Quantitative Estimates in Reiterated Homogenization

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Abstract

This paper investigates quantitative estimates in the homogenization of second-order elliptic systems with periodic coefficients that oscillate on multiple separated scales. We establish large-scale interior and boundary Lipschitz estimates down to the finest microscopic scale via iteration and rescaling arguments. We also obtain a convergence rate in the L^2 space by the reiterated homogenization method.

Keywords: Reiterated homogenization; Convergence rates; Large-scale regularity estimates

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1 Introduction

In this paper we investigate quantitative estimates in the homogenization of elliptic systems with periodic coefficients that oscillate on multiple separated scales. More precisely, consider the $m \times m$ elliptic system in divergence form,

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$, where

$$\mathcal{L}_\varepsilon = -\text{div}\big(A^\varepsilon(x)\nabla\big) = -\text{div}\big(A(x, x/\varepsilon_1, x/\varepsilon_2, \ldots, x/\varepsilon_n)\nabla\big), \tag{1.2}$$

and $\{0 < \varepsilon_n < \varepsilon_{n-1} < \cdots < \varepsilon_1 < 1\}$ represents a set of n ordered lengthscales, all depending on a single parameter ε. We assume that the coefficient tensor $A = A(x, y_1, y_2, \ldots, y_n)$ is real, bounded measurable, and satisfies the ellipticity condition,

$$\|A\|_{L^\infty(\mathbb{R}^{d\times(n+1)})} \leq \frac{1}{\mu} \quad \text{and} \quad \mu|\xi|^2 \leq \langle A\xi, \xi\rangle \tag{1.3}$$

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for any $\xi \in \mathbb{R}^{m\times d}$, where $\mu > 0$, and the periodicity condition

$$A(x, y_1 + z_1, \cdots, y_n + z_n) = A(x, y_1, \cdots, y_n) \quad \text{for any } (z_1, \cdots, z_n) \in \mathbb{Z}^{d\times n}. \tag{1.4}$$

We also impose the Hölder continuity condition on A: there exist constants $L \geq 0$ and $0 < \theta \leq 1$ such that

$$|A(x, y_1, \cdots, y_{n-1}, y_n) - A(x', y_1', \cdots, y_{n-1}', y_n)| \leq L\Big\{|x - x'| + \sum_{\ell=1}^{n-1} |y_\ell - y_\ell'|\Big\}^\theta \tag{1.5}$$

for $x, x', y_1, \ldots, y_n, y_1', \ldots, y_{n-1}' \in \mathbb{R}^d$. Note that no continuity condition is needed for the last variable y_n.

Homogenization problems with multiscale structures were first considered in the 1930s by Bruggeman [9]. In the case where $\varepsilon_k = \varepsilon^k$ for $1 \leq k \leq n$, the qualitative homogenization theory for $\mathcal{L}_\varepsilon$ in (1.2) was established in the 1970s by Bensoussan, Lions, and Papanicolaou [8]. Let u_ε be a weak solution of the Dirichlet problem,

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = f \quad \text{on } \partial\Omega. \tag{1.6}$$

Assume that A satisfies (1.3)-(1.4) and some continuity condition. It is known that u_ε converges weakly in $H^1(\Omega)$ to the solution u_0 of the homogenized problem,

$$\mathcal{L}_0(u_0) = F \quad \text{in } \Omega \quad \text{and} \quad u_0 = f \quad \text{on } \partial\Omega, \tag{1.7}$$

where $\mathcal{L}_0 = -\text{div}\big(\widehat{A}(x)\nabla\big)$ is a second-order elliptic operator. The effective tensor $\widehat{A}(x)$ is obtained by homogenizing separately and successively the different scales, starting from the finest one ε_n, as follows. One fixes $(x, y_1, \ldots, y_{n-1})$ and homogenizes the last variable $y_n = x/\varepsilon_n$ in $A_n = A(x, y_1, \ldots, y_n)$ to obtain $A_{n-1}(x, y_1, \ldots, y_{n-1})$. Repeat the same procedure on A_{n-1} to obtain A_{n-2}, and continue until one arrives at $A_0(x)$, which is $\widehat{A}(x)$. This process, in which at each step the standard homogenization is performed on an operator with a parameter, is referred in [8] as reiterated homogenization. For more recent work in the reiterated homogenization theory and its applications, we refer the reader to [6, 1, 14, 15, 17, 18, 22, 16, 20, 21] and their references. In particular, using the method of multiscale convergence, Allaire and Briane [1] obtained qualitative results for $\mathcal{L}_\varepsilon$ in a general case under the condition of separation of scales,

$$\varepsilon_1 \to 0 \quad \text{and} \quad \varepsilon_{k+1}/\varepsilon_k \to 0 \quad \text{for } 1 \leq k \leq n-1, \text{ as } \varepsilon \to 0. \tag{1.8}$$

This paper is devoted to the quantitative homogenization theory for the operator $\mathcal{L}_\varepsilon$ and concerns problems of convergence rates and large-scale regularity estimates. We point out that in the case $n = 1$, where $A^\varepsilon(x) = A(x/\varepsilon)$ or $A(x, x/\varepsilon)$, major progress has been made in quantitative homogenization in recent years. We refer the reader to [7, 26, 12, 13, 4, 23, 19, 24] and their references for the periodic case, and to [10, 5, 3, 11, 2] and their references for quantitative homogenization in the stochastic setting. The primary purpose of this paper is to extend quantitative estimates in periodic homogenization for $n = 1$ to the case $n > 1$, where the operator $\mathcal{L}_\varepsilon$ is used to model a composite medium with several microscopic scales.

Our main results are given in the following two theorems. We establish the large-scale interior and boundary Lipschitz estimates down to the finest scale ε_n, assuming that the scales $0 < \varepsilon_n < \varepsilon_{n-1} < \cdots < \varepsilon_1 < \varepsilon_0 = 1$ are well-separated in the sense that there exists a positive integer N such that

$$\left(\frac{\varepsilon_{k+1}}{\varepsilon_k}\right)^N \leq \frac{\varepsilon_k}{\varepsilon_{k-1}} \quad \text{for } 1 \leq k \leq n-1. \tag{1.9}$$

In particular, this includes the case where $\varepsilon_k = \varepsilon^{\lambda_k}$ with $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \infty$ and $0 < \varepsilon \leq 1$, but excludes the case $(\varepsilon_1, \varepsilon_2) = (\varepsilon, \varepsilon(|\log \varepsilon| + 1)^{-1})$. We point out that the condition (1.9) is equivalent to the following condition introduced in [1]: there exists $N \geq 1$ such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon_k} \left(\frac{\varepsilon_{k+1}}{\varepsilon_k}\right)^N = 0 \quad \text{for } 1 \leq k \leq n-1.$$

Theorem 1.1. *Suppose that A satisfies conditions (1.3), (1.4), and (1.5) for some $0 < \theta \leq 1$. Also assume that $0 < \varepsilon_n < \varepsilon_{n-1} < \cdots < \varepsilon_1 < \varepsilon_0 = 1$ and (1.9) holds. For $B_R = B(x_0, R)$ with $0 < \varepsilon_n < R \leq 1$, let $u_\varepsilon \in H^1(B_R; \mathbb{R}^m)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in B_R, where $F \in L^p(B_R; \mathbb{R}^m)$ for some $p > d$. Then for $0 < \varepsilon_n \leq r < R$,*

$$\left(\fint_{B_r} |\nabla u_\varepsilon|^2\right)^{1/2} \leq C \left\{ \left(\fint_{B_R} |\nabla u_\varepsilon|^2\right)^{1/2} + R \left(\fint_{B_R} |F|^p\right)^{1/p} \right\}, \tag{1.10}$$

where C depends at most on d, n, m, μ, p, (θ, L) in (1.5), and N in (1.9).

Let Ω be a bounded domain in $\mathbb{R}^d$. Define $D_r = D(x_0, r) = B(x_0, r) \cap \Omega$ and $\Delta_r = \Delta(x_0, r) = B(x_0, r) \cap \partial\Omega$, where $x_0 \in \partial\Omega$ and $0 < r < \operatorname{diam}(\Omega)$.

Theorem 1.2. *Assume that A and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ satisfy the same conditions as in Theorem 1.1. Let Ω be a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^d$ for some $\alpha > 0$. Let $u_\varepsilon \in H^1(D_R; \mathbb{R}^m)$ be a weak solution to $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in D_R and $u_\varepsilon = f$ on Δ_R, where $\varepsilon_n < R \leq 1$, $F \in L^p(D_R; \mathbb{R}^m)$ for some $p > d$, and $f \in C^{1,\nu}(\Delta_R; \mathbb{R}^m)$ for some $0 < \nu \leq \alpha$. Then for $0 < \varepsilon_n \leq r < R$,*

$$\left(\fint_{D_r} |\nabla u_\varepsilon|^2\right)^{1/2} \leq C \left\{ \left(\fint_{D_R} |\nabla u_\varepsilon|^2\right)^{1/2} + R \left(\fint_{D_R} |F|^p\right)^{1/p} + R^{-1} \|f\|_{C^{1,\nu}(\Delta_R)} \right\}, \tag{1.11}$$

where C depends at most on d, m, n, μ, p, ν, (θ, L) in (1.5), N in (1.9), and Ω.

Remark 1.1. Under the additional assumption that $A = A(x, y_1, \ldots, y_n)$ is also Hölder continuous in y_n, estimates (1.10) and (1.11) imply the uniform pointwise interior and boundary

Lipschitz estimates for u_ε, respectively. To see this, one introduces a dummy variable y_{n+1} and considers the tensor $\widetilde{A}(x, y_1, \dots, y_n, y_{n+1}) = A(x, y_1, \dots, y_n)$. Since ε_{n+1} may be arbitrarily small, it follows that the inequalities (1.10) and (1.11) hold for any $0 < r < R \le 1$. By letting $r \to 0$ we see that $|\nabla u_\varepsilon(x_0)|$ is bounded by the right-hand sides of the inequalities.

Remark 1.2. In the case $A^\varepsilon(x) = A(x/\varepsilon)$, Theorems 1.1 and 1.2 were proved by Avellaneda and Lin in a seminal paper [7] by using a compactness method. The boundary Lipschitz estimate in Theorem 1.2 was extended in [12] to solutions with Neumann conditions. Also see [4] for operators with almost-periodic coefficients and [5, 3] for large-scale Lipschitz estimates in stochastic homogenization. Our results for $n > 1$ are new even in the case $A^\varepsilon(x) = A(x/\varepsilon, x/\varepsilon^2)$.

We now describe our approach to the proof of Theorem 1.1; the same approach works equally well for Theorem 1.2. The proof is divided into two steps. In the first step we prove the estimate (1.10) for the case $\varepsilon_1 \le r < R \le 1$. To do this, we use a general approach developed in [5] by Armstrong and Smart (also see [4, 3]), which reduces the large-scale Lipschitz estimates to a problem of approximating solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ by solutions of $\mathcal{L}_0(u_0) = F$ in the L^2 norm. Given u_ε, to find a good approximation u_0, we use the idea of reiterated homogenization and introduce a (finite) sequence of approximations as follows. One first approximates u_ε by solutions of $-\text{div}\big(A^\varepsilon_{n-1}(x)\nabla u_{\varepsilon,n-1}\big) = F$, where $A^\varepsilon_{n-1}(x) = A_{n-1}(x, x/\varepsilon_1, \dots, x/\varepsilon_{n-1})$ and $A_{n-1}(x, y_1, \dots, y_{n-1})$ is the effective tensor for $A_n = A(x, y_1, \dots, y_{n-1}, y_n)$, with $(x, y_1, \dots, y_{n-1})$ fixed as parameters. The function $u_{\varepsilon,n-1}$ is then approximated by a solution of $-\text{div}\big(A^\varepsilon_{n-2}(x)\nabla u_{\varepsilon,n-2}\big) = F$, where $A^\varepsilon_{n-2}(x) = A_{n-2}(x, x/\varepsilon_1, \dots, x/\varepsilon_{n-2})$ and $A_{n-2}(x, y_1, \dots, y_{n-2})$ is the effective tensor for $A_{n-1}(x, y_1, \dots, y_{n-2}, y_{n-1})$, with $(x, y_1, \dots, y_{n-2})$ fixed. Continue the process until one reaches the tensor $A_0(x) = \widehat{A}(x)$. By an induction argument on n, to carry out the process above, it suffices to consider the special case where $n = 1$ and $A^\varepsilon(x) = A(x, x/\varepsilon)$. Moreover, by using a convolution in the x variable, one may assume that $A = A(x, y)$ is Lipschitz continuous in $x \in \mathbb{R}^d$. We point out that even though the case $A^\varepsilon(x) = A(x/\varepsilon)$ has been well studied, new techniques are needed for the case $A^\varepsilon(x) = A(x, x/\varepsilon)$ to derive estimates with sharp bounding constants depending explicitly on $\|\nabla_x A\|_\infty$. For otherwise, the results would not be useful in the induction argument.

In the second step, a rescaling argument, together with another induction argument, is used to reach the finest scale ε_n. We mention that the condition (1.9) of well-separation is only used in the first step. Without this condition, our argument yields estimates (1.10) and (1.11) for

$$\varepsilon_1 + \big(\varepsilon_2/\varepsilon_1 + \cdots + \varepsilon_n/\varepsilon_{n-1}\big)^N \le r < R \le 1, \tag{1.12}$$

where $N \ge 1$, with bounding constants C depending on N. See Remark 6.1. Although we do not know whether the condition (1.9) is necessary for Theorems 1.1 and 1.2, we believe that some well-separation condition stronger than (1.8) is required for the large-scale Lipschitz estimate down to the finest scale ε_n.

As a byproduct of the first step described above, we show that if $A^\varepsilon(x) = A(x, x/\varepsilon)$, then

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon \left\{1 + \|\nabla_x A\|_\infty + \varepsilon \|\nabla_x A\|_\infty^2\right\} \left(\|F\|_{L^2(\Omega)} + \|f\|_{H^{3/2}(\partial\Omega)}\right) \tag{1.13}$$

for $0 < \varepsilon < 1$, where C depends only on d, m, μ, and Ω (see Lemma 4.1). Estimate (1.13) improves a similar estimate in [28], where a general case $A^\varepsilon(x) = A(x, \rho(x)/\varepsilon)$ was considered by the first and third authors. It also leads to the following theorem on the L^2 convergence rate for the operator $\mathcal{L}_\varepsilon$. Note that in Theorem 1.3, we assume A satisfies (1.5) with $\theta = 1$, which, among other things, ensures that $\widehat{A}(x)$ is Lipschitz continuous.

Theorem 1.3. *Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$. Assume that A satisfies (1.3), (1.4), and (1.5) with $\theta = 1$. Let $\mathcal{L}_\varepsilon$ be given by (1.2) with $0 < \varepsilon_n < \varepsilon_{n-1} < \cdots < \varepsilon_1 < 1$. For $F \in L^2(\Omega; \mathbb{R}^m)$ and $f \in H^{3/2}(\partial\Omega; \mathbb{R}^m)$, let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^m)$ be the solution of (1.6) and u_0 the solution of the homogenized problem (1.7). Then*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\big\{\varepsilon_1 + \varepsilon_2/\varepsilon_1 + \cdots + \varepsilon_n/\varepsilon_{n-1}\big\}\|u_0\|_{H^2(\Omega)}, \tag{1.14}$$

where C depends at most on d, m, n, μ, L, and Ω.

In the case $A^\varepsilon = A(x/\varepsilon, x/\varepsilon^2)$, the estimate (1.14) was proved in [20] (also see [22, 21]). As indicated in [21], one may extend the proof to the general case considered in Theorem 1.3. However, the error estimates of the multiscale expansions for the case $n = 2$ in [20] are already quite involved, and their extension to the case $n > 2$ is not so obvious. Our proof of (1.14), which is based on the idea of reiterated homogenization, seems to be natural and is much simpler conceptually. Note that if $\varepsilon_1 = \varepsilon^\alpha$ and $\varepsilon_2 = \varepsilon$, where $0 < \alpha < 1$, the estimate (1.14) gives an $O(\varepsilon^\alpha + \varepsilon^{1-\alpha})$ convergence rate in $L^2(\Omega)$. The rate is sharp, at least in the case $d = 1$ by considering the example where

$$A^{-1}(y_1, y_2) = 2 + k^{-1}\Re\left(e^{2\pi i(ky_1 - y_2)}\right),$$

with $k \in \mathbb{N}$.

The paper is organized as follows. In Section 2 we give the definition of the effective tensor $\widehat{A}(x)$ as well as the tensors $A_k(x, y_1, \ldots, y_k)$ for $1 \le k \le n$, mentioned earlier. We also introduce a smoothing operator and prove two estimates needed in the following sections. The proof of (1.13) is given in Section 3 and that of Theorem 1.3 in Section 4. In Section 5 we establish an approximation theorem, using the results in Section 3. Sections 6 and 7 are devoted to the proofs of Theorems 1.1 and 1.2, respectively.

For notational simplicity we will assume $m = 1$ in the rest of the paper. However, no particular fact pertain to the scalar case is ever used. All results and proofs extend readily to the case $m > 1$ - the case of elliptic systems. We will use $\fint_E u$ to denote the L^1 average of u over the set E; i.e. $\fint_E u = \frac{1}{|E|}\int_E u$. A function is said to be 1-periodic in $y_k \in \mathbb{R}^d$ if it is periodic in y_k with respect to $\mathbb{Z}^d$. Finally, the summation convention is used throughout.

2 Preliminaries

2.1 Effective coefficients

Suppose $A = A(x, y_1, \ldots, y_n)$ satisfies conditions (1.3) and (1.4). To define the effective matrix $\widehat{A} = \widehat{A}(x)$ in the homogenized operator $\mathcal{L}_0 = -\mathrm{div}\big(\widehat{A}(x)\nabla\big)$, we introduce a sequence of $d \times d$ matrices,

$$A_\ell = A_\ell(x, y_1, \ldots, y_\ell) \quad \text{for } 0 \le \ell \le n, \tag{2.1}$$

which are 1-periodic in $(y_1, \ldots, y_\ell) \in \mathbb{R}^{d\times\ell}$ and satisfy the ellipticity condition,

$$\|A_\ell\|_{L^\infty(\mathbb{R}^{d\times(\ell+1)})} \le \mu_1 \quad \text{and} \quad \mu|\xi|^2 \le \langle A_\ell \xi, \xi\rangle \tag{2.2}$$

for $\xi \in \mathbb{R}^d$, where $\mu_1 > 0$ depends only on d, n and μ. To this end, we let $A_n(x, y_1, \cdots, y_n) = A(x, y_1, \ldots, y_n)$. Suppose A_ℓ has been given for some $1 \le \ell \le n$. For a.e. $(x, y_1, \ldots, y_{\ell-1}) \in \mathbb{R}^{d\times\ell}$ fixed, we solve the elliptic cell problem,

$$\begin{cases} -\,\mathrm{div}_y\big(A_\ell(x, y_1, \ldots, y_{\ell-1}, y)\nabla_y \chi_\ell^j\big) = \mathrm{div}_y\big(A_\ell(x, y_1, \ldots, y_{\ell-1}, y)\nabla_y y^j\big) \quad \text{in } \mathbb{T}^d, \\ \chi_\ell^j = \chi_\ell^j(x, y_1, \cdots, y_{\ell-1}, y) \text{ is 1-periodic in } y, \\ \displaystyle\int_{\mathbb{T}^d} \chi_\ell^j(x, y_1, \ldots, y_{\ell-1}, y)\, dy = 0 \end{cases} \tag{2.3}$$

for $1 \le j \le d$, where y^j denotes the jth component of $y \in \mathbb{R}^d$. Since A_ℓ is 1-periodic in $(y_1, \ldots, y_\ell)$, so is the corrector $\chi_\ell(x, y_1, \ldots, y_{\ell-1}, y_\ell) = (\chi_\ell^1, \cdots, \chi_\ell^d)$. We now define

$$A_{\ell-1}(x, y_1, \ldots, y_{\ell-1}) = \fint_{\mathbb{T}^d} \Big(A_\ell(x, y_1, \ldots, y_\ell) + A_\ell(x, y_1, \ldots, y_\ell)\nabla_{y_\ell}\chi_\ell\Big) dy_\ell. \tag{2.4}$$

Clearly, $A_{\ell-1}$ is 1-periodic in $(y_1, \ldots, y_{\ell-1})$. It is also well known that $A_{\ell-1}$ satisfies the ellipticity condition (2.2) [8]. As a result, by induction, we obtain the matrix A_ℓ for $0 \le \ell \le n$. In particular, $\widehat{A}(x) = A_0(x)$ is the effective matrix for the operator $\mathcal{L}_\varepsilon$ in (1.2).

Theorem 2.1. *Suppose A satisfies conditions (1.3) and (1.4). Also assume that as a function of $(x, y_1, \ldots, y_{n-1})$, $A \in C(\mathbb{R}^{d\times n}; L^\infty(\mathbb{R}^d))$. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let u_ε be a weak solution of the Dirichlet problem (1.6), with $F \in H^{-1}(\Omega)$ and $f \in H^{1/2}(\partial\Omega)$. Then, if $\varepsilon \to 0$ and $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ satisfies the condition (1.8), u_ε converges weakly in $H^1(\Omega)$ to the solution u_0 of the homogenized problem (1.7).*

Theorem 2.1, whose proof may be found in [8, 1], is not used in this paper. In fact, by approximating the coefficients, our quantitative result in Theorem 1.3, provides another proof of Theorem 2.1.

It follows by the energy estimate as well as Poincaré's inequality that

$$\fint_{\mathbb{T}^d} |\nabla_y \chi_\ell(x, y_1, \ldots, y_{\ell-1}, y_\ell)|^2\, dy_\ell + \fint_{\mathbb{T}^d} |\chi_\ell(x, y_1, \ldots, y_{\ell-1}, y_\ell)|^2\, dy_\ell \le C \tag{2.5}$$

for a.e. $(x, y_1, \ldots, y_{\ell-1}) \in \mathbb{R}^{d\times\ell}$, where $1 \leq \ell \leq n$ and C depends only on d, n and μ. The next lemma gives the Hölder estimates for χ_ℓ and A_ℓ under the Hölder continuity condition on A.

Lemma 2.1. *Suppose A satisfies conditions (1.3), (1.4), and (1.5) for some $\theta \in (0, 1]$ and $L \geq 0$. Then*

$$\begin{aligned} &\|\chi_\ell(x, y_1, \ldots, y_{\ell-1}, \cdot) - \chi_\ell(x', y_1', \ldots, y_{\ell-1}', \cdot)\|_{H^1(\mathbb{T}^d)} \\ &\qquad \leq CL\big(|x - x'| + |y_1 - y_1'| + \cdots + |y_{\ell-1} - y_{\ell-1}'|\big)^\theta, \\ &|A_{\ell-1}(x, y_1, \ldots, y_{\ell-1}) - A_{\ell-1}(x', y_1', \ldots, y_{\ell-1}')| \\ &\qquad \leq CL\big(|x - x'| + |y_1 - y_1'| + \cdots + |y_{\ell-1} - y_{\ell-1}'|\big)^\theta \end{aligned} \tag{2.6}$$

for $1 \leq \ell \leq n$, where C depends only on d, n, θ and μ. In particular, $|\widehat{A}(x) - \widehat{A}(x')| \leq C|x - x'|^\theta$, where C depends only on d, n, μ, θ and L.

Proof. It suffices to prove (2.6) for $\ell = n$. The rest follows by induction. Note that for $(x, y_1, \ldots, y_{n-1}), (x', y_1', \ldots, y_{n-1}') \in \mathbb{R}^{d\times n}$ fixed,

$$\begin{aligned} &-\operatorname{div}_y\Big(A(x, y_1, \ldots, y_{n-1}, y)\nabla_y\big(\chi_n^j(x, y_1, \ldots y_{n-1}, y) - \chi_n^j(x', y_1', \ldots, y_{n-1}', y)\big)\Big) \\ &= \operatorname{div}_y\Big(\big(A(x, y_1, \ldots, y_{n-1}, y) - A(x', y_1', \ldots, y_{n-1}', y)\big)\nabla_y\big(y^j + \chi_n^j(x', y_1', \ldots, y_{n-1}', y)\big)\Big). \end{aligned}$$

The estimate for the correct χ_n in (2.6) follows readily from the usual energy estimate and (1.5). In view of (2.4) we may deduce the estimate for A_{n-1} in (2.6) by using (1.5) and the estimate of χ_n in (2.6). $\square$

2.2 An ε-smoothing operator

Fix a function $\varphi \in C_0^\infty(B(0, 1/2))$ such that $\varphi \geq 0$ and $\int_{\mathbb{R}^d} \varphi dx = 1$. For functions of form $g^\varepsilon(x) = g(x, x/\varepsilon)$, we introduce a smoothing operator S_ε, defined by

$$S_\varepsilon(g^\varepsilon)(x) = \int_{\mathbb{R}^d} g(z, x/\varepsilon)\varphi_\varepsilon(x - z)dz, \tag{2.7}$$

where $\varphi_\varepsilon(z) = \varepsilon^{-d}\varphi(z/\varepsilon)$. Note that the smoothing is only done to the slow variable x.

Lemma 2.2. *Let $1 \leq p < \infty$. Suppose that $h = h(x, y)$ is 1-periodic in y and $h \in L^\infty(\mathbb{R}^d_x; L^p(\mathbb{T}^d_y))$. Then for any $f \in L^p(\mathbb{R}^d)$,*

$$\|S_\varepsilon(h^\varepsilon f)\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)} \sup_{x\in\mathbb{R}^d} \left(\fint_{\mathbb{T}^d} |h(x, y)|^p \, dy\right)^{1/p}, \tag{2.8}$$

where $h^\varepsilon(x) = h(x, x/\varepsilon)$ and C depends only on d and p.

Proof. It follows by Hölder's inequality and the assumption $\int_{\mathbb{R}^d} \varphi = 1$ that

$$|S_\varepsilon(h^\varepsilon f)(x)|^p \le \int_{\mathbb{R}^d} |h(z, x/\varepsilon)|^p |f(z)|^p \varphi_\varepsilon(x-z)\, dz.$$

This, together with Fubini's Theorem, gives

$$\begin{aligned}
\int_{\mathbb{R}^d} |S_\varepsilon(h^\varepsilon f)|^p\, dx &\le \int_{\mathbb{R}^d} |f(z)|^p \int_{\mathbb{R}^d} \varphi_\varepsilon(x-z)|h(z,x/\varepsilon)|^p\, dx\, dz\\
&\le \|f\|^p_{L^p(\mathbb{R}^d)} \sup_{z\in\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi_\varepsilon(x-z)|h(z,x/\varepsilon)|^p\, dx\\
&\le C\|f\|^p_{L^p(\mathbb{R}^d)} \sup_{z\in\mathbb{R}^d} \fint_{B(z,\varepsilon/2)} |h(z,x/\varepsilon)|^p\, dx.
\end{aligned}$$

Using the periodicity of $h(x,y)$ in the second variable, it is easy to see that

$$\sup_{z\in\mathbb{R}^d} \fint_{B(z,\varepsilon/2)} |h(z,x/\varepsilon)|^p\, dx \le C \sup_{x\in\mathbb{R}^d} \fint_{\mathbb{T}^d} |h(x,y)|^p\, dy,$$

which finishes the proof. ∎

Lemma 2.3. *Let $1 \le p \le \infty$. Suppose that $h = h(x,y) \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and $\nabla_x h \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. Then for any $f \in W^{1,p}(\mathbb{R}^d)$,*

$$\|h^\varepsilon f - S_\varepsilon(h^\varepsilon f)\|_{L^p(\mathbb{R}^d)} \le C\varepsilon \Big\{ \|\nabla_x h\|_\infty \|f\|_{L^p(\mathbb{R}^d)} + \|h\|_\infty \|\nabla f\|_{L^p(\mathbb{R}^d)} \Big\}, \tag{2.9}$$

where $h^\varepsilon(x) = h(x, x/\varepsilon)$ and C depends only on d and p.

Proof. Write

$$h^\varepsilon(x) f(x) - S_\varepsilon(h^\varepsilon f)(x) = \int_{\mathbb{R}^d} \big(h(x,x/\varepsilon) f(x) - h(z,x/\varepsilon) f(z) \big) \varphi_\varepsilon(x-z)\, dz,$$

which leads to

$$|h^\varepsilon(x) f(x) - S_\varepsilon(h^\varepsilon f)(x)| \le C \fint_{B(x,\varepsilon/2)} |h(x,x/\varepsilon) f(x) - h(z,x/\varepsilon) f(z)|\, dz.$$

We now apply the inequality,

$$\fint_{B(x,\varepsilon/2)} |u(z) - u(x)|\, dz \le C \int_{B(x,\varepsilon/2)} \frac{|\nabla u(z)|}{|z-x|^{d-1}}\, dz, \tag{2.10}$$

where C depends only on d. This gives

$$\begin{aligned}
&|h^\varepsilon(x) f(x) - S_\varepsilon(h^\varepsilon f)(x)|\\
&\le C\|\nabla_x h\|_\infty \int_{B(x,\varepsilon/2)} \frac{|f(z)|}{|z-x|^{d-1}}\, dz + C\|h\|_\infty \int_{B(x,\varepsilon/2)} \frac{|\nabla_z f(z)|}{|z-x|^{d-1}}\, dz.
\end{aligned}$$

It follows that

$$\begin{aligned}\int_{\mathbb{R}^d} |h^\varepsilon f - S_\varepsilon(h^\varepsilon f)||F|\, dx \le C\|\nabla_x h\|_\infty \int_{\mathbb{R}^d} \left(\int_{B(x,\varepsilon/2)} \frac{|f(z)||F(x)|}{|z-x|^{d-1}} dz \right) dx \\ + C\|h\|_\infty \int_{\mathbb{R}^d} \left(\int_{B(x,\varepsilon/2)} \frac{|\nabla_z f(z)||F(x)|}{|z-x|^{d-1}} dz \right) dx.\end{aligned} \tag{2.11}$$

Finally, we note that the operator defined by

$$Tg(x) = \int_{B(x,\varepsilon/2)} \frac{g(z)}{|z-x|^{d-1}}\, dz$$

is bounded on $L^p(\mathbb{R}^d)$ and $\|Tg\|_{L^p(\mathbb{R}^d)} \le C\varepsilon \|g\|_{L^p(\mathbb{R}^d)}$ for $1 \le p \le \infty$. Thus, if $1 \le p \le \infty$ and $q = p'$,

$$\int_{\mathbb{R}^d} |h^\varepsilon f - S_\varepsilon(h^\varepsilon f)||F|\, dx \le C\varepsilon \|F\|_{L^q(\mathbb{R}^d)} \Big\{ \|\nabla_x h\|_\infty \|f\|_{L^p(\mathbb{R}^d)} + \|h\|_\infty \|\nabla f\|_{L^p(\mathbb{R}^d)} \Big\},$$

from which the inequality (2.9) follows by duality. □

3 Convergence rate $(n = 1)$

In this section we consider a simple case, where $n = 1$ and

$$\mathcal{L}_\varepsilon = -\text{div}\big(A(x, x/\varepsilon)\nabla\big). \tag{3.1}$$

The matrix $A = A(x, y)$ satisfies the ellipticity condition (1.3) and is 1-periodic in $y \in \mathbb{R}^d$. We also assume that

$$\|\nabla_x A\|_\infty = \|\nabla_x A\|_{L^\infty(\mathbb{R}^d_x \times \mathbb{R}^d_y)} < \infty. \tag{3.2}$$

Recall that

$$\widehat{A}(x) = \fint_{\mathbb{T}^d} \Big(A(x, y) + A(x, y)\nabla_y \chi(x, y) \Big) dy,$$

where the corrector $\chi(x, y) = (\chi^1(x, y), \dots, \chi^d(x, y))$ is given by the cell problem (2.3) with $\ell = n = 1$. Note that by (2.6),

$$\|\nabla_x \widehat{A}\|_\infty \le C\|\nabla_x A\|_\infty, \tag{3.3}$$

and

$$\fint_{\mathbb{T}^d} \big(|\nabla_x \nabla_y \chi(x, y)|^2 + |\nabla_x \chi(x, y)|^2 \big) dy \le C\|\nabla_x A\|^2_\infty, \tag{3.4}$$

where C depends only on d and μ.

Define

$$B(x, y) = A(x, y) + A(x, y)\nabla_y \chi(x, y) - \widehat{A}(x). \tag{3.5}$$

The $d \times d$ matrix $B(x,y) = (b_{ij}(x,y))$ is 1-periodic in y and

$$\fint_{\mathbb{T}^d} |B(x,y)|^2\, dy \le C, \tag{3.6}$$

where C depends only on d and μ. In view of (3.3)-(3.4) we obtain

$$\fint_{\mathbb{T}^d} |\nabla_x B(x,y)|^2\, dy \le C\|\nabla_x A\|_\infty^2. \tag{3.7}$$

By the definitions of $\widehat{A}(x)$ and $\chi(x,y)$, it follows that

$$\int_{\mathbb{T}^d} b_{ij}(x,y)\, dy = 0 \quad \text{ and } \quad \frac{\partial}{\partial y^i} b_{ij}(x,y) = 0 \tag{3.8}$$

for each $x \in \mathbb{R}^d$ (the index i is summed from 1 to d), where we have used the notation $y = (y^1, \cdots, y^d) \in \mathbb{R}^d$.

Lemma 3.1. *There exist functions $\phi(x,y) = (\phi_{kij}(x,y))$ with $1 \le k,i,j \le d$ such that ϕ is 1-periodic in y,*

$$\phi_{kij} = -\phi_{ikj} \quad \textit{and} \quad b_{ij}(x,y) = \frac{\partial}{\partial y^k}\phi_{kij}(x,y). \tag{3.9}$$

Moreover, $\int_{\mathbb{T}^d} \phi(x,y) dy = 0$, and

$$\begin{aligned}\fint_{\mathbb{T}^d} |\nabla_y \phi(x,y)|^2\, dy + \fint_{\mathbb{T}^d} |\phi(x,y)|^2\, dy \le C,\\ \fint_{\mathbb{T}^d} |\nabla_x\nabla_y \phi(x,y)|^2\, dy + \fint_{\mathbb{T}^d} |\nabla_x\phi(x,y)|^2\, dy \le C\|\nabla_x A\|_\infty^2,\end{aligned} \tag{3.10}$$

where C depends only on d and μ.

Proof. Using (3.8), the flux correctors ϕ_{kij} are constructed in the same manner as in the case $A = A(y)$ (see e.g. [24]). Indeed, for each x fixed, one solves the cell problem

$$\begin{cases} \Delta_y f_{ij}(x,y) = b_{ij}(x,y) & \text{in } \mathbb{T}^d,\\ f_{ij}(x,y) \text{ is 1-periodic in } y, \end{cases}$$

and sets

$$\phi_{kij}(x,y) = \frac{\partial}{\partial y^k} f_{ij}(x,y) - \frac{\partial}{\partial y^i} f_{kj}(x,y).$$

The first inequality in (3.10) follows by using the L^2 estimate and (3.6). To see the second one uses (3.7). $\square$

Let u_ε be a weak solution of the Dirichlet problem (1.6) and u_0 the solution of the homogenized problem (1.7). Let

$$w_\varepsilon = u_\varepsilon - u_0 - \varepsilon S_\varepsilon(\eta_\varepsilon \chi^\varepsilon \nabla u_0), \tag{3.11}$$

where $\chi^\varepsilon(x) = \chi(x, x/\varepsilon)$ and the operator S_ε is defined by (2.7). The cut-off function η_ε in (3.11) is chosen so that $\eta_\varepsilon \in C_0^\infty(\Omega)$, $0 \le \eta_\varepsilon \le 1$,

$$\begin{aligned} &\eta_\varepsilon(x) = 1 \quad \text{if } x \in \Omega \text{ and } \operatorname{dist}(x, \partial\Omega) \ge 4\varepsilon,\\ &\eta_\varepsilon(x) = 0 \quad \text{if } \operatorname{dist}(x, \partial\Omega) \le 3\varepsilon, \end{aligned}$$

and $|\nabla \eta_\varepsilon| \le C\varepsilon^{-1}$. Define

$$\Omega_t = \big\{ x \in \Omega : \quad \operatorname{dist}(x, \partial\Omega) < t \big\}. \tag{3.12}$$

The following lemma was proved in [25] for the case $A^\varepsilon = A(x/\varepsilon)$. The case $A^\varepsilon = A(x, \rho(x)/\varepsilon)$ for stratified structures was considered in [28] by the first and third authors. Also see [27] for the nonlinear case. The estimate (3.13) is sharper than the similar estimates in [28, 27].

Lemma 3.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Let w_ε be defined by (3.11). Then for any $\psi \in H_0^1(\Omega)$,*

$$\begin{aligned} &\Big| \int_\Omega A^\varepsilon \nabla w_\varepsilon \cdot \nabla \psi \, dx \Big| \\ &\le C\varepsilon \|\nabla \psi\|_{L^2(\Omega)} \Big\{ \|\nabla_x A\|_\infty \|\nabla u_0\|_{L^2(\Omega)} + \|\nabla^2 u_0\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})} \Big\} \\ &\qquad + C \|\nabla \psi\|_{L^2(\Omega_{5\varepsilon})} \|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})}, \end{aligned} \tag{3.13}$$

where $A^\varepsilon = A(x, x/\varepsilon)$ and C depends only on d, μ, and Ω.

Proof. Using $\mathcal{L}_\varepsilon(u_\varepsilon) = \mathcal{L}_0(u_0)$, we obtain

$$\begin{aligned} \mathcal{L}_\varepsilon(w_\varepsilon) &= \operatorname{div}\big[(A^\varepsilon - \widehat{A})\nabla u_0\big] + \operatorname{div}\big[A^\varepsilon S_\varepsilon\big(\eta_\varepsilon (\nabla_y \chi)^\varepsilon \nabla u_0\big)\big] \\ &\quad + \varepsilon \operatorname{div}\big[A^\varepsilon S_\varepsilon\big((\nabla \eta_\varepsilon)\chi^\varepsilon \nabla u_0\big)\big] + \varepsilon \operatorname{div}\big[A^\varepsilon S_\varepsilon\big(\eta_\varepsilon (\nabla_x \chi)^\varepsilon \nabla u_0\big)\big] \\ &\quad + \varepsilon \operatorname{div}\big[A^\varepsilon S_\varepsilon\big(\eta_\varepsilon \chi^\varepsilon \nabla^2 u_0\big)\big]. \end{aligned} \tag{3.14}$$

The last three terms in the right-hand side of (3.14) are easy to handle. Let $B(x, y)$ be given by (3.5). To deal with the first two terms, we write the sum of them as

$$I_1 + I_2 + \operatorname{div}\big[S_\varepsilon\big(\eta_\varepsilon B^\varepsilon \nabla u_0\big)\big], \tag{3.15}$$

where $B^\varepsilon = B(x, x/\varepsilon)$, and

$$\begin{aligned} I_1 &= \operatorname{div}\big[(A^\varepsilon - \widehat{A})\nabla u_0 - S_\varepsilon\big((A^\varepsilon - \widehat{A})\eta_\varepsilon \nabla u_0\big)\big], \\ I_2 &= \operatorname{div}\big[A^\varepsilon S_\varepsilon\big(\eta_\varepsilon (\nabla_y \chi)^\varepsilon \nabla u_0\big) - S_\varepsilon\big(\eta_\varepsilon A^\varepsilon (\nabla_y \chi)^\varepsilon \nabla u_0\big)\big]. \end{aligned} \tag{3.16}$$

It follows from (3.14)-(3.16) that

$$
\begin{aligned}
&\Big|\int_\Omega A^\varepsilon \nabla w_\varepsilon \cdot \nabla \psi dx\Big|\\
&\le \int_\Omega \big|(A^\varepsilon - \widehat{A})\nabla u_0 - S_\varepsilon\big((A^\varepsilon - \widehat{A})\eta_\varepsilon \nabla u_0\big)\big| |\nabla \psi|\, dx\\
&\qquad + \int_\Omega \big|A^\varepsilon S_\varepsilon\big(\eta_\varepsilon (\nabla_y \chi)^\varepsilon \nabla u_0\big) - S_\varepsilon\big(\eta_\varepsilon A^\varepsilon (\nabla_y \chi)^\varepsilon \nabla u_0\big)\big| |\nabla \psi|\, dx\\
&\qquad + \Big|\int_\Omega S_\varepsilon\big(\eta_\varepsilon B^\varepsilon \nabla u_0\big)\cdot \nabla \psi\, dx\Big| \\
&\qquad + C\varepsilon \int_\Omega |S_\varepsilon\big((\nabla \eta_\varepsilon)\chi^\varepsilon \nabla u_0\big)| |\nabla \psi|\, dx\\
&\qquad + C\varepsilon \int_\Omega |S_\varepsilon\big(\eta_\varepsilon (\nabla_x \chi)^\varepsilon \nabla u_0\big)| |\nabla \psi|\, dx\\
&\qquad + C\varepsilon \int_\Omega |S_\varepsilon\big(\eta_\varepsilon \chi^\varepsilon \nabla^2 u_0\big)| |\nabla \psi|\, dx\\
&= J_1 + \cdots + J_6,
\end{aligned}
\tag{3.17}
$$

for any $\psi \in H_0^1(\Omega)$. We estimate $J_i, i = 1, \ldots, 6$ separately.

To bound J_4, we use the Cauchy inequality and (2.8) to obtain

$$
\begin{aligned}
J_4 &\le C\varepsilon \|S_\varepsilon\big((\nabla \eta_\varepsilon)\chi^\varepsilon \nabla u_0\big)\|_{L^2(\Omega)} \|\nabla \psi\|_{L^2(\Omega_{5\varepsilon})}\\
&\le C\varepsilon \|(\nabla \eta_\varepsilon)\nabla u_0\|_{L^2(\Omega)} \|\nabla \psi\|_{L^2(\Omega_{5\varepsilon})}\\
&\le C\|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})} \|\nabla \psi\|_{L^2(\Omega_{5\varepsilon})},
\end{aligned}
\tag{3.18}
$$

where we have used the estimate for $\chi(x, y)$ in (2.5). In view of the estimate for $\nabla_x \chi(x, y)$ in (3.4), the same argument also shows that

$$
J_5 + J_6 \le C\varepsilon \|\nabla \psi\|_{L^2(\Omega)} \big\{\|\nabla_x A\|_\infty \|\nabla u_0\|_{L^2(\Omega)} + \|\nabla^2 u_0\|_{L^2(\Omega \setminus \Omega_{3\varepsilon})}\big\}. \tag{3.19}
$$

Next, to bound J_3, we use the flux correctors ϕ_{kij} given by Lemma 3.1. Note that by using the second equation in (3.9),

$$
\begin{aligned}
&\eta_\varepsilon(x-z)b_{ij}(x-z,x/\varepsilon)\frac{\partial u_0}{\partial x^j}(x-z)\\
&=\varepsilon\eta_\varepsilon(x-z)\frac{\partial}{\partial x^k}\Big\{\phi_{kij}(x-z,x/\varepsilon)\Big\}\frac{\partial u_0}{\partial x^j}(x-z)\\
&\qquad-\varepsilon\eta_\varepsilon(x-z)\frac{\partial\phi_{kij}}{\partial x^k}(x-z,x/\varepsilon)\frac{\partial u_0}{\partial x^j}(x-z)\\
&=\varepsilon\frac{\partial}{\partial x^k}\Big\{\eta_\varepsilon(x-z)\phi_{kij}(x-z,x/\varepsilon)\frac{\partial u_0}{\partial x^j}(x-z)\Big\}\\
&\qquad-\varepsilon\frac{\partial}{\partial x^k}\Big\{\eta_\varepsilon(x-z)\Big\}\phi_{kij}(x-z,x/\varepsilon)\frac{\partial u_0}{\partial x^j}(x-z)\\
&\qquad-\varepsilon\eta_\varepsilon(x-z)\frac{\partial\phi_{kij}}{\partial x^k}(x-z,x/\varepsilon)\frac{\partial u_0}{\partial x^j}(x-z)\\
&\qquad-\varepsilon\eta_\varepsilon(x-z)\phi_{kij}(x-z,x/\varepsilon)\frac{\partial^2 u_0}{\partial x^j\partial x^k}(x-z).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
J_3=\varepsilon\Big|\int_\Omega\frac{\partial}{\partial x^k}S_\varepsilon\left(\eta_\varepsilon\phi^\varepsilon_{kij}\frac{\partial u_0}{\partial x^j}\right)\frac{\partial\psi}{\partial x_i}\,dx-\int_\Omega S_\varepsilon((\nabla\eta_\varepsilon)\phi^\varepsilon\nabla u_0)\cdot\nabla\psi\,dx\\
-\int_\Omega S_\varepsilon(\eta_\varepsilon(\nabla_x\phi)^\varepsilon\nabla u_0)\cdot\nabla\psi\,dx-\int_\Omega S_\varepsilon(\eta_\varepsilon\phi^\varepsilon\nabla^2u_0)\cdot\nabla\psi\,dx\Big|.
\end{aligned}
\tag{3.20}
$$

By using the skew-symmetry property of ϕ_{kij} in (3.9) and integration by parts we may show that the first term in the right-hand side of (3.20) is zero, if $\psi\in C_0^\infty(\Omega)$. The same is true for any $\psi\in H_0^1(\Omega)$ by a simple density argument. The remaining terms in the right-hand side of (3.20) may be handled as in the case of J_4, but using estimates of ϕ and $\nabla_x\phi$ in (3.10). As a result, we obtain

$$
\begin{aligned}
J_3&\le C\|\nabla\psi\|_{L^2(\Omega_{5\varepsilon})}\|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})}\\
&\qquad+C\varepsilon\|\nabla\psi\|_{L^2(\Omega)}\Big\{\|\nabla_xA\|_\infty\|\nabla u_0\|_{L^2(\Omega)}+\|\nabla^2u_0\|_{L^2(\Omega\setminus\Omega_{3\varepsilon})}\Big\}.
\end{aligned}
\tag{3.21}
$$

It remains to estimate J_1 and J_2. Note that

$$
\begin{aligned}
J_1&\le C\int_\Omega|\nabla u_0||1-\eta_\varepsilon||\nabla\psi|\,dx+\int_\Omega\big|(A^\varepsilon-\widehat{A})\eta_\varepsilon\nabla u_0-S_\varepsilon\big((A^\varepsilon-\widehat{A})\eta_\varepsilon\nabla u_0\big)\big|\,|\nabla\psi|\,dx\\
&=J_{11}+J_{12}.
\end{aligned}
\tag{3.22}
$$

By the Cauchy inequality,

$$
J_{11}\le C\|\nabla\psi\|_{L^2(\Omega_{4\varepsilon})}\|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})}.
\tag{3.23}
$$

To bound J_{12}, we use (2.11) to obtain

$$J_{12} \le C\|\nabla_x A\|_\infty \int_\Omega |\nabla\psi(x)| \int_{B(x,\varepsilon)} \frac{\eta_\varepsilon(z)|\nabla u_0(z)|}{|z-x|^{d-1}}\, dzdx + C\int_\Omega |\nabla\psi(x)| \int_{B(x,\varepsilon)} \frac{|\nabla\eta_\varepsilon||\nabla u_0(z)| + \eta_\varepsilon(z)|\nabla^2 u_0(z)|}{|z-x|^{d-1}}\, dzdx.$$

As in the proof of Lemma 2.3, this yields that

$$J_{12} \le C\varepsilon\|\nabla_x A\|_\infty \|\nabla\psi\|_{L^2(\Omega)}\|\nabla u_0\|_{L^2(\Omega)} + C\|\nabla\psi\|_{L^2(\Omega_{5\varepsilon})}\|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})} + C\varepsilon\|\nabla\psi\|_{L^2(\Omega)}\|\nabla^2 u_0\|_{L^2(\Omega\setminus\Omega_{3\varepsilon})}. \tag{3.24}$$

Finally, to bound J_2, we observe that

$$\begin{aligned}
J_2 &\le C\int_\Omega \fint_{B(x,\varepsilon)} |A(x,x/\varepsilon) - A(z,x/\varepsilon)|\eta_\varepsilon(z)|\nabla_y\chi(z,x/\varepsilon)||\nabla u_0(z)||\nabla\psi(x)|\, dzdx \\
&\le C\varepsilon\|\nabla_x A\|_\infty \int_\Omega \fint_{B(x,\varepsilon)} \eta_\varepsilon(z)|\nabla_y\chi(z,x/\varepsilon)||\nabla u_0(z)||\nabla\psi(x)|\, dzdx \\
&\le C\varepsilon\|\nabla_x A\|_\infty \|\nabla\psi\|_{L^2(\Omega)} \| \fint_{B(x,\varepsilon)} |\nabla_y\chi(z,x/\varepsilon)|\eta_\varepsilon(z)|\nabla u_0(z)|\, dz\|_{L^2(\Omega)} \\
&\le C\varepsilon\|\nabla_x A\|_\infty \|\nabla\psi\|_{L^2(\Omega)} \Big\| \left(\fint_{B(x,\varepsilon)} |\nabla_y\chi(z,x/\varepsilon)|^2\eta_\varepsilon(z)|\nabla u_0(z)|^2\, dz\right)^{1/2} \Big\|_{L^2(\Omega)},
\end{aligned}$$

where we have used the Cauchy inequality for the last two inequalities. By using Fubini's Theorem and (2.5) we see that

$$\Big\| \left(\fint_{B(x,\varepsilon)} |\nabla_y\chi(z,x/\varepsilon)|^2\eta_\varepsilon(z)|\nabla u_0(z)|^2\, dz\right)^{1/2} \Big\|_{L^2(\Omega)} \le C\|\nabla u_0\|_{L^2(\Omega)}.$$

This gives

$$J_2 \le C\varepsilon\|\nabla_x A\|_\infty \|\nabla\psi\|_{L^2(\Omega)}\|\nabla u_0\|_{L^2(\Omega)},$$

and completes the proof. □

The next theorem provides an error estimate in $H^1(\Omega)$.

Theorem 3.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Assume that A satisfies the same conditions as in Lemma 3.2. Let w_ε be defined by (3.11). Then*

$$\|w_\varepsilon\|_{H^1(\Omega)} \le C\varepsilon^{1/2}\|u_0\|^{1/2}_{H^2(\Omega)}\|\nabla u_0\|^{1/2}_{L^2(\Omega)} + C\varepsilon\|u_0\|_{H^2(\Omega)} + C\varepsilon\|\nabla_x A\|_\infty\|\nabla u_0\|_{L^2(\Omega)} \tag{3.25}$$

for $0 < \varepsilon < 1$, where C depends only on d, μ and Ω.

Proof. Note that $w_\varepsilon \in H_0^1(\Omega)$ and $\|w_\varepsilon\|_{H^1(\Omega)} \le C\|\nabla w_\varepsilon\|_{L^2(\Omega)}$. By taking $\psi = w_\varepsilon$ in (3.13) and using the ellipticity condition of A, we obtain

$$\|w_\varepsilon\|_{H^1(\Omega)} \le C\varepsilon\big\{\|\nabla_x A\|_\infty \|\nabla u_0\|_{L^2(\Omega)} + \|\nabla^2 u_0\|_{L^2(\Omega\setminus\Omega_{3\varepsilon})}\big\} + C\|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})}. \tag{3.26}$$

This, together with the inequality

$$\|v\|_{L^2(\Omega_t)} \le Ct^{1/2}\|v\|^{1/2}_{L^2(\Omega)}\|v\|^{1/2}_{H^1(\Omega)} \tag{3.27}$$

for $t > 0$ and $v \in H^1(\Omega)$, where Ω_t is defined by (3.12), gives (3.25). $\square$

Remark 3.1. Let Ω be a bounded Lipschitz domain. Let u_ε, u_0 and w_ε be the same as in Theorem 3.1. Observe that

$$\begin{aligned}\|u_\varepsilon - u_0\|_{L^2(\Omega)} &\le \|w_\varepsilon\|_{L^2(\Omega)} + \varepsilon\|S_\varepsilon\big(\eta_\varepsilon \chi^\varepsilon \nabla u_0\big)\|_{L^2(\Omega)}\\ &\le C\|w_\varepsilon\|_{H^1(\Omega)} + C\varepsilon\|\nabla u_0\|_{L^2(\Omega)},\end{aligned}$$

where we have used (2.8). This, together with (3.26), yields

$$\begin{aligned}\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon(\|\nabla_x A\|_\infty + 1)\|\nabla u_0\|_{L^2(\Omega)} &+ C\varepsilon\|\nabla^2 u_0\|_{L^2(\Omega\setminus\Omega_{3\varepsilon})}\\ &+ C\|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})},\end{aligned} \tag{3.28}$$

where C depends only on d, μ and Ω. Estimate (3.28) is not sharp, but will be useful in the proof of Theorems 1.1 and 1.2.

Remark 3.2. Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$. Let w_ε be defined by (3.11), where u_ε and u_0 have the same data F and f. Then

$$\|w_\varepsilon\|_{H^1(\Omega)} \le C\varepsilon^{1/2}\Big\{1 + \|\nabla_x A\|_\infty^{1/2} + \varepsilon^{1/2}\|\nabla_x A\|_\infty\Big\}\left(\|F\|_{L^2(\Omega)} + \|f\|_{H^{3/2}(\partial\Omega)}\right), \tag{3.29}$$

where C depends only on d, μ and Ω. This follows from (3.25), the energy estimate

$$\|u_0\|_{H^1(\Omega)} \le C\left(\|F\|_{L^2(\Omega)} + \|f\|_{H^{1/2}(\partial\Omega)}\right),$$

and the H^2 estimate for $\mathcal{L}_0$,

$$\|u_0\|_{H^2(\Omega)} \le C(\|\nabla_x A\|_\infty + 1)\left(\|F\|_{L^2(\Omega)} + \|f\|_{H^{3/2}(\partial\Omega)}\right), \tag{3.30}$$

where C depends only on d, μ and Ω.

4 Proof of Theorem 1.3

The proof of Theorem 1.3 is based on an approach of homogenization with a parameter. We start with the case $n = 1$ and $A^\varepsilon = A(x, x/\varepsilon)$, considered in the last section.

Lemma 4.1. *Let Ω be a bounded $C^{1,1}$ domain in $\mathbb{R}^d$. Assume that $A = A(x, y)$ is 1-periodic in y and satisfies conditions (1.3) and (3.2). Let u_ε be a weak solution of (1.6), with $\mathcal{L}_\varepsilon = -\text{div}\big(A(x, x/\varepsilon)\nabla\big)$, and u_0 the solution of (1.7) with the same data $F \in L^2(\Omega)$ and $f \in H^{3/2}(\partial\Omega)$. Then*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon\Big\{1 + \|\nabla_x A\|_\infty + \varepsilon\|\nabla_x A\|_\infty^2\Big\}\left(\|F\|_{L^2(\Omega)} + \|f\|_{H^{3/2}(\partial\Omega)}\right) \tag{4.1}$$

for $0 < \varepsilon < 1$, where C depends only on d, n, μ and Ω.

Proof. Let w_ε be given by (3.11). It follows from (2.8) that

$$\|S_\varepsilon(\eta_\varepsilon \chi^\varepsilon \nabla u_0)\|_{L^2(\Omega)} \le C\|\nabla u_0\|_{L^2(\Omega)}.$$

Thus it suffices to show that $\|w_\varepsilon\|_{L^2(\Omega)}$ is bounded by the right-hand side of (4.1). This is done by using (3.13) and a duality argument, as in [26]. Let $A^*(x, y)$ denote the adjoint of $A(x, y)$. Note that $A^*(x, y)$ satisfies the same conditions as $A(x, y)$. We denote the corresponding correctors and flux correctors by $\chi^*(x, y)$ and $\psi^*(x, y)$, respectively. Its matrix of effective coefficients is given by $\widehat{A^*} = (\widehat{A})^*$, the adjoint of $\widehat{A}$.

For $G \in C_c^\infty(\Omega)$, let v_ε be the weak solution of the following Dirichlet problem,

$$\begin{cases} -\text{div}\left(A^*(x, x/\varepsilon)\nabla v_\varepsilon(x)\right) = G & \text{in } \Omega, \\ v_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

and v_0 the homogenized solution. Define

$$\widetilde{w}_\varepsilon(x) = v_\varepsilon - v_0 - \varepsilon S_\varepsilon\big(\widetilde{\eta}_\varepsilon (\chi^*)^\varepsilon \nabla v_0\big),$$

where $(\chi^*)^\varepsilon = \chi^*(x, x/\varepsilon)$ and $\widetilde{\eta}_\varepsilon \in C_0^\infty(\Omega)$ is a cut-off function such that $0 \le \widetilde{\eta}_\varepsilon \le 1$,

$$\widetilde{\eta}_\varepsilon(x) = 1 \text{ in } \Omega \setminus \Omega_{10\varepsilon}, \quad \widetilde{\eta}_\varepsilon(x) = 0 \text{ in } \Omega_{8\varepsilon},$$

and $|\nabla \widetilde{\eta}_\varepsilon| \le C\varepsilon^{-1}$. Note that

$$\begin{aligned} \Big|\int_\Omega w_\varepsilon \cdot G\, dx\Big| &= \Big|\int_\Omega A^\varepsilon(x)\nabla w_\varepsilon \cdot \nabla v_\varepsilon\, dx\Big| \\ &\le \Big|\int_\Omega A^\varepsilon(x)\nabla w_\varepsilon \cdot \nabla \widetilde{w}_\varepsilon\, dx\Big| + \Big|\int_\Omega A^\varepsilon(x)\nabla w_\varepsilon \cdot \nabla v_0\, dx\Big| \\ &\quad + \varepsilon\Big|\int_\Omega A^\varepsilon(x)\nabla w_\varepsilon \cdot \nabla\big[S_\varepsilon\big(\widetilde{\eta}_\varepsilon (\chi^*)^\varepsilon \nabla v_0\big)\big]\, dx\Big| \end{aligned}$$

$$\doteq J_1 + J_2 + J_3. \tag{4.3}$$

We estimate J_1, J_2, and J_3 separately.

By using the Cauchy inequality and (3.29), we obtain

$$\begin{aligned} J_1 &\le C\|\nabla w_\varepsilon\|_{L^2(\Omega)}\|\nabla \widetilde{w}_\varepsilon\|_{L^2(\Omega)} \\ &\le C\varepsilon\Big\{1 + \|\nabla_x A\|_\infty + \varepsilon\|\nabla_x A\|_\infty^2\Big\}\left(\|F\|_{L^2(\Omega)} + \|f\|_{H^{3/2}(\partial\Omega)}\right)\|G\|_{L^2(\Omega)}, \end{aligned} \tag{4.4}$$

where we have also used the estimate

$$\|\widetilde{w}_\varepsilon\|_{H_0^1(\Omega)} \le C\varepsilon^{1/2}\Big\{1 + \|\nabla_x A\|_\infty^{1/2} + \varepsilon^{1/2}\|\nabla_x A\|_\infty\Big\}\|G\|_{L^2(\Omega)}. \tag{4.5}$$

The proof of (4.5) is the same as that of (3.29).

Next, we use (3.13) to obtain

$$\begin{aligned} J_2 \le C\varepsilon\|\nabla v_0\|_{L^2(\Omega)}&\Big\{\|\nabla_x A\|_\infty\|\nabla u_0\|_{L^2(\Omega)} + \|\nabla^2 u_0\|_{L^2(\Omega)}\Big\} \\ &+ C\|\nabla v_0\|_{L^2(\Omega_{5\varepsilon})}\|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})}. \end{aligned} \tag{4.6}$$

Note that by (3.27),

$$\|\nabla v_0\|_{L^2(\Omega_{5\varepsilon})}\|\nabla u_0\|_{L^2(\Omega_{4\varepsilon})} \le C\varepsilon\|\nabla v_0\|_{L^2(\Omega)}^{1/2}\|v_0\|_{H^2(\Omega)}^{1/2}\|\nabla u_0\|_{L^2(\Omega)}^{1/2}\|u_0\|_{H^2(\Omega)}^{1/2}.$$

This, together with (4.6) and the energy estimates and H^2 estimates for $\mathcal{L}_0$ and $\mathcal{L}_0^*$, gives

$$J_2 \le C\varepsilon(1 + \|\nabla_x A\|_\infty)\left(\|F\|_{L^2(\Omega)} + \|f\|_{H^{3/2}(\partial\Omega)}\right)\|G\|_{L^2(\Omega)}. \tag{4.7}$$

The estimate of J_3 is similar to that of J_2. By (3.13) we see that

$$J_3 \le C\varepsilon^2\|\nabla\big[S_\varepsilon\big(\widetilde{\eta}_\varepsilon(\chi^*)^\varepsilon\nabla v_0\big)\big]\|_{L^2(\Omega)}\Big\{\|\nabla_x A\|_\infty\|\nabla u_0\|_{L^2(\Omega)} + \|\nabla^2 u_0\|_{L^2(\Omega)}\Big\},$$

where we have used the fact $\widetilde{\eta}_\varepsilon = 0$ on $\Omega_{8\varepsilon}$. Note that by (2.8),

$$\begin{aligned} &\|\nabla\big[S_\varepsilon\big(\widetilde{\eta}_\varepsilon(\chi^*)^\varepsilon\nabla v_0\big)\big]\|_{L^2(\Omega)} \\ &\le \|S_\varepsilon\big[(\nabla\widetilde{\eta}_\varepsilon)(\chi^*)^\varepsilon\nabla v_0\big]\|_{L^2(\Omega)} + \|S_\varepsilon\big[\widetilde{\eta}_\varepsilon(\nabla_x\chi^*)^\varepsilon\nabla v_0\big]\|_{L^2(\Omega)} \\ &\qquad + \varepsilon^{-1}\|S_\varepsilon\big[\widetilde{\eta}_\varepsilon(\nabla_y\chi^*)^\varepsilon\nabla v_0\big]\|_{L^2(\Omega)} + \|S_\varepsilon\big[\widetilde{\eta}_\varepsilon(\chi^*)^\varepsilon\nabla^2 v_0\big]\|_{L^2(\Omega)} \\ &\le C\varepsilon^{-1}\|\nabla v_0\|_{L^2(\Omega)} + C\|\nabla^2 v_0\|_{L^2(\Omega)}. \end{aligned}$$

It follows that

$$\begin{aligned} J_3 &\le C\varepsilon\Big\{\|\nabla v_0\|_{L^2(\Omega)} + \varepsilon\|\nabla^2 v_0\|_{L^2(\Omega)}\Big\}\Big\{\|\nabla_x A\|_\infty\|\nabla u_0\|_{L^2(\Omega)} + \|\nabla^2 u_0\|_{L^2(\Omega)}\Big\} \\ &\le C\varepsilon(1 + \|\nabla_x A\|_\infty)(1 + \varepsilon\|\nabla_x A\|_\infty)\left(\|F\|_{L^2(\Omega)} + \|f\|_{H^{3/2}(\partial\Omega)}\right)\|G\|_{L^2(\Omega)}. \end{aligned}$$

By combining the estimates of J_1, J_2 and J_3 we obtain

$$\begin{aligned}
&\Big| \int_\Omega w_\varepsilon \cdot G \, dx \Big| \\
&\le C\varepsilon \Big\{ 1 + \|\nabla_x A\|_\infty + \varepsilon \|\nabla_x A\|_\infty^2 \Big\} \big(\|F\|_{L^2(\Omega)} + \|f\|_{H^{3/2}(\partial\Omega)} \big) \|G\|_{L^2(\Omega)},
\end{aligned}$$

from which the desired estimate for w_ε follows by duality. □

We are now in a position to give the proof of Theorem 1.3.

Proof of Theorem 1.3. We prove the theorem by using an induction argument on n. The case $n = 1$ follows directly from Lemma 4.1. Suppose that the theorem is true for some $n - 1$. To prove the theorem for n, let u_ε be a weak solution of the Dirichlet problem (1.6) and u_0 the solution of the homogenized problem (1.7) with the same data (F, f). Let v_ε be the weak solution to

$$-\text{div}\big(A_{n-1}(x, x/\varepsilon_1, \ldots, x/\varepsilon_{n-1}) \nabla v_\varepsilon\big) = F \quad \text{in } \Omega \quad \text{and} \quad v_\varepsilon = f \quad \text{on } \partial\Omega, \tag{4.8}$$

where A_{n-1} is defined by (2.4) with $\ell = n$ and $A_n = A$. Note that

$$\|\nabla_{x,y_1,\ldots,y_{n-2}} A_{n-1}\|_\infty \le C \|\nabla_{x,y_1,\ldots,y_{n-1}} A\|_\infty \le CL.$$

By the induction assumption,

$$\|v_\varepsilon - u_0\|_{L^2(\Omega)} \le C\big\{\varepsilon_1 + \varepsilon_2/\varepsilon_1 + \cdots \varepsilon_{n-1}/\varepsilon_{n-2}\big\}\big\{\|F\|_{L^2(\Omega)} + \|f\|_{H^{3/2}(\partial\Omega)}\big\}, \tag{4.9}$$

where C depends only on d, n, μ, L and Ω.

To bound $\|u_\varepsilon - v_\varepsilon\|_{L^2(\Omega)}$, we use Lemma 4.1. For each $0 < \varepsilon < 1$ fixed, we let

$$E(x, y) = A(x, x/\varepsilon_1, \ldots, x/\varepsilon_{n-1}, y).$$

Then

$$A(x, x/\varepsilon_1, \ldots, x/\varepsilon_n) = E(x, x/\varepsilon_n).$$

Note that

$$\|\nabla_x E\|_\infty \le CL\varepsilon_{n-1}^{-1},$$

where we have used the assumption that $0 < \varepsilon_n < \varepsilon_{n-1} < \cdots < \varepsilon_1 < 1$. By Lemma 4.1, we obtain

$$\begin{aligned}
\|u_\varepsilon - v_\varepsilon\| &\le C\varepsilon_n \left\{1 + \|\nabla_x E\|_\infty + \varepsilon_n \|\nabla_x E\|_\infty^2\right\} \left\{\|F\|_{L^2(\Omega)} + \|f\|_{H^{3/2}(\partial\Omega)}\right\} \\
&\le C\varepsilon_n \left\{1 + L\varepsilon_{n-1}^{-1} + L^2 \varepsilon_n \varepsilon_{n-1}^{-2}\right\} \left\{\|F\|_{L^2(\Omega)} + \|f\|_{H^{3/2}(\partial\Omega)}\right\} \\
&\le C(1 + L)^2 \varepsilon_n \varepsilon_{n-1}^{-1} \left\{\|F\|_{L^2(\Omega)} + \|f\|_{H^{3/2}(\partial\Omega)}\right\}.
\end{aligned}$$

This, together with (4.9), gives (1.14). □

5 Approximation

In preparation for the proofs of Theorems 1.1 and 1.2, we establish several results on the approximation of solutions of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ by solutions of $\mathcal{L}_0(u_0) = F$ in this section. We start with a simple case, where $n = 1$ and $A = A(x, y)$ is Lipschitz continuous in x.

Lemma 5.1. *Suppose $A = A(x, y)$ satisfies (1.3) and is 1-periodic in y. Also assume that $\|\nabla_x A\|_\infty < \infty$. Let $\mathcal{L}_\varepsilon = -\text{div}\big(A(x, x/\varepsilon)\nabla\big)$ and u_ε be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $B_{2r} = B(x_0, 2r)$, where $\varepsilon \le r \le 1$ and $F \in L^2(B_{2r})$. Then there exists a weak solution to $\mathcal{L}_0(u_0) = F$ in B_r such that*

$$\begin{aligned}
&\left(\fint_{B_r} |u_\varepsilon - u_0|^2\right)^{1/2} \\
&\le C\left\{\left(\frac{\varepsilon}{r}\right)^\sigma + \varepsilon\|\nabla_x A\|_\infty\right\}\left\{\left(\fint_{B_{2r}} |u_\varepsilon|^2\right)^{1/2} + r^2\left(\fint_{B_{2r}} |F|^2\right)^{1/2}\right\},
\end{aligned} \tag{5.1}$$

where $\sigma > 0$ and C depends only on d and μ.

Proof. By rescaling we may assume $r = 1$. To see this, we note that if $-\text{div}\big(A(x, x/\varepsilon)\nabla u_\varepsilon\big) = F$ in B_{2r} and $v(x) = u_\varepsilon(rx)$, then $-\text{div}\big(\widetilde{A}(x, x/\delta)\nabla v\big) = G$ in B_2, where $\widetilde{A}(x, y) = A(rx, y)$, $\delta = \varepsilon/r$, and $G(x) = r^2 F(rx)$. Also, observe that $\|\nabla_x \widetilde{A}\|_\infty = r\|\nabla_x A\|_\infty$.

Now, suppose that $-\text{div}\big(A(x, x/\varepsilon)\nabla u_\varepsilon\big) = F$ in B_2. Let $u_0 \in H^1(B_{3/2})$ be the weak solution to

$$\mathcal{L}_0(u_0) = F \quad \text{in } B_{3/2} \quad \text{and} \quad u_0 = u_\varepsilon \quad \text{on } \partial B_{3/2}.$$

Note that $u_0 - u_\varepsilon \in H_0^1(B_{3/2})$ and

$$\mathcal{L}_\varepsilon(u_0 - u_\varepsilon) = \text{div}\big((\widehat{A} - A^\varepsilon)\nabla u_\varepsilon\big) \quad \text{in } B_{3/2}.$$

It follows from the Meyers' estimates that

$$\fint_{B_{3/2}} |\nabla(u_\varepsilon - u_0)|^q \le C \fint_{B_{3/2}} |\nabla u_\varepsilon|^q$$

for some $q > 2$ and $C > 0$, depending only on d and μ. This, together with the Meyers' estimate,

$$\left(\fint_{B_{3/2}} |\nabla u_\varepsilon|^q\right)^{1/q} \le C\left(\fint_{B_2} |u_\varepsilon|^2\right)^{1/2} + C\left(\fint_{B_2} |F|^2\right)^{1/2},$$

gives

$$\left(\fint_{B_{3/2}} |\nabla u_0|^q\right)^{1/q} \le C\left(\fint_{B_2} |u_\varepsilon|^2\right)^{1/2} + C\left(\fint_{B_2} |F|^2\right)^{1/2}. \tag{5.2}$$

Also, by the interior H^2 estimate for $\mathcal{L}_0$,

$$\fint_{B(z,\rho)} |\nabla^2 u_0|^2 \le C \fint_{B(z,2\rho)} |F|^2 + C\big(\|\nabla_x A\|_\infty^2 + \rho^{-2}\big) \fint_{B(z,2\rho)} |\nabla u_0|^2, \tag{5.3}$$

where $B(z,2\rho) \subset B_2$, we may deduce that

$$\begin{aligned} \int_{B_{(3/2)-t}} |\nabla^2 u_0|^2 \, dx &\le C \int_{B_{3/2}} |F|^2 \, dx + C\|\nabla_x A\|_\infty^2 \int_{B_{3/2}} |\nabla u_0|^2 \, dx \\ &\qquad + C \int_{B_{(3/2)-(t/2)}} \frac{|\nabla u_0(x)|^2 \, dx}{|\text{dist}(x, \partial B_{3/2})|^2} \end{aligned} \tag{5.4}$$

for $0 < t < 1$. By Hölder's inequality, the last term in the right-hand side of (5.4) is bounded by

$$C t^{-\frac{2}{q}-1} \left(\int_{B_{3/2}} |\nabla u_0|^q \right)^{2/q}.$$

In view of (5.2) and (5.4) we obtain

$$\int_{B_{(3/2)-t}} |\nabla^2 u_0|^2 \, dx \le C \left\{ t^{-\frac{2}{q}-1} + \|\nabla_x A\|_\infty^2 \right\} \left\{ \fint_{B_2} |F|^2 + \fint_{B_2} |u_\varepsilon|^2 \right\} \tag{5.5}$$

for $0 < t < 1$, where C depends only on d and μ.

Finally, to finish the proof, we use the estimate (3.28) to obtain

$$\begin{aligned} \int_{B_{3/2}} |u_\varepsilon - u_0|^2 \le C\varepsilon^2 (\|\nabla_x A\|_\infty^2 + 1) \int_{B_{3/2}} |\nabla u_0|^2 + C\varepsilon^2 \int_{B_{|x|<\frac{3}{2}-3\varepsilon}} |\nabla^2 u_0|^2 \\ + C \int_{\frac{3}{2}-4\varepsilon<|x|<\frac{3}{2}} |\nabla u_0|^2. \end{aligned}$$

We bound the second term in the right-hand side of the inequality above by using (5.5), and the third term by using Hölder inequality and (5.2). It follows that

$$\int_{B_{3/2}} |u_\varepsilon - u_0|^2 \le C \Big\{ \varepsilon^{1-\frac{2}{q}} + \varepsilon^2 \|\nabla_x A\|_\infty^2 \Big\} \left\{ \int_{B_2} |u_\varepsilon|^2 + \int_{B_2} |F|^2 \right\}.$$

This gives the estimate (5.1) with $r = 1$ and $\sigma = \frac{1}{2} - \frac{1}{q} > 0$. $\square$

The next lemma deals with the case $n = 1$ and $A = A(x,y)$ is Hölder continuous in x,

$$|A(x,y) - A(x',y)| \le L|x - x'|^\theta \quad \text{for any } x, x' \in \mathbb{R}^d, \tag{5.6}$$

where $L \ge 0$ and $\theta \in (0,1)$.

Lemma 5.2. *Suppose $A = A(x, y)$ satisfies (1.3), (5.6), and is 1-periodic in y. Let $\mathcal{L}_\varepsilon = -\text{div}\big(A(x, x/\varepsilon)\nabla\big)$ and u_ε be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $B_{2r} = B(x_0, 2r)$, where $\varepsilon \le r \le 1$ and $F \in L^2(B_{2r})$. Then there exists a weak solution to $\mathcal{L}_0(u_0) = F$ in B_r such that*

$$\begin{aligned} &\left(\fint_{B_r} |u_\varepsilon - u_0|^2\right)^{1/2} \\ &\le C\left\{\left(\frac{\varepsilon}{r}\right)^\sigma + \varepsilon^\theta L\right\}\left\{\left(\fint_{B_{2r}} |u_\varepsilon|^2\right)^{1/2} + r^2\left(\fint_{B_{2r}} |F|^2\right)^{1/2}\right\}, \end{aligned} \tag{5.7}$$

where $\sigma > 0$ depends only on d and μ. The constant C depends only on d, μ and θ.

Proof. As in the proof of Lemma 5.1, by rescaling, we may assume $r = 1$. We also assume that $\varepsilon^\theta L < 1$; for otherwise the inequality is trivial.

By using a convolution in the x variable we may find a matrix $\widetilde{A} = \widetilde{A}(x, y)$ such that $\widetilde{A}$ satisfies the ellipticity condition (1.3), is 1-periodic in y, and

$$\|A - \widetilde{A}\|_\infty \le CL\varepsilon^\theta \quad \text{and} \quad \|\nabla_x \widetilde{A}\|_\infty \le CL\varepsilon^{\theta-1}, \tag{5.8}$$

where C depends only on d and θ. Let v_ε be the weak solution to

$$-\text{div}\big(\widetilde{A}(x, x/\varepsilon)\nabla v_\varepsilon\big) = F \quad \text{in } B_{3/2} \quad \text{and} \quad v_\varepsilon = u_\varepsilon \quad \text{on } \partial B_{3/2}. \tag{5.9}$$

By the energy estimate as well as the first inequality in (5.8),

$$\begin{aligned} \fint_{B_{3/2}} |\nabla(u_\varepsilon - v_\varepsilon)|^2 &\le C(L\varepsilon^\theta)^2 \fint_{B_{3/2}} |\nabla u_\varepsilon|^2 \\ &\le C(L\varepsilon^\theta)^2 \left\{\fint_{B_2} |u_\varepsilon|^2 + \fint_{B_2} |F|^2\right\}, \end{aligned}$$

where we have used the Caccioppoli inequality for the last step. This, together with Poincaré's inequality, gives

$$\left(\fint_{B_{3/2}} |u_\varepsilon - v_\varepsilon|^2\right)^{1/2} \le CL\varepsilon^\theta \left\{\left(\fint_{B_2} |u_\varepsilon|^2\right)^{1/2} + \left(\fint_{B_2} |F|^2\right)^{1/2}\right\}. \tag{5.10}$$

Next, we apply Lemma 5.1 (and its proof) to the operator $-\text{div}\big(\widetilde{A}(x, x/\varepsilon)\nabla\big)$. Let $\widetilde{A}_0(x)$ denote the matrix of effective coefficients for $\widetilde{A}(x, y)$. It follows that there exists $v_0 \in H^1(B_{5/4})$ such that $-\text{div}\big(\widetilde{A}_0(x)\nabla v_0\big) = F$ in $B_{5/4}$, and

$$\begin{aligned} \left(\fint_{B_{5/4}} |v_\varepsilon - v_0|^2\right)^{1/2} &\le C\big\{\varepsilon^\sigma + \varepsilon^\theta L\big\}\left\{\left(\fint_{B_{3/2}} |v_\varepsilon|^2\right)^{1/2} + \left(\fint_{B_{3/2}} |F|^2\right)^{1/2}\right\} \\ &\le C\big\{\varepsilon^\sigma + \varepsilon^\theta L\big\}\left\{\left(\fint_{B_2} |u_\varepsilon|^2\right)^{1/2} + \left(\fint_{B_2} |F|^2\right)^{1/2}\right\}, \end{aligned} \tag{5.11}$$

where we have used the second inequality in (5.8) as well as (5.10).

Finally, let u_0 be the weak solution to $\mathcal{L}_0(u_0) = F$ in B_1 and $u_0 = v_0$ on ∂B_1. Observe that by the first inequality in (5.8),

$$\|\widetilde{A}_0 - \widehat{A}\|_\infty \le C\varepsilon^\theta L,$$

where C depends only on d and μ. It follows that by Poincaré's inequality,

$$\begin{aligned}
\int_{B_1} |u_0 - v_0|^2 &\le C \int_{B_1} |\nabla(u_0 - v_0)|^2 \\
&\le C(\varepsilon^\theta L)^2 \int_{B_1} |\nabla v_0|^2 \\
&\le C(\varepsilon^\theta L)^2 \left\{ \int_{B_{5/4}} |v_0|^2 + \int_{B_2} |F|^2 \right\} \\
&\le C(\varepsilon^\theta L)^2 \left\{ \int_{B_2} |u_\varepsilon|^2 + \int_{B_2} |F|^2 \right\},
\end{aligned}$$

where we have used Cacciopoli's inequality for the third inequality and (5.11) for the fourth. This, together with (5.10) and 5.11), gives (5.7) for $r = 1$. □

We are now ready to handle the general case, where $n \ge 1$ and

$$\mathcal{L}_\varepsilon = -\text{div}\big(A(x, x/\varepsilon_1, \ldots, x/\varepsilon_n)\nabla\big) \tag{5.12}$$

with $0 < \varepsilon_n < \varepsilon_{n-1} < \cdots < \varepsilon_1 < 1$.

Theorem 5.1. *Suppose that $A = A(x, y_1, \ldots, y_n)$ satisfies conditions (1.3), (1.4), and (1.5) for some $\theta \in (0, 1]$ and $L \ge 0$. Let $\mathcal{L}_\varepsilon$ be given by (5.12) and u_ε a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $B_{tr} = B(x_0, tr)$ for some $t > 1$, where $\varepsilon_1 \le r \le 1$ and $F \in L^2(B_{tr})$. Then there exists $u_0 \in H^1(B_r)$ such that $\mathcal{L}_0(u_0) = F$ in B_r and*

$$\begin{aligned}
\left(\fint_{B_r} |u_\varepsilon - u_0|^2 \right)^{1/2} &\le C \left\{ \left(\frac{\varepsilon_1}{r} \right)^\sigma + (\varepsilon_1 + \varepsilon_2/\varepsilon_1 + \cdots + \varepsilon_n/\varepsilon_{n-1})^\theta L \right\} \\
&\qquad \cdot \left\{ \left(\fint_{B_{tr}} |u_\varepsilon|^2 \right)^{1/2} + r^2 \left(\fint_{B_{tr}} |F|^2 \right)^{1/2} \right\},
\end{aligned} \tag{5.13}$$

where $\sigma > 0$ depends only on d and μ. The constant C depends only on d, n, μ, t, and θ.

Proof. We prove the theorem by an induction argument on n. The case $n = 1$ with $t = 2$ is given by Lemma 5.2. The proof for the general case $t > 1$ is similar. Now suppose the theorem is true for $n - 1$. To show it is true for n, let u_ε be a weak solution to $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in B_{tr}, where $\mathcal{L}_\varepsilon$ is given by (5.12). Fix $\varepsilon > 0$ and consider the matrix

$$E(x, y) = A(x, x/\varepsilon_1, \ldots, x/\varepsilon_{n-1}, y).$$

Note that E satisfies the ellipticity condition (1.3) and is 1-periodic in y. Moreover, we have

$$|E(x,y) - E(x',y)| \le C\varepsilon_{n-1}^{-\theta} L|x-x'|^\theta \quad \text{for any } x, x' \in \mathbb{R}^d, \tag{5.14}$$

where C depends only on d and n. Also recall that the matrix of effective coefficients for $E(x,y)$ is given by

$$A_{n-1}(x, x/\varepsilon_1, \cdots, x/\varepsilon_{n-1}),$$

where $A_{n-1}(x, y_1, \cdots, y_{n-1})$ is given by (2.4) with $\ell = n$ and $A_n = A$. Let $1 < s < t$. By the theorem for the case $n = 1$, there exists $v_\varepsilon \in H^1(B_{sr})$ such that

$$-\text{div}\big(A_{n-1}(x, x/\varepsilon_1, \ldots, x/\varepsilon_{n-1})\nabla v_\varepsilon\big) = F \quad \text{in } B_{sr},$$

and

$$\begin{aligned}\left(\fint_{B_{sr}} |u_\varepsilon - v_\varepsilon|^2\right)^{1/2} &\le C\left\{\left(\frac{\varepsilon_n}{r}\right)^\sigma + \left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right)^\theta L\right\} \\ &\quad \cdot \left\{\left(\fint_{B_{tr}} |u_\varepsilon|^2\right)^{1/2} + r^2\left(\fint_{B_{tr}} |F|^2\right)^{1/2}\right\}.\end{aligned} \tag{5.15}$$

By induction assumption there exists $u_0 \in H^1(B_r)$ such that $\mathcal{L}_0(u_0) = F$ in B_r and

$$\begin{aligned}\left(\fint_{B_r} |v_\varepsilon - u_0|^2\right)^{1/2} &\le C\left\{\left(\frac{\varepsilon_1}{r}\right)^\sigma + (\varepsilon_1 + \varepsilon_2/\varepsilon_1 + \cdots + \varepsilon_{n-1}/\varepsilon_{n-2})^\theta L\right\} \\ &\quad \cdot \left\{\left(\fint_{B_{sr}} |v_\varepsilon|^2\right)^{1/2} + r^2\left(\fint_{B_{sr}} |F|^2\right)^{1/2}\right\}.\end{aligned} \tag{5.16}$$

Estimate (5.13) follows readily from (5.15) and (5.16). $\square$

Remark 5.1. Let $\delta = \varepsilon_1 + \varepsilon_2/\varepsilon_1 + \cdots + \varepsilon_n/\varepsilon_{n-1}$. It follows from Theorem 5.1 (with $t = 2$) that for $\delta \le r < 1$,

$$\left(\fint_{B_r} |u_\varepsilon - u_0|^2\right)^{1/2} \le C\left(\frac{\delta}{r}\right)^\sigma \left\{\left(\fint_{B_{2r}} |u_\varepsilon|^2\right)^{1/2} + r^2\left(\fint_{B_{2r}} |F|^2\right)^{1/2}\right\}, \tag{5.17}$$

where $\sigma > 0$ depends only on d, μ and θ. The constant C depends at most on d, n, μ and (θ, L). Suppose $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ satisfies the condition (1.9). Then $\delta \le C\varepsilon_1^\beta$ for some $\beta > 0$ depending only on n and N. This, together with (5.17), implies that for $\varepsilon_1 \le r < 1$,

$$\left(\fint_{B_r} |u_\varepsilon - u_0|^2\right)^{1/2} \le C\left(\frac{\varepsilon_1}{r}\right)^\rho \left\{\left(\fint_{B_{2r}} |u_\varepsilon|^2\right)^{1/2} + r^2\left(\fint_{B_{2r}} |F|^2\right)^{1/2}\right\}, \tag{5.18}$$

where $\rho > 0$ depends only on d, n, μ, θ, and N.

6 Large-scale interior estimates

This section focuses on large-scale interior estimates for $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ and gives the proof of Theorem 1.1. Throughout this section we assume that $\mathcal{L}_\varepsilon$ is given by (1.2) and $A = A(x, y_1, \ldots, y_n)$ satisfies (1.3), (1.4), and (1.5) for some $\theta \in (0, 1]$ and $L \geq 0$. We also assume that $0 < \varepsilon_n < \varepsilon_{n-1} < \cdots < \varepsilon_1 < 1$ and the condition (1.9) of well-separation is satisfied.

We start with estimates of solutions of $\mathcal{L}_0(u_0) = F$. Let $\mathcal{P}$ denote the set of linear functions.

Lemma 6.1. *Let $u_0 \in H^1(B_r)$ be a solution to $\mathcal{L}_0(u_0) = F$ in $B_r = B(0, r)$, where $0 < r \leq 1$ and $F \in L^p(B_r)$ for some $p > d$. Define*

$$G(r; u_0) = \frac{1}{r} \inf_{P \in \mathcal{P}} \left\{ \left(\fint_{B_r} |u_0 - P|^2 \right)^{1/2} + r^{1+\vartheta} |\nabla P| \right\} + r \left(\fint_{B_r} |F|^p \right)^{1/p}, \tag{6.1}$$

where $\vartheta = \min\{\theta, 1 - d/p\}$. Then there exists $t \in (0, 1/8)$, depending only on d, μ, p and (θ, L) in (1.5), such that

$$G(tr; u_0) \leq \frac{1}{2} G(r; u_0).$$

Proof. Let $P_0 = x \cdot \nabla u_0(0) + u_0(0)$. Then

$$\begin{aligned} G(tr; u_0) &\leq \frac{1}{tr} \|u_0 - P_0\|_{L^\infty(B_{tr})} + tr \left(\fint_{B_{tr}} |F|^p \right)^{1/p} + (tr)^\vartheta |\nabla u_0(0)| \\ &\leq (tr)^\vartheta \|\nabla u_0\|_{C^{0,\vartheta}(B_{tr})} + tr \left(\fint_{B_{tr}} |F|^p \right)^{1/p} + (tr)^\vartheta |\nabla u_0(0)| \\ &= (tr)^\vartheta \|\nabla (u_0 - P)\|_{C^{0,\vartheta}(B_{tr})} + tr \left(\fint_{B_{tr}} |F|^p \right)^{1/p} + (tr)^\vartheta |\nabla u_0(0)| \end{aligned} \tag{6.2}$$

for any $P \in \mathcal{P}$. Note that

$$tr \left(\fint_{B_{rt}} |F|^p \right)^{1/p} \leq C t^{1-d/p} r \left(\fint_{B_r} |F|^p \right)^{1/p}. \tag{6.3}$$

By interior Lipschitz estimates for u_0, we may deduce that

$$\begin{aligned} |\nabla u_0(0)| &\leq \frac{C}{r} \left(\fint_{B_r} |u_0 - b|^2 \right)^{1/2} + Cr \left(\fint_{B_r} |F|^p \right)^{1/p} \\ &\leq \frac{C}{r} \left(\fint_{B_r} |u_0 - P|^2 \right)^{1/2} + \frac{C}{r} \left(\fint_{B_r} |P - b|^2 \right)^{1/2} + Cr \left(\fint_{B_r} |F|^p \right)^{1/p} \\ &\leq \frac{C}{r} \left(\fint_{B_r} |u_0 - P|^2 \right)^{1/2} + C |\nabla P| + Cr \left(\fint_{B_r} |F|^p \right)^{1/p}, \end{aligned} \tag{6.4}$$

where $b = P(0)$. Also, note that

$$-\text{div}\big(\widehat{A}\nabla(u_0 - P)\big) = F + \text{div}\big([\widehat{A} - \widehat{A}(0)]\nabla P\big) \quad \text{ in } B_r.$$

By $C^{1,\vartheta}$ estimates for the elliptic operator $\mathcal{L}_0$, we obtain that for $0 < t < 1/2$,

$$\begin{aligned}
\|\nabla(u_0 - P)\|_{C^{0,\vartheta}(B_{tr})} &\le \|\nabla(u_0 - P)\|_{C^{0,\vartheta}(B_{r/2})} \\
&\le \frac{C}{r^{1+\vartheta}} \Big(\fint_{B_r} |u_0 - P|^2 \Big)^{1/2} + C r^{-\vartheta} \|[\widehat{A} - \widehat{A}(0)]\nabla P\|_{L^\infty(B_r)} \\
&\quad + C \|[\widehat{A} - \widehat{A}(0)]\nabla P\|_{C^{0,\vartheta}(B_r)} + C r^{1-\vartheta} \Big(\fint_{B_r} |F|^p \Big)^{1/p} \\
&\le \frac{C}{r^{1+\vartheta}} \Big(\fint_{B_r} |u_0 - P|^2 \Big)^{1/2} + C|\nabla P| + C r^{1-\vartheta} \Big(\fint_{B_r} |F|^p \Big)^{1/p}.
\end{aligned} \tag{6.5}$$

By using (6.3)–(6.5) to bound the right-hand side of (6.2), it yields that

$$G(tr; u_0) \le C t^\vartheta G(r; u_0)$$

for some constant C depending only on d, μ, p and (θ, L) in (1.5). The desired result follows by choosing t so small that $C t^\vartheta \le 1/2$. □

Lemma 6.2. *Let $u_\varepsilon \in H^1(B_1)$ be a solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in B_1, where $F \in L^p(B_1)$ for some $p > d$ and $\varepsilon \in (0, 1/4)$. For $0 < r \le 1$, we define*

$$\begin{aligned}
H(r) &= \frac{1}{r} \inf_{P \in \mathcal{P}} \left\{ \left(\fint_{B_r} |u_\varepsilon - P|^2 \right)^{1/2} + r^{1+\vartheta} |\nabla P| \right\} + r \left(\fint_{B_r} |F|^p \right)^{1/p}, \\
\Phi(r) &= \frac{1}{r} \inf_{b \in \mathbb{R}} \left(\fint_{B_r} |u_\varepsilon - b|^2 \right)^{1/2} + r \left(\fint_{B_r} |F|^2 \right)^{1/2}.
\end{aligned} \tag{6.6}$$

Let $t \in (0, 1/8)$ be given by Lemma 6.1. Then for $r \in (\varepsilon_1, 1/2]$,

$$H(tr) \le \frac{1}{2} H(r) + C \left(\frac{\varepsilon_1}{r} \right)^\rho \Phi(2r), \tag{6.7}$$

where $\rho > 0$ and C depends at most on d, n, μ, p, (θ, L) in (1.5), and N in (1.9).

Proof. For any fixed $r \in (\varepsilon_1, 1/2]$, let u_0 be the solution to $\mathcal{L}_0(u_0) = F$ in B_r, given in Theorem 5.1. By the definitions of G, H and Φ, we have

$$\begin{aligned}
H(tr) &\le \frac{1}{tr} \left(\fint_{B_{tr}} |u_\varepsilon - u_0|^2 \right)^{1/2} + G(tr; u_0) \\
&\le \frac{1}{tr} \left(\fint_{B_{tr}} |u_\varepsilon - u_0|^2 \right)^{1/2} + \frac{1}{2} G(r; u_0)
\end{aligned}$$

$$\leq \frac{C}{r}\left(\fint_{B_r} |u_\varepsilon - u_0|^2\right)^{1/2} + \frac{1}{2}H(r)$$

$$\leq C\left(\frac{\varepsilon_1}{r}\right)^\rho \left\{\frac{1}{r}\left(\fint_{B_{2r}} |u_\varepsilon - b|^2\right)^{1/2} + r\left(\fint_{B_{2r}} |F|^2\right)^{1/2}\right\} + \frac{1}{2}H(r)$$

for any $b \in \mathbb{R}$, where we have used Lemma 6.1 and (5.18) in the second and last inequalities, respectively. $\square$

The following lemma can be found in [23, p.155].

Lemma 6.3. *Let $H(r)$ and $h(r)$ be two nonnegative continuous functions on the interval $(0,1]$ and let $t \in (0,1/4)$. Assume that*

$$\max_{r\leq t\leq 2r} H(t) \leq C_0 H(2r), \qquad \max_{r\leq t,s\leq 2r} |h(t) - h(s)| \leq C_0 H(2r), \tag{6.8}$$

for any $r \in [\delta, 1/2]$, and also

$$H(tr) \leq \frac{1}{2}H(r) + C_0\omega(\delta/r)\left\{H(2r) + h(2r)\right\}, \tag{6.9}$$

for any $r \in [\delta, 1/2]$, where ω is a nonnegative increasing function on $[0,1]$ such that $\omega(0) = 0$ and

$$\int_0^1 \frac{\omega(s)}{s} ds < \infty. \tag{6.10}$$

Then

$$\max_{\delta\leq r\leq 1} \left\{H(r) + h(r)\right\} \leq C\left\{H(1) + h(1)\right\}, \tag{6.11}$$

where C depends only on C_0, θ_0 and ω.

The next lemma gives the large-scale Lipschitz estimate down to the scale ε_1.

Lemma 6.4. *Let $u_\varepsilon \in H^1(B_1)$ be a solution to $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in B_1, where $B_1 = B(x_0, 1)$ and $F \in L^p(B_1)$ for some $p > d \geq 2$. Then for $\varepsilon_1 \leq r < 1$,*

$$\left(\fint_{B_r} |\nabla u_\varepsilon|^2\right)^{1/2} \leq C\left\{\left(\fint_{B_1} |\nabla u_\varepsilon|^2\right)^{1/2} + \left(\fint_{B_1} |F|^p\right)^{1/p}\right\}, \tag{6.12}$$

where C depends only on d, n, μ, p, (θ, L) in (1.5), and N in (1.9).

Proof. By translation we may assume $x_0 = 0$. Let P_r, b_r be a linear function and constant achieving the infimum in (6.6). In particular,

$$H(r) = \frac{1}{r}\left(\fint_{B_r} |u_\varepsilon - P_r|^2\right)^{1/2} + r^\vartheta |\nabla P_r| + r\Big(\fint_{B_r} |F|^p\Big)^{1/p}.$$

Let $h(r) = |\nabla P_r|$. It follows by Poincaré's inequality that

$$\Phi(2r) \le H(2r) + \frac{1}{r}\inf_{b\in\mathbb{R}}\left(\fint_{B_{2r}} |P_{2r} - b|^2\right)^{1/2} \le H(2r) + Ch(2r).$$

This, combined with (6.7), gives (6.9) with $\omega(t) = t^\rho$, which satisfies (6.10).

For $t \in [r, 2r]$, it is obvious that $H(t) \le CH(2r)$. Furthermore, observe that

$$\begin{aligned}
|h(t) - h(s)| = |\nabla(P_t - P_s)| &\le \frac{C}{r}\left(\fint_{B_r} |P_t - P_s|^2\right)^{1/2} \\
&\le \frac{C}{r}\left(\fint_{B_r} |u_\varepsilon - P_t|^2\right)^{1/2} + \frac{C}{r}\left(\fint_{B_r} |u_\varepsilon - P_s|^2\right)^{1/2} \\
&\le \frac{C}{t}\left(\fint_{B_t} |u_\varepsilon - P_t|^2\right)^{1/2} + \frac{C}{s}\left(\fint_{B_s} |u_\varepsilon - P_s|^2\right)^{1/2} \\
&\le C\{H(t) + H(s)\} \\
&\le CH(2r)
\end{aligned}$$

for all $t, s \in [r, 2r]$, which is exactly the condition (6.8).

Thanks to (6.11), we obtain that

$$\begin{aligned}
\frac{1}{r}\inf_{b\in\mathbb{R}}\left(\fint_{B_r} |u_\varepsilon - b|^2\right)^{1/2} &\le H(r) + \frac{1}{r}\inf_{b\in\mathbb{R}}\left(\fint_{B_r} |P_r - b|^2\right)^{1/2} \\
&\le C\{H(r) + h(r)\} \\
&\le C\{H(1) + h(1)\} \\
&\le C\left\{\left(\fint_{B_1} |u_\varepsilon|^2\right)^{1/2} + \left(\fint_{B_1} |F|^p\right)^{1/p}\right\},
\end{aligned} \tag{6.13}$$

for any $r \in [\varepsilon_1, 1/2]$, where for the last step the following observation is used,

$$\begin{aligned}
h(1) &\le C\left(\fint_{B_1} |P_1|^2\right)^{1/2} \\
&\le C\left(\fint_{B_1} |u_\varepsilon - P_1|^2\right)^{1/2} + C\left(\fint_{B_1} |u_\varepsilon|^2\right)^{1/2}
\end{aligned}$$

$$\leq CH(1) + C \left(\fint_{B_1} |u_\varepsilon|^2 \right)^{1/2}.$$

The estimate (6.12) follows readily from (6.13) by Poincaré and Caccioppoli's inequalities. $\square$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. The proof uses an induction on n and relies on Lemma 6.4 and a rescaling argument. The case $n = 1$ follows directly from Lemma 6.4 by translation and dilation. Assume the theorem is true for $n - 1$. Suppose

$$\operatorname{div}\big(A(x, x/\varepsilon_1, \dots, x/\varepsilon_n)\nabla u_\varepsilon\big) = F \quad \text{ in } B_R = B(x_0, R)$$

for some $0 < R \leq 1$. We need to show that

$$\left(\fint_{B_r} |\nabla u_\varepsilon|^2 \right)^{1/2} \leq C \left\{ \left(\fint_{B_R} |\nabla u_\varepsilon|^2 \right)^{1/2} + R \left(\fint_{B_R} |F|^p \right)^{1/p} \right\}, \tag{6.14}$$

for $\varepsilon_n \leq r < R \leq 1$. By translation and dilation we may assume that $x_0 = 0$ and $R = 1$. Note that the case $(1/8) \leq r < R = 1$ is trivial. If $\varepsilon_n < r \leq (1/8)$, we may cover the ball $B(0, r)$ with a finite number of balls $B(x_\ell, \varepsilon_n)$, where $x_\ell \in B(0, r)$. Consequently, it suffices to prove (6.14) for the case $r = \varepsilon_n$ and $R = 1$. We further note that by Lemma 6.4, the estimate (6.14) holds for $r = \varepsilon_1$ and $R = 1$.

To reach the finest scale ε_n, we let $w(x) = u_\varepsilon(\varepsilon_1 x)$. Then

$$-\operatorname{div}\big(E(x, x/(\varepsilon_2\varepsilon_1^{-1}), \dots, x/(\varepsilon_n\varepsilon_1^{-1}))\nabla w\big) = H \quad \text{ in } B_1,$$

where $H(x) = \varepsilon_1^2 F(\varepsilon_1 x)$ and

$$E(x, y_2, \dots, y_n) = A(\varepsilon_1 x, x, y_2, \dots, y_n).$$

Observe that the matrix E satisfies (1.3) and is 1-periodic in $(y_2, \dots, y_n)$. It also satisfies the smoothness condition (1.5) with the same constants θ and L as for A. Furthermore, the $(n-1)$ scales $(\varepsilon_2\varepsilon_1^{-1}, \dots, \varepsilon_n\varepsilon_1^{-1})$ satisfies the condition (1.9) of well-separation. Thus, by the induction assumption,

$$\left(\fint_{B_r} |\nabla w|^2 \right)^{1/2} \leq C \left\{ \left(\fint_{B_1} |\nabla w|^2 \right)^{1/2} + \left(\fint_{B_1} |H|^p \right)^{1/p} \right\}, \tag{6.15}$$

for $r = \varepsilon_n/\varepsilon_1$. By a change of variables it follows that (6.14) holds for $r = \varepsilon_n$ and $R = \varepsilon_1$. This, combined with the inequality for $r = \varepsilon_1$ and $R = 1$, implies that (6.14) holds for $r = \varepsilon_n$ and $R = 1$. The proof is complete. $\square$

Remark 6.1. It follows from the proof of Theorem 1.1 that without the condition (1.9), the estimate (1.10) continues to hold if

$$\varepsilon_1 + (\varepsilon_2/\varepsilon_1 + \cdots + \varepsilon_n/\varepsilon_{n-1})^N \leq r < R \leq 1,$$

for any $N \geq 1$. In this case the constant C in (1.10) also depends on N. The case $N = 1$ follows by using (5.17) in the place of (5.18). The general case is proved by an induction argument on N. Suppose the claim is true for some $N \geq 1$. Assume that $\beta = \varepsilon_2/\varepsilon_1 + \cdots + \varepsilon_n/\varepsilon_{n-1} \geq \varepsilon_1$ (for otherwise, there is nothing to prove). Let $w(x) = u_\varepsilon(\beta x)$. Then $-\text{div}\big(E(x, x/(\beta^{-1}\varepsilon_1), \ldots, x/(\beta^{-1}\varepsilon_n))\nabla w\big) = H$, where $E(x, y_1, \ldots y_n) = A(\beta x, y_1, \ldots, y_n)$. By the induction assumption, the inequality (6.15) holds for $\beta^{-1}\varepsilon_1 + \beta^N < r < 1$. By a change of variables we obtain (1.10) for $\varepsilon_1 + \beta^{N+1} \leq r < R = \beta$. This, together with the estimate for the case $N = 1$, gives (1.10) for $\varepsilon_1 + \beta^{N+1} \leq r < R \leq 1$.

7 Large-scale boundary Lipschitz estimates

This section is devoted to the large-scale boundary Lipschitz estimate and contains the proof of Theorem 1.2. Throughout the section we assume that $\mathcal{L}_\varepsilon$ is given by (1.2) with $A = A(x, y_1, \ldots, y_n)$ satisfying conditions (1.3), (1.4) and (1.5) for some $0 < \theta \leq 1$. The condition (1.9) is also imposed.

Let $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ be a $C^{1,\alpha}$ function with

$$\psi(0) = 0 \quad \text{ and } \quad \|\nabla\psi\|_\infty + \|\nabla\psi\|_{C^{0,\alpha}(\mathbb{R}^{d-1})} \leq M. \tag{7.1}$$

Set

$$\begin{aligned} Z_r &= Z(r, \psi) = \left\{(x', x_d) \in \mathbb{R}^d : |x'| < r \text{ and } \psi(x') < x_d < 10(M+10)r\right\}, \\ I_r &= I(r, \psi) = \left\{(x', \psi(x')) \in \mathbb{R}^d : |x'| < r\right\}. \end{aligned} \tag{7.2}$$

For $f \in C^{1,\alpha}(I_r)$ with $0 < \alpha < 1$, we introduce a scaling-invariant norm,

$$\|f\|_{C^{1,\alpha}(I_r)} = \|f\|_{L^\infty(I_r)} + r\|\nabla_{\tan} f\|_{L^\infty(I_r)} + r^{1+\alpha}\|\nabla_{\tan} f\|_{C^{0,\alpha}(I_r)}, \tag{7.3}$$

where $\nabla_{\tan} f$ denotes the tangential gradient of f and

$$\|g\|_{C^{0,\alpha}(I_r)} = \sup_{x,y \in I_r, x \neq y} \frac{|g(x) - g(y)|}{|x-y|^\alpha}.$$

Theorem 7.1. *Let $u_\varepsilon \in H^1(Z_R)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Z_R and $u_\varepsilon = f$ on I_R, where $0 < \varepsilon_n < R \leq 1$, $F \in L^p(Z_R)$ for some $p > d$, and $f \in C^{1,\alpha}(I_R)$. Then for $\varepsilon_n \leq r < R$,*

$$\left(\fint_{Z_r} |\nabla u_\varepsilon|^2\right)^{1/2} \leq C\left\{\left(\fint_{Z_R} |\nabla u_\varepsilon|^2\right)^{1/2} + R^{-1}\|f\|_{C^{1,\alpha}(I_R)} + R\left(\fint_{Z_R} |F|^p\right)^{1/p}\right\}, \tag{7.4}$$

where C depends at most on d, n, μ, p, (θ, L) in (1.5), N in (1.9), and (α, M) in (7.1).

Theorem 1.2 follows readily from Theorem 7.1 by translation and a suitable rotation of the coordinate system. To prove Theorem 7.1, we use the same approach as in the proof of Theorem 6.4. We will provide only a sketch of the proof for Theorem 7.1.

First, we point out that the rescaling argument, which is used extensively for interior estimates, works equally well in the case of boundary estimates. Indeed, suppose $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in $Z(r,\psi)$ and $u_\varepsilon = f$ on $I(r,\psi)$ for some $0 < r \le 1$. Let $v(x) = u_\varepsilon(rx)$. Then

$$-\mathrm{div}\big(\widetilde{A}(x, x/\varepsilon_1 r^{-1}, \ldots, x/\varepsilon_n r^{-1})\nabla v\big) = G \quad \text{in } Z(1,\psi_r) \quad \text{and} \quad v = g \quad \text{on } I(1,\psi_r),$$

where $\widetilde{A}(x, y_1, \ldots, y_n) = A(rx, y_1, \ldots, y_n)$, $G(x) = r^2 F(rx)$, $g(x) = f(rx)$, and $\psi_r(x') = r^{-1}\psi(rx')$. Since $\nabla\psi_r(x') = \nabla\psi(rx')$ and $0 < r \le 1$, the function ψ_r satisfies the condition (7.1) with the same M. Also, note that $\|f\|_{C^{1,\alpha}(I(r,\psi))} = \|g\|_{C^{1,\alpha}(I(1,\psi_r))}$. As a result, it suffices to prove Theorem 7.1 for $R = 1$.

Next, we establish an approximation result in the place of (5.18). Define

$$\|f\|_{C^1(I_r)} = \|f\|_{L^\infty(I_r)} + r\|\nabla_{\tan} f\|_{L^\infty(I_r)}.$$

Theorem 7.2. *Let $u_\varepsilon \in H^1(Z_{2r})$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Z_{2r} and $u_\varepsilon = f$ on I_{2r}, where $0 < \varepsilon \le r \le 1$. Then there exists $u_0 \in H^1(Z_r)$ such that $\mathcal{L}_0(u_0) = F$ in Z_r, $u_0 = f$ on I_r, and*

$$\begin{aligned} &\left(\fint_{Z_r} |u_\varepsilon - u_0|^2\right)^{1/2} \\ &\le C\left(\frac{\varepsilon_1}{r}\right)^\rho \left\{ \left(\fint_{Z_{2r}} |u_\varepsilon|^2\right)^{1/2} + r^2\left(\fint_{Z_{2r}} |F|^2\right)^{1/2} + \|f\|_{C^1(I_{2r})} \right\}. \end{aligned} \tag{7.5}$$

The constants $\rho \in (0,1)$ and $C > 0$ depend at most on d, n, μ, (θ, L) in (1.5), N in (1.9), and (α, M) in (7.1).

Proof. The proof of (7.5) is similar to that of (5.18).
Step 1. Assume that $n = 1$, $\mathcal{L}_\varepsilon = -\mathrm{div}\big(A(x, x/\varepsilon)\nabla\big)$ and $A(x,y)$ is Lipschitz continuous in x. Suppose that $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Z_{2r} and $u_\varepsilon = f$ on I_{2r}. Show that there exists $u_0 \in H^1(Z_r)$ such that $\mathcal{L}_0(u_0) = F$ in Z_r, $u_0 = f$ on I_r, and

$$\begin{aligned} &\left(\fint_{Z_r} |u_\varepsilon - u_0|^2\right)^{1/2} \\ &\le C\left\{\left(\frac{\varepsilon}{r}\right)^\sigma + \varepsilon\|\nabla_x A\|_\infty\right\} \left\{ \left(\fint_{Z_{2r}} |u_\varepsilon|^2\right)^{1/2} + r^2\left(\fint_{Z_{2r}} |F|^2\right)^{1/2} + \|f\|_{C^1(I_{2r})} \right\}. \end{aligned} \tag{7.6}$$

The proof of (7.6) is similar to (5.1). By rescaling we may assume $r = 1$. Let u_0 be the weak solution of

$$\mathcal{L}_0(u_0) = F \quad \text{in } \Omega \quad \text{and} \quad u_0 = u_\varepsilon \quad \text{on } \partial\Omega,$$

where $\Omega = Z_{3/2}$. By using (3.28), we obtain

$$\int_\Omega |u_\varepsilon - u_0|^2 \le C\varepsilon^2(\|\nabla_x A\|_\infty^2 + 1)\int_\Omega |\nabla u_0|^2 + C\varepsilon^2 \int_{\Omega\setminus\Omega_{3\varepsilon}} |\nabla^2 u_0|^2 + C\int_{\Omega_{4\varepsilon}} |\nabla u_0|^2.$$

The rest of the proof is the same as the proof of Lemma 5.1, using interior H^2 estimates for $\mathcal{L}_0$ as well as Meyers' estimates for $\mathcal{L}_\varepsilon$,

$$\left(\fint_{Z_{3/2}} |\nabla u_\varepsilon|^q\right)^{1/q} \le C\left\{\left(\fint_{Z_2} |u_\varepsilon|^2\right)^{1/2} + \|f\|_{C^1(I_2)} + \left(\fint_{Z_2} |F|^2\right)^{1/2}\right\} \tag{7.7}$$

for some $q > 2$, depending only on d, μ and M.

Step 2. Assume $n = 1$ and $A(x, y)$ is Hölder continuous in x. Suppose $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Z_{2r} and $u_\varepsilon = f$ on I_{2r}. Show that there exists $u_0 \in H^1(Z_r)$ such that $\mathcal{L}_0(u_0) = F$ in Z_r, $u_0 = f$ on I_r, and

$$\begin{aligned}&\left(\fint_{Z_r} |u_\varepsilon - u_0|^2\right)^{1/2}\\ &\le C\left\{\left(\frac{\varepsilon}{r}\right)^\sigma + \varepsilon^\theta L\right\}\left\{\left(\fint_{Z_{2r}} |u_\varepsilon|^2\right)^{1/2} + r^2\left(\fint_{Z_{2r}} |F|^2\right)^{1/2} + \|f\|_{C^1(I_{2r})}\right\}.\end{aligned} \tag{7.8}$$

As in the case of (5.7), the estimate (7.8) follows from (7.6) by approximating $A(x, y)$ in the x variable.

Step 3. As in the interior case, the case $n > 1$ follows from (7.8) by an induction argument on n. $\square$

The following two lemmas will be used in the place of Lemmas 6.1 and 6.2. Recall that $\mathcal{P}$ denotes the set of linear functions in $\mathbb{R}^d$.

Lemma 7.1. *Let $u_0 \in H^1(Z_r)$ be a weak solution of $\mathcal{L}_0(u_0) = F$ in Z_r and $u_0 = f$ on I_r, where $0 < r \le 1$, $F \in L^p(Z_r)$ for some $p > d$, and $f \in C^{1,\alpha}(I_r)$ for some $0 < \alpha < 1$. Define*

$$\begin{aligned}\mathcal{G}(r; u_0) = \inf_{P\in\mathcal{P}} \frac{1}{r}&\left\{\left(\fint_{Z_r} |u_0 - P|^2\right)^{1/2} + r^{1+\vartheta}|\nabla P| + \|f - P\|_{C^{1,\alpha}(I_r)}\right\}\\ &+ r\left(\fint_{Z_r} |F|^p\right)^{1/p},\end{aligned}$$

where $\vartheta = \min\{\theta, \alpha, 1 - d/p\}$. Then there exists $t \in (0, 1/8)$, depending only on d, n, μ, p, (θ, L) in (1.5), and (α, M) in (7.1), such that,

$$\mathcal{G}(tr; u_0) \le \frac{1}{2}\mathcal{G}(r; u_0).$$

Proof. The proof is similar to that of Lemma 6.1. Let $P_0(x) = \nabla u_0(0) \cdot x + u_0(0)$. Then for $0 < t < (1/8)$,

$$\begin{aligned}
\mathcal{G}(tr; u_0) &\le C(tr)^\vartheta \big\{ \|\nabla u_0\|_{C^{0,\vartheta}(Z_{tr})} + |\nabla u_0(0)| \big\} + tr \left(\fint_{Z_{tr}} |F|^p \right)^{1/p} \\
&\le C(tr)^\vartheta \big\{ \|\nabla (u_0 - P)\|_{C^{0,\vartheta}(Z_{tr})} + |\nabla u_0(0) - \nabla P| + |\nabla P| \big\} \\
&\qquad + tr \left(\fint_{Z_{tr}} |F|^p \right)^{1/p}
\end{aligned} \tag{7.9}$$

for any $P \in \mathcal{P}$, where we have used the fact ∇P is constant. Note that

$$-\mathrm{div}\big(\widehat{A} \nabla (u_0 - P)\big) = F + \mathrm{div}\big([\widehat{A} - \widehat{A}(0)] \nabla P \big) \quad \text{in } Z_r.$$

By boundary $C^{1,\vartheta}$ estimates for the operator $\mathcal{L}_0$ in $C^{1,\alpha}$ domains, it follows that for $0 < t < (1/8)$,

$$\begin{aligned}
&\|\nabla (u_0 - P)\|_{C^{0,\vartheta}(Z_{tr})} + \|\nabla (u_0 - P)\|_{L^\infty(Z_{tr})} \\
&\le \|\nabla (u_0 - P)\|_{C^{0,\vartheta}(Z_{r/2})} + \|\nabla (u_0 - P)\|_{L^\infty(Z_{r/2})} \\
&\le \frac{C}{r^{1+\vartheta}} \left(\fint_{Z_r} |u_0 - P|^2 \right)^{1/2} + C|\nabla P| + C r^{1-\vartheta} \left(\fint_{Z_r} |F|^p \right)^{1/p} + \frac{C}{r^{1+\vartheta}} \|f - P\|_{C^{1,\alpha}(I_r)}
\end{aligned} \tag{7.10}$$

for any $P \in \mathcal{P}$. This, together with (7.9), implies that $\mathcal{G}(tr; u_0) \le C t^\vartheta \mathcal{G}(r; u_0)$. To complete the proof, we choose t so small that $C t^\vartheta \le (1/2)$. □

Lemma 7.2. *Let $u_\varepsilon \in H^1(Z_1)$ be a weak solution of $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Z_1 and $u = f$ on I_1, where $0 < \varepsilon < (1/4)$, $F \in L^p(Z_1)$ for some $p > d$ and $f \in C^{1,\alpha}(I_1)$ for some $\alpha > 0$. For $0 < r \le 1$, define*

$$\begin{aligned}
\mathcal{H}(r) = \inf_{P \in \mathcal{P}} \frac{1}{r} &\left\{ \left(\fint_{Z_r} |u_\varepsilon - P|^2 \right)^{1/2} + r^{1+\vartheta} |\nabla P| + \|f - P\|_{C^{1,\alpha}(I_r)} \right\} \\
&\qquad + r \left(\fint_{Z_r} |F|^p \right)^{1/p},
\end{aligned} \tag{7.11}$$

$$\Upsilon(r) = \inf_{b \in \mathbb{R}} \frac{1}{r} \left\{ \left(\fint_{Z_r} |u_\varepsilon - b|^2 \right)^{1/2} + \|f - b\|_{C^{1,\alpha}(I_r)} \right\} + r \left(\fint_{Z_r} |F|^2 \right)^{1/2}.$$

Let $t \in (0, 1/8)$ be given by Lemma 6.1. Then for any $r \in [\varepsilon_1, 1/2]$,

$$\mathcal{H}(tr) \le \frac{1}{2} \mathcal{H}(r) + C \left(\frac{\varepsilon_1}{r} \right)^\rho \Upsilon(2r), \tag{7.12}$$

where $\rho > 0$ and $C > 0$ depends at most on d, n, μ, p, (θ, L) in (1.5), N in (1.9), and (α, M) in (7.1).

Proof. We omit the proof, which is the same as that of Lemma 6.2. □

Proof of Theorem 7.1. With Theorem 7.2, Lemmas 7.1 and 7.2 at our disposal, Theorem 7.1 follows from Lemma 6.3 in the same manner as in the case of Theorem 6.4. We omit the details. □

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