C^*-ALGEBRAS ISOMORPHICALLY REPRESENTABLE ON l^p

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Abstract. Let $p \in (1,\infty)\backslash\{2\}$. We show that every homomorphism from a C^*-algebra $\mathcal{A}$ into $B(l^p(J))$ satisfies a compactness property where J is any set. As a consequence, we show that a C^*-algebra $\mathcal{A}$ is isomorphic to a subalgebra of $B(l^p(J))$, for some set J, if and only if $\mathcal{A}$ is residually finite dimensional.

1. Introduction

For $1 \leq p < \infty$ and a set J, let $l^p(J)$ be the space $\{f\colon J \to \mathbb{C}\colon \sum_{j\in J}|f(j)|^p < \infty\}$ with norm $\|f\| = (\sum_{j\in J}|f(j)|^p)^{\frac{1}{p}}$. Two Banach algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are *isomorphic* if there exist a bijective homomorphism $\phi\colon \mathcal{A}_1 \to \mathcal{A}_2$ and $C > 0$ such that

$$\frac{1}{C}\|a\| \leq \|\phi(a)\| \leq C\|a\|,$$

for all $a \in \mathcal{A}_1$. The algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are *isometrically isomorphic* if moreover, ϕ can be chosen so that $\|\phi(a)\| = \|a\|$ for all $a \in \mathcal{A}_1$.

Gardella and Thiel [2] showed that for $p \in [1,\infty)\backslash\{2\}$, a C^*-algebra $\mathcal{A}$ is isometrically isomorphic to a subalgebra of $B(l^p(J))$, for some set J, if and only if $\mathcal{A}$ is commutative. So it is natural to consider the question whether this result holds if we relax the condition of isometrically isomorphic to isomorphic. In this paper, we show that for $p \in (1,\infty)\backslash\{2\}$, a C^*-algebra $\mathcal{A}$ is isomorphic to a subalgebra of $B(l^p(J))$, for some set J, if and only if $\mathcal{A}$ is residually finite dimensional (Corollary 2.2). We prove this by showing that every homomorphism from a C^*-algebra $\mathcal{A}$ into $B(l^p(J))$ satisfies a compactness property (Theorem 2.1).

The proofs of the main results Theorem 2.1 and Corollary 2.2 in this paper are quite different from the proof of Gardella-Thiel's result. Lamperti's characterization [5] of isometries on L^p, for $p \neq 2$, plays a crucial role in the proof of Gardella-Thiel's result, while uniform convexity of l^p, for $1 < p < \infty$, and an argument in probability that imitates the proof of Khintchine's inequality [6, Theorem 2.b.3], for $p = 1$, are used in the proof of Theorem 2.1.

2. Main results and proofs

Throughout this paper, the scalar field is $\mathbb{C}$; for algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, a *homomorphism* $\phi\colon \mathcal{A}_1 \to \mathcal{A}_2$ is a bounded linear map such that $\phi(a_1a_2) = \phi(a_1)\phi(a_2)$ for all $a_1, a_2 \in \mathcal{A}$; for an element a of a C^*-algebra, $|a| = \sqrt{a^*a}$; the algebra of bounded linear operators on a Banach space $\mathcal{X}$ is denoted by $B(\mathcal{X})$ and the dual of $\mathcal{X}$ is denoted by $\mathcal{X}^*$; for $1 \leq p \leq \infty$, the l^p direct sum of Banach spaces $\mathcal{X}_\alpha$, for $\alpha \in \Lambda$, is denoted by $(\oplus_{\alpha\in\Lambda}\mathcal{X}_\alpha)_{l^p}$. Two Banach spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ are *isomorphic* if there is an invertible operator $S\colon \mathcal{X}_1 \to$

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$\mathcal{X}_2$. A C^*-algebra $\mathcal{A}$ is *residually finite dimensional* if for every $a \in \mathcal{A}$, there is a $*$-representation ϕ of $\mathcal{A}$ on a finite dimensional space such that $\phi(a) \neq 0$.

Theorem 2.1. *Let $p \in (1, \infty)\backslash\{2\}$. Let J be a set. Let $\mathcal{A}$ be a C^*-algebra. Let $\phi\colon \mathcal{A} \to B(l^p(J))$ be a homomorphism. Then*

(i) the norm closure of $\{\phi(a)x\colon a \in \mathcal{A}, \|a\| \leq 1\}$ in $l^p(J)$ is norm compact for every $x \in l^p(J)$; and

(ii) $\mathcal{A}/ker\ \phi$ is a residually finite dimensional C^*-algebra.

Corollary 2.2. *Let $p \in (1, \infty)\backslash\{2\}$. A C^*-algebra $\mathcal{A}$ is isomorphic to a subalgebra of $B(l^p(J))$, for some set J, if and only if $\mathcal{A}$ is residually finite dimensional.*

Theorem 2.1 and Corollary 2.2 will be proved at the end of this section after a series of lemmas are proved. Theorem 2.1 has an easier proof when ϕ is contractive. Indeed, if $\phi\colon \mathcal{A} \to B(l^p(J))$ is a contractive homomorphism, then the range of ϕ is in the algebra of diagonal operators on $l^p(J)$ by [8, Proposition 2.12] (or by [2, Lemma 5.2] when J is countable). Thus, $\{\phi(a)x\colon a \in \mathcal{A}, \|a\| \leq 1\}$ is norm relatively compact, for every $x \in l^p(J)$, and $\mathcal{A}/\ker\ \phi$ is commutative.

It is not known if Theorem 2.1 and Corollary 2.2 hold for $p = 1$. However, throughout their proofs, we use, in an essential way, the assumption that p is in the reflexive range. For example, in the proof of Theorem 2.1(i), we use the fact that every bounded sequence in $l^p(J)$ has a weakly convergent subsequence. In the proof of Corollary 2.2, we use a classical result of Pełczyński that the l^p direct sum of finite dimensional Hilbert spaces is isomorphic to $l^p(J)$ for some set J. This result of Pełczyński holds only when p is in the reflexive range.

The structure of the proof of Theorem 2.1(i) goes as follows: If the closure of $\{\phi(a)x_0\colon a \in \mathcal{A}, \|a\| \leq 1\}$ is not compact for some $x_0 \in l^p(J)$, then we can find a bounded sequence in $(b_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ such that $\phi(b_k)x_0 \to 0$ weakly, as $k \to \infty$, and $\inf_{k\in\mathbb{N}} \|\phi(b_k)x_0\| > 0$. Assume that $p > 2$. In Lemma 2.5, we show that $\phi(b_k) \to 0$ weakly implies that $\omega(b_k^* b_k) \to 0$ for all positive linear functional $\omega : \mathcal{A} \to \mathbb{C}$ of the form $\omega(a) = y_0^*(\phi(a)x_0)$. This is proved by considering $\sum_{k=1}^n \delta_k b_k$ for random $\delta_1, \ldots, \delta_n$ in $\{-1, 1\}$ and by exploiting $p > 2$. Lemma 2.9 says that when $y_0^* \in (l^p(J))^*$ is suitably chosen, $\omega(b_k^* b_k) \to 0$ implies that $\|\phi(b_k)x_0\| \to 0$, which contradicts with $\inf_{k\in\mathbb{N}} \|\phi(b_k)x\| > 0$. This is proved by using uniform convexity of $l^p(J)$.

Theorem 2.1(ii) follows from Theorem 2.1(i) by using a GNS type construction and a classical result about compact unitary representations of groups on Hilbert spaces.

The following two lemmas are needed for the proof of Lemma 2.5.

Lemma 2.3. *Let $\mathcal{A}$ be a unital C^*-algebra. Let $a \in \mathcal{A}$. Then there exists a sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ such that $\|c_n\| \leq 1$ for all $n \in \mathbb{N}$ and $|a| = \lim_{n\to\infty} c_n a$.*

Proof. Without loss of generality, we may assume that $\|a\| \leq 1$. For $n \in \mathbb{N}$, define $g_n \in C[0, 1]$ by

$$g_n(x) = \begin{cases} \frac{1}{\sqrt{x}}, & \frac{1}{n} \leq x \leq 1 \\ n\sqrt{n}x, & 0 \leq x \leq \frac{1}{n} \end{cases}.$$

Take $c_n = g_n(a^*a)a^*$. Then $c_n c_n^* = g_n(a^*a)a^*ag_n(a^*a)$. Note that

$$xg_n(x)^2 = \begin{cases} 1, & \frac{1}{n} \leq x \leq 1 \\ n^3x^3, & 0 \leq x \leq \frac{1}{n} \end{cases}.$$

Thus, $0 \le xg_n(x)^2 \le 1$ for all $x \in [0,1]$ and so $0 \le c_n c_n^* \le 1$. Hence $\|c_n\| \le 1$.

We have

$$xg_n(x) = \begin{cases} \sqrt{x}, & \frac{1}{n} \le x \le 1 \\ n\sqrt{n}x^2, & 0 \le x \le \frac{1}{n} \end{cases}$$

and so $|xg_n(x) - \sqrt{x}| \le \frac{1}{\sqrt{n}}$ for all $x \in [0,1]$. Since $c_n a = g_n(a^*a)a^*a$, it follows that $\|c_n a - \sqrt{a^*a}\| \le \frac{1}{\sqrt{n}}$. Thus, the result follows. □

Lemma 2.4. *Let $\mathcal{A}$ be a unital C^*-algebra. Let ω be a positive linear functional on $\mathcal{A}$. Let $a \in \mathcal{A}$. If $a \ge 0$ then*

$$\omega(a^2) \le \omega(a)^{\frac{2}{3}} \omega(a^4)^{\frac{1}{3}}.$$

Proof. There exists a measure μ on $[0, \|a\|]$ such that

$$\omega(f(a)) = \int f(x)\, d\mu(x),$$

for all $f \in C[0, \|a\|]$. So

$$\omega(a^2) = \int x^2\, d\mu(x) \le \left(\int x\, d\mu(x)\right)^{\frac{2}{3}} \left(\int x^4\, d\mu(x)\right)^{\frac{1}{3}} = \omega(a)^{\frac{2}{3}} \omega(a^4)^{\frac{1}{3}}.$$

□

Lemma 2.5. *Let $2 < p < \infty$. Let J be a set. Let $\mathcal{A}$ be a unital C^*-algebra. Let $\phi\colon \mathcal{A} \to B(l^p(J))$ be a unital homomorphism. Let $x_0 \in l^p(J)$. Let y_0^* be a bounded linear functional on $l^p(J)$. Define $\omega\colon \mathcal{A} \to \mathbb{C}$ by*

$$\omega(a) = y_0^*(\phi(a)x_0),$$

for $a \in \mathcal{A}$. Assume that ω is a positive linear functional. Let $(b_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{A}$ such that $\|b_k\| \le 1$ for all $k \in \mathbb{N}$ and $\phi(b_k)x_0 \to 0$ weakly as $k \to \infty$. Then $\omega(b_k^* b_k) \to 0$ as $k \to \infty$.

Proof. By contradiction, suppose that $\omega(b_k^* b_k)$ does not converge to 0. Passing to a subsequence, we have that there exists $\gamma > 0$ such that $\omega(b_k^* b_k) \ge \gamma$ for all $k \in \mathbb{N}$.

Since $\|\phi(b_k)x_0\| \le \|\phi\|\|x_0\|$ and $\phi(b_k)x_0 \to 0$ weakly, passing to a further subsequence, we may assume that there are $z_1, z_2, \ldots$ in $l^p(J)$ with disjoint supports such that $\|z_k\| \le \|\phi\|\|x_0\|$ and $\|\phi(b_k)x_0 - z_k\| \le \frac{1}{2^k}$ for all $k \in \mathbb{N}$.

Let $n \in \mathbb{N}$. For each $\delta = (\delta_1, \ldots, \delta_n) \in \{-1,1\}^n$, let

$$a_\delta = \left|\sum_{k=1}^n \delta_k b_k\right| \in \mathcal{A}.$$

By Lemma 2.4,

$$\omega(a_\delta^2) \le \omega(a_\delta)^{\frac{2}{3}} \omega(a_\delta^4)^{\frac{1}{3}}.$$

Thus,

$$\mathbb{E}\omega(a_\delta^2) \le [\mathbb{E}\omega(a_\delta)]^{\frac{2}{3}} [\mathbb{E}\omega(a_\delta^4)]^{\frac{1}{3}},$$

where $\mathbb{E}$ denotes expectation over $\delta = (\delta_1, \ldots, \delta_n)$ uniformly distributed on $\{-1,1\}^n$.

Note that

$$\mathbb{E}\omega(a_\delta^2) \quad = \quad \mathbb{E}\omega\left(\left(\sum_{k=1}^n \delta_k b_k\right)^* \left(\sum_{k=1}^n \delta_k b_k\right)\right)$$

$$= \mathbb{E}\omega\left(\sum_{1\le j,k\le n}\delta_j\delta_k b_j^* b_k\right) = \sum_{1\le j,k\le n}\mathbb{E}(\delta_j\delta_k)\omega(b_j^* b_k) = \sum_{k=1}^n \omega(b_k^* b_k) \ge n\gamma.$$

Therefore,

$$n\gamma \le [\mathbb{E}\omega(a_\delta)]^{\frac{2}{3}}[\mathbb{E}\omega(a_\delta^4)]^{\frac{1}{3}}. \tag{2.1}$$

We have

$$a_\delta^4 = \left[\left(\sum_{k=1}^n \delta_k b_k\right)^*\left(\sum_{k=1}^n \delta_k b_k\right)\right]^2 = \sum_{1\le k_1,\ldots,k_4\le n}\delta_{k_1}\delta_{k_2}\delta_{k_3}\delta_{k_4}b_{k_1}^* b_{k_2} b_{k_3}^* b_{k_4}.$$

Since $\|b_k\|\le 1$, it follows that

$$\mathbb{E}\omega(a_\delta^4) = \sum_{1\le k_1,\ldots,k_4\le n}\mathbb{E}(\delta_{k_1}\delta_{k_2}\delta_{k_3}\delta_{k_4})\omega(b_{k_1}^* b_{k_2} b_{k_3}^* b_{k_4}) \le \sum_{1\le k_1,\ldots,k_4\le n}\mathbb{E}(\delta_{k_1}\delta_{k_2}\delta_{k_3}\delta_{k_4}).$$

Note that $\mathbb{E}(\delta_{k_1}\delta_{k_2}\delta_{k_3}\delta_{k_4}) = 0$ unless the following occurs:

$$(k_1 = k_2 \text{ and } k_3 = k_4) \text{ or } (k_1 = k_3 \text{ and } k_2 = k_4) \text{ or } (k_1 = k_4 \text{ and } k_2 = k_3).$$

Thus, $\mathbb{E}\omega(a_\delta^4) \le 3n^2$. So by (2.1), we have $n\gamma \le 3^{\frac{1}{3}}n^{\frac{2}{3}}[\mathbb{E}\omega(a_\delta)]^{\frac{2}{3}}$. Hence,

$$\mathbb{E}\omega(a_\delta) \ge \frac{\gamma^{\frac{3}{2}}}{3^{\frac{1}{2}}}n^{\frac{1}{2}}. \tag{2.2}$$

Fix $\delta \in \{-1,1\}^n$. By Lemma 2.3,

$$\omega(a_\delta) = \omega\left(\left|\sum_{k=1}^n \delta_k b_k\right|\right) \le \sup_{c\in\mathcal{A},\,\|c\|\le 1}\left|\omega\left(c\sum_{k=1}^n \delta_k b_k\right)\right|.$$

For $c \in \mathcal{A}$ with $\|c\|\le 1$,

$$\begin{aligned}
\left|\omega\left(c\sum_{k=1}^n \delta_k b_k\right)\right| &= \left|y_0^*\left(\phi(c)\left(\sum_{k=1}^n \delta_k\phi(b_k)x_0\right)\right)\right| \\
&\le \|y_0^*\|\|\phi\|\left\|\sum_{k=1}^n \delta_k\phi(b_k)x_0\right\| \\
&\le \|y_0^*\|\|\phi\|\left(\left\|\sum_{k=1}^n \delta_k z_k\right\| + \sum_{k=1}^n \frac{1}{2^k}\right) \\
&\le \|y_0^*\|\|\phi\|(\|\phi\|\|x_0\|n^{\frac{1}{p}} + 1),
\end{aligned}$$

where the last two inequalities follow from the fact that $z_1, z_2, \ldots$ have disjoint supports, $\|z_k\| \le \|\phi\|\|x_0\|$ and $\|\phi(b_k)x_0 - z_k\| \le \frac{1}{2^k}$. Thus, $\omega(a_\delta) \le \|y_0^*\|\|\phi\|(\|\phi\|\|x_0\|n^{\frac{1}{p}} + 1)$ for all $\delta \in \{-1,1\}^n$. So by (2.2),

$$\frac{\gamma^{\frac{3}{2}}}{3^{\frac{1}{2}}}n^{\frac{1}{2}} \le \|y_0^*\|\|\phi\|(\|\phi\|\|x_0\|n^{\frac{1}{p}} + 1).$$

Since n can be chosen to be arbitrarily large and $p > 2$, an absurdity follows. □

For $1 < p < 2$, we have the following result, where the order of b_k^* and b_k are switched, by using the dual l^p space in Lemma 2.5.

Lemma 2.6. *Let $1 < p < 2$. Let J be a set. Let $\mathcal{A}$ be a unital C^*-algebra. Let $\phi\colon \mathcal{A} \to B(l^p(J))$ be a unital homomorphism. Let $x_0 \in l^p(J)$. Let y_0^* be a bounded linear functional on l^p. Define $\omega\colon \mathcal{A} \to \mathbb{C}$ by*

$$\omega(a) = y_0^*(\phi(a)x_0),$$

for $a \in \mathcal{A}$. Let $(b_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{A}$ such that $\|b_k\| \le 1$ for all $k \in \mathbb{N}$ and that the sequence $y_0^* \circ \phi(b_k)$ of bounded linear functionals on $l^p(J)$ converges to 0 weakly as $k \to \infty$. Assume that ω is a positive linear functional. Then $\omega(b_k b_k^*) \to 0$ as $k \to \infty$.

Proof. Let $\mathcal{A}_1$ be the unital C^*-algebra consisting of the same elements as $\mathcal{A}$ but with reverse order multiplication

$$a \cdot b = ba.$$

Define a unital homomorphism $\phi_1\colon \mathcal{A}_1 \to B((l^p(J))^*)$ by

$$\phi_1(a)y^* = y^* \circ \phi(a),$$

for all $a \in \mathcal{A}_1$, $y^* \in (l^p(J))^*$. Define $\omega_1\colon \mathcal{A}_1 \to \mathbb{C}$ by

$$\omega_1(a) = \omega(a) = x_0^{**}(\phi(a)y_0^*),$$

for all $a \in \mathcal{A}_1$, where x_0^{**} is the image of x_0 in the bidual $(l^p)^{**}$. By Lemma 2.5, the result follows. ∎

The following two lemmas are needed for the proof of Lemma 2.9.

Lemma 2.7 ([1])**.** *Let $1 < p < \infty$. Let J be a set. For every $\epsilon > 0$, there exists $\gamma > 0$ such that for all $x, y \in l^p(J)$ satisfying $\|x\|, \|y\| \le 1$ and $\|x + y\| > 2 - \gamma$, we have $\|x - y\| < \epsilon$.*

Lemma 2.8 ([9])**.** *Let $\mathcal{A}$ be a unital C^*-algebra. Then the closed unital ball of $\mathcal{A}$ is the closed convex hull of the set of all unitary elements of $\mathcal{A}$.*

Lemma 2.9. *Let $1 < p < \infty$. Let J be a set. Let $\mathcal{A}$ be a unital C^*-algebra. Let $\phi\colon \mathcal{A} \to B(l^p(J))$ be a unital homomorphism. Let $x_0 \in l^p(J)$. Then there exists $y_0^* \in (l^p(J))^*$ such that $\omega\colon \mathcal{A} \to \mathbb{C}$,*

$$\omega(a) = y_0^*(\phi(a)x_0), \quad a \in \mathcal{A},$$

defines a positive linear functional and for every $\epsilon > 0$, there exists $\gamma > 0$ such that whenever $a \in \mathcal{A}$ satisfies $\|a\| \le 1$ and $\omega(a^*a) < \gamma$, we have $\|\phi(a)x_0\| < \epsilon$.

Proof. Let $\mathcal{U}(\mathcal{A})$ be the set of all unitary elements of $\mathcal{A}$. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{U}(\mathcal{A})$ such that

$$\lim_{n\to\infty} \|\phi(v_n)x_0\| = \sup_{u\in\mathcal{U}(\mathcal{A})} \|\phi(u)x_0\|.$$

For each $n \in \mathbb{N}$, let x_n^* be a bounded linear functional on $l^p(J)$ such that $\|x_n^*\| = 1$ and $x_n^*(\phi(v_n)x_0) = \|\phi(v_n)x_0\|$. Then $x_n^* \circ \phi(v_n)$ is a bounded sequence in $(l^p(J))^*$. Passing to a subsequence, we may assume that $x_n^* \circ \phi(v_n)$ converges weakly to a bounded linear functional $y_0^* \in (l^p(J))^*$ as $n \to \infty$. Thus, $\omega\colon \mathcal{A} \to \mathbb{C}$,

$$\omega(a) = y_0^*(\phi(a)x_0) = \lim_{n\to\infty} x_n^*(\phi(v_n a)x_0),$$

for $a \in \mathcal{A}$, defines a bounded linear functional on $\mathcal{A}$. Note that

$$\omega(1) = \lim_{n\to\infty} x_n^*(\phi(v_n)x_0) = \lim_{n\to\infty} \|\phi(v_n)x_0\| = \sup_{u\in\mathcal{U}(\mathcal{A})} \|\phi(u)x_0\|,$$

and for every $u_0 \in \mathcal{U}(\mathcal{A})$,

$$|\omega(u_0)| = \lim_{n\to\infty} |x_n^*(\phi(v_n u_0)x_0)| \le \sup_{u\in\mathcal{U}(\mathcal{A})} \|\phi(u)x_0\|.$$

So by Lemma 2.8, we have $\|\omega\| \le \sup_{u\in\mathcal{U}(\mathcal{A})} \|\phi(u)x_0\|$. Thus, $\omega(1) = \|\omega\|$ and hence ω is a positive linear functional.

By contradiction, suppose that there are $\epsilon > 0$ and a sequence $(a_k)_{k\in\mathbb{N}}$ in $\mathcal{A}$ such that $\|a_k\| \le 1$ and $\|\phi(a_k)x_0\| \ge \epsilon$ for all $k \in \mathbb{N}$ and $\omega(a_k^* a_k) \to 0$ as $k \to \infty$. We have

$$\|a_k\| \ge \frac{\|\phi(a_k)x_0\|}{\|\phi\|\|x_0\|} \ge \frac{\epsilon}{\|\phi\|\|x_0\|},$$

for all $k \in \mathbb{N}$. For $k \in \mathbb{N}$, let $b_k = \frac{a_k}{\|a_k\|}$. We have $\|b_k\| = 1$ and $\|\phi(b_k)x_0\| \ge \epsilon$ for all $k \in \mathbb{N}$ and $\omega(b_k^* b_k) \to 0$ as $k \to \infty$.

Since $\|x_n^*\| = 1$,

$$\begin{aligned}
&\liminf_{n\to\infty} \|\phi(v_n)\phi(1-|b_k|)x_0 + \phi(v_n)x_0\| \\
\ge&\liminf_{n\to\infty} [x_n^*(\phi(v_n)\phi(1-|b_k|)x_0) + x_n^*(\phi(v_n)x_0)] \\
=&\omega(1-|b_k|) + \omega(1) = 2\omega(1) - \omega(|b_k|).
\end{aligned}$$

Thus,

$$\liminf_{n\to\infty} \|\phi(v_n)\phi(1-|b_k|)x_0 + \phi(v_n)x_0\| \ge 2\omega(1) - \omega(|b_k|).$$

But

$$\|\phi(v_n)\phi(1-|b_k|)x_0\| \le \sup_{b\in\mathcal{A},\, \|b\|\le 1} \|\phi(b)x_0\|\|1-|b_k|\| \le \sup_{u\in\mathcal{U}(\mathcal{A})} \|\phi(u)x_0\| = \omega(1)$$

and $\|\phi(v_n)x_0\| \le \omega(1)$ for all $n \in \mathbb{N}$. Take

$$x = \frac{1}{\omega(1)}\phi(v_n)\phi(|b_k|)x_0 \text{ and } y = \frac{1}{\omega(1)}\phi(v_n)x_0$$

in Lemma 2.7 and note that $\omega(|b_k|) \le \omega(b_k^* b_k)^{\frac{1}{2}}\omega(1)^{\frac{1}{2}} \to 0$ as $k \to \infty$. We have

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|\phi(v_n)\phi(1-|b_k|)x_0 - \phi(v_n)x_0\| = 0.$$

Thus,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|\phi(v_n)\phi(|b_k|)x_0\| = 0.$$

So $\|\phi(|b_k|)x_0\| \to 0$ as $k \to \infty$. Since $b_k = b_k(|b_k| + \frac{1}{k})^{-1}(|b_k| + \frac{1}{k})$ and $\|b_k(|b_k| + \frac{1}{k})^{-1}\| \le 1$, it follows that $\|\phi(b_k)x_0\| \to 0$ as $k \to \infty$ which contradicts with $\|\phi(b_k)x_0\| \ge \epsilon$. □

Proof of Theorem 2.1(i). Without loss generality, we may assume that $\mathcal{A}$ is unital by extending ϕ to a homomorphism from the unitization of $\mathcal{A}$ into $B(l^p(J))$. We may also assume that ϕ is unital since $\phi(1)$ is an idempotent on $l^p(J)$ and the range of every idempotent on $l^p(J)$ is isomorphic to $l^p(J_0)$ for some set J_0 [7], [3].

Let $x_0 \in l^p$. Let $(a_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{A}$ such that $\|a_k\| \le \frac{1}{2}$ for all $k \in \mathbb{N}$. We need to show that $(\phi(a_k)x_0)_{k\in\mathbb{N}}$ has a norm convergent subsequence.

Case 1: $p > 2$

Passing to a subsequence, we may assume that $(\phi(a_k)x_0)_{k\in\mathbb{N}}$ converges weakly to an element of $l^p(J)$. Thus, $\phi(a_{k_1} - a_{k_2})x_0 \to 0$ weakly as $k_1, k_2 \to \infty$.

By Lemma 2.5, we have $\lim_{k_1,k_2\to\infty} \omega((a_{k_1} - a_{k_2})^*(a_{k_1} - a_{k_2})) = 0$ for every positive linear functional $\omega\colon \mathcal{A} \to \mathbb{C}$ of the form $\omega(a) = y_0^*(\phi(a)x_0)$ for $a \in \mathcal{A}$. By Lemma 2.9, we have $\lim_{k_1,k_2\to\infty} \|\phi(a_{k_1} - a_{k_2})x_0\| = 0$. So $(\phi(a_k)x_0)_{k\in\mathbb{N}}$ is norm convergent.

Case 2: $p < 2$

Passing to a subsequence, we may assume that $(y_0^* \circ \phi(a_k^*))_{k\in\mathbb{N}}$ converges weakly to an element of $(l^p(J))^*$. Thus, $y^* \circ \phi(a_{k_1}^* - a_{k_2}^*) \to 0$ weakly as $k_1, k_2 \to \infty$ for every $y^* \in (l^p(J))^*$.

By Lemma 2.6, we have $\lim_{k\to\infty} \omega((a_{k_1}^* - a_{k_2}^*)(a_{k_1}^* - a_{k_2}^*)^*) = 0$ for every positive linear functional $\omega\colon \mathcal{A} \to \mathbb{C}$ of the form $\omega(a) = y_0^*(\phi(a)x_0)$ for $a \in \mathcal{A}$. By Lemma 2.9, we have $\lim_{k_1,k_2\to\infty} \|\phi(a_{k_1} - a_{k_2})x_0\| = 0$. So $(\phi(a_k)x_0)_{k\in\mathbb{N}}$ is norm convergent. □

Lemma 2.10 ([4], Theorem 2.24)**.** *Let G be a group. Let $\mathcal{H}$ be a Hilbert space. Let $\varphi\colon G \to B(\mathcal{H})$ be a unital homomorphism such that $\varphi(g)$ is unitary for all $g \in G$. If $\{\varphi(g)x\colon g \in G\}$ is norm precompact in $\mathcal{H}$ for all $x \in \mathcal{H}$, then $\mathcal{H}$ is the direct sum of some finite dimensional subspaces $\mathcal{H}_\alpha$, for $\alpha \in \Lambda$, such that $\mathcal{H}_\alpha$ is invariant under $\varphi(g)$ for all $\alpha \in \Lambda$ and $g \in G$.*

Proof of Theorem 2.1(ii). As in the proof Theorem 2.1(i), we may assume that $\mathcal{A}$ is unital and ϕ is unital. We may also assume that $\ker \phi = \{0\}$. Let $a_0 \neq 0$. There exists $x_0 \in l^p(J)$ such that $\phi(a_0)x_0 \neq 0$. By Lemma 2.9, there exists $y_0^* \in (l^p(J))^*$ such that $\omega\colon \mathcal{A} \to \mathbb{C}$,

$$\omega(a) = y_0^*(\phi(a)x_0),$$

for $a \in \mathcal{A}$, defines a positive linear functional and $\omega(a_0^*a_0) \neq 0$.

Equip $\mathcal{A}$ with the positive semidefinite sesquilinear form

$$\langle a, b\rangle = \omega(b^*a),$$

for $a, b \in \mathcal{A}$. Consider the ideal $\mathcal{A}_0 = \{a \in \mathcal{A}\colon \langle a, a\rangle = 0\}$ of $\mathcal{A}$. Let $\mathcal{H}$ be the completion of the quotient space $\mathcal{A}/\mathcal{A}_0$. Then $\mathcal{H}$ is a Hilbert space. For each $a \in \mathcal{A}$, we can define a bounded linear operator on $\mathcal{H}$ by sending $b + \mathcal{A}_0$ to $ab + \mathcal{A}_0$ for $b \in \mathcal{A}$. So $\eta\colon \mathcal{A} \to B(\mathcal{H})$,

$$\eta(a)(b + \mathcal{A}_0) = ab + \mathcal{A}_0,$$

for $a, b \in \mathcal{A}$, defines a unital $*$-homomorphism. We have

$$\begin{aligned} \|\eta(a_1)(b + \mathcal{A}_0) - \eta(a_2)(b + \mathcal{A}_0)\| &= \omega(b^*(a_1 - a_2)^*(a_1 - a_2)b) \\ &= y_0^*(\phi(b^*(a_1 - a_2)^*(a_1 - a_2)b)x_0) \\ &\leq \|y_0^*\|\|\phi\|\|b^*\|\|a_1 - a_2\|\|\phi(a_1 - a_2)\phi(b)x_0\|, \end{aligned}$$

for all $a_1, a_2, b \in \mathcal{A}$. By Theorem 2.1(i), we have that $\{\phi(a)x_0\colon a \in \mathcal{A}, \|a\| \leq 1\}$ is norm precompact so $\{\eta(a)(b+\mathcal{A}_0)\colon a \in \mathcal{A}, \|a\| \leq 1\}$ is norm precompact for all $b \in \mathcal{A}$. Let $\mathcal{U}(\mathcal{A})$ be the set of all unitary elements of $\mathcal{A}$. By Lemma 2.10, we have that $\mathcal{H}$ is the direct sum of some finite dimensional subspaces $\mathcal{H}_\alpha$, for $\alpha \in \Lambda$, such that $\mathcal{H}_\alpha$ is invariant under $\eta(u)$ for all $\alpha \in \Lambda$ and $u \in \mathcal{U}(\mathcal{A})$. Note that $\mathcal{H}_\alpha$ is thus invariant under $\eta(a)$ for all $a \in \mathcal{A}$.

Since $\omega(a_0^*a_0) \neq 0$, we have $\eta(a_0) \neq 0$. So $\eta(a_0) \neq 0$ on $\mathcal{H}_{\alpha_0}$ for some $\alpha_0 \in \Lambda$. Thus, $\mathcal{A}$ is residually finite dimensional. □

Proof of Corollary 2.2. One direction follows from Theorem 2.1. For the other direction, suppose that $\mathcal{A}$ is a residually finite dimensional C^*-algebra. Then there is a collection $(\phi_\alpha)_{\alpha\in\Lambda}$ of $*$-representations of $\mathcal{A}$ on finite dimensional Hilbert spaces $\mathcal{H}_\alpha$ such that $\|a\| = \sup_{\alpha\in\Lambda} \|\phi_\alpha(a)\|$ for all $a \in \mathcal{A}$. Define $\phi\colon \mathcal{A} \to B((\oplus_{\alpha\in\Lambda}\mathcal{H}_\alpha)_{l^p})$ by

$\phi = \oplus_{\alpha\in\Lambda}\phi_\alpha$. Thus ϕ is a norm preserving homomorphism. However, it is a classical result of Pełczyński [7] that for $1 < p < \infty$, the l^p direct sum of finite dimensional Hilbert spaces is isomorphic to $l^p(J)$ for some set J. Therefore, $\mathcal{A}$ is isomorphic to a subalgebra of $B(l^p(J))$, via the map $a \mapsto S\phi(a)S^{-1}$ where $S\colon (\oplus_{\alpha\in\Lambda}\mathcal{H}_\alpha)_{l^p} \to l^p(J)$ is any invertible operator. □

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