A semianalytic Fisher matrix for precessing binaries with a single significant spin

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(Dated: December 19, 2019)

Gravitational waves from a binary with a single dynamically significant spin, notably including precessing black hole-neutron star (BH-NS) binaries, let us constrain that binary's properties: the two masses and the dominant black hole spin. Based on a straightforward fourier transform of $h(t)$ enabled by the corotating frame, we show the Fisher matrix for precessing binaries can be well-approximated by an extremely simple semianalytic approximation. This approximation can be easily understood as a weighted average of independent information channels, each associated with one gravitational wave harmonic. Generalizing previous studies of nonprecessing binaries to include critical symmetry-breaking precession effects required to understand plausible astrophysical sources, our ansatz can be applied to address how well gravitational wave measurements can address a wide range of astrophysical and fundamental questions. This Fisher matrix approach provides a simple method to assess what parameters gravitational wave detectors can measure, how well, and why. Our study is the first analytically-tractable Fisher matrix calculation for precessing binaries.

PACS numbers:

Introduction. – Ground-based instruments like LIGO [1] and Virgo [2] will soon regularly identify and measure the properties [3, 4, 4–17] of the relatively well-understood gravitational wave (GW) signal from the nearly adiabatic and quasicircular inspiral of the lowest-mass coalescing compact binaries (CBCs) [18–29]: binaries consisting of either black holes or neutron stars with total masses $M = m_1 + m_2 \le 16 M_\odot$ and intrinsic spins $\mathbf{S}_1, \mathbf{S}_2$ that satisfy the Kerr limit $|\mathbf{S}_i|/m_i^2 \le 1$. These measurements' accuracy determines the range of astrophysical and fundamental questions that can be addressed via gravitational waves, including but not limited to identifying how coalescing compact binaries form [30–36]; how the universe expands [37]; how high-density nuclear matter behaves and responds [38–47]; and even how reliably general relativity describes the inspiral, coalescence, and gravitational radiation from each event [48]. In general, astrophysical formation channels [36, 49–54] will populate generic spin orientations, not just high-symmetry, nonprecessing configurations with $\mathbf{S}_1, \mathbf{S}_2$ parallel to $\mathbf{L}$. For these most likely sources, spin-orbit and spin-spin couplings cause the misaligned angular momenta to precess [55–57], breaking degeneracies present in the high-symmetry case and thus enabling higher-precision measurements [3, 7, 8, 17, 58–62]. While powerful analytic techniques were developed to estimate the measurement accuracy for nonprecessing binaries [63–66], then broadly applied [30–36][37][38–47][48], for precessing binaries a comparable theoretical tool has remained unavailable. Instead, the measurement accuracy has been evaluated on a case-by-case basis numerically, usually by Bayesian methods that systematically compare the data with all possible candidate signals [3, 4, 4–17].

The main result of this work is a generalization of the classic analytic approach used to approximate measurement accuracy [65] to the case of precessing binaries with a single significant spin undergoing an extended adiabatic, quasicircular inspiral. We restrict to a single significant spin both for convenience – this limit is well-studied [55, 67, 68] – and without significant loss of generality – the smaller body's spin often has little dynamically significant impact on the angular momentum budget, orbit, or precessional dynamics ($|\mathbf{S}_2| \le m_2^2 \ll |\mathbf{S}_1|, |\mathbf{L}|$), allowing a single-spin model to adequately reproduce the dynamics and posterior [15]. To highlight the broad utility of our approach, we defer concrete but arbitrary implementation details until we evaluate numerical results. Our result is important because it provides the first powerful analytic tool to assess what can be measured using gravitational waves and why, includes the critical symmetry-breaking effects of spin precession.

Inference from GW. – Bayes theorem provides an unambiguous expression for the posterior CBC parameter distribution given instrumental data $\{d\}$; see, e.g, Pankow et al. [69], henceforth denoted PBOO, for a review using our notation. The signal and network response to a quasicircular CBC inspiral is characterized by eight intrinsic parameters λ [$= (m_1, m_2, \mathbf{S}_1, \mathbf{S}_2)$] that uniquely specify the binary's dynamics and seven extrinsic parameters θ [four spacetime coordinates and three

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Euler angles] that specify where, when, and with what orientation the coalescence occurred. Each detector, indexed henceforth by k, responds to an imposed strain as $d_k = n_k + \text{Re}F_k(\hat{\mathbf{N}})^* h(t + \hat{\mathbf{N}} \cdot \mathbf{x}_k|\lambda, \theta)$, where $\mathbf{x}_k$ are the detector positions, F_k are antenna response functions, $\hat{\mathbf{N}}$ is the line of sight to the source, (a member of θ), $h(t|\lambda, \theta)$ is the strain derived from far-field solution to Einstein's equations for the CBC; and n_k is some random realization of detector noise. The distribution of stationary gaussian noise $n_k(t)$ in the kth detector is completely characterized by its covariance or power spectrum $\langle \tilde{n}_k(f)^* \tilde{n}_k(f') \rangle = \frac{1}{2} S_k(|f|)\delta(f - f')$. Let us define inner products $\langle \cdot | \cdot \rangle_k$ on arbitrary complex functions a, b as $\langle a|b \rangle_k \equiv 2\int_{-\infty}^{\infty} df \frac{a^*(f)b(f)}{S_k(|f|)}$. The log likelihood ratio $\mathcal{L}(\theta, \theta)$ favoring one signal with parameters λ, θ versus no signals is $2\ln\mathcal{L} = \sum_k \langle d_k|d_k \rangle_k - \langle d_k - h_k(\theta, \lambda)|d_k - h_k(\theta, \lambda) \rangle_k$. If the data is known to contain a signal $h(t|\lambda_0, \theta_0)$ with parameters $\Lambda_0 \equiv (\lambda_0, \theta_0)$, we will denote the parameters by $\Lambda = (\lambda, \theta)$ and the likelihood by $\mathcal{L}(\Lambda|\Lambda_0, n)$. The signal amplitude ρ is set by the expected value of the log likelihood ($\rho^2 = 2\ln\mathcal{L}(\Lambda|\Lambda_0, 0)$). Using 15-dimensional posterior distribution $p_{\rm post}(\lambda, \theta) = \mathcal{L}p(\theta)p(\lambda) / \int d\lambda d\theta p(\theta)p(\lambda)\mathcal{L}(\lambda, \theta)$, the measurement accuracy in some parameter λ_1 follows from the 90% confidence interval derived from a one-dimensional marginal distribution $p_{\rm post}(\lambda_1) = \int d\theta d\lambda_2 \ldots d\lambda_8 p_{\rm post}(\lambda, \theta)$.

While straightforward but expensive numerical techniques exist to estimate the marginal posterior distribution and hence measurement accuracy for concrete sources Λ_0 and noise realizations n_k [3, 4, 4–17], an equally straightforward analytic approximation to the (average) log likelihood exists at high signal amplitude, the Fisher matrix [63–66, 70, 70, 71] Γ_{ab}. The Fisher information matrix arises in a quadratic-order approximation to the log-likelihood [$\ln\mathcal{L}(\Lambda|\Lambda_o) \simeq -\frac{1}{2}\Gamma_{ab}(\Lambda - \Lambda_*)_a(\Lambda - \Lambda_*)_b + \text{const}$, with Λ_* the location of the noise-realization-dependent maximum]; often depends weakly on the specific noise realization used, particularly at high amplitude; and can be evaluated by a simple expression involving inner products of derivatives. For example, for a source directly overhead a network with equal sensitivity to both polarizations [72], the Fisher matrix is

$$\Gamma_{ab} = \left\langle \frac{dh}{d\lambda_a} | \frac{dh}{d\lambda_b} \right\rangle \quad (1)$$

The Fisher matrix can always be evaluated numerically via direct differentiation [58, 59, 61], henceforth denoted Fisher-D. For nonprecessing binaries, however, Poisson and Will [65] introduced a powerful analytic technique, denoted here as Fisher-SPA: express the signal using a single dominant harmonic, with a necessarily-simple form; evaluate the fourier transform $\tilde{h}$ via a stationary phase approximation; thereby evaluate the derivatives $\partial_a h$ *analytically*; and, by reorganizing the necessary integrals analytically, reduce the evaluation of Eq. (1) to an analytic expression and a handful of tabulated integrals. Despite its limitations, this method remains the most powerful and widely-used theoretical tool to estimate what can be measured and why. In the remainder of this work, we will review and generalize Fisher-SPA to precessing binaries.

As a matrix in at least 11 dimensions (15 with two precessing spins), the Fisher matrix is both difficult to interpret and highly prone to numerical instability. For nonprecessing binaries, several studies have demonstrated that the intrinsic and extrinsic posterior distributions largely seperate [72, 73] and that the intrinsic distribution depends weakly if at all on the specific network geometry. Hence, to quantitatively assess what can be measured and why for a nonprecessing binary, it suffices to consider a source directly overhead and optimally aligned with a fiducial detector network [34, 72–74]. In the high-amplitude limit where the likelihood is well-approximated by a gaussian, the event time t_c and polarization ψ_c can be marginalized out analytically. Using this approximation, the log likelihood is approximated using an "overlap" P $P(\lambda, \lambda') \equiv \max_{t\psi} \langle h(\lambda|t_c, \psi_c)|h(\lambda|0, 0) \rangle / ||h(\lambda)|| ||h(\lambda')||$ where $||h|| \equiv \sqrt{\langle h|h \rangle}$, via Eq. (18) of [72]; the Fisher matrix arises as a quadratic approximation to the overlap. This approach and its relatives, denoted here by Fisher-O and in the literature by overlap, mismatch, or ambiguity function methods, has been widely adopted when analyzing numerical simulations [75–79] and circumvents the numerical challenges that plague brute-force 11-dimensional calculations.

Even for precessing binaries, several studies have suggested that the four spacetime coordinates decouple from intrinsic parameters and the binary's three Euler angles [60, 80]. To simplify subsequent analytic calculations, following prior work [72, 73] we will therefore adopt the ansatz that the source can be assumed directly overhead a network with equal sensitivity to both polarizations.

Stationary-phase approximation. – The outgoing GW signal $h(t, \hat{n}, \lambda)$ is modeled using a stationary phase approximation of the leading-order (corotating) quadrupole emission, assuming a single significant spin. Specifically, adopting the conventions of [67] and PBOO, we express the strain for a source with intrinsic parameters λ as a spin-weighted spherical harmonic expansion $h(t|\lambda, \theta) = h_+ - ih_\times = e^{-2i\psi_J}(M/d_L)\sum_{lm} h_{lm}(t - t_c|\lambda)Y^{(-2)}(\hat{\mathbf{n}})$, relative to a cartesian frame $\hat{\mathbf{z}}' = \hat{\mathbf{J}}, \hat{\mathbf{x}}', \hat{\mathbf{y}}'$ defined by the total angular momentum $\mathbf{J}$, where d_L is the luminosity distance to the source, M is the binary mass, $\hat{\mathbf{n}}$ is the propagation direction away from the source, t_c is the coalescence time, and ψ_J is the angle of $\mathbf{J}$ on the plane of the sky. Within the post-Newtonian approximation, both the amplitude and phase of these functions $h_{lm}(t)$ and their (stationary-phase) fourier transforms are slowy-varying and analytically-tractable functions [29]. For a nonprecessing binary, the sum is dominated by a single pair of complex-conjugate terms $h_{22} = h^*_{2,-2} = |h_{22}|e^{-2i\Phi}$, enabling efficient calculation of Eq. (1) [65] for an optimally-aligned source directly overhead a single detctor (Fisher-O).

A generic quasicircular binary will orbit, precess, and inspiral on three well-seperated timescales $1/f_{\rm orb}$, t_{prec}, and $t_{\rm insp}$. For this reason, to a good approximation, the gravitational radiation from a precessing, inspiralling binary [81, 82] can be approximated as if from an instantaneously nonprecessing binary: $h_{lm}(t|\lambda) = \sum_{\bar{m}} h^{(\rm C)}_{l\bar{m}}(t,\lambda) D^l_{m\bar{m}}(R(t))$, where $R(t)$ is a minimal rotation transforming $\hat{\mathbf{z}}$ into $\hat{\mathbf{L}}$ [67, 68, 83], where $h^{(C)}_{lm}$ are available in the literature [28, 81, 84] in terms of the spins, the orbital phase Φ_{orb}, and a post-Newtonian expansion parameter $v = (M d\Phi_{orb}/dt)^{1/3}$. The quantities appearing in these expressions $(\mathbf{L}, \mathbf{S}_i, \Phi_{orborb}, v)$ are determined by post-Newtonian approximations that prescribe the evolution of both spins $[\partial_t \mathbf{S}_i = \mathbf{\Omega}_i \times \mathbf{S}_i$ and the orbital phase $[\frac{dv}{dt} = -\frac{\mathcal{F}+\dot{M}}{E'(v)}]$, [29, 85, 86], where $\dot{M}$ is the rate at which the black holes' mass changes [87, 88], henceforth neglected; $\mathcal{F}(v)$ is the rate at which energy is radiated to infinity; and $E(v)$ is the energy of an instantaneously quasicircular orbit; all of which are provided in the literature. At leading amplitude order, corotating-frame strain satisfies $h^{(C)}_{lm}(t|\lambda) = |h_{lm}(t)| \exp(-i\Phi_{orb})$, where $|h_{lm}|$ is a slowly-varying function of v; substituting this form into the general expression implies

$$h_{lm}(t|\lambda) = \sum_{\bar{m}} e^{-i\bar{m}(\Phi_{\rm orb}+\gamma)} e^{-im\alpha} d^l_{m\bar{m}}(\beta) |h_{lm}(v)| \quad (2)$$

In this expression, we have expanded the rotation $R(t)$ using Euler angles, set by the orbital angular momentum direction $\hat{\mathbf{L}}$ expanded relative to the (assumed fixed) total angular momentum direction $\hat{\mathbf{J}}$ as $\hat{\mathbf{L}} = \sin\beta_{JL}\cos\alpha_{JL}\hat{\mathbf{x}}' + \sin\beta_{JL}\sin\alpha_{JL}\hat{\mathbf{y}}' + \cos\beta_{JL}\hat{\mathbf{J}}$; the remaining Euler angle $\gamma = -\int \cos\beta_{JL} d\alpha_{JL}$. In this work, we restrict to the leading-order gravitational-wave quadrupole $h^{(C)}_{2,\pm 2} = -8\sqrt{\pi/5}\eta v^2 \exp(\mp i\Phi_{\rm orb})$, so this sum has only two terms.

For a binary with a single dynamically significant spin, the spin-precession equations imply that the orbital angular momentum precesses simply around the total angular momentum: β changes slowly, on the inspiral timescale, while α and γ evolve on the precessional timescale [55, 67].

Because of seperation of timescales, because of the simple form of Eq. (2), and particularly because the phase terms $\bar{m}(\Phi+\gamma) + m\alpha$ are monotonic and well-behaved, the stationary-phase approximation to the fourier transform $\tilde{h}_{lm}(\omega) = \int dt h_{lm} \exp i\omega t$ can be carried out term by term [67]. For each term, the stationary-phase condition defines an $m, \bar{m}$-dependent time-frequency trajectory $\tau_{m\bar{m}}(\omega)$ set by solving

$$\omega \equiv \bar{m}(\dot{\Phi}_{\rm orb} - \dot{\alpha}\cos\beta_{JL}) + m\dot{\alpha} \quad (3)$$

or, equivalently, $v = (M\Phi'_{\rm orb})^{1/3} = \left[M \frac{\omega - \bar{m}\dot{\gamma}(\tau_{m\bar{m}}) - m\dot{\alpha}(\tau_{m\bar{m}})}{\bar{m}}\right]^{1/3}$. Using this time-frequency relationship to evaluate $\Psi_{m,\bar{m}} \equiv \omega t - \bar{m}(\Phi_{\rm orb} + \zeta) - m\alpha$ and the slowly-varing coefficients in each term in Eq. (2), the stationary-phase approximation is

$$\tilde{h}_{lm}(\omega) \equiv \sum_{\bar{m}} \begin{cases} \frac{d^l_{m\bar{m}}(\beta(\tau_{m\bar{m}}(\omega)))|h^{\rm ROT}_{l\bar{m}}(\tau_{m\bar{m}}(\omega))| e^{i\Psi_{m\bar{m}}(\omega)}}{\sqrt{i(\bar{m}(\Phi''_{\rm orb}+\zeta'')+m\alpha'')/2\pi}} & \bar{m}\omega > 0 \\ 0 & \bar{m}\omega < 0 \end{cases} \quad (4)$$

This expression for the SPA had been previously derived in the restricted PN approximation by precisely this method [67], for simplicity neglecting the distinct time-frequency trajectories implied by $\tau_{m,\bar{m}}$.

Inner products via a time-frequency ansatz. – Most of the terms in Eq. (2) are mutually orthogonal. For example, the modes with $\bar{m} > 0$ and $\bar{m} < 0$ have different helicity and have almost no overlap, indepdendent of the precession state [72, 89–92]. Additionally, for binaries with more than a few precession cycles in band (see, e.g., [93] for suitable conditions on $|\mathbf{S}_1|, m_1, m_2$), each term in Eq. (2) is associated with a unique time-frequency trajectory and hence is orthogonal to all others. Using this ansatz, the inner product $\langle h(\Lambda)|h(\Lambda')\rangle$ for $\Lambda \simeq \Lambda'$ can be approximated using a sum over 10 terms: 5 for each of the $l = 2$ modes $m = -2, -1, \ldots 2$ and, for each mode, one term for each helicity.

By contrast, to evaluate the overlap of terms that are *not* orthogonal, the specific time-frequency trajectory has relatively little impact. We therefore approximate $\tau_{m\bar{m}} \simeq \tau_{0\bar{m}}$ henceforth.

Fisher matrix. – We now use the time-frequency ansatz and the restricted PN approximation to evaluate a Fisher matrix for a source directly overhead an idealized network of two interferometers oriented to have equal sensitivity to both polarizations [Eq. (1)]. Because of the time-frequency ansatz, the overall Fisher matrix is a weighted sum of 10 individual Fisher matricies, associated with each harmonic:

$$\Gamma_{ab} = \sum_{m=2}^{2} \sum_{s=\pm 1} \rho_{2ms}^2 \hat{\Gamma}_{ab}^{ms} \tag{5}$$

$$\rho_{2ms}^2 \equiv |Y_{2m}^{(-2)}(\theta_{JN}) d^2_{m,2s}(\beta)|^2 \int_0^\infty \frac{df}{S_h(f)} \frac{4(\pi \mathcal{M}_c{}^2)^2}{3d_L^2} (\pi \mathcal{M}_c f)^{-7/3} \tag{6}$$

$$\hat{\Gamma}_{ab}^{(ms)} = \frac{\int_0^\infty \frac{df}{S_h(f)} (\pi \mathcal{M}_c f)^{-7/3} \partial_a(\Psi_2 - 2\zeta - ms\alpha) \partial_b(\Psi_2 - 2\zeta - ms\alpha)}{\int_0^\infty \frac{df}{S_h(f)} (\pi \mathcal{M}_c f)^{-7/3}} \tag{7}$$

In this expression, $\Psi_2 = \omega t - 2\Phi_{orb}$ is the stationary-phase approximation derived via $\omega = 2\dot{\Phi}_{orb}$. The weights ρ_{ms} are associated with the relative contribution of each model m and sign s to the detected amplitude, along this line of sight. The 10 individual Fisher matrices $\hat{\Gamma}_{ab}$ reflect the Fisher matrix implied by a *single harmonic*, with a modified phase versus time to reflect that harmonic's precession-induced *secular* phase change; each one reduces trivially to the well-known nonprecessing Fisher matrix in the absence of precession.

Because measurements often cannot tightly constrain all parameters, whether computed directly (Fisher-O) or via our approximation, the Fisher matrix is often degenerate, particularly in phase angles. Following prior work, when evaluating the Fisher matrix to produce our final numerical results, we adopt a very weak (regularizing) prior to break degeneracy: $\Gamma_{final} = \Gamma + K$ where K reflects a multivariate gaussian distribution with standard deviation 2π in phase angles, $1M_\odot$ in total mass, 1 second in time, and $1/4$ in mass ratio. These priors are much wider than the extent of any plausible posterior for our candidate sources, preventing potential unphysical degeneracies in the Fisher matrix from propagating into our results (e.g., η is bounded between 0 and $1/4$; the event time is determined within milliseconds; et cetera).

Implementation and results. – As described at greater detail in our Appendix, almost all expressions needed to evaluate our Fisher matrix are available in the literature. In this work, to minimize superfluous differences associated with uncontrolled post-Newtonian remainders, we evaluated the phase functions $\Psi_2(v), \gamma(v), \alpha(v)$ needed for our Fisher-SPA approximation *numerically*, using precisely the same post-Newtonian evolution model adopted in the other calculations shown [60] to evaluate $t(v)$ and $\Phi_2(t), \alpha(t), \gamma(t)$; see that work for the specific post-Newtonian approximation used. Similar results follow by using the explicit expressions for $\Psi_2(v), \gamma(v), \alpha(v)$ available the literature [67].

A detailed comparison between different Fisher matrix approximations and full Bayesian inference is far beyond the scope of this work. However, as an anecdote to illustrate that our approximation is reasonable and consistent with past work for plausible source parameters, Figure 1 shows a comparison beteween our approximation; a Fisher-O approximation; and a full MCMC posterior distribution. Despite not including the prior $p(\lambda)$ and despite adopting highly simplifying assumptions, our expression shows good agreement with the multidimensional posterior and previous numerical estimates of the marginalized Fisher-O matrix.

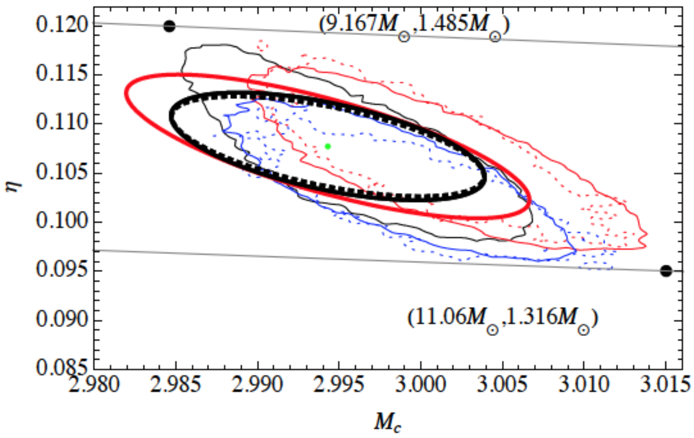

FIG. 1: **Sample comparison**: Comparison between marginalized Fisher matrix in $\mathcal{M}_c, \eta$ for a precessing binary evaluated using the ansatz in this paper (red solid curve: Fisher-SPA); via full 15-dimensional MCMC (thin curves); and via a quadratic approximation to the 7-dimensional overlap (Fisher-O). The latter two results, previously presented O'Shaughnessy et al. [60], are used with permission. As described in that work, the synthetic BH-NS source has masses $10M_\odot$ and $1.4M_\odot$, with $\mathbf{J}$ and $\mathbf{L}$ misaligned by $\pi/4$, with $\mathbf{J}$ inclined at angle $\theta_{JN} = 0.730$ radians relative to the line of sight. The thick solid black line shows a Fisher matrix evaluated by explicit differentiation of the mismatch; the thick dashed black line shows a corresponding result, evaluated including higher-order modes; the solid and dashed thin lines correspond to full MCMC posteriors evaluated without and with higher modes, respectively; and the thin lines' color indicates different noise realizations. The true source parameters are indicated by a light green dot.

Discussion. – An analytically-tractable Fisher matrix enables powerful, scalable insights into when and how precession break degeneracies and improves parameter measurement accuracy. Our Fisher matrix expression consists of the sum of 10 terms $\hat{\Gamma}_{ab}^{(ms)}$, each pair arising from information communicated through a single harmonic and weighted in proportion to the strength ρ_{2ms}

of that harmonic in the GW signal. First and foremost, these weights identify which source geometries correspond to observationally-significant precession. Because each individual term resembles the contribution from a nonprecessing binary, when the source geometry favors only one significant harmonic, precession does not break degeneracies. By contrast, the presence of several significant contributions ρ_{2ms} suggests that precession-induced modulations can produce an observationally-accessible imprint. Second, the individual terms' eigensystems describe the natural measurements enabled by each harmonic. Due to the offsets $ms\alpha$ in each phase derivative, the different terms do not have coaligned eigensystems, so their superposition generally breaks degeneracies present in any individual term. Our expression shows how these degeneracies are broken, and a path forward to calculating these degeneracy-breaking effects as a function of binary parameters.

While our SPA approximation in Eq. (4) is very general, the Fisher matrix approximation presented in this work relies on the strong assumption that each precession-induced sideband is orthogonal. Roughly speaking, this approximation requires α change significantly over the sensitive band of our instruments, which requires significant spin and mass ratio. That said, nothing in principle prevents an SPA-based Fisher matrix calculation in full generality. Our approximation also assumes simple precession (i.e., that β is nearly constant), an excellent approximation for configurations dominated by a single spin but less reliable for systems with two dynamically significant spins. Again, nothing in principle prevents using recently-developed solutions to the two-spin problem in a seperation-of-timescales-based SPA waveform model in thus Fisher matrix.

Future directions. – While only approximating the results of detailed Bayesian parameter estimation [16, 94, 95], Fisher-matrix calculations provide a powerful and analytically-tractable tool to assess what can be measured and why. In general, the eigenvectors of the Fisher matrix characterize what combinations of parameters are strongly correlated, while the eigenvalues characterize the measurement accuracy of each such combination. For analytically-tractable Fisher matrix calculations, each term Γ_{ab} can be understood analytically, giving substantially improved understanding over degeneracies present in the full and marginalized Fisher matrix; see, e.g., the discussion in [65]. Extending the Fisher-SPA method to include a single precessing spin will help rapidly interpret of real gravitational wave data, via improved methods to explore the model space and interpret the posterior; assess the impact of systematic errors from the waveform model; quantify the accuracy to which tidal effects and modifications of general relativity can be detected; and otherwise understand what can be measured and why.

Further investigation is needed to generalize our approach to account for a second significant spin, using recently-developed analytic solutions [8, 56, 57]; and to perform a large-scale comparison between our calculations and detailed Bayesian parameter estimates. Finally, to facilitate the immediate use of our approach and enhance its similarity to prior work, we have adopted extremely simple assumptions (e.g., the neglect of $\tau_{m,\bar{m}}$; the neglect of all but 10 overlaps; and the restricted PN expansion). These assumptions can easily be relaxed if more detailed calculations are required, since the simple form of Eq. (4) insures a theoretically-tractable analysis.

Acknowledgements. – ROS acknowledges support from NSF PHY-1505629 and PHY-0970074.

APPENDIX A: EVALUATING OUR EXPRESSION

Our model is assembled from well-understood ingredients. For example, expressions for α, ξ, and β are available in the electronic supplementary material of [67], while Ψ is available as Eq. (3.18) of [29]. Only the delay time $\tau_{m\bar{m}}$ was not previously presented. For $\bar{m}=2$ and $m=0$, this expression reduces to $M\omega = 2M\dot{\Phi}$, the same phase-frequency relation whose different methods of solution define the different Taylor approximations (T1, T2, T3, T4, ...) reviewed in [29]. In this work, for simplicity and to avoid ambiguities assocaited with series truncation, we eschew a true closed-form expression for $\tau_{m\bar{m}}(v)$ which invokes additional approximations and instead perform the Legendre-transformation inverse numerically, based on evaluating post-Newtonian expressions for the right-hand side as a function of time. Closed-form expressions can be produced by instead substituting and solving with power series. Just as different strategies to relate time and phase lead to different Taylor approximations, with different series truncation error, different strategies to solve for $\tau_{m,\bar{m}}$ will produce expressions with marginally different results.

Because the precession rate is small compared to the orbital frequency, we can usefully approximately solve for $\tau_{m\bar{m}}$ in terms of solutions $\tau(f)$ that arise for a nonprecessing binary when solving $\omega = 2\Phi'_{orb}(t)$ as $t=\tau(f)$. Depending on the approximation scheme adopted, the right-hand side is a familiar relationship $t(\omega)$ for a nonprecessing binary; for this work, the relevant expression follows from TaylorF2. Equation 3 has the form $y = f(x) + \epsilon g(x)$, which can inverted to leading order by $x = f^{-1}(y) - \epsilon g(f^{-1}(y))/f'(f^{-1}(y))$. Replacing $x \to t$ and $f^{-1} \to \tau$, we find

$$\tau_{m,\bar{m}}(\omega) = \tau\left(\omega\frac{2}{\bar{m}}\right) - (m - \bar{m}\cos\beta_{JL})\frac{\dot{\alpha}(\tau_{0,m})}{2\Phi''_{orb}(\tau_{0,m})} + \dots \tag{A1}$$

This expression has a clear geometric interpretation in terms of the frequencies of the individual harmonics versus time.

The inverse can also be identified by power-series methods. Closed-form power series solutions analagous to TaylorT/TaylorF can be generated by rewriting the expression explicitly in terms of post-Newtonian parameters:

$$\dot{\alpha} = \frac{d\alpha}{dv}\frac{dv}{dt} = \frac{d\alpha}{dv}\frac{\mathcal{F}(v)}{-dE/dv} = \frac{d\alpha}{dv}\frac{32}{5}\eta v^9[1+\ldots] \tag{A2}$$

(A3)

where $E(v)$ is the instantaneous binding energy and $\mathcal{F}(v)$ is the gravitational wave flux. With this substitution (and the definition $v^3/M = \dot{\Phi}_{orb}$) we express ωM as a power series in v, which can be inverted term by term.

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