Journal of Mathematical Behavior 61 (2021) 100834

Contents lists available at ScienceDirect

Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Backward transfer influences from quadratic functions instruction on students' prior ways of covariational reasoning about linear functions

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ARTICLE INFO

Keywords:
Backward transfer
Covariational reasoning
Linear functions
Quadratic functions

ABSTRACT

The study reported in this article examined the ways in which new mathematics learning influences students' prior ways of reasoning. We conceptualize this kind of influence as a form of transfer of learning called *backward transfer*. The focus of our study was on students' covariational reasoning about linear functions before and after they participated in a multi-lesson instructional unit on quadratic functions. The subjects were 57 students from two authentic algebra classrooms at two local high schools. Qualitative analysis suggested that quadratic functions instruction did influence students' covariational reasoning in terms of the number of quantities and the level of covariational reasoning they reasoned with. These results further the field's understanding of backward transfer and could inform how to better support students' abilities to engage in covariational reasoning.

1. Introduction

An abundance of mathematics education theory and research has shown the importance of the relationship between prior ways of reasoning and new learning (e.g., Bransford & Schwartz, 1999; Roschelle, 1995; Vosniadou & Brewer, 1987). For example, much research has shown that, as learners progress in their mathematical development, their prior ways of reasoning serve as a foundation on which new conceptions are constructed (e.g., Hiebert & Carpenter, 1992). However, it should be noted that most of this research focuses on influences that prior ways of reasoning have on new learning. What has yet to be well examined in the context of mathematics education and what we examined with our study is the other direction, namely the influences that new learning might have on prior ways of reasoning.

To distinguish between the two directions of this relationship, we refer to influences by prior ways of reasoning on new learning as influences in the *forward* direction and influences by new learning on prior ways of reasoning as influences in the *backward* direction. Furthermore, because influences in the forward direction have been conceived by several researchers as a form of transfer of learning (e.g., Diamond, 2018; Lobato, 2008; Lockwood, 2011), we in turn conceive of influences in the backward direction as a form of transfer we call *backward transfer*[1] . Specifically, we define backward transfer as the "influence that learning something new has on a learner's prior ways of reasoning about a different or related concept" (Hohensee, 2014, p. 136).

To our knowledge, mathematics education research that reports on backward transfer has been limited to nine studies across the

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[1] Marton (2006) refers to forward transfer as *prospective* and backward transfer as *retrospective*.

https://doi.org/10.1016/j.jmathb.2020.100834
Received 22 February 2020; Received in revised form 29 November 2020; Accepted 30 November 2020

following range of mathematics levels: middle school mathematics (Hohensee, 2016), grades 4–8 and 4–10 (Fernández, Llinares, Van Dooren, De Bock, & Verschaffel, 2012; Jiang, Li, Fernández, & Fu, 2017), secondary mathematics (Hohensee, Willoughby, & Gartland, 2020; Lima & Tall, 2008; Macgregor & Stacey, 1997; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2004), AP Calculus (Young, 2015), and undergraduate mathematics (Bagley, Rasmussen, & Zandieh, 2015; Moore, 2012). Despite the scarcity of backward transfer research in mathematics education, it should be noted that a well-established body of research has examined backward transfer in the context of second-language learning and definitively shown backward-transfer effects (e.g., Cook, 2003).

Backward transfer has been shown to either *enhance* or *undermine* prior ways of reasoning. An example of backward transfer that enhances prior ways of reasoning comes from Moore (2012), who found that between the time when engineering students finished their final mathematics course (Calculus III) and when they graduated, the students "exhibited deeper levels of understanding of 'function' and 'accumulation'" (p. 81). An example of backward transfer that undermines prior ways of reasoning comes from Macgregor and Stacey (1997), who found that when algebra students learned about notation that involved exponents (e.g., x^3), the new learning undermined their prior ways of reasoning about notation that involved multiplication (e.g., confusing $3x$ with x^3). The scarcity of research on backward transfer in mathematics education is problematic because it indicates that the field has not yet learned how to harness enhancing effects, nor how to counteract undermining effects.

The scarcity of research on backward transfer becomes yet more stark when considering studies whose primary focus was on backward transfer in authentic classrooms. Specifically, Moore (2012) and Young (2015) are the only two other reported studies where the main focus of the study was to examine backward transfer in authentic mathematics classrooms, both being situated in calculus classrooms. Thus, the field of mathematics education currently has little understanding about backward transfer in authentic high school algebra classrooms. This study represents beginning research efforts toward developing that understanding.

2. Theoretical framework

The theoretical framework for this study has two parts, (a) our theoretical perspective on transfer of learning, and (b) our theoretical perspective on ways of reasoning about functions. A common thread throughout our framework is the relationship between prior ways of reasoning and new learning.

2.1. *Transfer of learning*

In presenting our perspective on the transfer of learning, we first outline our perspective on transfer in general (also known as forward transfer). Then, we present our perspective on backward transfer in particular. Together, the perspectives suggest a bidirectional relationship between prior ways of reasoning and new learning experiences.

2.1.1. *Perspective on transfer of learning*

For our study, we adopted the *actor-oriented transfer* (AOT) perspective (Lobato, 2008), which defines transfer of learning as "the influence of prior experiences on learners' (actors') activity in novel situations" (Lobato, 2008, p. 437). In contrast to the AOT perspective, the traditional perspective is that transfer is about "whether people can apply something they have learned to a new problem or situation" (Bransford & Schwartz, 1999, p. 67).

We adopted the AOT perspective because this perspective counts *any and all* subjective influences by prior learning on learners' activities in novel situations as evidence of transfer, including those influences that are undermining (Lobato, 2008). In contrast, the traditional perspective on transfer typically only counts those instances when learners use "the *correct approach* in the transfer test" (Barnett & Ceci, 2002, p. 625, italics added) as evidence of transfer. In other words, from the AOT perspective, transfer is a subjective phenomenon that occurs within the learner, not an objective phenomenon in which the correct approach is predetermined by outside experts. Thus, AOT aligns well with the constructivist perspective that learners create their own knowledge (von Glasersfeld, 1995).

2.1.2. *Perspective on backward transfer*

We aligned our perspective on backward transfer with the AOT perspective on transfer by conceptualizing backward transfer as including *any and all* influences on prior ways of reasoning by new learning. Moreover, as explained above, we distinguish between backward transfer that *enhances* and that which *undermines* prior ways of reasoning. We define backward transfer that *enhances* prior ways of reasoning as when "a learner's reasoning about a previously-encountered mathematical concept becom[es] increasingly connected to the structural core of that concept" (Hohensee, 2014, p. 138). Similarly, we define backward transfer that *undermines* prior ways of reasoning as when a learner's prior ways of reasoning become less connected to the structural core of a concept.

Finally, according to our conceptualization, backward transfer involves new learning and prior ways of reasoning that pertain to different concepts. Such was the case in Young (2015), where new learning about integration influenced students' prior reasoning about differentiation, and in Van Dooren et al. (2004), where new learning about non-proportional relationships influenced students' prior ways of reasoning about proportional relationships. In our study, the new learning and prior ways of reasoning involved two different types of functions.

We make the additional point that while backward transfer *does* involve changes in individuals' prior ways of reasoning, it *does not necessarily* mean those changes will be permanent (although they could be). In other words, when learning about a new concept influences students to change their prior ways of reasoning, we would interpret that as backward transfer, regardless of whether or not the student at a later time reverted back to their original prior ways of reasoning. This aligns with our research goal to understand any and all complex ways that backward transfer occurs in authentic classrooms.

2.2. Perspective on ways of reasoning about functions

Part two of our theoretical framework focuses on two ways of reasoning about functions that were relevant for our study, namely (a) covariational reasoning, and (b) univariate reasoning. We also outline the specific functions that served as the context for our study.

2.2.1. Covariational reasoning

Covariational reasoning, which is how students reason with changes in both of a function's quantities, has been well-established as an important aspect of reasoning about functions (e.g., Saldanha & Thompson, 1998). Carlson, Jacobs, Coe, Larsen, and Hsu (2002) identified the following five empirically-based levels of covariational reasoning. L1 is reasoning with "an awareness that as one quantity changes, the other quantity also changes" (p. 361). L2 is reasoning with "an awareness of the direction of change of the output while considering changes in the input" (p. 357). L3 is reasoning with "an awareness of the amount of change of the output while considering changes in the input" (p. 357). L4 is reasoning with "an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input" (p. 357). Finally, L5 is reasoning with "an awareness of the instantaneous changes in the rate of change for the entire domain of the function" (p. 357).

For this study, we anticipated that backward transfer might influence students' levels of covariational reasoning. Furthermore, we interpreted moving up one or more levels as evidence of an enhancing effect and moving down one or more levels as evidence of an undermining effect. Thus, the Carlson et al. (2002) covariational reasoning framework informed our data collection and analysis.

2.2.2. Univariate reasoning

Research has shown that, prior to being able to reason with both of a function's quantities (i.e., reasoning covariationally), students may reason univariately (e.g., Harel, Behr, Lesh, & Post, 1994; Lobato, Ellis, & Munoz, 2003). Univariate reasoning is when students "reason with a single quantity" (Lobato & Ellis, 2010, p. 15). For example, Harel et al. found that when reasoning about the taste of an orange drink, some students only considered the volume of the drink, rather than considering both the volume and the amount of concentrate in the drink. For our study, we considered that a change toward or away from univariate reasoning could be evidence of potential backward transfer. Specifically, we interpreted moving away from univariate reasoning about linear functions toward reasoning covariationally as evidence of an enhancing effect, and moving away from covariational reasoning about linear functions toward univariate reasoning as evidence of an undermining effect.

2.2.3. Function contexts for this study

The specific functions that served as the mathematical contexts for this study were linear and quadratic functions. There were four criteria behind our decision to situate our backward transfer research in these mathematics topics. First, we wanted two mathematics topics that are well represented in high school algebra curricula, and that figure prominently in the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Second, we wanted topics that are distinct, so we could differentiate between students' ways of reasoning for each topic. Third, we wanted topics such that, across most algebra curricula, the topic covered first is the same (i.e., one of the topics typically serves as a context for prior ways of reasoning when the other serves as the context for new learning). Finally, we wanted the time between when the first and second topics are covered to be fairly short to minimize the intervening mathematics content that could influence students' prior ways of reasoning of the first topic before the second topic is covered. Linear and quadratic functions satisfy the first two criteria and, although it is possible for mathematics curricula to address linear and quadratic functions in ways that do not align well with the last two criteria, many curricula including the curriculum represented in our study do so.

2.3. Research questions

The two research questions that guided our study were the following: *After students engage in new learning in an instructional unit about quadratic functions,* (a) *What kinds of backward transfer influences on students' prior ways of covariational reasoning about linear functions are observed?*; and (b) *What conceptual connections exist between the foci of the quadratic functions lessons and any observed backward transfer influences?*

3. Methods

Because there currently exists little research on backward transfer in the context of mathematics education, an appropriate way to initiate research on backward transfer was with basic research. And as Sloane (2008) explained, a goal of basic research is to provide "the intellectual fodder, in the form of hypotheses, for more rigorous inquiries" (p. 627). Therefore, a goal of ours in conducting this study was to generate hypotheses that could drive more controlled studies in the future.

3.1. Setting and participants

Our study took place in two majority-minority urban high schools in the same district located in the Mid-Atlantic region of the United States. Approximately one-third of the students from each school received free or reduced-priced lunch. The participants in our study were 57 students from two tenth-grade algebra classes, one class from each high school. All students in each class agreed to participate. The students in the study reflected the ethnic diversity of their respective schools. The two teachers, Ms. Henry and Mr.

Anderson, had 8 years and 17 years of teaching experience, respectively. There were 24 students in Ms. Henry's class and 33 students in Mr. Anderson's class. Ms. Henry taught 70-minute periods and Mr. Anderson taught 45-minute periods, except for one 80-minute period per week. Each class had covered their linear functions unit prior to the start of our study.

3.2. Procedure

The study began with the administration of the linear functions pre-assessment to all students prior to the start of each class's quadratic functions unit. Next, four students from each class, who had been randomly selected, were interviewed to allow them to clarify, expand, and/or justify their reasoning on the pre-assessment. Then, each teacher taught their students a multi-lesson quadratic functions unit over a two- to three-week period. One research team member was always present to observe each lesson. After the unit, the linear functions post-assessment was administered to all students. Lastly, the same students who had been interviewed about their pre-assessment were interviewed about their post-assessment.

3.2.1. Assessment

Two versions, A and B, of a linear functions assessment were developed for this study. The versions varied in context and numerical values, but not in structure or mathematical intent. Students were randomly assigned to one version for their pre-assessment and the other version for their post-assessment. Because this study was basic research, we were unsure what kinds of backward transfer might be realized and so we decided to cast a wide net. Therefore, the assessment had several sections and the section evaluated for this study consisted of a four-question problem specifically designed to examine different aspects of students' levels of covariational reasoning. The other sections on the assessment were specifically designed to evaluate other aspects of reasoning about functions.[2]

The four-question covariational-reasoning problem on Version A was based on two linear graphs, set in the context of *gallons left in a gas tank* versus *miles travelled* for two different cars (see the Car Graphs in Fig. 1). The covariational-reasoning problem on Version B was also based on two linear graphs, but set in the context of *vacation days left* versus *dollars spent* for two different families planning their vacations (see the Family Vacation Graphs in Fig. 2).

On each graph, three coordinate points for a linear function were marked, labeled, and connected with a graph line. Note, that while the graphs for both versions had identical-looking negative slopes, in actuality the slopes were different, as could be determined with the labeled points. The four questions about the graphs were each specifically designed to examine a particular aspect of reasoning covariationally about the linear functions (see Table 1 for the questions and the reasoning they were designed to examine).

3.2.2. Instructional unit

The teachers in our study were not informed about the idea of backward transfer, nor did we attempt to influence their approach to teaching their quadratic functions unit in any way. Ms. Henry's and Mr. Anderson's quadratic functions units consisted of 16 lessons and 11 lessons, respectively. The adopted curriculum for both classrooms was Houghton Mifflin's *Integrated Mathematics 2*. However, neither teacher used that curriculum and instead used their own materials for the quadratic functions unit, indicating they regularly taught quadratic functions this way. In both classrooms, the quadratic functions unit followed the linear functions unit.

The formats for both classes were similar in nature. A typical lesson would progress as follows: class would begin with a warm-up involving previous material that students completed individually, followed by the teacher presenting new material using a lecture format. Next, students would work in small groups of two to six students, practicing what they had learned. Finally, the teacher would lead a whole class discussion, during which students would often be asked to share or present their answers. In both classes, the teachers were the primary mathematical authority.

Even though the instructional approaches were fairly traditional, both teachers created environments that were conducive to learning. For example, students seemed to feel comfortable asking questions. Also, they were held accountable for participating. For example, students were often asked to show their work on the board at the front of the class. Additionally, behavior disruptions were minimal. Despite these similarities, there were also differences in the instructional approaches to quadratic functions, which will be discussed later.

3.3. Data set

Our data set consisted of pre- and post-assessments, interview recordings, and classroom observation data. The pre- and post-assessment data came from the 57 students' written responses on the two assessments. Each response to a covariational-reasoning question was considered a separate data point. Therefore, we had a total of 228 potential responses that could be coded from the assessments (i.e., 57 students x four questions per assessment). However, sometimes students did not provide sufficient written responses to make it possible to determine if a change in their reasoning had occurred (e.g., a question was left blank on either the pre- or post-assessment, a numerical value was given without an accompanying explanation, etc.). The interview data were comprised of video and audio recordings of the eight randomly-selected students (four from each class). The classroom observation data consisted of detailed field notes of each lesson in both teachers' quadratic functions units.

[2] One of the other sections of the assessment examined students' view of functions, in terms of seeing functions as actions or processes (Breidenbach et al., 1992). The other section examined students' understanding of a correspondence view of functions (Confrey & Smith, 1995).

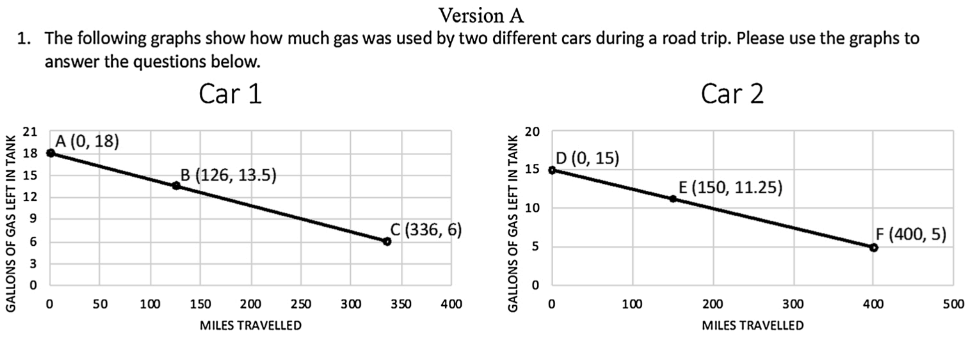


Fig. 1. Version A of the assessment (the Car Graphs).

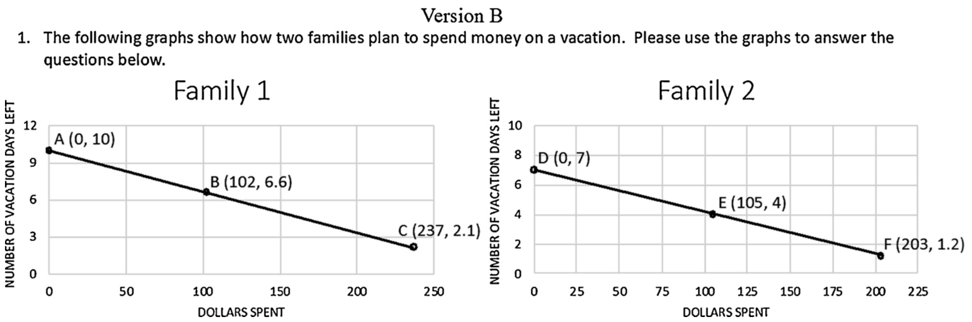


Fig. 2. Version B of the assessment (the Family Vacation Graphs).

Table 1
Descriptions of the Questions Designed to Examine Covariational Reasoning.

	Version A Car Problem Questions (see Fig. 1 for Graphs)	Version B Family Vacation Problem Questions (see Fig. 2 for Graphs)	Types of Reasoning the Questions Were Designed to Examine
(a)	For Car 1: If you drive for 182 miles, you will use ___ gallons of gas. Explain in words how you found your answer.	For Family 1: If Family 1 plans to spend $120, they will use ___ vacation days. Explain in words how you found your answer.	Reasoning based on estimating values, or reasoning covariationally to determining exact values.
(b)	For Car 2: Explain how the miles driven and the gallons of gas left in the tank change between points E and F.	For Family 2: Explain how the dollars spent and the number of vacation days left change between points E and F.	Reasoning based on the direction of changes, the magnitudes of changes, or the rate of change.
(c)	Does Car 1 use gas at the same rate between points A and B as it does between points B and C? Show all work and explain your answer.	Does Family 1 plan to spend money at the same rate between points A and B as it does between points B and C? Show all work and explain your answer.	Reasoning based on the spacing of points, the collinearity of the points, or the rates of change between points.
(d)	Using the rates of change for Car 1 and Car 2, explain which car is more fuel efficient. Show any work that helped you decide.	Using the rates of change for Family 1 and Family 2, explain which family plans to spend money at a slower rate. Show any work that helped you decide.	Reasoning based on the visual steepness of each line, the magnitudes of points on each line, or the rates of change for each line.

3.4. Data analysis

In our data analysis, we used the pre- and post-assessments as our primary data source for determining if there were changes in students' reasoning. We used the interviews as a secondary source of data to help us develop our initial coding scheme at the beginning of our analysis. Finally, we used the classroom observation data to explore what, during the quadratic functions units, might have influenced those changes.

In all, three stages of analyses were conducted: analyses of (a) assessments and interviews for the interviewed students, (b) assessments for the non-interviewed students, and (c) fieldnotes from the classroom observations. In the first stage of analysis, an initial set of codes was created by comparing written and interview responses on the four pre and post covariational-reasoning questions for the eight randomly-selected students who were interviewed. Specifically, the first author watched all interviews and read all the associated written responses for the eight interviewed students (i.e., 16 interviews in all), one question at a time, alternating between the pre- and post-assessments. During this process, the first author created a table for each interview, recording in each row the question number, a description without inference of what the students wrote and said, and any connections the first author saw to the first research question (i.e., any evidence of potential backward transfer effects from pre to post) and connections to Carlson et al.'s framework (2002). To facilitate comparisons between the pre- and post-assessment and interviews, each row of the table alternated between the pre and the post (i.e., Question (a) [pre], Question (a) [post], Question (b) [pre], Question (b) [post], and so on).

From the completed table for each student, the first author wrote two summary documents, one summary document about the four interviewed students from Ms. Henry's class, and one about the four interviewed students from Mr. Anderson's class. Each summary document explained and illustrated the categories of potential backward transfer effects that had been observed, both those that seemed certain and those that were more tentative. These categories then became the initial set of codes. These codes were informed by concepts in Carlson et al.'s framework (2002) (e.g., codes about magnitudes or direction of changes in quantities), but we did not yet code specifically for levels of covariational reasoning. The initial set of codes, and their supporting evidence, were then presented to and discussed by the research team to calibrate each team member's understanding of the codes.

In the second stage, the assessments from the 49 non-interviewed students were coded using the initial codes. The assessments were divided up among the three researchers so that each student's pre- and post-assessments were independently coded by two researchers. Codes were assigned to those problems which showed that a change in reasoning between the pre- and post-assessment had occurred. No code was assigned to a question if either the pre- or post-assessment lacked a response, or an interpretation could not be made from the given response.

The research team then met in pairs to discuss mutually-coded assessments to establish interrater reliability. When a consensus could not initially be reached between pairs of researchers, the third researcher was included in the discussion. Eventually consensus was reached on all coded responses. Throughout this process, *constant comparison* was used to refine codes and, when changes in students' reasoning were not adequately captured by the existing codes, new codes were established (Strauss & Corbin, 1994). The final set of codes was comprised of codes to capture changes in the number of quantities students reasoned with and codes to capture how reasoning changed (or did not change) between levels of Carlson et al.'s (2002) covariational reasoning framework (see Table 2 for the list of codes and see the results section for illustrations of the codes).

After the final set of codes was established, the researchers recoded the entire set of 57 assessments. The coded data were then scrutinized for patterns of changes in ways of reasoning from pre- to post-assessment that were consistent across both classes, as well as those that were unique to each class. Patterns of changes in reasoning were taken as potential evidence of backward transfer (i.e., evidence that the quadratic functions instruction had an influence on students' prior ways of reasoning about linear functions). We also reviewed each interview after having re-coded all the assessments as an additional layer of constant comparison to make sure our interview data fit our final coding scheme.

It should be noted that, when applying our codes, we counting it as changed reasoning even when students only changed their reasoning on one of the four questions. We had several reasons for this. First, because our study was exploratory, we wanted to identify all potential changes in reasoning that could subsequently be examined more directly in future studies. Second, it has been previously shown that neither increasing the number of quantities one reasons with or raising one's level of covariational reasoning are changes in reasoning that are so easily accomplished by students (e.g., Carlson et al., 2002; Lobato et al., 2003), so we viewed all such changes as potentially important. Third, we saw no examples in our data of students changing their reasoning randomly from pre to post (e.g., raising their level of covariational reasoning on one question and lowering it on another question), which added support for our decision to treat each change in reasoning as potentially important. Finally, because each question on the assessments addressed different aspects of reasoning about linear functions, we expected that backward transfer might influence a student's response to a particular question and not influence their response of a different question.

In the last stage of analysis, we analyzed the field notes to identify conceptual connections between the quadratic functions lesson foci and the linear functions backward transfer findings that could provide a partial account for why the backward transfer findings were observed. Initially, the first author read through the entire set of field notes for the two quadratic functions instructional units and created a comprehensive list of lesson foci. Second, the three-member team, who had all been observers of the lessons, met to discuss and refine the list of lesson foci until consensus on the list was reached. Third, the first author compared each lesson focus to each backward transfer finding from the written pre- and post-assessments, and ranked each lesson focus in terms of how closely the mathematical features represented in the lesson focus matched the mathematical features represented in the backward transfer findings. When there was a good match, the first author interpreted that lesson focus as conceptually connected to the backward transfer finding (i.e., they tentatively concluded that the lesson focus may partially account for the production of the backward transfer effect). The first author then presented the tentative conclusions to the other two members of the research team, for their feedback on whether, based on their knowledge of the assessments and the interview data and their observations of the lessons, the tentative conclusions seemed plausible. Making tentative conclusions such as these align with Sloane's (2008) characterization of basic

Table 2
List of Codes Related to Changes in Reasoning on the Covariational-Reasoning Questions.

Number of quantities reasoned with:	
a	Change in reasoning from 0 or 1 quantity to 2 quantities
b	Change in reasoning from 2 quantities to 0 or 1 quantity
c	Maintained reasoning with 0 or 1 quantity
d	Maintained reasoning with 2 quantities
Changes in levels of covariational reasoning (Carlson et al., 2002)*:	
a	Maintained reasoning with L2, L3, or L4
b	Change in reasoning by moving up one level (L2 to L3, L3 to L4)
c	Change in reasoning by moving down one or two levels (L3 to L1, L3 to L2, L4 to L3)

* Note that Levels 1 and 5 of Carlson et al.'s covariational reasoning framework were not observed in our study.

research.

4. Results

Findings presented here came from students' pre- and post-assessment responses to the four covariational-reasoning problems designed to examine students' level of covariational reasoning. As explained above, Version A involved the Car Graphs (see Fig. 1), and Version B involved the Family Vacation Graphs (see Fig. 2). To set the context for our findings of changes in reasoning, we first present observations about the quadratic functions instructional units that were relevant to our findings. Then, we present two main findings, regarding (a) changes in the numbers of quantities students reasoned with, and (b) changes in the level of covariational reasoning students exhibited.

4.1. Observations about the quadratic functions instructional units

As stated above, one person from the research team attended each quadratic functions lesson and took detailed fieldnotes during the entire lesson. When we examined the fieldnotes, we identified seven lesson topics for Ms. Henry and six lesson topics for Mr. Anderson (see Table 3). One relevant observation about the lessons is about the number of lessons that focused on graphs and graphing. Ms. Henry focused 9 of the 16 total lessons on graphs and graphing, with 3 of those lessons focused primarily on graphing not for solving, 2 lessons focused primarily on graphing for solving, and 4 lessons focused on modelling real-world contexts that included some graphs and graphing. In contrast, Mr. Anderson focused only 2 lessons on graphs and graphing.

This observation was relevant to our results because the four covariational-reasoning problems in our assessment involved graphs. Because Ms. Henry's students had more opportunities to reason about graphs of quadratic functions, one might anticipate greater influences by those experiences on Ms. Henry's students to change their reasoning from pre- to post-assessment. This observation was also relevant because the graphing lessons were the only lessons, with one exception, in which two quantities were focused on at the same time. Because Ms. Henry's students had more opportunities to reason with two quantities, one might anticipate there would be greater influences on Ms. Henry's students to reason with two quantities on the post-assessment.

A second observation about the lessons is that we did not find a single example in either class of a lesson focused on covariational reasoning. More specifically, we did not identify any aspect of any lesson on quadratic functions in either class that focused on how the two quantities involved in quadratic functions were changing in relation to each other. We looked for statements like *when x is getting bigger, y is getting bigger* to address the direction of how one quantity changes in relation to the other. We also looked for statements like *y is growing faster than x* to address how the change in one quantity was relatively bigger or smaller than the change in the other quantity, or statements like *as x grew by 3, y grew by 21* to address the specific amount the quantities changed between different points in the graph. However, we observed no such statements. Finally, there was also no evidence of either teacher mentioning average or instantaneous rates of change for quadratic functions.

What the teachers did talk to students a lot about was the shape of quadratic function graphs, which could have been an opportunity to address how the changes in quantities resulted in the shape. Additionally, the teachers did engage students in generating data tables for quadratic functions, which could have also been an opportunity to discuss how the two quantities were changing. However, neither of these ideas were pursued (or any others in relation to covariational reasoning).

Going into the observations, we were especially interested in any evidence of covariational reasoning and we even had a dedicated place on our observation fieldnotes protocol to record evidence of covariational reasoning. Therefore, if covariational reasoning did occur, it did so without us observing and thus must have been a rare occurrence. This was important for our results because one of our two main findings involve changes in students' level of covariational reasoning.

Table 3
Number of Lessons Per Topic and Number of Quantities Reasoned with Each Lesson.

Teacher	Lesson Topics	Number of Lessons for Each Topic	Number of Quantities Reasoned With During Lessons
Henry	Factoring not for solving quadratic equations	4	1
	Modelling quadratic equations with real-world contexts (involved graphs)	4	2
	Graphs/graphing not for solving quadratic equations	3	2
	Graphs/graphing for solving quadratic equations	2	2
	Factoring for solving quadratic equations	1.5	2
	Identifying quadratic expression forms	1	1
	Solving quadratic equations algebraically without factoring	0.5	2
Anderson	Solving quadratic equations algebraically (square rooting both sides of equation, completing the square, or quadratic formula)	5	1
	Graphs/graphing not for solving quadratic equations	2	2
	Factoring without solving quadratic equations	1	1
	Factoring to solve quadratic equations	1	1
	Modelling quadratic equations with real-world contexts	1	2
	Identifying quadratic expression forms	1	1

4.2. Changes in the numbers of quantities students reasoned with

In this section, we present evidence that students' reasoning changed in terms of the *number of quantities* they reasoned with on one of the four covariational-reasoning problems. Specifically, more students reasoned with fewer than two quantities on one assessment and with two quantities on the other assessment.

Three specific findings will be presented related to changes in the number of quantities students reasoned with. The first is a general finding across both classes. The second provides a comparison between the two classes. The third involves separate comparisons for students who reasoned with fewer than two quantities on the pre-assessment and for students who reasoned with two quantities on the pre-assessment. Before presenting the findings, we illustrate with two examples, changes in the number of quantities students reasoned with.

4.2.1. Examples illustrating changes in the number of quantities students reasoned with

We present two examples of changes in the number of quantities students reasoned with, by different students, from different classrooms, and on different problems. The first example comes from Cooper (Mr. Anderson's student), who was one of the eight interviewed students. Cooper went from reasoning with one quantity on the pre-assessment to reasoning with two quantities on the post-assessment. This example comes from corresponding parts of the two versions of the assessment that asked students to consider the Car 1 linear function graph (Version A) or the Family 1 linear function graph (Version B) and to explain whether the same rate was present between points A and B on the graph as it was between points B and C on the graph.

As can be seen in Fig. 3a, on the pre-assessment, Cooper considered the Car 1 graph and wrote "No, from A to B Car 1 drove 125 miles. From point B to C it drove 215 miles." During the interview, Cooper (C) explained his reasoning to the Interviewer (I):

> I:Can you tell me what you did for that problem?
> C:I saw how much miles it drove, and from point A to B it drove 126. Then, I solved for how much it drove from B to C, and that was about 215. So, I just said it doesn't travel at the same rate."
> I: Can you tell me again how you know that?
> C: I just subtracted C from B.

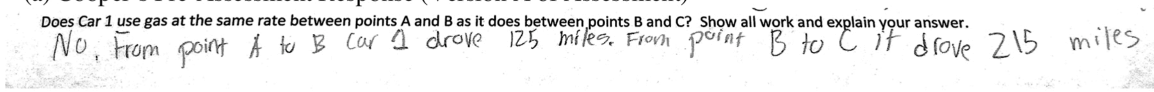
Notice that Cooper's reasoning appeared to be focused on the miles travelled from A to B and from B to C, in his written response ("Car 1 drove 125 miles") and in his interview explanation ("I saw how much miles it drove"). Cooper did not appear to factor the gallons left in the gas tank into his reasoning. Therefore, we concluded he was reasoning with one of the two quantities represented in the problem.

In contrast, on the post-assessment Cooper wrote, "From days A to B they spent \$102. From days B to C (4½ days = \$135) they spent \$135." He also completed the following calculations "6.6 – 2.1 = 4.5, \$237 – 142= \$135 and 10 – 6.6 = 3.4", and wrote "Every 3.4/5 days they spend \$102/5, .68 days = \$20.4" (see Fig. 3b).

During the interview, Cooper explained his reasoning:

> I:How were you thinking about that problem?
> C:The first thing I did for this was I found how much they spent from A to B, and they spent \$102. Then, I found out how much they spent from B to C, and I would say that they're not spending money at the same rate just looking at this, at the graph. From point A to B, which is like 3.4 days, they spent \$102. From B to C, which is like 4.5 days, they spent \$135, which is what I have. So, I think they spent more money from point A to B than point B to C.
> I:What was it, specifically, that made you say that?

(a) Cooper's Pre-Assessment Response (Version A of Assessment)


Does Car 1 use gas at the same rate between points A and B as it does between points B and C? Show all work and explain your answer.
No, From point A to B Car 1 drove 125 miles. From point B to C it drove 215 miles

(b) Cooper's Post-Assessment Response (Version B of Assessment)

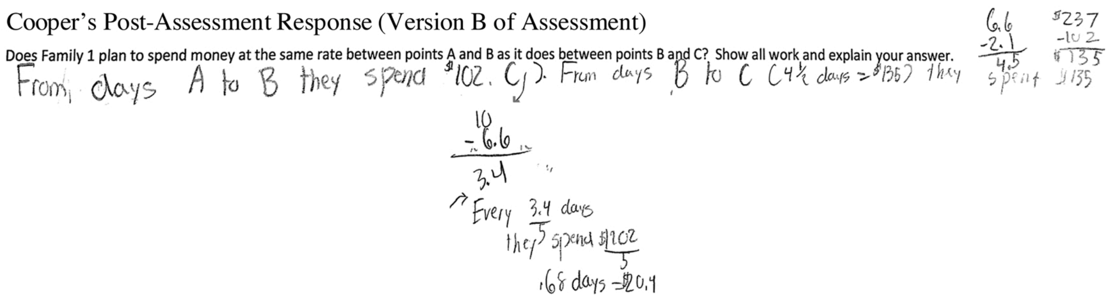

Does Family 1 plan to spend money at the same rate between points A and B as it does between points B and C? Show all work and explain your answer.
From days A to B they spend \$102. From days B to C (4½ days = \$135) they spent \$135

Fig. 3. Cooper's pre- and post-assessment responses.

C:It's like, because [pause] it's so hard to explain [pause] because A to B is a shorter time period and they spent almost the same amount.

In this post-assessment evidence, Cooper's reasoning no longer appeared solely focused on a single quantity. Instead, he factored into his reasoning both the dollars spent and the vacation days left. In his written response, Cooper referred to and calculated the days and dollars spent from A to B ("every 3.4 days they spend 102") and from B to C ("4 ½ days = $135"). In his verbal explanation he also referred to those amounts of both quantities and added "it's so hard to explain. . . because A to B is a shorter time period and they spent almost the same amount." This evidence led us to conclude that his reasoning from pre- to post-assessment had changed from reasoning with a single quantity to reasoning with two quantities, and we interpreted this as an enhancement of his reasoning.

The second example comes from Arjun (Ms. Henry's student), who was not one of the eight students we interviewed. This example comes from corresponding questions from the two versions of the assessment that asked students to consider both Car 1 and Car 2 Graphs (Version A) or both Family 1 and Family 2 Graphs (Version B) to determine which car was more fuel efficient or which family planned to spend money at a slower rate. Arjun went from reasoning with two quantities on the pre-assessment to reasoning with one quantity on the post-assessment.

As can be seen in Fig. 4a, on the pre-assessment, Arjun wrote "Car 2 is more fuel efficient. I say this because Car 2 used 3.75 gallons of gas in a distance of 150 miles while Car 1 used 4.5 gallons of gas in a distance of 126 miles. This experiment of mine proves that Car 2 was more fuel efficient." While the logic that led Arjun to the correct conclusion that Car 2 was more fuel efficient is not clear, what seemed clear to us was that he was factoring the gallons of gas left and the distance travelled into his reasoning. In other words, he was reasoning with two quantities.

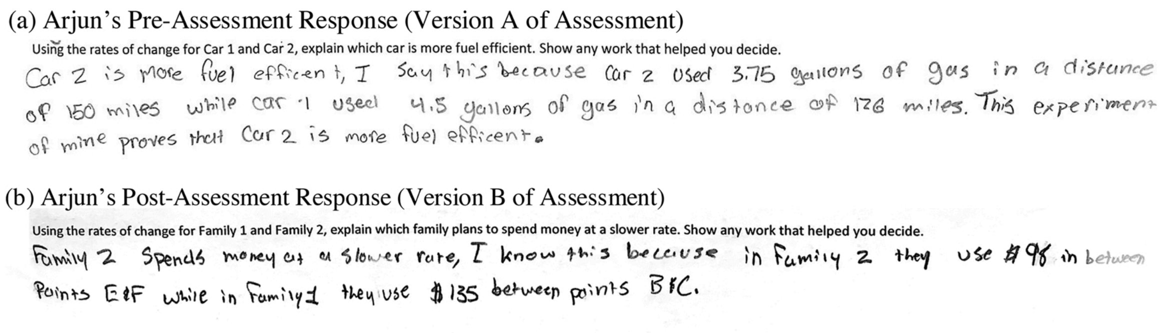
In contrast, as can be seen in Fig. 4b, on the post-assessment, Arjun wrote "Family 2 spends money at a slower rate, I [k]now this because in Family 2 they use $98 in between points E & F while in Family 1 they use $135 between points B & C." This response suggests that Arjun only reasoned with the money Family 1 and Family 2 planned to spend, and not with the vacation days left. We interpreted this change in reasoning as evidence that Arjun was now reasoning with a single quantity. We concluded that his reasoning had been somewhat undermined because, to compare the money spending rates correctly, students had to consider how the money spent and the vacation days left changed in relation to each other. We next present three findings with respect to students changing the number of quantities they reasoned with from pre- to post-assessment.

4.2.2. General finding across both classes about changes in the number of quantities students reasoned with

The first finding with respect to the number of quantities students reasoned with is that several students changed their reasoning in terms of how many quantities they reasoned with on the Car and Family Vacation covariational-reasoning problems. As shown in Table 4, 53 students provided sufficient evidence for us to interpret whether a change in the number of quantities they reasoned with had occurred. Of the 53 students, 20 exhibited this change in the number of quantities reasoned with (i.e., 12 students from Ms. Henry's class and 8 from Mr. Anderson's class). This change in reasoning reflected an important aspect of students' understanding of linear functions because as explained above, in order to reason correctly about the Car and Family Vacation problems, reasoning with two quantities was necessary.

We did consider whether, instead of being due to backward transfer influences, perhaps students arbitrarily varied the number of quantities they reasoned with from one assessment to the other. However, in most cases, students who changed their reasoning were fairly consistent within each assessment in the number of quantities they reasoned with. Also, the Car and Family Vacation problems were designed to address conceptual aspects of linear functions, they asked multiple questions about the same context, and they elicited explanations from students about their reasoning. In our experience, questions of this nature elicit how students understand, and even though understandings can change, they do not typically change randomly or on a whim (Smith, diSessa, & Roschelle, 1993). Instead, we interpreted this change in prior ways of reasoning as an indicator that something inherent in the quadratic functions content itself may have influenced students' ways of reasoning about linear function tasks. In other words, we interpreted this change in reasoning as potential backward transfer influences from the quadratics unit.

We are not asserting however that students who changed from reasoning with one to two quantities from pre- to post-assessment, fully understood on the post-assessment why reasoning with two quantities was required to solve the problem or problems, nor that students who changed from reasoning with two quantities to one, from pre- to post-assessment, no longer understood how to reason


(a) Arjun's Pre-Assessment Response (Version A of Assessment)

Using the rates of change for Car 1 and Car 2, explain which car is more fuel efficient. Show any work that helped you decide.

Car 2 is more fuel efficent, I say this because Car 2 used 3.75 gallons of gas in a distance of 150 miles while car 1 used 4.5 gallons of gas in a distonce of 126 miles. This experiment of mine proves that Car 2 is more fuel efficent.

(b) Arjun's Post-Assessment Response (Version B of Assessment)

Using the rates of change for Family 1 and Family 2, explain which family plans to spend money at a slower rate. Show any work that helped you decide.

Family 2 Spends money at a slower rate, I know this because in Family 2 they use $98 in between Points E&F while in Family 1 they use $135 between points B&C.

Fig. 4. Arjun's pre- and post-assessment responses.

Table 4
Students Who Maintained or Changed the Number of Quantities They Reasoned With.

From Pre-Assessment → Post-Assessment	Henry $n_1 = 24$	Anderson $n_2 = 33$	Total $N = 57$
Maintained (across both assessments, consistently reasoning with zero or one quantity, or consistently reasoning with 2 quantities)	17 % (4)	49 % (16)	35 % (20)
Changed (reasoned more with two quantities on the post-assessment than on the pre-assessment, or vice versa)*	50 % (12)	24 % (8)	35 % (20)
Mixed (consistently reasoned with zero or one quantity on at least one problem and with two quantities on the other problems)	25 % (6)	21 % (7)	23 % (13)
DNF (some aspect of responses to each problem on the pre- or post -assessment insufficiently answered for us to code for number of quantities reasoned with)	8% (2)	6% (2)	7% (4)

* Of the 20 students who changed their reasoning with two quantities, 12 students took Version A as the pre-assessment (7 from Henry, 5 from Anderson), and 8 students took Version B as the pre-assessment (5 from Henry, 3 from Anderson).

with two quantities. We are only asserting that the lessons on quadratic functions may have had an influence—in the former case an enhancing influence and in the latter case an undermining influence—on how students reasoned on particular problems at the particular time they took the assessment. It is the dynamics of these influences in authentic classrooms we are interested in learning more about.

4.2.3. Comparison between the two classes on changes in the number of quantities students reasoned with

The second finding is that more students changed their reasoning in Ms. Henry's than in Mr. Anderson's class. Specifically, of students who provided sufficient evidence for us to determine the number of quantities they reasoned with, 50 % of Ms. Henry's students, while only 24 % of Mr. Anderson's students changed the number of quantities they reasoned with from pre- to post-assessment (see Table 4).

We interpreted the differences in changes in reasoning between the two classes as an indication that the two teachers' approaches to teaching their respective units on quadratic functions could have had differential backward transfer influences. Ms. Henry dedicated more lessons to reasoning with two quantities (see Table 3), such as focusing more on graphing of quadratic functions than Mr. Anderson. We speculate that the extra time spent on reasoning with two quantities may have resulted in a greater influence on Ms. Henry's students' reasoning on their post-assessment responses.

However, it is important to note that more students in Mr. Anderson's class than in Ms. Henry's class were already reasoning with two quantities about linear functions on the pre-assessment. Thus, there were fewer potential students in Mr. Anderson's class who would be candidates for moving from reasoning with one quantity to reasoning with two quantities. For these reasons, we interpret this finding as partly attributable to the greater amount of graphing and reasoning with two quantities in Ms. Henry's class, and partly attributable to the different incoming abilities of the students in each class.

4.2.4. Separate considerations of students who reasoned with fewer than two quantities on the pre-assessment and of students who reasoned with two quantities on the pre-assessment

For the third finding, we separately considered the students who reasoned with fewer than two quantities on the pre-assessment and the students who reasoned with two quantities on the pre-assessment. For each group, we compared how many students maintained their reasoning to how many changed. We made this comparison to look for potential evidence of different amounts of influence on each group to change their reasoning.

There were 11 students who reasoned with fewer than two quantities on the pre-assessment (see Table 5). When we compared post-assessments for this group, we observed that all 11 students had changed to providing evidence of reasoning with two quantities on the post assessment. Thus, 100 % of the students who reasoned with fewer than two quantities on the pre-assessment, changed to reasoning with two quantities on the post-assessment.

There were 29 students who reasoned with two quantities on the pre-assessment (see Table 6). When we compared post-assessments for this group, we found that 9 students changed to reasoning with fewer than two quantities on the post assessment. Thus, 31 % of the students who reasoned with fewer than two quantities on the pre-assessment, changed to reasoning with fewer than two quantities on the post-assessment, and 69 % maintained reasoning with two quantities.

This evidence suggested to us there was potentially a stronger backward transfer influence on students who came into our study reasoning with fewer than two quantities to change to reasoning with two quantities, than on students who came into our study reasoning with two quantities to change to reasoning with fewer than two quantities. In other words, participation in the quadratic

Table 5
Comparison of Students Who Reasoned with Fewer than Two Quantities on the Pre-Assessment.

Students who reasoned with fewer than two quantities on the pre-assessment	Henry $n_1 = 7$	Anderson $n_2 = 4$	Total $n_T = 11$
Maintained reasoning with fewer than two quantities on the post-assessment	0% (0)	0% (0)	0% (0)
Changed to reasoning with two quantities on the post-assessment*	100 % (7)	100 % (4)	100 % (11)

* Of the 11 students who changed to reasoning with two quantities, 6 students took Version A as the pre-assessment, and 5 students took Version B as the pre-assessment.

Table 6
Comparison of Students Who Reasoned with Two Quantities on the Pre-Assessment.

Students who reasoned with two quantities on the pre-assessment	Henry $n_1 = 9$	Anderson $n_2 = 20$	Total $n_T = 29$
Maintained reasoning with two quantities on the post-assessment	44 % (4)	80 % (16)	69 % (20)
Changed to reasoning with fewer than two quantities on the post-assessment*	66 % (5)	20 % (4)	31 % (9)

* Of the 9 students who changed to reasoning with fewer than two quantities, 6 students took Version A as the pre-assessment, and 3 students took Version B as the pre-assessment.

functions unit may have had a stronger backward transfer influence on the first group.

Finally, we note that we did consider whether the order in which students took the assessments (i.e., pre/post = Version A/Version B or pre/post = Version B/Version A) provided a compelling explanation for why students changed the number of quantities they reasoned with. As Table 5 shows, of the 11 students who changed to reasoning with two quantities, 6 students took Version A, while the other 5 took Version B as the pre-assessment. Also, as Table 6 shows, of the 9 students who changed to reasoning with fewer than two quantities, 6 students took Version A, while the other 3 took Version B as the pre-assessment. Furthermore, when we scrutinized the students' assessment responses we did not find any additional supporting evidence that the order of the assessments was closely related to the changes in number of quantities reasoned with. We concluded that version order was not a very compelling explanation for this finding.

4.3. *Changes in students' levels of covariational reasoning*

In this section, we present evidence that some students changed how they reasoned covariationally. To establish these findings we applied codes based on Carlson et al.'s (2002) framework for levels of covariational reasoning to students' pre- and post-assessment responses (see Table 2 for the codes). To contextualize the findings and interpretations, we first illustrate Levels 2–4 because those were the most common levels of covariational reasoning students in our study exhibited (see Table 7). Students who reasoned with the direction of the changes of both quantities reasoned at L2, as illustrated by Ofilia's response. Students who reasoned with the direction *and* the magnitude of the changes of both quantities reasoned at L3, as illustrated by Ali's response. Students who reasoned with a calculated rate of change reasoned at L4, as illustrated by Wendy's response.

An analysis of students' levels of covariational reasoning on the covariational-reasoning problems revealed several trends across both classes in the ways their reasoning changed. In particular, three findings related to changes in students' levels of covariational reasoning will be presented. The first is a general finding across both classes. The second provides a comparison between the two classes. The third involves separate comparisons of students who changed their reasoning and students who did not change their reasoning. Before presenting the findings, we illustrate with two examples, how levels of covariational reasoning changed.

4.3.1. *Examples illustrating the changes in students' levels of covariational reasoning*

The first example comes from Amir (Ms. Henry's student), who was one of the eight students we interviewed, and who showed a change from L2 to L3 reasoning. When asked on the pre-assessment to explain how the dollars spent and the number of vacation days left change for Family 2 between points E and F, Amir left the problem blank. When asked about the problem in the interview, Amir (A)

Table 7
Illustrative Examples of L2-L4 Levels of Covariational Reasoning.

Level Illustrated	Illustrative Student Response Examples	Rationale for Designated Level
L2	Ofilia's L2 Covariational Reasoning For Car 2: Explain how the miles driven and the gallons of gas left in the tank change between points E and F. As car 2 drives further, the gas starts to decrease rapidly.	Reasons with direction of changes in quantities only
L3	Ali's L3 Covariational Reasoning Explain how the dollars spent and the number of vacation days left change between points E and F. As the $ went ~~down~~ up by a total of 98, the days went down by a total of 2.8	Reasons with direction *and* magnitude of changes in quantities
L4	Wendy's L4 Covariational Reasoning For Car 2: Explain how the miles driven and the gallons of gas left in the tank change between points E and F. With every mile driven, .025 of a gallon is used. At 150 miles 3.75 gallons were used. At 400, 10 gallons were used (miles).025 = gallons of gas used.	Coordinates changes in both quantities to reason with an average rate of change

said:

I:What did you think about that problem?

A:Well, so, I would say, that these numbers [points to *x*-coordinate for points E and F], they didn't double necessarily, but they kinda double. But then for the *y* [i.e., the *y*-coordinate], 4 decreases to 1.2. Even though they're like the dollars spent, it was higher, at the bottom, because it's a decreasing slope. So, yeah, therefore, the *y*, 4 and 1.2 would be decreasing as it goes.

I:And you were saying that these numbers, the first numbers there [points to *x*-coordinates for points E and F], what did you say about those?

A:They also increase.

In this excerpt, Amir noticed that the dollars spent was increasing ("kinda double. . . they also increase"), while the vacation days were decreasing ("the 'y'. . . would be decreasing"). We interpreted this as evidence that Amir was reasoning at L2 covariational reasoning.

On the corresponding post-assessment problem, which asked Amir to explain how the miles driven and the gallons of gas left in the tank for Car 2 changed between points E and F, Amir wrote, "the difference is 250 miles travels b/w (E, F) 6.25" (see Fig. 5). When asked about his response in the interview, Amir said:

I:Explain how the miles driven and the gallons of gas left in the changes between points E and F for Car 2?

A:So once I knew it said change, I was already gonna use subtraction, so I was gonna subtract E and F from each other, with the x and the y. And the difference is 250 miles travelled between the two, and then 6.25 gallons that were used. So yeah, 250 miles is the change and, well the change with the 6.25 gallons.

Here, Amir noticed the actual magnitudes of changes in miles travelled and gallons used, not just the direction of the changes. We interpreted this as evidence that Amir was reasoning at L3 covariational reasoning. Additionally, we interpreted Amir's post-assessment reasoning as an enhancement of his pre-assessment reasoning because incorporating the magnitudes of changes into one's reasoning is an important component of reasoning covariationally (Carlson et al., 2002). In other words, Amir exhibited a productive change from L2 to L3 reasoning, from pre- to post-assessment.

Our conclusion about Amir's reasoning is further supported by the inscriptions Amir made on the graphs. Whereas Amir made no inscriptions on the pre-assessment graphs, he drew arches between points and recorded changes in both quantities between points on the post-assessment (see Fig. 6). These inscriptions indicate a greater focus on magnitudes of changes in two quantities (i.e., L3 covariational reasoning).

The second example comes from Sumayah (Mr. Anderson's student, not interviewed), whose pre- and post-assessment reasoning also showed a change in covariational reasoning from L2 to L3. On the pre-assessment (Fig. 7a), Sumayah was asked if Family 1 planned to spend money at the same rate between points A and B as it did between points B and C. In her response, Sumayah identified the quantities that were changing and the direction of the changes: she wrote "In A $0 is spent and 10 vacation days is gotten, but in B $102 is spent and 6.6 vacation days, and in C $237 is spent with only 2.1 vacation days. So basically the more you spend the less amount of vaca days your getting." Although Sumayah identified the direction in which the quantities were changing ("the more you spend, the less amount of vaca days"), she did not reason with magnitudes of changes in the quantities. We coded her response as evidence of reasoning covariationally at L2 on this problem. It should also be noted that Sumayah did not directly answer the question about whether Family 1 planned to spend money from A to B and B to C at the same rate.

On the post-assessment, Sumayah was asked if Car 2 used gas at the same rate between points A and B as it does between points B and C. She recorded that the change in gallons between 18 and 13.5 gallons was 4.5 gallons, and between 13.5 gallons and 6 gallons was 7.5 gallons (see Fig. 7b), and that the change in miles between 126 miles and 336 miles was 210 miles. Also, in her written response to the prompt, she wrote, "No, at A it starts with 18 gallons, from that to B it goes 126 miles with 13.5 gallons left, using up 4.5 gallons. B to C is 210 miles with the gas usage of 7.5 gallons." Our interpretation of this response was that Sumayah was reasoning with the magnitudes of changes in the quantities. We coded her response as evidence of beginning to reason covariationally at L3. Even though Sumayah was still not correct in her conclusion, we interpreted this as a productive change in her reasoning because she started to incorporate elements of a higher-level covariational reasoning than were present on her pre-assessment.

4.3.2. General finding across both classes about changes in students' levels of covariational reasoning

For this finding, 51 students, 22 from Ms. Henry's class and 29 from Mr. Anderson's class, provided sufficient responses to determine whether or not a change in reasoning covariationally had occurred on one of the problems (see Table 8). There were 15 students in Ms. Henry's class and 21 students in Mr. Anderson's class that maintained their level of covariational reasoning from pre- to post-assessment, representing a majority of students across both classes. However, enough students exhibited a change in level to warrant a closer look at those changes. We found that, of the students who changed a level, twice as many moved up one or more levels as moved down one or more levels. Specifically, 10 students moved up at least one level, whereas 5 students moved down at least one

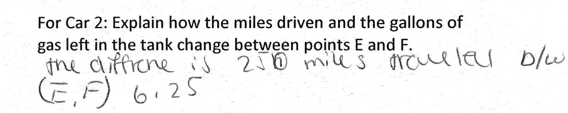


Fig. 5. Amir's L3 post-assessment responses.

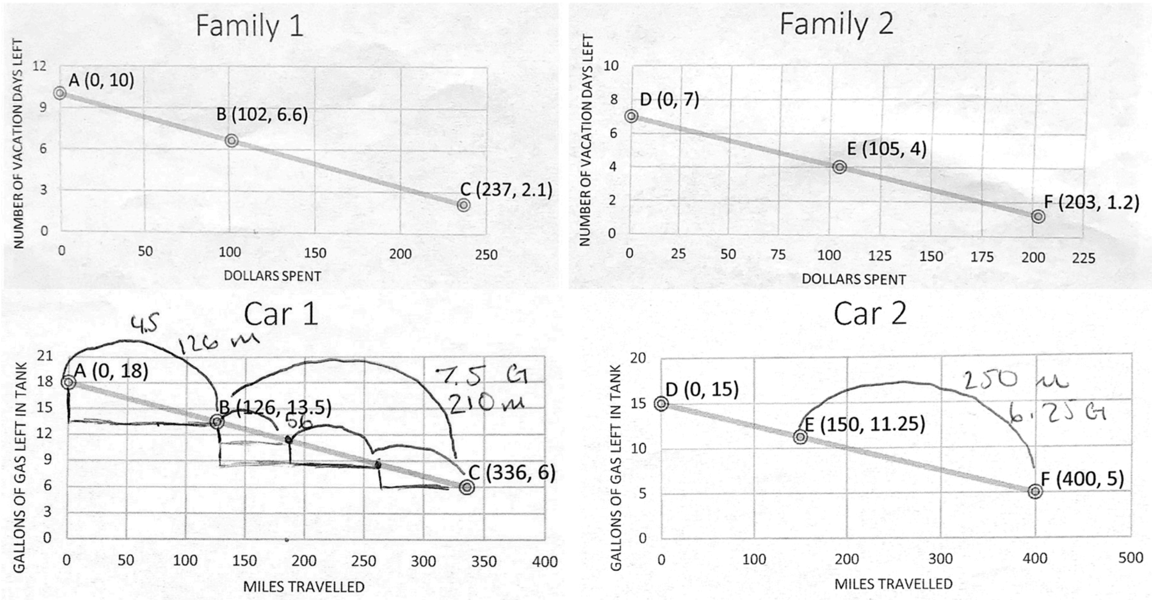
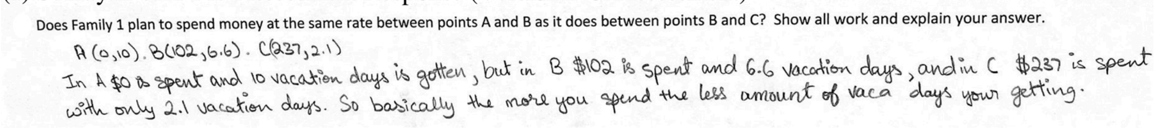


Fig. 6. Amir's inscriptions on the assessment graphs.

(a) Sumayah's L2 Pre-Assessment Response (Version B of Assessment)



(b) Sumayah's L2 Post-Assessment Response (Version A of Assessment)

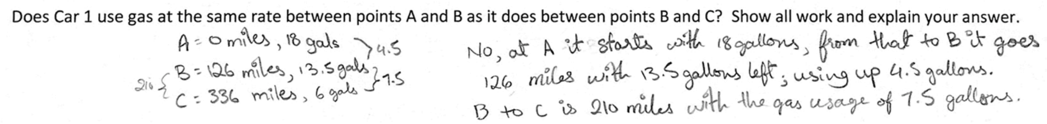


Fig. 7. Sumayah's L2 pre-assessment and L3 post-assessment responses.

Table 8
Students Maintained or Changed Their Levels of Covariational Reasoning.

From Pre-Assessment → Post-Assessment	Henry n = 24	Anderson n = 33	Total n = 57
Maintained (across both assessments, consistently reasoned with the same level of covariational reasoning)	63 % (15)	64 % (21)	63 % (36)
Change Upward (reasoning more with a higher level of covariational reasoning on the post-assessment than on the pre-assessment)*	21 % (5)	15 % (5)	18 % (10)
Change Downward (reasoning more with a higher level of covariational reasoning on the pre-assessment than on the post-assessment)**	8% (2)	9% (3)	9% (5)
DNF (some aspect of responses to each problem on the pre- or post -assessment insufficiently answered for us to code for the level of covariational reasoning)	8% (2)	12 % (4)	10 % (6)

* Of the 10 students who changed upward, 5 students took Version A as the pre-assessment and 5 students took Version B as the pre-assessment.
** Of the 5 students who changed downward, 4 students took Version A as the pre-assessment, and 1 student took Version B as the pre-assessment.

level.

We interpreted the changes in covariational reasoning as indicators of a potential backward transfer influence. In other words, when we considered changes in reasoning, such as Amir's and Sumayah's presented above, we interpreted those changes in reasoning as more than students arbitrarily changing their level of covariational reasoning from pre- to post-assessment. We have two reasons for this interpretation. First, as we explained earlier, the problems on the assessment were designed to address conceptual aspects of linear functions and asked students to explain their reasoning. Second, when students' covariational reasoning moved up a level, their responses improved. This is significant because, if a student simply changed strategies arbitrarily, it would not likely coincide with a move toward enhanced reasoning. For these reasons, we interpreted changes in covariational reasoning from pre- to post-assessment as a potential indicator of backward transfer.

As explained earlier, we cannot point to a single lesson in either class that explicitly examined covariational reasoning. However, we speculate that the lessons in which students were creating tables of values and graphing quadratic functions could have been a source of influence toward changes in covariational reasoning. We think this because, during moments when students were creating tables of values and graphing, we sometimes observed them reverting to linear patterns or expressing surprise by the characteristics of quadratic functions. This happened particularly when negative values were entered into quadratic functions, and this required that students correct and re-correct their tables and/or graphs. For example, during Lesson 2 in Mr. Albertson's class, students were graphing $x^2 - x + 6$. During this activity, there was significant confusion among students about the outputs for negative inputs, when they created a table, and surprise about where the points on the associated graph ended up being located. We suspect these moments of creating tables and graphs of quadratic functions, in which there was confusion about outputs, drew students' attention to unexpected ways the quantities were changing (in comparison to the more predictable ways linear function quantities change) that subsequently influenced some students to attend more to how quantities were changing in linear function contexts.

We did consider whether the order in which students took the assessments provided a better explanation for why students changed their level of covariational reasoning. As Table 8 shows, of the 10 students who changed to a higher level, 5 students took Version A, while the other 5 took Version B as the pre-assessment. In contrast, of the 5 students who changed to a lower level, 4 students took Version A, while the other 1 took Version B as the pre-assessment. Additionally, when we scrutinized the students' assessment responses we did not find any additional supporting evidence that the order of the assessments was closely related to the changes in number of quantities reasoned with. Therefore, we concluded that the version order was not a compelling explanation for this finding.

We again remind the reader that we do not interpret these changes in levels of covariational reasoning as necessarily permanent changes. Rather, we view the changes as potentially reflective of an influence on students' reasoning by the quadratic functions instruction, that were enhancing if the changes went from lower to higher levels, and undermining if the changes went from higher to lower levels. These influences may or may not be permanent, but regardless, we view the findings as revealing about the sometimes subtle (and sometimes not so subtle) dynamics at play between students' prior knowledge and their new learning.

4.3.3. Comparison between the two classes about changes in students' level of covariational reasoning

The second finding regarding covariational reasoning is that students in both Ms. Henry's and Mr. Anderson's classes, who were located in different high schools, changed in a similar manner from pre- to post-assessment. Looking back at Table 8, it shows that across both classes the majority of students maintained the same level of covariational reasoning (i.e., 63 % for Ms. Henry's class and 64 % for Mr. Anderson's class) and similar percentages of students changed their level of covariational reasoning (i.e., 29 % for Ms. Henry's class and 24 % for Mr. Anderson's class).

Again, we interpreted this finding to be more than students arbitrarily changing their level of covariational reasoning from pre- to post-assessment. We interpret the fact that the two classes were similar on this finding as an indication that something about exposure to quadratic functions, even in different instructional contexts, can influence students' levels of covariational reasoning in particular ways. On the one hand, the nearly identical changes across classes may indicate the two teachers did something that similarly influenced both classes. However, as stated above, none of the lesson foci in either class directly addressed covariational reasoning.

4.3.4. Separate comparisons of students who moved up one or more levels in covariational reasoning and students who moved down one or more levels

For the third finding, we only considered the 10 students who moved up one or more levels of covariational reasoning from pre- to post-assessment and the 5 students who moved down one or more levels. Our finding here is that of the changes that occurred, most ended at L3 covariational reasoning. Specifically, 4 students in each class moved up to L3 covariational reasoning between the pre- and post-assessment, while 1 student in each class moved down to L3 covariational reasoning (see Table 9). The 10 students who changed to L3 covariational reasoning represent 67 % of the students who changed their covariational reasoning from pre- to post-assessment. Also, Table 9 further highlights that changes in covariational reasoning were similar across both classes.

We also separately considered the 36 students who maintained the level of covariational reasoning they reasoned with from pre- to post-assessment. We found that, of the students who maintained their level of reasoning, more students, 16 of the 36 students across both classes, maintained L3, while fewer students maintained L2 and L4 (i.e., 11 and 9 students, respectively; see Table 10). We think

Table 9
Students Who Changed Their Level of Covariational Reasoning.

From Pre-Assessment → Post-Assessment	Henry n = 7	Anderson n = 8	Total n = 15
L1→ L3 or L2 → L3	27 % (4)	27 % (4)	54 % (8)
L2 → L4 or L3 → L4	7% (1)	7% (1)	13 % (2)
Total Moved Up	33 % **(5)**	33 % **(5)**	67 % (10)
L3 → L1 or L3 → L2	7% (1)	14 % (2)	20 % (3)
L4 → L3	7% (1)	7% (1)	13 % (2)
Total Moved Down	14 % **(2)**	20 % **(3)**	33 % (5)

this is significant because, if changing levels of covariation had been arbitrary, then given that more students started out in the pre-assessment at L3 than L2, we would have anticipated more students to change from L3 to L2 than from L2 to L3. The fact that there were more students who maintained at L3 than L2, and that more students changed from L2 to L3 than vice versa, suggests to us that there was a potential backward transfer influence in the direction of moving students from L2 to L3 covariational reasoning.

5. Discussion

This study produced the first known findings about how reasoning about linear functions by students in authentic classrooms changes after they learn about quadratic functions. We showed that many students changed the number of quantities they reasoned with. Furthermore, more students in Ms. Henry's class than in Mr. Anderson's class changed the number of quantities they reasoned with, and more students changed in the direction of reasoning with two quantities than changed in the other direction. We also showed that a number of students changed their level of reasoning on Carlson et al.'s (2002) covariational reasoning framework from pre- to post-assessment, that there were similar results across both classes on this dimension, and that most of the changes resulted in students reasoning at Level 3. Next, we discuss the significance of these findings and implications for research and practice, in terms of understanding backward transfer in general, and understanding backward transfer in the context of covariational reasoning specifically.

5.1. Significance and implications for understanding backward transfer in general

As stated earlier, this study represents a beginning effort by the field of mathematics education to understand the complexities of backward transfer. We view aspects of our findings as significant for gaining an understanding of those complexities in at least three ways. First, we think it is significant that some of our findings of changes and non-changes in students' linear function reasoning showed similar patterns across both classes (see Tables 5, 8, and 9). To us, this suggests that the quadratic functions instruction, regardless of the teacher, had particular kinds of influences on students' prior ways of reasoning about linear functions. This informs our understanding of backward transfer because it suggests that instruction in specific mathematics topics may be associated with particular backward transfer influences.

Second, we also think it is significant that some changes in reasoning were different across the two classes (see Tables 4 and 6). To us, this suggests that the unique features of each class's instructional unit on quadratic functions may have played a different role in the kind of influences students experienced. This is significant for our understanding of backward transfer because it suggests that certain instructional approaches, not just mathematics topics, may be associated with particular backward transfer influences.

Finally, we think it is significant that the changes we observed in students' ways of reasoning about linear functions in authentic classrooms were unplanned (i.e., they occurred without explicit attempts in the lessons to produce those changes). Moreover, it is significant that some unplanned changes were enhancing (e.g., moving up levels of covariational reasoning), while others were undermining (e.g., changing from reasoning with two quantities to reasoning with fewer than two quantities). This is significant because it provides an existence proof that it is possible that enhancing and undermining unplanned backward transfer is being produced in authentic classrooms. An implication for researchers is that research is needed to investigate how extensively, and for what mathematics topics, unplanned backward transfer is being produced in authentic classrooms.

An implication for teachers is that perhaps teachers should be made aware of the potential that their instruction may be producing unplanned enhancing and/or unplanned undermining backwards transfer effects. With that awareness, teachers might be in a position to identify when their instruction is having an enhancing or undermining influence on students' prior ways of reasoning and then, intentionally attempt to heighten or inhibit those effects, respectively. For example, a teacher could have a discussion with students to make explicit how an aspect of instruction may be having an enhancing or undermining influence on their prior ways of reasoning, or teachers could modify lessons to maximize enhancing influences and/or minimize undermining influences.

5.2. Significance and implications for understanding backward transfer in the context of covariational reasoning

We also view aspects of our findings as significant more specifically for understanding backward transfer in the context of covariational reasoning. The evidence indicated that more students exhibited enhanced abilities to reason covariationally about linear functions on the post-assessment, either by changing from lower to higher levels of covariational reasoning or by changing from reasoning with fewer than two quantities to two quantities. This is significant because it suggests that quadratic functions instruction could be used to leverage students towards higher levels of covariational reasoning about linear functions. On the other hand, our evidence also showed that a smaller group of students' prior ways of reasoning were undermined. An implication for researchers is that research is needed to better understand how to promote enhancing influences and inhibit undermining influences that quadratic functions instruction can have on students' levels of covariational reasoning about linear functions.

An implication for teachers is that perhaps when students are learning about quadratic functions, teachers should be watchful for students who might be experiencing undermining influences on their level of covariational reasoning about linear functions. For example, teachers could use formative assessments to monitor which students' abilities to reason covariationally are being undermined by the quadratic functions instruction and then provide targeted interventions for those identified students (e.g., special activities, small-group tutorials, etc.).

A second implication for teaching is that perhaps teachers should create activities that revisit covariational reasoning about linear functions after quadratic functions instruction. Specifically, teachers could consider backward transfer effects from quadratic functions instruction as new information that is used to revise their *hypothetical learning trajectories* (HTL) about furthering their students'

Table 10
Students Who Maintained Their Levels of Covariational Reasoning.

From Pre-Assessment → Post-Assessment	Henry n = 15	Anderson n = 21	Total n = 36
L2 → L2	33 % (5)	30 % (6)	31 % (11)
L3 → L3	53% (8)	40 % (8)	46 % (16)
L4 → L4	13 % (2)	30 % (7)	23 % (9)

understanding of covariational reasoning about linear functions (i.e., HTLs are the teacher's "learning goal, the learning activities, and the thinking and learning in which students might engage…to strengthen their understanding;" Simon, 1995, pp. 133–134). Then, they could use their revised HTL to revisit covariational reasoning about linear functions. The goal would be to reinforce enhancing influences that quadratic functions instruction may have had (e.g., promoting reasoning with two rather than one quantity, and reasoning with higher rather than lower levels of covariational reasoning), while the influences are fresh.

Finally, it is significant that most of the changes in covariational reasoning involved changes that ended at L3. This suggests that to influence students with quadratic functions instruction to move from L2 to L3 in linear functions contexts is easier than to influence students to move from L3 to L4. Thus, some extra level of intentionality on the part of the teacher and/or in the quadratic functions lessons may be required to achieve the latter. An implication for research would be to examine the kinds of instructional moves during quadratic functions instruction that support students in moving to L4 in linear function contexts.

An implication for teaching is that, even though exploring L4 covariational reasoning about quadratic functions (i.e., average rates of change) may not be a typical mathematical topic at beginning levels, a reason to address average rates of change at this level of quadratic functions instruction would be to help students move from L3 to L4 reasoning in linear function contexts. For example, during a quadratic functions unit a teacher could ask students to compare how x and y are changing at various points on the function and then lead a discussion about what the comparisons say about the quadratic function. This could be important because, as our study shows, many students still are not reasoning at L4 in linear function contexts after instruction on linear functions.

6. Conclusion

This study set out to examine how students' prior ways of reasoning about linear functions are influenced in authentic classrooms when they participate in a quadratic functions unit. We found that a number of students' ways of reasoning changed in terms of the numbers of quantities and the levels of covariational reasoning they reasoned with. This study and its findings provide new insights to the field about the characteristics of backward transfer in general, as well as about characteristics of backward transfer as they pertain to covariational reasoning about linear functions specifically. Our hope is that these findings further the understanding of backward transfer in the context of mathematics education.

CRediT authorship contribution statement

Charles Hohensee: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Writing - original draft, Writing - review & editing, Project administration, Funding acquisition. **Sara Gartland:** Methodology, Validation, Formal analysis, Investigation, Writing - original draft, Writing - review & editing. **Laura Willoughby:** Methodology, Validation, Formal analysis, Investigation, Writing - original draft, Writing - review & editing. **Matthew Melville:** Writing - original draft, Writing - review & editing.

Acknowledgments

The research reported in this article was funded by the National Science Foundation (DRL 1651571).

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