

# LONG EXIT TIMES NEAR A REPELLING EQUILIBRIUM

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**ABSTRACT.** For a smooth vector field in a neighborhood of a critical point with all positive eigenvalues of the linearization, we consider the associated dynamics perturbed by white noise. Using Malliavin calculus tools, we obtain polynomial asymptotics for probabilities of atypically long exit times in the vanishing noise limit.

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## 1. INTRODUCTION

In this paper, we continue the study of exit time distributions for diffusions obtained by small noisy perturbations of deterministic dynamical systems near unstable critical points. We are motivated by applications to the long-term dynamics in noisy heteroclinic networks and extensions of the work in [Bak11], [Bak10], [AMB11].

The most celebrated series of results on random perturbations of dynamical systems known as the Freidlin–Wentzell theory of metastability, see [FW12], is based on large deviation estimates and computes the asymptotics of probabilities associated with rare transitions between neighborhoods of stable equilibria. In these systems, the probability of a transition in a given finite time decays exponentially in  $\epsilon^{-2}$ , where  $\epsilon > 0$  is the noise magnitude, so it takes time of the order of  $\exp(c\epsilon^{-2})$ , to realize these transitions.

In the noisy heteroclinic network setting, it turns out that rare events of interest describing atypical transitions and determining the long-term behavior of the diffusion are tightly related to abnormally long stays in neighborhoods of unstable critical

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points. As a result, the probabilities of those events are related to the tails of the associated exit times, see a discussion of heteroclinic networks in [BPG18].

The probabilities we are interested in were shown to decay as a power of  $\epsilon$  if the starting point belongs to the stable manifold of the hyperbolic critical point (saddle) in [Mik95]. In the present paper, we provide much more precise asymptotics than the large deviation results of [Mik95] and prove a conjecture stated in that paper.

To be more precise, for  $\epsilon > 0$ , let us consider a diffusion process  $X^\epsilon$  solving an SDE in  $\mathbb{R}^d$ ,  $d \in \mathbb{N}$ :

$$(1.1) \quad dX_t^\epsilon = b(X_t^\epsilon)dt + \epsilon\sigma(X_t^\epsilon)dW_t$$

with noise given by the standard multi-dimensional Wiener process  $W$  and a smooth full-rank diffusion matrix  $\sigma$ , started at a distance of the order of  $\epsilon$  from the origin 0 which is assumed to be an unstable critical point of the smooth vector field  $b$ . Let  $\lambda_1 > 0$  be the leading simple eigenvalue of  $Db(0)$ , i.e., the real parts of all other eigenvalues are less than  $\lambda_1$ .

We are interested in the exit time  $\tau_\epsilon$  from a domain  $\mathbf{D}$  containing 0 and having a smooth boundary. The first results showing that the exit times typically behave like  $T_\epsilon = \frac{1}{\lambda_1} \log \frac{1}{\epsilon}$  plus  $O(1)$  corrections, were obtained in [Kif81] and [Day95]. Namely, it was shown in [Kif81] that  $\frac{\tau_\epsilon}{T_\epsilon} \xrightarrow{\mathbb{P}} 1$ ,  $\epsilon \rightarrow 0$ , and in [Day95], the limiting distribution of  $\tau_\epsilon - T_\epsilon$  as  $\epsilon \rightarrow 0$  was found. The distributions of exit locations were studied in [Eiz84], [Bak11], and (for the case where  $Db(0)$  is a Jordan block) in [BPG19b].

In [Mik95], probabilities of atypical deviations of  $\tau_\epsilon$  from  $T_\epsilon$  were studied. It was proved that in the 1-dimensional situation ( $d = 1$ ), for any  $h > 1$ ,

$$(1.2) \quad \lim_{\epsilon \rightarrow 0} \frac{\log \mathbb{P}\{\tau_\epsilon > hT_\epsilon\}}{\log \epsilon} = h - 1.$$

and a combination of results in [Kif81] and [Mik95] gives that for all  $d \geq 1$  and every  $h > 1$  there are finite positive numbers  $\mu_-(h), \mu_+(h) > 0$  such that

$$(1.3) \quad \mu_-(h) \leq \liminf_{\epsilon \rightarrow 0} \frac{\log \mathbb{P}\{\tau_\epsilon > hT_\epsilon\}}{\log \epsilon} \leq \limsup_{\epsilon \rightarrow 0} \frac{\log \mathbb{P}\{\tau_\epsilon > hT_\epsilon\}}{\log \epsilon} < \mu_+(h).$$

In [Mik95] it is actually conjectured that

$$(1.4) \quad \mu_-(h) = \mu_+(h) = \mu(h),$$

where

$$(1.5) \quad \mu(h) = \sum_{j=1}^d \left( \left( \frac{h \operatorname{Re} \lambda_j}{\lambda_1} - 1 \right) \vee 0 \right),$$

and  $\lambda_1, \dots, \lambda_d$  in this formula are the eigenvalues of  $Db(0)$ .

In [BPG19a] and [BPG18], the logarithmic asymptotics of (1.2) for the 1-dimensional situation was improved and it was shown that for any  $h > 1$ , for a range of deterministic initial conditions  $X_0^\epsilon = x$  near 0,

$$(1.6) \quad \mathbb{P}\{\tau_\epsilon > hT_\epsilon\} = \psi(x)\epsilon^{h-1}(1 + o(1)), \quad \epsilon \rightarrow 0,$$

and the coefficient  $\psi(x) > 0$  was computed explicitly. The paper [BPG19a] was based on Malliavin calculus techniques and [BPG18] used more elementary tools.

In the present paper, we consider the situation where  $d \in \mathbb{N}$  is arbitrary and the eigenvalues of  $\nabla b(0)$  are real and satisfy  $\lambda_1 > \lambda_2 > \dots > \lambda_d > 0$ . For this case, we prove the conjecture of [Mik95] showing that relations (1.3)–(1.5) hold true. In fact,

instead of the logarithmic equivalence in (1.3), we prove stronger estimates similar to (1.6) extending the latter to the higher-dimensional setting. For domains  $\mathbf{D}$  of a special type (preimages of rectangular domains under a linearizing conjugacy), our Theorem 2.2 states that there is  $p > 0$  such that, uniformly over deterministic initial conditions  $X_0^\epsilon = x$  at distance of the order of  $\epsilon$  from 0,

$$\mathbb{P}\{\tau_\epsilon > hT_\epsilon\} = \psi_h(x)\epsilon^{\mu(h)}(1 + o(\epsilon^p)),$$

with an explicit expression for the coefficient  $\psi_h(x) > 0$ . In fact, we prove a more general estimate on the tail of  $\tau_\epsilon$ .

The idea of the proof is the following. We treat the dynamics described by (1.1) as a perturbation of the linear dynamics given by the linearization of  $b$  at 0. For truly linear dynamics with additive noise the solution is given by stochastic Itô integrals of deterministic quantities. Thus it is a Gaussian process allowing for a direct computation which, in fact, was behind the conjecture (1.3)–(1.5) of [Mik95]. The main difficulty is to lift this computation to the general nonlinear situation. In particular, similarly to [BPG19a] we choose to work with Malliavin calculus tools in order to estimate densities of random variables that we want to treat as perturbations of Gaussian ones. Unlike [BPG19a], we use results of [BC14] to estimate the discrepancy between the Gaussian densities and the perturbed ones. These estimates are valid only for evolution times of the order of  $\theta \log \epsilon^{-1}$  with small values of  $\theta$ , so we have to apply them sequentially multiple times in order to get to  $hT_\epsilon$ , thus creating an iteration scheme similar to that of [BPG19a].

The analysis for more general domains can be partially reduced to the special domains defined above via the rectifying conjugacy. We can obtain, see Corollary 2.3, that there are constants  $\phi_\pm(x)$  such that

$$\phi_-(x)\epsilon^{\mu(h)}(1 + o(\epsilon^p)) \leq \mathbb{P}\{\tau_\epsilon > hT_\epsilon + r(\epsilon)\} \leq \phi_+(x)\epsilon^{\mu(h)}(1 + o(\epsilon^p)).$$

The slight discrepancy between the upper and lower estimates is due to the fact that the travel time along the drift vector field between the boundaries of domains immersed into one another depends on the starting point on the boundary. We give a slightly more precise result (Corollary 2.6) that takes these travel times into account and note here that further progress in understanding of exit times for general domains will be achieved as more information on the geometric properties of the exit location distribution becomes available. The asymptotics of the exit location distribution will be addressed in our forthcoming work.

The paper is organized as follows. In Section 2, we give a technical description of the setting and state our main results precisely. The proof is spread over Sections 3 through 5. The main result is derived from the comparison to the linearized problem in Section 3. An iterative scheme of sequential approximations that this comparison is based on is given in Section 4. Each step of this scheme is in turn based on a density discrepancy estimate that we derive using Malliavin calculus tools in Section 5.

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## 2. SETTING AND MAIN RESULTS

Let  $d \in \mathbb{N}$  and let simply connected domains  $D_1, D_2, \mathbf{D} \subset \mathbb{R}^d$  satisfy

$$(2.1) \quad 0 \in D_1 \subset \overline{D_1} \subset \mathbf{D} \subset \overline{\mathbf{D}} \subset D_2.$$

We consider a  $C^5$  vector field  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and the flow  $(S^t)$  generated by  $b$ :

$$(2.2) \quad \begin{aligned} \frac{d}{dt} S^t x &= b(S^t x), \\ S^0 x &= x. \end{aligned}$$

Since we are interested in the dynamics inside  $\mathbf{D}$ , by adjustments outside  $\mathbf{D}$ , we assume  $b$  and its derivatives are bounded. We assume that the following conditions hold:

- $b(x) = \mathbf{a}x + q(x)$  where
  - $|q(x)| \leq C_q|x|^2$  with a positive constant  $C_q$ ,
  - $\mathbf{a}$  is a  $d \times d$  diagonal matrix with real entries  $\lambda_1 > \lambda_2 > \dots > \lambda_d > 0$ ;
- for all open sets  $D_0$  satisfying  $0 \in D_0 \subset D_1$ ,

$$(2.3) \quad \sup_{x \in \partial D_0} t_{D_2}(x) < \infty,$$

where

$$(2.4) \quad t_D(x) = \inf\{t > 0 : S^t x \notin D\}, \quad D \subset \mathbb{R}^d, \quad x \in \mathbb{R}^d.$$

For brevity we will denote the vector field given by  $x \rightarrow \mathbf{a}x$  by  $\mathbf{a}$ . By the Hartman–Grobman Theorem (c.f. Theorem 6.3.1 from [KH95]), there is an open neighborhood  $O$  of 0 and a homeomorphism  $f : O \rightarrow f(O)$  conjugating the flow  $S$  generated by the vector field  $b$  to the linear flow generated by  $\mathbf{a}$ , namely,

$$(2.5) \quad \frac{d}{dt} f(S_t x) = \mathbf{a}f(S_t x).$$

- in addition, we assume that  $f$  is a  $C^5$  diffeomorphism.

**Remark 2.1.** Due to [Ste57], for this  $C^5$  conjugacy condition to hold in our setting, it suffices to require (i) a smoothness condition:  $b$  is  $C^k$  for some  $k \geq 5 \vee (\lambda_1/\lambda_n)$ , and (ii) a no-resonance condition:

$$\lambda_k \neq m_1\lambda_1 + \dots + m_d\lambda_d$$

for all  $k = 1, \dots, d$  and all nonnegative integer coefficients  $m_1, \dots, m_d$  satisfying  $m_1 + \dots + m_d \geq 2$ .

The vector field  $\mathbf{a}$  is the pushforward of  $b$  under  $f$ , and since  $\mathbf{a}$  is diagonal,  $f$  can be chosen to satisfy

$$(2.6) \quad f(0) = 0, \quad Df(0) = I,$$

where  $I$  is the identity matrix.

We are interested in the limiting behavior of random perturbations of the ODE (2.2) given by the SDE (1.1) as  $\epsilon$  tends to 0. In (1.1),

- $\epsilon \in (0, 1)$  is the noise amplitude parameter;
- $(W_t, \mathcal{F}_t)$  is a standard  $n$ -dimensional Wiener process with  $n \geq d$ ;
- $\sigma$  is a map from  $\mathbb{R}^d$  into the space of  $d \times n$  matrices satisfying
  - $\sigma$  is  $C^3$  (and, by adjustments outside  $\mathbf{D}$ , we may assume that  $\sigma$  and its derivatives are bounded),
  - $\sigma(0) : \mathbb{R}^n \rightarrow \mathbb{R}^d$  is surjective.

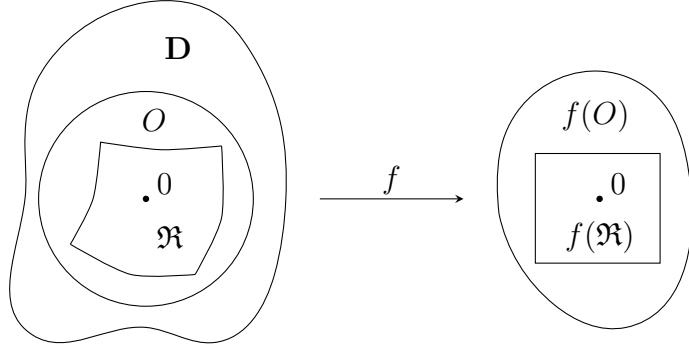


FIGURE 1. The diffeomorphism  $f : O \rightarrow f(O)$  maps  $\mathfrak{R}$  onto  $f(\mathfrak{R})$  which is a box containing 0.

To simplify the notation, we often suppress the dependence on  $\epsilon$ . In particular, we often write  $X_t$  instead of  $X_t^\epsilon$ .

We need some definitions to state our main result. We start by describing the exit event:

- for a measurable set  $A \subset \mathbb{R}^d$ , we define the exit time

$$(2.7) \quad \tau_A = \inf\{t > 0 : X_t \notin A\};$$

- for  $L_-^j, L_+^j \in \mathbb{R}$ ,  $j = 1, 2, \dots, d$ , we define  $\mathfrak{R} = f^{-1}(\prod_{j=1}^d [L_-^j, L_+^j]) \subset O$  with  $0 \in \mathring{\mathfrak{R}}$  to be such a set that its preimage under  $f$  is a box and that its interior contains the origin (see Figure 1);
- for any  $q > 0$ , let  $r : [0, 1] \rightarrow \mathbb{R}$  be any function satisfy

$$(2.8) \quad |r(\epsilon) - r(0)| = \mathcal{O}(\epsilon^q).$$

The theorem is concerned with events of the form

$$\{\tau_{\mathfrak{R}} > \alpha \log \epsilon^{-1} + r(\epsilon)\}$$

for some  $\alpha \geq 0$ . As  $\tau_{\mathfrak{R}}$  is of order  $\alpha \log \epsilon^{-1}$ , the term  $r(\epsilon)$  is interpreted as a small perturbation.

Next, we introduce definitions characterizing the decay rate:

- for  $\alpha > 0$ , let  $i(\alpha) \in \{1, 2, \dots, d+1\}$  satisfy

$$(2.9) \quad \frac{1}{\lambda_{i(\alpha)-1}} < \alpha \leq \frac{1}{\lambda_{i(\alpha)}}$$

where we agree that  $\lambda_0 = \infty$  and  $\lambda_{d+1} = 0$ ;

- the exponent determining the power decay, as a function of  $\alpha$ , is given by

$$(2.10) \quad \beta(\alpha) = \sum_{j=1}^d ((\lambda_j \alpha - 1) \vee 0) = \sum_{j=1}^{i(\alpha)-1} (\lambda_j \alpha - 1) = \mu(\lambda_1 \alpha),$$

where  $\mu(\cdot)$  was defined in (1.5).

We will consider initial conditions  $X_0 = \epsilon x$  satisfying  $|x| \leq K(\epsilon)$  for an admissible function  $K(\cdot)$ :

- for a fixed  $\alpha \geq 0$ , a function  $K : (0, 1] \rightarrow [0, +\infty)$  is said to be admissible if it satisfies, with  $i = i(\alpha)$ ,

$$\begin{cases} \lim_{\epsilon \rightarrow 0} \epsilon^{1-\lambda_i \alpha} K(\epsilon) = 0, & \text{if } i \leq d \text{ and } \alpha < \frac{1}{\lambda_i}, \\ \lim_{\epsilon \rightarrow 0} \epsilon^{1-\lambda_{i+1} \alpha} K(\epsilon) = 0, & \text{if } i \leq d, \text{ and } \alpha = \frac{1}{\lambda_i}, \\ \lim_{\epsilon \rightarrow 0} \epsilon^{1-c} K(\epsilon) = 0 \text{ for some } c \in (0, 1), & \text{if } i = d+1. \end{cases}$$

Lastly, we describe the limiting object:

- let  $d \times d$  matrix  $\mathcal{C}_0$  be given by

$$(2.11) \quad \mathcal{C}_0^{jk} = \sum_{l=1}^n \frac{\sigma_l^j(0)\sigma_l^k(0)}{\lambda_j + \lambda_k};$$

- for  $x \in \mathbb{R}^d$  and  $i = 1, \dots, d$ , we define

$$(2.12) \quad \begin{aligned} x^{<i} &= (x^1, x^2, \dots, x^{i-1}) \in \mathbb{R}^{i-1}, & x^{>i} &= (x^{i+1}, \dots, x^d) \in \mathbb{R}^{d-i}, \\ x^{\geq i} &= (x^i, x^{i+1}, \dots, x^d) \in \mathbb{R}^{d-i+1}; \end{aligned}$$

- for  $\alpha \geq 0$ , some small perturbation limit  $r(0) \in \mathbb{R}$ , a set  $\mathfrak{R}$  and  $x \in \mathbb{R}^d$ , we define, with  $i = i(\alpha)$ ,

$$(2.13) \quad \psi_{\alpha, r(0), \mathfrak{R}}(x) = \begin{cases} \frac{\prod_{j < i} (L_+^j - L_-^j) e^{-\lambda_j r(0)}}{\sqrt{(2\pi)^d \det \mathcal{C}_0}} \int_{\mathbb{R}^{d-i+1}} e^{-\frac{1}{2} z^\top \mathcal{C}_0^{-1} z} \Big|_{z^{<i} = -x^{<i}} dz^{\geq i}, & \alpha < \frac{1}{\lambda_i}, \\ \frac{\prod_{j < i} (L_+^j - L_-^j) e^{-\lambda_j r(0)}}{\sqrt{(2\pi)^d \det \mathcal{C}_0}} \int_{(e^{-\lambda_i r(0)} [L_-^i, L_+^i] - x^i) \times \mathbb{R}^{d-i}} e^{-\frac{1}{2} z^\top \mathcal{C}_0^{-1} z} \Big|_{z^{<i} = -x^{<i}} dz^{\geq i}, & \alpha = \frac{1}{\lambda_i}. \end{cases}$$

If  $i = d+1$ , then the integrals in (2.13) are understood to be simply  $e^{-\frac{1}{2} x^\top \mathcal{C}_0^{-1} x}$ .

We are now ready to state the main result.

**Theorem 2.2.** *Suppose  $X_t$  solves (1.1) with  $X_0 = \epsilon x$ , and  $r$  satisfies (2.8) for some  $q > 0$ .*

*There is a constant  $L_0 \geq 0$  such that, for every  $\alpha \geq 0$ , every admissible  $K$ , every  $\mathfrak{R} \subset f^{-1}([-L_0, L_0]^d)$ , the following holds*

$$(2.14) \quad \sup_{|x| \leq K(\epsilon)} |\epsilon^{-\beta(\alpha)} \mathbb{P}\{\tau_{\mathfrak{R}} > \alpha \log \epsilon^{-1} + r(\epsilon)\} - \psi(x)| = o(\epsilon^p)$$

for  $\psi = \psi_{\alpha, r(0), \mathfrak{R}}$  and some  $p = p(\alpha, q, \lambda, \sigma, f) \in (0, 1)$ .

For a general domain  $\mathbf{D}$ , we choose  $L_\pm^j$  small enough to guarantee  $\mathfrak{R} \subset D_1$ , where  $D_1$  was introduced in (2.1). Due to (2.3),  $T_- = \inf_{z \in \partial \mathfrak{R}} t_{D_1}(z)$  and  $T_+ = \sup_{z \in \partial \mathfrak{R}} t_{D_2}(z)$  are well-defined. Setting  $\phi_\pm(x) = \psi_{\alpha, r(0)-T_\pm, \mathfrak{R}}(x)$ , we obtain:

**Corollary 2.3.** *Under the conditions of Theorem 2.2,*

$$(2.15) \quad \phi_-(x) + o(\epsilon^p) \leq \epsilon^{-\beta(\alpha)} \mathbb{P}\{\tau_{\mathbf{D}} > \alpha \log \epsilon^{-1} + r(\epsilon)\} \leq \phi_+(x) + o(\epsilon^p)$$

uniformly over  $|x| \leq K(\epsilon)$ .

Taking the logarithm on both sides of (2.15), we obtain:

**Corollary 2.4.** *Under the conditions of Theorem 2.2, there is a constant  $C > 0$  such that*

$$\sup_{|x| \leq K(\epsilon)} \left| \frac{\log \mathbb{P}\{\tau_{\mathbf{D}} > \alpha \log \epsilon^{-1} + r(\epsilon)\}}{\log \epsilon} - \beta(\alpha) \right| \leq \frac{C}{|\log \epsilon|}.$$

**Remark 2.5.** (1) When  $d = 1$ , Proposition 2.2 is a slight improvement of the result in [BPG19a].

(2) If  $q = 0$ , then the above results still hold for  $p = 0$ .

(3) If  $X_0 = \epsilon \xi^\epsilon$  where the random variable  $\xi^\epsilon$  satisfies  $\mathbb{P}\{|\xi^\epsilon| > K(\epsilon)\} = o(\epsilon^{\beta(\alpha)})$ , then (2.14) and (2.15) imply, respectively,

$$\lim_{\epsilon \rightarrow 0} |\epsilon^{-\beta(\alpha)} \mathbb{P}\{\tau_{\mathfrak{R}} > \alpha \log \epsilon^{-1} + r(\epsilon)\} - \mathbb{E}\psi(\xi^\epsilon)| = 0;$$

$$\mathbb{E}\phi_-(\xi^\epsilon) + o(1) \leq \epsilon^{-\beta(\alpha)} \mathbb{P}\{\tau_{\mathbf{D}} > \alpha \log \epsilon^{-1} + r(\epsilon)\} \leq \mathbb{E}\phi_+(\xi^\epsilon) + o(1).$$

(4) In comparison with [Mik95], we make stronger smoothness assumptions on the coefficients and an additional assumption on the smoothness of the linearizing conjugacy. These assumptions are required for our Malliavin calculus approach. Namely, we must ensure that certain higher-order Malliavin derivatives of the diffusion process exist and admit useful bounds. In addition, we require the eigenvalues of linearization to be simple and positive. In this slightly more restrictive setting, our Corollary 2.4 improves and generalizes [Mik95, Theorem 1.3 and Proposition 1.4] and implies [Mik95, Conjecture 1.5].

Under additional geometric assumptions on  $\mathbf{D}$ , more precise results than Corollary 2.3 can be obtained. We assume that  $\mathbf{D}$  has  $C^1$  boundary and that  $b$  intersects  $\partial\mathbf{D}$  transversally in the sense that  $\langle n(x), b(x) \rangle > 0$  for every  $x \in \partial\mathbf{D}$ , where  $n(x)$  is the outer normal of  $\partial\mathbf{D}$ . Let us choose  $L_\pm^j$  small enough to ensure  $\mathfrak{R} \subset \mathbf{D}$  and recall (2.4).

**Corollary 2.6.** *Under the same conditions as Theorem 2.2 and the additional smoothness and transversality assumptions introduced in the above paragraph, we have*

$$\sup_{|x| \leq K(\epsilon)} |\epsilon^{-\beta(\alpha)} \mathbb{P}\{\tau_{\mathbf{D}} - t_{\mathbf{D}}(X_{\tau_{\mathfrak{R}}}) > \alpha \log \epsilon^{-1} + r(\epsilon)\} - \psi(x)| = o(\epsilon^p),$$

where  $\psi = \psi_{\alpha, r(0), \mathfrak{R}}$  is given in (2.13).

### 3. PROOF OF MAIN RESULTS

Corollaries 2.3 and 2.6 are direct consequences of Theorem 2.2, our geometric assumptions, and the following standard FW large deviation estimate which implies that, upon exiting  $\mathfrak{R}$ , the process  $X$  closely follows a deterministic trajectory:

**Lemma 3.1.** *For each fixed time  $T > 0$ , and each  $v \in [0, 1)$ , there are  $C, c > 0$  such that the following holds uniformly over all initial points  $X_0 = \mathbf{x}$ :*

$$\mathbb{P}\left\{ \sup_{0 \leq t \leq T} |X_t - S^t \mathbf{x}| > \epsilon^v \right\} \leq C \exp(-c\epsilon^{2(v-1)}).$$

This lemma can be proved using Lipschitzness of the vector field  $b$ , boundedness of  $\sigma$ , Gronwall's inequality, and the exponential martingale inequality (see [Bas11, Problem 12.10]). The key idea can be seen at the very beginning of [FW12, Chapter 3].

The rest of this section is our proof of Theorem 2.2.

From now on we will often use Einstein's convention of summation over matching upper and lower indices. Let us introduce a new process  $Y_t = f(X_t)$ , which by Itô's formula and (2.5) satisfies

$$(3.1) \quad dY_t^i = \lambda^i Y_t^i dt + \epsilon F_j^i(Y_t) dW_t^j + \epsilon^2 G^i(Y_t) dt,$$

where

$$\begin{aligned} F_j^i(y) &= \partial_k f^i(f^{-1}(y)) \sigma_j^k(f^{-1}(y)), \quad y \in f(O), \\ G^i(y) &= \frac{1}{2} \partial_{jk}^2 f^i(f^{-1}(y)) \langle \sigma^j(f^{-1}(y)), \sigma^k(f^{-1}(y)) \rangle, \quad y \in f(O), \end{aligned}$$

$\langle \cdot, \cdot \rangle$  denotes the inner product, and we set  $\lambda^i = \lambda_i$  to avoid the summation over  $i$ . Note that  $F, G \in C^3(f(O))$  and, due to (2.6), we have

$$(3.2) \quad F(0) = \sigma(0).$$

We shift our focus from the process  $X_t$  with  $X_0 = \epsilon x$  to  $Y_t = f(X_t)$  with  $Y_0 = \epsilon y = f(\epsilon x)$  by the following considerations. Due to (2.6), there is a constant  $C_f$  such that  $|z| \leq C_f |f(z)|$  for all  $z \in O$ . Set  $K'(\epsilon) = C_f^{-1} K(\epsilon)$ . Therefore, for  $\epsilon$  small with  $X_0 = \epsilon x \in O$ , we have that if  $|y| \leq K'(\epsilon)$ , then  $|x| \leq K(\epsilon)$ . Note that due to  $Y_t = f(X_t)$  the exit time  $\tau_{\mathfrak{R}'}$  defined in (2.7) in terms of the process  $X$  can be rewritten as

$$(3.3) \quad \tau = \inf\{t > 0 : Y_t \notin \mathfrak{R}'\},$$

where  $\mathfrak{R}' = \prod_{j=1}^d [L_-^j, L_+^j] = f(\mathfrak{R})$  (see Figure 1). Hence, Theorem 2.2 follows from the following result.

**Proposition 3.2.** *Suppose  $Y_t$  solves (3.1) with  $Y_0 = \epsilon y$  and let  $r$  satisfy (2.8). Then there is a constant  $L_0 \geq 0$  such that for each  $\alpha \geq 0$  and each  $K'(\epsilon)$  satisfying, with  $i = i(\alpha)$ ,*

$$(3.4) \quad \begin{cases} \lim_{\epsilon \rightarrow 0} \epsilon^{1-\lambda_i \alpha} K'(\epsilon) = 0, & \text{if } i \leq d \text{ and } \alpha < \frac{1}{\lambda_i}, \\ \lim_{\epsilon \rightarrow 0} \epsilon^{1-\lambda_{i+1} \alpha} K'(\epsilon) = 0, & \text{if } i \leq d \text{ and } \alpha = \frac{1}{\lambda_i}, \\ \lim_{\epsilon \rightarrow 0} \epsilon^{1-c} K'(\epsilon) = 0 \text{ for some } c \in (0, 1), & \text{if } i = d+1, \end{cases}$$

we have, for any set of the form  $\mathfrak{R}' = \prod_{j=1}^d [L_-^j, L_+^j] \subset O$  with  $0 \in \mathfrak{R}'$  and  $|L_\pm^j| \leq L_0$  for all  $j = 1, \dots, d$ ,

$$(3.5) \quad \sup_{|y| \leq K'(\epsilon)} \left| \epsilon^{-\beta(\alpha)} \mathbb{P}\{\tau > \alpha \log \epsilon^{-1} + r(\epsilon)\} - \psi\left(\frac{f^{-1}(\epsilon y)}{\epsilon}\right) \right| = o(\epsilon^p),$$

for some  $p = p(\alpha, q, \lambda, \sigma, f) \in (0, 1)$ .

Let us describe the plan to prove Proposition 3.2. The proof can be split into two main steps.

The first step is to show that  $\mathbb{P}\{\tau > \alpha \log \epsilon^{-1} + r(\epsilon)\}$  can be approximated by  $\mathbb{P}\{y + U_{T_0(\epsilon)}^\epsilon \in A_\epsilon\}$ , where  $T_0(\epsilon)$  is a deterministic time,  $U^\epsilon$  is a Gaussian-like process, and  $A_\epsilon \subset \mathbb{R}^d$  is a deterministic set. Namely, the probability of the exit event can be approximated by integrating over  $A_\epsilon$  with respect to a Gaussian-like density. This result is summarized in Lemma 3.3 below. The method is to find an explicit expression of  $\tau$  by using the fact that  $\mathfrak{R}'$ , the set to exit, is a box.

In the second step, we show that  $\mathbb{P}\{y + U_{T_0(\epsilon)}^\epsilon \in A_\epsilon\}$  is approximately  $\mathbb{P}\{y + \mathcal{Z} \in A_\epsilon\}$  for a centered Gaussian vector  $\mathcal{Z}$  with covariance (2.11). This is the content of Lemma 3.4. To show this, we apply tools from the Malliavin Calculus to deduce that the density of  $U_{T(\epsilon)}^\epsilon$  is close to that of  $Z_{T(\epsilon)}$ . Here  $T(\epsilon)$  is another deterministic time, which can be much smaller than  $T_0(\epsilon)$ , and  $Z$  is a Gaussian process independent of  $\epsilon$  with  $Z_\infty$  equal to  $\mathcal{Z}$  in distribution. This is done in Lemma 4.1. We use an iteration scheme to extend the Gaussian approximation to the larger time  $T_0(\epsilon)$ .

To conclude the proof of this proposition, we estimate the discrepancy between properly scaled  $\mathbb{P}\{y + \mathcal{Z} \in A_\epsilon\}$  and  $\psi(f^{-1}(\epsilon y)/\epsilon)$ , as  $\epsilon \rightarrow 0$ . This is done in this section after stating Lemma 3.3 and Lemma 3.4.

To state the two key lemmas, we start by introducing some useful objects.

Since  $F(0) = \sigma(0)$  is  $d \times n$  with full rank and  $F$  is continuous, we can choose  $L_0$  so small that there is  $c_0 > 0$  such that  $\min_{|u|=1, u \in \mathbb{R}^d} |u^\top F(x)|^2 \geq c_0$  for all  $x \in [-L_0, L_0]^d$ , where  $\top$  stands for matrix transpose. Since we only care about exiting from a subset of  $[-L_0, L_0]^d$ , we modify  $F, G$  outside  $[-L_0, L_0]^d$  so that

$$(3.6) \quad \begin{aligned} \min_{|u|=1, u \in \mathbb{R}^d} |u^\top F(x)|^2 &\geq c_0, \text{ for all } x \in \mathbb{R}^d; \\ F, G \text{ and their derivatives are bounded.} \end{aligned}$$

From now on, we fix this  $L_0$  and  $F, G$  modified according to (3.6). By Duhamel's principle, we can solve (3.1) with  $Y_0 = \epsilon y$  by

$$(3.7) \quad Y_t^j = \epsilon e^{\lambda_j t} (y^j + U_t^j),$$

where

$$(3.8) \quad U_t^j = M_t^j + \epsilon V_t^j$$

and

$$(3.9) \quad M_t^j = \int_0^t e^{-\lambda_j s} F_l^j(Y_s) dW_s^l,$$

$$(3.10) \quad V_t^j = \int_0^t e^{-\lambda_j s} G^j(Y_s) ds.$$

We emphasize that  $M_t, V_t$ , and  $U_t$  depend on  $y$  and  $\epsilon$ .

To make the notation less heavy we will assume that

$$(3.11) \quad \mathfrak{R}' = [-L, L]^d \quad \text{for some } L \in (0, L_0),$$

as it is easy to see that for general rectangles, all our arguments still hold.

**Lemma 3.3.** *Let*

$$(3.12) \quad T_0 = T_0(\epsilon) = \alpha \log \epsilon^{-1} + r(\epsilon).$$

For each  $\nu > 0$ , there are  $\epsilon_0 > 0$  and  $\gamma_j, j = 1, \dots, d$ , satisfying

$$(3.13) \quad (\lambda_j \alpha - 1) \vee 0 < \gamma_j < \lambda_j \alpha, \quad j = 1, \dots, d,$$

such that the following holds for all  $y$  satisfying  $|y| \leq K'(\epsilon)$  and all  $\epsilon \leq \epsilon_0$ :

$$-\epsilon^\nu + \mathbb{P}\{y + U_{T_0} \in A_-\} \leq \mathbb{P}\{\tau > \alpha \log \epsilon^{-1} + r(\epsilon)\} \leq \mathbb{P}\{y + U_{T_0} \in A_+\} + \epsilon^\nu,$$

where

$$(3.14) \quad A_\pm = \{x \in \mathbb{R}^d : |x^j| < \epsilon^{\lambda_j \alpha - 1} L e^{-\lambda_j r(\epsilon)} \pm \epsilon^{\gamma_j}, \quad j = 1, \dots, d\}.$$

**Lemma 3.4.** *Let  $T_0$  be defined in (3.12) and  $\mathcal{Z}$  be a centered Gaussian vector with covariance matrix given by (2.11). Then for each  $v \in (0, 1)$ , there are constants  $\epsilon_0, \delta > 0$  such that, for  $\epsilon \in (0, \epsilon_0]$*

$$\sup_{|y| \leq \epsilon^{v-1}} |\mathbb{P}\{y + U_{T_0} \in A_{\pm}\} - \mathbb{P}\{y + \mathcal{Z} \in A_{\pm}\}| = o(\epsilon^{\beta(\alpha)+\delta}).$$

These two lemmas are proved in Section 4.

*Proof of Proposition 3.2.* Let

$$(3.15) \quad \psi_{\epsilon}(y) = \epsilon^{-\beta(\alpha)} \mathbb{P}\{y + \mathcal{Z} \in A_{\pm}\} = \frac{\epsilon^{-\beta(\alpha)}}{\sqrt{(2\pi)^d \det \mathcal{C}_0}} \int_{A_{\pm}-y} e^{-\frac{1}{2}x^{\top} \mathcal{C}_0^{-1} x} dx.$$

Here and below, we use the same argument to treat the cases of  $A_+$  and  $A_-$  and often omit the dependence on the choice of  $+$  or  $-$ .

Since we have assumed (3.11), we have

$$\psi(y) = \begin{cases} \frac{\prod_{j < i} 2Le^{-\lambda_j r(0)}}{\sqrt{(2\pi)^d \det \mathcal{C}_0}} \int_{\mathbb{R}^{d-i+1}} e^{-\frac{1}{2}x^{\top} \mathcal{C}_0^{-1} x} \Big|_{x^{<i} = -y^{<i}} dx^{\geq i}, & \text{if } \alpha < \frac{1}{\lambda_i}, \\ \frac{\prod_{j < i} 2Le^{-\lambda_j r(0)}}{\sqrt{(2\pi)^d \det \mathcal{C}_0}} \int_{(e^{-\lambda_i r(0)}[-L, L] - y^i) \times \mathbb{R}^{d-i}} e^{-\frac{1}{2}x^{\top} \mathcal{C}_0^{-1} x} \Big|_{x^{<i} = -y^{<i}} dx^{\geq i}, & \text{if } \alpha = \frac{1}{\lambda_i}. \end{cases}$$

The key estimate is the following, to be proved later:

$$(3.16) \quad \sup_{|y| \leq K'(\epsilon)} |\psi_{\epsilon}(y) - \psi\left(\frac{f^{-1}(\epsilon y)}{\epsilon}\right)| \leq o(\epsilon^q), \text{ for some } q \in (0, 1).$$

By (3.15), (3.16), Lemma 3.3 and Lemma 3.4, we obtain (3.5). By the discussion above (3.5), the desired result (2.14) is attained.  $\square$

*Proof of (3.16).* We remind the notations introduced in (2.12). Let  $\Pi^i$ ,  $\Pi^{>i}$  and  $\Pi^{\geq i}$  be projection maps defined by  $\Pi^i x = x^i$ ,  $\Pi^{>i} x = x^{>i}$  and  $\Pi^{\geq i} x = x^{\geq i}$ . For any set  $E \subset \mathbb{R}^d$ , we define

$$(3.17) \quad E^i = \Pi^i E, \quad E^{>i} = \Pi^{>i} E, \quad E^{\geq i} = \Pi^{\geq i} E.$$

In addition, for a fixed  $y \in \mathbb{R}^d$  in (3.16), and each  $x \in \mathbb{R}^d$ , let  $\tilde{x} = (-y^{<i}, x^{\geq i}) \in \mathbb{R}^d$ .

Since  $\sigma(0)$  has full rank, by the definition of  $\mathcal{C}_0$  in (2.11), there is  $c > 0$  such that

$$(3.18) \quad e^{-\frac{1}{2}x^{\top} \mathcal{C}_0^{-1} x} \leq e^{-c|x|^2}.$$

Here and below the value of the constant  $C$  may vary from instance to instance. To estimate  $|\psi_{\epsilon}(y) - \psi\left(\frac{f^{-1}(\epsilon y)}{\epsilon}\right)|$ , we need the following intermediate quantities:

$$\begin{aligned} \mathbb{I} &= \frac{\epsilon^{-\beta(\alpha)}}{\sqrt{(2\pi)^d \det \mathcal{C}_0}} \int_{A_{\pm}-y} e^{-\frac{1}{2}\tilde{x}^{\top} \mathcal{C}_0^{-1} \tilde{x}} dx, \\ \mathbb{II} &= \frac{\prod_{j < i} 2Le^{-\lambda_j r(0)}}{\sqrt{(2\pi)^d \det \mathcal{C}_0}} \int_{(A_{\pm}-y)^{\geq i}} e^{-\frac{1}{2}\tilde{x}^{\top} \mathcal{C}_0^{-1} \tilde{x}} dx^{\geq i}. \end{aligned}$$

Let us write

$$(3.19) \quad \left| \psi_{\epsilon}(y) - \psi\left(\frac{f^{-1}(\epsilon y)}{\epsilon}\right) \right| \leq \left| \psi_{\epsilon}(y) - \mathbb{I} \right| + \left| \mathbb{I} - \mathbb{II} \right| + \left| \mathbb{II} - \psi(y) \right| + \left| \psi(y) - \psi\left(\frac{f^{-1}(\epsilon y)}{\epsilon}\right) \right|,$$

and estimate each term on the right of (3.19).

By the symmetry and positive definiteness of  $\mathcal{C}_0$ , we have, for any  $x, w \in \mathbb{R}^d$ ,

$$\begin{aligned}
(3.20) \quad & |e^{-\frac{1}{2}x^\top \mathcal{C}_0^{-1} x} - e^{-\frac{1}{2}w^\top \mathcal{C}_0^{-1} w}| \leq C(e^{-c|x|^2} \vee e^{-c|w|^2})|x + w||x - w| \\
& \leq Ce^{-c|x|^2}(|2x| + |x - w|)|x - w|\mathbb{1}_{\{|x| \leq |w|\}} + Ce^{-c|w|^2}(|2w| + |x - w|)|x - w|\mathbb{1}_{\{|x| > |w|\}} \\
& \leq C(e^{-c_1|x|^2} \vee e^{-c_1|w|^2})(|x - w| + |x - w|^2)
\end{aligned}$$

for some positive  $c_1 < c$ . Therefore, we have

$$\begin{aligned}
\sup_{x \in A_{\pm} - y} |e^{-\frac{1}{2}x^\top \mathcal{C}_0^{-1} x} - e^{-\frac{1}{2}\tilde{x}^\top \mathcal{C}_0^{-1} \tilde{x}}| & \leq \sup_{x \in A_{\pm} - y} Ce^{-c_1|x^{\geq i}|^2}(|x^{<i} + y^{<i}| + |x^{<i} + y^{<i}|^2) \\
& \leq Ce^{-c_1|x^{\geq i}|^2} \sum_{j < i} \left( (\epsilon^{\lambda_j \alpha - 1} L e^{-\lambda_j r(\epsilon)} + \epsilon^{\gamma_j}) + (\epsilon^{\lambda_j \alpha - 1} L e^{-\lambda_j r(\epsilon)} + \epsilon^{\gamma_j})^2 \right) \\
& \leq Ce^{-c_1|x^{\geq i}|^2} \epsilon^{q_1}
\end{aligned}$$

for some  $q_1 > 0$ . With this, we estimate

$$(3.21) \quad |\psi_\epsilon(y) - \mathbb{I}| \leq C\epsilon^{-\beta(\alpha)} \int_{A_{\pm} - y} e^{-c_1|x^{\geq i}|^2} \epsilon^{q_1} dx \leq C\epsilon^{q_1}.$$

Note that

$$\mathbb{I} = \frac{\prod_{j < i} 2(L e^{-\lambda_j r(\epsilon)} \pm \epsilon^{\gamma_j - (\lambda_j \alpha - 1)})}{\prod_{j < i} 2L e^{-\lambda_j r(0)}} \mathbb{II}.$$

Also, clearly we have  $|\mathbb{II}| \leq C$ . Hence, due to (2.8) and (3.13) we have, for some  $q_2 > 0$ ,

$$(3.22) \quad |\mathbb{I} - \mathbb{II}| \leq \left| \frac{\prod_{j < i} 2(L e^{-\lambda_j r(\epsilon)} \pm \epsilon^{\gamma_j - (\lambda_j \alpha - 1)})}{\prod_{j < i} 2L e^{-\lambda_j r(0)}} - 1 \right| |\mathbb{II}| \leq C\epsilon^{q_2}.$$

For the term  $|\mathbb{II} - \psi(y)|$ , note that if  $i = d + 1$ , then  $\mathbb{II} = \psi(y)$ . Let us consider the case  $i \leq d$ . Due to (3.4), we have that if either  $\alpha < \frac{1}{\lambda_i}$  and  $j \geq i$ , or  $\alpha = \frac{1}{\lambda_i}$  and  $j \geq i + 1$ , then

$$\begin{aligned}
(3.23) \quad & \int_{\mathbb{R} \setminus (A_{\pm} - y)^j} e^{-c|x^j|^2} dx^j = \int_{\epsilon^{\lambda_j \alpha - 1} L e^{-\lambda_j r(\epsilon)} \pm \epsilon^{\gamma_j} - y^j}^{\infty} e^{-c|x^j|^2} dx^j \\
& \quad + \int_{-\infty}^{-\epsilon^{\lambda_j \alpha - 1} L e^{-\lambda_j r(\epsilon)} \mp \epsilon^{\gamma_j} - y^j} e^{-c|x^j|^2} dx^j \\
& \leq 2 \int_{\epsilon^{\lambda_j \alpha - 1} L e^{-\lambda_j r(\epsilon)} \mp \epsilon^{\gamma_j} - K'(\epsilon)}^{\infty} e^{-c|x^j|^2} dx^j \leq C\epsilon^{q_2}.
\end{aligned}$$

For the case with  $\alpha < \frac{1}{\lambda_i}$ , by (3.18) and (3.23), we have

$$\begin{aligned}
(3.24) \quad & |\mathbb{II} - \psi(y)| \leq C \int_{\mathbb{R}^{d-i+1} \setminus (A_{\pm} - y)^{\geq i}} e^{-c|x^{\geq i}|^2} dx^{\geq i} \\
& \leq C \sum_{j \geq i} \int_{\mathbb{R} \setminus (A_{\pm} - y)^j} e^{-c|x^j|^2} dx^j \leq C\epsilon^{q_2}.
\end{aligned}$$

The case with  $\alpha = \frac{1}{\lambda_i}$  is more involved. Let

$$\text{III} = \frac{\prod_{j < i} 2Le^{-\lambda_j r(0)}}{\sqrt{(2\pi)^d \det \mathcal{C}_0}} \int_{[-Le^{-\lambda_i r(0)} - y, Le^{-\lambda_i r(0)} - y] \times (A_{\pm} - y)^{>i}} e^{-\frac{1}{2}\tilde{x}^T \mathcal{C}_\epsilon^{-1} \tilde{x}} dx^{\geq i}.$$

Then observe that, with  $\Delta$  denoting the symmetric difference of two sets, by (2.8),

$$\begin{aligned} |\text{II} - \text{III}| &\leq C \int_{(A_{\pm} - y)^{\geq i} \Delta ([-Le^{-\lambda_i r(0)} - y, Le^{-\lambda_i r(0)} - y] \times (A_{\pm} - y)^{>i})} e^{-c|x^{\geq i}|^2} dx^{\geq i} \\ &\leq C \int_{(A_{\pm} - y)^i \Delta [-Le^{-\lambda_i r(0)} - y, Le^{-\lambda_i r(0)} - y]} e^{-c|x^i|^2} dx^i \\ &\leq C(|Le^{-\lambda_i r(\epsilon)} - Le^{-\lambda_i r(0)}| + \epsilon^{\gamma_i}) \leq C\epsilon^{q_3} \end{aligned}$$

for some  $q_3 > 0$ . On the other hand, by (3.23), we have

$$\begin{aligned} |\text{III} - \psi(y)| &\leq C \int_{[-Le^{-\lambda_i r(0)} - y, Le^{-\lambda_i r(0)} - y] \times (\mathbb{R}^{d-i} \setminus (A_{\pm} - y)^{>i})} e^{-c|x^{\geq i}|^2} dx^{\geq i} \\ &\leq C \sum_{j > i} \int_{\mathbb{R} \setminus (A_{\pm} - y)^j} e^{-c|x^j|^2} dx^j \leq C\epsilon^{q_2}. \end{aligned}$$

The last two displays together give

$$(3.25) \quad \text{if } \alpha = \frac{1}{\lambda_i}, \quad \text{then } |\text{II} - \psi(y)| \leq C(\epsilon^{q_2} + \epsilon^{q_3}).$$

To estimate the last term  $|\psi(y) - \psi(\frac{f^{-1}(\epsilon y)}{\epsilon})|$ , first observe that by (3.4), there exists  $\epsilon_0$  such that for all  $\epsilon \leq \epsilon_0$ , if  $|y| \leq K'(\epsilon)$ , then  $\epsilon y \in f(O)$ . Due to (2.6), there is  $C > 0$  such that  $|\frac{f^{-1}(\epsilon y)}{\epsilon} - y| \leq C\epsilon|y|^2$  for all  $|y| \leq K'(\epsilon)$  with  $\epsilon \leq \epsilon_0$ . By this and (3.20), we have, using the exponential term to absorb powers of  $|y|$ ,

$$\begin{aligned} (3.26) \quad |\psi(y) - \psi(\frac{f^{-1}(\epsilon y)}{\epsilon})| &\leq C \int_{\mathbb{R}^{d-i+1}} e^{-c_1|\tilde{x}|^2} \left( \left| \frac{f^{-1}(\epsilon y)}{\epsilon} - y \right| + \left| \frac{f^{-1}(\epsilon y)}{\epsilon} - y \right|^2 \right) dx^{\geq i} \\ &\leq C \int_{\mathbb{R}^{d-i+1}} e^{-c_2|\tilde{x}|^2} (\epsilon + \epsilon^2) dx^{\geq i} \leq C\epsilon. \end{aligned}$$

Combining (3.19), (3.21), (3.22), (3.24), (3.25), and (3.26), we obtain (3.16).  $\square$

#### 4. APPROXIMATIONS

**4.1. Proof of Lemma 3.3.** Let us recall that  $\nu > 0$  is fixed and we work with processes defined in (3.7)–(3.10). We define an exit time along each direction:

$$(4.1) \quad \tau_j = \inf\{t > 0 : |Y_t^j| \geq L\}, \quad j = 1, 2, \dots, d.$$

Recalling (3.3) and (3.11), we obtain  $\tau = \min_{1 \leq j \leq d} \tau_j$ . By (3.4), there is  $\epsilon_0$  such that for  $\epsilon < \epsilon_0$ , we have  $|Y_0^j| = |\epsilon y^j| \leq L$  for all  $j$  and all  $y$  with  $|y| \leq K'(\epsilon)$ . This fact together with (3.7) and (4.1) implies that, for  $\epsilon < \epsilon_0$  and  $|y| \leq K'(\epsilon)$ ,

$$(4.2) \quad L = \epsilon e^{\lambda_j \tau_j} |y^j + U_{\tau_j}^j|, \quad \text{i.e., } \tau_j = \frac{1}{\lambda_j} \log \frac{L}{\epsilon |y^j + U_{\tau_j}^j|}.$$

Due to (3.12), on  $\{\tau > \alpha \log \epsilon^{-1} + r(\epsilon)\}$ , we have  $\tau_j > T_0$ , so (4.2) implies

$$\begin{aligned} (4.3) \quad \mathbb{P}\{\tau > \alpha \log \epsilon^{-1} + r(\epsilon)\} &= \mathbb{P}\{|y^j + U_{\tau_j}^j| < \epsilon^{\lambda_j \alpha - 1} L e^{-\lambda_j r(\epsilon)}, j = 1, \dots, d\} \\ &= \mathbb{P}\{|y^j + U_{\tau_j \vee T_0}^j| < \epsilon^{\lambda_j \alpha - 1} L e^{-\lambda_j r(\epsilon)} \text{ and } \tau_j > T_0, j = 1, \dots, d\}. \end{aligned}$$

Next, we approximate  $U_{\tau_j \vee T_0}^j$  by  $U_{T_0}^j$ . Using the definition of  $M_t$  given in (3.9) and the boundedness of  $F$  and  $r(\epsilon)$ , we get, for some  $C_1, C_2 > 0$ ,

$$\langle M^j \rangle_{\tau_j \vee T_0} - \langle M^j \rangle_{T_0} \leq C_1 e^{-2\lambda_j T_0} \leq C_2 \epsilon^{2\lambda_j \alpha}.$$

By the exponential martingale inequality (see [Bas11, Problem 12.10]), this leads to

$$\mathbb{P}\{|M_{\tau_j \vee T_0}^j - M_{T_0}^j| > \frac{1}{2}\epsilon^{\gamma_j}\} \leq 2 \exp(-\frac{1}{8C_2}\epsilon^{2\gamma_j - 2\lambda_j \alpha}),$$

where  $\gamma_j$  is chosen to satisfy (3.13). For the drift term  $V_t$ , by the boundedness of  $G$  and  $r(\epsilon)$ , we have the following estimate: for each  $q > 0$ , there is  $C_q > 0$  such that

$$\mathbb{P}\{|\epsilon V_{\tau_j \vee T_0}^j - \epsilon V_{T_0}^j| > \frac{1}{2}\epsilon^{\gamma_j}\} \leq (2\epsilon^{1-\gamma_j})^q \mathbb{E}|V_{\tau_j \vee T_0}^j - V_{T_0}^j|^q \leq C_q \epsilon^{(1-\gamma_j + \lambda_j \alpha)q}.$$

By choosing  $q$  large, we derive from the above two displays and (3.13) that

$$(4.4) \quad \mathbb{P}\{|U_{\tau_j \vee T_0}^j - U_{T_0}^j| > \epsilon^{\gamma_j}\} \leq \epsilon^\nu,$$

uniformly in  $y$  for  $\epsilon$  small.

Now (4.3) and (4.4) immediately imply the upper bound in Lemma 3.3.

To get the lower bound, first observe that by (4.3) we have

$$(4.5) \quad \begin{aligned} \mathbb{P}\{\tau > \log \epsilon^{-1} + r(\epsilon)\} &\geq \mathbb{P}\{|y^j + U_{\tau_j}^j| < \epsilon^{\lambda_j \alpha - 1} L e^{-\lambda_j r(\epsilon)}, \forall j; |U_{\tau_j}^j - U_{T_0}^j| \leq \epsilon^{\gamma_j}, \forall j\} \\ &\geq \mathbb{P}\{y + U_{T_0} \in A_-; |U_{\tau_j}^j - U_{T_0}^j| \leq \epsilon^{\gamma_j}, \forall j\} \\ &\geq \mathbb{P}\{y + U_{T_0} \in A_-\} - \mathbb{P}\{y + U_{T_0} \in A_-; |U_{\tau_j}^j - U_{T_0}^j| > \epsilon^{\gamma_j}, \exists j\}. \end{aligned}$$

To estimate the second term on the right-hand side, we bound it by

$$(4.6) \quad \mathbb{P}\{\tau \geq T_0; |U_{\tau_j}^j - U_{T_0}^j| > \epsilon^{\gamma_j}, \exists j\} + \mathbb{P}\{\tau < T_0; y + U_{T_0} \in A_-\}.$$

By (4.4), the first term can be bounded by  $d\epsilon^\nu$  for  $\epsilon$  small. For the second term, we first introduce the following notations. For  $x \in \mathbb{R}^d$ ,  $A \subset \mathbb{R}^d$ , and  $t \in \mathbb{R}$ , we write

$$(4.7) \quad e^{\lambda t} x = (e^{\lambda_j t} x^j)_{j=1}^d \in \mathbb{R}^d, \quad e^{\lambda t} A = \{e^{\lambda t} x : x \in A\} \subset \mathbb{R}^d.$$

We recall that if  $Y_0 = \epsilon y$ , then (3.7) holds. Using the strong Markov property of  $Y_t$  and the definition of  $A_-$  given in (3.14), we obtain

$$(4.8) \quad \begin{aligned} \mathbb{P}\{\tau < T_0; y + U_{T_0} \in A_-\} &= \mathbb{P}\{\tau < T_0; Y_{T_0} \in \epsilon e^{\lambda T_0} A_-\} \\ &\leq \sum_{j=1}^d \mathbb{P}\{\tau_j < T_0; |Y_{T_0}^j| \leq L - \epsilon^{1-\lambda_j \alpha + \gamma_j} e^{\lambda_j r(\epsilon)}\} \\ &\leq \sum_{j=1}^d \mathbb{E} \mathbb{P}^{Y_{\tau_j}} \left\{ \inf_{t \in [0, T_0]} |Y_t^j| \leq L - \epsilon^{1-\lambda_j \alpha + \gamma_j} e^{\lambda_j r(\epsilon)} \right\}, \end{aligned}$$

where  $\mathbb{P}^y$  denotes the probability measure under which  $Y_t$  satisfies (3.1) with  $Y_0 = \mathbf{y} \in \mathbb{R}^d$ . Note that if  $|Y_0^j| = L$ , then  $|Y_t^j| = |e^{\lambda_j t} (Y_0^j + \epsilon U_t^j)| \geq L - \epsilon |U_t^j|$ . By this, using  $-\lambda_j \alpha + \gamma_j < 0$  (which is due to (3.13)), the boundedness of  $V_t$ , and the exponential

martingale inequality, we have, for some  $c, c' > 0$  and small  $\epsilon$ ,

$$\begin{aligned}
(4.9) \quad & \mathbb{P}^{Y_{\tau_j}} \left\{ \inf_{t \in [0, T_0]} |Y_t^j| \leq L - \epsilon^{1-\lambda_j \alpha + \gamma_j} e^{\lambda_j r(\epsilon)} \right\} \\
& \leq \mathbb{P}^{Y_{\tau_j}} \left\{ \inf_{t \in [0, T_0]} (L - \epsilon |U_t^j|) \leq L - \epsilon^{1-\lambda_j \alpha + \gamma_j} e^{\lambda_j r(\epsilon)} \right\} \leq \mathbb{P}^{Y_{\tau_j}} \left\{ \epsilon^{-\lambda_j \alpha + \gamma_j} e^{\lambda_j r(\epsilon)} \leq \sup_{t \in [0, T_0]} |U_t^j| \right\} \\
& \leq \mathbb{P}^{Y_{\tau_j}} \left\{ \epsilon^{-\lambda_j \alpha + \gamma_j} e^{\lambda_j r(\epsilon)} - c\epsilon \leq \sup_{t \in [0, T_0]} |M_t^j| \right\} \leq 2 \exp(-c' \epsilon^{2(-\lambda_j \alpha + \gamma_j)}) \leq \epsilon^\nu.
\end{aligned}$$

Combining (4.5), (4.6), (4.8), and (4.9) leads to the desired lower bound.

**4.2. Proof of Lemma 3.4.** First of all, we state two density estimates that we need. For a random variable  $\mathcal{X}$  with values in  $\mathbb{R}^d$ , its Lebesgue density, if exists, is denoted by  $\rho_{\mathcal{X}}$ . Since  $U_t$  in (3.7) depends on  $y$ , we denote its density by  $\rho_{U_t}^y$ .

**Lemma 4.1.** *Consider (3.7) with  $Y_0 = \epsilon y$ . Let*

$$(4.10) \quad p(x) = \sum_{j,k=1}^d x^{\frac{\lambda_j}{\lambda_k}}, \quad \text{for } x \geq 0,$$

$$(4.11) \quad Z_t^j = \int_0^t e^{-\lambda_j s} F_l^j(0) dW_s^l.$$

Then

1) *there is  $\theta > 0$  such that for each  $\nu \in (0, 1)$  there are  $C, c, \delta > 0$  such that, for  $\epsilon$  sufficiently small,*

$$|\rho_{U_{T(\epsilon)}}^y(x) - \rho_{Z_{T(\epsilon)}}(x)| \leq C\epsilon^\delta (1 + p(\epsilon^{1-\nu}|y|)) e^{-c|x|^2}, \quad x, y \in \mathbb{R}^d,$$

*holds for all deterministic functions  $T(\cdot)$  satisfying  $1 \leq T(\epsilon) \leq \theta \log \epsilon^{-1}$ ,  $\epsilon \in (0, 1)$ ;*  
2) *for each  $\theta' > 0$ , there are  $C', c', \delta' > 0$  such that, for  $\epsilon$  sufficiently small,*

$$|\rho_{Z_{T(\epsilon)}}(x) - \rho_{Z_\infty}(x)| \leq C' \epsilon^{\delta'} e^{-c'|x|^2}, \quad x \in \mathbb{R}^d,$$

*holds for all deterministic functions  $T(\cdot)$  satisfying  $T(\epsilon) \geq \theta' \log \epsilon^{-1}$ ,  $\epsilon \in (0, 1)$ .*

This lemma will be proved in Section 5.

We recall the notation introduced in 4.7 and the definition of  $T_0 = T_0(\epsilon)$  in (3.12). We set  $N = \min\{n \in \mathbb{N} : \frac{T_0}{n} \leq \theta \log \epsilon^{-1}, \forall \epsilon \in (0, 1/2]\}$ , where  $\theta$  was introduced in Lemma 4.1, and  $t_k = \frac{k}{N} T_0$ . Hence, each increment  $t_k - t_{k-1}$  satisfies the condition imposed on time  $T(\epsilon)$  in part 1 of Lemma 4.1, so we can get the following iteration result.

**Lemma 4.2.** *For each  $\nu \in (0, 1)$ , there are constants  $\epsilon_k, C_k, \delta_k > 0$ ,  $k = 1, 2, \dots, N$ , and  $\nu' > 0$  such that*

$$\begin{aligned}
(4.12) \quad & \sup_{|y| \leq \epsilon^{\nu-1}} \sup_{|w| \leq \epsilon^{\nu'-1}} |\mathbb{P}^{\epsilon y} \{y + U_{t_k} + e^{-\lambda t_k} w \in A_\pm\} - \mathbb{P} \{y + Z_{t_k} + e^{-\lambda t_k} w \in A_\pm\}| \leq C_k \epsilon^{\beta(\alpha) + \delta_k}, \\
& \sup_{|y| \leq \epsilon^{\nu-1}} \sup_{|w| \leq \epsilon^{\nu'-1}} |\mathbb{P}^{\epsilon y} \{y + U_{t_k} + e^{-\lambda t_k} w \in A_\pm\} - \mathbb{P}^{\epsilon y} \{y + Z_{t_k} + e^{-\lambda t_k} w \in A_\pm\}| \leq C_k \epsilon^{\beta(\alpha) + \delta_k},
\end{aligned}$$

*holds for each  $k = 1, 2, \dots, N$  and for all  $\epsilon \in (0, \epsilon_k]$ .*

Let us first derive Lemma 3.4 from Lemmas 4.1 and 4.2, and then return to the proof of the latter.

*Proof of Lemma 3.4.* Set  $k = N$  and  $w = 0$  in Lemma 4.2. As  $t_N = T_0$ , we have that for each  $v \in (0, 1)$ , there is  $\delta > 0$  such that

$$\sup_{|y| \leq \epsilon^{v-1}} |\mathbb{P}^{\epsilon y} \{y + U_{T_0} \in A_{\pm}\} - \mathbb{P} \{y + Z_{T_0} \in A_{\pm}\}| = o(\epsilon^{\beta(\alpha)+\delta}).$$

It is easy to see from (4.11) that  $Z_{\infty}$  is defined (in the sense of a.s.-convergence) and has the same distribution as  $\mathcal{Z}$ : it is a centered Gaussian vector with covariance matrix (2.11) since  $F(0) = \sigma(0)$  by (3.2). Taking  $\theta' > 0$  such that  $T_0 \geq \theta' \log \epsilon^{-1}$  for all  $\epsilon$ , part 2 of Lemma 4.1 and the definition of  $A_{\pm}$  given in (3.14), imply that, for  $\epsilon$  sufficiently small,

$$|\mathbb{P} \{y + Z_{T_0} \in A_{\pm}\} - \mathbb{P} \{y + \mathcal{Z} \in A_{\pm}\}| = o(\epsilon^{\beta(\alpha)+\delta'}), \quad \forall y \in \mathbb{R}^d.$$

The above two displays together imply the desired result.  $\square$

*Proof of Lemma 4.2.* Let us choose  $v' \in (0, 1)$  to satisfy

$$(4.13) \quad \frac{\lambda_j \alpha}{N} \geq \frac{\lambda_d \alpha}{N} > v', \quad \text{for all } j = 1, 2, \dots, d.$$

For the case  $k = 1$ , (4.12) follows from Lemma 4.1 and the definition of  $A_{\pm}$  in (3.14). Then we proceed by induction. Assume (4.12) holds for  $k - 1$  with  $k \leq N$ .

Set  $z(u) = e^{\lambda t_{k-1}}(y + u)$ . The strong Markov property of  $Y_t$  implies that

$$(4.14) \quad \begin{aligned} \mathbb{P}^{\epsilon y} \{y + U_{t_k} + e^{-\lambda t_k} w \in A_{\pm}\} &= \mathbb{P}^{\epsilon y} \{Y_{t_k} + \epsilon w \in \epsilon e^{\lambda t_k} A_{\pm}\} \\ &= \mathbb{E}^{\epsilon y} \mathbb{P}^{Y_{t_{k-1}}} \{Y_{t_1} + \epsilon w \in \epsilon e^{\lambda t_k} A_{\pm}\} \\ &= \mathbb{E}^{\epsilon y} \left( \mathbb{P}^{\epsilon z(u)} \{z(u) + U_{t_1} + e^{-\lambda t_1} w \in e^{\lambda t_{k-1}} A_{\pm}\} \Big|_{u=U_{t_{k-1}}} \right). \end{aligned}$$

We will show the error of replacing  $U_{t_1}$  by  $Z_{t_1}$  and  $U_{t_{k-1}}$  by  $Z_{t_{k-1}}$  is small.

Using Lemma 4.1 (1) with  $v'$  in place of  $v$ , we see that there are  $\delta', C', c'$  such that

$$\begin{aligned} &|\mathbb{P}^{\epsilon z(u)} \{z(u) + U_{t_1} + e^{-\lambda t_1} w \in e^{\lambda t_{k-1}} A_{\pm}\} - \mathbb{P}^{\epsilon z(u)} \{z(u) + Z_{t_1} + e^{-\lambda t_1} w \in e^{\lambda t_{k-1}} A_{\pm}\}| \\ &\leq \int_{\{x \in \mathbb{R}^d : z(u) + x + e^{-\lambda t_1} w \in e^{\lambda t_{k-1}} A_{\pm}\}} C' \epsilon^{\delta'} (1 + p(\epsilon^{1-v'} |z(u)|)) e^{-c' |x|^2} dx. \end{aligned}$$

By (4.13),  $t_{k-1} = \frac{k-1}{N} T_0$ , and  $k \leq N$ , we have

$$e^{\lambda_j t_{k-1}} \epsilon^{\lambda_j \alpha - 1} \leq e^{\lambda_j t_{N-1}} \epsilon^{\lambda_j \alpha - 1} \leq \epsilon^{\frac{1}{N} \lambda_j \alpha - 1} e^{\frac{N-1}{N} r(\epsilon)} < \epsilon^{v'-1} e^{\frac{N-1}{N} r(\epsilon)}.$$

Together with the definition of  $A_{\pm}$  in (3.14), this implies that, for some  $C > 0$ ,

$$\epsilon^{1-v'} |z(u)| \leq C + \epsilon^{1-v'} |x|,$$

for  $z(u)$  satisfying  $z(u) + x + e^{-\lambda t_1} w \in e^{\lambda t_{k-1}} A_{\pm}$  and  $|w| \leq \epsilon^{v'-1}$ . Using  $e^{-c' |x|^2}$  to absorb powers of  $|x|$ , the above three displays give, for some  $C, c > 0$ ,

$$\begin{aligned} &|\mathbb{P}^{\epsilon z(u)} \{z(u) + U_{t_1} + e^{-\lambda t_1} w \in e^{\lambda t_{k-1}} A_{\pm}\} - \mathbb{P} \{z(u) + Z_{t_1} + e^{-\lambda t_1} w \in e^{\lambda t_{k-1}} A_{\pm}\}| \\ &\leq \epsilon^{\delta'} \int_{\{x \in \mathbb{R}^d : z(u) + x + e^{-\lambda t_1} w \in e^{\lambda t_{k-1}} A_{\pm}\}} C e^{-c |x|^2} dx, \quad |w| \leq \epsilon^{v'-1}. \end{aligned}$$

Let  $\mathcal{N}$  be a centered Gaussian with density proportional to  $e^{-c|x|^2}$  and independent of  $\mathcal{F}_{t_{k-1}}$ . Then the above display and (4.14) imply that

$$\begin{aligned} \mathbb{I} &= \left| \mathbb{P}^{\epsilon y} \{y + U_{t_k} + e^{-\lambda t_k} w \in A_{\pm}\} - \mathbb{E}^{\epsilon y} \left( \mathbb{P} \{z(u) + Z_{t_1} + e^{-\lambda t_k} w \in e^{\lambda t_{k-1}} A_{\pm}\} \mid u = U_{t_{k-1}} \right) \right| \\ &\leq C\epsilon^{\delta'} \mathbb{P}^{\epsilon y} \{y + U_{t_{k-1}} + e^{-\lambda t_k} w + e^{-\lambda t_{k-1}} \mathcal{N} \in A_{\pm}\}, \quad |w| \leq \epsilon^{v'-1}. \end{aligned}$$

Then we choose  $\rho$  large so that

$$(4.15) \quad \mathbb{P}\{|\mathcal{N}| > \rho \log \epsilon^{-1}\} = o(\epsilon^{\beta(\alpha)}), \quad \mathbb{P}\{|Z_{t_1}| > \rho \log \epsilon^{-1}\} = o(\epsilon^{\beta(\alpha)+\delta'}).$$

Note that  $e^{-\lambda t_1}$  decays like a small positive power of  $\epsilon$ . So, there is  $\epsilon_k$  such that

$$(4.16) \quad \text{if } |w| \leq \epsilon^{v'-1} \text{ for } \epsilon \leq \epsilon_k, \quad \text{then } |e^{-\lambda t_1} w| + \rho \log \epsilon^{-1} \leq \epsilon^{v'-1} \text{ for } \epsilon \leq \epsilon_{k-1}.$$

Then, the following holds uniformly over  $|y| \leq \epsilon^{v-1}$ ,  $|w| \leq \epsilon^{v'-1}$  and  $\epsilon \in (0, \epsilon_k]$ :

$$\begin{aligned} \mathbb{I} &\leq C\epsilon^{\delta'} \mathbb{P}^{\epsilon y} \{y + U_{t_{k-1}} + e^{-\lambda t_{k-1}} (e^{-\lambda t_1} w + \mathcal{N}) \in A_{\pm}; |\mathcal{N}| \leq \rho \log \epsilon^{-1}\} + o(\epsilon^{\beta(\alpha)+\delta'}) \\ &\leq C\epsilon^{\delta'} \mathbb{P} \{y + Z_{t_{k-1}} + e^{-\lambda t_{k-1}} (e^{-\lambda t_1} w + \mathcal{N}) \in A_{\pm}\} + o(\epsilon^{\beta(\alpha)+\delta_{k-1}+\delta'}) + o(\epsilon^{\beta(\alpha)+\delta'}), \end{aligned}$$

where in the second inequality we used the induction assumption allowed by (4.16), independence of  $\mathcal{N}$ , Fubini's theorem, and (4.15). One can check that for  $k-1 \geq 1$ , there are  $C, c > 0$  such that  $\rho_{Z_{t_{k-1}}}(x) \leq Ce^{-c|x|^2}$  for all  $x \in \mathbb{R}^d$ . We also recall that  $i = i(\alpha)$  is given in (2.9). Hence, we can estimate, using Fubini's theorem, the definition of  $A_{\pm}$  given in (3.14), the definition of  $\beta(\alpha)$  in (2.10) and notations given in (2.12)–(3.17),

$$\begin{aligned} \mathbb{P} \{y + Z_{t_{k-1}} + e^{-\lambda t_{k-1}} (e^{-\lambda t_1} w + \mathcal{N}) \in A_{\pm}\} &\leq \mathbb{E} \int_{A_{\pm} - y - e^{-\lambda t_{k-1}} (e^{-\lambda t_1} w + \mathcal{N})} Ce^{-c|x|^2} dx \\ &\leq C \int_{(A_{\pm})^{< i} \times \mathbb{R}^{d-i}} e^{-c|x^{\geq i}|^2} dx \leq C |(A_{\pm})^{< i}| \leq C' \epsilon^{\beta(\alpha)}. \end{aligned}$$

The above two displays indicate that, for some  $\delta'' > 0$

$$(4.17) \quad \mathbb{I} = o(\epsilon^{\beta(\alpha)+\delta''}), \quad \text{uniformly over } |y| \leq \epsilon^{v-1}, |w| \leq \epsilon^{v'-1}.$$

Then we estimate the error caused by replacing  $U_{t_{k-1}}$  by  $Z_{t_{k-1}}$ . Let  $\tilde{Z}_{t_1}$  be a copy of  $Z_{t_1}$  independent of  $\mathcal{F}_{t_{k-1}}$ . Using this independence and (4.15), we have that the following holds uniformly over  $|y| \leq \epsilon^{v-1}$  and  $|w| \leq \epsilon^{v'-1}$  with  $\epsilon \in (0, \epsilon_k]$ :

$$\begin{aligned} &\mathbb{E}^{\epsilon y} \left( \mathbb{P} \{z(u) + Z_{t_1} + e^{-\lambda t_1} w \in e^{\lambda t_{k-1}} A_{\pm}\} \mid u = U_{t_{k-1}} \right) \\ &= \mathbb{P}^{\epsilon y} \{y + U_{t_{k-1}} + e^{-\lambda t_{k-1}} (e^{-\lambda t_1} w + \tilde{Z}_{t_1}) \in A_{\pm}\} \\ &= \mathbb{P}^{\epsilon y} \{y + U_{t_{k-1}} + e^{-\lambda t_{k-1}} (e^{-\lambda t_1} w + \tilde{Z}_{t_1}) \in A_{\pm}; |\tilde{Z}_{t_1}| \leq \rho \log \epsilon^{-1}\} + o(\epsilon^{\beta(\alpha)+\delta'}) \\ &= \mathbb{P}^{\epsilon y} \{y + Z_{t_{k-1}} + e^{-\lambda t_{k-1}} (e^{-\lambda t_1} w + \tilde{Z}_{t_1}) \in A_{\pm}; |\tilde{Z}_{t_1}| \leq \rho \log \epsilon^{-1}\} + o(\epsilon^{\beta(\alpha)+\delta_{k-1}}) + o(\epsilon^{\beta(\alpha)+\delta'}) \\ &= \mathbb{P}^{\epsilon y} \{y + Z_{t_{k-1}} + e^{-\lambda t_{k-1}} \tilde{Z}_{t_1} + e^{-\lambda t_k} w \in A_{\pm}\} + o(\epsilon^{\beta(\alpha)+\delta_{k-1} \wedge \delta'}), \end{aligned}$$

where we used the induction assumption in the third identity allowed by (4.16), independence of  $\tilde{Z}_{t_1}$  and Fubini's theorem. By this independence again, a simple computation reveals that  $Z_{t_{k-1}} + e^{-\lambda t_{k-1}} \tilde{Z}_{t_1}$  has the same distribution as that of  $Z_{t_k}$ .

Then the above display implies that

$$\begin{aligned} & \left| \mathbb{E}^{\epsilon y} \left( \mathbb{P}\{z(u) + Z_{t_1} + e^{-\lambda t_1} w \in e^{\lambda t_{k-1}} A_{\pm}\} \mid u = U_{t_{k-1}} \right) - \mathbb{P}^{\epsilon y} \{y + Z_{t_k} + e^{-\lambda t_k} w \in A_{\pm}\} \right| \\ &= o(\epsilon^{\beta(\alpha)+\delta_{k-1}\wedge\delta'}), \quad \text{uniformly over } |y| \leq \epsilon^{v-1}, |w| \leq \epsilon^{v'-1} \text{ and } \epsilon \in (0, \epsilon_k]. \end{aligned}$$

From this and (4.17), we derive (4.12) for  $k$ , which completes our proof.  $\square$

## 5. DENSITY ESTIMATE

In this section, we prove Lemma 4.1.

We briefly introduce Malliavin calculus notations. For  $\mathcal{T} > 0$ , on  $(W_t, t \in [0, \mathcal{T}])$ , let  $\mathcal{D}$  be the derivative operator;  $\sigma_{\mathcal{X}}$  be the Malliavin covariance matrix for a random vector  $\mathcal{X} \in \mathcal{F}_{\mathcal{T}}$ ;  $\|\cdot\|_{k,p,\mathcal{T}}$  be the Sobolev norm defined in terms of derivatives up to the  $k$ th order with  $L^p$  integrability;  $\mathbb{D}^{k,p}(\mathcal{T})$  be the corresponding Sobolev space, in particular,  $\mathbb{D}^{k,\infty}(\mathcal{T}) = \cap_{p \geq 1} \mathbb{D}^{k,p}(\mathcal{T})$ . More details can be found in [Nua95].

Theorem 2.14.B from [BC14] estimates the difference between derivatives of two densities in terms of Sobolev norms and the covariance matrix. For our purposes, in our statement of this result, Theorem 5.1 below, we simplify the conditions of the original theorem by setting the localization random variable  $\Theta$  to be 1, the derivative order  $q = 0$  and using Meyer's inequality (c.f. [Nua95, Theorem 1.5.1]) to bound the Ornstein–Uhlenbeck operator. We stress that, although the conditions of Theorem 2.14.B as it is stated in [BC14] do not formally allow for  $q = 0$ , that theorem is still valid for this value of  $q$ . In fact, in [BC14], Theorem 2.14 is derived from Theorem 2.1 via an approximation argument. In turn, part B of Theorem 2.1 is restated and proved in the form of Theorem 3.10, where  $q$  is allowed to be 0.

**Theorem 5.1.** *For  $i = 1, 2$ , let  $\mathcal{X}_i \in \mathbb{D}^{3,\infty}(\mathcal{T})$  with values in  $\mathbb{R}^d$  satisfy  $\mathbb{E}(\det \sigma_{\mathcal{X}_i})^{-p} < \infty$  for every  $p > 1$ .*

*Then there exist positive constants  $C, a, b, \gamma$  only depending on  $d$  such that for all  $x \in \mathbb{R}^d$*

$$\begin{aligned} |\rho_{\mathcal{X}_1}(x) - \rho_{\mathcal{X}_2}(x)| &\leq C \|\mathcal{X}_1 - \mathcal{X}_2\|_{2,\gamma,\mathcal{T}} \left( \prod_{i=1,2} \left( 1 \vee \mathbb{E}(\det \sigma_{\mathcal{X}_i})^{-\gamma} \right) \left( 1 + \|\mathcal{X}_i\|_{3,\gamma,\mathcal{T}} \right) \right)^a \\ &\quad \cdot \left( \sum_{i=1,2} \mathbb{P}\{|\mathcal{X}_i - x| < 2\} \right)^b. \end{aligned}$$

The independence of  $C, a, p$  of  $\mathcal{T}$  is important because we will replace  $\mathcal{T}$  by a function of  $\epsilon$  converging to  $\infty$  as  $\epsilon \rightarrow 0$ .

Let us fix  $\theta$  and  $\epsilon_0$  such that

$$2\lambda_1\theta \leq 1, \quad \text{and} \quad \epsilon^{\frac{1}{2}}\theta \log(\epsilon^{-1}) \leq 1, \quad \epsilon \in (0, \epsilon_0].$$

For all deterministic  $T = T(\epsilon)$  satisfying  $1 \leq T \leq \theta \log(\epsilon^{-1})$ , we have

$$(5.1) \quad \epsilon e^{2\lambda_j T} \leq \epsilon e^{2\lambda_1 T} \leq 1 \text{ for all } j, \quad \text{and} \quad \epsilon^{\frac{1}{2}} T \leq 1, \quad \epsilon \in (0, \epsilon_0].$$

Now, arbitrarily fix such a  $T = T(\epsilon)$ . We will use  $\mathcal{T} = T(\epsilon)$  and simply write  $\|\cdot\|_{k,p} = \|\cdot\|_{k,p,T(\epsilon)}$ .

In the following, we use  $\lesssim$  to omit a positive multiplicative constant independent of  $\epsilon$  and  $T = T(\epsilon) \in [1, \theta \log(\epsilon^{-1})]$ . Sometimes such a constant will be denoted explicitly but generically as  $C$ . We also use the bracket  $[\cdot]_p = (\mathbb{E}|\cdot|^p)^{\frac{2}{p}}$  for  $p \geq 2$  and

note that this bracket satisfies the following properties, by BDG, Cauchy–Schwarz and Minkowski’s integral inequalities, for  $p \geq 2$ :

$$(5.2) \quad \begin{aligned} \left[ \int_{t_1}^{t_2} \mathcal{X}_{l,s} dW_s^l \right]_p &= \left( \mathbb{E} \left| \int_{t_1}^{t_2} \mathcal{X}_{l,s} dW_s^l \right|^p \right)^{\frac{2}{p}} \lesssim \left( \mathbb{E} \left| \int_{t_1}^{t_2} |\mathcal{X}_s|^2 ds \right|^{\frac{p}{2}} \right)^{\frac{2}{p}} \\ &\leq \int_{t_1}^{t_2} \left( \mathbb{E} |\mathcal{X}_s|^p \right)^{\frac{2}{p}} ds = \int_{t_1}^{t_2} [\mathcal{X}_s]_p ds, \end{aligned}$$

$$(5.3) \quad \left[ \int_{t_1}^{t_2} \mathcal{X}_s ds \right]_p = \left( \mathbb{E} \left| \int_{t_1}^{t_2} \mathcal{X}_s ds \right|^p \right)^{\frac{2}{p}} \leq \left( \int_{t_1}^{t_2} (\mathbb{E} |\mathcal{X}_s|^p)^{\frac{1}{p}} ds \right)^2 \leq |t_2 - t_1| \int_{t_1}^{t_2} [\mathcal{X}_s]_p ds,$$

$$\left[ \left( \int_{E \subset \mathbb{R}^n} |\mathcal{X}_{s^1, s^2, \dots, s^n}|^2 ds^1 ds^2 \dots ds^n \right)^{\frac{1}{2}} \right]_p \leq \int_{E \subset \mathbb{R}^n} [\mathcal{X}_{s^1, s^2, \dots, s^n}]_p ds^1 ds^2 \dots ds^n.$$

Let  $\mathcal{H} = L^2([0, T], \mathbb{R}^d)$ . The last property above implies

$$(5.4) \quad [\|\mathcal{X}\|_{\mathcal{H}^{\otimes n}}]_p \leq \int_{[0, T]^n} [\mathcal{X}_{s^1, s^2, \dots, s^n}]_p ds^1 ds^2 \dots ds^n, \quad n \in \mathbb{N} \setminus \{0\},$$

where  $\mathcal{H}^{\otimes n}$  is the  $n$ -fold tensor product of  $\mathcal{H}$  and  $\mathcal{X}$  is an  $\mathcal{H}^{\otimes n}$ -valued random variable.

In the following, we fix an arbitrary  $p \geq 2$ , and use the above properties.

**5.1. Estimates of Malliavin derivatives.** The formulae for Malliavin derivatives of a solution of an SDE can be found in [Nua95, Section 2.2.2]. We will use them without further notice.

**Remark 5.2.** In [Nua95, Section 2.2.2], the coefficients of the SDE are required to be  $C^\infty$  in order to compute Malliavin derivatives of all orders but here we need to work only with Malliavin derivatives up to order 3, and our assumptions on smoothness of the coefficients are sufficient.

Let  $N_t = U_t - Z_t$  and  $H(\cdot) = F(\cdot) - F(0)$ . By (3.6), there are  $C_{H,1}, C_{H,2} > 0$  such that,

$$(5.5) \quad |H(x)| \leq C_{H,1}|x|, \quad |H(x)| \leq C_{H,2}, \quad x \in \mathbb{R}^d.$$

For  $0 \leq r \leq t \leq T$ , by (3.7) and (4.11), easy calculations yield

$$(5.6) \quad \begin{aligned} N_t^i &= \int_0^t e^{-\lambda_i s} H_l^i(Y_s) dW_s^l + \epsilon V_t^i; \quad \mathcal{D}_r^j Z_t^i = e^{-\lambda_i r} F_j^i(0); \\ \mathcal{D}_r^j Y_t^i &= \epsilon e^{\lambda_i t} (\mathcal{D}_r^j U_t^i) = \epsilon e^{\lambda_i t} (\mathcal{D}_r^j N_t^i + \mathcal{D}_r^j Z_t^i). \end{aligned}$$

**5.1.1. 0th order derivatives.** For some  $\beta \in (0, 1)$  to be chosen later, we define the stopping times  $\eta_k = \inf\{t > 0 : |Y_t^k| \geq \epsilon^\beta\}$  and  $\eta = \min_{1 \leq k \leq d} \{\eta_k\}$

Using (5.5), (5.6), and the boundedness of  $V_t^i$  we have

$$\begin{aligned}
\mathbb{E}|N_T^i|^p &\lesssim \mathbb{E} \left| \int_0^T |e^{-\lambda_i s} H^i(Y_s)|^2 ds \right|^{\frac{p}{2}} + \epsilon^p \\
&\lesssim \mathbb{E} \left| \int_0^{T \wedge \eta} |e^{-\lambda_i s} H^i(Y_s)|^2 ds \right|^{\frac{p}{2}} + \mathbb{E} \left| \int_{T \wedge \eta}^T |e^{-\lambda_i s} H^i(Y_s)|^2 ds \right|^{\frac{p}{2}} + \epsilon^p \\
(5.7) \quad &\lesssim \sum_{k=1}^d \mathbb{E} \left| \int_0^{T \wedge \eta} |e^{-\lambda_i s} Y_s^k|^2 ds \right|^{\frac{p}{2}} + \mathbb{E} \left| \int_{T \wedge \eta}^T |e^{-\lambda_i s}|^2 ds \right|^{\frac{p}{2}} + \epsilon^p \\
&\lesssim \sum_{k=1}^d \mathbb{E} \left| \int_0^{T \wedge \eta} |e^{-\lambda_i s} \epsilon^\beta|^2 ds \right|^{\frac{p}{2}} + \mathbb{E} e^{-p\lambda_i \eta} + \epsilon^p \\
&\lesssim \epsilon^{p\beta} + \mathbb{E} e^{-p\lambda_i \eta} + \epsilon^p.
\end{aligned}$$

By the definition of  $\eta_k$  and the relation  $\epsilon^\beta \leq |Y_{\eta_k}^k| = \epsilon e^{\lambda_k \eta_k} |y^k + U_{\eta_k}^k|$ , we have  $\eta_k \geq \frac{1}{\lambda_k} \log(\epsilon^{\beta-1} |y^k + U_{\eta_k}^k|^{-1})$ , which implies that

$$\begin{aligned}
\mathbb{E} e^{-p\lambda_i \eta} &\leq \sum_{k=1}^d \mathbb{E} e^{-p\lambda_i \eta_k} \leq \sum_{k=1}^d \epsilon^{(1-\beta)p \frac{\lambda_i}{\lambda_k}} \mathbb{E} |y^k + U_{\eta_k}^k|^{p \frac{\lambda_i}{\lambda_k}} \\
&\lesssim \sum_{k=1}^d (\epsilon^{(1-\beta)} |y^k|)^{p \frac{\lambda_i}{\lambda_k}} + \sum_{k=1}^d \epsilon^{(1-\beta)p \frac{\lambda_i}{\lambda_k}} \mathbb{E} |U_{\eta_k}^k|^{p \frac{\lambda_i}{\lambda_k}}.
\end{aligned}$$

Note that any positive moment of  $U_{\eta_k}$  is bounded by an absolute constant independent of  $\epsilon$ . Recall the definition of  $p$  given in (4.10). Then, in view of the above display and (5.7), for an arbitrary  $v \in (0, 1)$  we can choose  $\beta = \frac{1}{2}v$  so that there is  $\delta_0$  independent of  $p$  such that

$$(5.8) \quad [N_T^i]_p \lesssim \epsilon^{\delta_0} (1 + p(\epsilon^{1-v} |y^k|))^2, \quad i = 1, 2, \dots, d, \quad \epsilon \in (0, \epsilon_0].$$

5.1.2. *1st order derivatives.* Consider  $r \leq t \leq T$ . Before estimating  $\mathcal{D}_r^j N_t^i$ , we first study  $\mathcal{D}_r^j U_t^i$ . Observe that

$$\begin{aligned}
(5.9) \quad \mathcal{D}_r^j U_t^i &= e^{-\lambda_i r} F_j^i(Y_r) + \epsilon \int_r^t e^{(\lambda_k - \lambda_i)s} \partial_k F_l^i(Y_s) \mathcal{D}_r^j U_s^k dW_s^l + \epsilon^2 \int_r^t e^{(\lambda_k - \lambda_i)s} \partial_k G^i(Y_s) \mathcal{D}_r^j U_s^k ds.
\end{aligned}$$

Hence, using the boundedness of the derivatives of  $F$  and  $G$  due to (3.7), the  $[\cdot]_p$  properties (5.2) and (5.3), and lastly (5.1), we have

$$\begin{aligned}
[\mathcal{D}_r^j U_t^i]_p &\lesssim [e^{-\lambda_i r} F_j^i(Y_r)]_p + \left[ \epsilon \int_r^t e^{(\lambda_k - \lambda_i)s} \partial_k F_l^i(Y_s) \mathcal{D}_r^j U_s^k dW_s^l \right]_p \\
&\quad + \left[ \epsilon^2 \int_r^t e^{(\lambda_k - \lambda_i)s} \partial_k G^i(Y_s) \mathcal{D}_r^j U_s^k ds \right]_p \\
&\lesssim e^{-2\lambda_i r} + (\epsilon^2 + \epsilon^4 T) \sum_{k=1}^d \int_r^t e^{2(\lambda_k - \lambda_i)s} [\mathcal{D}_r^j U_s^k]_p ds \\
&\lesssim e^{-2\lambda_i r} + \epsilon^2 \sum_{k=1}^d \int_r^t e^{2(\lambda_k - \lambda_i)s} [\mathcal{D}_r^j U_s^k]_p ds.
\end{aligned}$$

We fix  $j, r$  momentarily and set  $a^i(t) = [\mathcal{D}_r^j U_t^i]_p$ , obtaining a system of inequalities

$$(5.10) \quad a^i(t) \lesssim e^{-2\lambda_i r} + \epsilon^2 \sum_{k=1}^d \int_r^t e^{2(\lambda_k - \lambda_i)s} a^k(s) ds, \quad i = 1, 2, \dots, d.$$

This type of systems will occur a few more times. So, it is useful to state the following bound proved in Section 5.4:

**Lemma 5.3.** *Let  $d \in \mathbb{N}$  and  $m \geq 0$ . Then the system of inequalities*

$$(5.11) \quad 0 \leq a^i(t) \lesssim \epsilon^m e^{-2\lambda_i r} + \epsilon^2 \sum_{k=1}^d \int_r^t e^{2(\lambda_k - \lambda_i)s} a^k(s) ds, \quad t \in [r, T], \quad i = 1, 2, \dots, d,$$

with  $T$  and  $\epsilon \in (0, \epsilon_0]$  satisfying (5.1), implies that there is a constant  $C$  independent of  $\epsilon$ ,  $T$ , and  $r$  such that  $a^i(t) \leq C\epsilon^m e^{-\lambda_d r}$  for all  $t \in [r, T]$  and  $i = 1, 2, \dots, d$ .

Applying this lemma to (5.10), we obtain

$$(5.12) \quad [\mathcal{D}_r^j U_t^i]_p = a^i(t) \leq C e^{-2\lambda_d r}, \quad r \leq t \leq T, \quad p \geq 2, \quad \epsilon \in (0, \epsilon_0],$$

which gives, by (5.1) and (5.4),

$$(5.13) \quad [\|\mathcal{D}U_T\|_{\mathcal{H}}]_p \leq \int_0^T [\mathcal{D}_r U_T]_p dr \lesssim \sum_{i,j=1}^d \int_0^T [\mathcal{D}_r^j U_T^i]_p dr \lesssim 1, \quad p \geq 2.$$

The following estimate implied by (5.1) and (5.12) will be used later:

$$(5.14) \quad [\mathcal{D}_r^j Y_t^i]_p = \epsilon^2 e^{2\lambda_i t} [\mathcal{D}_r^j U_t^i]_p \lesssim \epsilon^2 e^{2\lambda_i t} e^{-2\lambda_d r} \lesssim \epsilon, \quad r \leq t \leq T, \quad p \geq 2, \quad \epsilon \in (0, \epsilon_0].$$

Then we proceed to estimating  $\mathcal{D}_r^j N_t^i$ . The calculation (5.6) gives, for  $r \leq t \leq T$ ,

$$\mathcal{D}_r^j N_t^i = e^{-\lambda_i r} H_j^i(Y_r) + \int_r^t e^{-\lambda_i s} \partial_k H_l^i(Y_s) \mathcal{D}_r^j Y_s^k dW_s^l + \epsilon \int_r^t e^{-\lambda_i s} \partial_k G^i(Y_s) \mathcal{D}_r^j Y_s^k ds,$$

which implies

$$\begin{aligned} [\|\mathcal{D}N_T\|_{\mathcal{H}}]_p &\lesssim \sum_{i=1}^d \left( \mathbb{E} \left| \int_0^T |e^{-\lambda_i r} H^i(Y_r)|^2 dr \right|^{\frac{p}{2}} \right)^{\frac{2}{p}} \\ &+ \sum_{i,j,k=1}^d \left[ \left\| \int_0^T e^{-\lambda_i s} \partial_k H_l^i(Y_s) \mathcal{D}_r^j Y_s^k dW_s^l \right\|_{\mathcal{H}} \right]_p \\ &+ \sum_{i,j,k=1}^d \epsilon^2 \left[ \left\| \int_0^T e^{-\lambda_i s} \partial_k G^i(Y_s) \mathcal{D}_r^j Y_s^k ds \right\|_{\mathcal{H}} \right]_p. \end{aligned}$$

The terms in the first sum of the above display appeared in (5.7), and thus are  $\lesssim \epsilon^{\delta_0} (1 + p(\epsilon^{1-v} |y|))^2$ . For the next two sums, we first invoke properties (5.2), (5.3) and (5.4), and then apply (5.14) and (5.1) to get

$$\begin{aligned} \left[ \left\| \int_0^T e^{-\lambda_i s} \partial_k H_l^i(Y_s) \mathcal{D}_r^j Y_s^k dW_s^l \right\|_{\mathcal{H}} \right]_p &\leq \int_0^T \left[ \int_r^T e^{-\lambda_i s} \partial_k H_l^i(Y_s) \mathcal{D}_r^j Y_s^k dW_s^l \right]_p dr \\ &\lesssim \int_0^T \int_r^T \sum_l [e^{-\lambda_i s} \partial_k H_l^i(Y_s) \mathcal{D}_r^j Y_s^k]_p ds dr \lesssim \int_0^T \int_r^T e^{-2\lambda_i s} [\mathcal{D}_r^j Y_s^k]_p ds dr \lesssim \epsilon T \leq \epsilon^{\frac{1}{2}} \end{aligned}$$

and similarly,

$$\epsilon^2 \left[ \left\| \int_0^T e^{-\lambda_i s} \partial_k G^i(Y_s) \mathcal{D}^j Y_s^k ds \right\|_{\mathcal{H}} \right]_p \lesssim \epsilon^2 T \int_0^T \int_r^T e^{-2\lambda_i s} [\mathcal{D}_r^j Y_s^k]_p ds dr \leq \epsilon^2.$$

Therefore, we conclude that for some  $\delta_1 > 0$ ,

$$(5.15) \quad [\|\mathcal{D}N_T\|_{\mathcal{H}}]_p \lesssim \epsilon^{\delta_1} (1 + p(\epsilon^{1-v} |y|))^2, \quad p \geq 2, \quad \epsilon \in (0, \epsilon_0].$$

5.1.3. *2nd order derivatives.* Since 2nd order derivatives of  $Z_t$  vanish as can be seen in (5.6), we have

$$(5.16) \quad \mathcal{D}^{(2)} N_t = \mathcal{D}^{(2)} U_t,$$

where the superscript indicates the order of differentiation. So we only need to study the latter.

Let us rewrite (3.7) as

$$\begin{aligned} U_t^j &= \int_0^t e^{-\lambda_j s} F_l^j(Y_s) dW_s^l + \epsilon \int_0^t e^{-\lambda_j s} G^j(Y_s) ds \\ &= \int_0^t e^{-\lambda_j s} F_l^j(\epsilon e^{\lambda s} (y + U_s)) dW_s^l + \epsilon \int_0^t e^{-\lambda_j s} G^j(\epsilon e^{\lambda s} (y + U_s)) ds \end{aligned}$$

and apply formula (2.54) in [Nua95, Section 2.2] to this equation in place of equation (2.37) therein. For  $r_1, r_2 \leq t \leq T$ , we obtain

$$\begin{aligned} \mathcal{D}_{r_1, r_2}^{j_1, j_2} U_t^i &= e^{-\lambda_i r_1} \partial_k F_{j_1}^i(Y_{r_1}) \mathcal{D}_{r_2}^{j_2} Y_{r_1}^k + e^{-\lambda_i r_2} \partial_k F_{j_2}^i(Y_{r_2}) \mathcal{D}_{r_1}^{j_1} Y_{r_2}^k \\ &+ \int_{r_1 \vee r_2}^t e^{-\lambda_i s} (\partial_{k_1, k_2}^2 F_l^i(Y_s)) (\mathcal{D}_{r_1}^{j_1} Y_s^{k_1}) (\mathcal{D}_{r_2}^{j_2} Y_s^{k_2}) dW_s^l + \epsilon \int_{r_1 \vee r_2}^t e^{(\lambda_k - \lambda_i)s} \partial_k F_l^i(Y_s) \mathcal{D}_{r_1, r_2}^{j_1, j_2} U_s^k dW_s^l \\ &+ \epsilon \int_{r_1 \vee r_2}^t e^{-\lambda_i s} (\partial_{k_1, k_2}^2 G^i(Y_s)) (\mathcal{D}_{r_1}^{j_1} Y_s^{k_1}) (\mathcal{D}_{r_2}^{j_2} Y_s^{k_2}) ds + \epsilon^2 \int_{r_1 \vee r_2}^t e^{(\lambda_k - \lambda_i)s} \partial_k G^i(Y_s) \mathcal{D}_{r_1, r_2}^{j_1, j_2} U_s^k ds. \end{aligned}$$

Here we choose to express some terms only in terms of the process  $Y$  while some terms are expressed in terms of both  $U$  and  $Y$  (we recall that by (3.7)  $Y_t^j = \epsilon e^{\lambda_j t} (y^j + U_t^j)$ ).

This, along with (5.2), (5.3), the Cauchy–Schwarz inequality and the boundedness of derivatives of  $F$  and  $G$ , implies that

$$\begin{aligned} (5.17) \quad [\mathcal{D}_{r_1, r_2}^{j_1, j_2} U_t^i]_p &\lesssim e^{-2\lambda_i r_1} [\mathcal{D}_{r_2}^{j_2} Y_{r_1}^k]_p + e^{-2\lambda_i r_2} [\mathcal{D}_{r_1}^{j_1} Y_{r_2}^k]_p \\ &+ (1 + \epsilon^2 T) \sum_{k_1, k_2=1}^d \int_{r_1 \vee r_2}^t e^{-2\lambda_i s} [\mathcal{D}_{r_1}^{j_1} Y_s^{k_1}]_{2p} [\mathcal{D}_{r_2}^{j_2} Y_s^{k_2}]_{2p} ds \\ &+ (\epsilon^2 + \epsilon^4 T) \sum_{k=1}^d \int_{r_1 \vee r_2}^t e^{2(\lambda_k - \lambda_i)s} [\mathcal{D}_{r_1, r_2}^{j_1, j_2} U_s^k]_p ds. \end{aligned}$$

Let us temporarily fix  $j_1, j_2, r_1, r_2$  and set  $a^i(t) = [\mathcal{D}_{r_1, r_2}^{j_1, j_2} U_t^i]_p$  and  $r = r_1 \wedge r_2$ . Then, using (5.1) and (5.14) for  $p$  and  $2p$ , from the above display we obtain

$$\begin{aligned} a^i(t) &\lesssim e^{-2\lambda_i r_1} \epsilon + e^{-2\lambda_i r_2} \epsilon + \int_{r_1 \vee r_2}^t e^{-2\lambda_i s} \epsilon^2 ds + \epsilon^2 \sum_{k=1}^d \int_{r_1 \vee r_2}^t e^{2(\lambda_k - \lambda_i)s} a^k(s) ds \\ &\lesssim \epsilon e^{-2\lambda_i r} + \epsilon^2 \sum_{k=1}^d \int_r^t e^{2(\lambda_k - \lambda_i)s} a^k(s) ds, \end{aligned}$$

which by Lemma 5.3 implies

$$(5.18) \quad [\mathcal{D}_{r_1, r_2}^{j_1, j_2} U_t^i]_p = a^i(t) \leq C\epsilon e^{-2\lambda_d r}, \quad r_1, r_2 \leq t \leq T, \quad p \geq 2, \quad \epsilon \in (0, \epsilon_0].$$

This, along with (3.7) and (5.1) implies the following estimate which will be used later:

$$(5.19) \quad [\mathcal{D}_{r_1, r_2}^{j_1, j_2} Y_t^i]_p = \epsilon^2 e^{2\lambda_i t} [\mathcal{D}_{r_1, r_2}^{j_1, j_2} U_t^2]_p \lesssim \epsilon^3 e^{2\lambda_1 T} \lesssim \epsilon^2, \quad r_1, r_2 \leq t \leq T, \quad p \geq 2, \quad \epsilon \in (0, \epsilon_0].$$

Lastly we obtain, by (5.4), (5.16) and (5.18),

$$(5.20) \quad \begin{aligned} [\|\mathcal{D}^{(2)} N_T\|_{\mathcal{H}^{\otimes 2}}]_p &= [\|\mathcal{D}^{(2)} U_T\|_{\mathcal{H}^{\otimes 2}}]_p \lesssim \sum_{i, j_1, j_2=1}^d \int_{[0, T]^2} [\mathcal{D}_{r_1, r_2}^{j_1, j_2} U_T^i]_p dr_1 dr_2 \\ &\lesssim \epsilon \sum_{i=1}^d \int_{[0, T]^2} e^{-2\lambda_d r_1 \wedge r_2} dr_1 dr_2 \lesssim \epsilon T \leq \epsilon^{\frac{1}{2}}, \quad p \geq 2. \end{aligned}$$

5.1.4. *3rd order derivatives.* Similarly to the above argument for second order derivatives, we apply (2.54) from [Nua95, Section 2.2] to obtain that for  $r_1, r_2, r_3 \leq t \leq T$ ,

$$\begin{aligned} &\mathcal{D}_{r_1, r_2, r_3}^{j_1, j_2, j_3} U_t^i \\ &= \frac{1}{2} \sum_{\{n_0, n_1, n_2\}=\{1, 2, 3\}} e^{-\lambda_i r_{n_0}} \left( \partial_{k_1, k_2}^2 F_{j_{n_0}}^i(Y_{r_{n_0}}) \prod_{m=1}^2 \mathcal{D}_{r_{n_m}}^{j_{n_m}} Y_{r_{n_0}}^{k_m} + \partial_k F_{j_{n_0}}^i(Y_{r_{n_0}}) \mathcal{D}_{r_{n_1}, n_2}^{j_{n_1}, j_{n_2}} Y_{r_{n_0}}^k \right) \\ &+ \int_{r_1 \vee r_2 \vee r_3}^t e^{-\lambda_i s} \left( \partial_{k_1, k_2, k_3}^3 F_l^i(Y_s) \prod_{m=1}^3 \mathcal{D}_{r_m}^{j_m} Y_s^{k_m} \right. \\ &+ \frac{1}{2} \sum_{\{n_1, n_2, n_3\}=\{1, 2, 3\}} \partial_{k_1, k_2}^2 F_l^i(Y_s) (\mathcal{D}_{r_{n_1}, r_{n_2}}^{j_{n_1}, j_{n_2}} Y_s^{k_1}) (\mathcal{D}_{r_{n_3}}^{j_{n_3}} Y_s^{k_2}) + \epsilon e^{\lambda_k s} \partial_k F_l^i(Y_s) \mathcal{D}_{r_1, r_2, r_3}^{j_1, j_2, j_3} U_s^k \left. \right) dW_s^l \\ &+ \epsilon \left( \text{a similar integral with } F_l^i \text{ and } dW_s^l \text{ replaced by } G^i \text{ and } ds, \text{ respectively} \right), \end{aligned}$$

where the factor of 1/2 comes from counting certain terms twice. Let us temporarily fix  $j_1, j_2, j_3, r_1, r_2, r_3$  and set  $a^i(t) = [\mathcal{D}_{r_1, r_2, r_3}^{j_1, j_2, j_3} U_t^i]_p$  and  $r = r_1 \wedge r_2 \wedge r_3$ . Then, similarly to (5.17), using Hölder's inequality, the  $[\cdot]_p$  properties (5.2), (5.3), estimates (5.14) and (5.19) for  $p, 2p, 3p$ , and lastly (5.1), we obtain

$$\begin{aligned} a^i(t) &\lesssim e^{-2\lambda_i r} \left( \epsilon^2 + \epsilon^2 \right) + \int_r^t e^{-2\lambda_i s} \left( \epsilon^3 + \epsilon^3 + \sum_{k=1}^d \epsilon^2 e^{2\lambda_k s} a^k(s) \right) ds \\ &\lesssim \epsilon^2 e^{-2\lambda_i r} + \epsilon^2 \sum_{k=1}^d \int_r^t e^{2(\lambda_k - \lambda_i)s} a^k(s) ds, \end{aligned}$$

which by Lemma 5.3 yields

$$[\mathcal{D}_{r_1, r_2, r_3}^{j_1, j_2, j_3} U_t^i]_p = a^i(t) \leq C\epsilon^2 e^{-2\lambda_d r}, \quad r_1, r_2, r_3 \leq t \leq T, \quad p \geq 2.$$

Finally, by (5.4) and (5.1) we have, with  $r = r_1 \wedge r_2 \wedge r_3$ ,

$$(5.21) \quad \begin{aligned} [\|\mathcal{D}^{(3)}U_T\|_{\mathcal{H}^{\otimes 3}}]_p &\lesssim \sum_{i,j_1,j_2,j_3=1}^d \int_{[0,T]^3} [\mathcal{D}_{r_1,r_2,r_3}^{j_1,j_2,j_3} U_T^i]_p dr_1 dr_2 dr_3 \\ &\lesssim \epsilon^2 \sum_{i=1}^d \int_{[0,T]^3} e^{-2\lambda_d r} dr_1 dr_2 dr_3 \lesssim \epsilon^2 T^2 \leq \epsilon, \quad p \geq 2, \quad \epsilon \in (0, \epsilon_0]. \end{aligned}$$

5.1.5. *Conclusion of derivative estimates.* Combining estimates (5.8), (5.15), (5.20), and Jensen's inequality, we obtain, for each  $p \geq 1$ , all  $\epsilon \in (0, \epsilon_0]$ ,

$$(5.22) \quad \|U_T - Z_T\|_{2,p} = \|N_T\|_{2,p} = [N_T]_p^{\frac{1}{2}} + \sum_{k=1}^2 [\|\mathcal{D}^k N_T\|_{\mathcal{H}^{\otimes k}}]_p^{\frac{1}{2}} \lesssim \epsilon^\delta (1 + p(\epsilon^{1-v}|y|))$$

for some  $\delta > 0$ .

By (5.13), (5.18), (5.21), Jensen's inequality and the easy observation that all moments of  $U_t$  are bounded uniformly in  $t$ , we have

$$(5.23) \quad \|U_T\|_{3,p} \lesssim 1, \quad p \geq 1, \quad \epsilon \in (0, \epsilon_0].$$

Lastly, a simple calculation shows that

$$(5.24) \quad \|Z_T\|_{3,p} \lesssim 1, \quad p \geq 1, \quad \epsilon \in (0, \epsilon_0].$$

5.2. **Negative moments for determinants of Malliavin matrices  $\sigma_{U_T}$  and  $\sigma_{Z_T}$ .** The goal is to show for each  $p \geq 1$  there is a  $C_p > 0$  such that

$$(5.25) \quad \mathbb{E}|\det \sigma_{U_T}|^{-p}, \quad \mathbb{E}|\det \sigma_{Z_T}|^{-p} \leq C_p, \quad \epsilon \in (0, \epsilon_0].$$

Using the formula of  $\mathcal{D}Z_t$  in (5.6) and that  $F(0) = \sigma(0)$  is of full rank, it is easy to verify (5.25) for  $\sigma_{Z_T}$  as it is deterministic. For  $\sigma_{U_T}$ , we first simplify the expression for  $\mathcal{D}_r^j U_t^i$  in (5.9). Let

$$(5.26) \quad \begin{aligned} A_j^i(r) &= e^{-\lambda_i r} F_j^i(Y_r), \quad \overline{A}_{k,l}^i(s) = \epsilon e^{(\lambda_k - \lambda_l)s} \partial_k F_l^i(Y_s), \quad \overline{B}_k^i(s) = \epsilon^2 e^{(\lambda_k - \lambda_l)s} \partial_k G^i(Y_s). \end{aligned}$$

Then, we can rewrite (5.9) as

$$\mathcal{D}_r^j U_t^i = A_j^i(r) + \int_r^t \overline{A}_{k,l}^i(s) \mathcal{D}_r^j U_s^k dW_s^l + \int_r^t \overline{B}_k^i(s) \mathcal{D}_r^j U_s^k ds.$$

By the boundedness of derivatives of  $F$ ,  $G$  and (5.1), we have, for some  $C > 0$ ,

$$(5.27) \quad |A_j^i(s)| \leq C e^{-\lambda_i s}, \quad |\overline{A}_{k,l}^i(s)| \leq C \epsilon^{\frac{1}{2}}, \quad |\overline{B}_k^i(s)| \leq C \epsilon^{\frac{3}{2}}, \quad s \leq T, \quad \epsilon \in (0, \epsilon_0].$$

Let us introduce two  $d \times d$ -matrix-valued processes, where  $\delta_j^i$  is the Kronecker delta,

$$(5.28) \quad \begin{aligned} \mathbf{Y}_j^i(t) &= \delta_j^i + \int_0^t \left( \overline{A}_{k,l}^i(s) \mathbf{Y}_j^k(s) dW_s^l + \overline{B}_k^i(s) \mathbf{Y}_j^k(s) ds \right), \\ \mathbf{Z}_j^i(t) &= \delta_j^i - \int_0^t \left( \overline{A}_{j,l}^k \mathbf{Z}_k^i(s) dW_s^l + (\overline{B}_j^k(s) - \sum_{l=1}^d \overline{A}_{m,l}^k(s) \overline{A}_{j,l}^m(s)) \mathbf{Z}_k^i(s) ds \right). \end{aligned}$$

These two processes correspond to (2.57) and (2.58) in [Nua95, Section 2.3.1]. The computations below (2.58) there show  $\mathbf{Z}(t)\mathbf{Y}(t) = \mathbf{Y}(t)\mathbf{Z}(t) = I$  the identity matrix. In addition, (2.60) and (2.61) from [Nua95, Section 2.3.1] state that  $\sigma_{U_t}$  satisfies

$$(5.29) \quad \sigma_{U_t} = \mathbf{Y}(t) \mathbf{C}_t \mathbf{Y}(t)^\top$$

where  $\top$  denotes the matrix transpose operation and

$$(5.30) \quad \mathbf{C}_t^{ij} = \sum_{l=1}^d \int_0^t \mathbf{Z}_k^i(s) A_l^k(s) \mathbf{Z}_m^j(s) A_l^m(s) ds.$$

Then, observe that, by (5.29) and Hölder's inequality,

$$(5.31) \quad \mathbb{E} |\det \sigma_{U_T}|^{-p} \leq (\mathbb{E} |\det \mathbf{C}_T|^{-2p})^{\frac{1}{2}} (\mathbb{E} |\det \mathbf{Z}(T)|^{4p})^{\frac{1}{2}}, \quad p \geq 1.$$

Therefore, to prove boundedness of  $\mathbb{E} |\det \sigma_{U_T}|^{-p}$ , it suffices to prove that it holds for  $\mathbb{E} |\det \mathbf{C}_T|^{-2p}$  and  $\mathbb{E} |\det \mathbf{Z}(T)|^{4p}$ . We first bound the latter.

Although it is more than what we need here, we shall find a bound on moments of  $\tilde{\mathbf{Z}}(T)$  with  $\tilde{\mathbf{Z}}_j^i(t) = \sup_{0 \leq r \leq t} |\mathbf{Z}_j^i(r)|$ , which will be used later. By (5.27), we have

$$\tilde{\mathbf{Z}}_j^i(T) \lesssim 1 + \sup_{0 \leq r \leq T} \left| \int_0^r \bar{A}_{j,l}^k \mathbf{Z}_k^i(s) dW_s^l \right| + \int_0^T \epsilon \sum_{k=1}^d \tilde{\mathbf{Z}}_k^i(s) ds.$$

Then, using BDG inequality, the  $[\cdot]_p$  properties (5.2) and (5.3), (5.1) and (5.27), we obtain, for all  $p \geq 2$  and  $\epsilon \in (0, \epsilon_0]$ ,

$$[\tilde{\mathbf{Z}}_j^i(T)]_p \lesssim 1 + \sum_{k=1}^d \int_0^T (\epsilon + \epsilon^2 T) [\tilde{\mathbf{Z}}_k^i(s)]_p ds \lesssim 1 + \epsilon \sum_{k=1}^d \int_0^T [\tilde{\mathbf{Z}}_k^i(s)]_p ds.$$

Summing up the above in  $j$  and using Gronwall's inequality, we get, for some  $c > 0$ ,

$$(5.32) \quad [\tilde{\mathbf{Z}}_j^i(T)]_p \leq \sum_{k=1}^d [\tilde{\mathbf{Z}}_k^i(T)]_p \lesssim e^{c\epsilon T} \lesssim 1, \quad p \geq 2, \quad \epsilon \in (0, \epsilon_0].$$

Using this and the expression of the matrix determinant as a polynomial of the entries, we apply Hölder's inequality to conclude that for each  $p \geq 1$  there is  $C_p > 0$  such that  $(\mathbb{E} |\det \mathbf{Z}(T)|^{4p})^{\frac{1}{2}} \leq C_p$ ,  $\epsilon \in (0, \epsilon_0]$ .

To bound  $\mathbb{E} |\det \mathbf{C}_T|^{-2p}$  for all  $p \geq 1$ , it suffices to show that, for each  $p \geq 1$ , there is  $C_p > 0$  such that  $\mathbb{P}\{\nu \leq \zeta\} \leq C_p \zeta^p$ , where  $\nu$  is the smallest eigenvalue of  $\mathbf{C}_T$ . Note that  $\nu \geq 0$ , because  $\mathbf{C}_T$  is positive semi-definite, which can be derived from (5.29) since  $\sigma_{U_T}$  is positive semi-definite and  $\mathbf{Y}(T)$  is invertible. We need the following lemma which will be proved in Section 5.4.

**Lemma 5.4.** *Let  $\mathcal{A}$  be a symmetric positive semi-definite random  $d \times d$  matrix. Let  $\nu$  be its smallest eigenvalue. Then for each  $p \geq 1$ , there is a constant  $C_{p,d} > 0$  such that*

$$(5.33) \quad \mathbb{P}\{\nu \leq \zeta\} \leq C_{p,d} \left( \sup_{|v|=1} \mathbb{E} |\langle v, \mathcal{A}v \rangle|^{-(p+2d)} + \mathbb{E} \left| \sum_{i,j=1}^d |\mathcal{A}^{ij}|^2 \right|^{\frac{p}{2}} \right) \zeta^p, \quad \zeta \geq 0.$$

For each  $p > 1$ , by (5.27), (5.32) and Hölder's inequality, we have

$$\begin{aligned} (\mathbb{E} |\mathbf{C}_T^{ij}|^p)^{\frac{1}{p}} &\leq \int_0^T (\mathbb{E} |\mathbf{Z}_k^i(s) A_l^k(s) \mathbf{Z}_m^j(s) A_l^m(s)|^p)^{\frac{1}{p}} ds \\ &\lesssim \sum_{k,m=1}^d \int_0^T e^{-\lambda_k s} e^{-\lambda_m s} ds \lesssim 1, \quad \epsilon \in (0, \epsilon_0]. \end{aligned}$$

Hence, for each  $p \geq 1$ , there is  $c_p > 0$  such that  $\mathbb{E} \left| \sum_{i,j=1}^d |\mathbf{C}_T^{ij}|^2 \right|^{\frac{p}{2}} \leq c_p$ ,  $\epsilon \in (0, \epsilon_0]$ . Therefore, if we can show that for each  $p \geq 1$  there is  $C_p$  such that

$$(5.34) \quad \sup_{|v|=1} \mathbb{E} |\langle v, \mathbf{C}_T v \rangle|^{-p} \leq C_p, \quad \epsilon \in (0, \epsilon_0],$$

then Lemma 5.4 and the discussion above imply that  $\mathbb{E} |\det \mathbf{C}_T|^{-p}$  is bounded for each  $p \geq 1$ , when  $\epsilon$  is small (in comparison with [Nua95, Lemma 2.3.1], we need a bound that is uniform in  $\epsilon \in (0, \epsilon_0]$ ). Consequently, this and (5.31) will imply the desired result (5.25). Therefore, it remains to show (5.34).

*Proof of (5.34).* Let us fix an arbitrary  $v \in \mathbb{S}^{d-1}$ , the  $(d-1)$ -sphere. By the definition of  $\mathbf{C}_t$  given in (5.30),

$$\langle v, \mathbf{C}_T v \rangle = \int_0^T |v^\top \mathbf{Z}(s) A(s)|^2 ds.$$

Recall  $A_j^i(s)$  given in (5.26). By (3.6), we have, in the sense of positive semi-definite matrices,  $F(Y_s)F(Y_s)^\top \geq c_0 I$ . Therefore, we get

$$(5.35) \quad \langle v, \mathbf{C}_T v \rangle \geq c_0 \int_0^T \left| \left( \sum_{i=1}^d v_i \mathbf{Z}_j^i(s) e^{-\lambda_j s} \right)_{j=1}^d \right|^2 ds, \quad \epsilon \in (0, \epsilon_0].$$

Let us define

$$(5.36) \quad \begin{aligned} R_t^j &= \sqrt{c_0} \sum_{i=1}^d v_i \mathbf{Z}_j^i(t) e^{-\lambda_j t} = r_0^j + M_t^j + A_t^j \\ &= r_0^j + \int_0^t u_l^j(s) dW_s^l + \int_0^t a^j(s) ds, \quad j = 1, 2, \dots, d, \end{aligned}$$

and additionally  $N_t^j = \int_0^t R^j(s) u_l^j(s) dW_s^l$ ,  $j = 1, 2, \dots, d$ , where, by the Itô formula and the expression for  $\mathbf{Z}(t)$  given in (5.28),

$$(5.37) \quad \begin{aligned} r_0^j &= \sqrt{c_0} v_j, & u_l^j(s) &= -\sqrt{c_0} \sum_{i,k=1}^d v_i e^{-\lambda_j s} \bar{A}_{j,l}^k(s) \mathbf{Z}_k^i(s), \\ a^j(s) &= - \left( \sqrt{c_0} \sum_{i=1}^d v_i \lambda_j e^{-\lambda_j s} \mathbf{Z}_j^i(s) \right) - \left( \sqrt{c_0} \sum_{i,k,m,l} v_i e^{-\lambda_j s} (\bar{B}_j^k(s) - \bar{A}_{m,l}^k(s) \bar{A}_{j,l}^m(s)) \mathbf{Z}_k^i(s) \right). \end{aligned}$$

Then, (5.35) and (5.36) imply

$$\mathbb{P} \{ \langle v, \mathbf{C}_T v \rangle \leq \zeta \} \leq \mathbb{P} \left\{ \int_0^T |R_s|^2 ds \leq \zeta \right\}, \quad \epsilon \in (0, \epsilon_0].$$

Recall that  $T = T(\epsilon) \geq 1$  is assumed. Since  $v \in \mathbb{S}^{d-1}$  is arbitrary, Lemma 5.5 that we state and prove below implies (5.34).  $\square$

**Lemma 5.5.** *Let  $\epsilon_0$  be given in (5.1), and  $R_s$  be given in (5.36) which depends on the choice of  $v \in \mathbb{S}^{d-1}$ . For each  $p \geq 1$ , there is  $C_p > 0$  independent of  $v$  such that*

$$(5.38) \quad \mathbb{P} \left\{ \int_0^T |R_s|^2 ds \leq \zeta \right\} \leq C_p \zeta^{\frac{1}{8}p}, \quad \epsilon \in (0, \epsilon_0].$$

This lemma is a variation of [Nua95, Lemma 2.3.2].

*Proof of Lemma 5.5.* By (5.27), (5.32) and (5.37), there is  $c_p > 0$  independent of  $v$  such that

$$\mathbb{E} \sup_{0 \leq s \leq T} |u(s)|^p \leq c_p; \quad \mathbb{E} \left( \int_0^T |a(s)|^2 ds \right)^p \leq c_p, \quad \epsilon \in (0, \epsilon_0].$$

This and Markov's inequality imply that for some  $c_p > 0$  independent of  $v \in \mathbb{S}^{d-1}$ ,

$$(5.39) \quad \mathbb{P} \left\{ \sup_{0 \leq s \leq T} \left( |u(s)| + \int_0^s |a(r)|^2 dr \right) > \zeta^{-\frac{1}{8}} \right\} \leq c_p \zeta^{\frac{1}{8}p}, \quad \zeta > 0, \quad \epsilon \in (0, \epsilon_0].$$

Recalling the definitions of  $M_t$  and  $N_t$  in (5.36), we define, for each  $\zeta > 0$  and each  $\epsilon \in (0, \epsilon_0]$ ,

$$\begin{aligned} B_0^{\zeta, \epsilon} &= \left\{ \int_0^T |R_s|^2 ds \leq \zeta, \sup_{0 \leq s \leq T} \left( |u(s)| + \int_0^s |a(r)|^2 dr \right) \leq \zeta^{-\frac{1}{8}} \right\}, \\ B_{1,j}^{\zeta, \epsilon} &= \{ \langle M^j \rangle_T \leq 2\zeta^{\frac{1}{4}}, \sup_{0 \leq t \leq T} |M_t^j| \geq d^{-1}\zeta^{\frac{1}{16}} \}, \\ B_{2,j}^{\zeta, \epsilon} &= \{ \langle N^j \rangle_T \leq \zeta^{\frac{3}{4}}, \sup_{0 \leq t \leq T} |N_t^j| \geq \zeta^{\frac{1}{4}} \}, \end{aligned}$$

where the dependence on  $\epsilon$  comes from  $T = T(\epsilon)$ ,  $R_s$ ,  $u(s)$ ,  $a(s)$ ,  $M_s$ , and  $N_s$ . The exponential martingale inequality implies that, for some  $c_p > 0$  independent of  $v$ ,

$$\mathbb{P} \{ (\cup_{j=1}^d B_{1,j}^{\zeta, \epsilon}) \cup (\cup_{j=1}^d B_{2,j}^{\zeta, \epsilon}) \} \leq 2d \exp(-\frac{\zeta^{-\frac{1}{8}}}{4d^2}) + 2d \exp(-\frac{\zeta^{-\frac{1}{4}}}{2}) \leq c_p \zeta^{\frac{1}{8}p}, \quad \zeta > 0, \quad \epsilon \in (0, \epsilon_0].$$

Observe that by this and (5.39), we can attain the desired result (5.38) if we can show there is a  $\zeta_0 > 0$  such that

$$(5.40) \quad B_0^{\zeta, \epsilon} \subset (\cup_{j=1}^d B_{1,j}^{\zeta, \epsilon}) \cup (\cup_{j=1}^d B_{2,j}^{\zeta, \epsilon}), \quad \zeta \in (0, \zeta_0), \quad \epsilon \in (0, \epsilon_0].$$

Hence, it remains to show (5.40). Choose a  $\zeta_0$  to satisfy, with  $c_0, r_0$  given in (5.37),

$$(5.41) \quad 2d(\zeta_0^{\frac{1}{4}} + \zeta_0^{\frac{7}{16}}) \leq c_0 = |r_0|^2, \quad 4\zeta_0^{\frac{1}{16}} \leq \sqrt{c_0}, \quad \text{and} \quad \zeta_0^{\frac{1}{3}} \leq \frac{1}{2}.$$

We show (5.40) with this chosen  $\zeta_0$ . Argue by contradiction. Suppose (5.40) is false. Then, for some  $\zeta \in (0, \zeta_0)$  and some  $\epsilon \in (0, \epsilon_0]$ , there is

$$(5.42) \quad \omega \in B_0^{\zeta, \epsilon} - \left( (\cup_{j=1}^d B_{1,j}^{\zeta, \epsilon}) \cup (\cup_{j=1}^d B_{2,j}^{\zeta, \epsilon}) \right).$$

From now on, fix this pair of  $\zeta$  and  $\epsilon$ , and evaluate all random variables at this  $\omega$ .

By  $\omega \in B_0^{\zeta, \epsilon}$  due to (5.42), we clearly have

$$\langle N^j \rangle_T \leq \int_0^T |R_s^j u^j(s)|^2 ds \leq \left( \sup_{0 \leq s \leq T} |u(s)|^2 \right) \int_0^T |R_s^j|^2 ds \leq \zeta^{-\frac{2}{8}+1} = \zeta^{\frac{3}{4}}.$$

Then, since  $\omega \notin B_{2,j}^{\zeta, \epsilon}$ ,  $j = 1, 2, \dots, d$ , due to (5.42), we deduce

$$\sup_{0 \leq t \leq T} \left| \int_0^t R_s^j u^j(s) dW_s^l \right| = \sup_{0 \leq t \leq T} |N_t^j| < \zeta^{\frac{1}{4}}, \quad j = 1, 2, \dots, d.$$

By  $\omega \in B_0^{\zeta, \epsilon}$  due to (5.42) and the Cauchy-Schwarz inequality, we have

$$\sup_{0 \leq t \leq T} \left| \int_0^t R_s^j a^j(s) ds \right| \leq \left( \int_0^T |R_s^j|^2 ds \right)^{\frac{1}{2}} \left( \int_0^T |a^j(s)|^2 ds \right)^{\frac{1}{2}} \leq \zeta^{\frac{1}{2}-\frac{1}{16}} = \zeta^{\frac{7}{16}}.$$

Itô formula applied to (5.36) gives  $|R_t|^2 = |r_0|^2 + \sum_{j=1}^d 2(\int_0^t R_s^j u_l^j dW_s^l + \int_0^t R_s^j a^j(s) ds) + \sum_{j=1}^d \langle M^j \rangle_t$ . The above two displays, (5.41), and  $\omega \in B_0^{\zeta, \epsilon}$  due to (5.42) imply

$$\begin{aligned} \int_0^T \sum_{j=1}^d \langle M^j \rangle_t dt &= \int_0^T |R_s|^2 dt - T|r_0|^2 - \int_0^T \left( \sum_{j=1}^d 2 \int_0^t R_s^j dR_s^j \right) dt \\ &\leq \zeta - T|r_0|^2 + 2dT(\zeta^{\frac{1}{4}} + \zeta^{\frac{7}{16}}) \leq \zeta. \end{aligned}$$

Because  $t \mapsto \sum_{j=1}^d \langle M^j \rangle_t$  is nondecreasing, the above display indicates

$$\gamma \sum_{j=1}^d \langle M^j \rangle_{T-\gamma} \leq \zeta, \quad \gamma \leq 1 \leq T.$$

Since  $\omega \in B_0^{\zeta, \epsilon}$  implies  $\sup_{0 \leq s \leq T} |u(s)| \leq \zeta^{-\frac{1}{8}}$ , by the definition of  $M_t$  in (5.36), we get

$$\sum_{j=1}^d (\langle M^j \rangle_T - \langle M^j \rangle_{T-\gamma}) \leq \gamma \zeta^{-\frac{1}{4}}.$$

The above two displays yield  $\sum_{j=1}^d \langle M^j \rangle_T \leq \gamma^{-1} \zeta + \gamma \zeta^{-\frac{1}{4}}$ . By (5.41), we can set  $\gamma = \zeta^{\frac{1}{2}} < \zeta_0^{\frac{1}{2}} < 1$  to obtain

$$\langle M^j \rangle_T \leq \sum_{j=1}^d \langle M^j \rangle_T \leq \zeta^{-\frac{1}{2}+1} + \zeta^{\frac{1}{2}-\frac{1}{4}} \leq 2\zeta^{\frac{1}{4}}.$$

Since  $\omega \notin B_{1,j}^{\zeta, \epsilon}$ ,  $j = 1, 2, \dots, d$ , due to (5.42), we have

$$(5.43) \quad \sup_{0 \leq t \leq T} |M_t| < \sum_{j=1}^d \sup_{0 \leq t \leq T} |M_t^j| < dd^{-1} \zeta^{\frac{1}{16}} = \zeta^{\frac{1}{16}}.$$

On the other hand, Markov's inequality and  $\omega \in B_0^{\zeta, \epsilon}$  imply,

$$\mathbf{m}\{t \in [0, T] : |R_t| \geq \zeta^{\frac{1}{3}}\} \leq \frac{1}{\zeta^{\frac{2}{3}}} \int_0^T |R_t|^2 dt \leq \zeta^{\frac{1}{3}},$$

where  $\mathbf{m}$  is the Lebesgue measure on the real line. By (5.43) and (5.36), we thus have

$$\mathbf{m}\{t \in [0, T] : |r_0 + A_t| \geq \zeta^{\frac{1}{3}} + \zeta^{\frac{1}{16}}\} \leq \zeta^{\frac{1}{3}}.$$

Note that  $\zeta^{\frac{1}{3}} < \zeta_0^{\frac{1}{3}} \leq \frac{1}{2} \leq \frac{1}{2}T$  due to (5.41) and  $T \geq 1$ . Hence, for each  $t \in [0, T]$ , there is  $t' \in [0, T]$  satisfying  $|t - t'| \leq 2\zeta^{\frac{1}{3}}$  and  $|r_0 + A_{t'}| < \zeta^{\frac{1}{3}} + \zeta^{\frac{1}{16}}$ . Therefore, for each  $t \in [0, T]$ , it holds that, by the definition of  $A_t$  in (5.36),

$$\begin{aligned} |r_0 + A_t| &\leq |r_0 + A_{t'}| + \left| \int_{t'}^t a(s) ds \right| < \zeta^{\frac{1}{3}} + \zeta^{\frac{1}{16}} + \left| \int_{t'}^t |a(s)|^2 ds \right|^{\frac{1}{2}} |t - t'|^{\frac{1}{2}} \\ &\leq \zeta^{\frac{1}{3}} + \zeta^{\frac{1}{16}} + \sqrt{2} \zeta^{-\frac{1}{16} + \frac{1}{6}} \leq 4\zeta^{\frac{1}{16}}. \end{aligned}$$

Set  $t = 0$  to obtain  $|r_0| < 4\zeta^{\frac{1}{16}}$ . However,  $\sqrt{c_0} = |r_0|$ , due to (5.37), and (5.41) imply that

$$\sqrt{c_0} = |r_0| < 4\zeta^{\frac{1}{16}} < 4\zeta_0^{\frac{1}{16}} \leq \sqrt{c_0}.$$

By contradiction, (5.40) holds for  $\zeta_0$  satisfying (5.41).  $\square$

### 5.3. Proof of Lemma 4.1.

5.3.1. *Part (1).* We will apply Theorem 5.1 to  $U_T$  and  $Z_T$ . First note that, since  $U_t, Z_t$  are solutions of SDEs, by [Nua95, Theorem 2.2.2], we know they belong to  $\mathbb{D}^{3,\infty}$ , see Remark 5.2. Since  $U_T = M_T + \epsilon V_T$ , using boundedness of  $V_T$  and applying exponential martingale inequality to  $M_T$  and  $Z_T$ , after a simple computation, we have that there are constants  $C, c > 0$  such that

$$(5.44) \quad \mathbb{P}\{|U_T - x| < 2\}, \quad \mathbb{P}\{|Z_T - x| < 2\} \leq Ce^{-c|x|^2}.$$

Theorem 5.1, (5.22), (5.23), (5.24), (5.25) and (5.44) give rise to, for some  $C', c' > 0$ ,

$$\begin{aligned} & |\rho_{U_T}^y(x) - \rho_{Z_T}(x)| \\ & \leq C\|U_T - Z_T\|_{2,\gamma,T} \left( (1 \vee \mathbb{E}|\det \sigma_{U_T}|^{-\gamma}) (1 + \|U_T\|_{3,\gamma,T}) \right)^a \\ & \quad \cdot \left( (1 \vee \mathbb{E}|\det \sigma_{Z_T}|^{-\gamma}) (1 + \|Z_T\|_{3,\gamma,T}) \right)^a \cdot (\mathbb{P}\{|U_T - x| < 2\} + \mathbb{P}\{|Z_T - x| < 2\})^b \\ & \leq C'\epsilon^\delta (1 + p(\epsilon^{1-v}|y|)) e^{-c'|x|^2}. \end{aligned}$$

5.3.2. *Part (2).* We estimate the difference  $|\rho_{Z_T}(x) - \rho_{Z_\infty}(x)|$ . The covariance matrix of  $Z_T$  is given by

$$\mathcal{C}_\epsilon^{jk} = \mathbb{E}Z_T^j Z_T^k = \sum_{l=1}^d \sigma_l^j(0)\sigma_l^k(0) \frac{1 - e^{-(\lambda_j + \lambda_k)T}}{\lambda_j + \lambda_k}.$$

By  $T \geq \theta'_0 \log \epsilon^{-1}$ , we have  $\lim_{\epsilon \rightarrow 0} \mathcal{C}_\epsilon^{jk} = \mathcal{C}_0^{jk}$ . Therefore, there is a constant  $c > 0$  such that

$$(5.45) \quad e^{-\frac{1}{2}x^\top \mathcal{C}_\epsilon^{-1} x}, \quad e^{-\frac{1}{2}x^\top \mathcal{C}_0^{-1} x} \leq e^{-c|x|^2}.$$

We can write

$$|\rho_{Z_T}(x) - \rho_{Z_\infty}(x)| \leq \left| \rho_{Z_T} - \sqrt{\frac{\det \mathcal{C}_\epsilon}{\det \mathcal{C}_0}} \rho_{Z_\infty} \right| + \left| \sqrt{\frac{\det \mathcal{C}_\epsilon}{\det \mathcal{C}_0}} \rho_{Z_\infty} - \rho_{Z_\infty} \right|.$$

Since  $\sqrt{\frac{\det \mathcal{C}_\epsilon}{\det \mathcal{C}_0}}$  can be viewed as the square root of a polynomial of  $e^{-T}$  with positive fractional powers, one can see that  $|1 - \sqrt{\frac{\det \mathcal{C}_\epsilon}{\det \mathcal{C}_0}}| \leq C_1(e^{-T})^{q_1}$  for some  $C_1, q_1 > 0$ . Therefore, using the hypothesis  $\theta \log \epsilon^{-1} \leq T$  and (5.45), we obtain

$$\left| \rho_{Z_T} - \sqrt{\frac{\det \mathcal{C}_\epsilon}{\det \mathcal{C}_0}} \rho_{Z_\infty} \right| \leq C_1 \epsilon^{q_1 \theta'_0} e^{-c|x|^2}.$$

For any matrix, we use  $|\cdot|$  to denote its Frobenius norm. Then observe that, for some  $q_2 > 0$ , we have, for some  $C_2, q_2 > 0$ ,

$$|\mathcal{C}_\epsilon^{-1} - \mathcal{C}_0^{-1}| \leq |\mathcal{C}_0^{-1}| |\mathcal{C}_0 - \mathcal{C}_\epsilon| |\mathcal{C}_\epsilon^{-1}| \leq C_2(e^{-T})^{q_2} \leq C_2 \epsilon^{q_2 \theta'_0}.$$

As  $\mathcal{C}_\epsilon$  and  $\mathcal{C}_0$  are positive definite, so are their inverses. Then by (5.45), we can get

$$\begin{aligned} & \left| e^{-\frac{1}{2}x^\top \mathcal{C}_\epsilon^{-1} x} - e^{-\frac{1}{2}x^\top \mathcal{C}_0^{-1} x} \right| \leq \left( e^{-\frac{1}{2}x^\top \mathcal{C}_\epsilon^{-1} x} \vee e^{-\frac{1}{2}x^\top \mathcal{C}_0^{-1} x} \right) \left| e^{-\frac{1}{2}|x^\top (\mathcal{C}_\epsilon^{-1} - \mathcal{C}_0^{-1}) x|} - 1 \right| \\ & \leq \frac{1}{2} e^{-c|x|^2} |x|^2 |\mathcal{C}_\epsilon^{-1} - \mathcal{C}_0^{-1}| \leq C_3 \epsilon^{q_2 \theta'_0} e^{-c'|x|^2} |x|^2. \end{aligned}$$

Therefore, we have

$$\left| \sqrt{\frac{\det \mathcal{C}_\epsilon}{\det \mathcal{C}_0}} \rho_{Z_\infty} - \rho_{Z_\infty} \right| \leq C_4 \epsilon^{q_2 \theta'_0} e^{-c''|x|^2}.$$

In conclusion,  $|\rho_{Z_T}(x) - \rho_{Z_\infty}(x)| \leq C'' \epsilon^{\delta'} e^{-c'''|x|^2}$  which completes the proof of Lemma 4.1.

#### 5.4. Proofs of auxiliary lemmas.

*Proof of Lemma 5.3.* Let  $b(t) = \sum_{i=1}^d a^i(t)$ . Summing up the inequalities (5.11) in  $i$  and using  $\lambda_1 > \lambda_2 > \dots > \lambda_d$ , we get

$$0 \leq b(t) \lesssim \epsilon^m e^{-2\lambda_d r} + \epsilon^2 \int_r^t e^{2\lambda_1 s} b(s) ds.$$

Now Gronwall's inequality implies that, for some constant  $c$  independent of  $\epsilon$ ,

$$0 \leq b(t) \lesssim \epsilon^m e^{-2\lambda_d r} e^{c\epsilon^2 e^{2\lambda_1 T}}.$$

Finally, we use (5.1) and the fact  $a^i(t) \geq 0$  to derive  $a^i(t) \leq b(t) \lesssim \epsilon^m e^{-2\lambda_d r}$ , and it is clear from this computation that all the constants involved do not depend on  $r$ .  $\square$

*Proof of Lemma 5.4.* This proof is a modification of the proof of [Nua95, Lemma 2.3.1].

Let us fix  $\zeta > 0$ . Let  $u_1, u_2, \dots, u_{N_d}$  be unit vectors in  $\mathbb{R}^d$  such that

$$(5.46) \quad \mathbb{S}^{d-1} \subset \bigcup_{k=1}^{N_d} \{x \in \mathbb{R}^d : |x - u_k| < \frac{\zeta^2}{4}\},$$

where  $\mathbb{S}^{d-1}$  is the unit sphere and  $N_d$  is chosen so that

$$(5.47) \quad N_d \leq C_d \zeta^{-2d}$$

for a positive constant  $C_d$  only depending on the dimension  $d$ . Writing  $|\mathcal{A}| = (\sum_{i,j=1}^d |\mathcal{A}^{ij}|^2)^{\frac{1}{2}}$ , we obtain

$$(5.48) \quad \begin{aligned} \mathbb{P}\{\nu \leq \zeta\} &= \mathbb{P}\{\inf_{|v|=1} \langle v, \mathcal{A}v \rangle \leq \zeta\} \\ &\leq \mathbb{P}\{\inf_{|v|=1} \langle v, \mathcal{A}v \rangle \leq \zeta; |\mathcal{A}| \leq \frac{1}{\zeta}\} + \mathbb{P}\{|\mathcal{A}| > \frac{1}{\zeta}\}. \end{aligned}$$

The second term can be estimated using Markov's inequality as

$$(5.49) \quad \mathbb{P}\{|\mathcal{A}| > \frac{1}{\zeta}\} \leq \zeta^p \mathbb{E}|\mathcal{A}|^p = \zeta^p \mathbb{E} \left| \sum_{i,j=1}^d |\mathcal{A}^{ij}|^2 \right|^{\frac{p}{2}}.$$

For the first term, more effort is needed. On the set

$$B = \left\{ \inf_{|v|=1} \langle v, \mathcal{A}v \rangle \leq \zeta; |\mathcal{A}| \leq \frac{1}{\zeta} \right\},$$

suppose  $\langle u_k, \mathcal{A}u_k \rangle \geq 2\zeta$  for all  $k = 1, \dots, N_d$ . For any  $v$  with  $|v| = 1$ , by (5.46), there is  $u_k$  such that  $|v - u_k| < \frac{\zeta^2}{4}$ . Then observe that, on  $B$ ,

$$\begin{aligned} \langle v, \mathcal{A}v \rangle &\geq \langle u_k, \mathcal{A}u_k \rangle - |\langle v, \mathcal{A}v \rangle - \langle u_k, \mathcal{A}u_k \rangle| \\ &\geq 2\zeta - (|\langle v, \mathcal{A}v \rangle - \langle v, \mathcal{A}u_k \rangle| + |\langle v, \mathcal{A}u_k \rangle - \langle u_k, \mathcal{A}u_k \rangle|) \\ &\geq 2\zeta - 2|\mathcal{A}||v - u_k| > 2\zeta - 2\frac{1}{\zeta} \frac{\zeta^2}{4} = \frac{3}{2}\zeta. \end{aligned}$$

But on  $B$ , we necessarily have  $\inf_{|v|=1} \langle v, \mathcal{A}v \rangle \leq \zeta$ . Hence, by contradiction, we must have  $B \subset \bigcup_{k=1}^{N_d} \{ \langle u_k, \mathcal{A}u_k \rangle < 2\zeta \}$ . This fact together with (5.47) implies

$$\begin{aligned} \mathbb{P}\left\{ \inf_{|v|=1} \langle v, \mathcal{A}v \rangle \leq \zeta; |\mathcal{A}| \leq \frac{1}{\zeta} \right\} &\leq \mathbb{P}\left( \bigcup_{k=1}^{N_d} \{ \langle u_k, \mathcal{A}u_k \rangle < 2\zeta \} \right) \\ &\leq \sum_{k=1}^{N_d} (2\zeta)^{p+2d} \mathbb{E} |\langle u_k, \mathcal{A}u_k \rangle|^{-(p+2d)} \\ &\leq N_d (2\zeta)^{p+2d} \sup_{|v|=1} \mathbb{E} |\langle v, \mathcal{A}v \rangle|^{-(p+2d)} \\ &\leq 2^{p+2d} C_d \zeta^p \sup_{|v|=1} \mathbb{E} |\langle v, \mathcal{A}v \rangle|^{-(p+2d)}. \end{aligned}$$

The above display, (5.48) and (5.49) show that there is  $C_{p,d} > 0$  depending only on  $p$  and  $d$  such that (5.33) holds.  $\square$

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