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# PROJECTIVITY OF THE MODULI SPACE OF STABLE LOG-VARIETIES AND SUBADDITVITY OF LOG-KODAIRA DIMENSION

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## Contents



## 1. Summary of results

Throughout the article we are working over an algebraically closed base field $k$ of characteristic zero. The main result of is the following.

**Theorem 1.1** (=Corollary 7.3). *Any algebraic space that is the coarse moduli space of a moduli functor of stable log-varieties with fixed volume, dimension and coefficient set (as defined in Definition 6.2) is a projective scheme over $k$.*

In the non-logarithmic case an essentially equivalent statement was proved in [Kol90] and [Fuj12], but in the logarithmic case a similar result has only been known in dimension 1, that is, this is a new result for moduli spaces of log-varieties of dimension at least 2. Of course, it also proves projectivity of $\overline{\mathscr{M}}_{g,n}$, although admittedly it is not an efficient proof in that case.

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In Section 6.A we also present one particular functor satisfying the above condition, based on a functor suggested by Kollár [Kol13a, §6]. In particular, the above result is not vacuous.

Kollár's proof of the non-logarithmic case is based on his celebrated Ampleness Lemma. The naive application of the Ampleness Lemma in the logarithmic case yields that for a family $f:(X,D)\to Y$ of stable log-varieties, the functorial line bundle $\det f_*\mathscr{O}_X(m(K_{X/Y}+D))$ is big provided that the ambient varieties $X_y$ of the fibers have maximal variation (here $m$ is sufficiently divisible). However, a family of log-varieties may have positive variation even if the ambient variety stays the same for general fibers, since the boundary divisors can still change. This simple roadblock has been a formidable obstacle in proving this result for many years. By generalizing the Ampleness Lemma (see Section 5) and then using a somewhat complex argument (see Section 2.A and Section 7) we prove that in fact the line bundle $\det f_*\mathscr{O}_X(m(K_{X/Y}+D))$ is big if the pairs $(X_y, D_y)$ vary maximally, even if the ambient varieties $X_y$ do not (see point (2) of Theorem 7.1). This result provides the main ingredient of the proof of Theorem 1.1.

For certain applications, e.g., Theorem 1.3, one needs to understand the positivity of $f_*\mathscr{O}_X(m(K_{X/Y}+D))$, that is, the direct image of the relative pluricanonical sheaf before taking the determinant. Positivity of this sheaf is much stronger than that of its determinant. Our second main result is concerned with this:

**Theorem 1.2** (=Theorem 8.1). *If $f:(X,D)\to Y$ is a family of stable log-varieties of maximal variation over a normal, projective variety $Y$ with a klt general fiber, then $f_*\mathscr{O}_X(r(K_{X/Y}+D))$ is big for every sufficiently divisible integer $r>0$.*

This is a direct generalization of [Kol87] and [EV90, Thm 3.1] to the logarithmic case. Note that it fails without the klt assumption, see Examples 8.5, 8.6, 8.7.

Theorem 1.2 yields numerous applications, including ampleness of the CM line bundle on the moduli space of stable varieties by Patakfalvi and Xu [PX15], and a log-version of [Abr97] by Ascher and Turchet [AT16]. We also prove Theorem 1.2 and our other positivity results over almost projective bases in Section 10, that is, over bases that are big open sets in projective varieties.

For us the main importance of Theorem 1.2 is its application to the Iitaka-Viehweg conjecture on subadditivity of log-Kodaira dimension. We prove this conjecture assuming that the general fiber is of log general type in Theorem 9.5. This generalizes to the logarithmic case the celebrated results of Kawamata, Viehweg and Kollár on the subadditivity of Kodaira dimension [Kaw81, Kaw85, Vie83a, Vie83b, Kol87], also known as Iitaka's conjecture $C_{n,m}$ and its strengthening by Viehweg, known as $C^+_{n,m}$. For the strongest statement we are proving the reader is referred to Section 9. Here we only state two corollaries that need less preparation.

**Theorem 1.3** (=Theorem 9.6 and Corollary 9.8).

*(1) Let $f:(X,D)\to(Y,B)$ be a surjective map of projective log canonical snc pairs such that $\operatorname{supp} D\supseteq \operatorname{supp} f^*B$. Assume that $K_{X_\eta}+D_\eta$ is big, where $\eta$ is the generic point of $Y$, and that either both $B$ and $D$ are reduced or $D\geq f^*B$, then*

$$\kappa(K_X+D)\geq \kappa(K_Y+B)+\kappa\left(K_{X_\eta}+D_\eta\right).$$

*(2) Let $f : X \to Y$ be a dominant map of (not necessarily proper) algebraic varieties such that the generic fiber has maximal Kodaira dimension. Then*

$$\kappa(X) \geq \kappa(Y) + \kappa(X_\eta).$$

In the non-logarithmic case recent related results include [Bir09, CH11, Fuj03, Lai11, Fuj13, CP16]. In the logarithmic case Fujino obtained results similar to the above assuming that the base has maximal Kodaira dimension [Fuj14a, Thm 1.7], or that the family is relative 1-dimensional [Fuj15]. Recently, Cao and Păun covered the case when the base is an abelian variety.

In a slightly different direction Fujino obtained another related result on subadditivity of the *numerical* log-Kodaira dimension [Fuj14b]. A version of the latter, under some additional assumptions, had been proved by Nakayama [Nak04, V.4.1]. The numerical log-Kodaira dimension is expected to be equal to the usual log-Kodaira dimension by the Abundance Conjecture. However, that conjecture is arguably one of the most difficult open problems in birational geometry currently. Our proof does not use either the Abundance Conjecture or the notion of numerical log-Kodaira dimension.

Further note that our proof of Theorem 1.3 is primarily algebraic. That is, we obtain our positivity results, from which Theorem 1.3 is deduced, in a purely algebraic way, starting from the semi-positivity results of Fujino [Fuj12, Fuj14a]. Hence, our approach has a good chance to be portable to positive characteristic when the appropriate semi-positivity results (and other ingredients such as the minimal model program) become available in that setting. See [Pat14] for the currently available semi-positivity results, and [CZ13, Pat16b] for results on subadditivity of Kodaira-dimension in positive characteristic.

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## 2. OVERVIEW

Since Mumford's seminal work on the subject, $\mathscr{M}_g$, the moduli space of smooth projective curves of genus $g \geq 2$, has occupied a central place in algebraic geometry and the study of $\mathscr{M}_g$ has yielded numerous applications. An important aspect of the applicability of the theory is that these moduli spaces are naturally contained as open sets in $\overline{\mathscr{M}}_g$ the moduli space of stable curves of genus $g$, and the fact that this latter space admits a projective coarse moduli scheme.

Even more applications stem from the generalization of this moduli space, $\mathscr{M}_{g,n}$, the moduli space of $n$-pointed smooth projective curves of genus $g$ and its projective compactification, $\overline{\mathscr{M}}_{g,n}$, the moduli space of $n$-pointed stable curves of genus $g$.

It is no surprise that after the success of the moduli theory of curves huge efforts were devoted to develop a similar theory for higher dimensional varieties. However, the methods used in the curve case, most notably GIT, proved inadequate for the higher dimensional case. Gieseker [Gie77] proved using GIT that the moduli space of smooth projective surfaces of general type is quasi-projective, but the proof did not provide a modular projective compactification. Subsequently several people wrote down examples of natural limits of families of surfaces of general type that were not asymptotically stable (e.g., [SB83]). This led to the natural question, first asked by Kollár, whether asymptotically stable limits of surfaces of general type exits at all. Recently Wang and Xu [WX14] showed that in fact such limits do not exist in general, at least when one considers asymptotic Chow-stability and the standard linearization on the Chow variety.

The right definition of stable surfaces only emerged after the development of the minimal model program allowed bypassing the GIT approach [KSB88]. The existence and projectivity of the moduli space of stable surfaces and higher dimensional varieties have only been proved very recently as the combined result of the effort of several people over many years [KSB88, Kol90, Ale94, Vie95, HK04, AH11, Kol08, Kol13a, Kol13b, Fuj12, HMX14, Kol14].

Naturally, one would also like to have a higher dimensional analogue of $n$-pointed curves and to extend the existing results to that case [Ale96]. The obvious analogue of an $n$-pointed smooth projective curve is a smooth projective log-variety, that is, a pair $(X, D)$ consisting of a smooth projective variety $X$ and a simple normal crossing divisor $D \subseteq X$. For reasons originating in the minimal model theory of higher dimensional varieties, one would also like to allow some mild singularities of $X$ and $D$ and fractional coefficients in $D$, but we will defer the discussion of the precise definition to a later point in the paper (see Definition 3.9). We note here that the introduction of fractional coefficients for higher dimensional pairs led Hassett to go back to the case of $n$-pointed curves and study a weighted version in [Has03]. These moduli spaces are more numerous and have greater flexibility than the traditional ones. In fact, they admit natural birational transformations among each other and demonstrate the workings of the minimal model program in concrete highly non-trivial examples. Furthermore, the log canonical models of these moduli spaces of weighted stable curves may be considered to approximate the canonical model of $\overline{\mathscr{M}}_{g,n}$ [HH09, HH13].

The theory of *moduli of stable log-varieties*, also known as *moduli of semi-log canonical models* or *KSBA stable pairs*, which may be regarded as the higher dimensional analogues of Hassett's moduli spaces above, is still very much in the making. It is clear what a stable log-variety should be: the correct class (for surfaces) was identified in [KSB88] and further developed in [Ale96]. This notion generalizes to arbitrary dimension [Kol13a]. On the other hand, at the time of the writing of this article it is not entirely obvious what the right definition of the corresponding moduli functor is over non reduced bases. For a discussion of this issue we refer to [Kol13a, §6]. A major difficulty is that in higher dimensions when the coefficients of $D$ are not all greater than $1/2$ a deformation of a log-variety cannot be simplified to studying deformations of the ambient variety $X$ and then deformations of the divisor $D$. An example of this phenomenon, due to Hassett, is presented in Section 2.B, where a family $(X, D) \to \mathbb{P}^1$ of stable log varieties is given such that $D \to \mathbb{P}^1$ does not form a flat family of pure codimension 1 subvarieties. In fact, the flat limit $D_0$ acquires an embedded point, or equivalently, the scheme theoretic restriction of $D$ onto a fiber is not equal to the divisorial restriction. Therefore, in the moduli functor of stable log-varieties one should allow both deformations that acquire and also ones that do not acquire embedded points on the boundary divisors. This is easy to phrase over nice (e.g., normal) bases see Definition 6.2 for details. However, at this point it is not completely clear how it should be presented in more intricate cases, such as for instance over a non-reduced base. Loosely speaking the optimal infinitesimal structure of the moduli space is not determined yet (see Remark 6.15 for a discussion on this), although there are also issues about the implementation of labels or markings on the components of the boundary divisor (cf. Remark 7.17).

By the above reasons, several functors have been suggested, but none of them emerged yet as the obvious "best". However, our results apply to any moduli functor for which the objects are stable log-varieties (see Definition 6.2 for the precise condition on the functors). In particular, our results apply to any moduli space that is sometimes called a *KSBA compactification* of the moduli space of log-canonical models.

As mentioned Mumford's GIT method used in the case of moduli of stable curves does not work in higher dimensions and so we study the question of projectivity in a different manner. The properness of any algebraic space as in Theorem 1.1 is shown in [Kol14]. For the precise statement see Proposition 6.4. Hence, to prove projectivity over $k$ one only has to exhibit an ample line bundle on any such algebraic space. Variants of this approach have been already used in [Knu83, Kol90, Has03]. Generalizing Kollár's method to our setting [Kol90], we use the polarizing line bundle $\det f_* \mathscr{O}_X(r(K_{X/Y} + D))$, where $f : (X, D) \to Y$ is a stable family and $r > 0$ is a sufficiently divisible integer. Following Kollár's idea using the Nakai-Moishezon criterion it is enough to prove that this line bundle is big for a maximal variation family over a normal base. However, Kollár's Ampleness Lemma [Kol90, 3.9,3.13] is unfortunately not strong enough for our purposes and hence we prove a stronger version in Theorem 5.1. There, we also manage to drop an inconvenient condition on the stabilizers from [Kol90, 3.9,3.13], which is not necessary for the current application, but we hope will be useful in the future. Applying Theorem 5.1 and some other arguments outlined in Section 2.A we prove that the above line bundle is big in Theorem 7.1.

### 2.A. ***Outline of the proof***

As mentioned above, using the Nakai-Moishezon criterion for ampleness, Theorem 1.1 reduces to the following statement (= Proposition 7.16): given a family of stable log-varieties $f : (X, D) \to Y$ with maximal variation over a smooth, projective variety, $\det f_* \mathscr{O}_X(q(K_{X/Y} + D))$ is big for every sufficiently divisible integer $q > 0$. This follows relatively easily from the bigness of $K_{X/Y} + D$. To be precise it also follows from the bigness of the log canonical divisor $K_{X^{(r)}/Y} + D_{X^{(r)}}$ of some large enough fiber power for some integer $r > 0$ (see Notation 3.12 and the proof of Proposition 7.16). In fact, one cannot expect to do better for higher dimensional bases, see Remark 7.2 for details. Here we review the proof of the bigness of these relative canonical divisors, going from the simpler cases to the harder ones.

2.A(i). *The case* $\dim Y = 1$ *and* $\dim X = 2$. In this situation, roughly speaking, we have a family of weighted stable curves as defined by Hassett [Has03]. The only difference is that in our notion of a family of stable varieties there is no marking (that is, the points are not ordered). This means that the marked points are allowed to form not only sections but multisections as well. However, over a finite cover of $Y$ these multisections become unions of sections, and hence we may indeed assume that we have a family of weighted stable curves. Denote by $s_i : Y \to X$ $(1, \dots, m)$ the sections given by the marking and let $D_i$ be the images of these sections. Hassett proved projectivity [Has03, Thm 2.1, Prop 3.9] by showing that the following line bundle is ample:

$$\det f_* \mathscr{O}_X(r(K_{X/Y} + D)) \otimes \left( \bigotimes_{i=1}^{m} s_i^* \mathscr{O}_X(r(K_{X/Y} + D)) \right). \tag{2.A.1}$$

Unfortunately, this approach does not work for higher dimensional fibers. This is demonstrated in the example of Section 2.B, where the sheaves corresponding to $s_i^* \mathscr{O}_X(r(K_{X/Y} + D))$ which are the same as $(f|_{D_i})_* \mathscr{O}_{D_i}(r(K_{X/Y} + D)|_{D_i})$ are not functorial in higher dimensions. In fact, the function

$$y \mapsto h^0 \left( (D_i)_y , \mathscr{O}_{(D_i)_y} \left( r(K_{X/Y} + D)|_{(D_i)_y} \right) \right)$$

jumps down in the limit in the case of example of Section 2.B, which means that there is no possibility to collect the corresponding space of sections on the fibers into a direct image sheaf. Note that here it is important that $(D_i)_y$ means the divisorial restriction of $D_i$ onto $X_y$. Indeed, with the scheme theoretic restriction there would be no jumping down, since $D_i$ is flat as a scheme over $Y$. However, the scheme theoretic restriction of $D_i$ onto $X_y$ contains an embedded point and therefore the space of sections on the divisorial restriction is one less dimensional than on the scheme theoretic restriction.

So, the idea is to try to prove the ampleness of $\det f_* \mathscr{O}_X(r(K_{X/Y} + D))$ in the setup of the previous paragraph, hoping that that argument would generalize to higher dimensions. Assume that $\det f_* \mathscr{O}_X(r(K_{X/Y} + D))$ is not ample. Then by the ampleness of (2.A.1), for some $1 \leq i \leq m$, $s_i^* \mathscr{O}_X(r(K_{X/Y} + D))$ must be ample. Therefore, for this value of $i$, $D_i \cdot (K_{X/Y} + D) > 0$. Furthermore, by decreasing the coefficients slightly, the family is still a family of weighted stable curves. Hence $K_{X/Y} + D - \varepsilon D_i$ is nef for every $0 \leq \varepsilon \ll 1$ (see Lemma 7.7, although this has been known by other methods for curves). Putting these two facts together yields that

$$(K_{X/Y} + D)^2 = \underbrace{(K_{X/Y} + D) \cdot (K_{X/Y} + D - \varepsilon D_i)}_{\geq 0, \text{ because } K_{X/Y}+D \text{ and } K_{X/Y}+D-\varepsilon D_i \text{ are nef}} + \underbrace{(K_{X/Y} + D) \cdot \varepsilon D_i}_{>0} > 0.$$

This proves the bigness of $K_{X/Y} + D$, and the argument indeed generalizes to higher dimensions as explained below.

2.A(ii). *The case where* $\dim Y = 1$ *and* $\dim X$ *is arbitrary.* Let $f : (X, D) \to Y$ be an arbitrary family of non-isotrivial stable log-varieties over a smooth projective curve. Let $D_i$ $(i = 1, \dots, m)$ be the union of the divisors (with reduced structure) of the same coefficient (cf. Definition 7.4). The argument in the previous case suggests that the key is to obtain an inequality of the form

$$\left( (K_{X/Y} + D)|_{D_i} \right)^{\dim D_i} > 0. \tag{2.A.2}$$

Note that it is considerably harder to reach the same conclusion from this inequality, than in the previous case, because the $D_i$ are not necessarily $\mathbb{Q}$-Cartier and then $(X, D - \varepsilon D_i)$ might not be a stable family. To remedy this issue we pass to a $\mathbb{Q}$-factorial dlt-blowup. For details see Lemma 7.13.

Let us now turn to how one might obtain (2.A.2). First, we prove using our generalization (see Theorem 5.1) of the Ampleness Lemma a higher dimensional analogue of (2.A.1) in Proposition 7.8, namely, that the following line bundle is ample:

$$\det f_* \mathscr{O}_X(r(K_{X/Y}+D)) \otimes \left( \bigotimes_{i=1}^{m} \det (f|_{D_i})_* \mathscr{O}_{D_i}(r(K_{X/Y}+D)|_{D_i}) \right). \tag{2.A.3}$$

The main difference compared to (2.A.1) is that $f|_{D_i}$ is no longer an isomorphism between $D_i$ and $Y$ as it was in the previous case where the $D_i$ were sections. In fact, $D_i \to Y$ has positive dimensional fibers and hence $\mathscr{E}_i := (f|_{D_i})_* \mathscr{O}_{D_i}(r(K_{X/Y}+D)|_{D_i})$ is a vector bundle of higher rank. As before, if $\det f_* \mathscr{O}_X(r(K_{X/Y}+D))$ is not ample, then for some $i$, $\det \mathscr{E}_i$ has to be. However, since $\mathscr{E}_i$ is higher rank now, it is not as easy to obtain intersection theoretic information as earlier.

As a result one has to utilize a classic trick of Viehweg which leads to working with fibered powers. Viehweg's trick uses the fact that there is an inclusion

$$\det \mathscr{E}_i \hookrightarrow \bigotimes_{j=1}^{d} \mathscr{E}_i, \tag{2.A.4}$$

where $d := \operatorname{rk} \mathscr{E}_i$. Here the latter sheaf can be identified with a direct image sheaf from the fiber product space $D_i^{(d)} \to Y$ (see Notation 3.12). This way one obtains that

$$\left( \left(K_{X^{(d)}/Y} + D_{X^{(d)}}\right)\big|_{D_i^{(d)}} \right)^{\dim D_i^{(d)}} > 0,$$

from which it is an easy computation to prove (2.A.2)

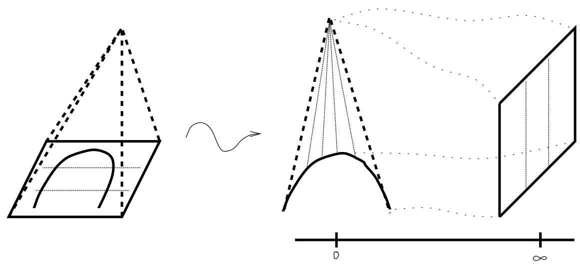
2.A(iii). *The case where both* $\dim Y$ *and* $\dim X$ *are arbitrary.* We only mention briefly what goes wrong here compared to the previous case, and what the solution is. The argument is very similar to the previous case until we show that (2.A.3) is big. However, it is no longer true that if $\det f_* \mathscr{O}_X(r(K_{X/Y}+D))$ is not big, then one of the $\det \mathscr{E}_i$ is big. So, the solution is to treat all the sheaves at once via an embedding as in (2.A.4) of the whole sheaf from (2.A.3) into a tensor-product sheaf that can be identified with a direct image from an appropriate fiber product (see (7.12.1)). The downside of this approach is that one then has to work on $X^{(l)}$ for some big $l$, but we still obtain an equation of the type (2.A.2), although with $D_i$ replaced with a somewhat cumbersome subvariety of fiber product type.

After that an enhanced version of the previous arguments yields that $K_{X^{(l)}/Y} + D_{X^{(l)}}$ is big on at least one component, which is enough for our purposes. In fact, in this case we cannot expect that $K_{X/Y}+D$ would be big on any particular component, cf. Remark 7.2. However, the bigness of $K_{X^{(l)}/Y}+D^{(l)}$ on a component already implies the bigness of $\det f_* \mathscr{O}_X(r(K_{X/Y}+D))$ (see Proposition 7.16). This argument is worked out in Section 7.

2.A(iv). *Subadditivity of log-Kodaira dimension.* First we prove Theorem 1.2 in Section 8 using ideas originating in the works of Viehweg. This implies that although in Section 7 we were not able to prove the bigness of $K_{X/Y}+D$ (only of $K_{X^{(l)}/Y}+D^{(l)}$), it actually does hold for stable families of maximal variation with klt general fibers (cf. Corollary 8.3). Then with a comparison process (see the proof of Theorem 9.9) of an arbitrary log-fiber space $f' : (X', D') \to Y'$ and of the image in moduli of the log-canonical model of its generic fiber, we are able to obtain enough positivity of $K_{X'/Y'} + D'$ to deduce subadditivity of log-Kodaira dimension if the log-canonical divisor of the general fiber is big.

### 2.B. *An important example*

The following example is due to Hassett (cf. [Kol13a, Example 42]), and has been referenced at a couple of places in the introduction.




Let $\mathscr{X}$ be the cone over $\mathbb{P}^1 \times \mathbb{P}^1$ with polarization $\mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 1)$ and let $\mathscr{D}$ be the conic divisor $\frac{1}{2} p_2^* P + \frac{1}{2} p_2^* Q$, where $p_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection to the second factor, and $P$ and $Q$ are general points. Let $H_0$ be a cone over a hyperplane section $C$ of $\mathbb{P}^1 \times \mathbb{P}^1$ with the given polarization, and $H_\infty$ a general hyperplane section of $\mathscr{X}$ (which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$). Note that since $\deg \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,1)|_C = 4$, $H_0$ is a cone over a rational normal curve of degree 4. Let $f : \mathscr{H} \to \mathbb{P}^1$ be the pencil of $H_0$ and $H_\infty$. It is naturally a subscheme of the blowup $\mathscr{X}'$ of $\mathscr{X}$ along $H_0 \cap H_\infty$. Furthermore, the pullback of $\mathscr{D}$ to $\mathscr{X}'$ induces a divisor $\mathscr{D}'$ on $\mathscr{H}$, such that

(1) its reduced fiber over 0 is a cone over the intersection of $\frac{1}{2} p_2^* P + \frac{1}{2} p_2^* Q$ with $C$, that is, over 4 distinct points on $\mathbb{P}^1$ with coefficients $\frac{1}{2}$, and
(2) its fiber over $\infty$ is two members of one of the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ with coefficients $\frac{1}{2}$. In the limit both of these lines degenerate to a singular conic, and they are glued together at their singular points.

In case the reader is wondering how this is relevant to stable log-varieties of general type, we note that this is actually a local model of a degeneration of stable log-varieties, but one can globalize it by taking a cyclic cover branched over a large enough degree general hyperplane section of $\mathscr{X}$. For us only the local behaviour matters, so we will stick to the above setup. Note that since $\chi(\mathscr{O}_{\mathscr{D}'_\infty}) = 2$, the above described reduced structure cannot agree with the scheme theoretic restriction $\mathscr{D}'_{0,\mathrm{sch}}$ of $\mathscr{D}'$ over 0, since then $\chi(\mathscr{O}_{\mathscr{D}'_{0,\mathrm{sch}}}) = 1$ would hold. Therefore $\mathscr{D}'_{0,\mathrm{sch}}$ is non-reduced at the cone point. Furthermore, note that the log canonical divisor of $(\mathscr{X}, \mathscr{D})$ is the cone over a divisor corresponding to $\mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}\left(-2+2, -2+1+\frac{1}{2}+\frac{1}{2}\right) \simeq \mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$. In particular, this log canonical class is $\mathbb{Q}$-Cartier, and hence $(\mathscr{H}, \mathscr{D}')$ does yield a local model of a degeneration of stable log-varieties.

### 2.C. *Organization*

We introduce the basic notions on general and on almost proper varieties in Section 3 and Section 4. In Section 5 we state our version of the Ampleness Lemma. In Section 6 we define moduli functors of stable log-varieties and we also give an example of a concrete moduli functor for auxiliary use. Section 7 contains the proof of Theorem 1.1 as well as of the necessary positivity of $\det f_* \mathscr{O}_X(r(K_{X/Y} + D))$. Section 8 is devoted to the proof of Theorem 1.2. Section 9 contains the statements and the proofs of the subadditivity statements including Theorem 1.3. Finally, in Section 10 we shortly deduce almost projective base versions of the previously proven positivity statements.

## 3. Basic tools and definitions

We will be working over an algebraically closed base field $k$ of characteristic zero in the entire article. In this section we give those definitions and auxiliary statements that are used in multiple sections of the article. Most importantly we define stable log-varieties and their families here.

**Definition 3.1.** A *variety* will mean a reduced but possibly reducible separated scheme of finite type over $k$. A *vector bundle* $W$ on a variety $Z$ in this article will mean a locally free sheaf. Its dual is denoted by $W^*$.

*Remark* 3.2. It will always be assumed that the support of a divisor does not contain any irreducible component of the conductor subscheme. Obviously this is only relevant on non-normal schemes. The theory of Weil, Cartier, and $\mathbb{Q}$-Cartier divisors works essentially the same on *demi-normal* schemes, i.e., on schemes that satisfy Serre's condition $S_2$ and are semi-normal and Gorenstein in codimension 1. For more details on demi-normal schemes and their properties, including the definition and basic properties of divisors on demi-normal schemes see [Kol13b, §5.1].

**Definition 3.3.** Let $Z$ be a scheme. A *big open subset* $U$ of $Z$ is an open subset $U \subseteq Z$ such that $\operatorname{depth}_{Z\setminus U} \mathscr{O}_Z \geq 2$. If $Z$ is $S_2$, e.g., if it is normal, then this is equivalent to the condition that $\operatorname{codim}_Z(Z \setminus U) \geq 2$.

**Definition 3.4.** The dual of a coherent sheaf $\mathscr{F}$ on a scheme $Z$ will be denoted by $\mathscr{F}^*$ and the sheaf $\mathscr{F}^{**}$ is called the *reflexive hull* of $\mathscr{F}$. If the natural map $\mathscr{F} \to \mathscr{F}^{**}$ is an isomorphism, then $\mathscr{F}$ is called *reflexive*. For the basic properties of reflexive sheaves see [Har80, §1].

Let $Z$ be an $S_2$ scheme and $\mathscr{F}$ a coherent sheaf on $Z$. Then the *reflexive powers* of $\mathscr{F}$ are the reflexive hulls of tensor powers of $\mathscr{F}$ and are denoted the following way:

$$\mathscr{F}^{[m]} := \left(\mathscr{F}^{\otimes m}\right)^{**}$$

Obviously, $\mathscr{F}$ is reflexive if and only if $\mathscr{F} \simeq \mathscr{F}^{[1]}$. Let $\mathscr{G}$ be coherent sheaf on $Z$. Then the *reflexive product* of $\mathscr{F}$ and $\mathscr{G}$ (resp. reflexive symmetric power of $\mathscr{F}$) is the reflexive hull of their tensor product (resp. of the symmetric power of $\mathscr{F}$) and is denoted the following way:

$$\mathscr{F}[\otimes]\mathscr{G} := (\mathscr{F} \otimes \mathscr{G})^{**} \qquad \operatorname{Sym}^{[a]}(\mathscr{F}) := (\operatorname{Sym}^a(\mathscr{F}))^{**}$$

**Notation 3.5.** Let $f : X \to Y$ and $Z \to Y$ be morphisms of schemes. Then the base change to $Z$ will be denoted by

$$f_Z : X_Z \to Z,$$

where $X_Z := X \times_Y Z$ and $f_Z := f \times_Y \operatorname{id}_Z$. If $Z = \{y\}$ for a point $y \in Y$, then we will use $X_y$ and $f_y$ to denote $X_{\{y\}}$ and $f_{\{y\}}$.

**Lemma 3.6.** *Let $f : X \to Y$ and $g : Z \to Y$ be surjective morphisms such that $Y$ is normal and let $\mathscr{L}$ and $\mathscr{N}$ be line bundles on $X$ and $Z$ respectively. Assume that there is a big open set of $Y$ over which $X$ and $Z$ are flat and $f_*\mathscr{L}$ and $g_*\mathscr{N}$ are* *locally free. Then*

$$((g \circ p_Z)_*(p_X^* \mathscr{L} \otimes p_Z^* \mathscr{N}))^{**} \simeq f_* \mathscr{L} \, [\otimes] \, g_* \mathscr{N}.$$

*Furthermore, if $X$ and $Z$ are flat and $f_*\mathscr{L}$ and $g_*\mathscr{N}$ are locally free over the entire $Y$, then the above isomorphism is true without taking reflexive hulls.*

*Proof.* Since the statement is about reflexive sheaves, we may freely pass to big open sets. In particular, we may assume that $f$ and $g$ are flat and $f_*\mathscr{L}$ and $g_*\mathscr{N}$ are locally free. Then

$$(g \circ p_Z)_*(p_X^* \mathscr{L} \otimes p_Z^* \mathscr{N}) \underset{\text{projection formula for } p_Z}{\simeq} g_*\left((p_Z)_* p_X^* \mathscr{L} \otimes \mathscr{N}\right) \underset{\text{flat base-change}}{\simeq} g_*\left(g^* f_* \mathscr{L} \otimes \mathscr{N}\right) \underset{\text{projection formula for } g}{\simeq} f_* \mathscr{L} \otimes g_* \mathscr{N}.$$

□

**Notation 3.7.** Let $f : X \to Y$ be a flat equidimensional family of demi-normal schemes, and $Z \to Y$ a morphism between normal varieties. Then for a $\mathbb{Q}$-divisor $D$ on $X$ that avoids the generic and codimension 1 singular points of the fibers of $f$, we will denote by $D_Z$ the *divisorial pull-back* of $D$ to $X_Z$, which is defined as follows: As $D$ avoids the singular codimension 1 points of the fibers, there is a big open set $U \subseteq X$ such that $D|_U$ is $\mathbb{Q}$-Cartier. Clearly, $U_Z$ is also a big open set in $X_Z$ and we define $D_Z$ to be the unique divisor on $X_Z$ whose restriction to $U_Z$ is $(D|_U)_Z$.

*Remark* 3.8. Note that this construction agrees with the usual pullback if $D$ itself is $\mathbb{Q}$-Cartier, because the two divisors agree on $U_Z$.

Also note that $D_Z$ is not necessarily the (scheme theoretic) base change of $D$ as a subscheme of $X$. In particular, for a point $y \in Y$, $D_y$ is not necessarily equal to the scheme theoretic fiber of $D$ over $y$. The latter may contain smaller dimensional embedded components, but we restrict our attention to the divisorial part of this scheme theoretic fiber. This issue has already come up multiple times in Section 1, in particular in the example of Section 2.B.

Finally, note that if $q(K_{X/Y} + D)$ is Cartier, then using this definition the line bundle $\mathscr{O}_X(q(K_{X/Y} + D))$ is compatible with base-change, that is, for a morphism $Z \to Y$,

$$\left(\mathscr{O}_X(q(K_{X/Y} + D))\right)_Z \simeq \mathscr{O}_Z(q(K_{X_Z/Z} + D_Z)).$$

To see this, recall that this holds over $U_Z$ by definition and both sheaves are reflexive on $Z$. (See Definition 3.10 for the precise definition of $K_{X/Y}$.)

**Definition 3.9.** A *pair* $(Z, \Gamma)$ consist of an equidimensional demi-normal variety $Z$ and an effective $\mathbb{Q}$-divisor $\Gamma \subset Z$. An *snc pair* (or *log-smooth log-variety*) is a pair $(Z, \Gamma)$ such that $Z$ is smooth and $\operatorname{supp} \Gamma$ is a simple normal crossing divisor. Notice that for an snc pair we are not placing any bounds on the coefficients of the boundary divisor $\Gamma$. A *stable log-variety* $(Z, \Gamma)$ is a pair such that

(1) $Z$ is proper,
(2) $(Z, \Gamma)$ has *slc singularities*, and
(3) the $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $K_Z + \Gamma$ is *ample*.

For the definition of slc singularities the reader is referred to [Kol13b, 5.10]

**Definition 3.10.** Let $f : X \to Y$ be a dominant morphism of relative dimension $d$. If $f$ is either

(1) a flat projective family of equidimensional demi-normal varieties, or
(2) a surjective morphism between normal projective varieties,

then $\omega_{X/Y}$ is defined to be $h^{-d}(f^! \mathscr{O}_Y)$. In particular, if $Y$ is Gorenstein (e.g., $Y$ is smooth), then $\omega_{X/Y} \simeq \omega_X \otimes f^* \omega_Y^{-1}$. In any case, $\omega_{X/Y}$ is a reflexive sheaf (c.f., [PS14, Lemma 4.9]) of rank 1. Furthermore, if either in the first case $Y$ is also normal or in the second case $Y$ is smooth, then $\omega_{X/Y}$ is trivial at the codimension one points, and hence it corresponds to a Weil divisor that avoids the singular codimension one points [Kol13b, 5.6]. This divisor can be obtained by fixing a big open set $U \subseteq X$ over which $\omega_{X/Y}$ is a line bundle, and hence over which it corresponds to a Cartier divisor, and then extending this Cartier divisor to the unique Weil-divisor extension on $X$. Note that in the first case $U$ can be chosen to be the relative Gorenstein locus of $f$, and in the second case the regular locus of $X$. Furthermore, in the first case, we have $K_{X/Y}|_V \sim K_{X_V/V}$ for any $V \to Y$ base-change from a normal variety (here restriction is taken in the sense of Notation 3.7).

**Definition 3.11.** A *family of stable log-varieties*, $f : (X, D) \to Y$ over a normal variety consists of a pair $(X, D)$ and a flat proper surjective morphism $f : X \to Y$ such that

(1) $D$ avoids the generic and codimension 1 singular points of every fiber,
(2) $K_{X/Y} + D$ is $\mathbb{Q}$-Cartier, and
(3) $(X_y, D_y)$ is a connected stable log-variety for all $y \in Y$.

**Notation 3.12.** For a morphism $f : X \to Y$ of schemes and $m \in \mathbb{N}_+$, define

$$X_Y^{(m)} := \bigtimes_{1}^{m}{}_{Y} X = \underbrace{X \times_Y X \times_Y \cdots \times_Y X}_{m \text{ times}},$$

and let $f_Y^{(m)} : X_Y^{(m)} \to Y$ be the induced natural map. For a sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ define

$$\mathscr{F}_Y^{(m)} := \bigotimes_{i=1}^{m} p_i^* \mathscr{F},$$

where $p_i$ is the $i$-th projection $X_Y^{(m)} \to X$. Similarly, if $f$ is flat, equidimensional with demi-normal fibers, then for a divisor $\Gamma$ on $X$ define

$$\Gamma_{X_Y^{(m)}} := \sum_{i=1}^{m} p_i^* \Gamma,$$

a divisor on $X_Y^{(m)}$.

Finally, for a subscheme $Z \subseteq X$, $Z_Y^{(m)}$ is naturally a subscheme of $X_Y^{(m)}$. Notice however that if $m > 1$ and $Z$ has positive codimension in $X$, then $Z_Y^{(m)}$ is never a divisor in $X_Y^{(m)}$. In particular, if $Y$ is normal, $f$ is flat, equidimensional and has demi-normal fibers, and $\Gamma$ is an effective divisor that does not contain any generic or singular codimension 1 points of the fibers of $f$, then

$$\left(\Gamma_Y^{(m)}\right)_{\mathrm{red}} = \left(\bigcap_{i=1}^{m} p_i^* \Gamma\right)_{\mathrm{red}}. \tag{3.12.1}$$

Notice the difference between $\Gamma_{X_Y^{(m)}}$ and $\Gamma_Y^{(m)}$. The former corresponds to taking the $(m)^{\text{th}}$ box-power of a divisor as a sheaf, while the latter to taking fiber power as a subscheme. In particular,

$$\mathscr{O}_{X_Y^{(m)}}(\Gamma_{X_Y^{(m)}}) \simeq (\mathscr{O}_X(\Gamma))_Y^{(m)},$$

while $\Gamma_Y^{(m)}$ is not even a divisor if $m > 1$.

In most cases, we omit $Y$ from the notation. I.e., we use $X^{(m)}$, $\Gamma_{X^{(m)}}$, $\Gamma^{(m)}$, $f^{(m)}$ and $\mathscr{F}^{(m)}$ instead of $X_Y^{(m)}$, $\Gamma_{X_Y^{(m)}}$, $\Gamma_Y^{(m)}$, $f_Y^{(m)}$ and $\mathscr{F}_Y^{(m)}$, respectively.

## 4. Almost proper varieties and big line bundles

**Definition 4.1.** An *almost proper* variety is a variety $Y$ that admits an embedding as a big open set into a proper variety $Y \hookrightarrow \overline{Y}$. If $Y$ is almost proper, then a *proper closure* will mean a proper variety with such an embedding. The proper closure is not unique, but also, obviously, an almost proper variety is not necessarily a big open set for an arbitrary embedding into a proper (or other) variety. An almost proper variety $Y$ is called *almost projective* when it has a proper closure $\overline{Y}$ which is projective. Such a proper closure will be called a *projective closure*.

**Lemma 4.2.** *Let $Y$ be an almost projective variety of dimension $n$ and $B$ a Cartier divisor on $Y$. Then there exists a constant $c > 0$ such that for all $m > 0$*

$$h^0(Y, \mathscr{O}_Y(mB)) \leq c \cdot m^n$$

*Proof.* Let $\iota : Y \hookrightarrow \overline{Y}$ be a projective closure of $Y$ and set $\mathscr{B}_m = \iota_* \mathscr{O}_Y(mB)$. Let $\mathscr{H}$ be a very ample invertible sheaf on $\overline{Y}$ such that $H^0(\overline{Y}, \mathscr{H} \otimes (\mathscr{B}_1)^*) \neq 0$ where $(\mathscr{B}_1)^*$ is the dual of $\mathscr{B}_1$. It follows that there exists an embedding $\mathscr{O}_Y(B) \hookrightarrow \mathscr{H}|_Y$ and hence for all $m > 0$ another embedding $\mathscr{O}_Y(mB) \hookrightarrow \mathscr{H}^m|_Y$. Pushing this forward to $\overline{Y}$ one obtains that $\mathscr{B}_m \subseteq \iota_* \mathscr{H}^m|_Y \simeq \mathscr{H}^m$. Note that the last isomorphism follows by the condition of $Y$ being almost projective/proper, that is, because $\operatorname{depth}_{\overline{Y} \setminus Y} \mathscr{O}_{\overline{Y}} \geq 2$. Finally this implies that

$$h^0(Y, \mathscr{O}_Y(mB)) = h^0(\overline{Y}, \mathscr{B}_m) \leq h^0(\overline{Y}, \mathscr{H}^m) \sim c \cdot m^n,$$

where the last inequality follows from [Har77, I.7.5]. □

**Definition 4.3.** Let $Y$ be an almost proper variety of dimension $n$. A Cartier divisor $B$ on $Y$ is called *big* if $h^0(Y, \mathscr{O}_Y(mB)) > c \cdot m^n$ for some $c > 0$ constant and $m \gg 1$ integer. A line bundle $\mathscr{L}$ is called *big* if the associated Cartier divisor is big.

**Lemma 4.4.** *Let $Y$ be an almost proper variety of dimension $n$ and $\iota : Y \hookrightarrow \overline{Y}$ a projective closure of $Y$. Let $\overline{B}$ be a Cartier divisor on $\overline{Y}$ and denote its restriction to $Y$ by $B = \overline{B}|_Y$. Then $B$ is big if and only if $\overline{B}$ is big.*

*Proof.* Clear from the definition and the fact that $\iota_* \mathscr{O}_Y(mB) \simeq \mathscr{O}_{\overline{Y}}(m\overline{B})$ for every $m \in \mathbb{Z}$. □

*Remark* 4.5. Note that it is generally not assumed that $B$ extends to $\overline{Y}$ as a Cartier divisor.

**Lemma 4.6.** *Let $Y$ be an almost projective variety of dimension $n$ and $B$ a Cartier divisor on $Y$. Then the following are equivalent:*

*(1) $mB \sim A + E$ where $A$ is ample and $E$ is effective for some $m > 0$,*

*(2) the rational map $\phi_{|mB|}$ associated to the linear system $|mB|$ is birational for some $m > 0$,*
*(3) the projective closure of the image $\phi_{|mB|}$ has dimension $n$ for some $m > 0$, and*
*(4) $B$ is big.*

*Proof.* The proof included in [KM98, 2.60] works almost verbatim. We include it for the benefit of the reader since we are applying it in a somewhat unusual setup.

Clearly, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. To prove (3) $\Rightarrow$ (4), let $T = \overline{\phi_{|mB|}(Y)} \subseteq \mathbb{P}^N$. By assumption $\dim T = n$, so by [Har77, I.7.5] the Hilbert polynomial of $T$ is $h^0(T, \mathscr{O}_T(l)) = (\deg T/n!) \cdot l^n +$ (lower order terms). By definition of the associated rational map $\phi_{|mB|}$ induces an injection $H^0(T, \mathscr{O}_T(l)) \subseteq H^0(Y, \mathscr{O}_Y(lmB))$, which proves (3) $\Rightarrow$ (4).

To prove (4) $\Rightarrow$ (1), let $B$ be a Cartier divisor on $Y$ and let $\iota : Y \hookrightarrow \overline{Y}$ be a projective closure of $Y$. Further let $\overline{A}$ be a general member of a very ample linear system on $\overline{Y}$. Then $A := \overline{A} \cap Y$ is an almost projective variety by [Fle77, 5.2]. It follows by Lemma 4.2 that $h^0(A, \mathscr{O}_A(mB|_A)) \leq c \cdot m^{n-1}$, which, combined with the exact sequence

$$0 \to H^0(Y, \mathscr{O}_Y(mB - A)) \to H^0(Y, \mathscr{O}_Y(mB)) \to H^0(A, \mathscr{O}_A(mB|_A)),$$

shows that if $B$ is big, then $H^0(Y, \mathscr{O}_Y(mB - A)) \neq 0$ for $m \gg 0$ which implies (1) as desired. □

The notion of weak-positivity used in this article is somewhat weaker than that of [Vie95]. The main difference is that we do not require being global generated on a fixed open set for every $b > 0$ in the next definition. This is a minor technical issue and proofs of the basic properties work just as for the definitions of [Vie95], after disregarding the fixed open set. The reason why this weaker form is enough for us is that we use it only as a tool to prove bigness, where there is no difference between our definition and that of [Vie95].

**Definition 4.7.** Let $X$ be a normal, almost projective variety and $\mathscr{H}$ an ample line bundle on $X$.

(1) A coherent sheaf $\mathscr{F}$ on $X$ is weakly-positive, if for every integer $a > 0$ there is an integer $b > 0$, such that $\mathrm{Sym}^{[ab]}(\mathscr{F}) \otimes \mathscr{H}^b$ is generically globally generated. Note that this does not depend on the choice of $\mathscr{H}$ [Vie95, Lem 2.14.a].
(2) A coherent sheaf $\mathscr{F}$ on $X$ is big if there is an integer $a > 0$ such that $\mathrm{Sym}^{[a]}(\mathscr{F}) \otimes \mathscr{H}^{-1}$ is generically globally generated. This definition also does not depend on the choice of $\mathscr{H}$ by a similar argument as for the previous point. Furthermore, this definition is compatible with the above definition of bigness for divisors and the correspondence between divisors and rank one reflexive sheaves.

**Lemma 4.8.** *Let $X$ be a normal, almost projective variety, $\mathscr{F}$ a weakly-positive and $\mathscr{G}$ a big coherent sheaf. Then*

*(1) $\bigoplus_{i=1}^{a} \mathscr{F}$, $\mathrm{Sym}^{[a]}(\mathscr{F})$, $\left[\bigotimes_{i=1}^{a}\right] \mathscr{F}$, $\det \mathscr{F}$ are weakly-positive,*
*(2) generically surjective images of $\mathscr{F}$ are weakly-positive, and those of $\mathscr{G}$ are big,*

*(3) if $\mathscr{A}$ is an ample line bundle, then $\mathscr{F} \otimes \mathscr{A}$ is big, and*
*(4) if $\mathscr{G}$ is of rank 1, then $\mathscr{F}[\otimes]\mathscr{G}$ is big.*

*Proof.* Let us fix an ample line bundle $\mathscr{H}$. (1) follows verbatim from [Vie95, 2.16(b) and 2.20], and (2) follows immediately from the definition. Indeed, given generically surjective morphisms $\mathscr{F} \to \mathscr{F}'$ and $\mathscr{G} \to \mathscr{G}'$, there are generically surjective morphisms $\operatorname{Sym}^{[ab]}(\mathscr{F}) \otimes \mathscr{H}^b \to \operatorname{Sym}^{[ab]}(\mathscr{F}') \otimes \mathscr{H}^b$ and $\operatorname{Sym}^{[a]}(\mathscr{G}) \otimes \mathscr{H}^{-1} \to \operatorname{Sym}^{[a]}(\mathscr{G}') \otimes \mathscr{H}^{-1}$ proving the required generic global generation.

To prove (3), take an $a > 0$, such that $\mathscr{A}^a \otimes \mathscr{H}^{-1}$ is effective and $\mathscr{A}^c$ is very ample for $c > a$. Then for a $b > a$ such that $\operatorname{Sym}^{[3b]}(\mathscr{F}) \otimes \mathscr{A}^b$ is globally generated, the embedding

$$\begin{aligned}\operatorname{Sym}^{[3b]}(\mathscr{F}) \otimes \mathscr{A}^b \hookrightarrow \operatorname{Sym}^{[3b]}(\mathscr{F}) \otimes \mathscr{A}^{3b-a} \simeq \\ \simeq \operatorname{Sym}^{[3b]}(\mathscr{F} \otimes \mathscr{A}) \otimes \mathscr{A}^{-a} \hookrightarrow \operatorname{Sym}^{[3b]}(\mathscr{F} \otimes \mathscr{A}) \otimes \mathscr{H}^{-1}\end{aligned}$$

is generically surjective which implies the statement.

To prove (4) take an $a$, such that $\mathscr{H}^{-1} \otimes \mathscr{G}^{[a]}$ is generically globally generated. This corresponds to a generically surjective embedding $\mathscr{H} \to \mathscr{G}^{[a]}$. According to (1) and (3), $\left(\operatorname{Sym}^{[a]}(\mathscr{F}) \otimes \mathscr{H}\right)$ is big. Hence, by (2), $\operatorname{Sym}^{[a]}(\mathscr{F}) \otimes \mathscr{G}^{[a]} \simeq \operatorname{Sym}^{[a]}(\mathscr{F}[\otimes]\mathscr{G})$ is also big. Therefore, for some $b > 0$, $\operatorname{Sym}^{[b]}(\operatorname{Sym}^{[a]}(\mathscr{F}[\otimes]\mathscr{G})) \otimes \mathscr{H}^{-1}$ is generically globally generated and then the surjection

$$\operatorname{Sym}^{[b]}(\operatorname{Sym}^{[a]}(\mathscr{F}[\otimes]\mathscr{G})) \to \operatorname{Sym}^{[ab]}(\mathscr{F}[\otimes]\mathscr{G})$$

concludes the proof. □

## 5. Ampleness Lemma

**Theorem 5.1.** *Let $W$ be a weakly-positive vector bundle of rank $w$ on a normal almost projective variety $Y$ over the field $k$ with a reductive structure group $G \subseteq \operatorname{GL}(k,w)$ the closure of the image of which in the projectivization $\mathbb{P}(\operatorname{Mat}(k,w))$ of the space of $w \times w$ matrices is normal. Further let $Q_i$ be vector bundles of rank $q_i$ on $Y$ admitting generically surjective homomorphisms $\alpha_i : W \to Q_i$ for $i = 0, \dots, n$ and $\lambda = \bigtimes_{i=0}^{n} \lambda(\alpha_i) : Y(k) \to \bigtimes_{i=0}^{n} \operatorname{Gr}(w, q_i)(k)/G(k)$ the induced classifying map of sets. Assume that $\lambda$ has finite fibers on a dense open set of $Y$. Then $\bigotimes_{i=0}^{n} \det Q_i$ is big.*

*Remark* 5.2. One way to define the above classifying map $\lambda$ is to choose a basis on every fiber of $W$ over every closed point up to the action of $G(k)$. For this it is enough to fix a basis on one fiber of $W$ over a closed point, and transport it around using the $G$-structure. In fact, a little less is enough. Given a basis, multiplying every basis vector by an element of $k^{\times}$ does not change the corresponding rank $q$ quotient space, and hence the classifying map, so we only need to fix a basis up to scaling by an element of $k^{\times}$. To make it easier to talk about these in the sequel we will call a basis which is determined up to scaling by an element of $k^{\times}$ a *homogeneous basis.*

*Remark* 5.3. The normality assumption in Theorem 5.1 is satisfied if $W = \operatorname{Sym}^d V$ with $v := \operatorname{rk} V$ and $G := \operatorname{GL}(k,v)$ acting via the representation $\operatorname{Sym}^d$. Indeed, in this case the closure of the image of $G$ in $\mathbb{P}(\operatorname{Mat}(k,w))$ agrees with the image of the embedding $\mathrm{Sym}^d : \mathbb{P}(\mathrm{Mat}(k,v)) \to \mathbb{P}(\mathrm{Mat}(k,w))$. In particular, it is isomorphic to $\mathbb{P}(\mathrm{Mat}(k,v))$, which is smooth.

For more results regarding when this normality assumption is satisfied in more general situations see [Tim03, DC04, BGMR11] and other references in those papers.

*Remark* 5.4. Theorem 5.1 is a direct generalization of the core statement [Kol90, 3.13] of Kollár's Ampleness Lemma [Kol90, 3.9]. This statement is more general in several ways:

- The finiteness assumption on the classifying map is weaker (no assumption on the stabilizers).
- The ambient variety $Y$ is only assumed to be almost projective instead of projective.

Some aspects of our proof are based on Kollár's original idea, but the generality that we need requires several modifications and other ideas to allow for weakening the finiteness assumptions.

Note that if $Y$ is projective and $W$ is nef on $Y$, then it is also weakly positive [Vie95, Prop. 2.9.e].

We will start by making a number of reduction steps to simplify the statement. The goal of these reductions is to show that it is enough to prove the following theorem which contains the essential statement.

**Theorem 5.5.** *Let $W$ be a weakly-positive vector bundle of rank $w$ on a normal almost projective variety $Y$ with a reductive structure group $G \subseteq \mathrm{GL}(k,w)$ the closure of the image of which in the projectivization $\mathbb{P}(\mathrm{Mat}(k,w))$ of the space of $w \times w$ matrices is normal. Further let $\alpha : W \twoheadrightarrow Q$ be a surjective morphism onto a vector bundle of rank $q$ and $\lambda(\alpha) : Y(k) \to \mathrm{Gr}(w,q)(k)/G(k)$ the induced classifying map. If $\lambda(\alpha)$ has finite fibers on a dense open set of $Y$, then the line bundle $\det Q$ is big.*

**Lemma 5.6.** *Theorem 5.5 implies Theorem 5.1.*

*Proof.* **Step 1.** *We may assume that the $\alpha_i$ are surjective.* Let $Q_i^- = \mathrm{im}\, \alpha_i \subseteq Q_i$. Then there exists a big open subset $\iota : U \hookrightarrow Y$ such that $Q_i^-|_U$ is locally free of rank $q_i$. If $\bigotimes_{i=1}^n \det(Q_i^-|_U)$ is big, then so is $[\bigotimes_{i=1}^n] \det Q_i^- = \iota_* \left(\bigotimes_{i=1}^n \det(Q_i^-|_U)\right)$ and hence so is $\bigotimes_{i=1}^n \det Q_i$. Therefore we may replace $Y$ with $U$ and $Q_i$ with $Q_i^-|_U$.

**Step 2.** *It is enough to prove the statement for one quotient bundle.* Indeed, let $W' = \bigoplus_{i=0}^n W$ with the diagonal $G$-action, $Q' = \bigoplus_{i=0}^n Q_i$, and $\alpha := \bigoplus_{i=0}^n \alpha_i : W' \to Q'$ the induced morphism. If all the $\alpha_i$ are surjective, then so is $\alpha$.

Furthermore, there is a natural injective $G$-invariant morphism

$$\begin{array}{ccc} \displaystyle\bigtimes_{i=0}^n Gr(w,q_i) & \hookrightarrow & Gr\left(rw, \displaystyle\sum_{i=0}^n q_i\right) \\ (L_1,\dots,L_r) & \longmapsto & \displaystyle\bigoplus_{i=0}^n L_i. \end{array}$$

The $G$-action on $\bigtimes_{i=0}^n Gr(w, q_i)$ is the restiction of the $G$-action on $Gr\left(rw, \sum_{i=0}^n q_i\right)$ via this embedding and hence the induced map on the quotients remain injective:

$$\bigtimes_{i=0}^{n} Gr(w, q_i) \Big/ G \hookrightarrow Gr\left(rw, \sum_{i=0}^{n} q_i\right) \Big/ G.$$

It follows that the classifying map of $\alpha' : W' \to Q'$ also has finite fibers and then the statement follows because $\det Q \simeq \bigotimes_{i=0}^n \det Q_i$. □

**Lemma 5.7.** *If $V \subseteq W$ is a $G$-invariant sub-vector bundle of the $G$-vector bundle $W$ on a normal almost projective variety $X$, and $W$ is weakly positive, then so is $V$.*

*Proof.* $V$ corresponds to a subrepresentation of $G$, and by the characteristic zero and reductivity assumptions it follows that $V$ is a direct summand of $W$, so $V$ is also weakly positive. □

*Remark* 5.8. The above lemma, which is used in the last paragraph of the proof, is the only place where the characteristic zero assumption is used in the proof of Theorem 5.1. In particular, the statement holds in positive characteristic for a given $W$ if the $G$-subbundles of $W$ are weakly-positive whenever $W$ is. According to [Kol90, Prop 3.5] this holds for example if $Y$ is projective and $W$ is nef satisfying the assumption $(\Delta)$ of [Kol90, Prop 3.6]. The latter is satisfied for example if $W = \mathrm{Sym}^d(W')$ for a nef vector bundle $W'$ of rank $w'$ and $G = \mathrm{GL}(k, w')$.

*Proof of Theorem 5.5.* We start with the same setup as in [Kol90, 3.13]. Let $\pi : \mathbb{P} = \mathbb{P}(\oplus_{i=1}^w W^*) \to Y$, which can be viewed as the space of matrices with columns in $W$, and consider the universal basis map

$$\varsigma : \bigoplus_{j=1}^{w} \mathscr{O}_{\mathbb{P}}(-1) \to \pi^* W,$$

formally given via the identification $H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(1) \otimes \pi^* W) \simeq H^0(Y, \bigoplus_{j=1}^w W^* \otimes W)$ by the identity sections of the different summands of the form $W^* \otimes W$. Informally, the closed points of $\mathbb{P}$ over $y \in Y$ can be thought of as $w$-tuples $(x_1, \dots, x_w) \in W_y$ and hence a dense open subset of $\mathbb{P}_y$ corresponds to the choice of a basis of $W_y$ up to scaling by an element of $k^\times$, i.e., to a homogenous basis. Similarly, the map $\varsigma$ gives $w$ local sections of $\pi^* W$ which over $(x_1, \dots, x_w)$ take the values $x_1, \dots, x_w$, up to scaling by an element of $k^\times$ where this scaling corresponds to the transition functions of $\mathscr{O}_{\mathbb{P}}(-1)$.

As explained in Remark 5.2, to define the classifying map we need to fix a homogenous basis of a fiber over a fixed closed point. Let us fix such a point $y_0 \in Y$ and a homogenous basis on $W_{y_0}$ and keep these fixed throughout the proof. This choice yields an identification of $\mathbb{P}_{y_0}$ with $\mathbb{P}(\mathrm{Mat}(k, w))$. Notice that the dense open set of $\mathbb{P}_{y_0}$ corresponding to the different choices of a homogenous basis of $W_{y_0}$ is identified with the image of $\mathrm{GL}(k, w)$ in $\mathbb{P}(\mathrm{Mat}(k, w))$ and the point in $\mathbb{P}_{y_0}$ representing the fixed homogenous basis above is identified with the image of the identity matrix in $\mathbb{P}(\mathrm{Mat}(k, w))$. Now we want to restrict to a $G$ orbit inside all the choices of homogenous bases. Let $\widetilde{G}$ denote the closure of the image of $G \subseteq \mathrm{GL}(k, w)$ in $\mathbb{P}(\mathrm{Mat}(k, w))$. Via the identification of $\mathbb{P}_{y_0}$ and $\mathbb{P}(\mathrm{Mat}(k, w))$, $\widetilde{G}$ corresponds to a $G$-invariant closed subscheme of $\mathbb{P}_{y_0}$, which carried around by the $G$-action defines a $G$-invariant closed subscheme $\mathbf{P} \subseteq \mathbb{P}$. Note that since $\widetilde{G}$ is assumed to be normal, so is $\mathbf{P}$ by [EGA-IV, II 6.5.4]. To simplify notation let us denote the restriction $\pi|_{\mathbf{P}}$ also by $\pi$. Restricting the universal basis map to $\mathbf{P}$ and twisting by $\mathscr{O}_{\mathbf{P}}(1)$ gives

$$\beta := \varsigma|_{\mathbf{P}} \otimes \mathrm{id}_{\mathscr{O}_{\mathbf{P}}(1)} : \bigoplus_{j=1}^{w} \mathscr{O}_{\mathbf{P}} \to \pi^* W \otimes \mathscr{O}_{\mathbf{P}}(1).$$

Let $\Upsilon \subset \mathbf{P}$ be the divisor where this map is not surjective, i.e., those points that correspond to non-invertible matrices via the above identification of $\mathbb{P}_{y_0}$ and $\mathbb{P}(\mathrm{Mat}(k,w))$. By construction, $\beta$ gives a trivialization of $\pi^* W \otimes \mathscr{O}_{\mathbf{P}}(1)$ over $\mathbf{P} \setminus \Upsilon$. It is important to note the following fact about this trivialization: let $p \in \mathbf{P}_{y_0}$ be the closed point that via the above identification of $\mathbb{P}_{y_0}$ and $\mathbb{P}(\mathrm{Mat}(k,w))$ corresponds to the image of the identity matrix in $\mathbb{P}(\mathrm{Mat}(k,w))$. Then the trivialization of $\pi^* W \otimes \mathscr{O}_{\mathbf{P}}(1)$ given by $\beta$ gives a basis on $(\pi^* W_{y_0})_p$ which is compatible with our fixed homogenous basis on $W_{y_0}$. Furthermore, for any $p' \in (\mathbf{P} \setminus \Upsilon)_{y_0}$ the basis on $(\pi^* W_{y_0})_{p'}$ given by $\beta$ corresponds to the fixed homogenous basis of $W_{y_0}$ twisted by the matrix (which is only given up to scaling by an element of $k^{\times}$) corresponding to the point $p' \in \mathbf{P}_{y_0}$ via the identification of $\mathbb{P}_{y_0}$ and $\mathbb{P}(\mathrm{Mat}(k,w))$. Note that as $G$ is reductive, it is closed in $\mathrm{GL}(k,w)$ and hence $G(k)$ is transitive on $(\mathbf{P} \setminus \Upsilon)_{y_0}$. It follows that then the choices of homogenous bases of $W_{y_0}$ given by $\beta$ on $(\pi^* W_{y_0})_{p'}$ for $p' \in (\mathbf{P} \setminus \Upsilon)_{y_0}$ form a $G(k)$-orbit, and this orbit may be identified with $(\mathbf{P} \setminus \Upsilon)_{y_0}$. Transporting this identification around $Y$ using the $G$-action we obtain:

(5.5.1) For every $y \in Y(k)$, $(\mathbf{P} \setminus \Upsilon)_y$ may be identified with the $G(k)$-orbit of homogenous bases of $W_y$ containing the homogenous basis obtained from the fixed homogenous basis of $W_{y_0}$ via the $G$-structure.

Next consider the composition of $\widetilde{\alpha} = \pi^* \alpha \otimes \mathrm{id}_{\mathscr{O}_{\mathbf{P}}(1)}$ and $\beta$:

$$\gamma : \bigoplus_{j=1}^{w} \mathscr{O}_{\mathbf{P}} \xrightarrow{\beta} \pi^* W \otimes \mathscr{O}_{\mathbf{P}}(1) \xrightarrow{\widetilde{\alpha}} \pi^* Q \otimes \mathscr{O}_{\mathbf{P}}(1)$$

which is surjective on $\mathbf{P} \setminus \Upsilon$. Taking $q^{\mathrm{th}}$ wedge products yields

$$\gamma^q : \bigoplus_{j=1}^{\binom{w}{q}} \mathscr{O}_{\mathbf{P}} \xrightarrow{\beta^q} \pi^*(\wedge^q W) \otimes \mathscr{O}_{\mathbf{P}}(q) \xrightarrow{\widetilde{\alpha}^q} \pi^* \det Q \otimes \mathscr{O}_{\mathbf{P}}(q)$$

which is still surjective outside $\Upsilon$ and hence gives a morphism

$$\nu : \mathbf{P} \setminus \Upsilon \to \underbrace{\mathrm{Gr}(w,q) \subseteq \mathbf{P}\left(\bigwedge^{q}(k^{\oplus w})\right) =: \mathbf{P}_{Gr}}_{\text{Plücker embedding}},$$

such that

- ($\star$) by (5.5.1), on the $k$-points $\nu$ is a lift of the classifying map $\lambda(\alpha) : Y \to \mathrm{Gr}/G$, where $\mathrm{Gr} := \mathrm{Gr}(w,q)$ is the Grassmannian of rank $q$ quotients of a rank $w$ vectorspace, and
- ($\star\star$) $\nu^* \mathscr{O}_{\mathrm{Gr}}(1) \simeq (\pi^* \det Q \otimes \mathscr{O}_{\mathbf{P}}(q))|_{\mathbf{P} \setminus \Upsilon}$, where $\mathscr{O}_{\mathrm{Gr}}(1)$ is the restriction of $\mathscr{O}_{\mathbf{P}_{Gr}}(1)$ via the Plücker embedding.

We will also view $\nu$ as a rational map $\nu : \mathbf{P} \dashrightarrow \mathrm{Gr}$. Let $\sigma : \widetilde{\mathbf{P}} \to \mathbf{P}$ be the blow up of $(\operatorname{im} \gamma^q) \otimes (\pi^* \det Q \otimes \mathscr{O}_{\mathbf{P}}(q))^{-1} \subseteq \mathscr{O}_{\mathbf{P}}$ and set $\widetilde{\pi} := \pi \circ \sigma$. It follows that

$\widetilde{\nu} = \nu \circ \sigma : \widetilde{\mathbf{P}} \to \mathrm{Gr}$ is well-defined everywhere on $\widetilde{\mathbf{P}}$ and there exists an effective Cartier divisor $E$ on $\widetilde{\mathbf{P}}$ such that

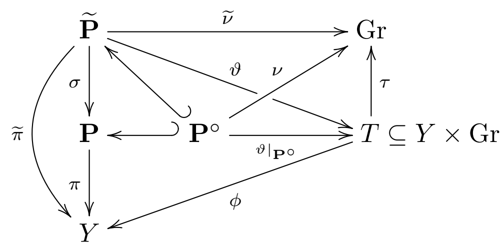
$$\sigma^*(\pi^* \det Q \otimes \mathscr{O}_{\mathbf{P}}(q)) \simeq \widetilde{\nu}^* \mathscr{O}_{\mathrm{Gr}}(1) \otimes \mathscr{O}_{\widetilde{\mathbf{P}}}(E). \tag{5.5.2}$$

Let $Y^\circ \subseteq Y$ be the dense open set where the classifying map $\lambda(\alpha)$ has finite fibers and let $\mathbf{P}^\circ := \widetilde{\pi}^{-1}(Y^\circ) \setminus \sigma^{-1}(\Upsilon) \subset \widetilde{\mathbf{P}}$. Observe that $\mathbf{P}^\circ \simeq \pi^{-1}(Y^0) \setminus \Upsilon$ via $\sigma$.

Next let $T$ be the image of the product map $(\widetilde{\pi} \times \widetilde{\nu}) : \widetilde{\mathbf{P}} \to Y \times \mathrm{Gr}$:

$$T := \mathrm{im}\, [\, (\widetilde{\pi} \times \widetilde{\nu})\, ] \subseteq Y \times \mathrm{Gr},$$

and let $\tau : T \to \mathrm{Gr}$ and $\phi : T \to Y$ be the projection. Furthermore, let $\vartheta : \widetilde{\mathbf{P}} \to T$ denote the induced morphism. We summarize our notation in the following diagram. Note that although $Y$ is only almost proper, every scheme in the diagram (except Gr which is proper over $k$) is proper over $Y$.



*Claim* 5.5.3. The map $\tau|_{\vartheta(\mathbf{P}^\circ)}$ has finite fibers.

*Proof.* Since $k$ is assumed to be algebraically closed, it is enough to show that for every $k$-point $x$ of Gr there are finitely many $k$-points of $\vartheta(\mathbf{P}^\circ)$ mapping onto $x$. Let $(y, x)$ be such a $k$-point, where $y \in Y(k)$. Choose then $z \in \mathbf{P}^\circ(k)$ such that $\vartheta(z) = (y, x)$. Then $\pi(z) = y$ and $\nu(z) = x$. Furthermore, if $\xi$ denotes the quotient map $\mathrm{Gr}(k) \to \mathrm{Gr}(k)/G(k)$ and we set $\lambda$ to denote the classifying map $\lambda(\alpha) : Y(k) \to \mathrm{Gr}(w, q)(k)/G(k)$, then by $(\star)$,

$$\lambda(y) = \lambda(\pi(z)) = \xi(\nu(z)) = \xi(x).$$

Therefore, $y \in \lambda^{-1}(\xi(x))$. However, by the finiteness of $\lambda$ there are only finitely many such $y$. $\square$

By construction $\vartheta(\mathbf{P}^\circ)$ is dense in $T$ and it is constructible by Chevalley's Theorem. Then the dimension of the generic fiber of $\tau$ equals the dimension of the generic fiber of $\tau|_{\vartheta(\mathbf{P}^\circ)}$ and hence $\tau$ is generically finite.

Next consider a projective closure $Y \hookrightarrow \overline{Y}$ of $Y$ and let $\overline{T} \subseteq \overline{Y} \times \mathrm{Gr}$ denote the closure of $T$ in $\overline{Y} \times \mathrm{Gr}$. Let $\overline{\phi} : \overline{T} \to \overline{Y}$ and $\overline{\tau} : \overline{T} \to \mathrm{Gr}$ denote the projections. Clearly, $\overline{\phi}|_T = \phi$, $\overline{\tau}|_T = \tau$, and $\overline{\tau}$ is also generically finite. Let $\overline{H}$ be an ample Cartier divisor on $\overline{Y}$. Since $\overline{\tau}^* \mathscr{O}_{\mathrm{Gr}}(1)$ is big, there is an $m$, such that $\overline{\tau}^* \mathscr{O}_{\mathrm{Gr}}(m) \otimes \phi^* \mathscr{O}_{\overline{Y}}(-\overline{H})$ has a non-zero section. Let $H = \overline{H}|_{\overline{Y}}$ and restrict this section to $T$. It follows that the line bundle

$$\vartheta^* \left(\tau^* \mathscr{O}_{\mathrm{Gr}}(m) \otimes \phi^* \mathscr{O}_Y(-H)\right) \simeq \widetilde{\nu}^* \mathscr{O}_{\mathrm{Gr}}(m) \otimes \widetilde{\pi}^* \mathscr{O}_Y(-H) \tag{5.5.4}$$

also has a non-zero section, and then by (5.5.2) and (5.5.4) there is also a non-zero section of

$$\sigma^* \left(\pi^* (\det Q)^m \otimes \mathscr{O}_{\mathbf{P}}\,(mq)\right) \otimes \widetilde{\pi}^* \mathscr{O}_Y(-H) \simeq \sigma^* \left(\pi^* (\det Q)^m \otimes \pi^* \mathscr{O}_Y(-H) \otimes \mathscr{O}_{\mathbf{P}}\,(mq)\right).$$

Pushing this section forward by $\sigma$ and using the projection formula we obtain a section of

$$(\pi^*(\det Q)^m \otimes \pi^* \mathscr{O}_Y(-H) \otimes \mathscr{O}_{\mathbf{P}}(mq)) \otimes \sigma_* \mathscr{O}_{\widetilde{\mathbf{P}}} \simeq \\ \simeq \underbrace{\pi^*(\det Q)^m \otimes \pi^* \mathscr{O}_Y(-H) \otimes \mathscr{O}_{\mathbf{P}}(mq)}_{\sigma \text{ is birational and } \mathbf{P} \text{ is normal}}.$$

Pushing this section down via $\pi$ and rearranging the sheaves on the two sides of the arrow we obtain a non-zero morphism

$$(\pi_* \mathscr{O}_{\mathbf{P}}(mq))^* \otimes \mathscr{O}_Y(H) \to (\det Q)^m. \tag{5.5.5}$$

Now observe, that by construction

$$(\pi_* \mathscr{O}_{\mathbb{P}}(mq))^* \simeq \left( \operatorname{Sym}^{mq} \left( \bigoplus_{i=1}^{w} W^* \right) \right)^* \simeq \operatorname{Sym}^{mq} \left( \bigoplus_{i=1}^{w} W \right)$$

is weakly-positive and $(\pi_* \mathscr{O}_{\mathbf{P}}(mq))^*$ is a $G$-invariant subbundle of $(\pi_* \mathscr{O}_{\mathbb{P}}(mq))^*$ for $m \gg 0$. In particular, by Lemma 5.7, $(\pi_* \mathscr{O}_{\mathbf{P}}(mq))^*$ is weakly positive as well. Then by (5.5.5) and Lemma 4.8 it follows that $\det Q$ is big. □

## 6. MODULI SPACES OF STABLE LOG-VARIETIES

**Definition 6.1.** A set $I \subseteq [0,1]$ of coefficients is said to be *closed under addition*, if for every integer $s > 0$ and every $x_1, \dots, x_s \in I$ such that $\sum_{i=1}^s x_i \leq 1$ it holds that $\sum_{i=1}^s x_i \in I$.

**Definition 6.2.** Fix $0 < v \in \mathbb{Q}$, $0 < n \in \mathbb{Z}$ and a finite set of coefficients $I \subseteq [0,1]$ closed under addition. A functor $\mathscr{M} : \mathfrak{Sch}_k \to \mathfrak{Sets}$ (or to groupoids) is *a moduli (pseudo-)functor of stable log-varieties of dimension* $n$*, volume* $v$ *and coefficient set* $I$, if for each normal $Y$,

$$\mathscr{M}(Y) = \left\{ \begin{array}{c|l} (X,D) & (1)\ f \text{ is a flat morphism,} \\ \Big\downarrow f & (2)\ D \text{ is a Weil-divisor on } X \text{ avoiding the generic} \\ Y & \quad \text{and the codimension 1 singular points of } X_y \\ & \quad \text{for all } y \in Y, \\ & (3)\ \text{for each } y \in Y,\ (X_y, D_y) \text{ is a stable log-variety} \\ & \quad \text{of dimension } n, \text{ such that the coefficients of} \\ & \quad D_y \text{ are in } I, \text{ and } (K_{X_y} + D_y)^n = v, \text{ and} \\ & (4)\ K_{X/Y} + D \text{ is } \mathbb{Q}\text{-Cartier.} \end{array} \right\}, \tag{6.2.1}$$

and the line bundle $Y \mapsto \det f_* \mathscr{O}_X(r(K_{X/Y} + D))$ associated to every family as above extends to a functorial line bundle on the entire (pseudo-)functor for every sufficiently divisible integer $r > 0$.

Also note that if $\mathscr{M}$ is regarded as a functor in groupoids, then in (6.2.1) instead of equality only equivalence of categories should be required.

*Remark* 6.3. (1) Note that the definition implies that for any $(X_0, D_0)$ stable log-variety of dimension $n$, such that the coefficients of $D_0$ are in $I$, and $(K_{X_0} + D_0)^n = v$, $(X_0, D_0) \in \mathscr{M}(\operatorname{Spec} k)$.

(2) The condition "$D$ is a Weil-divisor on $X$ avoiding the generic and the codimension 1 singular points of $X_y$ for all $y \in Y$" guarantees that $D_y$ can be defined sensibly. Indeed, according to this condition, there is a big open set of $X_y$, over which $D$ is $\mathbb{Q}$-Cartier.

(3) The condition "$K_{X/Y} + D$ is $\mathbb{Q}$-Cartier" is superfluous by a recent, yet unpublished result of Kollár stating that for a flat family with stable fibers if $y \mapsto (K_{X_y} + D_y)^n$ is constant, then $K_{X/Y} + D$ is automatically $\mathbb{Q}$-Cartier.
(4) To guarantee properness $I$ has to be closed under addition as divisors with coefficients $c_1, \dots, c_s$, can come together in the limit to form a divisor with coefficient $\sum_{i=1}^{s} c_i$.
(5) By [HMX14, Thm 1.1], after fixing $n$, $v$ and a DCC set $I \subseteq [0,1]$, there exist
   (a) a finite set $I_0 \subseteq I$ containing all the possible coefficients of stable log-varieties of dimension $n$, volume $v$ and coefficient set $I$, and
   (b) a uniform $m$ such that $m(K_X + D)$ is Cartier for all stable log-varieties $(X, D)$ of dimension $n$, volume $v$ and coefficient set $I$.

   In particular, $m$ may also be fixed in the above definition if it is chosen to be sufficiently divisible after fixing the other three numerical invariants.

**Proposition 6.4.** *Let $n > 0$ be an integer, $v > 0$ a rational number and $I \subseteq [0,1]$ a finite coefficient set closed under addition. Then any moduli (pseudo-)functor of stable log-varieties of dimension n, volume v and coefficient set I is proper. That is, if it admits a coarse moduli space which is an algebraic space, then that coarse moduli space is proper over k. If in addition the pseudo-functor itself is a DM-stack , then it is a proper DM-stack over k (from which the existence of the coarse moduli space as above follows* [KeM97, Con05]*).*

*Proof.* This is shown in [Kol14, Thm 12.11]. □

The following is a simple consequence of [Iit82, Thm 11.12]. We include an argument for the convenience of the reader.

**Proposition 6.5** (Iitaka). *If $(X, D)$ is a stable log-variety then* $\operatorname{Aut}(X, D)$ *is finite.*

*Proof.* Let $\pi : \overline{X} \to X$ be the normalization of $X$ and $\overline{D}$ is defined via

$$K_{\overline{X}} + \overline{D} = \pi^*(K_X + D)$$

where $K_X$ and $K_{\overline{X}}$ are chosen compatibly such that $K_X$ avoids the singular codimension one points of $X$. Note that $\overline{D} \geq 0$ by [Kol13b, (5.75)]. Any automorphism of $(X, D)$ extends to an automorphism of $\left(\overline{X}, \overline{D}\right)$, hence we may assume that $(X, D)$ is normal. Furthermore, since $X$ has finitely many irreducible components, the automorphisms fixing each component form a finite index subgroup. Therefore, we may also assume that $X$ is irreducible. Let $U \subseteq X$ be the regular locus of $X \setminus \operatorname{Supp} D$. Note that $U$ is $\operatorname{Aut}(X, D)$-invariant, hence there is an embedding $\operatorname{Aut}(X, D) \hookrightarrow \operatorname{Aut}(U)$. In particular, it is enough to show that $\operatorname{Aut}(U)$ is finite. Next let $g : (Y, E) \to (X, D)$ be a log-crepant resolution that is an isomorphism over $U$ and for which $g^{-1}(X \setminus U)$ is a normal-crossing divisor. Let $F$ be the reduced divisor with support equal to $g^{-1}(X \setminus U)$. Then $(Y, E)$ is log-canonical, and $E \leq F$. Therefore, $g^*(K_X + D) = K_Y + E \leq K_Y + F$ and hence $(Y, F)$ is of log general type. However, $U = Y \setminus \operatorname{Supp} F$, and hence $U$ itself is of general type. Then by [Iit82, Thm 11.12] a group (which is called $\operatorname{SBir}(U)$ there) containing $\operatorname{Aut}(U)$ is finite. □

### 6.A. *A particular functor of stable log-varieties*

In what follows we describe a particular functor of stable log-varieties introduced by János Kollár [Kol13a, (3) of page 155]. The main reason we do so is to be able to give Definition 6.16 and prove Corollary 6.18 and Corollary 6.19. These are used in the following sections.

In fact, our method will be somewhat non-standard: we define a pseudo-functor $\mathscr{M}_{n,m,h}$ which is larger than needed in Definition 6.6. We show that $\mathscr{M}_{n,m,h}$ is a DM-stack (Proposition 6.11) and if $m$ is sufficiently divisible (after fixing $n$ and $v$), the locus of stable log-varieties of dimension $n$, volume $v$ and coefficient set $I$ is proper and closed in $\mathscr{M}_{n,m,h}$. Hence the reduced closed substack on this locus is a functor of stable log-varieties as in Definition 6.2. We emphasize that our construction is not a functor that we propose to use in the long run. For example, we are not describing the values it takes on Artinian non-reduced schemes. However, it does allow us to make Definition 6.16 and prove Corollary 6.18 and Corollary 6.19, which is our goal here. Finding a reasonably good functor(s) is an extremely important, central question which is postponed for future endeavors.

The issue in general about functors of stable log-varieties is that, as Definition 6.2 suggests, it is not clear what their values should be over non-reduced schemes. The main problem is to understand the nature and behavior of $D$ in those situations. Kollár's solution to this is that instead of trying to figure out how $D$ should be defined over non-reduced schemes, let us replace $D$ as part of the data with some other data equivalent to (6.2.1) that has an obvious extension to non-reduced schemes. This "other" data is as follows: instead of remembering $D$, fix an integer $m > 0$ such that $m(K_X + D)$ is Cartier, and remember instead of $D$ the map $\omega_X^{\otimes m} \to \mathscr{O}_X(m(K_X + D)) =: \mathscr{L}$. There are two things we note before proceeding to the precise definition.

(1) A global choice of $m$ as above is possible according to Remark 6.3.
(2) Fixing $\left(X, \phi : \omega_X^{\otimes m} \to \mathscr{L}\right)$ is slightly more than just fixing $(X, D)$, since composing $\phi$ with an automorphism $\xi$ of $\mathscr{L}$ is formally different, but yields the same $D$. In particular, we have to remember that different pairs $(X, \phi)$ that only differ by an automorphism $\xi$ of $\mathscr{L}$ should be identified eventually.

We define our auxiliary functor $\mathscr{M}_{n,m,h}$ according to the above considerations.

**Definition 6.6.** Fix an integer $n > 0$, a polynomial $h : \mathbb{Z} \to \mathbb{Z}$ and an integer $m > 0$ divisible enough (after fixing $n$ and $h$). We define the auxiliary pseudo-functor $\mathscr{M}_{n,m,h}$ as

(6.6.1)
$$\mathscr{M}_{n,m,h}(Y) = \left\{ \left( \begin{array}{c} X \\ \downarrow \\ Y \end{array}, \phi : \omega_{X/Y}^{\otimes m} \to \mathscr{L} \right) \middle| \begin{array}{l} (1)\ f \text{ is a flat morphism of pure relative dimension } n, \\ (2)\ \mathscr{L} \text{ is a relatively very ample line bundle on } X \text{ such} \\ \quad \text{that } R^i f_*(\mathscr{L}^r) = 0 \text{ for every } r > 0, \text{ and} \\ (3)\ \text{for all } y \in Y: \\ \quad \text{i. } \phi \text{ is an isomorphism at the generic points and} \\ \qquad \text{at the codimension 1 singular points of } X_y, \text{ and} \\ \qquad \text{hence it determines a divisor } D_y, \text{ such that} \\ \qquad \mathscr{L}_y \simeq \mathscr{O}_y(m(K_{X_y} + D_y)), \\ \quad \text{ii. } (X_y, D_y) \text{ is slc, and} \\ \quad \text{iii. } h(r) = \chi(X_y, \mathscr{L}_y^r) \text{ for every integer } r > 0. \end{array} \right\}$$

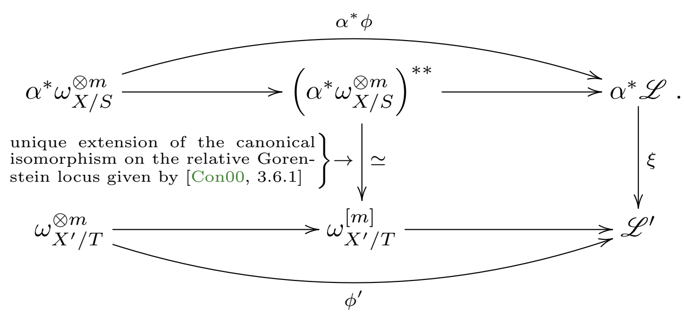
where

(a) as indicated earlier, if $Y$ is normal, $\phi$ corresponds to an actual divisor $D$ such that $\mathscr{O}_X(m(K_{X/Y}+D)) \simeq \mathscr{L}$. Explicitly, $D$ is the closure of $\frac{E}{m}$, where $E$ is the divisor determined by $\phi$ on the relatively Gorenstein locus $U$.
(b) The arrows in $\mathscr{M}_{n,m,h}$ between

$$\left(X \to S, \phi : \omega_{X/S}^{\otimes m} \to \mathscr{L}\right) \in \mathscr{M}_{n,m,h}(S),$$

and
$$\left(X' \to T, \phi' : \omega^{\otimes m}_{X'/T} \to \mathscr{L}'\right) \in \mathscr{M}_{n,m,h}(T),$$
over a fixed $T \to S$ are of the form $(\alpha : X' \to X, \xi : \alpha^*\mathscr{L} \to \mathscr{L}')$, such that the square
$$\begin{array}{ccc} X' & \xrightarrow{\alpha} & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$
is Cartesian, and $\xi$ is an isomorphism such that the following diagram is commutative.

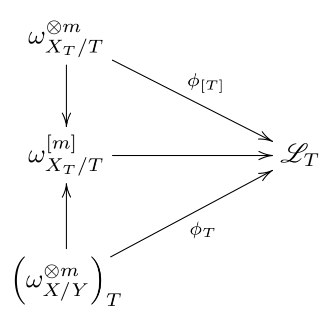
(6.6.2)




In other words, $\phi'$ corresponds to $\xi \circ \alpha^*\phi$ via the natural identification
$$\operatorname{Hom}\left(\alpha^*\omega^{\otimes m}_{X/S}, \mathscr{L}'\right) = \operatorname{Hom}\left(\omega^{\otimes m}_{X'/T}, \mathscr{L}'\right).$$

(c) An arrow as above is an isomorphism if $T \to S$ is the identity and $\alpha$ is an isomorphisms.

(d) We fix the following pullback construction. It features subtleties similar to that of (6.6.2) stemming from the fact that only the hull $\omega^{[m]}_{X/Y}$ of $\omega^{\otimes m}_{X/Y}$ is compatible with base-change. So, let us consider $(X, \phi) := \left(X \to S, \phi : \omega^{\otimes m}_{X/S} \to \mathscr{L}\right) \in \mathscr{M}_{n,m,h}(S)$ and a $k$-morphism $T \to S$. Then $(X, \phi)_T := \left(X_T \to T, \phi_{[T]} : \omega^{\otimes m}_{X_T/T} \to \mathscr{L}_T\right)$, where $\phi_{[T]}$ is defined via the following commutative diagram.




In other words, via the natural identification $\operatorname{Hom}\left(\left(\omega^{\otimes m}_{X/Y}\right)_T, \mathscr{L}_T\right) = \operatorname{Hom}\left(\omega^{\otimes m}_{X_T/T}, \mathscr{L}_T\right)$, $\phi_T$ corresponds to $\phi_{[T]}$.

We leave the proof of the following statement to the reader. We only note that the main reason it holds is that the presence of the very ample line bundle $\mathscr{L}$ makes descent work.

**Proposition 6.7.** *When viewed as a pseudo-functor (or equivalently as a category fibered in groupoids) $\mathscr{M}_{n,m,h}$ is an étale (or even fppf) stack.*

**Proposition 6.8.** *Consider two objects*

$$\left(f : X \to Y, \phi : \omega_{X/Y}^{\otimes m} \to \mathscr{L}\right), \left(f' : X' \to Y, \phi' : \omega_{X'/Y}^{\otimes m} \to \mathscr{L}'\right) \in \mathscr{M}_{n,m,h}(Y).$$

*Then the isomorphism functor of these two families* $\mathsf{Isom}_Y((X,\phi),(X',\phi'))$ *is representable by a quasi-projective scheme over $Y$, denoted by* $\mathrm{Isom}_Y((X,\phi),(X',\phi'))$. *Furthermore, this isomorphism scheme is unramified over $Y$.*

*Remark* 6.9. Recall that, by definition, $\mathsf{Isom}_Y((X,\phi),(X',\phi'))(T)$ is the set of $T$-isomorphisms between $(X,\phi)_T$ and $(X',\phi')_T$ for any scheme $T$ over $k$.

*Proof.* First, we show the representability part of the statement. Denote the connected components of the Isom scheme $\mathrm{Isom}_Y(X,X')$ parametrizing isomorphisms $\gamma : X_T \to X'_T$ such that $\gamma^*\mathscr{L}'_T \equiv_T \mathscr{L}_T$ (cf. [Kol96, Exercise 1.10.2]) by $I := \mathrm{Isom}^*_Y(X,X') \to Y$. It comes equipped with a universal isomorphism $\alpha : X_I \to X'_I$. Now, let $J := \underline{\mathrm{Isom}}_I(\alpha^*\mathscr{L}'_I, \mathscr{L}_I)$ be the open part of $\underline{\mathrm{Hom}}_I(\alpha^*\mathscr{L}'_I, \mathscr{L}_I)$ [Kol08, 33] parametrizing isomorphisms. This space also comes equipped with a universal isomorphism $\xi : \alpha_J^*\mathscr{L}'_J \to \mathscr{L}_J$. This space $J$, with the universal family $\alpha_J : X_J \to X'_J$ and $\xi : \alpha_J^*\mathscr{L}'_J \to \mathscr{L}_J$ is a fine moduli space for the functor

$$T \mapsto \{(\beta,\zeta) | \beta : X_T \to X'_T \text{ and } \zeta : \beta^*\mathscr{L}'_T \to \mathscr{L}_T \text{ are isomorphisms}\}.$$

This is almost the functor $\mathsf{Isom}_Y((X,\phi),(X',\phi'))$, except in the latter there is an extra condition that the following diagram commutes:

$$\begin{array}{ccc} \beta^*\omega^{\otimes m}_{X'_T/T} & \xrightarrow{\beta^*\phi'_{[T]}} & \beta^*\mathscr{L}'_T \\ \downarrow \simeq & & \downarrow \zeta \\ \omega^{\otimes m}_{X_T/T} & \xrightarrow{\phi_{[T]}} & \mathscr{L}_T \end{array} \tag{6.9.1}$$

Note that here we do not have to take hulls. Indeed, $\beta^*\omega^{\otimes m}_{X'_T/T}$ itself is isomorphic to $\omega^{\otimes m}_{X_T/T}$ via the $m$-th tensor power of the unique extension of the canonical map of [Con00, Thm 3.6.1] from the relative Gorenstein locus, since $\beta$ is an isomorphism and hence $\beta^*\omega_{X'_T/T}$ is reflexive.

Hence we are left to show that the condition of the commutativity of (6.9.1) is a closed condition. That is, there is a closed subscheme $S \subseteq J$, such that the condition of (6.9.1) holds if and only if the induced map $T \to J$ factors through $S$.

Set $\psi := \phi_{[J]}$ and let $\psi'$ be the composition

$$\omega^{\otimes m}_{X_J/J} \simeq \alpha^*\omega^{\otimes m}_{X'_J/J} \xrightarrow{\alpha^*\phi'_{[J]}} \alpha^*\mathscr{L}'_J \xrightarrow{\xi} \mathscr{L}_J .$$

Consider $M := \underline{\mathrm{Hom}}\left(\omega^{\otimes m}_{X_J/J}, \mathscr{L}_J\right)$ [Kol08, 33]. The homomorphisms $\psi$ and $\psi'$ give two sections $s, s' : J \to M$. Let $S := s'^{-1}(s(J))$.

In the remainder of the proof we show the above claimed universal property of $S$. Take a scheme $T$ over $k$ and a pair of isomorphisms $(\beta,\zeta)$, where $\beta$ is a morphism $X_T \to X'_T$ and $\xi$ is a homomorphism $\beta^*\mathscr{L}'_T \to \mathscr{L}_T$. Let $\mu : T \to J$ be the moduli map, that is via this map $\beta = \alpha_T$ and $\zeta = \xi_T$. We have to show that the commutativity of (6.9.1) holds if and only if $\mu$ factors through the closed subscheme $S \subseteq J$.

First, by the natural identification $\underline{\mathrm{Hom}}\left(\omega_{X_T/T}^{\otimes m}, \mathscr{L}_T\right) = \underline{\mathrm{Hom}}\left(\left(\omega_{X_J/J}^{\otimes m}\right)_T, \mathscr{L}_T\right)$ the commutativity of (6.9.1) is equivalent to $\psi_T = \psi'_T$. Second, by functoriality of Mor, the latter condition is equivalent to $s_T = s'_T$ (as sections of $M_T \to T$). However, the latter is equivalent to the factorization of $T \to J$ through $S$, which shows that indeed $\mathrm{Isom}_Y((X,\phi),(X',\phi')) := S$ represents the functor $\mathsf{Isom}_Y((X,\phi),(X',\phi'))$.

For the addendum, note that $\mathrm{Isom}_Y((X,\phi),(X',\phi'))$ is a group scheme over $Y$. Since char$k = 0$, the characteristics of all the geometric points is 0 and hence all the geometric fibers are smooth. This implies that $\mathrm{Isom}_Y((X,\phi),(X',\phi'))$ is unramified over $Y$ [StacksProject, Tag 02G8], since its geometric fibers are finite by Proposition 6.5. □

**Lemma 6.10.** *Let* $\left(f : X \to Y, \omega_{X/Y}^{\otimes m} \to \mathscr{L}\right)$ *satisfy conditions* (1), (2), (3i) *and* (3iii) *in (6.6.1), i.e., do not assume that* $(X_y, D_y)$ *is slc. Further assume that* $X_y$ *is demi-normal for all* $y \in Y$ *and* $Y$ *is essentially of finite type over* $k$*. Then the subset* $Y^\circ := \{y \in Y \mid (X_y, D_y) \text{ is slc}\} \subseteq Y$ *is open.*

*Proof.* Let $\tau : Y' \to Y$ be a resolution. As $\tau$ is proper, we may replace the original family with the pullback to $Y'$ and so we may assume that $Y$ is smooth. Next we show that *the slc locus* $\{y \in Y | (X_y, D_y) \text{ is slc}\}$ *is constructible*. For that it is enough to show that there is a non-empty open set $U$ of $Y$ such that either $(X_y, D_y)$ is slc for all $y \in U$ or $(X_y, D_y)$ is not slc for all $y \in U$ and conclude by noetherian induction. To prove the existence of such a $U$, we may assume that $Y$ is irreducible. Let $\rho : X' \to X$ be a semi log-resolution and $U \subseteq Y$ an open set for which

- $\rho^{-1}f^{-1}U \to U$ is flat,
- $X'_y \to X_y$ is a semi log-resolution for all $y \in U$, and
- for any exceptional divisor $E$ of $\rho$ that does not dominate $Y$ (i.e., which is $f$-vertical) $f(\rho(E)) \cap U = \emptyset$.

It follows that for $y \in U$, the discrepancies of $(X_y, D_y)$ are independent of $y$. Hence, either every such $(X_y, D_y)$ is slc or all of them are not slc.

Next, we prove that *the locus* $\{y \in Y | (X_y, D_y) \text{ is slc}\}$ *is closed under generalization*, which will conclude our proof by [Har77, Exc I.3.18.c]. For, this we should prove that if $Y$ is a DVR, essentially of finite type over $k$, and $(X_\xi, D_\xi)$ is slc for the closed point $\xi \in Y$, then so is $(X_\eta, D_\eta)$ for the generic point $\eta \in Y$. However, this follows immeditaley by inversion of adjunction for slc varieties [Pat16a, Cor 2.11], since that implies that $(X, D + X_\xi)$ is slc and then by localizing at $\eta$ we obtain that $(X_\eta, D_\eta)$ is slc. □

**Proposition 6.11.** $\mathscr{M}_{n,m,h}$ *is a DM-stack of finite type over* $k$*.*

*Proof.* For simplicity let us denote $\mathscr{M}_{n,m,h}$ by $\mathscr{M}$. By [DM69, 4.21] we have to show that $\mathscr{M}$ has representable and unramified diagonal, and there is a smooth surjection onto $\mathscr{M}$ from a scheme of finite type over $k$. For any stack $\mathscr{X}$ and a morphism from a scheme $T \to \mathscr{X} \times_k \mathscr{X}$ corresponding to $s, t \in \mathscr{X}(T)$, the fiber product $\mathscr{X} \times_{\mathscr{X} \times_k \mathscr{X}} T$ can be identified with $\mathrm{Isom}_T(s,t)$. Hence the first condition follows from Proposition 6.8. For the second condition we are to construct a cover $S$ of $\mathscr{M}$ by a scheme such that $S \to \mathscr{M}$ is formally smooth. The rest of the proof is devoted to this.

Set $N := p(1) - 1$. Then, $\mathfrak{Hilb}^h_{\mathbb{P}^N}$ contains every $\left(X, \phi : \omega_X^{\otimes m} \to \mathscr{L}\right) \in \mathscr{M}(k)$, where $X$ is embedded into $\mathbb{P}^N_k$ using $H^0(X, \mathscr{L})$. Let $\mathscr{H}_1 := \mathfrak{Hilb}^h_{\mathbb{P}^N}$ be the open subscheme corresponding to $X \subseteq \mathbb{P}^N$, such that $H^i(X, \mathscr{O}_X(r)) = 0$ for all integers $i > 0$ and $r > 0$. According to [EGA-IV, III.12.2.1], there is an open subscheme $\mathscr{H}_2 \subseteq \mathscr{H}_1$ parametrizing the reduced equidimensional and $S_2$ varieties. Since small deformations of nodes are either nodes or regular points, we see that there is an open subscheme $\mathscr{H}_3 \subseteq \mathscr{H}_2$ parametrizing the demi-normal varieties (where reducedness and equidimensionality is included in demi-normality). Let $\mathscr{U}_3$ be the universal family over $\mathscr{H}_3$. According to [Kol08, 33] there is a fine moduli scheme $M_4 := \underline{\mathrm{Hom}}_{\mathscr{H}_3}\left(\omega^{\otimes m}_{\mathscr{U}_3/\mathscr{H}_3}, \mathscr{O}_{\mathscr{U}_3}(1)\right)$. Define $\mathscr{U}_4$ and $\mathscr{O}_{\mathscr{U}_4}(1)$ to be the pullback of $\mathscr{U}_3$ and of $\mathscr{O}_{\mathscr{U}_3}(1)$ over $M_4$. Then there is a universal homomorphism $\gamma : \omega^{\otimes m}_{\mathscr{U}_4/M_4} \to \mathscr{O}_{\mathscr{U}_4}(1)$. Let $M_5 \subseteq M_4$ be the open locus where $\gamma$ is an isomorphism at every generic point and singular codimension one point of each fiber is open. Let $\mathscr{U}_5$ and $\mathscr{O}_{\mathscr{U}_5}(1)$ the restrictions of $\mathscr{U}_4$ and $\mathscr{O}_{\mathscr{U}_4}(1)$ over $M_5$. According to Lemma 6.10, there is an even smaller open locus $M_6 \subseteq M_5$ defined by

$$M_6 := \left\{ t \in M_5 \,\middle|\, \omega^{\otimes m}_{(\mathscr{U}_5)_t} \to \mathscr{O}_{(\mathscr{U}_5)_t}(1) \text{ corresponds to an slc pair } \right\}.$$

Then define $S := M_6$ and $g : U \to S$ and $\phi : \omega^{\otimes m}_{U/S} \to \mathscr{O}_U(1)$ to be respectively the restrictions of $\mathscr{U}_5 \to M_5$ and of $\gamma$ over $M_6$. From Definition 6.6 and by cohomology and base-change it follows that for each $\left(h : X_T \to T, \phi' : \omega^{\otimes m}_{X_T/T} \to \mathscr{L}_T\right) \in \mathscr{M}(T)$ such that $T$ is Noetherian,

(1) the sheaf $h_* \mathscr{L}_T$ is locally free, and
(2) giving a map $\nu : T \to S$ and an isomorphism $(\alpha, \xi)$ between $(h : X_T \to T, \phi' : \omega^{\otimes m}_{X_T/T} \to \mathscr{L}_T)$ and $\left(U_T \to T, \phi_{[T]} : \omega^{\otimes m}_{U_T/T} \to \mathscr{O}_{U_T}(1)\right)$ is equivalent to fixing a set of free generators $s_0, \ldots, s_n \in h_* \mathscr{L}_T$.

Indeed, for the second statement, fixing such a generator set is equivalent to giving a closed embedding $\iota : X_T \to \mathbb{P}^N_T$ with Hilbert polynomial $h$ together with an isomorphism $\zeta : \mathscr{L}_T \to \iota^* \mathscr{O}_{\mathbb{P}^N_T}(1)$. Furthermore, the latter is equivalent to a map $\nu_{\mathrm{pre}} : T \to \mathscr{H}_3$ together with isomorphisms $\alpha : X_T \to (\mathscr{U}_3)_T$ and $\xi : \mathscr{L}_T \to \alpha^* \mathscr{O}_{(\mathscr{U}_3)_T}(1)$. Then the composition

$$\alpha^* \omega^{\otimes m}_{(\mathscr{U}_3)_T/T} \xrightarrow{\simeq} \omega^{\otimes m}_{X_T/T} \xrightarrow{\phi'} \mathscr{L}_T \xrightarrow{\xi} \iota^* \mathscr{O}_{(\mathscr{U}_3)_T}(1)$$

yields a lifting of $\nu_{\mathrm{pre}}$ to a morphism $\nu : T \to S$, such that $(\alpha, \xi^{-1})$ is an isomorphism between $(X_T, \phi')$ and $\left(U_T, \phi_{[T]}\right)$.

Now, we show that the map $S \to \mathscr{M}$ induced by the universal family over $S$ is smooth. It is of finite type by construction, so we have to show that it is formally smooth. Let $\delta : (A', \mathfrak{m}') \twoheadrightarrow (A, \mathfrak{m})$ be a surjection of Artinian local rings over $k$ such that $\mathfrak{m}(\ker \delta) = 0$. Set $T := \operatorname{Spec} A$ and $T' := \operatorname{Spec} A'$. According to [EGA-IV, IV.17.14.2], we need to show that if there is a 2-commutative diagram of solid arrows as follows, then one can find a dashed arrow keeping the diagram 2-commutative.

$$\begin{array}{ccc} S & \longleftarrow & T \\ \downarrow & \nwarrow & \downarrow \\ \mathscr{M} & \longleftarrow & T' \end{array}$$

In other words, given a family $\left(h : X_{T'} \to T', \phi' : \omega_{X_{T'}/T'}^{\otimes m} \to \mathscr{L}\right) \in \mathscr{M}(T')$, with an isomorphism $(\beta, \zeta)$ between $(X_T, \phi'_T)$ and $(U_T, \phi_T)$. We are supposed to prove that $(\beta, \zeta)$ extends over $T'$. However, as explained above, $(\beta, \zeta)$ corresponds to free generators of $(h_T)_* \mathscr{L}_T$, which can be lifted over $T'$ since $T \to T'$ is an infinitesimal extension of Artinian local schemes. ☐

**Lemma 6.12.** *Let* $\left(f : X \to Y, \omega_{X/Y}^{\otimes m} \to \mathscr{L}\right) \in \mathscr{M}_{n,m,h}(T)$ *for some* $T$ *essentially of finite type over* $k$ *and* $I \subseteq [0, 1]$ *a finite coefficient set closed under addition. Then the locus*

(6.12.1) $$\{t \in T | (X_t, D_t) \text{ has coefficients in } I\}$$

*is closed (here* $D_t$ *is the divisor corresponding to* $\phi_t$*). Furthermore, if* $m$ *is sufficiently divisible (after fixing* $n$*,* $v$ *and* $I$*), then the above locus is proper over* $k$*.*

*Proof.* For the first statement, by [Har77, Exc II.3.18.c] we need to prove that the above locus is constructible and closed under specialization. Both of these follow from the fact that if $T$ is normal, and $D_T$ is the divisor corresponding to $\phi_T$, then there is a dense open set $U \subseteq T$ such that the coefficients of $D_T$ and of $D_t$ agree for all $t \in U$. For the "closed under specialization" part one should also add that if $T$ is a DVR with generic point $\eta$ and special point $\varepsilon$, then the coefficient set of $D_\eta$ agrees with the coefficient set of $D$, and the coefficients of $D_\varepsilon$ are sums formed from coefficients of $D$. Since $I$ is closed under addition, if $D_\eta$ has coefficients in $I$, so does $D_\varepsilon$.

The properness statement follows from [Kol14, Thm 12.11] and [HMX14, Thm 1.1]. ☐

**Notation 6.13.** Fix an integer $n > 0$, a rational number $v > 0$ and a finite coefficient set $I \subseteq [0, 1]$ closed under addition. After this choose an $m$ that is divisible enough. For stable log-varieties $(X, D)$ over $k$ for which $\dim X = n$, $(K_X + D)^n = v$ and the coefficient set is in $I$, there are finitely many possibilities for the Hilbert polynomial $h(r) = \chi(X, rm(K_X + D))$ by [HMX14, Thm 1.1]. Let $h_1, \dots, h_s$ be these values. For each integer $1 \le i \le s$, let $\mathscr{M}_i$ denote the reduced structure on the locus (6.12.1) of $\mathscr{M}_{n,m,h_i}$ and let $\mathscr{M}_{n,v,I} := \amalg_{i=1}^s \mathscr{M}_i$ (where $\amalg$ denotes disjoint union).

**Proposition 6.14.** $\mathscr{M}_{n,v,I}$ *is a pseudo-functor for stable log-varieties of dimension* $n$*, volume* $v$ *and coefficient set* $I$*.*

*Proof.* Given a normal variety $T$, $\mathscr{M}_{n,v,I}(T) = \amalg_{i=1}^s \mathscr{M}_i(T)$. Since in Notation 6.13, $\mathscr{M}_i$ were defined by taking reduced structures, for reduced schemes $T$, there are no infinitesmial conditions on $\mathscr{M}_i(T)$. That is it is equivalent to the sub-groupoid of $\mathscr{M}_{n,m,h_i}(T)$ consisting of $\left(X \to T, \phi : \omega_{X/T}^{\otimes m} \to \mathscr{L}\right)$, such that the coefficients of $(X_t, D_t)$ is in $I$. Then it follows by construction that the disjoint union of these is equivalent to the groupoid given in (6.2.1) and that the line bundle $\det f_* \mathscr{L}^j$

associated to $\left(X \to T, \phi : \omega_{X/T}^{\otimes m} \to \mathscr{L}\right) \in \mathscr{M}_{n,v,I}(T)$ yields a polarization for every integer $j > 0$. $\square$

*Remark* 6.15. $\mathscr{M}_{n,v,I}$ a-priori depends on the choice of $m$, which will not matter for our applications. However, one can show by exhibiting isomorphic groupoid representations that in fact the normalization of any DM-stack $\mathscr{M}$ which is a pseudo-functor of stable log-varieties of dimenion $n$, volume $v$ and coefficient set $I$ is isomorphic to the normalization of $\mathscr{M}_{n,v,I}$.

**Definition 6.16.** Given a family $f : (X, D) \to Y$ of stable log-varieties over an irreducible normal variety, such that the dimension $\dim X_y = n$ and the volume $(K_{X_y} + D_y)^n$ of the fibers are fixed. Let $I$ be the set of all possible sums, at most 1, formed from the coefficients of $D$. Then, there is an associated moduli map $\mu : Y \to \mathscr{M}_{n,v,I}$. The *variation* $\operatorname{Var} f$ *of* $f$ is defined as the dimension of the image of $\mu$.

Note that this does not depend on the choice of $m$ or $I$ (see Remark 6.15), since it is $\dim Y - d$, where $d$ is the general dimension of the isomorphism equivalence classes of the fibers $(X_y, D_y)$. This general dimension exists, because it can also be expressed as the general fiber dimension of $\mathrm{Isom}_Y((X, \phi), (X, \phi))$, where $(X, \phi) \in \mathscr{M}_{n,m,h_i}(Y)$ corresponds to $(X, D)$.

Further note that it follows from the above discussion that using any pseudo-functor of stable log-varieties of dimension $n$, volume $v$ and coefficient set $I$ instead of $\mathscr{M}_{n,v,I}$ leads to the same definition of variation.

*Remark* 6.17. Corollary 6.20 gives another alternative definition of variation: it is the smallest number $d$ such that there exists a diagram as in Corollary 6.20 with $d = \dim Y'$.

**Corollary 6.18.** *Given $f : (X, D) \to Y$ a family of stable log-varieties over a normal variety $Y$, and a compactification $\overline{Y} \supseteq Y$, there is a generically finite proper morphism $\tau : \overline{Y}' \to \overline{Y}$ from a normal variety, and a family $f : \left(\overline{X}, \overline{D}\right) \to \overline{Y}'$ of stable log-varieties, such that $\left(\overline{X}_{Y'}, \overline{D}_{Y'}\right) \simeq (X_{Y'}, D_{Y'})$, where $Y' := \tau^{-1} Y$.*

*Proof.* Let $n$ be the dimension and $v$ the volume of the fibers of $f$. Let $I \subseteq [0, 1]$ be a finite coefficient set closed under addition that contains the coefficients of $D$. Denote for simplicity $\mathscr{M}_{n,v,I}$ by $\mathscr{M}$. According to [LMB00, Thm 16.6], there is a finite, generically étale surjective map $S \to \mathscr{M}$, and $f : (X, D) \to Y$ induces another one $Y \to \mathscr{M}$. Let $Y'$ be a component of the normalization of $Y \times_{\mathscr{M}} S$ dominating $Y$. Note that since $\mathscr{M}$ is a DM-stack, $Y$ is a scheme and $Y' \to Y$ is finite and surjective. Hence, we may compactify $Y'$ to obtain a normal projective variety $\overline{Y}'$, such that the maps $Y' \to S$ and $Y' \to Y$ extend to morphisms $\overline{Y}' \to S$ and $\overline{Y}' \to \overline{Y}$ (note that both $S$ and $\overline{Y}'$ are proper over $k$). Hence, we have a 2-commutative diagram

$$\begin{array}{ccccc} Y' & \hookrightarrow & \overline{Y}' & \longrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle \tau} & & \downarrow \\ Y & \hookrightarrow & \overline{Y} & & \mathscr{M} \end{array} \quad (Y \to \mathscr{M}),$$

which shows that the induced family on $\overline{Y}'$ has the property as required, that is, by pulling back to $Y'$ it becomes isomorphic to the pullback of $(X, D)$ to $Y'$. $\square$

**Corollary 6.19.** *If $\mathscr{M}$ is a moduli (pseudo-)functor of stable log-varieties of dimension $n$, volume $v$ and coefficient set $I$ admitting a coarse moduli space $\mathsf{M}$ which is an algebraic space, then there is a finite cover $S \to \mathsf{M}$ from a normal scheme $S$ induced by a family $\mathsf{f} \in \mathscr{M}(S)$.*

*Proof.* Since for every moduli (pseudo-)functor $\mathscr{M}$ of stable log-varieties of dimension $n$, volume $v$ and coefficient set $I$, $\mathscr{M}(k)$ is the same (as a set or as a groupoid), and furthermore $\mathsf{M}$ is proper over $k$ according to Proposition 6.4, it is enough to show that there is a proper $k$-scheme $S$, such that $S$ supports a family $\mathsf{f} \in \mathscr{M}(S)$ for which

(1) the isomorphism equivalence classes of the fibers of $\mathsf{f}$ are finite, and
(2) every isomorphism class in $\mathscr{M}(k)$ appears as a fiber of $\mathsf{f}$.

However, the existence of this follows by [LMB00, Thm 16.6] and Proposition 6.11. ☐

**Corollary 6.20.** *Given a family $f : (X, D) \to Y$ of stable log-varieties over a normal variety, there is diagram*

$$\begin{array}{ccccc} (X', D') & \longleftarrow & (X'', D'') & \longrightarrow & (X, D) \\ \downarrow{\scriptstyle f'} & & \downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longleftarrow & Y'' & \longrightarrow & Y \end{array}$$

*with Cartesian squares, such that*

*(1) $Y'$ and $Y''$ are normal,*
*(2) $\operatorname{Var} f = \dim Y'$,*
*(3) $Y'' \to Y$ is finite, surjective, and*
*(4) $f' : (X', D') \to Y'$ is a family of stable log-varieties for which the induced moduli map is finite. In particular, the fiber isomorphism classes of $f' : (X', D') \to Y'$ are finite.*

*Proof.* Set $n := \dim X_y$ and $v := (K_{X_y} + D_y)^n$. Let $I$ be the set of all possible sums, at most 1, formed from the coefficients of $D$. Then there is an induced moduli map $\nu : Y \to \mathscr{M}_{n,v,I}$. Let $S \to \mathscr{M}_{n,v,I}$ be the finite cover given by Corollary 6.19. The map $Y \times_{\mathscr{M}_{n,v,I}} S \to Y$ is finite and surjective. Define $Y''$ to be the normalization of an irreducible component of $Y \times_{\mathscr{M}_{n,v,I}} S$ that dominates $Y$ and define $Y'$ to be the normalization of the image of $Y''$ in $S$. That is, we obtain a 2-commutative diagram

$$\begin{array}{ccc} Y'' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\nu} & \mathscr{M}_{n,v,I} \end{array} .$$

This yields families over $Y'$ and $Y''$ as required by the statement. ☐

## 7. Determinants of pushforwards

The main results of this section are the following theorem and its corollary. For the definition of stable families see Definition 3.11 and for the definition of variation see Definition 6.16 and Remark 6.17. We also use Notation 3.12 in the next statement.

**Theorem 7.1.** *If $f : (X, D) \to Y$ is a family of stable log-varieties of maximal variation over a smooth projective variety, then*

(1) *there exists an $r > 0$ such that $K_{X^{(r)}/Y} + D_{X^{(r)}}$ is big on at least one component of $X^{(r)}$, or equivalently*
$$\left(K_{X^{(r)}/Y} + D_{X^{(r)}}\right)^{\dim X^{(r)}} > 0,$$
*and*

(2) *for every sufficiently divisible $q > 0$, $\det f_* \mathscr{O}_X(q(K_{X/Y} + D))$ is big.*

*Remark* 7.2. The $r$-th fiber power in point (1) of Theorem 7.1 cannot be dropped. This is because there exist families $f : X \to Y$ of maximal variation that are not varying maximally on any of the components of $X$. Note the following about such a family:

(1) $K_{X/Y}$ cannot be big on any component $X_i$ of $X$. Indeed, since the variation of $f|_{X_i}$ is not maximal, after passing to a generically finite cover of $X_i$, $K_{X/Y}|_{X_i}$ is a pull back from a lower dimensional variety.

(2) On the other hand, $X^{(r)} \to Y$ will have a component of maximal variation for $r \gg 0$. In particular, $K_{X^{(r)}/Y}$ does have a chance to be big on at least one component.

To construct a family as above, start with two non-isotrivial smooth families $g_i : Z_i \to C_i$ ($i = 1, 2$) of curves of different genera, both at least two [BPVdV84, Sec V.14]. Take a multisection on each of these. By taking a base-change via the multisections, we may assume that in fact each $g_i$ is endowed with a section $s_i : C_i \to Z_i$. Now define $f_1 := g_1 \times \mathrm{id}_{C_2} : X_1 := Z_1 \times C_2 \to Y := C_1 \times C_2$ and $f_2 := \mathrm{id}_{C_1} \times g_2 : X_2 := C_1 \times Z_2 \to Y$. The section $s_i$ of $g_i$ induce sections of $f_i$ as well. Let $D_i$ be the images of these. Then, according to [Kol13b, Thm 5.13], $(X_1, D_1)$ and $(X_2, D_2)$ glue along $D_1$ and $D_2$ to form a stable family $f : X \to Y$ as desired. Also notice that in this example $f^{(2)} : X^{(2)} \to Y$ has a component of maximal variation.

**Corollary 7.3.** *Any algebraic space that is the coarse moduli space of a functor of stable log-varieties with fixed volume, dimension and coefficient set (see Definition 6.2) is a projective variety over $k$.*

The rest of the section contains the proofs of Theorem 7.1 and Corollary 7.3. The first major step is Proposition 7.8, which needs a significant amount of notation to be introduced.

**Definition 7.4.** For a $\mathbb{Q}$-Weil divisor $D$ on a demi-normal variety and for a $c \in \mathbb{Q}$ we define the $c$-coefficient part of $D$ to be the reduced effective divisor
$$D_c := \sum_{\mathrm{coeff}_E D = c} E,$$
where the sum runs over all prime divisors. Clearly
$$D = \sum_{c \in \mathbb{Q}} c D_c.$$
Notice that $D_c$ is invariant under any automorphism of the pair $(X, D)$, that is, under any automorphism of $X$ that leaves $D$ invariant. In fact, an automorphism of $X$ is an automorphism of the pair $(X, D)$ if and only if it leaves $D_c$ invariant for every $c \in \mathbb{Q}$.

**Definition 7.5.** Let $f : (X, D) \to Y$ be a family of stable log-varieties. We will say that the *coefficients of $D$ are compatible with base-change* if for each $c \in \mathbb{Q}$ and $y \in Y$,

$$D_c|_{X_y} = (D_y)_c.$$

Note that this condition is automatically satisfied if all the coefficients are greater than $\frac{1}{2}$.

**Notation 7.6.** Let $f : (X, D) \to Y$ be a family of stable log-varieties over a smooth projective variety. For a fixed $m \in \mathbb{Z}$ that is divisible by the Cartier index of $K_{X/Y} + D$, and an arbitrary $d \in \mathbb{Z}$ set $\mathscr{L}_d := \mathscr{O}_X(dm(K_{X/Y} + D))$.

Observe that there exists a dense big open subset $U \subseteq Y$ over which all the possible unions of the components of $D$ (with the reduced structure) are flat. Our goal is to apply Theorem 5.1 for $f_U : X_U \to U$ (we allow shrinking $U$ after fixing $d$ and $m$, while keeping $U$ a big open set).

Next we will group the components of $D$ according to their coefficients. Recall the definition of $D_c$ from Definition 7.4 where $c \in \mathbb{Q}$ and observe that there is an open set $V \subseteq U$ over which

(A) $D_c$ is compatible with base-change as in Definition 7.5 for all $c \in \mathbb{Q}$, and
(B) the scheme theoretic fiber of $D_c$ over $v \in V$ is reduced and therefore is equal to its divisorial restriction (see the definition of the latter in Notation 3.7).

To simplify notation we will make the following definitions: Let $\{c_1, \dots, c_n\} := \{c \in \mathbb{Q} \mid D_c \neq \emptyset\}$ be the set of coefficients appearing in $D$ and let $D_i := D_{c_i}$, for $i = 1, \dots, n$.

Next we choose an $m \in \mathbb{Z}$ satisfying the following conditions for every integers $j, d > 0$ and $i = 1, \dots, n$:

(C) $m(K_{X/Y} + D)$ is Cartier,
(D) $\mathscr{L}_d = \mathscr{O}_X(dm(K_{X/Y} + D))$ is $f$-very ample,
(E) $R^j f_* \mathscr{L}_d = 0$,
(F) $\left(R^j \left(f|_{D_i}\right)_* \mathscr{L}_d|_{D_i}\right)\big|_V = 0$, and
(G) $(f_* \mathscr{L}_1)|_V \to \left(\left(f|_{D_i}\right)_* \mathscr{L}_1|_{D_i}\right)|_V$ is surjective.

These conditions imply that

(H) $\mathbb{N} \ni N := h^0(\mathscr{L}_1|_{X_y}) - 1$ is independent of $y \in Y$, and in fact
(I) $f_* \mathscr{L}_d$ and $\left(\left(f|_{D_i}\right)_* \mathscr{L}_d|_{D_i}\right)|_V$ are locally free and compatible with base-change.

By possibly increasing $m$ we may also assume that

(J) the multiplication maps

$$\mathrm{Sym}^d(f_* \mathscr{L}_1) \to (f_* \mathscr{L}_d) \quad \text{and} \quad \mathrm{Sym}^d(f_* \mathscr{L}_1)|_V \to \left(\left(f|_{D_i}\right)_* \mathscr{L}_d|_{D_i}\right)|_V$$

are surjective.

For the surjectivity of the map $\mathrm{Sym}^d(f_* \mathscr{L}_1)|_V \to \left(\left(f|_{D_i}\right)_* \mathscr{L}_d|_{D_i}\right)|_V$ we write it as the composition of the restriction map $\mathrm{Sym}^d (f_* \mathscr{L}_1)|_V \to \mathrm{Sym}^d \left(\left(f|_{D_i}\right)_* \mathscr{L}_1|_{D_i}\right)|_V$ and the multiplication map $\mathrm{Sym}^d \left(\left(f|_{D_i}\right)_* \mathscr{L}_1|_{D_i}\right)|_V \to \left(\left(f|_{D_i}\right)_* \mathscr{L}_d|_{D_i}\right)|_V$. The former is surjective by the choice of $m$ and condition (G) while the surjectivity of the latter follows by the finite generation of the relative section ring, after an adequate increase of $m$.

We fix an $m$ satisfying the above requirements for the rest of the section and use the global sections of $\mathscr{L}_1|_{X_y}$ to embed $X_y$ (and hence $D_i|_{X_y}$ as well) into the fixed projective space $\mathbb{P}^N_k$ for every closed point $y \in V$. The ideal sheaves corresponding to these embeddings will be denoted by $\mathscr{I}_{X_y}$ and $\mathscr{I}_{D_i|_{X_y}}$ respectively. As the embedding of $X_y$ is well-defined only up to the action of $\mathrm{GL}(N+1,k)$, the corresponding ideal sheaf is also well-defined only up to this action. Furthermore, in what follows we deal with only such properties of $X_y$, $D_i|_{X_y}$, $\mathscr{I}_{X_y}$ and $\mathscr{I}_{D_i|_{X_y}}$ that are invariant under the $\mathrm{GL}(N+1,k)$ action.

So, finally, we choose a $d>0$ such that

(K) for all $y \in V$, $X_y$ as well as $D_i|_{X_y}$ are defined by degree $d$ equations.

From now on we keep $d$ fixed with the above chosen value and we supress it from the notation. We make the following definitions:

(L) $W := \mathrm{Sym}^d(f_*\mathscr{L}_1)|_U$, and
(M) $Q_0 := (f_*\mathscr{L}_d)|_U$.

Further note that $(f|_{D_i})_*\mathscr{L}_d|_{D_i}$ is torsion-free, since $f|_{D_i}$ is surjective on all components and $D_i$ is reduced. Hence by possibly shrinking $U$, but keeping it still a big open set, we may assume that

(N) $Q_i := ((f|_{D_i})_*\mathscr{L}_d|_{D_i})|_U$ is locally free for all $i = 1, \dots, n$.

Our setup ensures that we have natural homomorphisms $\alpha_i : W \to Q_i$ which are surjective over $V$ and we may make the following identifications for all closed points $y \in V$ up to the above explained $\mathrm{GL}(N+1,k)$ action:

$$W \otimes k(y) \longleftrightarrow H^0\left(\mathbb{P}^N, \mathscr{O}_{\mathbb{P}^N}(d)\right)$$

$$Q_0 \otimes k(y) \longleftrightarrow H^0\left(X_y, \mathscr{O}_{X_y}(d)\right)$$

$$Q_i \otimes k(y) \longleftrightarrow H^0\left(D_i|_{X_y}, \mathscr{O}_{D_i|_{X_y}}(d)\right), \text{ for } i>0$$

$$\ker\left[W \otimes k(y) \to Q_0 \otimes k(y)\right] \longleftrightarrow H^0\left(\mathbb{P}^N, \mathscr{I}_{X_y}(d)\right)$$

$$\ker\left[W \otimes k(y) \to Q_i \otimes k(y)\right] \longleftrightarrow H^0\left(\mathbb{P}^N, \mathscr{I}_{D_i|_{X_y}}(d)\right), \text{ for } i>0.$$

We will use this setup and notation for the rest of the present section.

**Lemma 7.7.** *Let $f : (X,D) \to Y$ be a family of stable log-varieties over a normal proper variety $Y$, and let $m>0$ be an integer such that*

*(1) $m(K_{X/Y}+D)$ is Cartier,*
*(2) $m(K_{X/Y}+D)$ is relatively basepoint-free with respect to $f$, and*
*(3) $R^i f_*\mathscr{O}_X(m(K_{X/Y}+D)) = 0$ for all $i>0$.*

*Then $f_*\mathscr{O}_X(m(K_{X/Y}+D))$ is a nef locally free sheaf. Further note, that the above conditions and hence the statement hold for every sufficiently divisible $m$. In particular, it applies for the $m$ chosen in Notation 7.6, and hence $f_*\mathscr{L}_d$ is weakly positive for all $d>0$.*

*Proof.* The assumptions guarantee that $f_*\mathscr{O}_X(m(K_{X/Y}+D))$ is compatible with base-change. As being nef is decided on curves, we may assume that $Y$ is a smooth curve. Note that then by the slc version of inversion of adjunction (e.g., [Pat16a, Cor 2.11]) $(X,D)$ itself is slc. Hence, [Fuj12, Theorem 1.13] applies and yields the statement. □

**Proposition 7.8.** *In addition to Notation 7.6, assume that* $\operatorname{Var} f$ *is maximal. Then for all* $d \gg 0$,

$$\det f_* \mathscr{L}_d \otimes \left(\otimes_{i=1}^n \det\left((f|_{D_i})_* \mathscr{L}_d|_{D_i}\right)\right) \text{ is big.}$$

*Proof.* Note that $f_*\mathscr{L}_1$ is weakly positive by Lemma 7.7 and hence so is $W = \operatorname{Sym}^d f_*\mathscr{L}_1$. This will allow us to use Theorem 5.1 in the situation of Notation 7.6 by setting $G := \mathrm{GL}(N+1,k)$ (see Remark 5.3) with the natural action on $W$ if we prove that the restriction over $V$ of the classifying map of the morphisms $\alpha_i$ for $i = 0, \dots, n$ have finite fibers.

Translating this required finiteness to geometric terms means that fixing a general $y \in V(k)$ and the fiber $X_y$, there are only finitely many other general $z \in V(k)$, such that for the fiber $X_z$ the degree $d$ forms in the ideals of $X_y$ and $D_i|_{X_y}$ can be taken by the same $\phi \in \mathrm{GL}(N+1,k)$ to the degree $d$ forms in the ideals of $X_z$ and $D_i|_{X_z}$. However, if such a $\phi$ exists, then $(X_y, D_y) \simeq (X_z, D_z)$ meaning that $y$ and $z$ lie in the same fiber of the associated moduli map $\mu : Y \to \mathscr{M}_{m,v,I}$ (see Section 6.A). The maximal variation assumption implies that $\mu$ is generically finite, so there is an open $Y^0 \subseteq Y$, over which $\mu$ has finite fibers, which is exactly what we need. By shrinking $V$, we may assume that $V \subseteq Y^0$ and applying Theorem 5.1 yields the statement. □

**Lemma 7.9.** *Let* $f : (X, D) \to Y$ *be a family of stable log-varieties over a smooth variety. Then* $D_c|_T$ *is flat for all* $c \in \mathbb{Q}$, *where* $T$ *is the locus over which* $D_c$ *is Cartier. Note that* $T|_{X_y}$ *is a big open set for every* $y \in Y$.

*Proof.* As $\mathscr{O}_{D_c|_T}$ is the cokernel of $\varepsilon : \mathscr{O}_T(-D_c) \to \mathscr{O}_T$, it is enough to prove that $\varepsilon_y : \mathscr{O}_T(-D_c) \otimes \mathscr{O}_{T_y} \to \mathscr{O}_{T_y}$ is injective for every $y \in Y$ [StacksProject, Tag 00MD]. However, as $\mathscr{O}_T(-D_c) \otimes \mathscr{O}_{T_y}$ is a line bundle on $T_y$, and hence $S_2$, and the map $\varepsilon_y$ is an isomorphism, in particular injective, at every generic point of $T_y$, it is in fact injective everywhere. □

**Lemma 7.10.** *Let* $f : (X, D) \to Y$ *be a family of stable log-varieties over a smooth variety. Then* $D_c \to Y$ *is an equidimensional morphism for all* $c \in \mathbb{Q}$.

*Proof.* By assumption $D_c$ has codimension 1 in $X$ and it does not contain any irreducible components of any fiber. It follows that the general fiber of $D_c$ over $Y$ has codimension 1 in the corresponding fiber of $X$ and that this is the maximum dimension any of its fibers may achieve. Since the dimension of the fibers is semicontinuous this implies that all fibers of $D_c$ have the same dimension. □

**Lemma 7.11.** *Let* $f : (X, D) \to Y$ *be a family of stable log-varieties over a smooth variety. Let* $Z$ *be the fiber product* $X^{(r)}$ *over* $Y$ *of some copies of* $X$ *and of the* $D_i = D_{c_i}$*'s. Then*

*(1) every irreducible component of* $Z$ *dominates* $Y$,
*(2) there is a big open set of* $Y$ *over which* $Z$ *is flat and reduced,*
*(3)* $Z$ *is equidimensional over* $X$, *and*
*(4)* $X^{(r)}$ *is regular at every generic point of* $Z$.

*Proof.* First notice that (3) follows directly from Lemma 7.10.

Next recall that we have already noted in Notation 7.6 that there exists a big open set $U \subseteq Y$, over which $X$ and all the possible unions of the components of $D$ are flat, and hence so is $Z$. It follows that all the embedded points of $Z$ over $U$

map to the generic point $\eta$ of $Y$. However $Z_\eta$ is reduced, so $Z$ is not only flat, but also reduced over $U$. This proves (2).

On the other hand, $Z$ can definitely have multiple irreducible or even connected components. Assume that there exists an irreducible component $S$ that does not dominate $Y$. Then $S$ is contained in the non-flat locus of $Z$. However, according to Lemma 7.9, the non-flat locus of $D_i$ has codimension at least one in each fiber of $D_i \to Y$ for all $i$'s. Therefore, the non-flat locus of $Z$ also has codimension at least one in each fiber. Hence, the existence of $S$ would contradict (3) (and ultimately Lemma 7.10). This proves (1).

By (1) the generic points of $Z$ are dominating the generic points of $D_i$. At these points the corresponding fibers of $X$ are regular and so (4) follows. $\square$

Notation 7.12 is used in the proof of Theorem 7.1.1, which is presented right after it.

**Notation 7.12.** Assume that we are in the situation of Notation 7.6, in particular, recall the definition $D_i = D_{c_i}$. To simplify the notation we also set $D_0 := X$. For a fixed positive natural number $r \in \mathbb{N}_+$ consider a partition of $r$: i.e., a set of natural numbers $r_i \in \mathbb{N}$ for $i = 0, \dots, n$ such that $\sum_{i=0}^n r_i = r$. We will denote a partition by $[r_0, r_1, \dots, r_n]$. For $[r_0, r_1, \dots, r_n]$ we define the following *mixed product* (we omit $Y$ from the notation for sanity):

$$D_\bullet^{(r_0,r_1,\dots,r_n)} := \left( \bigtimes_{i=0}^{n}{}_{Y} D_i^{(r_i)} \right)_{\mathrm{red}} = \left( D_0^{(r_0)} \times_Y \cdots \times_Y D_n^{(r_n)} \right)_{\mathrm{red}}.$$

Observe that $D_\bullet^{(r_0,r_1,\dots,r_n)}$ is naturally a closed subscheme of $X_Y^{(r)}$.

Let us assume now that $r_j > 0$ for some $j$. Then $[r_0+1, r_1, \dots, r_j - 1, \dots, r_n]$ is another partition of the same $r$ and up to reordering the terms of the products

$$D_\bullet^{(r_0,r_1,\dots,r_n)} \subset D_\bullet^{(r_0+1,r_1,\dots,r_j-1,\dots,r_n)}$$

is a reduced effective Weil divisor. It follows from Lemma 7.11 that no component of $D_\bullet^{(r_0,r_1,\dots,r_n)}$ is contained in the singular locus of $D_\bullet^{(r_0+1,r_1,\dots,r_j-1,\dots,r_n)}$. In particular, for a sequence of partitions,

$$\begin{gathered}[r_0, r_1, r_2, \dots, r_n], [r_0+1, r_1-1, r_2, \dots, r_n], \dots, [r_0+r_1, 0, r_2, \dots, r_n],\\ [r_0+r_1+1, 0, r_2-1, \dots, r_n], \dots, [r_0+r_1+r_2, 0, 0, \dots, r_n], \dots\\ [r_0+\dots+r_{n-1}, 0, \dots, 0, r_n], [r_0+\dots+r_{n-1}+1, 0, \dots, 0, r_n-1], \dots, [r, 0, \dots, 0, 0]\end{gathered}$$

we obtain a filtration of $X^{(r)}$ where each consecutive embedding is a reduced effective Weil divisor in the subsequent member of the filtration and furthermore no component of the former is contained in the singular locus of the latter:

$$\begin{gathered}D_\bullet^{(r_0,r_1,r_2,\dots,r_n)} \subset D_\bullet^{(r_0+1,r_1-1,r_2,\dots,r_n)} \subset \dots \subset D_\bullet^{(r_0+r_1,0,r_2,\dots,r_n)} \subset\\ \subset D_\bullet^{(r_0+r_1+1,0,r_2-1,\dots,r_n)} \subset \dots \subset D_\bullet^{(r_0+r_1+r_2,0,0,\dots,r_n)} \subset \dots\\ \subset D_\bullet^{(r_0+\dots+r_{n-1},0,\dots,0,r_n)} \subset D_\bullet^{(r_0+\dots+r_{n-1}+1,0,\dots,0,r_n-1)} \subset \dots \subset D_\bullet^{(r,0,\dots,0,0)} = X^{(r)}.\end{gathered}$$

In fact, using Lemma 7.11, one can see that for every (not necessarily subsequent) pair $D' \subseteq D''$ of schemes appearing in the above filtration, $D''$ is regular at the generic points of $D'$. Indeed, according to Lemma 7.11 every generic point $\xi$ of $D'$ is over the generic point $\eta$ of $Y$. Hence it is enough to see that $D''_\eta$ is regular at $\xi$. Observe, that $D''_\eta$ is a product over $\operatorname{Spec} k(\eta)$, and not over a positive dimensional scheme as $D''$ is. Hence it is enough to see that all the components of $D''_\eta$ are regular at the image of $\xi$ under the appropriate projection $X^{(r)} \to X$. However, this follows immediately from our definition of stable families (Definition 3.11), that is, by the assumption that $D_i$ avoid the codimension one singular points of the fibers and hence in particular of $X_\eta$.

*Proof of Theorem 7.1.1.* We will use the setup established in Notation 7.6 and 7.12. As before, $f_*\mathscr{L}_d$ is a nef vector bundle by Lemma 7.7. Therefore, by the surjective natural map $f^*f_*\mathscr{L}_d \to \mathscr{L}_d$, $K_{X/Y} + D$ is nef as well. Clearly the same holds for $K_{X^{(j)}/Y} + D_{X^{(j)}}$ for any integer $j > 0$.

Now, let $r_0 := \operatorname{rk} f_*\mathscr{L}_d$ and for $i > 0$ let $r_i := \operatorname{rk} (f|_{D_i})_* \mathscr{L}_d|_{D_i}$. Furthermore, set $r := \sum_{i=0}^n r_i$, $Z := D_\bullet^{(r_0,r_1,\dots,r_n)}$ and $\eta : \widetilde{Z} \to Z$ the normalization of $Z$. Note that $Z$ can be reducible and a priori even non-reduced, but it is a closed subscheme of $X^{(r)}$, its irreducible components dominate $Y$ and non-reducedness on $Z$ may happen only in codimension greater than 2 by Lemma 7.11.

Consider the natural injection below, which can be defined first over the big open set $U \subseteq Y$ of Notation 7.6, and then extended reflexively to $Y$,

$$\iota_d : \mathscr{A}_d := \det(f_*\mathscr{L}_d) \otimes \left( \bigotimes_{i=1}^n \det\left((f|_{D_i})_* \mathscr{L}_d|_{D_i}\right) \right) \hookrightarrow$$

$$\hookrightarrow \bigotimes_1^{r_0} f_*\mathscr{L}_d \otimes \left[ \bigotimes_{i=1}^n \right] \left( \bigotimes_{j=1}^{r_i} (f|_{D_i})_* \mathscr{L}_d|_{D_i} \right) \simeq \tag{7.12.1}$$

$$\simeq \underbrace{\left( \left( f^{(r)}\Big|_Z \right)_* \mathscr{L}_d^{(r)}\Big|_Z \right)^{**}}_{\text{iterated use of Lemma 3.6}}.$$

By a slight abuse of notation we will denote the composition of restriction from $X^{(r)}$ to $Z$ and the pull-back via the normalization morphism $\eta : \widetilde{Z} \to Z$ by restriction to $\widetilde{Z}$. In other words we make the following definition:

$$(\_)|_{\widetilde{Z}} := \eta^* \circ (\_)|_Z$$

So, for instance, $\left(f^{(r)}|_{\widetilde{Z}}\right)^*$ denotes the pulling back by the composition

$$\widetilde{Z} \xrightarrow{\eta} Z \hookrightarrow X^{(r)} \xrightarrow{f^{(r)}} Y .$$

As in its definition above, if we restrict $\iota_d$ to $U$, then the reflexive hulls are unecessary on the right hand side of (7.12.1). Then by adjointness we obtain a non-zero homomorphism

$$\left( f^{(r)}\Big|_Z \right)^* \mathscr{A}_d\Big|_U \to \mathscr{L}_d^{(r)}\Big|_{\left(f^{(r)}|_Z\right)^{-1}U} .$$

Pulling this further back over $\widetilde{Z}$ yields a non-zero homomorphism

$$\left( f^{(r)}\Big|_{\widetilde{Z}} \right)^* \mathscr{A}_d\Big|_U \to \mathscr{L}_d^{(r)}\Big|_{\left(f^{(r)}|_{\widetilde{Z}}\right)^{-1}U} . \tag{7.12.2}$$

Since $Z \to Y$ and hence also $\widetilde{Z} \to Y$ is an equidimensional morphism, $\left( f^{(r)}\big|_{\widetilde{Z}} \right)^{-1} U$ is also a big open set in $\widetilde{Z}$ and hence (7.12.2) induces a non-zero homomorphism

$$\left( f^{(r)}\Big|_{\widetilde{Z}} \right)^{*} \mathscr{A}_d \to \mathscr{L}_d^{(r)}\Big|_{\widetilde{Z}} . \tag{7.12.3}$$

The non-zero map (7.12.3) induces another non-zero map

$$\mathscr{L}_d^{(r)}\Big|_{\widetilde{Z}} \otimes \left( f^{(r)}\Big|_{\widetilde{Z}} \right)^{*} \mathscr{A}_d \to \left( \mathscr{L}_d^{(r)} \right)^{\otimes 2}\Big|_{\widetilde{Z}} ,$$

where on the left hand side we have a relatively ample and nef line bundle tensored with the pullback of a big line bundle. Hence the line bundle on the left hand side is big on every component of $\widetilde{Z}$. Therefore the line bundle on the right hand side is big on at least one component. Let $L^{(r)}$ denote a Cartier divisor corresponding to $\mathscr{L}_d^{(r)}$. For the self-intersection of $L^{(r)}$ on (the normalization) of a subvariety, say $Z'$, we use the notation $L^{(r)}\big|_{Z'}^{\dim Z'}$, which formally means $\left( L^{(r)}\big|_{Z'} \right)^{\dim Z'}$. Then by the nefness of $L^{(r)}$ it follows that

$$0 < L^{(r)}\Big|_{\widetilde{Z}}^{\dim \widetilde{Z}} ,$$

and then also

$$0 < L^{(r)}\Big|_{Z}^{\dim Z} . \tag{7.12.4}$$

Next we will define a filtration starting with $X^{(r)}$ and ending with $Z$ where each consecutive member is a reduced divisor in the previous member. Recall that $r = \sum_{i=0}^{n} r_i$ and observe that for any integer $r_0 \leq t < r$ there is a unique $0 \leq j < n$ such that

$$\sum_{i=0}^{j} r_i \leq t < \sum_{i=0}^{j+1} r_i .$$

and hence

$$0 \leq t_{j+1} := t - \sum_{i=0}^{j} r_i < r_{j+1} .$$

Now recall Notation 7.12 and let us define $Z_r := X^{(r)}$ and for any $t$, $r_0 \leq t < r$,

$$Z_t := D_{\bullet}^{\left(\sum_{i=0}^{j} r_i + t_{j+1}, \overbrace{0, \dots, 0}^{j \text{ times}}, r_{j+1} - t_{j+1}, r_{j+2}, \dots, r_n\right)} .$$

Notice that $Z_{r_0} = Z$ and that for all $t$, $r_0 \leq t < r$, $Z_t \subset Z_{t+1}$ is a reduced effective divisor without components contained in the singular locus of $Z_{t+1}$ (see Notation 7.12 for the explanation). Note that set theoretically $Z_t$ is the intersection of $Z_{t+1}$ with $p_t^* D_{j+1}$. We claim that this is in fact true also divisorially. Indeed, $Z_t$ is reduced and by Lemma 7.11 it is equidimensional. So, it is enough to check that $Z_t$ and the divisorial restriction $p_t^* D_{j+1}$ agrees at all codimension one points $\xi$ of $Z_{t+1}$. If $p_t^* D_{j+1}$ contains $\xi$ in its support, then $D_{j+1}$ contains $p_t(\xi)$, hence $p_t(\xi)$ has to be a codimension 1 regular point of $X$ lying over the generic point $\eta$ of $Y$. Note $\operatorname{mult}_\xi p_t^* D_{j+1} = \operatorname{mult}_{p_t(\xi)} D_{j+1} = 1$, and that $Z_{j+1}$ contains exactly the same codimension one points of $Z_{t+1}$, which concludes our claim that

$$Z_t = p_t^* D_{j+1}|_{Z_{t+1}} . \tag{7.12.5}$$

Our goal is to show that

$$0 < \left(L^{(r)}\right)^{\dim X^{(r)}} \left( = \left(L^{(r)}\right)^{\dim Z_r} \right).$$

For any rational number $1 \gg \varepsilon > 0$ we have

$$\left(L^{(r)}\right)^{\dim Z_r} = \left(L^{(r)}\right)^{\dim Z_r} + \sum_{t=r_0}^{r-1} \varepsilon^{r-t} \left( L^{(r)}\Big|_{Z_t}^{\dim Z_t} - L^{(r)}\Big|_{Z_t}^{\dim Z_t} \right) =$$
$$= \varepsilon^{r-r_0} \, L^{(r)}\Big|_{Z}^{\dim Z} + \sum_{t=r_0}^{r-1} \varepsilon^{r-t-1} \left( L^{(r)}\Big|_{Z_{t+1}}^{\dim Z_{t+1}} - \varepsilon \, L^{(r)}\Big|_{Z_t}^{\dim Z_t} \right).$$

Thus, according to (7.12.4), it is enough to prove that for each integer $r_0 \leq t < r$,

$$0 \leq \, L^{(r)}\Big|_{Z_{t+1}}^{\dim Z_{t+1}} - \varepsilon \, L^{(r)}\Big|_{Z_t}^{\dim Z_t}. \tag{7.12.6}$$

In the rest of the proof we fix an integer $r_0 \leq t < r$, and prove (7.12.6) for that value of $t$. Let $\widetilde{Z}_{t+1}$ be the normalization of $Z_{t+1}$, and let $S$ be the strict transform of $Z_t$ in $\widetilde{Z}_{t+1}$. Denote by $\rho$ the composition $\widetilde{Z}_{t+1} \to Z_{t+1} \to X^{(r)}$. According to the discussion in Notation 7.12, $\widetilde{Z}_{t+1} \to Z_{t+1}$ is an isomorphism at the generic point of $Z_t$. Hence it is enough to prove that

$$0 \leq \left(\rho^* L^{(r)}\right)^{\dim \widetilde{Z}_{t+1}} - \varepsilon \left(\rho^* L^{(r)}\Big|_{S}\right)^{\dim \widetilde{Z}_{t+1}-1} = \left(\rho^* L^{(r)}\right)^{\dim \widetilde{Z}_{t+1}-1} \cdot \left(\rho^* L^{(r)} - \varepsilon S\right).$$

Note that the right most expression is the intersection of several Cartier divisors with a Weil $\mathbb{Q}$-divisor, and hence it is well-defined. Furthermore, since $\rho^* L^{(r)}$ is nef, to prove the above inequality it is enough to prove that the $\mathbb{Q}$-divisor $\left(\rho^* L^{(r)} - \varepsilon S\right)$ is pseudo-effective on every component of $\widetilde{Z}_{t+1}$. This follows if we apply Lemma 7.13 by setting $Z := Z_{t+1}$, $\widetilde{Z} := \widetilde{Z}_{t+1}$, $E := p_t^* D_{j+1}$ and by using (7.12.5) (and its implication that $S = p_t^* D_{j+1}|_{\widetilde{Z}_{t+1}}$). □

Recall that a $\mathbb{Q}$-Weil divisor $D$ is called $\mathbb{Q}$-effective if $mD$ is linearly equivalent to an effective divisor for some integer $m > 0$.

**Lemma 7.13.**

*(1) Let $f : (X, D) \to Y$ be an equidimensional, surjective, projective morphism from a semi-log canonical pair onto a smooth projective variety, such that $K_{X/Y} + D$ is $f$-ample and all irreducible components of $X$ dominate $Y$.*

*(2) Let $Z$ be a closed subscheme of $X$, which is equidimensional over $Y$, reduced, and all its irreducible components dominate $Y$.*

*(3) Let $E$ be a reduced effective divisor on $X$ with support in* Supp $D$*, in particular, no component of $E$ is contained in the singular locus of $X$. Assume that $E$ does not contain any component of $Z$ and that both $Z$ and $X$ are regular at the generic points of $Z$ and at the codimension one points of $Z$ that are contained in $E$.*

*(4) Let $\widetilde{Z} \to Z$ be the normalization.*

*Then $(K_{X/Y} + D - \varepsilon E)|_{\widetilde{Z}}$ is pseudo-effective for every $\varepsilon \in \mathbb{Q}$, $0 < \varepsilon \ll 1$, meaning that for any fixed ample divisor $A$ on $\widetilde{Z}$, $(K_{X/Y} + D - \varepsilon E)|_{\widetilde{Z}} + \delta A$ is $\mathbb{Q}$-effective on every component of $\widetilde{Z}$ for every $\delta \in \mathbb{Q}, 0 < \delta \ll 1$.*

*Remark* 7.14. In the above statement $E|_{\widetilde{Z}}$ is defined by considering the (big) open locus in $Z$, where $E$ is Cartier, pulling back to $\widetilde{Z}$ and taking the closure there using that the complement has codimension at least 2.

*Proof.* **Reduction step:** Let $\pi : (\overline{X}, \overline{D}) \to (X, D)$ be the normalization and $\overline{Z}$ and $\overline{E}$ the strict transforms (by the regularity assumptions $\pi$ is an isomorphism at all generic points of $\overline{Z}$ and $\overline{E}$ so these strict transforms are meaningful). Since $\widetilde{Z} \to Z$ factors through $\overline{Z} \to Z$, this setup shows that we may assume that $(X, D)$ is log canonical.

**Summary of assumptions after the reduction step:**

(1) $f : (X, D) \to Y$ is an equidimensional, surjective, projective morphism from a log canonical pair onto a smooth projective variety, such that $K_{X/Y} + D$ is $f$-ample,
(2) $Z$ is equidimensional over $Y$, reduced, and all its irreducible components dominate $Y$,
(3) $\operatorname{Supp} E \subseteq \operatorname{Supp} D$,
(4) no irreducible component of $Z$ is contained in the support of $E$, and
(5) regularity assumptions: $X$ is regular at the generic points of $Z$ and both $E$ and $Z$ are regular at the codimension one points of $Z$ that are contained in $E$.

**The argument.** Set $L := K_{X/Y} + D$, $\mathscr{L} := \mathscr{O}_X(L)$ and $S := E|_{\widetilde{Z}}$ and let $\rho$ be the composition $\widetilde{Z} \to Z \to X$. Note that to establish that $\rho^*L - \varepsilon S$ is pseudo-effective one may use an arbitrary Cartier divisor $A$ on $\widetilde{Z}$, and show that $\rho^*L - \varepsilon S + \delta A$ is $\mathbb{Q}$-effective on every component for every $0 < \delta \ll 1$. Indeed, choosing an ample $A'$, it follows that $tA' - A$ is effective on every component for some $t \gg 0$, and hence then

$$\rho^*L - \varepsilon S + \delta t A' = \rho^*L - \varepsilon S + \delta A + \delta(tA' - A)$$

is also $\mathbb{Q}$-effective on every component as well. Here we will choose $A$ to be the pullback of an appropriate ample line bundle on $Y$.

Let us take a $\mathbb{Q}$-factorial dlt model $\tau : (T, \Theta) \to (X, D)$ such that $K_T + \Theta = \tau^*(K_X + D)$ (cf. [KK10, 3.1]) and define $g := f \circ \tau$. Note that $\tau$ is an isomorphism both at the generic points of $Z$ and at the codimension one points of $Z$ that are contained in $E$, since $X$ is regular at all these points. Set $\Gamma := \tau_*^{-1}E$. Consider

$$q\tau^*L - \Gamma = q\left(K_{T/Y} + \Theta - \frac{1}{q}\Gamma\right).$$

for a sufficiently divisible integer $q > 0$. There are two important facts about the above divisor. On one hand,

$$\tau_*\mathscr{O}_T(q\tau^*L - \Gamma) \subseteq \mathscr{O}_X(qL - E), \tag{7.13.1}$$

on the other hand, the above divisor is the $q^{\text{th}}$ multiple of the relative log-canonical divisor of a dlt pair. Hence according to [Fuj14a, Thm 1.1], for every sufficiently divisible $q$,

$$g_*\mathscr{O}_T(q\tau^*L - \Gamma)$$

is weakly positive. Therefore after fixing an ample line bundle $H$ on $Y$, for each $a > 0$, there is a $b > 0$, such that

$$\operatorname{Sym}^{[ab]}(g_*\mathscr{O}_T(q\tau^*L - \Gamma)) \otimes H^b$$

is generically globally generated.

Let $U$ be the open set where both $g_*\mathscr{O}_T(q\tau^*L-\Gamma)$ and $f_*\mathscr{O}_X(qL-E)$ are locally free. Over $U$ consider the composition of the following homomorphisms, where the left most one is the push-forward of the embedding in (7.13.1):

$$\begin{aligned} f^*\operatorname{Sym}^{ab}(g_*\mathscr{O}_T(q\tau^*L-\Gamma)) \to f^*\operatorname{Sym}^{ab}(f_*\mathscr{O}_X(qL-E)) \to \\ \to f^*f_*\mathscr{O}_X(ab(qL-E)) \to \mathscr{O}_X(ab(qL-E)). \end{aligned} \tag{7.13.2}$$

Let us pause for a moment and recall that $qL-E$ is not necessarily Cartier in general. However, it is Cartier over a big open set of $f^*U$, so the natural map $\operatorname{Sym}^{ab}(f_*\mathscr{O}_X(qL-E)) \to f_*\mathscr{O}_X(ab(qL-E))$, which yields the middle arrow above, can still be constructed over that big open set and then extended uniquely, since $X$ is normal.

Setting $h := f\circ\rho$, still over $U$, we obtain the following natural morphism by pulling back the composition of (7.13.2) via $\rho$.

$$h^*\operatorname{Sym}^{ab}(g_*\mathscr{O}_T(q\tau^*L-\Gamma)) \to \mathscr{O}_{\widetilde{Z}}(ab(q\rho^*L-S)).$$

Again, note that $qL-E$ is not necessarily Cartier over $Z$. However, by our regularity assumption it is Cartier over a big open set $U_Z$ of $Z$. So the above map is constructed first over $\rho^{-1}(U_Z\cap f^{-1}U)$ and then extended uniquely using that $\widetilde{Z}$ is normal.

So, since $\widetilde{Z}\to Y$ is equidimensional, $h^{-1}U$ is a big open set of $\widetilde{Z}$. In particular, we obtain a homomorphism

$$h^{[*]}\operatorname{Sym}^{[ab]}(g_*\mathscr{O}_T(q\tau^*L-\Gamma))\otimes h^*H^b \to \mathscr{O}_{\widetilde{Z}}(ab(q\rho^*L-S))\otimes h^*H^b. \tag{7.13.3}$$

Now choose $q$ sufficiently divisible so that $\tau_*\mathscr{O}_T(q\tau^*L-\Gamma)\simeq\mathscr{O}_X(qL)\otimes\tau_*\mathscr{O}_T(-\Gamma)$ is $f$-globally generated (recall that $L$ is $f$-ample). Note that the ideal $\tau_*\mathscr{O}_T(-\Gamma)$ is supported on $\operatorname{Supp}E$ and $\operatorname{Supp}E$ does not contain any component of $Z$ by assumption. Hence, it follows that the natural map

$$h^*g_*\mathscr{O}_T(q\tau^*L-\Gamma)\to\mathscr{O}_{\widetilde{Z}}(q\rho^*L-S)$$

is surjective at all generic points of $\widetilde{Z}$ and then the same holds for the map in (7.13.3). Furthermore, the sheaf on the left hand side in (7.13.3) is globally generated at every generic point of $\widetilde{Z}$. This gives us the desired sections of $\mathscr{O}_{\widetilde{Z}}(ab(q\rho^*L-S))\otimes h^*H^b$ and concludes the proof. □

We will need the following analog of Lemma 4.6 for reducible schemes.

**Lemma 7.15.** *If $X$ is a projective scheme of pure dimension $n$ over $k$ and $L$ a nef Cartier divisor which is big on at least one component (that is, $L^n>0$), then for every Cartier divisor $D$ that does not contain any component of $X$, $L-\varepsilon D$ is $\mathbb{Q}$-effective for every rational number $0<\varepsilon\ll 1$ (however the corresponding effective divisor may be zero on every irreducible component but one).*

*Proof.* Let $\mathscr{L}:=\mathscr{O}_X(L)$. Consider the exact sequence,

$$0\longrightarrow\mathscr{L}^a(-D)\longrightarrow\mathscr{L}^a\longrightarrow\mathscr{L}^a|_D\longrightarrow 0$$

Since $L$ is nef, by the asymptotic Riemann-Roch Theorem [Laz04a, Corollary 1.4.41], $h^0(L^a)=\frac{a^n}{n!}L^n+O(a^{n-1})$. Furthermore, $h^0(\mathscr{L}^a|_D)=O(a^{n-1})$. Hence, for every $a\gg 0$ $H^0(\mathscr{L}^a(-D))\neq 0$. □

Theorem 7.1.2 is an immediate consequence of the following statement.

**Proposition 7.16.** *If $f : (X, D) \to Y$ is a family of stable log-varieties of maximal variation over a normal proper variety, then there exists an integer $q > 0$ and a proper closed subvariety $S \subseteq Y$, such that for every integer $a > 0$, and closed irreducible subvariety $T \subseteq Y$ not contained in $S$, $\det f_* \mathscr{O}_X(aq(K_{X/Y} + D))|_{\widetilde{T}}$ is big, where $\widetilde{T}$ is the normalization of $T$.*

*Proof.* First note that $q$ may be chosen sufficiently divisible, so $f_* \mathscr{O}_X(aq(K_{X/Y} + \Delta))$ commutes with base-change, and hence we may replace $Y$ by any of its resolutions. That is, we may assume that $Y$ is smooth and projective. We may also replace $\widetilde{T}$ by a resolution of $T$ in the statement.

Let $H$ be any ample effective Cartier divisor on $Y$, and let $\mathscr{H} := \mathscr{O}_Y(H)$ be the associated line bundle. Let $r > 0$ be the integer given by Theorem 7.1.1. Since every component of $X^{(r)}$ dominates $Y$, according to Lemma 7.15, $q(K_{X^{(r)}/Y} + D_{X^{(r)}}) - \left(f^{(r)}\right)^* H$ is linearly equivalent to an effective divisor for some multiple $q$ of $dm$. Equivalently, there is a non-zero map

$$\left(f^{(r)}\right)^* \mathscr{H} \to \mathscr{O}_{X^{(r)}} \left(q \left(K_{X^{(r)}/Y} + D_{X^{(r)}}\right)\right). \tag{7.16.1}$$

Let $S \subseteq Y$ be the (proper) closed set over which (7.16.1) is zero. For any integer $a > 0$ consider the following non-zero map induced by the $a^{\text{th}}$ tensor power of (7.16.1).

$$\begin{aligned} \mathscr{H}^a \simeq f^{(r)}_* \left(f^{(r)}\right)^* \mathscr{H}^a \to f^{(r)}_* \mathscr{O}_{X^{(r)}} \left(aq \left(K_{X^{(r)}/Y} + D_{X^{(r)}}\right)\right) \simeq \\ \simeq \underbrace{\bigotimes^r f_* \mathscr{O}_X(aq(K_{X/Y} + D))}_{\text{Lemma 3.6}} \end{aligned} \tag{7.16.2}$$

This is necessarily an embedding, because $Y$ is integral. Let $\sigma : \widetilde{T} \to Y$ be a resolution of an irreducible closed subset $T$ of $Y$ that is not contained in $S$. Then, the induced map

$$\sigma^* \mathscr{H}^a \to \bigotimes^r \sigma^* f_* \mathscr{O}_X \left(aq \left(K_{X/Y} + D\right)\right) \simeq \bigotimes^r \left(f_{\widetilde{T}}\right)_* \mathscr{O}_{X_{\widetilde{T}}} \left(aq \left(K_{X_{\widetilde{T}}/\widetilde{T}} + D_{\widetilde{T}}\right)\right)$$

is not zero and therefore it is actually an embedding.

Let $\mathscr{B}$ denote the saturation of $\sigma^* \mathscr{H}^a$ in $\bigotimes^r \left(f_{\widetilde{T}}\right)_* \mathscr{O}_{X_{\widetilde{T}}/\widetilde{T}}(aq(K_{X_{\widetilde{T}}/\widetilde{T}} + D_{\widetilde{T}}))$. Then $\mathscr{B}$ is big since $\mathscr{H}$ is ample and it induces another exact sequence

$$0 \longrightarrow \mathscr{B} \longrightarrow \bigotimes^r \left(f_{\widetilde{T}}\right)_* \mathscr{O}_{X_{\widetilde{T}}} \left(aq \left(K_{X_{\widetilde{T}}/\widetilde{T}} + D_{\widetilde{T}}\right)\right) \longrightarrow \mathscr{G} \longrightarrow 0,$$

where $\mathscr{G}$ is locally free in codimension one. $\left(f_{\widetilde{T}}\right)_* \mathscr{O}_{X_{\widetilde{T}}}(aq(K_{X_{\widetilde{T}}/\widetilde{T}} + D_{\widetilde{T}}))$ is nef by Lemma 7.7, so $\mathscr{G}$ is weakly-positive by [Vie95, prop 2.9.e] and point (2) of Lemma 4.8. Note that we cannot infer that $\mathscr{G}$ is nef, since $\mathscr{G}$ is not necessarily locally free. However, we can infer that $\det \mathscr{G}$ is weakly-positive as well by (1) of Lemma 4.8 and then for some $N > 0$,

$$\begin{aligned} \mathscr{B} \otimes \det \mathscr{G} \simeq \det \left( \bigotimes^r \left(f_{\widetilde{T}}\right)_* \mathscr{O}_{X_{\widetilde{T}}} \left(aq \left(K_{X_{\widetilde{T}}/\widetilde{T}} + D_{\widetilde{T}}\right)\right)\right) \simeq \\ \simeq \left(\det \left(f_{\widetilde{T}}\right)_* \mathscr{O}_X \left(aq \left(K_{X_{\widetilde{T}}/\widetilde{T}} + D_{\widetilde{T}}\right)\right)\right)^N \end{aligned}$$

is big by (4) of Lemma 4.8. This concludes the proof. ☐

*Proof of Corollary 7.3.* Let $M$ be the algebraic space in the statement, and $\mathscr{M}$ the (pseudo-)functor that it coarsely represents. First note that by finiteness of the automorphism groups (Proposition 6.5), an appropriate power of the functorial polarization required in Definition 6.2 descends to $M$. Since $M$ is proper by Proposition 6.4, according to the Nakai-Moishezon criterion we only need to show that the highest self-intersection of this polarization on every proper irreducible subspace of $M$ is positive. However, by Corollary 6.19 it is enough to show this, instead of $M$, for a proper, normal scheme $Z$, that supports a family $f : (X_Z, D_Z) \to Z$ with the property that each fiber of $f$ is isomorphic to only finitely many others.

Let us state our goal precisely at this point: we are supposed to exhibit an $r > 0$ such that for any closed irreducible subvariety $V \subseteq Z$,

$$c_1 \left(\det (f_V)_* \mathscr{O}_{X_V} \left(r \left(K_{X_V/V} + D_V\right)\right)\right)^{\dim V} > 0.$$

In fact we, are proving something slightly stronger. We claim that *there exist an integer $q > 0$, such that for every integer $a > 0$ and closed irreducible subvariety $V \subseteq Z$,*

$$c_1 \left(\det (f_V)_* \mathscr{O}_{X_V} \left(aq \left(K_{X_V/V} + D_V\right)\right)\right)^{\dim V} > 0.$$

We prove this statement by induction. For $\dim Z = 0$ it is vacuous, so we may assume that $\dim Z > 0$. By Proposition 7.16 there exist a $q_Z > 0$ and a closed subset $S \subsetneq Z$ that does not contain any component of $Z$, such that for every $a > 0$ and every irreducible closed subset $T \subseteq Z$ not contained in $S$, if we set $\mathscr{N}_{aq_Z} := \det f_* \mathscr{O}_{X_Z} \left(aq_Z \left(K_{X_Z/Z} + D_Z\right)\right)$, then $c_1 \left(\mathscr{N}_{aq_Z}|_T\right)^{\dim T} > 0$. Let $\widetilde{S}$ denote the normalization of $S$. Then by induction, since $\dim S < \dim Z$, there exists a $q_{\widetilde{S}} > 0$, such that for every $a > 0$ and every irreducible closed subset $V \subseteq \widetilde{S}$, $c_1\left(\mathscr{N}_{aq_{\widetilde{S}}}|_V\right)^{\dim V} > 0$. Taking $q := \max\{q_Z, q_{\widetilde{S}}\}$ concludes the proofs of the claim and of Corollary 7.3. □

*Remark* 7.17. If one allows labeling of the components as well, which was excluded up to this point from Definition 6.2 for simplicity, then Theorem 7.1 still yields projectivity as in Corollary 7.3 for the unlabeled case. This follows from the fact that each stable log-variety admits at most finitely many labelings. Hence, forgetting the labeling of a labeled family with finite isomorphism equivalence classes yields a non-labeled family with finite isomorphism equivalence classes. In particular, the proof of Corollary 7.3 implies that the polarization by $\det f_* \mathscr{O}_X(dm(K_{X/Y} + D))$ yields an ample line bundle on the base of the labeled family as well.

## 8. Pushforwards without determinants

The main goal of this section is to prove the following result.

**Theorem 8.1.** *If $f : (X, D) \to Y$ is a family of stable log-varieties of maximal variation over a normal, projective variety $Y$ with klt general fiber, then $f_* \mathscr{O}_X(q(K_{X/Y} + D))$ is big for every sufficiently divisible integer $q > 0$.*

*Remark* 8.2. One might wonder if this could be true without assuming that the general fiber is klt. We will show below that that assumption is in fact necessary.

**Corollary 8.3.** *If $f : (X, D) \to Y$ is a family of stable log-varieties of maximal variation over a normal, projective variety $Y$ with klt general fiber, then $K_{X/Y} + D$ is big.*

This corollary follows from Theorem 8.1 by a rather general argument which we present in the following lemma.

**Lemma 8.4.** *Let $f : X \to Y$ be a surjective morphism between normal proper varieties and assume also that $Y$ is projective. Let $\mathscr{L}$ be an $f$-big line bundle on $X$ such that $f_*\mathscr{L}$ is a big vector bundle. Then $\mathscr{L}$ itself is big.*

*Proof.* Choose an ample line bundle $\mathscr{A}$ on $Y$ such that $f^*\mathscr{A} \otimes \mathscr{L}$ is big. Then by Definition 4.7 there is a generically isomorphic inclusion for some integer $a > 0$:

$$\bigoplus \mathscr{A} \hookrightarrow \operatorname{Sym}^a(f_*\mathscr{L})$$

This induces the following non-zero composition of homomorphisms, which concludes the proof:

$$\bigoplus \underbrace{f^*\mathscr{A} \otimes \mathscr{L}}_{\text{big}} \hookrightarrow f^* \operatorname{Sym}^a(f_*\mathscr{L}) \otimes \mathscr{L} \to f^* f_*(\mathscr{L}^a) \otimes \mathscr{L} \to \mathscr{L}^{a+1}. \qquad \square$$

*Proof of Corollary 8.3.* Take $\mathscr{L} = \mathscr{O}_X(q(K_{X/Y} + D))$ for a sufficiently divisible $q > 0$. □

Next we show that the klt assumption in Theorem 8.1 is necessary.

**Example 8.5.** Let $f : X \to Y$ be an arbitrary non-isotrivial smooth projective family of curves over a smooth projective curve. Assume that it admits a section $\sigma : Y \to X$ (this can be easily achieved after a base change) and let $D = \operatorname{im} \sigma \subset X$. This is one of the simplest examples of a family of stable log-varieties. Notice that the fibers are log canonical, but not klt. By adjunction $K_D = (K_X + D)|_D$ and as $f|_D : D \to Y$ is an isomorphism, it follows that $\mathscr{O}_X(K_{X/Y} + D)|_D \simeq \mathscr{O}_D$. The following claim implies that $f_*\mathscr{O}_X(r(K_{X/Y} + D))$ cannot be big for for any integer $r > 0$.

*Claim 8.5.1.* Let $f : X \to Y$ be a flat morphism, $\mathscr{L}$ a torsion-free sheaf on $X$, and $\mathscr{E}$ a locally free sheaf on $Y$. Further let $D \subset X$ be the image of a section $\sigma : Y \to X$ and assume that $Y$ is irreducible, that $\mathscr{L}|_D \subseteq \mathscr{O}_D$, and that there exists a homomorphism $\varrho : f^*\mathscr{E} \to \mathscr{L}$ such that $\varrho|_D \neq 0$. Then $\mathscr{E}$ cannot be big.

*Proof.* Since $f|_D$ is an isomorphism, if $\mathscr{E}$ were big, so would be $(f^*\mathscr{E})|_D$ and then $\varrho|_D \neq 0$ would imply that $\mathscr{O}_D$ is big. This is a contradiction which proves the statement. □

A variant of Example 8.5 shows that even assuming that $D = 0$ would not be enough to get the statement of Theorem 8.1 without the klt assumption:

**Example 8.6.** Let $f : X \to Y$ be an arbitrary non-isotrivial smooth projective family of curves over a smooth projective curve. Assume that it admits two disjoint sections $\sigma_i : Y \to X$ for $i = 1, 2$ and let $D_i = \operatorname{im} \sigma \subset X$. Next glue $X$ to itself by identifying $D_1$ and $D_2$ via the isomorphism $\sigma_1 \circ \sigma_2^{-1}$ and call the resulting variety $X'$. Then the induced $f' : X' \to Y$ is a family of stable varieties. The same computation as above shows that $f'_*\mathscr{O}_{X'}(rK_{X'/Y})$ cannot be big for any $r > 0$ for this example as well. For computing the canonical class of non-normal varieties see [Kol13b, 5.7].

A variant of the above examples can be found in [Kee99, Thm. 3.0], for which not only $K_{X/Y} + D$ is numerically trivial on a curve $C$ contained in $D$ (and hence other ones can be constructed where the same happens over the double locus), but $K_{X/Y} + D|_C$ is not even semi-ample.

One might complain that in Example 8.6 the fibers are not normal. One can construct a similar example of a family of stable varieties where the general fiber is log canonical (and hence normal) that shows that the klt assumption is necessary, but this is a little bit more complicated.

**Example 8.7.** Let $Z$ be a projective cone over a genus 1 curve $C$. Assume that $Z \subseteq \mathbb{P}^3$ is embedded compatibly with this cone structure, that is, via this embedding, $Z \cap \mathbb{P}^2 = C$ for some fixed $\mathbb{P}^2 \subseteq \mathbb{P}^3$. Fix also coordinates $x_0, \dots, x_3$ such that $x_1, x_2, x_3$ are coordinates for $\mathbb{P}^2$ and the cone point is $P := [1, 0, 0, 0]$. Choose two general polynomials $f(x_1, x_2, x_3)$ and $g(x_0, x_1, x_2, x_3)$. Consider the pencil of hypersurfaces in $Z$ defined by these two equations. This yields a hypersurface $\mathscr{D} \subseteq Z \times \mathbb{P}^1$ with $\mathscr{D}_0 = V(f) \cap Z$ a general conic hypersurface section of $Z$ and $\mathscr{D}_\infty = V(g) \cap Z$ a general hypersurface section of $Z$. Since $g$ was chosen generally, $P \notin \mathscr{D}_\infty$. On the other hand, $P \in \mathscr{D}_0$, and hence $P \notin \mathscr{D}_t$ for $t \neq 0$. Furthermore, since in codimension 1 hypersurface sections of $Z$ disjoint from $P$ acquire only nodes $\mathscr{D}_t$ is either smooth or has only nodes for $t \neq 0$. Hence, for $d \gg 0$ the family $(Z \times \mathbb{P}^1, \mathscr{D}) \to \mathbb{P}^1$ is a family of stable log-varieties outside $t = 0$. For $t = 0$ we run stable reduction. Since the stable limit is unique, we may figure out the stable limit without going through the meticulous process by hand: it is enough to exhibit one family that is isomorphic in a neighborhood of 0 to the original family after a base-change and which does have a stable limit. The pencil $\mathscr{D}$ around $t = 0$ is described by the equation $f(x_1, x_2, x_3) + tg(x_0, x_1, x_2, x_3)$. Extract a $d$-th root from $t$ and denote the new family also by $(Z \times \operatorname{Spec} k[t], \mathscr{D})$ (i.e., we keep the same notation for the boundary). Then $\mathscr{D}$ around $t = 0$ is described by the equation

$$F_1(t, x_0, x_1, x_2, x_3) := f(x_1, x_2, x_3) + t^d g(x_0, x_1, x_2, x_3).$$

Now set

$$F_2(t, x_0, x_1, x_2, x_3) := f(x_1, x_2, x_3) + t^d g(x_0/t, x_1, x_2, x_3),$$

and let $\mathscr{D}'$ be the hypersurface of $Z \times \operatorname{Spec} k[t]$ defined by $F_2$. Then in a punctured neighborhood of $t = 0$, $(Z \times \operatorname{Spec} k[t], \mathscr{D})$ is isomorphic to $(Z \times \operatorname{Spec} k[t], \mathscr{D}')$, via the map

$$x_i \mapsto x_i (i \neq 0) \qquad t \mapsto t \qquad x_0 \mapsto t \cdot x_0.$$

Here the key is that Z, being a cone, is invariant under scaling by $x_0$. Note that since $g$ is general, $x_0^d$ has a non-zero coefficient, say $c$. Then it is easy to see that $F_2(0, x_1, x_2, x_3) = f(x_1, x_2, x_3) + cx_0^d$. That is, $\mathscr{D}'_0$ is a $d$-th cyclic cover of $V(f) \cap C \subseteq \mathbb{P}^2$ in $Z$. Since $f$ is general, $V(f) \cap C$ is smooth (i.e., a union of reduced points), and hence $\mathscr{D}'_0$ is also smooth. Furthermore, $\mathscr{D}'_0$ avoids $P$. It follows that $(Z, \mathscr{D}'_0)$ is log canonical, whence stable and therefore it has to be the central fiber of the stable reduction.

Summarizing, after the stable reduction, we obtain a family $(\mathscr{Z}, \mathscr{D}) \to Y$ of stable log-pairs over a smooth projective curve (we denote the divisor by $\mathscr{D}$ here as well for simplicity), such that $\mathscr{Z}_y \simeq Z$ and $\mathscr{D}_y$ avoids the cone point in $\mathscr{Z}_y$ for each $y \in Y$. Note that $\mathscr{Z}$ cannot be isomorphic to $Y \times Z$ anymore (not even after a proper base-change), since then $\mathscr{D}_y$ would give a proper family of moving divisors in $Z$ that does not contain $P$. This is impossible, since a proper family covers a proper image, which would have to be the entire $Z$.

In any case, after possibly a finite base-change, we are able to take the cyclic cover of $\mathscr{X}$ of degree $d$ ramified along $\mathscr{D}$. For $d \gg 0$ the obtained family $X \to Y$ is stable of maximal variation over the projective curve $Y$. It has elliptic singularities along a curve $B$ that covers $d$ times the singularity locus of $\mathscr{X} \to Y$. Hence, $B \to Y$ is proper and has $d$ preimages over each point. In particular it is étale (though $B$ might be reducible). If we blow-up $B$, and resolve the other singular points as well (which are necessarily disjoint from $B$, since they originate from the nodal fibers of $\mathscr{D} \to Y$), we obtain a resolution $\pi : V \to X$. Let $E$ be the (reduced) preimage of $B$. Then we have that $K_{V/Y} + E + F \equiv \pi^* K_{X/Y}$, where $F$ is exceptional and disjoint from $E$. In particular then

$$K_{E/Y} \equiv (K_{V/Y} + E)|_E \equiv (K_{V/Y} + E + F)|_E \equiv \pi^* K_{X/Y}|_E \equiv (\pi|_E)^* \left(K_{X/Y}|_B\right).$$

Hence it is enough to show that $K_{E/Y} \equiv 0$ (since then we have found a horizontal curve over which $K_{X/Y}$ is numerically trivial). Since $B \to Y$ is étale, it is enough to show that $K_{E/B} \equiv 0$. However $E \to B$ is a smooth family of isomorphic genus one curves. In particular, after a finite base-change we may also assume that it has a section, in which case we do know that its relative canonical sheaf is numerically trivial. However, then it is numerically trivial even without the base-change. It follows that $K_{X/Y}|_B$ is numerically trivial and the same argument as above shows that then it cannot be big.

Recall that if $(X, D)$ is a klt pair and $\Gamma$ a $\mathbb{Q}$-Cartier divisor, then the log canonical threshold is defined as

$$\operatorname{lct}(\Gamma; X, D) := \sup\{t | (X, D + t\Gamma) \text{ is log canonical }\}.$$

**Lemma 8.8.** *The log canonical threshold is lower semi-continuous in projective, flat families with $\mathbb{Q}$-Cartier relative log canonical bundle. That is, if $f : (X, D) \to S$ is a projective, flat morphism with $S$ normal and essentially of finite type over $k$ such that $K_{X/S} + D$ is $\mathbb{Q}$-Cartier, $(X_s, D_s)$ is klt for all $s \in S$ and $\Gamma \geq 0$ is a $\mathbb{Q}$-Cartier divisor on $X$ not containing any fibers, then $\operatorname{lct}(\Gamma_s; X_s, D_s)$ is lower semi-continuous.*

*Furthermore, if $S$ is regular, then for every $s \in S$ there is a neighborhood $U$ of $s$, such that*

$$\operatorname{lct}(\Gamma|_{f^{-1}U}; f^{-1}U, D|_{f^{-1}U}) \geq \operatorname{lct}(\Gamma_s; X_s, D_s).$$

*Proof.* Let us first show the second statement, which is an application of inversion of adjunction. Let $A = f^{-1}H$ for some very ample reduced effective divisor $H$. Then

$$\begin{aligned}(A, D|_A + t\Gamma|_A) \text{ is lc } &\Rightarrow (X, D + t\Gamma + A) \text{ is lc in a neighborhood of } A \Rightarrow \\ &\Rightarrow (X, D + t\Gamma) \text{ is lc in a neighborhood of } A.\end{aligned}$$

Applying this inductively gives the second statement, since for regular schemes every point can be (locally) displayed as the intersection of hyperplanes.

Next, let us prove that $s \mapsto \operatorname{lct}(\Gamma_s; X_s, D_s)$ is constant on a dense open set $U$ and that $U$ can be chosen such that $\operatorname{lct}(\Gamma|_{f^{-1}U}; f^{-1}U, D|_{f^{-1}U})$ agrees with this constant value. For this we may assume that $S$ is smooth. Take a log-resolution $\pi : Y \to X$ of $(X, D + \Gamma)$. By replacing $S$ with a dense Zariski open set we may assume that $(Y, \operatorname{Exc} \pi + \pi^* D + \pi^* \Gamma)$ is relative simple normal crossing over $S$. That is, every stratum is smooth over $S$, where a stratum is either $Y$ or the intersection of any collection of divisors showing up. However, then the discrepancies of $(X_s, D_s +$

$t\Gamma_s$) agree for all $s \in S$ and $t \in \mathbb{Q}$ and furthermore, this is the same set as the discrepancies of $(X, D + t\Gamma)$. This concludes our claim.

The above two claims show that we have semi-continuity over smooth curves, and also that the function is constructible. These together show that the function is semi-continuous in general. ☐

**Definition 8.9.** We define the *log canonical threshold of a line bundle* $\mathscr{L}$ on a projective pair $(X, \Delta)$ as the minimum of the log canonical thresholds of the effective divisors in $\mathbb{P}\left(H^0(X, \mathscr{L})^*\right)$, the complete linear system of $\mathscr{L}$:

$$\operatorname{lct}(\mathscr{L}; X, \Delta) := \min\left\{\operatorname{lct}(\Gamma; X, \Delta) \,\middle|\, \Gamma \in \mathbb{P}\left(H^0(X, \mathscr{L})^*\right)\right\}.$$

By the above lemma this minimum exists.

**Lemma 8.10.** *The log canonical threshold of a line bundle is bounded in projective, flat families. That is, let $f : (X, D) \to T$ be a projective flat morphism with $T$ normal and essentially of finite type over $k$ and $\mathscr{L}$ a line bundle on $X$. Assume that $(X_t, D_t)$ is klt for all $t \in T$ and $K_{X/T} + D$ is $\mathbb{Q}$-Cartier. Then there exists a real number $c$, such that* $\operatorname{lct}(\mathscr{L}_t; X_t, D_t) > c$ *for all $t \in T$.*

*Proof.* First assume that $f_*\mathscr{L}$ commutes with base-change (and it is consequently locally free) and let $\mathbb{P} := \operatorname{Proj}_T((f_*\mathscr{L})^*)$. Notice that the points of $\mathbb{P}_t$ for $t \in T$ may be identified with elements of the linear systems $\mathbb{P}\left(H^0(X, \mathscr{L})^*\right)$. Further let $\Gamma$ be the universal divisor on $X \times_T \mathbb{P}$ corresponding to $\mathscr{L}$, that is, $(x, [D]) \in \Gamma$ iff $x \in D$. Now, applying Lemma 8.8 to $X \times_T \mathbb{P} \to \mathbb{P}$ and $\Gamma$ yields the statement.

In the general case, we work by induction on the dimension of $T$. We can find a dense open set over which $f_*\mathscr{L}$ commutes with base change. So, there is a lower bound as above over this open set, and there is another lower bound on the complement. Combining the two gives a lower bound over the entire $T$. ☐

**Proposition 8.11.** *Let $(X, D_X)$ and $(Y, D_Y)$ be two klt pairs and $\mathscr{L}$ and $\mathscr{N}$ line bundles on $X$ and $Y$ respectively. Then*

$$\operatorname{lct}\left(p_X^*\mathscr{L} \otimes p_Y^*\mathscr{N}; X \times Y, p_X^* D_X + p_Y^* D_Y\right) = \min\{\operatorname{lct}(\mathscr{L}; X, D_X), \operatorname{lct}(\mathscr{N}; Y, D_Y)\}$$

*Proof.* It is obvious that

$$\operatorname{lct}(p_X^*\mathscr{L} \otimes p_Y^*\mathscr{N}; X \times Y, p_X^* D_X + p_Y^* D_Y) \leq \min\{\operatorname{lct}(\mathscr{L}; X, D_X), \operatorname{lct}(\mathscr{N}; Y, D_Y)\}.$$

We have to prove the opposite inequality. To do that, choose $\Gamma \in |p_X^*\mathscr{L} \otimes p_Y^*\mathscr{N}|$ and

$$t < \min\{\operatorname{lct}(\mathscr{L}; X, D_X), \operatorname{lct}(\mathscr{N}; Y, D_Y)\}.$$

We have to show that $(X \times Y, p_X^* D_X + p_Y^* D_Y + t\Gamma)$ is log canonical.

Let $\tau : \widetilde{Y} \to (Y, D_Y)$ be a log-resolution, and define $D_{\widetilde{Y}}$ via the equality

$$K_{\widetilde{Y}} + D_{\widetilde{Y}} := \tau^*(K_Y + D_Y).$$

Let $\rho : X \times \widetilde{Y} \to X \times Y$ be the product morphism, and denote by $\pi_X$ and $\pi_{\widetilde{Y}}$ the two projections $X \times \widetilde{Y} \to X$ and $X \times \widetilde{Y} \to \widetilde{Y}$, respectively. According to [Vie95, Claim 5.20], $\widetilde{Y}$ can be chosen such that $\rho^*\Gamma = \Gamma' + \pi_{\widetilde{Y}}^*\Delta$ where $\Delta$ is simple normal crossing on $\widetilde{Y}$ and $\Gamma'$ contains no fibers of $\pi_{\widetilde{Y}}$. By further blowing up $\widetilde{Y}$ if necessary, we may also assume that $\Delta + D_{\widetilde{Y}}$ is simple normal crossing.

To prove that $(X\times Y, p_X^*D_X + p_Y^*D_Y + t\Gamma)$ is log canonical, it is enough to prove that $X\times\widetilde{Y}$ is log canonical with the following boundary (where we are allowing the boundary to have negative coefficients):

$$\pi_X^*D_X + \pi_{\widetilde{Y}}^*D_{\widetilde{Y}} + t\rho^*\Gamma = \pi_X^*D_X + \pi_{\widetilde{Y}}^*(D_{\widetilde{Y}} + t\Delta) + t\Gamma'$$

Let $E$ be the reduced divisor supported on $\operatorname{Supp}\left(D_{\widetilde{Y}} + \Delta\right)$.

*Claim* 8.11.1. It is enough to show that $(X\times\widetilde{Y}, \pi_X^*D_X + \pi_{\widetilde{Y}}^*E + t\Gamma')$ is log canonical.

Indeed, (8.11.1) follows as soon as the coefficients of $D_{\widetilde{Y}} + \Delta$ are at most 1. To see that, let $x\in X$ be a general closed point and let $Y_x = \{x\}\times Y\subseteq X\times Y$ and $\widetilde{Y}_x = \{x\}\times\widetilde{Y}\subseteq X\times Y$. Further let $D_{Y_x}$ and $D_{\widetilde{Y}_x}$ denote the divisors corresponding to $D_Y$ and $D_{\widetilde{Y}}$ respectively via the obvious isomorphisms $Y_x\simeq Y$ and $\widetilde{Y}_x\simeq\widetilde{Y}$. Then, $(Y_x, D_{Y_x} + t\,(\Gamma|_{Y_x}))$ is klt by the assumption $t < \operatorname{lct}(\mathscr{N}; Y, D_Y)$. However, then $\widetilde{Y}_x$ is also klt with the boundary:

$$D_{\widetilde{Y}_x} + t\left(\rho^*\Gamma|_{\widetilde{Y}_x}\right) = D_{\widetilde{Y}_x} + t\left(\pi_{\widetilde{Y}}^*\Delta + \Gamma'|_{\widetilde{Y}_x}\right).$$

Then, it follows that $(\widetilde{Y}, D_{\widetilde{Y}} + t\Delta)$ is klt as well. Since, the support of $D_{\widetilde{Y}} + t\Delta$ is a simple normal crossing divisor this implies that the coefficients are at most 1, which in turn implies (8.11.1).

Next, let $G$ be an arbitrary fiber of $\pi_{\widetilde{Y}} : X\times\widetilde{Y}\to\widetilde{Y}$. Then $G\simeq X$ and via this isomorphism $(\pi_X^*D_X + t\Gamma')\,|_G$ corresponds to $D_X + t\Gamma''$, where $\Gamma''\in|\mathscr{L}|$. In particular, it follows that $(G, (\pi_X^*D_X + t\Gamma')|_G)$ is klt.

Let $y\in\widetilde{Y}$ be a closed point and if necessary, add new components to $E$ such that the intersection of the components $E_1,\dots,E_{\dim Y}$ of $E$ containing $y$ is equal to $\{y\}$. Then

$$\left(\bigcap_{i=1}^{\dim Y}\pi_{\widetilde{Y}}^*E_i\ ,\left(\pi_X^*D_X + \pi_{\widetilde{Y}}^*\sum_{i>\dim Y}E_i + t\Gamma'\right)\Bigg|_{\bigcap_{i=1}^{\dim Y}\pi_{\widetilde{Y}}^*E_i}\right) = (G, (\pi_X^*D_X + t\Gamma')|_G)$$

is klt, so in particular log canonical. Then by inversion of adjunction [Kaw07] and downward induction on $j$, we obtain that

$$\left(\bigcap_{i\le j}\pi_{\widetilde{Y}}^*E_i\ ,\left(\pi_X^*D_X + \pi_{\widetilde{Y}}^*\sum_{i>j}E_i + t\Gamma'\right)\Bigg|_{\bigcap_{i\le j}\pi_{\widetilde{Y}}^*E_i}\right)$$

is also log canonical for every $j = \dim Y, \dim Y - 1,\dots,0$. In particular,

$$(X\times\widetilde{Y}, \pi_X^*D_X + \pi_{\widetilde{Y}}^*E + t\Gamma') = \left(\bigcap_{i\le 0}\pi_{\widetilde{Y}}^*E_i\ ,\left(\pi_X^*D_X + \pi_{\widetilde{Y}}^*\sum_{i>0}E_i + t\Gamma'\right)\Bigg|_{\bigcap_{i\le 0}\pi_{\widetilde{Y}}^*E_i}\right)$$

is also log canonical, which is what we needed to prove according to (8.11.1). □

For the next statement recall Notation 3.12.

**Corollary 8.12.** *If $(X,D)$ is a projective klt pair, $\mathscr{L}$ a line bundle on $X$, then for all integers $m>0$,*

$$\operatorname{lct}\left(\mathscr{L}^{(m)}; X^{(m)}, D_{X^{(m)}}\right) = \operatorname{lct}(\mathscr{L}; X, D).$$

In the next statement multiplier ideals are used. Recall that the *multiplier ideal* of a pair $(X, D)$ consisting of a normal variety and an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier is $\mathscr{J}(X, D) := \tau_* \mathscr{O}_Z(\lceil K_{Z/X} - \tau^* D \rceil) \subseteq \mathscr{O}_X$.

**Lemma 8.13.** *Let $(X, D)$ be a pair, and $A$ a general element of a base-point free linear system $V$ on $X$. Then $\mathscr{J}(X, D) = \mathscr{J}(A, D|_A)$.*

*Proof.* Let $\tau : Z \to X$ be a log resolution of $(X, D)$. The key observation is that defining $H := \tau_*^{-1} A = \tau^* A$, the restriction $\rho := \tau|_H : H \to A$ is also a log-resolution and that $H + \tau_*^{-1} D + \operatorname{Exc} \tau$ is simple normal crossing. Consider the the following exact sequence

(8.13.1)

$$0 \longrightarrow \mathscr{O}_Z(\lceil K_Z - \tau^*(K_X + A + D)\rceil) \longrightarrow$$
$$\longrightarrow \mathscr{O}_Z(\lceil K_Z + H - \tau^*(K_X + A + D)\rceil) \longrightarrow$$
$$\longrightarrow \mathscr{O}_H(\lceil K_H - \rho^*(K_A + D|_A)\rceil) \longrightarrow 0$$

According to [Laz04b, Theorem 9.4.15(i)] $R^1 \tau_* \mathscr{O}_Z(\lceil K_Z - \tau^*(K_X + A + D)\rceil) = 0$, and by the projection formula,

$$\begin{aligned} &\tau_* \mathscr{O}_Z(\lceil K_Z - \tau^*(K_X + A + D)\rceil) \\ &\qquad \simeq \mathscr{O}_X(-A) \otimes \tau_* \mathscr{O}_Z(\lceil K_Z - \tau^*(K_X + D)\rceil) \\ &\qquad\qquad \simeq \mathscr{O}_X(-A) \otimes \tau_* \mathscr{O}_Z(\lceil K_Z + H - \tau^*(K_X + A + D)\rceil). \end{aligned} \tag{8.13.2}$$

Applying $f_*(\_\_)$ to (8.13.1) and using (8.13.2) and the above vanishing yields the statement. ☐

**Proposition 8.14.** *Let $f : X \to Y$ be a surjective morphism between projective, normal varieties with equidimensional, reduced $S_2$ fibers, $L$ a Cartier divisor and $\Delta \geq 0$ an effective divisor on $X$ such that $\Delta$ contains no general fibers, $(X_y, \Delta_y)$ is klt for general $y \in Y$ and $L - K_{X/Y} - \Delta$ is a nef and f-ample $\mathbb{Q}$-Cartier divisor. Assume further that $K_Y$ is Cartier. Then $f_* \mathscr{O}_X(L)$ is weakly-positive (in the weak sense).*

*Proof.* Set $\mathscr{L} := \mathscr{O}_X(L)$. Let $A$ be a general very ample effective divisor on $Y$ and $m > 0$ an integer. In this proof a subscript $(\_)_A$ will denote a base change to $A$.

*Claim* 8.14.1. For any nef Cartier divisor $N$ on $Y$ the natural restriction map,

$$H^0\left(X^{(m)}, \mathscr{J}\left(X^{(m)}, \Delta_{X^{(m)}}\right) \otimes \mathscr{L}^{(m)}\left(\left(f^{(m)}\right)^*(K_Y + 2A + N)\right)\right) \longrightarrow$$
$$\longrightarrow H^0\left(X_A^{(m)}, \mathscr{J}\left(X_A^{(m)}, \Delta_{X_A^{(m)}}\right) \otimes \mathscr{L}^{(m)}\left(\left(f^{(m)}\right)^*(K_Y + 2A + N)\right)\Big|_{X_A^{(m)}}\right)$$

is surjective.

*Proof.* Note that in the statement we are already using the fact that

$$\mathscr{J}\left(X_A^{(m)}, \Delta_{X_A^{(m)}}\right) \simeq \mathscr{O}_{X_A^{(m)}} \otimes \mathscr{J}\left(X^{(m)}, \Delta_{X^{(m)}}\right),$$

which follows from Lemma 8.13. For the above homomorphism to be surjective, it is enough to prove that

$$H^1\left(X^{(m)}, \mathscr{J}\left(X^{(m)}, \Delta_{X^{(m)}}\right) \otimes \mathscr{L}^{(m)}\left(\left(f^{(m)}\right)^*(K_Y + A + N)\right)\right) = 0 \tag{8.14.2}$$

However,

$$L_{X^{(m)}} + \left(f^{(m)}\right)^* (K_Y + A + N) - (K_{X^{(m)}} + \Delta_{X^{(m)}}) =$$

$$= \underbrace{(L - K_{X/Y} - \Delta)_{X^{(m)}}}_{\substack{\text{nef and} \\ \text{relatively ample}}} + \left(f^{(m)}\right)^* (\underbrace{A + N}_{\text{ample}})$$

is ample, hence (8.14.2) holds by Nadel-vanishing. This proves the claim. □

Note that the assumptions of the proposition remain valid for $f|_{X_A} : X_A \to A$ and $\Delta|_{X_A}$. Hence we may use (8.14.1) iteratively. By the klt assumption on the general fiber, we may further leave out the multiplier ideal in the last term. Thus we obtain a surjective homomorphism

$$H^0\left(X^{(m)}, \mathscr{J}\left(X^{(m)}, \Delta_{X^{(m)}}\right) \otimes \mathscr{L}^{(m)}\left(\left(f^{(m)}\right)^*\left(K_Y + A + \sum_{i=1}^n A_i\right)\right)\right) \longrightarrow$$

$$\longrightarrow H^0\left(X_y^{(m)}, \mathscr{L}_y^{(m)}\right)$$

where $A_1, \dots, A_n \in |A|$ are general, $y \in \bigcap_{i=1}^n A_i$ is arbitrary and $n := \dim Y$. Since the left hand side of this homomorphism is a subspace of

$$H^0\left(Y, \mathscr{O}_Y\left(K_Y + A + \sum_{i=1}^n A_i\right) \otimes f_*^{(m)} \mathscr{L}^{(m)}\right)$$

and the right hand side can be identified with $f_*^{(m)} \mathscr{L}^{(m)} \otimes k(y)$ (recall $y \in Y$ is general), we obtain that

$$H^0\left(Y, \mathscr{O}_Y\left(K_Y + A + \sum_{i=1}^n A_i\right) \otimes f_*^{(m)} \mathscr{L}^{(m)}\right) \to f_*^{(m)} \mathscr{L}^{(m)} \otimes k(y)$$

is surjective. Therefore,

$$\mathscr{O}_Y \left(K_Y + A + \textstyle\sum_{i=1}^n A_i\right) \otimes f_*^{(m)} \mathscr{L}^{(m)} \overset{\text{Lemma 3.6}}{\simeq} \mathscr{O}_Y \left(K_Y + A + \textstyle\sum_{i=1}^n A_i\right) \otimes \left[\bigotimes_{j=1}^m\right] f_* \mathscr{L}$$

is generically globally generated for all $m > 0$. However, then it follows that so is $\mathscr{O}_Y\left(K_Y + A + \sum_{i=1}^n A_i\right) \otimes \operatorname{Sym}^{[a]}(f_* \mathscr{L})$, since there is a generically surjective homomorphism from the former to the latter. This yields weak positivity (in the weak sense). □

*Proof of Theorem 8.1.* Let $\tau : \widetilde{Y} \to Y$ be a resolution of singularities. We claim that we may replace $f : (X, D) \to Y$ with $\widetilde{f} : \left(X_{\widetilde{Y}}, D_{\widetilde{Y}}\right) \to \widetilde{Y}$, and hence we may assume that $Y$ is smooth. Indeed, the pullback of $f_* \mathscr{O}_X(q(K_{X/Y} + D))$ to $\widetilde{Y}$ is isomorphic to $\widetilde{f}_* \mathscr{O}_{X_{\widetilde{Y}}}\left(q\left(K_{X_{\widetilde{Y}}/\widetilde{Y}} + D_{\widetilde{Y}}\right)\right)$ for every sufficiently divisible $q$ by the relative ampleness of $K_{X/Y} + D$. In particular, if we know the theorem for $\widetilde{f} : \left(X_{\widetilde{Y}}, D_{\widetilde{Y}}\right) \to \widetilde{Y}$, then we know that the pullback of $f_* \mathscr{O}_X(q(K_{X/Y} + D))$ to $\widetilde{Y}$ is big. This in turn implies that $f_* \mathscr{O}_X(q(K_{X/Y} + D))$ is also big (c.f., [Vie83a, 1.4.4]).

So, from now we assume that $Y$ is smooth. According to Theorem 7.1, for all sufficiently divisible $q > 0$, $\det f_* \mathscr{O}_X(q(K_{X/Y} + D))$ is big. Fix such a $q$. According to Lemma 8.10 there is a real number $c > 0$ such that

$$c < \operatorname{lct}\left(\mathscr{O}_{X_y}\left(q\left(K_{X_y} + D_y\right)\right); X_y, D_y\right)$$

for every $y \in U$, where $U$ is the open locus over which the fibers $(X_y, D_y)$ are klt. Fix also such a $c$ and let $l := \left\lceil \frac{1}{c} \right\rceil$. By replacing $Y$ with a finite cover, we may assume that $\det f_* \mathscr{O}_X(q(K_{X/Y} + D)) = \mathscr{O}_Y(lA)$ for some Cartier divisor $A$. Define $m := \operatorname{rk} f_* \mathscr{O}_X(q(K_{X/Y} + D))$ and consider the natural homomorphism,
(8.14.3)

$$\mathscr{O}_Y(lA) = \det f_* \mathscr{O}_X(q(K_{X/Y} + D)) \hookrightarrow \hookrightarrow \bigotimes_{i=1}^m f_* \mathscr{O}_X(q(K_{X/Y} + D)) \simeq \underbrace{f_*^{(m)} \mathscr{O}_X\left(q\left(K_{X^{(m)}/Y} + D_{X^{(m)}}\right)\right)}_{\text{Lemma 3.6}}$$

which implies that

$$\left(f^{(m)}\right)^* lA + \Gamma \sim q\left(K_{X^{(m)}/Y} + D_{X^{(m)}}\right) \tag{8.14.4}$$

for some appropriate effective divisor $\Gamma$ on $X^{(m)}$. Note that since (8.14.3) has a local splitting, $\Gamma_y \neq 0$ for any $y \in Y$. In particular, $\Gamma$ does not contain any $X_y^{(m)}$ for any $y \in U$, since fibers over $U$ are irreducible.

By (8.14.4) we obtain that
(8.14.5)

$$\frac{1}{l}\Gamma + \frac{q(2l-1)}{l}\left(K_{X^{(m)}/Y} + D_{X^{(m)}}\right) \sim_{\mathbb{Q}} 2q\left(K_{X^{(m)}/Y} + D_{X^{(m)}}\right) - \left(f^{(m)}\right)^* A.$$

Note that for each $y \in U$,

$$\operatorname{lct}\left(\frac{1}{l}\Gamma_y; X_y^{(m)}, \left(D_{X^{(m)}}\right)_y\right) \geq$$
$$\geq l \cdot \operatorname{lct}\left(\mathscr{O}_{X_y^{(m)}}\left(q\left(K_{X_y^{(m)}} + \left(D_{X^{(m)}}\right)_y\right)\right); X_y^{(m)}, \left(D_{X^{(m)}}\right)_y\right) =$$
$$= \underbrace{l \cdot \operatorname{lct}\left(\mathscr{O}_{X_y}\left(q\left(K_{X_y} + D_y\right)\right); X_y, D_y\right)}_{\text{Corollary 8.12}} > \left\lceil \frac{1}{c} \right\rceil c \geq 1$$

Therefore, $\left(X_y^{(m)}, \frac{1}{l}\Gamma_y + \left(D_{X^{(m)}}\right)_y\right)$ is klt for all $y \in U$. Then by (8.14.5) and Lemma 7.7 we may apply Proposition 8.14 to show that

$$f_*^{(m)} \mathscr{O}_{X^{(m)}}\left(2q\left(K_{X^{(m)}/Y} + D_{X^{(m)}}\right) - \left(f^{(m)}\right)^* A\right) \simeq$$
$$\simeq f_*^{(m)} \mathscr{O}_{X^{(m)}}\left(2q\left(K_{X^{(m)}/Y} + D_{X^{(m)}}\right)\right) \otimes \mathscr{O}_Y(-A) \simeq$$
$$\simeq \underbrace{\mathscr{O}_Y(-A) \otimes \bigotimes_{i=1}^m f_* \mathscr{O}_X\left(2q\left(K_{X/Y} + D\right)\right)}_{\text{Lemma 3.6}}$$

is weakly-positive. Therefore there exists an integer $b > 0$ such that

$$\mathscr{O}_Y(bA) \otimes \operatorname{Sym}^{2b}\left(\mathscr{O}_Y(-A) \otimes \bigotimes_{i=1}^{m} f_*\mathscr{O}_X\left(2q\left(K_{X/Y}+D\right)\right)\right) \simeq$$
$$\simeq \mathscr{O}_Y(-bA) \otimes \operatorname{Sym}^{2b}\left(\bigotimes_{i=1}^{m} f_*\mathscr{O}_X\left(2q\left(K_{X/Y}+D\right)\right)\right) \twoheadrightarrow$$
$$\twoheadrightarrow \mathscr{O}_Y(-bA) \otimes \operatorname{Sym}^{2bm}\left( f_*\mathscr{O}_X\left(2q\left(K_{X/Y}+D\right)\right)\right)$$

is generically globally generated. Hence $f_*\mathscr{O}_X\left(2q\left(K_{X/Y}+D\right)\right)$ is big. □

## 9. Subadditivity of log-Kodaira dimension

In this section we will consider the question of subadditivity of log-Kodaira dimension. Since, at this point, there are multiple non-equivalent statements of this conjecture in the literature, we state a couple of them. All of these follow from Theorem 9.9.

**Definition 9.1.** A *log canonical fiber space* is a surjective morphism $f : (X, D) \to Y$ such that

(1) both $X$ and $Y$ are irreducible, normal and projective,
(2) $K_X + D$ is $\mathbb{Q}$-Cartier and
(3) $(X_\eta, D_\eta)$ has log canonical singularities, where $\eta$ is the generic point of $Y$.

**Notation 9.2.** We will use the notation introduced above for the present section. In particular, $f : (X, D) \to Y$ will denote a log canonical fiber space and $\eta$ the generic point of $Y$.

Next we define the notion of variation for log canonical fiber spaces. Unfortunately, at this time we have to put a restriction on the log canonical fiber spaces on which the definition works. The main issue is that in Definition 6.16, variation is defined only for families of stable log-varieties. For general log canonical fiber spaces as in Definition 9.1 the reasonable expectation is that we would define variation as the variation of the relative log canonical model of $(X, D)$ (restricted to the open locus where it is a stable family). However, for log canonical singularities, the existence of a log canonical model is not known at this time even in the log-general type case. Hence, in order to make this definition, we assume that a relative log canonical model exists. This is known for example if the general fiber is klt.

**Definition 9.3.** Let $f : (X, D) \to Y$ be a log canonical fiber space such that $K_{X_\eta}+D_\eta$ is big and $(X_\eta, D_\eta)$ admits a log canonical model. Then set $\operatorname{Var} f$ to be the variation of the log canonical model of $(X_\eta, D_\eta)$ as defined in Definition 6.16.

*Remark* 9.4. If $(X_\eta, D_\eta)$ is klt and $K_{X_\eta} + D_\eta$ is big, then $(X_\eta, D_\eta)$ admits a log canonical model by [BCHM10, Thm 1.2]) and hence in this case $\operatorname{Var} f$ is defined.

**Theorem 9.5.** *Let $f : (X, D) \to Y$ be a log canonical fiber space with $K_{X_\eta} + D_\eta$ big. Then subadditivity of log-Kodaira dimension holds. That is,*

$$\kappa(K_X + D) \geq \kappa(Y) + \kappa(K_{X_\eta} + D_\eta).$$

*If in addition $(X_\eta, D_\eta)$ is klt and $\kappa(Y) \geq 0$, then*

$$\kappa(K_X + D) \geq \max\{\kappa(Y), \operatorname{Var} f\} + \kappa(K_{X_\eta} + D_\eta).$$

**Theorem 9.6.** *Let $f : (X, D) \to (Y, B)$ be a surjective map of projective log canonical snc pairs (cf. Definition 3.9). Assume that $K_{X_\eta} + D_\eta$ is big and that either*

*(1) both $B$ and $D$ are reduced and $\operatorname{supp} D \supseteq \operatorname{supp} f^*B$, or*
*(2) $D \geq f^*B$.*

*Then*

$$\kappa(K_X + D) \geq \kappa(K_Y + B) + \kappa\left(K_{X_\eta} + D_\eta\right).$$

In the next corollary we use the notion of Kodaira dimension of an arbitrary algebraic variety $X$.

**Definition 9.7.** Let $X$ be an algebraic variety, $\overline{X}$ a proper compactification of $X$, and $\pi : \widetilde{X} \to \overline{X}$ a resolution of singularities such that $X$ is projective and $\widetilde{D} := \left(\widetilde{X} \setminus \pi^{-1}(X)\right)$ has simple normal crossings, i.e., $(\widetilde{X}, \widetilde{D})$ is an snc pair. Then set

$$\kappa(X) := \kappa(K_{\widetilde{X}} + \widetilde{D}).$$

This is independent of the projective compactification or its resolution chosen [Iit82, §11.2].

**Corollary 9.8.** *Let $f : X \to Y$ be a dominant map of (not necessarily proper) algebraic varieties such that the generic fiber has maximal Kodaira dimension. Then*

$$\kappa(X) \geq \kappa(Y) + \kappa\left(X_\eta\right).$$

*Proof.* Let $\overline{Y}$ be a proper compactification of $Y$ and $\sigma : \widetilde{Y} \to \overline{Y}$ a resolution of singularities such that $\widetilde{Y}$ is projective and $B = \widetilde{Y} \setminus \sigma^{-1}(Y)$ is a simple normal crossing divisor. Next let $\overline{X}$ be a proper compactification of $X$ such that $f$ extends to a morphism $\overline{f} : \overline{X} \to \overline{Y}$. Further let $\pi_W : \widetilde{X} \to W$ be a resolution of singularities where $W \subseteq \overline{X} \times_{\overline{Y}} \widetilde{Y}$ is the component dominating $\overline{Y}$. Note that the induced morphism $\pi : \widetilde{X} \to \overline{X}$ is also a resolution of singularities and we may choose $\widetilde{X}$ to be projective and such that $\widetilde{D} := \left(\widetilde{X} \setminus \pi^{-1}(X)\right)$ has simple normal crossings. Let $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ denote the induced morphism. By construction $\operatorname{supp} \widetilde{f}^*B \subseteq \operatorname{supp} D$ and $\widetilde{X}_\eta$ is the resolution of a compactification of $X_\eta$ and $\widetilde{D}_\eta$ is the corresponding boundary divisor, it follows that $K_{X'_\eta} + D_\eta$ is big, and hence Theorem 9.6 implies the statement. $\square$

We will prove Theorems 9.5 and 9.6 via the following more general statement.

**Theorem 9.9.** *Let $f : (X, D) \to Y$ be a log canonical fiber space such that*

*(1) $Y$ is smooth,*
*(2) $(X, D)$ is an snc pair, and*
*(3) $K_{X_\eta} + D_\eta$ is big.*

*Further let $M$ be a $\mathbb{Q}$-Cartier divisor on $Y$ with $\kappa(M) \geq 0$. Then*

$$\kappa(K_{X/Y} + D + f^*M) \geq \kappa(M) + \kappa(K_{X_\eta} + D_\eta).$$

*If in addition $(X_\eta, D_\eta)$ is klt, then*

$$\kappa(K_{X/Y} + D + f^*M) \geq \max\{\kappa(M), \operatorname{Var} f\} + \kappa(K_{X_\eta} + D_\eta).$$

### 9.A. *Proof of Theorem 9.5*

Let $\tau : Y' \to Y$ be a resolution of $Y$, and let $X'$ be a resolution of the component of $X \times_Y Y'$ dominating $Y'$ that is also a log-resolution of $(X, D)$. Let $\varrho : X' \to X$ be the induced map. Choose canonical divisors $K_X$ and $K_{X'}$ such that they agree on the locus where $\varrho$ is an isomorphism and define $D' \geq 0$ and $R \geq 0$ without common components via

$$K_{X'} + D' = \varrho^*(K_X + D) + R. \tag{9.A.1}$$

Then $\kappa(K_X + D) = \kappa(K_{X'} + D')$, and similarly $\kappa\left(K_{X_\eta} + D_\eta\right) = \kappa\left(K_{X'_\eta} + D'_\eta\right)$ by restricting (9.A.1) to $X_\eta$. If in addition $(X_\eta, D_\eta)$ is klt, then $\operatorname{Var} f = \operatorname{Var} f'$ as the log canonical models of $(X_\eta, D_\eta)$ and of $(X'_\eta, D'_\eta)$ agree. Then Theorem 9.9 applied to $f' : (X', D') \to Y'$, which clearly satisfies the assumptions required, with $M = K_{Y'}$ completes the proof.

### 9.B. *Proof of Theorem 9.6*

**Lemma 9.10** (Simple normal crossing in codimension 1). *Let $f : (X, D) \to Y$ be a log canonical fiber space such that $Y$ is a smooth projective variety and $(X, D)$ is an snc pair. Let $D_h$ be the horizontal part of $D$. Then there is a log resolution $\varrho : X' \to (X, D)$, such that for $D'_h = \varrho_*^{-1} D_h$, the strict transform of $D_h$ on $X'$, and for each prime divisor $E$ in $Y$, $(X, D'_h + f^*E)$ is simple normal crossing over a neighborhood of the generic point of $E$.*

*Proof.* Let $U \subseteq Y$ be the largest open subset such that every stratum of $(X, D)$ (see [Kol13b, 1.7] for *stratum*) is smooth over $U$. Let $E_1, \dots, E_s$ be the components of the divisorial part of $Y \setminus U$ and define $X'$ to be a log-resolution of $(X, D + \sum_{i=1}^s f^*E_i)$ that is an isomorphism over $f^{-1}U$. □

**Lemma 9.11** (Reduced fibers in codimension 1). *Let $f : (X, D) \to Y$ be a log canonical fiber space such that $Y$ is a smooth projective variety and $(X, D)$ is an snc pair. Let $D_h$ be the horizontal part of $D$ and assume that for each prime divisor $E$ in $Y$, $(X, D_h + f^*E)$ is an snc pair over a neighborhood of the generic point of $E$. Then there exist a finite surjective morphism $\tau : Y' \to Y$ of smooth projective (irreducible) varieties and a resolution $X'$ of the normalization $X^n$ of $X \times_Y Y'$ such that:*

- *(1) If $\zeta : X^n \to X$ is the induced morphism and $D^n$ the horizontal part of $\zeta^*D$, then $(X^n, D^n)$ is a* locally stable family *at every codimension one point $y \in Y'$. That is, at every such $y \in Y'$ the following two equivalent conditions hold:*
  - *$(X^n, \overline{X^n_y} + D^n)$ is lc around $X^n_y$, where $\overline{X^n_y}$ is the closure of $X^n_y$, or equivalently*
  - *$(X^n_y, D^n_y)$ is slc and $K_{X^n} + D^n$ is $\mathbb{Q}$-Cartier around $X^n_y$.*
- *(2) The induced $f' : X' \to Y'$ agrees with the pullback of $f$ via an étale morphism over a dense open subset of $Y$.*
- *(3) If in addition there exists an effective simple normal crossing divisor $B$ on $Y$ such that both $B$ and $D$ are reduced and $\operatorname{Supp} D \supseteq \operatorname{Supp} f^*B$, then there exists an effective simple normal crossing divisor $D'$ on $X'$, which agrees with the pull-back of $D$ over a dense open subset of $Y$, and for which the inequality $\kappa(K_{X'/Y'} + D' + f'^*\tau^*(K_Y + B)) \leq \kappa(K_X + D)$ holds.*

*Proof.* Let $\tau : Y' \to Y$ be a surjective finite morphism, guaranteed by [Kaw81, Thm. 17], such that for any codimension 1 point $y \in Y'$, the multiplicity of every irreducible component of $X_{\tau(y)}$ divides the ramification index of $\tau$ at $y$. Notice that according to [Kol14, first 6 paragraphs in the proof of Thm 12.11], this choice of $\tau$ satisfies the requirements of (1). In fact, to apply [Kol14, Thm 12.11] our assumption that $(X, D_h + f^*E)$ is simple normal crossing in a neighborhood of the generic point of $E$, and hence $(X, D_h + (f^*E)_{\mathrm{red}})$ is log canonical in this neighborhood, is vital. It is also used that $D^n = \zeta^* D_h$, since $\zeta$ is finite. In any case, (1) in particular implies that the fibers of $f^n : X^n \to Y'$ are reduced in codimension 1 of $Y'$.

Next, let $\gamma : X' \to X^n$ be a log resolution of $(X^n, \zeta^* D)$ that does not change the general fibers of $f^n$. In this case, point (2) holds automatically.

We are left to show (3). Let $B' = (\tau^* B)_{\mathrm{red}}$ and $S := (\zeta^* D)_{\mathrm{red}}$. Note that by the construction of [Kaw81, Thm. 17], $\tau$ can be chosen such that $B'$ is a simple normal crossing divisor. The assumption $\operatorname{Supp} D \supseteq \operatorname{Supp} f^* B$ implies that $\operatorname{Supp} S \supseteq \operatorname{Supp}(f^n)^* B'$. Using that the codimension one fibers of $f^n$ are reduced this implies that $(f^n)^* B' \leq S$. Denote by $\mathscr{Q}$ the set of codimension 1 points of $X^n$ that are not contained in $\operatorname{supp} S$ and at which $\zeta$ is ramified. For every $x \in \mathscr{Q}$ denote by $Q_x$ the corresponding prime divisor and $q_x$ the ramification index. Similarly, let $\mathscr{R}$ be the set of codimension 1 points of $Y'$ that are not contained in $\operatorname{supp} B'$ and at which $\tau$ is ramified. Also, for every $y \in \mathscr{R}$ denote by $R_y$ the corresponding prime divisor and by $r_y$ the ramification index. Then we obtain the following formulas:

$$K_{Y'} + B' = \tau^*(K_Y + B) + \sum_{y \in \mathscr{R}} (r_y - 1) R_y \tag{9.B.1}$$

and

$$K_{X^n} + S = \zeta^*(K_X + D) + \sum_{x \in \mathscr{Q}} (q_x - 1) Q_x \tag{9.B.2}$$

There are two important facts, we use in the next step. First, if $x \in X^n$ is a codimension 1 point mapping onto a codimension 1 point $f^n(x)$, then $q_x | r_{f^n(x)}$. Indeed, if $z$, $v$, $t$ and $u$ are the local parameters at the codimension one points $x$, $\zeta(x)$, $f^n(x)$ and $\tau(f^n(x))$, then up to multiplication by units $z = t = u^{r_{f^n(x)}}$ and $z = v^{q_x} = (u^a)^{q_x} = u^{a \cdot q_x}$, where $a$ is the multiplicity of $\zeta(x)$ in the fiber over $\tau(f^n(x))$. Second, if $x \in \mathscr{Q}$, then $f^n(x) \in \mathscr{R}$, since $(f^n)^* B' \leq S$ and by the previous fact if $\zeta$ is ramified at $x$ then so is $\tau$ at $f^n(x)$. In particular, (9.B.1) and (9.B.2) yields

$$\begin{aligned} &\zeta^*(K_X + D) - \zeta^* f^*(K_Y + B) = \\ &= K_{X^n/Y'} + \underbrace{(S - (f^n)^* B') + \sum_{x \in \mathscr{Q}} (r_{f(x)} - q_x) Q_x + \sum_{\substack{x \text{ is a codimension 1} \\ \text{point of } X^n \text{ not in } \mathscr{Q}, \\ \text{such that } f^n(x) \in \mathscr{R}}} (r_{f(x)} - 1) R_{f(x)}}_{T :=} \end{aligned} \tag{9.B.3}$$

where the divisor $T$ is effective.

Define $D'$ and $T'$ to be the strict transforms of $S - (f^n)^* B$ and $T$, respectively, on $X'$. Then point (3) follows from the following inequality that holds for every integer $m > 0$.

$$h^0\left(m\left(K_{X'/Y'} + D' + f'^*\tau^*(K_Y+B)\right)\right) \underset{\text{since } D' \leq T'}{\leq}$$

$$\leq h^0\left(m\left(K_{X'/Y'} + T' + f'^*\tau^*(K_Y+B)\right)\right) \underset{\text{since } \gamma_*(K_{X'/Y'}+T') = K_{X^n/Y}+T}{\leq}$$

$$\leq h^0\left(m\left(K_{X^n/Y'} + T + (f^n)^*\tau^*(K_Y+B)\right)\right) \underset{\text{by (9.B.3)}}{=}$$

$$= h^0(m\zeta^*(K_X+D)). \quad \square$$

**Proposition 9.12.** *Theorem 9.9 implies Theorem 9.6.*

*Proof.* If $f^*B \leq D$, then apply Theorem 9.9 for $(X, D - f^*B)$ and $M := K_Y + B$. Otherwise we prove the statement below.

STEP 1: *Normal crossing in codimension* 1. Apply Lemma 9.10 to $f : (X, D) \to Y$ and define $D' := (\varrho^* D)_{\mathrm{red}}$. With this choice, $\operatorname{Supp} f'^* B \subseteq \operatorname{Supp} D'$, where $f' : X' \to Y$ is the induced morphism. Furthermore, $\kappa\left(K_{X_\eta} + D_\eta\right) = \kappa\left(K_{X'_\eta} + D'_\eta\right)$ since $\varrho_*(K_{X'} + D') = K_X + D$, $\kappa(K_{X'} + D') \leq \kappa(K_X + D)$, and the generic fiber is unchanged. That is, we may assume that the consequences of Lemma 9.10 hold for $f : (X, D) \to Y$.

STEP 2: *Reduced fibers in codimension* 1. By Step 1 we may apply Lemma 9.11 for $f : (X, D) \to Y$ to obtain $f' : (X', D') \to Y'$ as stated there (same notation, but different from the the one in the previous step). However, then if $\eta$ and $\eta'$ are the generic points of $Y$ and $Y'$, respectively, then the following computation concludes the proof.

$$\kappa(K_X + D) \geq \kappa\left(K_{X'/Y'} + D' + f'^*\tau^*(K_Y+B)\right) \underset{\text{by Lemma 9.11}}{\geq}$$

$$\underset{\text{by Theorem 9.9 with } M = \tau^*(K_Y+B)}{\geq} \kappa\left(K_{X'_{\eta'}} + D'_{\eta'}\right) + \kappa(\tau^*(K_Y+B)) = \underbrace{\kappa\left(K_{X_\eta} + D_\eta\right)}_{\substack{\text{by Lemma 9.11} \\ (X'_{\eta'}, D'_{\eta'}) = (X,D)_{\eta'}}} + \underbrace{\kappa(K_Y+B)}_{\tau \text{ is finite}}. \quad \square$$

## 9.C. *Proof of Theorem 9.9*

**Lemma 9.13.** *Consider the following commutative diagram of normal varieties, where $f$ is flat and Gorenstein, $\tau$ is surjective, $\overline{X} := X \times_Y Y'$ and $X^n$ is the normalization of the component of $X \times_Y Y'$ dominating $Y'$.*

$$\begin{array}{ccccc} X & \xleftarrow{\alpha} & \overline{X} & \xleftarrow{\beta} & X^n \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} & \swarrow{\scriptstyle f_n} & \\ Y & \xleftarrow{\tau} & Y' & & \end{array}$$

*Then, there is a natural embedding $\omega_{X^n/Y'} \hookrightarrow \beta^*\alpha^*\omega_{X/Y}$.*

*Proof.* Since $f$ is flat and Gorenstein, $\omega_{\overline{X}/Y'} \simeq \alpha^*\omega_{X/Y}$ by [Con00, Thm 3.6.1]. In particular, $\omega_{\overline{X}/Y'}$ is a line bundle. Consider the pull-back of the Gorthendieck trace of $\beta$, $\phi : \beta^*\beta_*\omega_{X^n/Y'} \to \beta^*\omega_{\overline{X}/Y'}$. *We claim that $\phi$ factors through the natural map $\xi : \beta^*\beta_*\omega_{X^n/Y'} \to \omega_{X^n/Y'}$.* For this, first note that $\xi$ is surjective, since $\beta$ is affine and for any morphism of rings $A \to B$ and $B$-module $M$, the natural morphism $M \otimes_A B \to M$ is surjective. Next note that $\beta^*\omega_{\overline{X}/Y'}$ is a line bundle, in particular torsion-free and hence $\phi$ factors through the natural map $\beta^*\beta_*\omega_{X^n/Y'} \to \beta^*\beta_*\omega_{X^n/Y'}/\mathscr{T}$, where $\mathscr{T}$ is the torsion part of $\beta^*\beta_*\omega_{X^n/Y'}$. Therefore, it is enough to show that the latter map is isomorphic to $\xi$, that is, that $\ker\xi = \mathscr{T}$. To show that $\ker\xi \subseteq \mathscr{T}$ simply observe that $\beta$ is generically an isomorphism, and hence so is $\xi$. The opposite containment, $\ker\xi \supseteq \mathscr{T}$, follows from the fact that $\omega_{X^n/Y'}$ is torsion-free. This proves the claim, and hence we obtain an embedding $\omega_{X^n/Y'} \hookrightarrow \beta^*\omega_{\overline{X}/Y'} \simeq \beta^*\alpha^*\omega_{X/Y}$. ☐

**Lemma 9.14.** *Let $\pi : Z \to W$ be a birational morphism of normal projective varieties with $K_Z$ and $K_W$ are $\mathbb{Q}$-Cartier, $L$ a $\mathbb{Q}$-Cartier divisor on $W$, and $B$ a (not necessarily effective) $\pi$-exceptional $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $Z$. Then*

$$\kappa(K_W + L) \geq \kappa(K_Z + \pi^*L + B)$$

*Proof.* Let $\iota : W^\circ \hookrightarrow W$ be a dense open embedding over which $\pi$ is an isomorphism. Then for any sufficiently divisible $m \in \mathbb{Z}$ there is an injective map as follows. This proves the statement.

$$\pi_*\mathscr{O}_Z(m(K_Z + \pi^*L + E)) \hookrightarrow \iota_*\left((\pi_*\mathscr{O}_Z(mK_Z) \otimes \mathscr{O}_W(mL))\,|_{W^\circ}\right) \simeq$$
$$\simeq \iota_*\left(\mathscr{O}_{W^\circ}(mK_{W^\circ} + mL|_{W^\circ})\right) \simeq \mathscr{O}_W(m(K_W + L)). \quad ☐$$

*Proof of Theorem 9.9.*

Step 0: *Assuming klt.* If $(X_\eta, D_\eta)$ is not klt, then by decreasing the coefficients of $D$ a little all our assumptions remain true, and so we may assume that $(X_\eta, D_\eta)$ is klt.

Step 1: *Allowing an extra divisor avoiding a big open set of the base.* According to [Vie83a, Lemma 7.3], there is a birational morphism $\widetilde{Y} \to Y$ from a smooth projective variety, and another one from $\widetilde{X}$ onto the component of $X \times_Y \widetilde{Y}$ dominating $\widetilde{Y}$, such that for the induced map $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ and for every prime divisor $E \subseteq \widetilde{X}$, if $\operatorname{codim}_{\widetilde{Y}} \widetilde{f}(E) \geq 2$, then $E$ is $\widetilde{X} \to X$ exceptional. Furthermore, it follows from the proof of [Vie83a, Lemma 7.3] that we may choose $\widetilde{X} \to X$ and $\widetilde{Y} \to Y$ to be isomorphisms over the smooth locus of $f$ on $Y$ and also $\widetilde{X} \to X$ to be a log-resolution of $(X, D)$. Let $\varrho : \widetilde{X} \to X$ and $\tau : \widetilde{Y} \to Y$ be the induced maps and set $\widetilde{D} := \varrho^*D$ and $\widetilde{M} := \tau^*M$.

*Claim* 9.C.1. Let $B \geq 0$ be an effective divisor on $\widetilde{X}$ for which $\operatorname{codim}_{\widetilde{Y}} \widetilde{f}(B) \geq 2$. Then

$$\kappa(K_{X/Y} + D + f^*M) \geq \kappa(K_{\widetilde{X}/\widetilde{Y}} + \widetilde{D} + \widetilde{f}^*\widetilde{M} + B).$$

*Proof of (9.C.1).* Since both $Y$ and $\widetilde{Y}$ are smooth, there is an effective divisor $F$ on $\widetilde{Y}$ such that $K_{\widetilde{Y}} = \tau^*K_Y + F$. In particular, the following holds:

$$K_{\widetilde{X}/\widetilde{Y}} = K_{\widetilde{X}} - \widetilde{f}^*K_{\widetilde{Y}} = K_{\widetilde{X}} - \widetilde{f}^*(\tau^*K_Y + F) = K_{\widetilde{X}} - \varrho^*f^*K_Y - \widetilde{f}^*F.$$

Now apply Lemma 9.14 with $\pi = \varrho$ and $L = -f^*K_Y + D + f^*M$. ☐

It follows that it is enough to prove that for some $0 \leq B$ for which $\operatorname{codim}_Y f(B) \geq 2$,

$$\kappa(K_{X/Y} + D + f^*M + B) \geq \max\{\kappa(M), \operatorname{Var} f\} + \kappa(K_{X_\eta} + D_\eta). \tag{9.C.2}$$

STEP 2: *Discarding vertical components of $D$.* Notice that vertical components of $D$ may be discarded at any time because their presence may only decrease the log-Kodaira dimension (e.g., by (9.C.1)). We will apply this principle repeatedly in the sequel.

STEP 3: *Replacing* $\operatorname{Var} f$ *by* $\operatorname{Var} f_{\mathrm{can}}$. Let $f_{\mathrm{can}} : (X_{\mathrm{can}}, D_{\mathrm{can}}) \to Y_0$ be the log canonical model of $(X, D)$ over some dense open set $Y_0 \subseteq Y$ over which $(X, D)$ is klt. By shrinking $Y_0$ we may assume that $f_{\mathrm{can}}$ is a stable family. Note that since $(X, D)$ is klt over $Y_0$, $\operatorname{Var} f = \operatorname{Var} f_{\mathrm{can}}$ (where the latter is taken as the variation as a stable family of log-varieties), hence, in order to obtain (9.C.2) it is enough to prove the following inequality:

$$\kappa(K_{X/Y} + D + f^*M + B) \geq \max\{\kappa(M), \operatorname{Var} f_{\mathrm{can}}\} + \kappa(K_{X_\eta} + D_\eta). \tag{9.C.3}$$

Throughout the rest of the proof we will define several new objects and morphisms. Table 1 on page 56 indicates the interrelations of these.

STEP 4: *An auxilliary base change.* Set $n := \dim X - \dim Y$, $v := \operatorname{vol}\left(K_{X_\eta} + D_\eta\right)$, where $\eta$ is the generic point of $Y$. Let $I \subseteq [0,1]$ be a finite coefficient set closed under addition (Definition 6.1) that contains the coefficients of $D$. Let $\mu : Y_0 \to \mathscr{M}_{n,v,I}$ be the moduli map associated to $f_{\mathrm{can}}$ and let $S \to \mathscr{M}_{n,v,I}$ be the finite cover guaranteed by Corollary 6.19. Further let $Y^{\mathrm{aux}}$ be a resolution of a compactification of a component of $Y_0 \times_{\mathscr{M}_{n,v,I}} S$ that dominates $Y_0$. We may assume that the rational maps $\delta : Y^{\mathrm{aux}} \dashrightarrow Y$ and $Y^{\mathrm{aux}} \dashrightarrow S$ are morphisms. Let $Y''$ be the normalization of the image of $Y^{\mathrm{aux}}$ in $S$ and $f'' : (X'', D'') \to Y''$ the family over $Y''$ induced by $\mathsf{f} \in \mathscr{M}(S)$ given in Corollary 6.19. Then the pullback of this family over $\delta^{-1}(Y_0)$ is isomorphic to the pullback of $(X_{\mathrm{can}}, D_{\mathrm{can}})$ and hence $\dim Y'' = \operatorname{Var} f_{\mathrm{can}}$.

STEP 5: *Local stable reduction over a big open set.* Let $X^{\mathrm{aux}}$ be a resolution of the main component of $X \times_Y Y^{\mathrm{aux}}$ such that the pullback of $D$ to $X^{\mathrm{aux}}$ is simple normal crossing. Let $D^{\mathrm{aux}}$ be the horizontal part of this pullback. Now, we apply Lemma 9.10 and then Lemma 9.11 to $(X^{\mathrm{aux}}, D^{\mathrm{aux}}) \to Y^{\mathrm{aux}}$. Let $Y'$ be the one obtained in Lemma 9.11 and similarly to the notation used there let

- $X^n \to X^{\mathrm{aux}} \times_{Y^{\mathrm{aux}}} Y'$ be the normalization,
- $D^n$ be the horizontal part of the pullback of $D^{\mathrm{aux}}$ to $X^n$,
- $X'$ be a log resolution of $(X^n, D^n)$, and
- $f' : X' \to Y'$, $\tau : Y' \to Y$ and $f^n : X^n \to Y'$ be the induced natural morphisms.

STEP 6: *Choosing nice big open sets.* Let $Y'_0 \subseteq Y'$ be a big open set such that

(1) $(X^n, D^n)$ is klt and forms a flat locally stable family of log-varieties over $Y'_0$.

Let $X^n_0 := (f^n)^{-1} Y'_0$, $D^n_0 := D^n|_{X^n_0}$ and let $f'_{\mathrm{can}} : (X'_{\mathrm{can}}, D'_{\mathrm{can}}) \to Y'_0$ be the log canonical model of $(X^n_0, D^n_0)$ over $Y'_0$. By shrinking $Y'_0$ (but keeping it big in $Y'$) we may further assume that

(2) $f'_{\mathrm{can}}$ is flat (and hence it is a family of stable log-varieties).

Let $\eta'$ be the generic point of $Y'$. Then $\left(X^n_{\eta'}, D^n_{\eta'}\right) \simeq (X_\eta, D_\eta)_{\eta'}$, since over the locus (in $Y$) over which $f$ is smooth and $(X, D)$ is a relative normal crossing divisor,

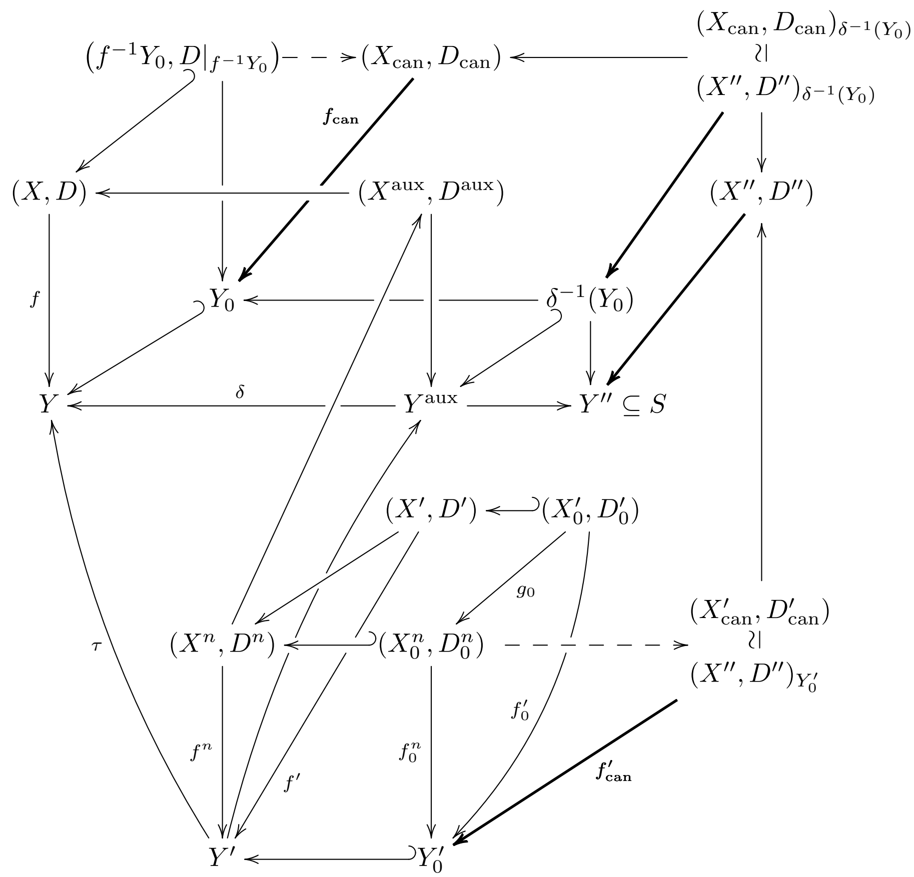



TABLE 1. The interrelations of the spaces defined throughout the proof. The bold arrows denote stable log-families, the hook arrows denote open embeddings. Note that not all squares are Cartesian.

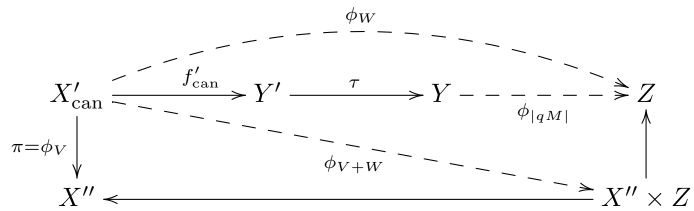
$(X^n, D^n)$ is isomorphic to $(X, D) \times_Y Y'$. Therefore

$$(X'_{\mathrm{can}}, D'_{\mathrm{can}})_{\eta'} \simeq (X_{\mathrm{can}}, D_{\mathrm{can}})_{\eta'} \simeq (X'', D'')_{\eta'} .$$

That is, $(X'', D'') \times_{Y''} Y'_0$ is isomorphic to $(X'_{\mathrm{can}}, D'_{\mathrm{can}})$ over $\eta'$. Equivalently, their Isom scheme has a rational point over $\eta'$ whose closure yields a section of the Isom scheme over a big open set of $Y'_0$. Therefore, by further restricting $Y'_0$ (and still keeping it big) we may assume that

(3) $(X'', D'') \times_{Y''} Y'_0$ is isomorphic to $(X'_{\mathrm{can}}, D'_{\mathrm{can}})$.

STEP 7: *Bounding* $\kappa\left(K_{X'_{\mathrm{can}}/Y'_0} + D'_{\mathrm{can}} + f'^*_{\mathrm{can}}\tau^*M\right)$. By Corollary 8.3 $K_{X''/Y''} + D''$ is big. In particular, there is a very ample divisor $H$ and an effective divisor $E$ on $X''$, such that $H + E \sim q(K_{X''/Y''} + D'')$ for some sufficiently divisible $q > 0$. Let $\pi : X'_{\mathrm{can}} \to X''$ be the induced map and let $V \subseteq |\pi^*H|$ be a linear system inducing $\pi$. Further let $W \subseteq |qf'^*_{\mathrm{can}}\tau^*M|$ be the linear system corresponding to the natural embedding $H^0(Y, \mathscr{O}_Y(qM)) \hookrightarrow H^0(X'_{\mathrm{can}}, f'^*_{\mathrm{can}}\tau^*\mathscr{O}_{X'_{\mathrm{can}}}(qM))$.




We compute the dimension of a general fiber of $\phi_{V+W}$. For that choose an open set $U' \subseteq X'_{\mathrm{can}}$, such that $\phi_{V+W}$ is a morphism over $U'$ and $\phi_{|qM|}$ is a morphism over $\tau\left(f'_{\mathrm{can}}(U')\right)$. In the next few sentences, when computing fibers of $\phi_{V+W}$, $\phi_W$ and $\phi_{|qM|}$, we take $U'$ and $\tau_*\left(f'_{\mathrm{can}}\right)_* U'$ as the domain. So, choose $z \in Z$ and $x \in X''$ general, where $Z$ is the image of $\phi_{|qM|}$. We have $\phi^{-1}_{V+W}((x,z)) = \phi^{-1}_W(z) \cap \phi^{-1}_V(x)$. Furthermore, $\phi^{-1}_W(z)$ is of the form $f'^{-1}_{\mathrm{can}}(Z')$ for a variety $Z'$ of dimension $\dim Y - \kappa(M)$. On the other hand, $\phi^{-1}_V(x)$ intersects each fiber of $f'_{\mathrm{can}}$ in at most one point and has dimension $\dim Y - \operatorname{Var} f_{\mathrm{can}}$. Therefore,

$$\dim \phi^{-1}_{V+W}((x,z)) \leq \min\{\dim Y - \operatorname{Var} f_{\mathrm{can}}, \dim Y - \kappa(M)\}. \tag{9.C.4}$$

Hence,

(9.C.5)

$$\kappa\left(K_{X'_{\mathrm{can}}/Y'_0} + D'_{\mathrm{can}} + f'^*_{\mathrm{can}}\tau^*M\right) \overset{\text{since } \pi^*H+\pi^*E \sim q\left(K_{X'_{\mathrm{can}}/Y'_0}+D'_{\mathrm{can}}\right) \text{ and } E \geq 0}{\geq} \kappa\left(\pi^*H + qf'^*_{\mathrm{can}}\tau^*M\right) \geq \dim \operatorname{im} \phi_{V+W} \geq$$
$$\underset{\text{by (9.C.4)}}{\geq} n + \dim Y - \min\{\dim Y - \operatorname{Var} f_{\mathrm{can}}, \dim Y - \kappa(M)\} = n + \max\{\operatorname{Var} f_{\mathrm{can}}, \kappa(M)\}.$$

Step 8: *Conclusion.* Let $X'_0 := f'^{-1}Y'_0$, $f'_0 : X'_0 \to Y'_0$ the induced morphism and define on $X'_0$ the effective divisor $D'_0$ via the equality

$$K_{X'_0} + D'_0 = g_0^*(K_{X^n_0} + D^n_0) + G,$$

where $g_0 : X'_0 \to X^n_0$ is the induced morphism and $G$ is any effective $\mathbb{Q}$-divisor that makes the equality hold. Let $D'$ be the smallest extension of $D'_0$ from $X'_0$ to $X'$.

Note then that for every integer $q > 0$,

$$H^0\left(X'_0, \mathscr{O}_{X'_0}\left(q\left(K_{X'_0/Y'_0} + D'_0 + \left(f'_0\right)^*\tau^*M\right)\right)\right) \simeq$$
$$\simeq H^0\left(X^n_0, \mathscr{O}_{X^n_0}\left(q\left(K_{X^n_0/Y'_0} + D^n_0 + (f^n_0)^*\tau^*M\right)\right)\right) \simeq$$
$$\simeq H^0\left(X'_{\mathrm{can}}, \mathscr{O}_{X'_{\mathrm{can}}}\left(q\left(K_{X'_{\mathrm{can}}/Y'_0} + D'_{\mathrm{can}} + f'^*_{\mathrm{can}}\tau^*M\right)\right)\right).$$

Hence, by (9.C.5) there is an effective divisior $B'$ in $X'$ supported on $X' \setminus X'_0$, such that

$$\kappa(K_{X'/Y'} + D' + f'^*\tau^*M + B') \geq n + \max\{\operatorname{Var} f_{\mathrm{can}}, \kappa(M)\}. \tag{9.C.6}$$

Then define the following.

- Let $\widetilde{X}$ be the normalization of the main component of $X \times_Y Y'$ (different from $X^n$, which is the normalization of the main component of $X^{\mathrm{aux}} \times_{Y^{\mathrm{aux}}} Y'$).
- Let $\gamma : X' \to \widetilde{X}$ and $\varrho : \widetilde{X} \to X$ be the induced morphisms.

- Let $\widetilde{X} \xrightarrow{\xi} \widetilde{X}^s \xrightarrow{\zeta} X$ be the Stein-factorization of $\varrho$. Note that $\varrho$ is not necessarily finite since $\delta$ is not finite in general, so taking Stein-factorization is not void.
- Let $T$ be an effective Weil-divisor on $\widetilde{X}$ given by Lemma 9.13 such that $K_{\widetilde{X}/Y'} \leq \varrho^* K_{X/Y} + T$, and $T$ is mapped into the non-flat locus of $f$ on $Y$ (which has codimension at least 2).
- Let $(\gamma^* D')_v$ and $(\gamma^* D')_h$ be the vertical and the horizontal parts of $\gamma_* D'$. Note that since $\gamma$ factors through $X^n$, where the pushforward of $D'$ is $D^n$, which does not have vertical components over $Y'_0$, $(\gamma_* D')_v$ is supported over $Y' \setminus Y'_0$. Furthermore, by the same factorization $(\gamma_* D')_v$ is the horizontal part of $\varrho^* D$.
- Let $B$ be an effective divisor on $X$ such that $\operatorname{codim}_Y f(\operatorname{Supp} B) \geq 2$ and $\zeta^* B \geq \xi_* T + \xi_* \gamma_* B' + (\gamma_* D')_v$. Such a choice of $B$ is possible by the choice of $T$ and the fact that $\operatorname{codim}_{Y'} Y' \setminus Y'_0 \geq 2$.

Then the following holds for every $q > 0$:

$$h^0(q\zeta^*(K_{X/Y}+D+f^*M+B)) \geq h^0\left(q\left(\zeta^*(K_{X/Y}+D+f^*M)+\xi_*T+\xi_*\gamma_*B'\right)\right) \geq$$

since $\zeta^* B \geq \xi_* T + \xi_* \gamma_* B' + \xi_*(\gamma_* D')_v$

$$\geq h^0\left(q\left(\varrho^*(K_{X/Y}+D+f^*M)+T+\gamma_*B'+(\gamma_*D')_v\right)\right) \geq$$

since $\xi$ is birational

$$\geq h^0\left(q\left(K_{\widetilde{X}/Y'}+\varrho^*D+\varrho^*f^*M+\gamma_*B'+(\gamma_*D')_v\right)\right) =$$

by Lemma 9.13 and the choice of $T$

$$\geq h^0\left(q\left(K_{\widetilde{X}/Y'}+(\gamma_*D')_h+\varrho^*f^*M+\gamma_*B'+(\gamma_*D')_v\right)\right) =$$

$(\gamma_* D')_h \leq \varrho^* D$ because $(\gamma_* D')_h$ is the horizontal part of $\varrho^* D'$

$$= h^0\left(q\left(K_{X'/Y'}+D'+f'_*\tau^*M+B'\right)\right)$$

$(\gamma_* D')_h + (\gamma_* D')_v = \gamma_* D'$ and $\gamma_* K_{X'/Y'} = K_{\widetilde{X}/Y'}$

In particular, by (9.C.6),

$$\kappa\left(\zeta^*(K_{X/Y}+D+f^*M+B)\right) \geq n + \max\{\operatorname{Var} f_{\mathrm{can}}, \kappa(M)\} =$$
$$= \kappa\left(K_{X_\eta}+D_\eta\right) + \max\{\operatorname{Var} f_{\mathrm{can}}, \kappa(M)\}.$$

Hence, since Kodaira-dimension of a line bundle is invariant under finite pullback [Uen75, Thm 5.13], (9.C.3) holds. □

## 10. Almost proper bases

**Lemma 10.1.** *Consider the following commutative diagram of normal, irreducible varieties, where*

$$\begin{array}{ccc} Y' & \hookrightarrow & \overline{Y}' \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\overline{\tau}} \\ Y & \hookrightarrow & \overline{Y} \end{array}$$

*(1) $\overline{Y}$ and $\overline{Y}'$ are projective over $k$*
*(2) $\tau$ is generically finite,*
*(3) $Y' = \tau^{-1} Y$,*
*(4) $Y$ is a big open set of $\overline{Y}$,*

*Let $\mathscr{G}$ be a big vector bundle on $\overline{Y}'$ and assume that there is a vector bundle $\mathscr{F}$ on $Y$, such that $\tau^* \mathscr{F} \simeq \mathscr{G}|_{Y'}$. Then $\mathscr{F}$ is big as well.*

*Proof.* Choose ample line bundles $\mathscr{H}$ and $\mathscr{A}$ on $\overline{Y}$ and $\overline{Y}'$, respectively. Let $b > 0$ be an integer such that there is an injection $\overline{\tau}^*\mathscr{H} \hookrightarrow \mathscr{A}^b$. Since $\mathscr{G}$ is big, there is an integer $a > 0$ such that $\mathrm{Sym}^a(\mathscr{G}) \otimes \mathscr{A}^{-1}$ is generically globally generated. Hence, so is $\mathrm{Sym}^{ab}(\mathscr{G}) \otimes \mathscr{A}^{-b}$. So, by the embedding $\mathrm{Sym}^{ab}(\mathscr{G}) \otimes \mathscr{A}^{-b} \hookrightarrow \mathrm{Sym}^{ab}(\mathscr{G}) \otimes \overline{\tau}^*\mathscr{H}^{-1}$, the latter sheaf is generically globally generated as well. In particular, so is

$$\mathrm{Sym}^{ab}(\mathscr{G}) \otimes \overline{\tau}^*\mathscr{H}^{-1}\Big|_{Y'} \simeq \mathrm{Sym}^{ab}(\tau^*\mathscr{F}) \otimes \tau^*\mathscr{H}|_Y^{-1}.$$

Let

$$Y' \xrightarrow{\ \nu\ } Z \xrightarrow{\ \rho\ } Y, \qquad \tau = \rho \circ \nu$$

be the Stein factorization of $\tau$. Then since $\nu$ is birational,

$$\nu_*\left(\mathrm{Sym}^{ab}(\tau^*\mathscr{F}) \otimes \tau^*\mathscr{H}|_Y^{-1}\right) \simeq \mathrm{Sym}^{ab}(\rho^*\mathscr{F}) \otimes \rho^*\mathscr{H}|_Y^{-1}$$

is also generically globally generated. Then [VZ02, Lem 1.3] shows that $\mathrm{Sym}^{ab}(\mathscr{F}) \otimes \mathscr{H}^{-1}|_Y$ is generically globally generated, and hence $\mathscr{F}$ is big indeed. $\square$

Using Lemma 10.1 and Corollary 6.18 immediately follow versions of point (2) of Theorem 7.1 and of Theorem 8.1 for the almost projective base case.

**Corollary 10.2.** *If $f : (X, D) \to Y$ is a family of stable log-varieties of maximal variation over a normal almost projective variety, then*

*(1) for every sufficiently divisible $q > 0$, $\det f_*\mathscr{O}_X(q(K_X + \Delta))$ is big.*
*(2) $f_*\mathscr{O}_X(q(K_{X/Y} + D))$ is big for every sufficiently divisible integer $q > 0$, provided that $(X, D)$ has klt general fibers over $Y$.*

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Abstract. We prove that any coarse moduli space of stable log-varieties of general type is projective. We also prove subadditivity of log-Kodaira dimension for fiber spaces whose general fiber is of log general type.

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