

# Quantum modular forms and singular combinatorial series with repeated roots of unity

by

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## 1. Introduction and statement of results

**1.1. Background.** Let  $p(n)$  denote the number of partitions of a positive integer  $n$ , where a *partition* of  $n$  is a non-increasing sequence of positive integers whose sum is  $n$ . As an example, we see there are five partitions of 4, namely 4,  $3 + 1$ ,  $2 + 2$ ,  $2 + 1 + 1$ ,  $1 + 1 + 1 + 1$ , and therefore  $p(4) = 5$ . The generating function of  $p(n)$  is given by

$$1 + \sum_{n=1}^{\infty} p(n)q^n = \prod_{k=1}^{\infty} \frac{1}{1 - q^k} = \frac{q^{1/24}}{\eta(\tau)},$$

where

$$(1.1) \quad \eta(\tau) := q^{1/24} \prod_{k=1}^{\infty} (1 - q^k)$$

is *Dedekind's  $\eta$ -function*, a weight  $1/2$  modular form. Here and throughout this section we are setting  $q = e^{2\pi i\tau}$ , where

$$\tau \in \mathbb{H} := \{x + iy \mid x, y \in \mathbb{R}, y > 0\},$$

the upper half of the complex plane.

In order to provide a combinatorial proof of Ramanujan's remarkable partition congruences, Dyson [9] defined the *rank* of a partition as the largest part of the partition minus the number of parts. He also defined the *partition rank function*  $N(m, n)$  to be the number of partitions of  $n$  with rank equal to  $m$ . If we set  $N(m, 0) := \delta_{m0}$  with  $\delta_{ij}$  the Kronecker delta, and define the

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$q$ -Pochhammer symbol for  $n \in \mathbb{N}_0 \cup \{\infty\}$  by

$$(a)_n = (a; q)_n := \prod_{j=1}^n (1 - aq^{j-1}),$$

then the generating function for  $N(m, n)$  is given by

$$(1.2) \quad \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N(m, n) w^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(wq; q)_n (w^{-1}q; q)_n} =: R_1(w; q).$$

Due to the deep connection between the rank generating function and the theory of modular forms, there have been many studies on the  $q$ -hypergeometric series defined in (1.2). For example, when  $w = 1$ , one recovers the partition generating function, namely

$$(1.3) \quad R_1(1; q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n^2} = 1 + \sum_{n=1}^{\infty} p(n) q^n = \frac{q^{1/24}}{\eta(\tau)},$$

(essentially <sup>(1)</sup>) the reciprocal of the Dedekind  $\eta$ -function, the modular form of weight  $1/2$  defined in (1.1). When  $w = -1$ , we have

$$(1.4) \quad R_1(-1; q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n^2} =: f(q),$$

where  $f(q)$  is one of Ramanujan's third order mock theta functions [3].

Mock theta functions, and more generally mock modular forms and harmonic Maass forms, have played central roles in modern number theory. In particular, for several decades after Ramanujan's death in 1920, no one understood how Ramanujan's mock theta functions fit into the theory of modular forms until the groundbreaking 2002 thesis of Zwegers [21]: we now know that Ramanujan's mock theta functions, a finite list of curious  $q$ -hypergeometric functions including  $f(q)$ , are examples of *mock modular forms*, the holomorphic parts of *harmonic Maass forms*. In other words, they exhibit suitable modular transformation properties after they are *completed* by the addition of certain non-holomorphic functions. Briefly speaking, harmonic Maass forms, first defined by Bruinier and Funke [7], are non-holomorphic generalizations of ordinary modular forms that, in addition to satisfying appropriate modular transformations, must be eigenfunctions of the weight  $k$  hyperbolic Laplacian operator, and satisfy suitable growth conditions in cusps (see [3, 7, 16, 18] for more).

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<sup>(1)</sup> Here and throughout, as is standard in this subject, for simplicity's sake we may slightly abuse terminology and refer to a function as a modular form or other modular object when in reality it must first be multiplied by a suitable power of  $q$  to transform appropriately.

Motivated by the fact that specializing  $R_1$  at  $w = \pm 1$  yields two different modular objects, namely an ordinary modular form and a mock modular form as described in (1.3) and (1.4), Bringmann and Ono [5] proved more generally that upon specialization of the parameter  $w$  to complex roots of unity not equal to 1, the rank generating function  $R_1$  is also a mock modular form. (See also [18] for related work.)

THEOREM ([5, Theorem 1.1]). *For positive integers  $a$  and  $c$  satisfying  $0 < a < c$ , the function*

$$q^{-\ell_c/24} R_1(\zeta_c^a; q^{\ell_c}) + \frac{i \sin(\pi a/c) \ell_c^{1/2}}{\sqrt{3}} \int_{-\bar{\tau}}^{i\infty} \frac{\Theta(a/c; \ell_c \rho)}{\sqrt{-i(\tau + \rho)}} d\rho$$

*is a harmonic Maass form of weight  $1/2$  on  $\Gamma_c$ .*

Here,  $\zeta_c^a := e^{2\pi i a/c}$  is a  $c$ th root of unity,  $\Theta(a/c; \ell_c \tau)$  is a sum of weight  $3/2$  unary theta functions,  $\ell_c := \text{lcm}(2c^2, 24)$ , and

$$(1.5) \quad \Gamma_c := \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \ell_c^2 & 1 \end{pmatrix} \right\rangle.$$

In this paper we investigate modularity properties for a related combinatorial  $q$ -hypergeometric series, namely the *rank generating function for  $n$ -marked Durfee symbols*, as defined by Andrews [1]. Our results here extend our prior work on this topic [13, 11].

We will not give details of the combinatoric objects called  $n$ -marked Durfee symbols themselves here, and instead refer the reader to [1] for a full treatment, or [11] for a brief overview. However, we note that the  $n$ -marked Durfee symbols are generalizations, using  $n$  copies of the integers, of simpler objects called *Durfee symbols*. The latter represent a partition's Ferrers diagram by indicating the size of the Durfee square, as well as the columns to the right of and below the Durfee square. For example, the Durfee symbol

$$\begin{pmatrix} 2 & \\ 2 & 1 \end{pmatrix}_3$$

represents the partition  $4 + 4 + 3 + 2 + 1$  of 14. Andrews defined the *rank* of a Durfee symbol to be the number of parts in the top row minus the number in the bottom row, which recovers Dyson's rank of the associated partition when  $n = 1$ . For the more general  $n$ -marked Durfee symbols, Andrews define a notion of rank for each of the  $n$  copies of the integers used.

Let  $\mathcal{D}_n(m_1, \dots, m_n; r)$  denote the number of  $n$ -marked Durfee symbols arising from partitions of  $r$  with  $j$ th rank equal to  $m_j$ . In [1], Andrews showed that the  $(n + 1)$ -variable generating function for Durfee symbols may be expressed in terms of certain  $q$ -hypergeometric series, analogous to (1.2). To

describe this, for  $n \geq 2$ , define

$$R_n(\mathbf{x}; q) := \sum_{\substack{m_1 > 0 \\ m_2, \dots, m_n \geq 0}} \frac{q^{(m_1 + \dots + m_n)^2 + (m_1 + \dots + m_{n-1}) + (m_1 + \dots + m_{n-2}) + \dots + m_1}}{(x_1 q; q)_{m_1} \left(\frac{q}{x_1}; q\right)_{m_1}} \\ \times \frac{1}{(x_2 q^{m_1}; q)_{m_2+1} \left(\frac{q^{m_1}}{x_2}; q\right)_{m_2+1} \cdots (x_n q^{m_1 + \dots + m_{n-1}}; q)_{m_n+1} \left(\frac{q^{m_1 + \dots + m_{n-1}}}{x_n}; q\right)_{m_n+1}},$$

where  $\mathbf{x} = \mathbf{x}_n := (x_1, \dots, x_n)$ . For  $n = 1$ , the function  $R_1(x; q)$  is defined as the  $q$ -hypergeometric series in (1.2). In what follows, for ease of notation, we may also write  $R_1(\mathbf{x}; q)$  to denote  $R_1(x; q)$ , with the understanding that  $\mathbf{x} := x$ . In [1], Andrews established the following result, generalizing (1.2).

THEOREM ([1, Theorem 10]). *For  $n \geq 1$  we have*

$$(1.6) \quad \sum_{m_1, \dots, m_n = -\infty}^{\infty} \sum_{r=0}^{\infty} \mathcal{D}_n(m_1, \dots, m_n; r) x_1^{m_1} \cdots x_n^{m_n} q^r = R_n(\mathbf{x}; q).$$

When  $n = 1$ , one recovers Dyson's rank, in that  $\mathcal{D}_1(m_1; r) = N(m_1, r)$ , so we see that (1.6) reduces to (1.2) in this case. The mock modularity of the associated two-variable generating function  $R_1(x; q)$  was established in [5] as described in the theorem above. In [2], Bringmann showed that  $R_2(1, 1; q)$  is a *quasimock theta function*, and a year later Bringmann, Garvan, and Mahlburg [4] proved that more generally  $R_n(1, \dots, 1; q)$  is a quasimock theta function for  $n \geq 2$ . Precise statements of these results can be found in [2, 4].

Two of the authors [13] established the automorphic properties of  $R_n(\mathbf{x}; q)$  for more arbitrary parameters  $\mathbf{x} = (x_1, \dots, x_n)$ , thus treating families of the rank generating functions for  $n$ -marked Durfee symbols with additional singularities, as compared to  $R_n(1, \dots, 1; q)$ . The techniques of Andrews [1] and Bringmann [2] were not directly applicable in this instance due to the presence of such additional singularities. These singular combinatorial families are essentially mixed mock and quasimock modular forms. Using this result, the authors [11] established quantum modular properties of  $R_n(\mathbf{x}; q)$  with distinct roots of unity  $x_1, \dots, x_n$  as stated in the Theorem in Section 1.3 below. (See [11] for more details.) To precisely state the result from [13], we first introduce some notation, which we also use for the remainder of this paper. Namely, we consider functions evaluated at certain length  $n$  vectors  $\zeta_{n,N}$  of roots of unity defined as follows (as in [13]).

Let  $n$  and  $N$  be fixed integers satisfying  $0 \leq N \leq \lfloor n/2 \rfloor$ , and  $n \geq 2$ . Suppose for  $1 \leq j \leq n - N$  that  $\alpha_j \in \mathbb{Z}$  and  $\beta_j \in \mathbb{N}$ , where  $\beta_j \nmid \alpha_j$ ,  $\beta_j \nmid 2\alpha_j$ , and that  $\alpha_r/\beta_r \pm \alpha_s/\beta_s \notin \mathbb{Z}$  if  $1 \leq r \neq s \leq n - N$ . Let

$$\begin{aligned}
 \alpha_{n,N} &:= \left( \underbrace{\frac{\alpha_1}{\beta_1}, \frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_2}, \frac{\alpha_2}{\beta_2}, \dots, \frac{\alpha_N}{\beta_N}, \frac{\alpha_N}{\beta_N}}_{2N}, \underbrace{\frac{\alpha_{N+1}}{\beta_{N+1}}, \frac{\alpha_{N+2}}{\beta_{N+2}}, \dots, \frac{\alpha_{n-N}}{\beta_{n-N}}}_{n-2N} \right) \in \mathbb{Q}^n, \\
 \zeta_{n,N} &:= \left( \underbrace{\zeta_{\beta_1}^{\alpha_1}, \zeta_{\beta_1}^{\alpha_1}, \zeta_{\beta_2}^{\alpha_2}, \zeta_{\beta_2}^{\alpha_2}, \dots, \zeta_{\beta_N}^{\alpha_N}, \zeta_{\beta_N}^{\alpha_N}}_{2N}, \underbrace{\zeta_{\beta_{N+1}}^{\alpha_{N+1}}, \zeta_{\beta_{N+2}}^{\alpha_{N+2}}, \dots, \zeta_{\beta_{n-N}}^{\alpha_{n-N}}}_{n-2N} \right) \in \mathbb{C}^n.
 \end{aligned}
 \tag{1.7}$$

Here,  $\zeta_\beta^\alpha = e^{2\pi i \alpha / \beta}$  as before.

REMARK 1.1. We point out that the dependence of the vector  $\zeta_{n,N}$  on  $n$  is reflected only in the length of the vector, and not (necessarily) in the roots of unity that comprise its components. In particular, the vector components may be chosen to be  $m$ th roots of unity for different values of  $m$ .

REMARK 1.2. The conditions given in [13] do not require  $\gcd(\alpha_j, \beta_j) = 1$ . Instead, they merely require that  $\alpha_j / \beta_j \neq \frac{1}{2}\mathbb{Z}$ . Without loss of generality, we will assume here that  $\gcd(\alpha_j, \beta_j) = 1$ . Then, requiring that  $\beta_j \nmid 2\alpha_j$  is the same as saying  $\beta_j \neq 2$ .

To complete the function  $R_n(\zeta_{n,N}; q)$  we first define the holomorphic function

$$B_n^+(\zeta_{n,N}; q) := R_n(\zeta_{n,N}; q) + b_n(\zeta_{n,N}; q),$$

with

$$\begin{aligned}
 &b_n(\zeta_{n,N}; q) \\
 &:= \frac{1}{(q)_\infty} \sum_{j=1}^N \zeta_{2\beta_j}^{-\alpha_j} \frac{\zeta_{\beta_j}^{-\alpha_j}}{2} \left( \frac{3}{\Pi_j(\alpha_{n,N}, 0)} + \frac{\frac{d}{dw} \Pi_j(\alpha_{n,N}, w)|_{w=0}}{\pi i (\Pi_j(\alpha_{n,N}, 0))^2} \right) A_3\left(\frac{\alpha_j}{\beta_j}, -2\tau; \tau\right) \\
 &\quad - \frac{1}{(q)_\infty} \sum_{j=1}^N \zeta_{2\beta_j}^{-3\alpha_j} \frac{\zeta_{\beta_j}^{-\alpha_j}}{2} \left( \frac{1}{\Pi_j(\alpha_{n,N}, 0)} + \frac{\frac{d}{dw} \Pi_j(\alpha_{n,N}, w)|_{w=0}}{\pi i (\Pi_j(\alpha_{n,N}, 0))^2} \right) A_3\left(\frac{\alpha_j}{\beta_j}, -2\tau; \tau\right).
 \end{aligned}$$

Here,  $\Pi_j$  is a constant depending only on  $\zeta_{n,N}$  as defined in [13] (see also (2.8) below), and  $A_3$  is the level 3 Appell function (see [3] or [22])

$$A_3(u, v; \tau) := e^{3\pi i u} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{3n(n+1)/2} e^{2\pi i v}}{1 - e^{2\pi i u} q^n},
 \tag{1.8}$$

where  $u, v \in \mathbb{C}$ . In [22], Zwegers showed that  $A_3(u, v; \tau)$  can be completed using the non-holomorphic function  $\mathcal{R}_3$  in (2.5) to transform like a non-holomorphic Jacobi form. Using these functions, as in [13] we let

$$\hat{B}_n(\zeta_{n,N}; q) := q^{-1/24} (B_n^+(\zeta_{n,N}; q) + B_n^-(\zeta_{n,N}; q)),$$

where the function  $B_n^-$  is given explicitly in terms of sums of functions involving  $F_{m,3}^-$  (see (2.10)) and  $\mathcal{R}_3$  (see (2.5)) in [13, equation (4.3)]. We have the following theorem, established by two of the authors in [13].

THEOREM ([13, Theorem 1.1]). If  $n \geq 2$  is an integer, and  $N$  is an integer satisfying  $0 \leq N \leq \lfloor n/2 \rfloor$ , then  $\widehat{B}_n(\zeta_{n,N}; q) = \widehat{\mathcal{H}}(\zeta_{n,N}; q) + \widehat{\mathcal{A}}(\zeta_{n,N}; q)$ , where  $\widehat{\mathcal{H}}(\zeta_{n,N}; q)$  and  $\widehat{\mathcal{A}}(\zeta_{n,N}; q)$  are non-holomorphic modular forms of weights  $3/2$  and  $1/2$ , respectively, on  $\Gamma_{n,N}$ , with character  $\chi_\gamma^{-1}$ .

Here, the functions  $\widehat{\mathcal{H}}(\zeta_{n,N}; q)$  and  $\widehat{\mathcal{A}}(\zeta_{n,N}; q)$ , as well as their holomorphic parts  $\mathcal{H}(\zeta_{n,N}; q)$  and  $\mathcal{A}(\zeta_{n,N}; q)$ , are defined in (2.11) and (2.12), respectively. The subgroup  $\Gamma_{n,N} \subseteq \mathrm{SL}_2(\mathbb{Z})$  under which  $\widehat{B}_n(\zeta_{n,N}; q)$  transforms is defined by

$$\Gamma_{n,N} := \bigcap_{j=1}^{n-N} \Gamma_0(2\beta_j^2) \cap \Gamma_1(2\beta_j),$$

and the Nebentypus character  $\chi_\gamma$  is given in Lemma 2.1.

REMARK 1.3. Zagier defined a *mixed mock modular form* [3, 19] to be the product of a mock modular form and a modular form. Here, the holomorphic parts of  $\widehat{B}_n$  consist of linear combinations of mixed mock modular forms, and also terms consisting of derivatives  $\frac{d}{du}\phi(u, \tau)|_{u=0}$  of mock Jacobi forms  $\phi(u, \tau)$  in the Jacobi  $u$  variable evaluated at  $u = 0$ , multiplied by modular forms. For simplicity, we may still refer to holomorphic parts of  $\widehat{B}_n(\zeta_{n,N}; q)$  as *mixed mock modular forms*.

**1.2. Quantum modular forms.** In this paper, we extend results from [11], which establish quantum modular properties for the  $(n+1)$ -variable rank generating function for  $n$ -marked Durfee symbols  $R_n(\mathbf{x}; q)$  with distinct roots of unity  $x_1, \dots, x_n$ , by determining quantum modular properties for  $R_n(\mathbf{x}; q)$  when there are repeated roots of unity.

Loosely speaking, a quantum modular form is similar to a mock modular form in that it exhibits a modular-like transformation with respect to the action of a suitable subgroup of  $\mathrm{SL}_2(\mathbb{Z})$ ; however, rather than the upper half-plane  $\mathbb{H}$ , the domain of a quantum modular form is the set of rationals  $\mathbb{Q}$  or an appropriate subset. The formal definition of a quantum modular form was originally introduced by Zagier [20] and has since been slightly modified to allow for half-integral weights, subgroups of  $\mathrm{SL}_2(\mathbb{Z})$ , etc. (see [3]).

DEFINITION 1.4. A weight  $k \in \frac{1}{2}\mathbb{Z}$  *quantum modular form* is a complex-valued function  $f$  on  $\mathbb{Q}$  such that, for all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , the functions  $h_\gamma : \mathbb{Q} \setminus \gamma^{-1}(i\infty) \rightarrow \mathbb{C}$  defined by

$$h_\gamma(x) := f(x) - \varepsilon^{-1}(\gamma)(cx + d)^{-k} f\left(\frac{ax + b}{cx + d}\right)$$

satisfy a “suitable” property of continuity or analyticity in a subset of  $\mathbb{R}$ .

REMARKS. (1) The complex numbers  $\varepsilon(\gamma)$ , which satisfy  $|\varepsilon(\gamma)| = 1$ , are such as those appearing in the theory of half-integral weight modular forms.

(2) We may modify Definition 1.4 appropriately to allow transformations on appropriate subgroups of  $\mathrm{SL}_2(\mathbb{Z})$ . We may also restrict the domains of the functions  $h_\gamma$  to be suitable subsets of  $\mathbb{Q}$ .

Since Zagier's initial definition, the subject of quantum modular forms has been widely studied (see [3] and references therein for a number of examples and applications). In particular, the notion of a quantum modular form is now known to have a direct connection to Ramanujan's original definition of a mock theta function [6, 14] and more generally to that of a mock modular form [8].

**1.3. Results.** Although automorphic properties of the rank generating function for  $n$ -marked Durfee symbols  $R_n$  in (1.6) on  $\mathbb{H}$  have been established by two of the authors (see [13, Theorem 1.1] above) and  $\mathbb{Q}$  is a natural boundary to  $\mathbb{H}$ , a priori there is no reason to expect  $R_n$  to converge on  $\mathbb{Q}$ , let alone exhibit quantum-automorphic properties there. However, here (as well as in previous work [11]) we do in fact establish quantum-automorphic properties for  $R_n$ .

For the remainder of this paper, we use the notation

$$\mathcal{V}_{n,N}(\tau) := \mathcal{V}(\zeta_{n,N}; q),$$

where  $\mathcal{V}$  may refer to any one of the functions

$$\widehat{\mathcal{A}}, \mathcal{A}, \widehat{\mathcal{H}}, \mathcal{H}, \widehat{B}_n, R_n, b_n, B_n^+, B_n^-.$$

(We omit repetitive subscripts and write  $\mathcal{V}_{n,N}(\tau)$  for  $(\mathcal{V}_n)_{n,N}(\tau)$  as well.) Note that when  $N = 0$ , these functions are equal to the ones in [11], that is,  $\mathcal{V}_{n,0}(\tau) = \mathcal{V}_n(\tau)$ .

In [11], we established the quantum modular properties of  $R_n$  in the special case when  $N = 0$ . More precisely, we showed that for  $N = 0$ ,  $\mathcal{A}_{n,N}(\tau) = q^{-1/24} R_n(\zeta_{n,N}; q)$  is a quantum modular form under the action of a subgroup of  $\Gamma_{n,0}$ , with quantum set

$$(1.9) \quad Q_{\zeta_{n,N}} := \left\{ \frac{h}{k} \in \mathbb{Q} \left| \begin{array}{l} h \in \mathbb{Z}, k \in \mathbb{N}, \gcd(h, k) = 1, \beta_j \nmid k \ \forall 1 \leq j \leq n, \\ \left| \frac{\alpha_j}{\beta_j} k - \left\lfloor \frac{\alpha_j}{\beta_j} k \right\rfloor \right| > \frac{1}{6} \ \forall 1 \leq j \leq n \end{array} \right. \right\},$$

where  $[x]$  denotes the closest integer to  $x$ .

REMARK 1.5. For  $x \in 1/2 + \mathbb{Z}$ , different sources define  $[x]$  to mean either  $x - 1/2$  or  $x + 1/2$ . The definition of  $Q_{\zeta_{n,N}}$  involving  $[\cdot]$  is well-defined for either of these conventions in the case of  $x \in 1/2 + \mathbb{Z}$ , as  $|x - [x]| = 1/2$ .

Here, we consider the complementary case of  $N > 0$ , and ultimately establish quantum modular properties for the function  $q^{-1/24}B_{n,N}^+$  in this setting. When  $N > 0$ , one has repeated roots of unity in (1.7). This leads to additional singularities, rendering the study of the modular properties of  $q^{-1/24}B_{n,N}^+$  in the case  $N > 0$  significantly more complex than in the case  $N = 0$ . Before stating our main result, we first define

$$(1.10) \quad \ell = \ell(\zeta_{n,N}) := \begin{cases} 6[\text{lcm}(\beta_1, \dots, \beta_n)]^2 & \text{if } 3 \nmid \beta_j \text{ for all } 1 \leq j \leq n, \\ 2[\text{lcm}(\beta_1, \dots, \beta_n)]^2 & \text{if } 3 \mid \beta_j \text{ for some } 1 \leq j \leq n, \end{cases}$$

and let  $S_\ell := \begin{pmatrix} 1 & 0 \\ \ell & 1 \end{pmatrix}$ ,  $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . We define the group generated by these two matrices as

$$\Gamma_{\zeta_{n,N}} := \langle S_\ell, T \rangle,$$

which we note is analogous to the group given in (1.5) used by Bringmann and Ono [5] to prove the mock modularity of the partition rank function  $R_1$ . It is likely an infinite index subgroup of  $\text{SL}_2(\mathbb{Z})$  rather than a congruence subgroup, say. Moreover, the constant  $\prod_j^\dagger(\alpha_{n,N})$  (a finite product) is as defined in [13, (4.2)] (where one must replace  $n \mapsto j$  and  $k \mapsto n$ , see also (2.9) below). Throughout the paper we let  $e(x) := e^{2\pi i x}$ .

**THEOREM ([11, Theorem 1.7]).** *Let  $N = 0$ . For all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\zeta_{n,N}}$ , and  $x \in \mathbb{Q}_{\zeta_{n,N}}$ ,*

$$H_{n,\gamma}(x) := \mathcal{A}_n(x) - \chi_\gamma(cx + d)^{-1/2} \mathcal{A}_n(\gamma x)$$

*is defined, and extends to an analytic function in  $x$  on  $\mathbb{R} - \{-c/d\}$ . In particular, for the matrix  $S_\ell$ ,*

$$(1.11) \quad \begin{aligned} & H_{n,S_\ell}(x) \\ &= \frac{\sqrt{3}}{2} \sum_{j=1}^n \frac{(\zeta_{2\beta_j}^{-3\alpha_j} - \zeta_{2\beta_j}^{-\alpha_j})}{\prod_j^\dagger(\alpha_{n,N})} e\left(\frac{2\alpha_j}{\beta_j}\right) \left[ \sum_{\pm} \zeta_6^{\pm 1} \int_{1/\ell}^{i\infty} \frac{g_{\pm 1/3+1/2, -3\alpha_j/\beta_j+1/2}(3\rho)}{\sqrt{-i}(\rho+x)} d\rho \right] \\ & \quad + \sum_{j=1}^n \frac{(\zeta_{2\beta_j}^{-3\alpha_j} - \zeta_{2\beta_j}^{-\alpha_j})}{\prod_j^\dagger(\alpha_{n,N})} (\ell x + 1)^{-1/2} \zeta_{24}^{-\ell} \mathcal{E}_1\left(\frac{\alpha_j}{\beta_j}, \ell; x\right), \end{aligned}$$

*where the weight  $3/2$  theta functions  $g_{a,b}$  are defined in (2.7), and  $\mathcal{E}_1$  is defined in Lemma 4.2.*

As described above, for the case of  $N > 0$ , there is an additional holomorphic function  $b_n(\zeta_{n,N}; q)$  which is added to  $R_n(\zeta_{n,N}; q)$  to obtain a “modular” object (see [13, Theorem 1.1] recalled above.) For  $N \geq 0$ , we have the following result which generalizes [11, Theorem 1.7] above.

THEOREM 1.6. For any integer  $N \geq 0$  we have

$$e^{-\pi i x/12} B_{n,N}^+(x) = \mathcal{H}_{n,N}(x) + \mathcal{A}_{n,N}(x),$$

where  $\mathcal{H}_{n,N}$  is a quantum modular form of weight  $3/2$ , and  $\mathcal{A}_{n,N}$  is a quantum modular form of weight  $1/2$ , both defined on the quantum set  $Q_{\zeta_{n,N}}$  with respect to the group  $\Gamma_{\zeta_{n,N}}$  and with character  $\chi_\gamma^{-1}$ . That is, for all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\zeta_{n,N}}$  and  $x \in Q_{\zeta_{n,N}}$ ,

$$H_{n,N,\gamma}^{(1)}(x) := \mathcal{A}_{n,N}(x) - \chi_\gamma(cx + d)^{-1/2} \mathcal{A}_{n,N}(\gamma x),$$

$$H_{n,N,\gamma}^{(2)}(x) := \mathcal{H}_{n,N}(x) - \chi_\gamma(cx + d)^{-3/2} \mathcal{H}_{n,N}(\gamma x)$$

are defined, and extend to analytic functions in  $x$  on  $\mathbb{R} - \{-c/d\}$ .

In particular, for the matrix  $S_\ell$ , we have  $H_{n,N,S_\ell}^{(1)}(x) = H_{n,S_\ell}(x)$ , where  $H_{n,S_\ell}(x)$  is as in (1.11), and

$$H_{n,N,S_\ell}^{(2)}(x) = -\zeta_{24}^{-\ell}(\ell x + 1)^{-3/2} \left( \sum_{j=1}^N \frac{\zeta_{2\beta_j}^{\alpha_j} - \zeta_{2\beta_j}^{-\alpha_j}}{2\pi_j(0)} \right. \\ \left. \times \left[ \sqrt{\ell x + 1} \zeta_{24}^\ell \left( \left( \frac{\ell}{2} - 3 \frac{\alpha_j}{\beta_j} \ell \right) H_{\alpha_j, \beta_j}(x) - \frac{1}{2\pi i} D_{\alpha_j, \beta_j}(x) \right) + \mathcal{E}_2 \left( \frac{\alpha_j}{\beta_j}, \ell; x \right) \right] \right),$$

where  $H_{\alpha, \beta}$  is as in (4.3),  $D_{\alpha, \beta}$  is defined in (4.18), and  $\mathcal{E}_2$  is defined in Proposition 4.6.

REMARK 1.7. Our results reveal that  $e^{-\pi i x/12} B_{n,N}^+(x)$  is a mixed weight quantum modular form. From this one also obtains the analytic nature of

$$H_{n,N,\gamma}^{(1)}(x)(cx + d)^{-1} + H_{n,N,\gamma}^{(2)}(x),$$

which showcases the transformation of  $R_{n,N}(x)$ .

REMARK 1.8. By combining the explicit closed-form evaluation of the function  $R_n(\zeta_{n,N}; \zeta_k^h)$  as a rational polynomial in roots of unity given in Section 3 with the quantum modular transformations from Theorem 1.6, we obtain explicit evaluations of Eichler integrals of (derivatives of) modular forms. Similar corollaries have been explicitly established in [10, 12].

## 2. Preliminaries

**2.1. Modular, mock modular and Jacobi forms.** The Dedekind  $\eta$ -function, defined in (1.1), is a well-known modular form of weight  $1/2$ . It transforms with character  $\chi_\gamma$  (see [15, Ch. 4, Thm. 2]):

$$\chi_\gamma = \begin{cases} \left( \frac{d}{|c|} \right) e\left( \frac{1}{24}((a+d)c - bd(c^2 - 1) - 3c) \right) & \text{if } c \equiv 1 \pmod{2}, \\ \left( \frac{c}{d} \right) e\left( \frac{1}{24}((a+d)c - bd(c^2 - 1) + 3d - 3 - 3cd) \right) & \text{if } d \equiv 1 \pmod{2}, \end{cases}$$

where  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , and  $\left(\frac{a}{\beta}\right)$  is the generalized Legendre symbol. Precisely,  $\eta$  satisfies the following transformation property [17].

LEMMA 2.1. *For  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ , we have  $\eta(\gamma\tau) = \chi_\gamma(c\tau + d)^{1/2}\eta(\tau)$ .*

We require two additional “modular” objects, namely the Jacobi theta function  $\vartheta(u; \tau)$ , an ordinary Jacobi form, and a non-holomorphic function  $R(u; \tau)$  used by Zwegers [21]. In what follows, we will also need certain transformation properties of these functions.

PROPOSITION 2.2. *For  $u \in \mathbb{C}$  and  $\tau \in \mathbb{H}$ , define*

$$(2.1) \quad \vartheta(u; \tau) := \sum_{\nu \in 1/2 + \mathbb{Z}} e^{\pi i \nu^2 \tau + 2\pi i \nu(u + 1/2)}.$$

*Then*

$$\begin{aligned} (1) \quad & \vartheta(u + 1; \tau) = -\vartheta(u; \tau), \\ (2) \quad & \vartheta(u + \tau; \tau) = -e^{-\pi i \tau - 2\pi i u} \vartheta(u; \tau), \\ (3) \quad & \vartheta(u; \tau) \\ &= -ie^{\pi i \tau/4} e^{-\pi i u} \prod_{m=1}^{\infty} (1 - e^{2\pi i m \tau})(1 - e^{2\pi i u} e^{2\pi i \tau(m-1)})(1 - e^{-2\pi i u} e^{2\pi i m \tau}). \end{aligned}$$

The non-holomorphic function  $R(u; \tau)$  is defined in [21] by

$$(2.2) \quad R(u; \tau) := \sum_{\nu \in 1/2 + \mathbb{Z}} \left\{ \operatorname{sgn}(\nu) - E\left(\left(\nu + \frac{\operatorname{Im}(u)}{\operatorname{Im}(\tau)}\right) \sqrt{2\operatorname{Im}(\tau)}\right) \right\} (-1)^{\nu-1/2} e^{-\pi i \nu^2 \tau - 2\pi i \nu u},$$

where

$$E(z) := 2 \int_0^z e^{-\pi t^2} dt.$$

The function  $R$  transforms like a (non-holomorphic) mock Jacobi form:

PROPOSITION 2.3 ([21, Propositions 1.9 and 1.10]). *The function  $R$  has the following transformation properties:*

$$\begin{aligned} (1) \quad & R(u + 1; \tau) = -R(u; \tau), \\ (2) \quad & R(u; \tau) + e^{-2\pi i u - \pi i \tau} R(u + \tau; \tau) = 2e^{-\pi i u - \pi i \tau/4}, \\ (3) \quad & R(u; \tau) = R(-u; \tau), \\ (4) \quad & R(u; \tau + 1) = e^{-\pi i/4} R(u; \tau), \\ (5) \quad & \frac{1}{\sqrt{-i\tau}} e^{\pi i u^2/\tau} R(u/\tau; -1/\tau) + R(u; \tau) = h(u; \tau), \text{ where the Mordell integral} \\ & \text{is defined by} \end{aligned}$$

$$(2.3) \quad h(u; \tau) := \int_{\mathbb{R}} \frac{e^{\pi i \tau t^2 - 2\pi i u t}}{\cosh \pi t} dt.$$

Using the functions  $\vartheta$  and  $R$ , Zwegers defined the completion of  $A_3(u, v; \tau)$  (see (1.8)) by

$$(2.4) \quad \hat{A}_3(u, v; \tau) := A_3(u, v; \tau) + \mathcal{R}_3(u, v; \tau),$$

with

$$(2.5) \quad \begin{aligned} \mathcal{R}_3(u, v; \tau) &:= \frac{i}{2} \sum_{j=0}^2 e^{2\pi i j u} \vartheta(v + j\tau + 1; 3\tau) R(3u - v - j\tau - 1; 3\tau) \\ &= \frac{i}{2} \sum_{j=0}^2 e^{2\pi i j u} \vartheta(v + j\tau; 3\tau) R(3u - v - j\tau; 3\tau), \end{aligned}$$

where the equality is justified by Propositions 2.2 and 2.3. This completed function transforms like a (non-holomorphic) Jacobi form, and in particular satisfies the following elliptic transformation.

**THEOREM 2.4** ([22, Theorem 2.2]). *For  $n_1, n_2, m_1, m_2 \in \mathbb{Z}$ , the completed level 3 Appell function  $\hat{A}_3$  satisfies*

$$(2.6) \quad \begin{aligned} \hat{A}_3(u + n_1\tau + m_1, v + n_2\tau + m_2; \tau) \\ = (-1)^{n_1+m_1} e^{2\pi i(u(3n_1-n_2)-vn_1)} q^{3n_1^2/2-n_1n_2} \hat{A}_3(u, v; \tau). \end{aligned}$$

We will also make use of the following results on the Mordell integral defined in (2.3).

**THEOREM 2.5** ([21, Theorem 1.2(1, 2, 4)]). *Let  $z \in \mathbb{C}$ ,  $\tau \in \mathbb{H}$ . Then*

- (1)  $h(z; \tau) + h(z + 1; \tau) = \frac{2}{\sqrt{-i\tau}} e^{\pi i(z+1/2)^2/\tau},$
- (2)  $h(z; \tau) + e(-z - \tau/2)h(z + \tau; \tau) = 2e(-z/2 - \tau/8),$
- (3)  $h$  is an even function of  $z$ .

Zwegers also showed how under certain hypotheses, the functions  $h$  and  $R$  can be written in terms of integrals involving the weight  $3/2$  modular forms  $g_{a,b}(\tau)$ , defined for  $a, b \in \mathbb{R}$  and  $\tau \in \mathbb{H}$  by

$$(2.7) \quad g_{a,b}(\tau) := \sum_{\nu \in a+\mathbb{Z}} \nu e^{\pi i \nu^2 \tau + 2\pi i \nu b}.$$

We have the following properties of  $g_{a,b}$ .

**PROPOSITION 2.6** ([21, Proposition 1.15(1, 2, 4, 5)]). *The function  $g_{a,b}$  satisfies*

- (1)  $g_{a+1,b}(\tau) = g_{a,b}(\tau),$
- (2)  $g_{a,b+1}(\tau) = e^{2\pi i a} g_{a,b}(\tau),$
- (3)  $g_{a,b}(\tau + 1) = e^{-\pi i a(a+1)} g_{a,a+b+1/2}(\tau),$
- (4)  $g_{a,b}(-1/\tau) = i e^{2\pi i a b} (-i\tau)^{3/2} g_{b,-a}(\tau).$

THEOREM 2.7 ([21, Theorem 1.16(2)]). Let  $\tau \in \mathbb{H}$ . For  $a, b \in (-1/2, 1/2)$ ,

$$h(a\tau - b; \tau) = -e\left(\frac{a^2\tau}{2} - a\left(b + \frac{1}{2}\right)\right) \int_0^{i\infty} \frac{g_{a+1/2, b+1/2}(\rho)}{\sqrt{-i(\rho + \tau)}} d\rho.$$

**2.2. Completing the function  $R_{n,N}$ .** Here we review some preliminary results and functions from [13]. Recall that  $n$  and  $N$  are fixed integers satisfying  $0 \leq N \leq \lfloor n/2 \rfloor$ , and  $n \geq 2$ . The vectors  $\zeta_{n,N}$  are defined in (1.7). For reference, we state here the explicit definitions of the constants  $\Pi$  and  $\Pi^\dagger$ , which were originally defined in [13]. Let  $\mathbf{z} := (z_1, \dots, z_n) \in \mathbb{R}^n$ . For fixed pairs  $(n, N)$  as above we define, for each  $1 \leq m \leq N$  and  $N+1 \leq j \leq n-N$  respectively,

$$(2.8) \quad \begin{aligned} \Pi_m(\mathbf{z}, w) &:= (1 - e(-2z_m)) \prod_{\substack{t=1 \\ t \neq m}}^N (e(w + z_m) - e(z_t))^2 \left(1 - \frac{1}{e(w + z_m + z_t)}\right)^2 \\ &\quad \times \prod_{\ell=N+1}^{n-N} (e(w + z_m) - e(z_\ell)) \left(1 - \frac{1}{e(w + z_m + z_\ell)}\right) \end{aligned}$$

and

$$(2.9) \quad \begin{aligned} \Pi_j^\dagger(\mathbf{z}) &:= \prod_{t=1}^N (e(z_j) - e(z_t))^2 \left(1 - \frac{1}{e(z_j + z_t)}\right)^2 \\ &\quad \times \prod_{\substack{\ell=N+1 \\ \ell \neq j}}^{n-N} (e(z_j) - e(z_\ell)) \left(1 - \frac{1}{e(z_j + z_\ell)}\right), \end{aligned}$$

where  $w \in \mathbb{R}$ . (As usual, we take the empty product to equal 1.)

To complete the function  $R_{n,N}(\tau) := R(\zeta_{n,N}; q)$  as described in §1, we use the following functions from [13]:

$$(2.10) \quad \begin{aligned} F_{m,s}^+(\mathbf{x}; \tau) &:= \lim_{w \rightarrow 0} \left( \frac{e(-x_m)}{e(w) - e(-w)} \right. \\ &\quad \times \left. \left( e^{s\pi i w} \frac{A_3(-w + x_m, -2\tau; \tau)}{\Pi_m(\mathbf{x}, -w)} - e^{-s\pi i w} \frac{A_3(w + x_m, -2\tau; \tau)}{\Pi_m(\mathbf{x}, w)} \right) \right), \end{aligned}$$

$$\begin{aligned} F_{m,s}^-(\mathbf{x}; \tau) &:= \lim_{w \rightarrow 0} \left( \frac{e(-x_m)}{e(w) - e(-w)} \right. \\ &\quad \times \left. \left( e^{s\pi i w} \frac{\mathcal{R}_3(-w + x_m, -2\tau; \tau)}{\Pi_m(\mathbf{x}, -w)} - e^{-s\pi i w} \frac{\mathcal{R}_3(w + x_m, -2\tau; \tau)}{\Pi_m(\mathbf{x}, w)} \right) \right), \end{aligned}$$

with  $A_3$  as defined in (1.8),  $\mathcal{R}_3$  as defined in (2.5), and  $\Pi_m(\mathbf{x}, w)$  as defined

explicitly in [13]. The corresponding completed function is

$$\hat{F}_{m,s}(\mathbf{x}; \tau) := F_{m,s}^+(\mathbf{x}; \tau) + F_{m,s}^-(\mathbf{x}; \tau).$$

Using  $\hat{A}_3$  (see (2.4)), we also define

$$\begin{aligned} \hat{G}_{m,s}(\boldsymbol{\alpha}_{n,N}; \tau) \\ := \frac{\zeta_{\beta_m}^{-\alpha_m}}{2} \left( \frac{4-s}{\Pi_m(\boldsymbol{\alpha}_{n,N}, 0)} + \frac{\frac{d}{dw} \Pi_m(\boldsymbol{\alpha}_{n,N}, w)|_{w=0}}{\pi i (\Pi_m(\boldsymbol{\alpha}_{n,N}, 0))^2} \right) \hat{A}_3 \left( \frac{\alpha_m}{\beta_m}, -2\tau; \tau \right) \end{aligned}$$

and

$$\hat{H}_{m,s}(\boldsymbol{\alpha}_{n,N}; \tau) := \hat{F}_{m,s}(\boldsymbol{\alpha}_{n,N}; \tau) + \hat{G}_{m,s}(\boldsymbol{\alpha}_{n,N}; \tau).$$

The non-holomorphic functions from [13, Theorem 1.1] (see §1) are defined by

$$\begin{aligned} (2.11) \quad \hat{\mathcal{H}}(\boldsymbol{\zeta}_{n,N}; q) &= \hat{\mathcal{H}}_{n,N}(\boldsymbol{\zeta}_{n,N}; q) \\ &:= \frac{1}{\eta(\tau)} \left( \sum_{j=1}^N (\zeta_{2\beta_j}^{-\alpha_j} \hat{H}_{j,1}(\boldsymbol{\alpha}_{n,N}; \tau) - \zeta_{2\beta_j}^{-3\alpha_j} \hat{H}_{j,3}(\boldsymbol{\alpha}_{n,N}; \tau)) \right), \end{aligned}$$

$$\begin{aligned} (2.12) \quad \hat{\mathcal{A}}(\boldsymbol{\zeta}_{n,N}; q) &= \hat{\mathcal{A}}_{n,N}(\boldsymbol{\zeta}_{n,N}; q) \\ &:= \frac{1}{\eta(\tau)} \left( \sum_{j=N+1}^{n-N} (\zeta_{2\beta_j}^{-3\alpha_j} - \zeta_{2\beta_j}^{-\alpha_j}) \frac{\hat{A}_3(\alpha_j/\beta_j, -2\tau; \tau)}{\Pi_j^\dagger(\boldsymbol{\alpha}_{n,N})} \right). \end{aligned}$$

We recall that the constant (a finite product)  $\Pi_j^\dagger(\boldsymbol{\alpha}_{n,N})$  is defined explicitly in [13, (4.2)] (where one must replace  $n \mapsto j$  and  $k \mapsto n$ ). The holomorphic parts  $\mathcal{H}$  and  $\mathcal{A}$  of the functions  $\hat{\mathcal{H}}$  and  $\hat{\mathcal{A}}$  are defined by replacing the non-holomorphic functions  $\hat{H}_{j,1}$ ,  $\hat{H}_{j,3}$  and  $\hat{A}_3$  with their respective holomorphic parts  $H_{j,1}$ ,  $H_{j,3}$  and  $A_3$  in (2.11) and (2.12) above.

**3. The quantum set.** We call a subset  $S \subseteq \mathbb{Q}$  a *quantum set* for a function  $F$  with respect to a group  $G \subseteq \mathrm{SL}_2(\mathbb{Z})$  if both  $F(x)$  and  $F(Mx)$  exist (are non-singular) for all  $x \in S$  and  $M \in G$ .

In this section, we will show that  $Q_{\boldsymbol{\zeta}_{n,N}}$  as defined in (1.9) is a quantum set for  $\mathcal{A}_{n,N}$  and  $\mathcal{H}_{n,N}$  with respect to the group  $\Gamma_{\boldsymbol{\zeta}_{n,N}}$ .

Recall that the holomorphic part of our “modular object” (see Section 1) is  $R_{n,N} + b_{n,N}$ . We now analyze the convergence of  $R_{n,N}$  and  $b_{n,N}$  separately.

It was shown in [11, Section 3] that  $Q_{\boldsymbol{\zeta}_{n,N}}$  is a quantum set for  $\mathcal{A}_{n,N}(\tau) = q^{-1/24} R_{n,N}(\tau)$ . Moreover, the following theorem establishes the convergence of  $R_{n,N}$  on  $Q_{\boldsymbol{\zeta}_{n,N}}$ .

**THEOREM ([11, Theorem 3.2]).** *For  $\boldsymbol{\zeta}_{n,N}$  as in (1.7), if  $h/k \in Q_{\boldsymbol{\zeta}_{n,N}}$ , then  $R_n(\boldsymbol{\zeta}_{n,N}; \zeta_k^h)$  converges and can be evaluated as a finite sum. In particular,*

$$\begin{aligned}
R_n(\zeta_{n,N}; \zeta_k^h) &= \prod_{j=1}^n \frac{1}{1 - ((1 - x_j^k)(1 - x_j^{-k}))^{-1}} \\
&\times \sum_{\substack{0 \leq m_1 \leq k \\ 0 \leq m_2, \dots, m_n < k}} \frac{\zeta_k^{h[(m_1 + \dots + m_n)^2 + (m_1 + \dots + m_{n-1}) + (m_1 + \dots + m_{n-2}) + \dots + m_1]}}{(x_1 \zeta_k^h; \zeta_k^h)_{m_1} \left(\frac{\zeta_k^h}{x_1}; \zeta_k^h\right)_{m_1} (x_2 \zeta_k^{hm_1}; \zeta_k^h)_{m_2+1} \left(\frac{\zeta_k^{hm_1}}{x_2}; \zeta_k^h\right)_{m_2+1}} \\
&\times \frac{1}{(x_3 \zeta_k^{h(m_1+m_2)}; \zeta_k^h)_{m_3+1} \left(\frac{\zeta_k^{h(m_1+m_2)}}{x_3}; \zeta_k^h\right)_{m_3+1}} \times \dots \\
&\times \frac{1}{(x_n \zeta_k^{h(m_1+\dots+m_{n-1})}; \zeta_k^h)_{m_n+1} \left(\frac{\zeta_k^{h(m_1+\dots+m_{n-1})}}{x_n}; \zeta_k^h\right)_{m_n+1}},
\end{aligned}$$

where  $\zeta_{n,N} = (x_1, \dots, x_n)$ .

We now turn our attention to  $b_{n,N}$ . In what follows, as in the definition of  $Q_{\zeta_{n,N}}$ , we take  $h \in \mathbb{Z}$  and  $k \in \mathbb{N}$  such that  $\gcd(h, k) = 1$ . To show that  $b_n(\zeta_{n,N}; \zeta)$  is defined for  $\zeta = e^{2\pi i h/k}$  with  $h/k \in Q_{\zeta_{n,N}}$ , it is enough to show that

$$(3.1) \quad \frac{1}{(\zeta)_\infty} A_3\left(x_j, -\frac{2h}{k}; \frac{h}{k}\right)$$

is defined, with  $A_3(u, v; \tau)$  as in (1.8). For this proof, we will make use of the following transformation formula for  $A_3(u, v; \tau)$ .

PROPOSITION 3.1. *For  $u, v \in \mathbb{C}$  and  $\tau \in \mathbb{H}$  we have*

$$(3.2) \quad A_3(u, v + \tau; \tau) = e^{-2\pi i u} A_3(u, v; \tau) + i e^{\pi i u - \pi i v - 3\pi i \tau/4} \vartheta(v; 3\tau),$$

where  $\vartheta$  is as defined in (2.1).

*Proof.* We will first rewrite  $\mathcal{R}_3(u, v + \tau; \tau)$  in terms of  $\mathcal{R}_3(u, v; \tau)$ . By definition (2.5),

$$\begin{aligned}
(3.3) \quad \mathcal{R}_3(u, v + \tau; \tau) &= \frac{i}{2} \vartheta(v + \tau; 3\tau) R(3u - v - \tau; 3\tau) \\
&\quad + \frac{i}{2} e^{2\pi i u} \vartheta(v + 2\tau; 3\tau) R(3u - v - 2\tau; 3\tau) \\
&\quad + \frac{i}{2} e^{4\pi i u} \vartheta(v + 3\tau; 3\tau) R(3u - v - 3\tau; 3\tau).
\end{aligned}$$

Letting  $\tau \mapsto 3\tau$  in Proposition 2.2, we can rewrite the third summand in (3.3) as

$$\begin{aligned}
&\frac{i}{2} e^{4\pi i u} \vartheta(v + 3\tau; 3\tau) R(3u - v - 3\tau; 3\tau) \\
&= -\frac{i}{2} e^{-3\pi i \tau - 2\pi i v + 4\pi i u} \vartheta(v; 3\tau) R(3u - v - 3\tau; 3\tau).
\end{aligned}$$

We now let  $\tau \mapsto 3\tau$  and  $u \mapsto 3u - v - 3\tau$  in the second transformation in Proposition 2.3 to obtain

$$(3.4) \quad -\frac{i}{2}e^{-3\pi i\tau-2\pi iv+4\pi iu}\vartheta(v;3\tau)R(3u-v-3\tau;3\tau) \\ = -ie^{\pi iu-\pi iv-3\pi i\tau/4}\vartheta(v;3\tau) + \frac{i}{2}e^{-2\pi iu}\vartheta(v;3\tau)R(3u-v;3\tau).$$

Plugging (3.4) into (3.3) and using the definition of  $\mathcal{R}_3(u, v; \tau)$ , we see that

$$(3.5) \quad \mathcal{R}_3(u, v + \tau; \tau) = e^{-2\pi iu}\mathcal{R}_3(u, v; \tau) - ie^{\pi iu-\pi iv-3\pi i\tau/4}\vartheta(v;3\tau).$$

By Theorem 2.4, we have

$$A_3(u, v + \tau; \tau) + \mathcal{R}_3(u, v + \tau; \tau) = e^{-2\pi iu}A_3(u, v; \tau) + e^{-2\pi iu}\mathcal{R}_3(u, v; \tau).$$

Using (3.5), we then achieve the desired result. ■

We are now ready to show that the function in (3.1) converges.

**THEOREM 3.2.** *For  $h/k \in Q_{\zeta_{n,N}}$ ,  $\zeta = e^{2\pi ih/k}$ , and  $x_j = e^{2\pi i\alpha_j/\beta_j}$  the  $j$ th component in  $\zeta_{n,N}$  (as in (1.7)), the function  $\frac{1}{(\zeta)_\infty}A_3(\alpha_j/\beta_j, -2h/k; h/k)$  converges. In particular,*

$$\frac{1}{(\zeta)_\infty}A_3\left(\frac{\alpha_j}{\beta_j}, -\frac{2h}{k}; \frac{h}{k}\right) = \frac{x_j^{5/2}}{1-x_j}R_1(x_j; \zeta) + x_j^{3/2} \\ = \frac{x_j^{5/2}}{(1-x_j)(1-((1-x_j^k)(1-x_j^{-k}))^{-1})} \sum_{0 \leq s < k} \frac{\zeta^{s^2}}{(x_j\zeta; \zeta)_s(x_j^{-1}\zeta; \zeta)_s} + x_j^{3/2}.$$

*Proof.* Equations (1.7) to (1.10) in [1] show that for  $u \in \mathbb{C}$ ,  $\tau \in \mathbb{H}$ , and  $q = e^{2\pi i\tau}$ ,

$$R_1(e^{2\pi iu}; q) = \sum_{m=0}^{\infty} \frac{q^{m^2}}{(e^{2\pi iu}q; q)_m(e^{-2\pi iu}q; q)_m} \\ = \frac{1 - e^{2\pi iu}}{(q)_\infty} \sum_{m \in \mathbb{Z}} \frac{(-1)^m q^{m(3m+1)/2}}{1 - e^{2\pi iu}q^m} \\ = \frac{1 - e^{2\pi iu}}{(q)_\infty} e^{-3\pi iu} A_3(u, -\tau; \tau).$$

Taking  $v = -2\tau$  in (3.2) and rearranging gives

$$(3.6) \quad \frac{1}{(q)_\infty} A_3(u, -2\tau; \tau) = \frac{e^{5\pi iu}}{1 - e^{2\pi iu}} R_1(e^{2\pi iu}; q) - \frac{ie^{3\pi iu}e^{5\pi i\tau/4}}{(q)_\infty} \vartheta(-2\tau; 3\tau).$$

Applying the Jacobi triple product from Proposition 2.2, we can simplify the

second term in (3.6) as

$$\begin{aligned} \frac{ie^{3\pi iu}e^{5\pi i\tau/4}}{(q)_\infty}\vartheta(-2\tau;3\tau) &= \frac{e^{3\pi iu}e^{4\pi i\tau}}{(q)_\infty} \prod_{m=1}^{\infty} (1-q^{3m})(1-q^{3m-5})(1-q^{3m+2}) \\ &= \frac{e^{3\pi iu}q^2}{(1-q)(1-q^2)\prod_{m=3}^{\infty}(1-q^m)}(1-q^{-2})(1-q)\prod_{m=3}^{\infty}(1-q^m) \\ &= \frac{e^{3\pi iu}q^2(1-q^{-2})}{(1-q^2)} = -e^{3\pi iu}. \end{aligned}$$

This simplification allows us to see that

$$\frac{1}{(q)_\infty}A_3(u, -2\tau; \tau) = \frac{e^{5\pi iu}}{1 - e^{2\pi iu}}R_1(e^{2\pi iu}; q) + e^{3\pi iu}.$$

In [11, Theorem 3.2] recalled above, it was shown that  $R_1(e^{2\pi iu}; \zeta)$  is defined for  $u = \alpha_j/\beta_j$  and  $\zeta = e^{2\pi ih/k}$  with  $h/k \in Q_{\zeta_{n,N}}$ . By definition,  $\alpha_j/\beta_j \notin \mathbb{Z}$ , meaning  $1 - x_j \neq 0$ . Thus, we have shown that  $\frac{1}{(\zeta)_\infty}A_3(\alpha_j/\beta_j, -2h/k; h/k)$  is defined for  $h/k \in Q_{\zeta_{n,N}}$ , as desired. To obtain the exact formula for it, we let  $n = 1$  in the exact formula of [11, Theorem 3.2] given above. ■

We obtain the following corollary from Theorem 3.2.

**COROLLARY 3.3.** *The function  $b_n(\zeta_{n,N}; \zeta)$  is defined for  $\zeta = e^{2\pi ih/k}$  with  $h/k \in Q_{\zeta_{n,N}}$ .*

We also obtain the following corollary, which we will need in the proof of Theorem 1.6.

**COROLLARY 3.4.** *For  $x \in Q_{\zeta_{n,N}}$ , the functions  $\mathcal{H}_{n,N}(x)$  and  $\mathcal{A}_{n,N}(x)$  converge.*

*Proof.* First we consider  $\mathcal{A}_{n,N}(x)$ . From (2.12), we have

$$\mathcal{A}_{n,N}(\tau) = \frac{1}{\eta(\tau)} \sum_{j=N+1}^{n-N} \frac{\zeta_{2\beta_j}^{-3\alpha_j} - \zeta_{2\beta_j}^{-\alpha_j}}{\Pi_j^\dagger(\alpha_{n,N})} A_3\left(\frac{\alpha_j}{\beta_j}, -2\tau; \tau\right),$$

a linear combination of  $A_3(\alpha_j/\beta_j, -2\tau; \tau)/\eta(\tau)$ . Thus, the convergence of  $\mathcal{A}_{n,N}(x)$  for  $x \in Q_{\zeta_{n,N}}$  follows directly from Theorem 3.2.

In order to show the convergence of  $\mathcal{H}_{n,N}(x)$ , we consider the holomorphic part of  $\widehat{B}_{n,N}(\tau)$  at  $x \in Q_{\zeta_{n,N}}$ , namely

$$\begin{aligned} (3.7) \quad e^{-\pi ix/12} B_{n,N}^+(x) &= \mathcal{H}_{n,N}(x) + \mathcal{A}_{n,N}(x) \\ &= \zeta^{-1/24}(R_n(\zeta_{n,N}; \zeta) + b_n(\zeta_{n,N}; \zeta)), \end{aligned}$$

where  $\zeta = e^{2\pi ix}$ . From [11, Theorem 3.2] and Corollary 3.3, the left-hand side of (3.7) converges on  $Q_{\zeta_{n,N}}$ , which yields the claim. ■

**4. Proof of Theorem 1.6.** Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\zeta_{n,N}}$  and  $x \in Q_{\zeta_{n,N}}$  as before. From [13, Theorem 1.1] (see §1), we deduce that

$$\begin{aligned}
 (4.1) \quad & (\mathcal{H}_{n,N}(x) - \chi_\gamma(cx + d)^{-3/2} \mathcal{H}_{n,N}(\gamma x)) \\
 & + (\mathcal{A}_{n,N}(x) - \chi_\gamma(cx + d)^{-1/2} \mathcal{A}_{n,N}(\gamma x)) \\
 & = -(\mathcal{H}_{n,N}^-(x) - \chi_\gamma(cx + d)^{-3/2} \mathcal{H}_{n,N}^-(\gamma x)) \\
 & - (\mathcal{A}_{n,N}^-(x) - \chi_\gamma(cx + d)^{-1/2} \mathcal{A}_{n,N}^-(\gamma x)),
 \end{aligned}$$

where we write  $\mathcal{H}_{n,N}^-$  and  $\mathcal{A}_{n,N}^-$  to denote the non-holomorphic parts of the functions  $\hat{\mathcal{H}}_{n,N}$  and  $\hat{\mathcal{A}}_{n,N}$ , respectively (see (2.11) and (2.12)).

In this section, we prove  $\mathcal{H}_{n,N} + \mathcal{A}_{n,N}$  is a mixed weight quantum modular form on  $Q_{\zeta_{n,N}}$ . To do so, we first show that the left-hand side of (4.1) is defined on  $Q_{\zeta_{n,N}}$ . This follows directly from Corollary 3.4. Therefore, to prove Theorem 1.6, it remains to be seen that the right-hand side of (4.1) extends to an analytic function on  $\mathbb{R} - \{-c/d\}$ .

It is shown in [11, Theorem 1.7] that  $\mathcal{A}_{n,N}^-(x) - \chi_\gamma(cx + d)^{-1/2} \mathcal{A}_{n,N}^-(\gamma x)$  is analytic in  $x$  on  $\mathbb{R} - \{-c/d\}$ . Thus, we turn to the function  $\mathcal{H}_{n,N}^-$ . A short calculation using the definition of  $\mathcal{H}_{n,N}^-$ , as well as (2.11) and [13, eqs. (4.5), (4.6), and (4.18)], leads to the following result.

LEMMA 4.1. *With notation and hypotheses as above,*

$$\begin{aligned}
 \mathcal{H}_{n,N}^-(\tau) &= \frac{2}{\eta(\tau)} \sum_{j=1}^N \frac{\zeta_{2\beta_j}^{-3\alpha_j} - \zeta_{2\beta_j}^{-5\alpha_j}}{\Pi_j(0)} \mathcal{R}_3\left(\frac{\alpha_j}{\beta_j}\right) \\
 &\quad - \frac{1}{2\pi i \eta(\tau)} \sum_{j=1}^N \frac{\zeta_{2\beta_j}^{-3\alpha_j} - \zeta_{2\beta_j}^{-5\alpha_j}}{\Pi_j(0)} \frac{d}{du} \mathcal{R}_3(u) \Big|_{u=\alpha_j/\beta_j} \\
 &= 2 \sum_{j=1}^N \frac{\zeta_{2\beta_j}^{-3\alpha_j} - \zeta_{2\beta_j}^{-5\alpha_j}}{\Pi_j(0)} S\left(\frac{\alpha_j}{\beta_j}; \tau\right) \\
 &\quad - \frac{1}{2\pi i} \sum_{j=1}^N \frac{\zeta_{2\beta_j}^{-3\alpha_j} - \zeta_{2\beta_j}^{-5\alpha_j}}{\Pi_j(0)} \frac{d}{du} S(u; \tau) \Big|_{u=\alpha_j/\beta_j},
 \end{aligned}$$

where

$$S(u; \tau) := \frac{\mathcal{R}_3(u, -2\tau; \tau)}{\eta(\tau)}.$$

In order to examine  $\mathcal{H}_{n,N}^-(\gamma x)$ , we transform the functions  $S(\alpha/\beta; \tau)$  and  $\frac{d}{du} S(u; \tau) \Big|_{u=\alpha/\beta}$  separately. For ease of notation, we sometimes suppress dependence on  $j$ , and write, for example,  $\alpha/\beta$  for  $\alpha_j/\beta_j$  when the context is clear. Note that it suffices to consider the generators  $T$  and  $S_\ell$  of  $\Gamma_{\zeta_{n,N}}$  as

before. Using the definition (2.5) of  $\mathcal{R}_3$ , we begin by rewriting  $S(u; \tau)$  as

$$(4.2) \quad \begin{aligned} S(u; \tau) &= \frac{i}{2\eta(\tau)} \sum_{j=0}^2 e(ju) \vartheta((j-2)\tau; 3\tau) R(3u + (2-j)\tau; 3\tau) \\ &= \frac{q^{-1/6}}{2} \sum_{\pm} \mp e(u(2 \mp 1)) R(3u \pm \tau; 3\tau) - q^{-1/24} e\left(\frac{3}{2}u\right). \end{aligned}$$

The second equality follows directly from Propositions 2.2(3) and 2.3(2). More precisely, from Proposition 2.2(3) we have  $\vartheta(-2\tau; 3\tau) = iq^{2/3}\eta(\tau)$ ,  $\vartheta(-\tau; 3\tau) = iq^{-1/6}\eta(\tau)$ , and  $\vartheta(0; 3\tau) = 0$ , and from Proposition 2.3(2),  $R(3u + 2\tau; 3\tau) = 2e(3/2u)q^{5/8} - e(3u)q^{1/2}R(3u - \tau; 3\tau)$ .

We deduce the following transformation properties of  $S(\alpha/\beta; \tau)$ .

LEMMA 4.2. *With notation and hypotheses as above, we have*

$$S\left(\frac{\alpha}{\beta}; \tau + 1\right) = \zeta_{24}^{-1} S\left(\frac{\alpha}{\beta}; \tau\right)$$

and

$$\begin{aligned} S\left(\frac{\alpha}{\beta}; S_{\ell}\tau\right) &= (\ell\tau + 1)^{1/2} \zeta_{24}^{\ell} S\left(\frac{\alpha}{\beta}; \tau\right) + \frac{1}{2}(\ell\tau + 1)^{1/2} \zeta_{24}^{\ell} e\left(2\frac{\alpha}{\beta}\right) H_{\alpha, \beta}(\tau) + \mathcal{E}_1\left(\frac{\alpha}{\beta}, \ell; \tau\right), \end{aligned}$$

where  $\mathcal{E}_1\left(\frac{\alpha}{\beta}, \ell; \tau\right) := (\ell\tau + 1)^{1/2} \zeta_{24}^{\ell} q^{-1/24} e\left(\frac{3}{2}\frac{\alpha}{\beta}\right) - e\left(-\frac{S_{\ell}\tau}{24}\right) e\left(\frac{3}{2}\frac{\alpha}{\beta}\right)$ . The function  $H_{\alpha, \beta}(\tau)$  is defined in (4.6) and equals

$$(4.3) \quad H_{\alpha, \beta}(\tau) = \sqrt{3} \sum_{\pm} \mp e\left(\mp \frac{1}{6}\right) \int_{1/\ell}^{i\infty} \frac{g_{\pm 1/3 + 1/2, -3\alpha/\beta + 1/2}(3\rho)}{\sqrt{-i(\rho + \tau)}} d\rho.$$

*Proof.* Letting  $\tau \mapsto \tau + 1$  in (4.2) gives

$$(4.4) \quad \begin{aligned} S(u; \tau + 1) &= \frac{\zeta_6^{-1} q^{-1/6}}{2} \sum_{\pm} \mp e(u(2 \mp 1)) R(3u \pm \tau \pm 1; 3\tau + 3) - \zeta_{24}^{-1} q^{-1/24} e\left(\frac{3}{2}u\right). \end{aligned}$$

Using the transformation properties (1) and (4) from Proposition 2.3, we have

$$R(3u \pm \tau \pm 1; 3\tau + 3) = -e^{-3\pi i/4} R(3u \pm \tau; 3\tau).$$

Substituting this into (4.4) yields

$$\begin{aligned} S(u; \tau + 1) &= \frac{\zeta_{24}^{-1} q^{-1/6}}{2} \sum_{\pm} \mp e(u(2 \mp 1)) R(3u \pm \tau; 3\tau) - \zeta_{24}^{-1} q^{-1/24} e\left(\frac{3}{2}u\right) \\ &= \zeta_{24}^{-1} S(u; \tau). \end{aligned}$$

We now turn to the  $S_\ell$  transformation. Recalling the definition of  $F_{\alpha,\beta}$  in [11]:

$$F_{\alpha,\beta}(\tau) := q^{-1/6} \sum_{\pm} \pm e\left(\mp \frac{\alpha}{\beta}\right) R\left(\frac{3\alpha}{\beta} \pm \tau; 3\tau\right),$$

we rewrite  $S(\alpha/\beta; \tau)$  in terms of  $F_{\alpha,\beta}(\tau)$  as

$$S\left(\frac{\alpha}{\beta}; \tau\right) = -\frac{1}{2}e\left(\frac{2\alpha}{\beta}\right)F_{\alpha,\beta}(\tau) - q^{-1/24}e\left(\frac{3}{2}\frac{\alpha}{\beta}\right),$$

and thus

$$(4.5) \quad S\left(\frac{\alpha}{\beta}; S_\ell\tau\right) = -\frac{1}{2}e\left(\frac{2\alpha}{\beta}\right)F_{\alpha,\beta}(S_\ell\tau) - e\left(-\frac{S_\ell\tau}{24}\right)e\left(\frac{3}{2}\frac{\alpha}{\beta}\right).$$

We further recall the definition of  $H_{\alpha,\beta}$  from [11]:

$$(4.6) \quad H_{\alpha,\beta}(\tau) := F_{\alpha,\beta}(\tau) - \zeta_{24}^{-\ell}(\ell\tau + 1)^{-1/2}F_{\alpha,\beta}(S_\ell\tau).$$

Inserting (4.6) into (4.5) with a direct calculation reveals that

$$\begin{aligned} S\left(\frac{\alpha}{\beta}; S_\ell\tau\right) &= -\frac{1}{2}(\ell\tau + 1)^{1/2}\zeta_{24}^\ell e\left(\frac{2\alpha}{\beta}\right)(F_{\alpha,\beta}(\tau) - H_{\alpha,\beta}(\tau)) - e\left(-\frac{S_\ell\tau}{24}\right)e\left(\frac{3}{2}\frac{\alpha}{\beta}\right) \\ &= (\ell\tau + 1)^{1/2}\zeta_{24}^\ell S\left(\frac{\alpha}{\beta}; \tau\right) + \frac{1}{2}(\ell\tau + 1)^{1/2}\zeta_{24}^\ell e\left(\frac{2\alpha}{\beta}\right)H_{\alpha,\beta}(\tau) \\ &\quad + (\ell\tau + 1)^{1/2}\zeta_{24}^\ell q^{-1/24}e\left(\frac{3}{2}\frac{\alpha}{\beta}\right) - e\left(-\frac{S_\ell\tau}{24}\right)e\left(\frac{3}{2}\frac{\alpha}{\beta}\right), \end{aligned}$$

as claimed. ■

In order to establish the transformation properties of  $\frac{d}{du}S(u; \tau)|_{u=\alpha/\beta}$ , we first deduce the following, using (4.2):

$$\frac{d}{du}S(u; \tau)\Big|_{u=\alpha/\beta} = -3\pi i q^{-1/24}e\left(\frac{3}{2}\frac{\alpha}{\beta}\right) + W_1(\tau) + W_2(\tau),$$

where

$$(4.7) \quad W_1(\tau) := \frac{q^{-1/6}}{2} \sum_{\pm} \mp e\left(\frac{\alpha}{\beta}(2 \mp 1)\right) \frac{d}{du}R(3u \pm \tau; 3\tau)\Big|_{u=\alpha/\beta},$$

$$(4.8) \quad W_2(\tau) := \pi i q^{-1/6} \sum_{\pm} (1 \mp 2)e\left(\frac{\alpha}{\beta}(2 \mp 1)\right) R\left(3\frac{\alpha}{\beta} \pm \tau; 3\tau\right).$$

First we establish the following transformation properties of  $W_2(\tau)$ .

LEMMA 4.3. *With notation and hypotheses as above, we have*

$$\begin{aligned} W_2(\tau + 1) &= \zeta_{24}^{-1} W_2(\tau), \\ W_2(S_\ell \tau) &= (\ell \tau + 1)^{1/2} \zeta_{24}^\ell (W_2(\tau) + \tilde{H}_{\alpha, \beta}(\tau)). \end{aligned}$$

*Proof.* As before, shifting  $\tau \mapsto \tau + 1$  in (4.8) and using the transformation properties (1) and (4) from Proposition 2.3 directly yields the first claim.

On the other hand, letting  $\tau \mapsto S_\ell \tau = -1/\tau_\ell$  in (4.8) with  $\tau_\ell = -1/\tau - \ell$  as before, we have

$$(4.9) \quad W_2(S_\ell \tau) = \pi i e \left( \frac{1}{6\tau_\ell} \right) \sum_{\pm} (1 \mp 2) e \left( \frac{\alpha}{\beta} (2 \mp 1) \right) R \left( 3 \frac{\alpha}{\beta} \mp \frac{1}{\tau_\ell}; \frac{-3}{\tau_\ell} \right).$$

From [11, proof of Proposition 4.1], we know that

$$\begin{aligned} (4.10) \quad R \left( 3 \frac{\alpha}{\beta} \mp \frac{1}{\tau_\ell}; \frac{-3}{\tau_\ell} \right) &= (\ell \tau + 1)^{1/2} \zeta_{24}^\ell e \left( \frac{-1}{6\tau_\ell} - \frac{\tau}{6} \right) R \left( 3 \frac{\alpha}{\beta} \pm \tau; 3\tau \right) \\ &\quad + \sqrt{3} (\ell \tau + 1)^{1/2} \zeta_{24}^\ell e \left( \frac{-1}{6\tau_\ell} \pm \frac{\alpha}{\beta} \mp \frac{1}{6} \right) \int_{1/\ell}^{i\infty} \frac{g_{\pm 1/3+1/2, -3\alpha/\beta+1/2}(3\rho)}{\sqrt{-i(\rho + \tau)}} d\rho. \end{aligned}$$

Inserting (4.10) into (4.9) gives us

$$\begin{aligned} W_2(S_\ell \tau) &= \pi i (\ell \tau + 1)^{1/2} \zeta_{24}^\ell q^{-1/6} \sum_{\pm} (1 \mp 2) e \left( \frac{\alpha}{\beta} (2 \mp 1) \right) R \left( 3 \frac{\alpha}{\beta} \pm \tau; 3\tau \right) \\ &\quad + \sqrt{3} \pi i (\ell \tau + 1)^{1/2} \zeta_{24}^\ell e \left( 2 \frac{\alpha}{\beta} \right) \\ &\quad \times \sum_{\pm} (1 \mp 2) e \left( \mp \frac{1}{6} \right) \int_{1/\ell}^{i\infty} \frac{g_{\pm 1/3+1/2, -3\alpha/\beta+1/2}(3\rho)}{\sqrt{-i(\rho + \tau)}} d\rho \\ &= (\ell \tau + 1)^{1/2} \zeta_{24}^\ell W_2(\tau) + (\ell \tau + 1)^{1/2} \zeta_{24}^\ell \tilde{H}_{\alpha, \beta}(\tau), \end{aligned}$$

where

$$(4.11) \quad \tilde{H}_{\alpha, \beta}(\tau) := \sqrt{3} \pi i e \left( 2 \frac{\alpha}{\beta} \right) \sum_{\pm} (1 \mp 2) e \left( \mp \frac{1}{6} \right) \int_{1/\ell}^{i\infty} \frac{g_{\pm 1/3+1/2, -3\alpha/\beta+1/2}(3\rho)}{\sqrt{-i(\rho + \tau)}} d\rho. \blacksquare$$

We may now deduce the following transformation properties of  $W_1(\tau)$ .

LEMMA 4.4. *Let  $m := [3\alpha/\beta]$  so that  $3\alpha/\beta = m + r$  with  $r \in (-1/2, 1/2)$ . With notation and hypotheses as above, we have*

$$W_1(\tau + 1) = \zeta_{24}^{-1} W_1(\tau)$$

and

$$\begin{aligned} W_1(S_\ell \tau) &= (\ell\tau + 1)^{3/2} \zeta_{24}^\ell W_1(\tau) \\ &\quad + \pi i \ell \tau (\ell\tau + 1)^{1/2} \zeta_{24}^\ell q^{-1/6} \sum_{\pm} e\left(\frac{\alpha}{\beta}(2 \mp 1)\right) R\left(3\frac{\alpha}{\beta} \pm \tau; 3\tau\right) \\ &\quad + \frac{3(-1)^m}{2} e\left(-\frac{S_\ell \tau}{6}\right) \sum_{\pm} \mp e\left(\frac{\alpha}{\beta}(2 \mp 1)\right) \frac{d}{dv} (T_1(v; \tau) + T_2(v; \tau)) \Big|_{v=r}, \end{aligned}$$

where  $T_1$  and  $T_2$  are defined in (4.13) and (4.14) below, respectively.

*Proof.* The first claim follows again by letting  $\tau \mapsto \tau + 1$  in (4.7) and using the transformation properties (1) and (4) in Proposition 2.3

To show the second claim, we first consider

$$\frac{d}{du} R(3u \pm \tau; 3\tau) \Big|_{u=\alpha/\beta}.$$

By the chain rule with  $v = 3u - m$  and Proposition 2.3(1), this derivative becomes

$$3 \frac{d}{dv} R(v + m \pm \tau; 3\tau) \Big|_{v=r} = 3(-1)^m \frac{d}{dv} R(v \pm \tau; 3\tau) \Big|_{v=r}.$$

We then transform this function by using transformation properties in Proposition 2.3. More precisely, we start with

$$R(v \mp S_\ell \tau; 3S_\ell \tau) = R\left(v \mp \frac{1}{\tau_\ell}; -\frac{3}{\tau_\ell}\right),$$

and apply (5) and (4) of Proposition 2.3. We then use (1) to shift the  $R$  function by  $-r\ell/3$ . Note that  $\alpha\ell/\beta \in 2\mathbb{Z}$  by the definition of  $\ell$  in (1.10) and  $r = 3\alpha/\beta - [3\alpha/\beta]$ , which yields  $-r\ell/3 \in 2\mathbb{Z}$ . Lastly, we apply (3) and (5) again to obtain

$$\begin{aligned} (4.12) \quad R\left(v \mp \frac{1}{\tau_\ell}; -\frac{3}{\tau_\ell}\right) &= T_1(v; \tau) + T_2(v; \tau) \\ &\quad + \left[ (\ell\tau + 1)^{1/2} \zeta_{24}^\ell q^{-r^2\ell^2/6} e\left(-\ell \frac{(v(\ell\tau + 1) \pm \tau)^2}{6(\ell\tau + 1)} + \frac{r\ell}{3} (v(\ell\tau + 1) \pm \tau)\right) \right. \\ &\quad \left. \times R(v(\ell\tau + 1) - r\ell\tau \pm \tau; 3\tau) \right], \end{aligned}$$

where

$$(4.13) \quad T_1(v; \tau) := \sqrt{\frac{i}{3} \left( \frac{1}{\tau} + \ell \right)} e\left(\frac{(v(\ell\tau + 1) \pm \tau)^2}{6\tau(\ell\tau + 1)}\right) h\left(\frac{v\tau_\ell}{3} \mp \frac{1}{3}; \frac{\tau_\ell}{3}\right),$$

$$\begin{aligned}
(4.14) \quad T_2(v; \tau) &:= -(\ell\tau + 1)^{1/2} \zeta_{24}^\ell q^{-r^2\ell^2/6} \\
&\quad \times e\left(-\ell \frac{(v(\ell\tau + 1) \pm \tau)^2}{6(\ell\tau + 1)} + \frac{r\ell}{3}(v(\ell\tau + 1) \pm \tau)\right) \\
&\quad \times h(v(\ell\tau + 1) - r\ell\tau \pm \tau; 3\tau).
\end{aligned}$$

Next we calculate the derivative of  $R(v \mp 1/\tau_\ell; -3/\tau_\ell)$ . To do so, we first consider the derivative of the exponential term on the right-hand side of (4.12). A short calculation shows that

$$\begin{aligned}
(4.15) \quad \frac{d}{dv} e\left(-\ell \frac{(v(\ell\tau + 1) \pm \tau)^2}{6(\ell\tau + 1)} + \frac{r\ell}{3}(v(\ell\tau + 1) \pm \tau)\right) \Big|_{v=r} \\
= \mp \frac{2\pi i \ell \tau}{3} q^{r^2\ell^2/6} e\left(\frac{-\ell\tau^2}{6(\ell\tau + 1)}\right).
\end{aligned}$$

We further examine the derivative of the  $R$  function in (4.12). Applying the chain rule with  $u = (v(\ell\tau + 1) - r\ell\tau + m)/3$  and then using Proposition 2.3(1) gives

$$\begin{aligned}
(4.16) \quad \frac{d}{dv} R(v(\ell\tau + 1) - r\ell\tau \pm \tau; 3\tau) \Big|_{v=r} \\
= \frac{\ell\tau + 1}{3} (-1)^m \frac{d}{du} R(3u \pm \tau; 3\tau) \Big|_{u=\alpha/\beta}.
\end{aligned}$$

Therefore, by (4.15), (4.16), and a direct calculation with Proposition 2.3(1), we have

$$\begin{aligned}
(4.17) \quad 3(-1)^m \frac{d}{dv} R\left(v \mp \frac{1}{\tau_\ell}; -\frac{3}{\tau_\ell}\right) \Big|_{v=r} \\
= 3(-1)^m \frac{d}{dv} (T_1(v; \tau) + T_2(v; \tau)) \Big|_{v=r} \\
+ (\ell\tau + 1)^{1/2} \zeta_{24}^\ell (\mp 2\pi i \ell \tau) e\left(\frac{-\ell\tau^2}{6(\ell\tau + 1)}\right) R\left(\frac{3\alpha}{\beta} \pm \tau; 3\tau\right) \\
+ (\ell\tau + 1)^{3/2} \zeta_{24}^\ell e\left(\frac{-\ell\tau^2}{6(\ell\tau + 1)}\right) \frac{d}{du} R(3u \pm \tau; 3\tau) \Big|_{u=\alpha/\beta}.
\end{aligned}$$

We are now able to prove the second claim of the lemma. By the definition of  $W_1$  in (4.7), and by (4.17), we find that

$$\begin{aligned}
W_1(S_\ell\tau) \\
= \frac{1}{2} e\left(-\frac{S_\ell\tau}{6}\right) \sum_{\pm} \mp e\left(\frac{\alpha}{\beta}(2 \mp 1)\right) \left(3(-1)^m \frac{d}{dv} R\left(v \mp \frac{1}{\tau_\ell}; -\frac{3}{\tau_\ell}\right) \Big|_{v=r}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{3(-1)^m}{2} e\left(-\frac{S_\ell \tau}{6}\right) \sum_{\pm} \mp e\left(\frac{\alpha}{\beta}(2 \mp 1)\right) \frac{d}{dv} (T_1(v; \tau) + T_2(v; \tau)) \Big|_{v=r} \\
&\quad + \pi i \ell \tau (\ell \tau + 1)^{1/2} \zeta_{24}^\ell q^{-1/6} \sum_{\pm} e\left(\frac{\alpha}{\beta}(2 \mp 1)\right) R\left(3 \frac{\alpha}{\beta} \pm \tau; 3\tau\right) \\
&\quad + (\ell \tau + 1)^{3/2} \zeta_{24}^\ell W_1(\tau). \blacksquare
\end{aligned}$$

We require the following lemma.

LEMMA 4.5. *Suppose that  $v \in (-1/2, 1/2)$ , and  $|v - r| < \epsilon$  for some sufficiently small  $\epsilon > 0$ . Then*

$$\begin{aligned}
&e\left(-\frac{S_\ell \tau}{6}\right) T_1(v; \tau) \\
&= \sqrt{3} (\ell \tau + 1)^{1/2} \zeta_{24}^\ell e\left(\mp \frac{1}{6} \pm \frac{v}{3} - \frac{v\ell}{6} + \frac{v^2 \ell}{6}\right) \int_{1/\ell}^0 \frac{g_{v\ell/3 \pm 1/3 + 1/2, -v+1/2}(3\rho)}{\sqrt{-i(\rho + \tau)}} d\rho, \\
&e\left(-\frac{S_\ell \tau}{6}\right) T_2(v; \tau) \\
&= \sqrt{3} (\ell \tau + 1)^{1/2} \zeta_{24}^\ell e\left(\mp \frac{1}{6} \pm \frac{v}{3} - \frac{v\ell}{6} + \frac{v^2 \ell}{6}\right) \int_0^{i\infty} \frac{g_{v\ell/3 \pm 1/3 + 1/2, -v+1/2}(3\rho)}{\sqrt{-i(\rho + \tau)}} d\rho.
\end{aligned}$$

*Proof.* Since  $v, \pm 1/3 \in (-1/2, 1/2)$ , we may apply Theorem 2.7 to the function  $h(v\tau_\ell/3 \mp 1/3; \tau_\ell/3)$  in the definition of  $T_1(v; \tau)$  in (4.13). Using Proposition 2.6(3, 4), we proceed as in [11] (4.6) and (4.7)] with  $v$  instead of  $r$ . A straightforward calculation yields the first equality asserted.

Similarly, the second equality follows directly by applying Theorem 2.7 to the function  $h(v(\ell\tau + 1) - r\ell\tau \pm \tau; 3\tau)$  in  $T_2(v; \tau)$  defined in (4.14) with  $a = \ell/3(v - r) \pm 1/3$ ,  $b = -v$ , and  $\tau \mapsto 3\tau$ . This is allowed because  $-v \in (-1/2, 1/2)$ , and since  $|v - r| < \epsilon$  for sufficiently small  $\epsilon > 0$ , we have  $\frac{\ell}{3}(v - r) \pm 1/3 \in (-1/2, 1/2)$ .  $\blacksquare$

We define (in parallel to  $H_{\alpha, \beta}(\tau)$ )

$$\begin{aligned}
(4.18) \quad D_{\alpha, \beta}(\tau) \\
&:= \sqrt{3} \sum_{\pm} \mp e\left(\mp \frac{1}{6}\right) \int_{1/\ell}^{i\infty} \frac{\frac{d}{du} g_{\ell u \pm 1/3 + 1/2, -3u+1/2}(3\rho)|_{u=\alpha/\beta}}{\sqrt{-i(\rho + \tau)}} d\rho.
\end{aligned}$$

By the lemmas above, we finally have the following result.

PROPOSITION 4.6. *Assume the notation and hypotheses as above. Then*

$$\mathcal{H}_{n, N}^-(\tau + 1) - \zeta_{24}^\ell \mathcal{H}_{n, N}^-(\tau) = 0$$

and

$$\begin{aligned} & \mathcal{H}_{n,N}^-(S_\ell\tau) - (\ell\tau + 1)^{3/2} \zeta_{24}^\ell \mathcal{H}_{n,N}^-(\tau) \\ &= \sum_{j=1}^N \frac{\zeta_{2\beta_j}^{\alpha_j} - \zeta_{2\beta_j}^{-\alpha_j}}{2\Pi_j(0)} \left[ (\ell\tau + 1)^{1/2} \zeta_{24}^\ell \left( \left( \frac{\ell}{2} - 3\frac{\alpha_j}{\beta_j} \ell \right) H_{\alpha_j, \beta_j}(\tau) \right. \right. \\ & \quad \left. \left. - \frac{1}{2\pi i} D_{\alpha_j, \beta_j}(\tau) \right) + \mathcal{E}_2\left(\frac{\alpha_j}{\beta_j}, \ell; x\right) \right], \end{aligned}$$

where  $\mathcal{E}_2(\alpha/\beta, \ell; \tau) := (\ell\tau + 1)^{3/2} \zeta_{24}^\ell q^{-1/24} e(-\frac{\alpha}{2\beta}) - e(-\frac{S_\ell\tau}{24}) e(-\frac{\alpha}{2\beta})$ .

*Proof.* We begin by recalling Lemma 4.1, that is,

$$\begin{aligned} (4.19) \quad \mathcal{H}_{n,N}^-(\tau) &= 2 \sum_{j=1}^N \frac{\zeta_{2\beta_j}^{-3\alpha_j} - \zeta_{2\beta_j}^{-5\alpha_j}}{\Pi_j(0)} S\left(\frac{\alpha_j}{\beta_j}; \tau\right) \\ &\quad - \frac{1}{2\pi i} \sum_{j=1}^N \frac{\zeta_{2\beta_j}^{-3\alpha_j} - \zeta_{2\beta_j}^{-5\alpha_j}}{\Pi_j(0)} \frac{d}{du} S(u; \tau) \Big|_{u=\alpha_j/\beta_j}, \end{aligned}$$

where

$$\frac{d}{du} S(u; \tau) \Big|_{u=\alpha/\beta} = -3\pi i q^{-1/24} e\left(\frac{3\alpha}{2\beta}\right) + W_1(\tau) + W_2(\tau).$$

The first claim follows directly from Lemmas 4.2 and 4.4.

For the second claim, we first rewrite  $W_1(S_\ell\tau)$  of Lemma 4.4 using  $S(\alpha/\beta; \tau)$  and  $W_2(\tau)$ , that is,

$$\begin{aligned} (4.20) \quad W_1(S_\ell\tau) &= (\ell\tau + 1)^{3/2} \zeta_{24}^\ell W_1(\tau) \\ &\quad + \ell\tau (\ell\tau + 1)^{1/2} \zeta_{24}^\ell \left( W_2(\tau) - 4\pi i S\left(\frac{\alpha}{\beta}; \tau\right) - 4\pi i q^{-1/24} e\left(\frac{3}{2} \frac{\alpha}{\beta}\right) \right) \\ &\quad + \frac{3(-1)^m}{2} e\left(-\frac{S_\ell\tau}{6}\right) \sum_{\pm} \mp e\left(\frac{\alpha}{\beta} (2 \mp 1)\right) \frac{d}{dv} (T_1(v; \tau) + T_2(v; \tau)) \Big|_{v=r}. \end{aligned}$$

We now consider  $\mathcal{H}_{n,N}^-(S_\ell\tau)$ . Combining (4.19), (4.20) and the second claims of Lemmas 4.2 and 4.3, we have

$$\begin{aligned} \mathcal{H}_{n,N}^-(S_\ell\tau) &= 2 \sum_{j=1}^N \frac{\zeta_{2\beta_j}^{-3\alpha_j} - \zeta_{2\beta_j}^{-5\alpha_j}}{\Pi_j(0)} \left( (\ell\tau + 1)^{1/2} \zeta_{24}^\ell \left( S\left(\frac{\alpha_j}{\beta_j}; \tau\right) \right. \right. \\ & \quad \left. \left. + \frac{1}{2} e\left(2\frac{\alpha_j}{\beta_j}\right) H_{\alpha_j, \beta_j}(\tau) \right) + \mathcal{E}_1\left(\frac{\alpha_j}{\beta_j}, \ell; \tau\right) \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2\pi i} \sum_{j=1}^N \frac{\zeta_{2\beta_j}^{-3\alpha_j} - \zeta_{2\beta_j}^{-5\alpha_j}}{\Pi_j(0)} \left[ -3\pi i e\left(-\frac{S_\ell \tau}{24}\right) e\left(\frac{3}{2} \frac{\alpha_j}{\beta_j}\right) \right. \\
& - 4\pi i \ell \tau (\ell \tau + 1)^{1/2} \zeta_{24}^\ell q^{-1/24} e\left(\frac{3}{2} \frac{\alpha_j}{\beta_j}\right) + (\ell \tau + 1)^{3/2} \zeta_{24}^\ell W_1(\tau) \\
& + (\ell \tau + 1)^{3/2} \zeta_{24}^\ell W_2(\tau) - 4\pi i \ell \tau (\ell \tau + 1)^{1/2} \zeta_{24}^\ell S\left(\frac{\alpha_j}{\beta_j}; \tau\right) \\
& + \frac{3(-1)^{m_j}}{2} e\left(-\frac{S_\ell \tau}{6}\right) \sum_{\pm} \mp e\left(\frac{\alpha}{\beta}(2 \mp 1)\right) \frac{d}{dv} (T_1(v; \tau) + T_2(v; \tau)) \Big|_{v=r_j} \\
& \left. + (\ell \tau + 1)^{1/2} \zeta_{24}^\ell \tilde{H}_{\alpha_j, \beta_j}(\tau) \right] \\
& = 2 \sum_{j=1}^N \frac{\zeta_{2\beta_j}^{-3\alpha_j} - \zeta_{2\beta_j}^{-5\alpha_j}}{\Pi_j(0)} \left[ (\ell \tau + 1)^{3/2} \zeta_{24}^\ell S\left(\frac{\alpha_j}{\beta_j}; \tau\right) \right. \\
& + \frac{1}{2} (\ell \tau + 1)^{1/2} \zeta_{24}^\ell e\left(2 \frac{\alpha_j}{\beta_j}\right) H_{\alpha_j, \beta_j}(\tau) \\
& + \frac{1}{4} (\ell \tau + 1)^{3/2} \zeta_{24}^\ell q^{-1/24} e\left(\frac{3}{2} \frac{\alpha_j}{\beta_j}\right) - \frac{1}{4} e\left(-\frac{S_\ell \tau}{24}\right) e\left(\frac{3}{2} \frac{\alpha_j}{\beta_j}\right) \Big] \\
& - \frac{1}{2\pi i} \sum_{j=1}^N \frac{\zeta_{2\beta_j}^{-3\alpha_j} - \zeta_{2\beta_j}^{-5\alpha_j}}{\Pi_j(0)} \left[ -3\pi i (\ell \tau + 1)^{3/2} \zeta_{24}^\ell q^{-1/24} e\left(\frac{3}{2} \frac{\alpha_j}{\beta_j}\right) \right. \\
& + (\ell \tau + 1)^{3/2} \zeta_{24}^\ell W_1(\tau) + (\ell \tau + 1)^{3/2} \zeta_{24}^\ell W_2(\tau) + (\ell \tau + 1)^{1/2} \zeta_{24}^\ell \tilde{H}_{\alpha_j, \beta_j}(\tau) \\
& + \frac{3(-1)^{m_j}}{2} e\left(-\frac{S_\ell \tau}{6}\right) \sum_{\pm} \mp e\left(\frac{\alpha}{\beta}(2 \mp 1)\right) \frac{d}{dv} (T_1(v; \tau) + T_2(v; \tau)) \Big|_{v=r_j} \Big] \\
& = (\ell \tau + 1)^{3/2} \zeta_{24}^\ell \mathcal{H}_{n, N}^-(\tau) \\
& + \sum_{j=1}^N \frac{\zeta_{2\beta_j}^{-3\alpha_j} - \zeta_{2\beta_j}^{-5\alpha_j}}{\Pi_j(0)} \left[ (\ell \tau + 1)^{1/2} \zeta_{24}^\ell \left( e\left(2 \frac{\alpha_j}{\beta_j}\right) H_{\alpha_j, \beta_j}(\tau) - \frac{1}{2\pi i} \tilde{H}_{\alpha_j, \beta_j}(\tau) \right) \right. \\
& + \frac{1}{2} (\ell \tau + 1)^{3/2} \zeta_{24}^\ell q^{-1/24} e\left(\frac{3}{2} \frac{\alpha_j}{\beta_j}\right) - \frac{1}{2} e\left(-\frac{S_\ell \tau}{24}\right) e\left(\frac{3}{2} \frac{\alpha_j}{\beta_j}\right) \\
& \left. - \frac{3(-1)^{m_j}}{4\pi i} e\left(-\frac{S_\ell \tau}{6}\right) \sum_{\pm} \mp e\left(\frac{\alpha}{\beta}(2 \mp 1)\right) \frac{d}{dv} (T_1(v; \tau) + T_2(v; \tau)) \Big|_{v=r_j} \right].
\end{aligned}$$

We continue to simplify the term in the parenthesis [ ] above. Using (4.3) and (4.11), we have

$$\begin{aligned}
& e\left(2\frac{\alpha}{\beta}\right)H_{\alpha,\beta}(\tau) - \frac{1}{2\pi i}\tilde{H}_{\alpha,\beta}(\tau) \\
&= \frac{\sqrt{3}}{2}e\left(2\frac{\alpha}{\beta}\right)\sum_{\pm}e\left(\mp\frac{1}{6}\right)\int_{1/\ell}^{i\infty}\frac{g_{\pm 1/3+1/2,-3\alpha/\beta+1/2}(3\rho)}{\sqrt{-i(\rho+\tau)}}d\rho.
\end{aligned}$$

Moreover, since in what follows we take the derivative in  $v$  at the points  $v = r_j$  and  $r_j \in (-1/2, 1/2)$ , we may assume  $|v - r_j| < \epsilon$  for sufficiently small  $\epsilon > 0$ . We further note from Proposition 2.6(2) that for  $m \in \mathbb{Z}$ ,

$$g_{a,b} = e(ma)g_{a,b-m}.$$

Applying this to Lemma 4.5, we obtain

$$\begin{aligned}
(4.21) \quad & 3(-1)^m e\left(-\frac{S_\ell \tau}{6}\right)[T_1(v; \tau) + T_2(v; \tau)] \\
&= 3\sqrt{3}(\ell\tau + 1)^{1/2}\zeta_{24}^\ell e\left(\mp\frac{1}{6} \pm \frac{v+m}{3} - \frac{v\ell}{6} + \frac{v^2\ell}{6} + \frac{vm\ell}{3}\right) \\
&\quad \times \int_{1/\ell}^{i\infty} \frac{g_{v\ell/3 \pm 1/3+1/2,-v-m+1/2}(3\rho)}{\sqrt{-i(\rho+\tau)}} d\rho.
\end{aligned}$$

Differentiating (4.21) yields

$$\begin{aligned}
& 3(-1)^m e\left(-\frac{S_\ell \tau}{6}\right) \frac{d}{dv} [T_1(v; \tau) + T_2(v; \tau)] \Big|_{v=r} \\
&= 2\pi i \sqrt{3}(\ell\tau + 1)^{1/2}\zeta_{24}^\ell \left(\pm 1 - \frac{\ell}{2} + 3\frac{\alpha}{\beta}\ell\right) e\left(\mp\frac{1}{6} \pm \frac{\alpha}{\beta}\right) \\
&\quad \times \int_{1/\ell}^{i\infty} \frac{g_{\pm 1/3+1/2,-3\alpha/\beta+1/2}(3\rho)}{\sqrt{-i(\rho+\tau)}} d\rho \\
&\quad + \sqrt{3}(\ell\tau + 1)^{1/2}\zeta_{24}^\ell e\left(\mp\frac{1}{6} \pm \frac{\alpha}{\beta}\right) \frac{d}{du} \int_{1/\ell}^{i\infty} \frac{g_{u \pm 1/3+1/2,-3u+1/2}(3\rho)}{\sqrt{-i(\rho+\tau)}} d\rho \Big|_{u=\alpha/\beta}.
\end{aligned}$$

Here we use Proposition 2.6(1) and the chain rule with  $u = (v + m)/3$ .

Alltogether, we finally have

$$\begin{aligned}
(4.22) \quad & \mathcal{H}_{n,N}^-(S_\ell \tau) - \zeta_{24}^\ell (\ell\tau + 1)^{3/2} \mathcal{H}_{n,N}^-(\tau) \\
&= \sum_{j=1}^N \frac{\zeta_{2\beta_j}^{-3\alpha_j} - \zeta_{2\beta_j}^{-5\alpha_j}}{\Pi_j(0)} \left[ \sqrt{3}(\ell\tau + 1)^{1/2}\zeta_{24}^\ell \left(\frac{\ell}{4} - \frac{3}{2}\frac{\alpha_j}{\beta_j}\ell\right) e\left(2\frac{\alpha_j}{\beta_j}\right) \right. \\
&\quad \times \sum_{\pm} \mp e\left(\mp\frac{1}{6}\right) \int_{1/\ell}^{i\infty} \frac{g_{\pm 1/3+1/2,-3\alpha_j/\beta_j+1/2}(3\rho)}{\sqrt{-i(\rho+\tau)}} d\rho
\end{aligned}$$

$$\begin{aligned}
& - \frac{\sqrt{3}(\ell\tau+1)^{1/2}\zeta_{24}^\ell}{4\pi i} e\left(2\frac{\alpha_j}{\beta_j}\right) \sum_{\pm} \mp e\left(\mp\frac{1}{6}\right) \\
& \times \frac{d}{du} \int_{1/\ell}^{i\infty} \frac{g_{\ell u \pm 1/3 + 1/2, -3u + 1/2}(3\rho)}{\sqrt{-i(\rho + \tau)}} d\rho \Big|_{u=\alpha_j/\beta_j} \\
& + \frac{1}{2}(\ell\tau+1)^{3/2}\zeta_{24}^\ell q^{-1/24} e\left(\frac{3}{2}\frac{\alpha_j}{\beta_j}\right) - \frac{1}{2}e\left(-\frac{S_\ell\tau}{24}\right) e\left(\frac{3}{2}\frac{\alpha_j}{\beta_j}\right) \Big] \\
& = \sum_{j=1}^N \frac{\zeta_{2\beta_j}^{\alpha_j} - \zeta_{2\beta_j}^{-\alpha_j}}{2\Pi_j(0)} \left[ (\ell\tau+1)^{1/2}\zeta_{24}^\ell \left(\frac{\ell}{2} - 3\frac{\alpha_j}{\beta_j}\ell\right) H_{\alpha_j, \beta_j}(\tau) \right. \\
& \quad \left. - \frac{(\ell\tau+1)^{1/2}\zeta_{24}^\ell}{2\pi i} D_{\alpha_j, \beta_j}(\tau) + \zeta_{2\beta_j}^{-\alpha_j} \left( (\ell\tau+1)^{3/2}\zeta_{24}^\ell q^{-1/24} - e\left(-\frac{S_\ell\tau}{24}\right) \right) \right],
\end{aligned}$$

where we justify bringing the derivative inside the integral defining  $D_{\alpha, \beta}$  in the proof of Proposition 4.7 below. ■

To finish the proof of Theorem 1.6, it remains to show the following.

PROPOSITION 4.7. *The function  $\mathcal{H}_{n, N}^-(S_\ell\tau) - (\ell\tau+1)^{3/2}\zeta_{24}^\ell \mathcal{H}_{n, N}^-(\tau)$  is analytic on  $\mathbb{R} - \{-1/\ell\}$ .*

*Proof.* We use Proposition 4.6. Clearly, the function  $\mathcal{E}_2$  is analytic on  $\mathbb{R} - \{-1/\ell\}$ . Moreover, [11, Proposition 4.1] establishes the same for the function  $H_{\alpha, \beta}$ . Thus, it suffices to show that the function  $D_{\alpha, \beta}(\tau)$  is analytic on  $\mathbb{R} - \{-1/\ell\}$ . We begin by computing

$$\begin{aligned}
& \frac{d}{du} g_{\ell u \pm 1/3 + 1/2, -3u + 1/2}(3\rho) \\
& = \ell \sum_{n \in \mathbb{Z}} e\left(\frac{3}{2}\rho(n + \ell u \pm \frac{1}{3} + \frac{1}{2})^2\right) e\left((n + \ell u \pm \frac{1}{3} + \frac{1}{2})(-3u + \frac{1}{2})\right) \\
& \quad + 2\pi i \ell \left(\frac{1}{2} - 3u\right) \sum_{n \in \mathbb{Z}} (n + \ell u \pm \frac{1}{3} + \frac{1}{2}) e\left(\frac{3}{2}\rho(n + \ell u \pm \frac{1}{3} + \frac{1}{2})^2\right) \\
& \quad \quad \quad \times e\left((n + \ell u \pm \frac{1}{3} + \frac{1}{2})(-3u + \frac{1}{2})\right) \\
& \quad + 6\pi i (\ell\rho - 1) \sum_{n \in \mathbb{Z}} (n + \ell u \pm \frac{1}{3} + \frac{1}{2})^2 e\left(\frac{3}{2}\rho(n + \ell u \pm \frac{1}{3} + \frac{1}{2})^2\right) \\
& \quad \quad \quad \times e\left((n + \ell u \pm \frac{1}{3} + \frac{1}{2})(-3u + \frac{1}{2})\right).
\end{aligned}$$

Since we will take the derivative in  $u$  at  $u = \alpha/\beta$ , it suffices to assume  $|u - \alpha/\beta| < \epsilon$  for some sufficiently small  $\epsilon > 0$  as before. Hence, we have  $\ell(u - \alpha/\beta) \pm 1/3 \in (-1/2, 1/2)$ , so that

$$\frac{\partial}{\partial u} g_{\ell u \pm 1/3 + 1/2, -3u + 1/2}(3\rho) \ll_u |\rho| e^{-3\pi \operatorname{Im}(\rho)(N + \ell u \pm 1/3 + 1/2)^2}$$

for some fixed  $N \in \mathbb{Z}$ . Thus, we may apply the Leibniz Rule for indefinite

integrals to the sum of derivatives (in (4.22)):

$$(4.23) \quad \sqrt{3} \sum_{\pm} e\left(\mp \frac{1}{6}\right) \frac{d}{du} \int_{1/\ell}^{i\infty} \frac{g_{\ell u \pm 1/3 + 1/2, -3u + 1/2}(3\rho)}{\sqrt{-i(\rho + \tau)}} d\rho \Big|_{u=\alpha/\beta},$$

and deduce that  $D_{\alpha,\beta}$  is analytic for  $\tau \in \mathbb{R} - \{-1/\ell\}$ . ■

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