WEAKLY MIXING SMOOTH PLANAR VECTOR FIELD WITHOUT ASYMPTOTIC DIRECTIONS

YURI BAKHTIN AND LIYING LI

ABSTRACT. We construct a planar smooth weakly mixing stationary random vector field with nonnegative components such that, with probability 1, the flow generated by this vector field does not have an asymptotic direction. Moreover, for all individual trajectories, the set of partial limiting directions coincides with those spanning the positive quadrant. A modified example shows that a particle in space-time weakly mixing positive velocity field does not necessarily have an asymptotic average velocity.

1. INTRODUCTION AND THE MAIN RESULTS

Homogenization problems for stochastic Hamilton–Jacobi (HJ) type equations (see [Sou99],[RT00],[NN11],[CS13],[JESVT18]) and limit shape problems in First Passage Percolation (FPP) type models (see a recent book [ADH17]) are tightly related to asymptotic properties of optimal paths in random environments. In several interesting situations where the setup involves stationarity and fast decorrelation of the environment, one can prove that optimal paths solving the control problem in the variational characterization of solutions in the HJ case and the random geodesics in the FPP case have some kind of straightness property (see [LN96],[HN01],[Wüt02],[CP11],[CS13],[BCK14],[Bak16]). In particular, for a one-sided semi-infinite minimizer or geodesic γ, existence of a well-defined asymptotic direction $\lim_{t\to\infty}(\gamma(t)/t)$ has been established for several models. The FPP limit shape and the effective Lagrangian (aka shape function) in stationary control problems are always convex. In the literature cited above, it is shown that stronger assumptions on curvature imply quantitative estimates on deviations from straightness, and it is believed that the asymptotic behavior of these deviations are often governed by KPZ scalings (see, e.g., [BK18]).

It is tempting to conjecture that in a closely related and simpler passive tracer setting where the control is eliminated and the particles simply flow along an ergodic stationary random vector field, each of the resulting trajectories will have an asymptotic direction. However, the generality of this picture is limited, and the main goal of this note is to construct a weakly mixing stationary random field v on $\mathbb{R}^2$ such that none of its integral curves possesses a limiting direction. We refer to Section 4 for reminders on weak mixing and related notions and recall here only that weak mixing is stronger than ergodicity.

The construction in our main result is based on lifting the discrete $\mathbb{Z}^2$-ergodic example recently introduced in [CK19] onto $\mathbb{R}^2$ with the help of appropriate tilings, smoothing, and additional randomizations. Although the method seems natural and general to us, we are not able to locate analogs in the existing literature. In fact, the Poissonization that we use can also be used to construct a $\mathbb{Z}^2$-weak mixing example out of the ergodic example of [CK19].

Our vector field (along with the discrete arrow field of [CK19]) traps the integral curves in long corridors each stretched along one of the two prescribed extreme directions, and the random length of these corridors has heavy tails. The result is that the integral curves oscillate between these two directions never settling on any specific one. This idea is similar to that of [HM95], where it is shown how to construct an FPP model with any given convex symmetric limit shape, by employing long random properly directed corridors that are easy to percolate along. As noted in [CK19], the absence of a well-defined average velocity is a manifestation of the fact that there is no averaging of the environment as seen from the particle moving along the random realization of the vector field.

Let us be more precise now. For every bounded smooth vector field v on $\mathbb{R}^2$ and every initial condition $z \in \mathbb{R}^2$, we can define the integral curve $\gamma_z : \mathbb{R}_+ \to \mathbb{R}^2$ (here $\mathbb{R}_+ = [0, \infty)$) as a unique solution of the autonomous ODE

$$\dot{\gamma}_z(t) = v\big(\gamma_z(t)\big), \tag{1.1}$$

satisfying

$$\gamma_z(0) = z. \tag{1.2}$$

We denote the two components of $v \in \mathbb{R}^2$ by v^1 and v^2.

Theorem 1.1. *There is a weakly mixing stationary random vector field $v \in C^\infty(\mathbb{R}^2)$ such that for all $z \in \mathbb{R}$, $v^1(z), v^2(z) \geq 0$, $v^1(z) + v^2(z) > 0$, and with probability 1 the following holds for all $z \in \mathbb{R}^2$:*

$$\lim_{t\to\infty} |\gamma_z(t)| = \infty, \tag{1.3}$$

$$\liminf_{t\to\infty} \frac{\gamma_z^2(t)}{\gamma_z^1(t)} = 0, \quad \limsup_{t\to\infty} \frac{\gamma_z^2(t)}{\gamma_z^1(t)} = \infty. \tag{1.4}$$

In other words, the random vector field that we construct in this theorem generates integral curves γ_z that do not have a well-defined asymptotic average slope because their finite time horizon slopes oscillate between 0 and ∞.

One can interpret our result in variational terms. The integral curves generated by the vector field v are minimizers of the action associated with the Lagrangian given by $L(\gamma, \dot{\gamma}) = |\dot{\gamma} - v(\gamma)|^2$. They are also the characteristics for the corresponding HJ equation. Our result means that the infinite one-side minimizers of the action associated with this Lagrangian have no asymptotic slope.

Using Theorem 1.1, we can also construct a time-dependent space-time stationary, weakly mixing, and smooth one-dimensional positive velocity field $u(t, x) \in \mathbb{R}$, $(t, x) \in \mathbb{R} \times \mathbb{R}$ that does not give rise to a well-defined asymptotic speed.

Given a velocity field u and a starting point (t_0, x_0), we define $(x_{(t_0,x_0)}(t))_{t\geq t_0}$ via

$$\dot{x}_{(t_0,x_0)}(t) = u(t, x_{(t_0,x_0)}(t)), \quad t \geq t_0, \tag{1.5}$$

$$x_{(t_0,x_0)}(t_0) = x_0. \tag{1.6}$$

Theorem 1.2. *If $0 \leq u_0 < u_1 < \infty$, then there is a weakly mixing space-time stationary random velocity field $u \in C^\infty(\mathbb{R} \times \mathbb{R})$ and such that $u(t, x) \in [u_0, u_1]$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}$ and, with probability 1, for every (t_0, x_0) the trajectory $x_{(t_0,x_0)}$ solving (1.5) –(1.6) satisfies*

$$\liminf_{t\to+\infty} \frac{x_{(t_0,x_0)}(t)}{t} = u_0, \qquad \limsup_{t\to+\infty} \frac{x_{(t_0,x_0)}(t)}{t} = u_1. \tag{1.7}$$

We can construct a velocity field u with properties claimed in Theorem 1.2 using the vector field v constructed in Theorem 1.1 and pushing it forward by the linear map defined by the invertible matrix

$$A = \begin{pmatrix} 1 & 1 \\ u_0 & u_1 \end{pmatrix}$$

mapping the horizontal and vertical axes into lines defined by $x = u_0 t$ and $x = u_1 t$. More precisely, we first define

$$\hat{v}(t, x) = Av\left(A^{-1}\begin{pmatrix} t \\ x \end{pmatrix}\right), \quad (t, x) \in \mathbb{R} \times \mathbb{R}.$$

The flow lines generated by $\hat{v}$ on the (t, x)-plane are the images of the flow lines generated by v, so they oscillate between the images of the horizontal and vertical axes. To interpret the first coordinate as time, we must reparametrize the flow lines so that the motion along the t-axis happens with speed 1. Thus, for Theorem 1.2 we define u as the second component of $\hat{v}$ normalized by $\hat{v}^1$:

$$u(t, x) = \hat{v}^2(t, x)/\hat{v}^1(t, x).$$

This normalization does not affect the weak mixing property.

The rest of this note is organized as follows. We extend the construction in [CK19] in two stages. First, in Section 2, we introduce two vector fields on the unit square associated with vertical and horizontal arrows, respectively, and then obtain a smooth vector field on $\mathbb{R}^2$ tesselating it by square tiles in agreement with the random arrow fields introduced in [CK19]. The trajectories generated by the resulting vector field do not have asymptotic directions but the vector field itself lacks stationarity with respect to shifts in $\mathbb{R}^2$, so one cannot even speak about weak mixing. To fix this and finish the proof, a random Poissonian deformation of this vector field is introduced in Section 3.

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2. Constructing a smooth vector field from an arrow field on $\mathbb{Z}^2$

Let $r = (1, 0)$ and $u = (0, 1)$ be the standard coordinate vectors on the plane pointing right and up, respectively. On $\mathbb{Z}^2$, an (up-right) arrow field is a function $\alpha : \mathbb{Z}^2 \to \{r, u\}$, and the random walk $X_z : \mathbb{N} \to \mathbb{Z}^2$ that starts at z and follows the arrow field α is defined by

$$X_z(0) = z, \quad X_z(n) = X_z(n-1) + \alpha\big(X_z(n-1)\big).$$

In [CK19], the authors constructed an ergodic up-right random walk on $\mathbb{Z}^2$ such that no trajectories have asymptotic directions, and hence by the result therein all random walks must coalesce. More precisely, they proved the following:

Theorem 2.1. *There is a $\mathbb{Z}^2$-ergodic dynamical system $((T_z)_{z\in\mathbb{Z}^2}, \Omega, \mathcal{F}, \nu)$ and a measurable function $\bar{\alpha} : \Omega \to \{r, u\}$ that defines a stationary $\mathbb{Z}^2$-arrow field by*

$$\alpha^\omega(z) = \bar{\alpha}(T_z\omega), \quad \omega \in \Omega, \quad z \in \mathbb{Z}^2,$$

such that none of the corresponding family of random walks $(X_z^\omega)_{z\in\mathbb{Z}^2}$ have an asymptotic direction and all the random walks $(X_z^\omega)_{z\in\mathbb{Z}^2}$ coalesce. More precisely, *for ν-a.e. $\omega \in \Omega$,*

$$\liminf_{n\to\infty} \frac{X_z^\omega(n)\cdot u}{X_z^\omega(n)\cdot r} = 0, \quad \limsup_{n\to\infty} \frac{X_z^\omega(n)\cdot u}{X_z^\omega(n)\cdot r} = \infty, \qquad z \in \mathbb{Z}^2, \tag{2.1}$$

and

$$\forall z_1, z_2 \in \mathbb{Z}^2,\ \exists k_1, k_2 \text{ such that } X_{z_1}^\omega(k_1) = X_{z_2}^\omega(k_2). \tag{2.2}$$

In fact, the authors in [CK19] constructed the $\mathbb{Z}^2$-system as the product of two appropriately chosen $\mathbb{Z}$-systems $(S_1, X, \mathcal{B}, \lambda)$ and $(S_2, Y, \mathcal{B}, \lambda)$, with $X = Y = [0,1)$, $\mathcal{B}$ being the Borel σ-algebra, and λ the Lebesgue measure. The product $\mathbb{Z}^2$-action is defined by $T_{(a,b)}(x,y) = (S_1^a x, S_2^b y)$. This $\mathbb{Z}^2$-system is weakly mixing since both $\mathbb{Z}$-systems are. (See Section 4 for a collection of definitions and statements in ergodic theory that will be used in this paper.)

In this section, we will demonstrate how to create a smooth vector field Ψ_α from any given up-right $\mathbb{Z}^2$-arrow field α, such that the integral curves of Ψ_α have similar behavior as the random walks following the arrow field α. When α is given by Theorem 2.1, Ψ_α will satisfy (1.4) (Theorem 2.2).

Suppose V_u and V_r are two smooth fixed vector fields on $[0,1]^2$ roughly behaving like "up arrow" and "right arrow" that will be specified later. The vector field Ψ_α, as a functional of α, is defined by piecing together copies of V_r and V_u:

$$\Psi_\alpha(x+i, y+j) = V_{\alpha(i,j)}(x,y), \quad (i,j) \in \mathbb{Z}^2, (x,y) \in [0,1)^2. \tag{2.3}$$

Naturally, we assume that V_u and V_r are diagonally symmetric to each other, i.e.,

$$\big(V_u \circ \tau\big)(z) = \big(\tau \circ V_r\big)(z), \quad z \in [0,1]^2, \tag{2.4}$$

where $\tau\big((x,y)\big) = (y,x)$ is the reflection w.r.t. the diagonal $\{x = y\}$. To simplify the construction, we also require that that V_r (and hence V_u) is itself diagonally symmetric near the boundary, that is, there exists $\delta > 0$ such that

$$\big(V_r \circ \tau\big)(z) = \big(\tau \circ V_r\big)(z), \quad z \in \Gamma_\delta, \tag{2.5}$$

where for $h \ge 0$, Γ_h is the region

$$\Gamma_h = \big\{(x,y) \in [0,1]^2 : \min\{x, 1-x, y, 1-y\} \le h\big\}, \quad h \ge 0.$$

The construction of V_r and V_u is as follows. Let us take any $\delta < 1/10$. Let $\tilde{F}_r$ be a potential function in $[0,1]^2$ as defined in Fig 1. The potential $\tilde{F}_r$ is a piece-wise linear function so that $\nabla \tilde{F}_r$ is constant in each polygon region. At the four (dotted) pentagon regions at the corners $\tilde{F}_r$ is given by the following:

$$\tilde{F}_r(x,y) = \begin{cases} 3(x+y), & (x,y) \text{ at the SW corner,} \\ 3(x+y) - 1, & (x,y) \text{ at the SE and NW corners} \\ 3(x+y) - 2, & (x,y) \text{ at the NE corner.} \end{cases},$$

And at the middle (shaded) non-convex pentagon $\tilde{F}_r(x,y) = \frac{2-12\delta}{1-4\delta}(x - 2\delta) + (1 + 6\delta)$. The values of $\tilde{F}_r$ at all the vertices are then determined, given in boldface, and $\tilde{F}_r$ in the remaining triangle and rectangle regions are given by the linear interpolation of its values at the vertex.

We extend $\tilde{F}_r$ to $\mathbb{R}^2$ by

$$\tilde{F}_r(x+i, y+j) = \tilde{F}_r(x,y) + 2(i+j), \quad (i,j) \in \mathbb{Z}^2, (x,y) \in [0,1)^2, \tag{2.6}$$

and then by smoothing it we define $F_r = \eta * \tilde{F}_r$, where $\eta \in C^\infty$ is a radially symmetric kernel supported on $B_0(\delta) = \{(x,y) : x^2 + y^2 \le \delta^2\}$. Finally, we define V_r as the restriction of the gradient field ∇F_r to $[0,1]^2$:

$$V_r(x,y) = (\nabla F_r)(x,y), \quad (x,y) \in [0,1]^2.$$

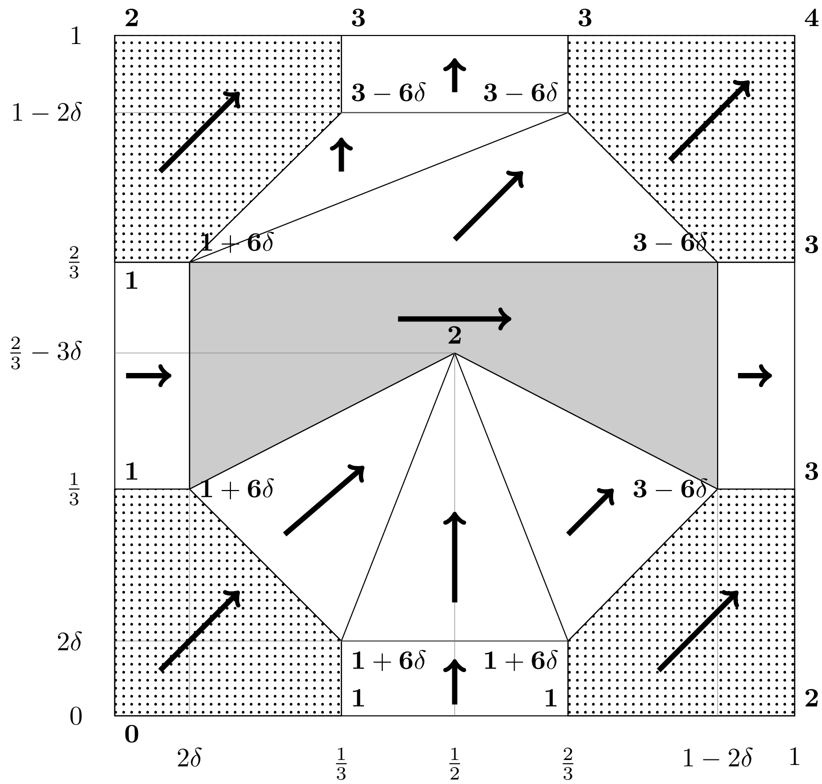


FIGURE 1. Definition of $\tilde{F}_r$ in the unit square $[0,1]^2$. This potential is continuous on $[0,1]^2$ and linear in every polygonal cell. The values of $\tilde{F}_r$ at the tesselation vertices are given in boldface. The arrows indicate the direction of $\nabla\tilde{F}_r$.

We define V_u through diagonal symmetry (2.4).

Lemma 2.1. *Let V_r and V_u be defined as above. For any arrow field α, the vector field Ψ_α as defined in (2.3) is smooth and bounded. Moreover,*

$$\Psi^1_\alpha \geq 0, \quad \Psi^2_\alpha \geq 0, \quad \Psi^1_\alpha + \Psi^2_\alpha \geq c > 0, \tag{2.7}$$

for some constant c.

PROOF: By (2.6), $\nabla\tilde{F}_r$ is $\mathbb{Z}^2$-periodic, i.e.,

$$\nabla\tilde{F}_r(x+i, y+j) = \nabla\tilde{F}_r(x,y), \quad (i,j) \in \mathbb{Z}^2,$$

Hence $\nabla F_r = \eta * \nabla\tilde{F}_r$ is also $\mathbb{Z}^2$-periodic. This implies $\nabla F_r = \Psi_{\alpha_r}$, where α_r is the $\mathbb{Z}^2$-arrow field with right arrows only. From the $\mathbb{Z}^2$-periodicity of $\nabla\tilde{F}_r$ and Fig. 1, it is also easy to see that

$$\tau\big(\nabla\tilde{F}_r(z)\big) = \nabla\tilde{F}_r\big(\tau(z)\big), \quad z \in \bar{\Gamma}_{2\delta},$$

where

$$\bar{\Gamma}_h = \bigcup_{(i,j)\in\mathbb{Z}^2} \{(x+i, y+j) : (x,y) \in \Gamma_h\}, \quad h \geq 0.$$

Since the smoothing kernel η is supported on $B_0(\delta)$ and satisfies $\eta(z) = \eta\big(\tau(z)\big)$ due to the radial symmetry, $\nabla F_r = \eta * \nabla \tilde{F}_r$ will satisfy

$$\nabla F_r\big(\tau(z)\big) = \tau\big(\nabla F_r(z)\big), \quad z \in \bar{\Gamma}_\delta.$$

Therefore, V_r satisfies (2.5).

Let α be any arrow field. Due to (2.5), we have $\Psi_\alpha = \Psi_{\alpha_r}$ in $\bar{\Gamma}_\delta$, which implies that Ψ_α is smooth in a neighborhood of $\bar{\Gamma}_0$. Since, in addition, V_r and V_u are smooth in $(0,1)^2$, Ψ_α is smooth everywhere.

Finally, the condition (2.7) holds for Ψ since it holds for $\nabla \tilde{F}_r$. □

It is also easy to see that we have the following corollary:

Corollary 2.1. *For any arrow field α, there is a potential F_α such that $\Psi_\alpha = \nabla F_\alpha$.*

Theorem 2.2. *Let α be the stationary arrow field introduced in Theorem 2.1 and Ψ_α be the corresponding vector field defined by (2.3). Then, with probability one, all integral curves γ_z of Ψ_α will satisfy (1.4).*

Proof: By Lemma 2.1, Ψ_α is smooth, bounded and nondegenerate, so the integral curves of Ψ_α are well-defined.

We can partition $\mathbb{R}^2$ into the union of unit squares:

$$\mathbb{R}^2 = \bigcup_{(i,j)\in\mathbb{Z}^2} S_{(i,j)}, \quad S_{(i,j)} = [i, i+1) \times [j, j+1).$$

We say that $z \in S_{(i,j)}$ is regular, if the curve γ_z visit these squares in the order given by the random walks $X_{(i,j)}$. It suffices to show that with probability one, every curve of Ψ_α passes through a regular point. The conclusion of the theorem follows from (2.1).

We notice that $e^2 \cdot V_r(x,y) \equiv 0$ in the strip

$$\{(x,y) : \ 0 \le x \le 1, \ 2/3 - 2\delta \le y \le 2/3 - \delta\}.$$

This follows from the fact that $e^2 \cdot \nabla \tilde{F}_r \equiv 0$ in the strip

$$\{(x,y) : x \in \mathbb{R}, 2/3 - 3\delta \le y \le 2/3\}$$

and that η is a kernel supported on $B_0(\delta)$. Therefore, all the integral curves of V_r entering the unit square through the set

$$s_1 = \{(0,y) : 0 \le y \le 2/3 - \delta\} \cup \{(x,0) : 0 \le x \le 1\}$$

have to exit through

$$s_2 = \{(1,y) : 0 \le y \le 2/3 - \delta\}.$$

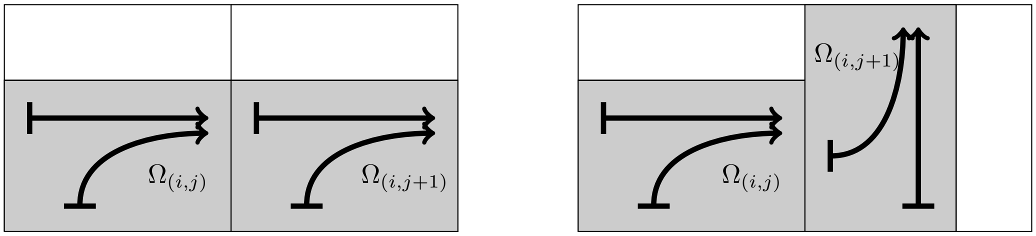
Let us define $\Omega_{(i,j)} \subset S_{(i,j)}$ to be

$$\Omega_{(i,j)} = \begin{cases} \{(x,y) : i \le x < i+1, j \le y \le j + 2/3 - \delta\}, & \alpha(i,j) = r, \\ \{(x,y) : i \le x \le i + 2/3 - \delta, j \le y < j+1\}, & \alpha(i,j) = u. \end{cases}$$

We now claim that any point in $\Omega = \bigcup_{(i,j)\in\mathbb{Z}^2} \Omega_{(i,j)}$ is regular.

Suppose $(i_0, j_0) \in \mathbb{Z}^2$ and $z \in \Omega_{(i_0,j_0)}$. If $\alpha(i_0,j_0) = r$, then our construction implies that after exiting $S(i_0,j_0)$, γ_z enters $\Omega_{(i_0+1,j_0)} \subset S_{(i_0+1,j_0)}$. If $\alpha(i_0,j_0) = u$, then after exiting $S(i_0,j_0)$, γ_z enters $\Omega_{(i_0,j_0+1)} \subset S_{(i_0,j_0+1)}$, see Fig. 2. Applying these steps inductively, we see that γ_z indeed "follows the arrows", so z is regular. This proves the claim.

Furthermore, since all walks coalesce due to Theorem 2.1, any up-right curve (i.e., $\gamma(t)$ such that $\gamma'(t) \cdot r \ge 0, \gamma'(t) \cdot u \ge 0, \gamma'(t) \cdot (r+u) > 0$) must intersect Ω. This implies that any integral curve of Ψ_α passes through some regular point. The proof is complete. □



(A) Case 1: $\alpha(i,j+1)=r$ (B) Case 2: $\alpha(i,j+1)=u$

FIGURE 2. Illustration of the flow when $\alpha(i,j)=r$.

3. Weakly mixing vector field

The vector field Ψ_α constructed in the previous section has all the properties that are required in Theorem 1.1 except $\mathbb{R}^2$-stationarity and weak mixing, although its distribution is invariant under $\mathbb{Z}^2$-shifts. The goal of this section is to modify the vector field and gain those properties.

To obtain an $\mathbb{R}^2$-stationary and ergodic random vector field without requiring the weak mixing property, we could introduce a simple randomization by adding an independent $[0,1]^2$-uniformly distributed random shift to Ψ_α. To obtain a weakly mixing vector field we need to apply an additional random deformation that we proceed to describe.

Let $\mu=\sum_i \delta_{a_i}$ and $\nu=\sum_j \delta_{b_j}$ be two Poissonian point processes on $\mathbb{R}$. They can be regarded as elements of $\mathcal{M}$, the space of locally finite configurations of points on $\mathbb{R}$ (which can be identified with integer-valued measures such that masses of all atoms equal 1) equipped with appropriate topology. We also fix a family of positive C^∞-functions $(\phi_\Delta)_{\Delta>0}$ with the following properties:

1. $\phi_\Delta(x)\equiv 1$ near $x=0$ and $x=\Delta$,
2. $\int_0^\Delta \phi_\Delta(x)\,dx=1$,
3. $(\Delta,x)\mapsto\phi_\Delta(x)$ is continuous (and hence measurable).

We define

$$\varphi_{\mu,\nu}(x,y) \;=\; \Big(\mu((0,x]) + \int_0^{x-\underline{a}} \phi_{\bar a-\underline{a}}(t)\,dt,\quad \nu((0,y]) + \int_0^{y-\underline{b}} \phi_{\bar b-\underline{b}}(t)\,dt\Big),$$

where

$$\bar a=\bar a(x)=\inf\{a_i : a_i>x\}, \qquad \underline{a}=\underline{a}(x)=\sup\{a_i : a_i\le x\},$$
$$\bar b=\bar b(y)=\inf\{b_j : b_j>x\}, \qquad \underline{b}=\underline{b}(y)=\sup\{b_j : b_j\le x\},$$

and $\mu((0,x])$ (resp. $\nu((0,y])$) is the number of Poissonian points in the interval $(0,x]$ (resp. $(0,y]$), with a "$-$" sign if $x<0$ (resp. $y<0$). Let us order the Poisson points in the following way:

$$a:\cdots<a_{-1}<a_0\le 0<a_1<\cdots, \quad b:\cdots<b_{-1}<b_0\le 0<b_1<\cdots.$$

Lemma 3.1. *The map $\varphi_{\mu,\nu}$ is a C^∞-automorphism of $\mathbb{R}^2$ and satisfies*

$$\varphi_{\mu,\nu}(\{a_i\}\times\mathbb{R})=\{i\}\times\mathbb{R}, \quad \varphi_{\mu,\nu}(\mathbb{R}\times\{b_j\})=\mathbb{R}\times\{j\}, \qquad i,j\in\mathbb{Z}. \tag{3.1}$$

In particular, $\varphi_{\mu,\nu}$ maps the rectangle $R_{(i,j)}=[a_i,a_{i+1})\times[b_j,b_{j+1})$ to the unit square $S_{(i,j)}$. Moreover, the map $(\mu,\nu,x,y)\mapsto\varphi_{\mu,\nu}(x,y)$ is measurable from $\mathcal{M}^2\times\mathbb{R}^2$ to $\mathbb{R}^2$.

PROOF: We will show that the first coordinate $\varphi^1_{\mu,\nu}$ is a strictly increasing smooth function on $\mathbb{R}$ and $\varphi^1_{\mu,\nu}(a_i) = i$, and that the map $(\mu,\nu,x,y) \mapsto \varphi^1_{\mu,\nu}(x,y)$ is measurable. Similar statements hold for $\varphi^2_{\mu,\nu}$. These will prove the lemma.

From definition we have $\varphi^1_{\mu,\nu}(a_i) = i$, and that $\varphi^1_{\mu,\nu}$ is continuous and strictly increasing on $[a_i, a_{i+1})$. The left-continuity of $\varphi^1_{\mu,\nu}$ at a_i is guaranteed by the second condition of ϕ_Δ, so $\varphi^1_{\mu,\nu}$ is indeed continuous on $\mathbb{R}$.

In each interval (a_i, a_{i+1}), $\varphi^1_{\mu,\nu}$ is C^∞ since $\phi_\Delta(\cdot)$ are smooth. In the neighborhood of each a_i, $\varphi^1_{\mu,\nu}$ is a linear function with slope 1 due to the first condition of ϕ_Δ. This proves the smoothness of $\varphi^1_{\mu,\nu}(\cdot)$.

Lastly, we notice that the map

$$(\mu,\nu,x,y) \mapsto \Big(\mu((0,x]), \bar{a}(x), \underline{a}(x)\Big)$$

is measurable. The measurability statement follows from this and the third condition on ϕ_Δ. □

Let us consider the pushforward of Ψ_α under the map φ^{-1}, i.e., the vector field

$$\Phi(\mathbf{x}) = D\varphi^{-1}_{\mu,\nu}\big(\varphi(\mathbf{x})\big) \cdot \Psi_\alpha\big(\varphi_{\mu,\nu}(\mathbf{x})\big) = \Big(D\varphi_{\mu,\nu}(\mathbf{x})\Big)^{-1} \Psi_\alpha\big(\varphi_{\mu,\nu}(\mathbf{x})\big), \quad \mathbf{x} \in \mathbb{R}^2,$$

where Df denotes the Jacobian matrix of f and Ψ_α is introduced in section 2. Due to (3.1), in each rectangle $R_{(i,j)}$, the vector field Φ is a "deformation" of either V_r or V_u, depending on whether $\alpha(i,j) = u$ or r.

We will show that if α, μ and ν are independent, then Φ is stationary and weakly mixing. We start by a formal construction of an appropriate $\mathbb{R}^2$-system. Let $((L_v)_{v\in\mathbb{R}}, \mathcal{M}, \mathrm{P}_\mathcal{M})$ be a $\mathbb{R}^1$-system where $\mathcal{M}$ is introduced as above, $\mathrm{P}_\mathcal{M}$ is the Poisson measure on $\mathcal{M}$ with intensity 1, and the $\mathbb{R}^1$-action L_v acting on $\mu = \sum \delta_{a_i}$ by $L_v\mu = \sum \delta_{a_i - v}$. We also recall the $\mathbb{Z}^1$-systems (S_1, X, λ) and (S_2, Y, λ) from Section 2. Let us consider the following skew-products

$$((L_v)_{v\in\mathbb{R}}, \mathcal{M}\times X, \mathrm{P}_\mathcal{M}\otimes\lambda), \qquad L_v(\mu,x) = (L_v\mu, S_1^{\mu((0,v])}x), \tag{3.2}$$

and

$$((L_v)_{v\in\mathbb{R}}, \mathcal{M}\times Y, \mathrm{P}_\mathcal{M}\otimes\lambda), \qquad L_v(\nu,y) = (L_v\nu, S_2^{\nu((0,y])}y). \tag{3.3}$$

Let us take the product of (3.2) and (3.3):

$$\big((\hat{L}_{v,w})_{(v,w)\in\mathbb{R}^2}, \hat{\Omega}, \mathsf{P}\big) = \big((L_v \times L_w)_{(v,w)\in\mathbb{R}^2}, \mathcal{M}^2 \times X \times Y, \mathrm{P}^2_\mathcal{M} \otimes \lambda^2\big). \tag{3.4}$$

For $\hat{\Omega} \ni \hat{\omega} = (\mu,\nu,x,y)$, one can check that the vector field Φ satisfies

$$\Phi^{\hat{\omega}}(v,w) = \Big(D\varphi_{\mu,\nu}(v,w)\Big)^{-1} \Psi_{\alpha(x,y)}\big(\varphi_{\mu,\nu}(v,w)\big) = \hat{\alpha}(\hat{L}_{v,w}\ \hat{\omega}), \tag{3.5}$$

where

$$\hat{\alpha}(\mu,\nu,x,y) = \Big(D\varphi_{\mu,\nu}(0,0)\Big)^{-1} V_{\bar{\alpha}(x,y)}(\varphi_{\mu,\nu}(0,0)).$$

The definition (3.5) implies that Φ is stationary. The following theorem states that it is weakly mixing.

Theorem 3.1. *The $\mathbb{R}^2$-system (3.4) is weakly mixing. Moreover, with probability one, all integral curves of the vector field $\Phi^{\hat{\omega}}$ satisfy (1.4).*

The fact that (3.4) is weakly mixing is implied by the following and Theorem 4.2.

Lemma 3.2. *The $\mathbb{R}^1$-systems (3.2) and (3.3) are weakly mixing.*

Proof: We will only show that (3.2) is weakly mixing. By Definition 4.2, this is equivalent to the ergodicity of its direct product with itself, i.e., the $\mathbb{R}^1$-system

$$((L_v^2)_{v\in\mathbb{R}}, \mathcal{M}^2 \times X^2, \mathrm{P}_{\mathcal{M}}^2 \otimes \lambda^2). \tag{3.6}$$

For $(\mu, \mu', x, x') \in \mathcal{M}^2 \times X^2$, let us write $L_v^2(\mu, \mu', x, x') = (\mu_v, \mu'_v, x_v, x'_v)$. We notice that under the measure $\mathrm{P}_{\mathcal{M}}^2 \times \lambda^2$, $(x_v, x'_v)_{v\in\mathbb{R}}$ is a Markov jump process on X^2 starting from λ^2, jumping from (x, x') to $(x, S_1 x')$ with rate 1 at times recorded by μ' and from (x, x') to $(S_1 x, x')$ with rate 1 at times recorded by μ. The $\mathbb{R}^1$-action L_v^2 acting on $\mathcal{M}^2 \times X^2$ is the time shift of this Markov process.

Therefore, the ergodicity of (3.6) is equivalent to the ergodicity of the stationary Markov process $(x_v, x'_v)_{v\in\mathbb{R}}$. The ergodicity of a stationary Markov process can be described in terms of the associated semigroup and invariant measure. We recall that for a Markov semigroup $\mathrm{P} = (\mathrm{P}_t)_{t\geq 0}$ and a P-invariant measure ν (i.e., satisfying $\nu \mathrm{P}^t = \nu$ for all $t \geq 0$), a set A is called (almost) P-invariant if for all t, $\mathrm{P}^t \mathbf{1}_A = \mathbf{1}_A$ ν-a.s. The pair (P, ν) is ergodic if and only if $\nu(A) = 0$ or 1 for all invariant sets A.

Suppose that $A \subset X^2$ is an invariant set for the Markov semigroup P associated with the process $(x_v, x'_v)_{v\in\mathbb{R}}$. Then, for any $t > 0$,

$$\mathrm{P}^t \mathbf{1}_A(x, x') = \sum_{a,b=0}^{\infty} p_t(a, b) \mathbf{1}_A(S_1^a x, S_1^b x'),$$

where $p_t(a, b)$ is the probability that the two independent rate 1 Poisson processes make a and b jumps respectively between times 0 and t. This implies that A is an invariant set for the $\mathbb{Z}^2$-system

$$((S_1^a \times S_1^b)_{(a,b)\in\mathbb{Z}^2}, X^2, \lambda^2).$$

By Theorem 4.1, since (S_1, X) is ergodic, this product system is also ergodic. This implies that $\lambda^2(A) = 0$ or 1 and completes the proof. □

Proof of Theorem 3.1: The weak mixing follows from Definition 4.2 and Lemma 3.2. Since all integral curves of Φ are images of those of Ψ_α under the map $\varphi_{\mu,\nu}^{-1}$, (1.4) follows from Theorem 2.2 and SLLN for i.i.d. exponential random variables. □

4. Appendix

Here we give some standard definitions and facts from the ergodic theory.

Let G be a group. We call $((T_g)_{g\in G}, X, \mathcal{B}, \mu)$ a *G-system* if $(T_g)_{g\in G}$ is a measure preserving action of the group G on a probability space space $(X, \mathcal{B}, \mu)$. When $G = \mathbb{Z}$, we will write $(S, X, \mathcal{B}, \mu)$ where $S = T_1$. We may omit the σ-algebra $\mathcal{B}$ along with the measure μ if the context is clear.

The *product* of two systems, $((T_g)_{g\in G}, X, \mathcal{B}, \mu)$ and $((T'_h)_{h\in H}, Y, \mathcal{B}', \nu)$, is a $(G \times H)$-system $((T_g \times T'_h)_{(g,h)\in G\times H}, X \times Y, \mathcal{B} \otimes \mathcal{B}', \mu \otimes \nu)$. The group action is defined by

$$(T_g \times T'_h)(x, y) = (T_g x, T'_h y), \quad g \in G,\ h \in H. \tag{4.1}$$

The *direct product* of two G-systems $((T_g)_{g\in G}, X, \mathcal{B}, \mu)$ and $((T'_{g'})_{g'\in G}, Y, \mathcal{B}', \nu)$ is again a G-system $((T_g \times T'_g)_{g\in G}, X \times Y, \mathcal{B} \otimes \mathcal{B}', \mu \otimes \nu)$, where $T_g \times T'_g$ is defined according to (4.1) with $h = g \in G$, so this is the diagonal group action of G on $X \times Y$.

In the rest of the section and in the paper, the group we are dealing with will always be $\mathbb{R}^d$ or $\mathbb{Z}^d$, $d \in \mathbb{N}$. For $g = (g_1, ..., g_d) \in G$, $|g| = \max_{1\leq i\leq d} |g_i|$ its L^∞-norm.

We use dg to denote the Haar measure, i.e., the Lebesgue measure if $G = \mathbb{R}^d$ and counting measure if $G = \mathbb{Z}^d$.

The following are standard definitions on ergodicity and weak mixing for group actions (see [BG04]).

Definition 4.1. *We say that a G-system $((T_g)_{g\in G}, X, \mathcal{B}, \mu)$ is ergodic if and only if one of the following equivalent conditions holds true:*

1) If a set A is almost G-invariant, i.e., $\mu(A\Delta T_g A) = 0$ for all $g \in G$, then $\mu(A) = 0$ or $\mu(A) = 1$.

2) For any bounded measurable function f,

$$\lim_{R\to\infty} \frac{1}{(2R)^d} \int_{|g|\le R} f(T_g x)\, dg = \int f(x)\, \mu(dx), \quad \mu\text{-a.s. } x. \tag{4.2}$$

Definition 4.2. *We say that a G-system $((T_g)_{g\in G}, X, \mathcal{B}, \mu)$ is weakly mixing if and only if one of the following equivalent conditions holds true:*

1) For any two sets A and B,

$$\lim_{R\to\infty} \frac{1}{(2R)^d} \int_{|g|\le R} |\mu(T_g A \cap B) - \mu(A)\mu(B)|\, dg = 0.$$

2) The direct product $((T_g \times T_g)_{g\in G}, X \times X)$ is ergodic.

Theorem 4.1. *The product of two ergodic systems is ergodic.*

Proof: Let $((T_g)_{g\in G}, X, \mathcal{B}, \mu)$ and $((T'_h)_{h\in H}, Y, \mathcal{B}', \nu)$ be two ergodic systems. It suffices to show that (4.2) holds true for the product system with $f(x, y) = \mathbf{1}_{A\times B}(x, y)$ for any $A \in \mathcal{B}$ and $B \in \mathcal{B}'$.

We can use the ergodicity of $((T_g)_{g\in G}, X)$ and $((T'_h)_{h\in H}$ to see that

$$\begin{aligned}
&\lim_{R\to\infty} \frac{1}{(2R)^{2d}} \int_{|(g,h)|\le R} \mathbf{1}_{A\times B}(T_g x, T'_h y)\, dg\, dh \\
&= \lim_{R\to\infty} \left(\frac{1}{(2R)^d} \int_{|g|\le R} \mathbf{1}_A(T_g x)\, dg \ \cdot \frac{1}{(2R)^d} \int_{|h|\le R} \mathbf{1}_B(T'_h y)\, dh \right) = \mu(A)\nu(B)
\end{aligned}$$

holds for μ-a.e. x and ν-a.e. y, i.e., for $\mu \times \nu$-a.e. (x, y). The proof is complete. □

Theorem 4.2. *The product of two weakly mixing systems is weakly mixing.*

Proof: Let $((T_g)_{g\in G}, X)$ and $((T'_h)_{h\in H}, Y)$ be two weakly mixing systems. Their product $((T_g \times T'_h)_{(g,h)\in G\times H}, X \times Y)$ is weakly mixing if and only if

$$(((T_g \times T'_h) \times (T_g \times T'_h))_{(g,h)\in G\times H}, (X \times Y) \times (X \times Y)) \tag{4.3}$$

is ergodic. The latter is isomorphic to the product of $((T_g \times T_g)_{g\in G}, X \times X)$ and $((T'_h \times T'_h)_{h\in H}, Y \times Y)$, and both of these systems are ergodic. So (4.3) is ergodic by Theorem 4.1 and this completes the proof. □

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Courant Institute of Mathematical Sciences, New York University, 251 Mercer St, New York, NY 10012

Email address: bakhtin@cims.nyu.edu, liying@cims.nyu.edu