in $\Psi_\alpha(A_G)$ even when z is not in X^w_i. The number x associated with m is one of these numbers. The strategy σ ensures that $x \in \Psi_\alpha(A_G)$ remains true at further true stages by dumping relevant elements, and so in this case, $x \in \Psi_\alpha(A_G)$.

If σ is a $\mathcal{T}_k$-strategy such that $k \notin G$, then m cannot end up in D^j_z. So suppose that $k \in G$ and $F \not\subseteq F_k$. Let m' be an axiom location from the same promise as m but which is targeted for some X^a_l with $l \in F \setminus F_k$. If the axiom associated with m and m' is invalid, then m' is not enumerated into X^a_l via the function g_ρ by any strategy ρ along the true path. We need to show that m' cannot enter $D^l_{z'}$ via a different strategy $\tau \prec f$. Assuming that m is the axiom location associated with x that enters its corresponding set first, it follows that $\tau \not\preceq \sigma$. Strategies extending $\sigma\hat{}d$ are visited for the first time after m' is already defined, so their witnesses are larger than m' and cannot influence whether m' is enumerated or not, so m' can never enter the set $D^l_{z'}$ for a witness z'. It follows that m' remains out of X^a_l, contradicting the assumptions.

Thus $x \in \Gamma_\alpha(A^a_F)$ must imply that $x \in \Psi_\alpha(A_G)$, and so $\mathcal{M}^{F,G}$ is once again satisfied. □

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