

Instantaneous braids and Dehn twists in topologically ordered states

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A defining feature of topologically ordered states of matter is the existence of locally indistinguishable states on spaces with nontrivial topology. These degenerate states form a representation of the mapping class group (MCG) of the space, which is generated by braids of defects or anyons, and by Dehn twists along noncontractible cycles. These operations can be viewed as fault-tolerant logical gates in the context of topological quantum error correcting codes and topological quantum computation. Here, we show that braids and Dehn twists can in general be implemented by a constant depth quantum circuit, with a depth that is *independent of code distance d and system size*. The circuit consists of a constant depth *local* quantum circuit (LQC) implementing a local geometry deformation of the quantum state, followed by a permutation on (relabeling of) the qubits. The permutation requires permuting qubits that are separated by a distance of order d ; it can be implemented by collective classical motion of mobile qubits or as a constant depth circuit using long-range SWAP operations (with a range set by d) on immobile qubits. We further show that (i) applying a given braid or Dehn twist k times can be achieved with $\mathcal{O}(\log k)$ time overhead, independent of code distance and system size, which implies an exponential speedup for certain logical gate sequences by trading space for time, and (ii) an *arbitrary* element of the MCG can be implemented by a constant depth (independent of d) LQC followed by a permutation, where in this case the range of interactions of the LQC grows with the number of generators in the presentation of the group element. Applying these results to certain non-Abelian quantum error correcting codes demonstrates how universal logical gate sets can be implemented on encoded qubits using only constant depth unitary circuits.

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I. INTRODUCTION

A profound property of topologically ordered states of matter is the possibility of topologically degenerate ground states, which arise when the system exists on a topologically nontrivial space [1–3]. The degeneracy is protected by the fact that the states are indistinguishable by any local operators, up to exponentially small corrections in system size.

This local indistinguishability of topological states is the key feature underlying quantum error correction and the possibility of creating a fault-tolerant quantum memory [2–4]. Many well-known quantum error correcting codes (QECCs), such as Shor’s nine-qubit code, the Steane code, and the Reed-Muller code, can all be interpreted in terms of the ground-state subspace of a topologically ordered state defined on a cellulation of a topologically nontrivial manifold [5,6]. More generally, a large class of QECCs, known as topological QECCs, are associated with a particular class of topologically ordered states of matter. These include the surface and toric codes and their generalizations: the Kitaev quantum double models [4] and Turaev-Viro-Levin-Wen models [7–11]. Topological QECCs play an important role in the theory of quantum error correction, as they provide the only approach to decrease the logical error rate arbitrarily while maintaining local interactions among the microscopic degrees of freedom.

In general, topologically ordered states can be realized in two distinct ways. In the “passive” approach, they can be realized in equilibrium as ground states of an appropriate many-body Hamiltonian. The topological protection derives

from $T/\Delta \ll 1$, where T is the temperature and Δ is the energy gap. In the “active” approach to quantum error correction (QEC), the topologically ordered states arise as eigenstates of commuting local operators [12,13]. The state can be maintained actively by continuously measuring these local operators. In the language of QECCs, the topologically degenerate ground-state subspace is known as the *code subspace*, and the minimum length of a string operator that acts nontrivially in the code subspace is known as the *code distance d* .

An important question is to understand how to perform robust, nontrivial operations on the code subspace. It is well known that nontrivial operations can be obtained by braiding non-Abelian anyons [2], twist defects [14], and holes with gapped boundaries [12,15]. Alternatively, when topological degeneracies arise in a closed genus- g surface, nontrivial operations can be obtained by performing Dehn twists [16,17]. In other words, the code subspace forms a representation of the mapping class group (MCG) of the space; the braid group on n strands corresponds to the MCG of a disk with n punctures, while the MCG of a closed genus g surface is generated by Dehn twists along noncontractible cycles.

In the passive approach, elements of the mapping class group have been proposed to be implemented through adiabatic evolution with a local Hamiltonian, leading to a non-Abelian Berry phase [2,18–20]. To be adiabatic, the time to implement such transformations must be large compared with $1/\Delta$, and increases at least linearly with the code distance (or system size) d . In the active approach, known methods to implement braids (of holes, twist defects, or non-Abelian

anyons) through unitary circuits also increase at least linearly with the code distance d . Alternatively, mapping class group elements can be effectively achieved through measurement-based approaches [21–24], which also take a time that diverges as $d \rightarrow \infty$. For example, in active approaches, fault-tolerant readout of a measurement requires either (i) d rounds of measurements [12], or (ii) a single round of measurement with classical processing time that diverges as $d \rightarrow \infty$, and an extra factor of $\mathcal{O}(d)$ in space overhead [25,26]. Therefore, in all approaches proposed to date, there is a fundamental tradeoff between space-time overhead and accuracy of a fault-tolerant quantum computation; in the limit where the logical error rate goes to zero, $d \rightarrow \infty$, and therefore the time to implement logical gates by braiding or Dehn twists also goes to infinity.

Ideally, it is of interest to implement logical gates in a time that is independent of the code distance, and without increasing the asymptotic scaling of the space overhead. A certain class of such logical gates are known as transversal logical gates [27]. Transversal gates consist of nontrivial unitary operations on the code subspace that decompose as a tensor product of local unitary transformations that do not couple different sites within the same code block. Transversal gates are special cases of local constant depth quantum circuits, which are intrinsically fault tolerant as the code distance $d \rightarrow \infty$ due to the locality of error propagation. The Eastin-Knill theorem establishes that a universal encoded gate set cannot be implemented transversally [28]. Related theorems impose strong restrictions on logical gates implemented with local, finite depth quantum circuits [29,30].

Recently, it was discovered that certain elements of the MCG of a generic topological state can be effectively implemented in one shot as a transversal gate [31]. These elements correspond to certain finite-order (torsion) elements of the MCG, which correspond to specific combinations of braids or Dehn twists. While these results do not violate the Eastin-Knill theorem, they do circumvent the assumptions of Ref. [29] for implementing nontrivial logical gates in topological QECCs using constant depth local quantum circuits, and hence the corresponding no-go theorem for non-Abelian codes.

In this paper, we demonstrate that braids and Dehn twists in a wide class of topological states can always be achieved by a quantum circuit of *constant depth, independent of the code distance d* . Specifically, we demonstrate that elementary braids and Dehn twists can be implemented by (i) a local constant depth quantum circuit (LQC), followed by (ii) a permutation on the qubits. The permutation requires qubits that have a separation of order d to be permuted. If the qubits of the system are mobile, the permutation can be physically implemented by shuttling the qubits around, as has been done experimentally in ion trap systems [32–34]. If the qubits are immobile, the permutation can be achieved in one step by utilizing long-range SWAP operations and ancilla qubits.

Our results imply that by utilizing non-Abelian topological QECCs together with our braiding and Dehn twist protocols, a universal logical gate set can be implemented on encoded qubits through a constant depth unitary quantum circuit, without increasing the scaling of the space overhead. We note that the implementation of these quantum circuits does

not require any additional classical computational resources nor does it depend on the result of measurement outcomes at intermediate steps in the computation. Furthermore, our protocols for topological codes with local syndromes require $\mathcal{O}(d^2)$ space overhead per logical qubit.

Our constant depth circuits maps a local operator with support in a region $\mathcal{R}$ to another local operator with support in a region $\mathcal{R}'$, such that the areas of $\mathcal{R}$ and $\mathcal{R}'$ are related by a constant factor, independent of code distance. Consequently, our logical gates are naturally topologically protected and can be made fault tolerant since they only change the length of an error string by an $\mathcal{O}(1)$ constant factor independent of code distance. The extra time overhead for decoding and error correction after applying these constant depth circuits depends on the detailed properties of the logical circuits, i.e., the logical gate sequence. In the presence of noisy syndrome measurements, in the worst-case scenario of the logical circuit (logical gate sequence) we expect a sublinear time overhead where at least $\mathcal{O}(d/\log d)$ rounds of syndrome measurements need to be performed for each application of the logical gate in order to successfully decode error strings. Therefore, our estimated average space-time overhead per logical qubit per logical gate for the most general logical circuits is lower bounded by $\mathcal{O}(d^3/\log d)$ in the presence of noisy syndrome measurements.

This result demonstrates that universal logical gate sets can be implemented on encoded qubits with constant depth circuits, and without increasing the scaling of the space overhead. Other proposals for implementing universal logical gate sets, such as those which utilize magic state distillation or code switching, all require $\mathcal{O}(d^3)$ space-time overhead per logical qubit, per logical gate, in the presence of noisy syndrome measurements. This space-time overhead can come from either (1) a time overhead that diverges at least linearly with the code distance d and $\mathcal{O}(d^2)$ space overhead per logical qubit, or (2) polylog time overhead (including classical computational resources) and $\mathcal{O}(d^3)$ space overhead per logical qubit [25,26,35–42].¹

We note that some of the results discussed in this paper with respect to braiding with constant depth circuits have also been summarized by us in a short paper [43].

A. Summary of results

Our specific technical results are summarized below. Let us consider $N + N_a$ physical qubits arranged on a lattice. We further consider a state

$$|\Psi\rangle = |\Phi\rangle \otimes |\Pi\rangle_a. \quad (1)$$

Here, $|\Pi\rangle_a = \otimes_j |\psi_j\rangle_a$ is an arbitrary product state for the N_a ancilla qubits. $|\Phi\rangle$ is a topologically ordered state on N qubits on a genus g surface with p punctures $\Sigma_{g,p}$. The punctures could correspond to holes with gapped boundaries or anyons.

¹Note that the proposed code switching protocols have intermediate steps that depend on outcomes of measurements during the protocol. For the measurements to be fault tolerant, they must either be performed d times or require a factor of d increase in space overhead [12,25,26].

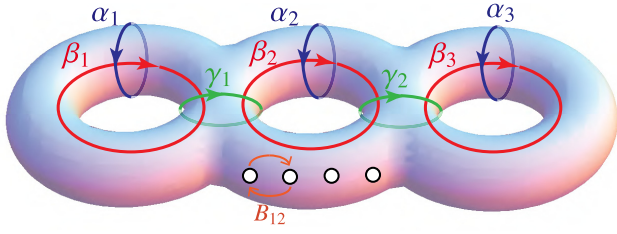


FIG. 1. Noncontractible cycles and braid operations on a genus $g = 3$ surface with $p = 4$ punctures.

$|\Phi\rangle$ is an arbitrary (Abelian or non-Abelian), nonchiral topologically ordered state. Such states are always related, by a constant depth local quantum circuit (alternatively, by adiabatic evolution), to an exact ground state of a commuting projector Hamiltonian, such as the Kitaev quantum double or Levin-Wen models [4,10,11]. Alternatively, such nonchiral topological orders can be described within a path-integral state-sum construction, as described in Refs. [7–9,17,44].

$|\Phi\rangle$ belongs to a representation of $\text{MCG}(\Sigma_{g,p})$, the mapping class group of $\Sigma_{g,p}$. It is well known that $\text{MCG}(\Sigma_{g,p})$ can be generated by $3g - 1$ Dehn twists along the simple curves $\alpha_i, \beta_i, \gamma_j$, where $i = 1, \dots, g$ and $j = 1, \dots, g - 1$, together with elementary (half-) braids between neighboring punctures [45]. See Sec. II and Fig. 1 for detailed discussions and illustration.

Let us denote σ to be a permutation on the $N + N_a$ qubits and $\mathcal{P}_\sigma$ the unitary representation of that permutation. Furthermore, let $\mathcal{LU}$ denote a local, constant depth quantum circuit. In particular, $\mathcal{LU}$ is local in the sense that the range r of interactions is independent of system size and code distance. Similarly, constant depth means that the depth of the circuit is also independent of system size and code distance.

We first demonstrate the following result:

Theorem 1. Let $\kappa \in \text{MCG}(\Sigma_{g,p})$ denote either a Dehn twist along a simple curve $\alpha_i, \beta_i, \gamma_j$, where $i = 1, \dots, g$ and $j = 1, \dots, g - 1$, or an elementary braid between neighboring punctures. We let $\mathcal{V}_\kappa$ be the unitary representation of κ on the topological ground-state subspace (i.e., the code subspace). Then,

$$\mathcal{V}_\kappa \otimes I|\Psi\rangle = (\mathcal{V}_\kappa|\Phi\rangle) \otimes |\Pi\rangle_a = \mathcal{P}_\sigma \mathcal{LU}_\kappa(|\Phi\rangle \otimes |\Pi\rangle_a). \quad (2)$$

I is the identity operator on the ancilla qubits and $\mathcal{LU}_\kappa$ is a constant depth local unitary that depends on κ . The permutation $\mathcal{P}_\sigma$ (which depends on κ) can be implemented in constant time by utilizing ancilla qubits. For example, one first performs a SWAP operation between each qubit and an ancilla qubit, followed by a second SWAP operation between the ancilla qubit and the target location of the SWAPs. (The second SWAP is actually unnecessary, as explained in more detail in Sec. VII). It is crucial to note that these SWAP operations are *long-range* operations. In general, the range of the SWAPs is set by the code distance d .

It is useful to note that, depending on the physical implementation, the permutation $\mathcal{P}_\sigma$ can also be performed by physically moving the location of the qubits in physical space. For example, if the qubits are associated with ions in an ion-trap quantum computer, the ions can be physically moved to their target locations [32–34,46–48].

A corollary of the above theorem is with respect to the space-time overhead for universal fault-tolerant quantum computation. It is well known that mapping class group elements, such as braiding of anyons, in the Fibonacci topological state is universal for topological quantum computation [3,49,50]. We can thus consider the Turaev-Viro code [11,13] (and associated Levin-Wen model [10]) based on the Fibonacci fusion category, whose topological order corresponds to two time-reversed copies of the Fibonacci state. Applying Theorem 1 to such a code thus implies that a universal fault-tolerant gate set can be achieved through constant time braiding of Fibonacci anyons, without changing the asymptotic scaling of the space overhead.

More specifically, in a two-dimensional topological code with local interactions (alternatively, in an active error correction approach, with local syndrome measurements), the space overhead is $\mathcal{O}(d^2)$ per logical qubit. The result of Theorem 1 thus implies that universal fault-tolerant gate sets can be achieved with time overhead that is independent of code distance d , while keeping the space overhead at $\mathcal{O}(d^2)$ per logical qubit.

Theorem 2. Let $\kappa \in \text{MCG}(\Sigma_{g,p})$ denote either a Dehn twist along a simple curve $\alpha_i, \beta_i, \gamma_j$, where $i = 1, \dots, g$ and $j = 1, \dots, g - 1$, or an elementary braid between neighboring punctures. Furthermore, let $\mathcal{V}_{\kappa^n} = \mathcal{V}_\kappa^n$ be the unitary representation of κ^n on the topological ground-state subspace (i.e., the code subspace), where n is an arbitrary integer. Then,

$$\mathcal{V}_{\kappa^n} \otimes I|\Psi\rangle = (\mathcal{V}_{\kappa^n}|\Phi\rangle) \otimes |\Pi\rangle_a = \prod_{i=1}^k \mathcal{LU}_{i,\kappa} \mathcal{P}_{\sigma_i}(|\Phi\rangle \otimes |\Pi\rangle_a). \quad (3)$$

Here, $k = \mathcal{O}(\log n)$, $\mathcal{P}_{\sigma_i}$ is a qubit permutation, which permutes qubits over a range of $\mathcal{O}(d)$, and $\mathcal{LU}_{i,\kappa}$ are local, finite depth quantum circuits, where the range r of the gates and depth are independent of n , code distance d , and system size. In the case of the $\mathbb{Z}_N$ toric code, we also have

$$\mathcal{V}_{\kappa^n} \otimes I|\Psi\rangle = (\mathcal{V}_{\kappa^n}|\Phi\rangle) \otimes |\Pi\rangle_a = \mathcal{P}_\sigma \mathcal{LU}_\kappa(|\Phi\rangle \otimes |\Pi\rangle_a), \quad (4)$$

where now $\mathcal{LU}_\kappa$ is a local quantum circuit with maximum range $r = \mathcal{O}(n)$ and fixed depth independent of n , code distance d , and system size.

Theorem 3. Let $\zeta \in \text{MCG}(\Sigma_{g,p})$ be an arbitrary group element, and $\mathcal{V}_\zeta$ its representation on the quantum state. ζ has a presentation in terms of a string of k Dehn twists and braids, for some integer k . Then,

$$\mathcal{V}_\zeta \otimes I|\Psi\rangle = (\mathcal{V}_\zeta|\Phi\rangle) \otimes |\Pi\rangle_a = \mathcal{P}_\sigma \mathcal{LU}_\zeta|\Phi\rangle \otimes |\Pi\rangle_a. \quad (5)$$

Here, $\mathcal{LU}_\zeta$ is a constant (independent of code distance and system size) depth local quantum circuit. The range r of gates in $\mathcal{LU}_\zeta$ increases with k , such that $r = \mathcal{O}(c^k)$ where c is a constant independent of ζ and code distance and system size.

We note that in all the above theorems, the order of $\mathcal{LU}$ and $\mathcal{P}_\sigma$ can in principle be switched (with the concrete circuits being modified), and does not affect the final results.

We provide proofs of these statements by explicit construction for ground states of exactly solvable commuting projector models. As noted above, any nonchiral topologically ordered state can be transformed to the ground state of an exactly

solvable commuting projector model by a finite depth local quantum circuit.

An important by-product of our analysis is to demonstrate how non-Abelian anyons and holes can be moved by distances of order the code distance by a constant depth local circuit followed by a permutation on the qubits. This overturns the general belief that moving non-Abelian anyons by a distance ℓ always requires a quantum circuit of depth $\propto \ell$. While this belief is correct when restricted to purely local interactions, we see that the use of long-range permutations allows us to implement the motion by a constant depth circuit. Interestingly, however, our protocols can only move anyons by a distance ℓ that is a constant factor of the minimum separation between anyons. We therefore arrive at an interesting version of Zeno's paradox in the context of non-Abelian topological quantum order: the time it takes to create two non-Abelian anyons out of the vacuum and separate them a distance of order ℓ requires $\mathcal{O}(\log \ell)$ steps (ignoring the presence of any other anyons). However, if the anyons are already a distance ℓ apart, they can be moved by a distance of order ℓ in a time that is independent of ℓ .

B. Structure of the paper

This paper is structured as follows. We begin in Sec. II by providing a brief review of the mapping class group of surfaces, Dehn twists, and their relation to braiding. In Sec. III, we focus on the case of $\mathbb{Z}_N$ topological order, proving Theorems 1 and 2 by explicitly constructing the quantum circuits with the desired properties. In Sec. IV, we then generalize this discussion to encompass arbitrary nonchiral topological orders, which include both arbitrary Abelian and non-Abelian topological orders. We prove Theorem 3 separately in Sec. V. We further show the fault-tolerance aspects of our schemes in Sec. VI, and conclude our paper with a discussion of the central results in Sec. VII.

II. BASIC CONCEPTS: DEHN TWISTS, BRAIDS, AND MAPPING CLASS GROUPS

A. Mapping class group and its representation

We start with the definition of the mapping class group of a surface, which includes all the central concepts discussed in this paper such as Dehn twists and braids. Consider a surface of genus g with p punctures, denoted $\Sigma_{g,p}$. The mapping class group of $\Sigma_{g,p}$, denoted $\text{MCG}(\Sigma_{g,p})$, is defined to be the group of orientation-preserving diffeomorphisms of $\Sigma_{g,p}$ modulo those which can be continuously connected to identity:

$$\text{MCG}(\Sigma_{g,p}) = \text{Diff}^+(\Sigma_{g,p}, \partial \Sigma_{g,p}) / \text{Diff}_0(\Sigma_{g,p}, \partial \Sigma_{g,p}), \quad (6)$$

with the diffeomorphisms being restricted to the identity on the boundary $\partial \Sigma_{g,p}$ [45]. Here, $\text{Diff}_0(\Sigma_{g,p}, \partial \Sigma_{g,p})$ is the subgroup of $\text{Diff}^+(\Sigma_{g,p}, \partial \Sigma_{g,p})$ which consists of elements that are isotopic (continuously connected) to the identity. As a consequence of this definition, any element of the mapping class group, $[\zeta] \in \text{MCG}(\Sigma_{g,p})$, is an equivalence class of a diffeomorphism of manifold $\Sigma_{g,p}$ which maps the manifold back to itself, i.e., $\zeta : \Sigma_{g,p} \mapsto \Sigma_{g,p}$.

As illustrated in Fig. 1, $\text{MCG}(\Sigma_{g,p})$ can be generated by $3g - 1$ Dehn twists along the noncontractible cycles denoted

α , β , and γ (see Fig. 1), together with braids between neighboring punctures $B_{1,2}$, $B_{2,3}, \dots$, and $B_{p-1,p}$. In the following subsections, we will provide more background on Dehn twists and their relation to braids.

In the context of topological states or codes supported on a manifold Σ , the unitary representations of $\text{MCG}(\Sigma)$ are topologically protected (i.e., fault tolerant) unitary transformations acting on the ground-state subspace or equivalently the code space $\mathcal{H}_\Sigma$. We denote the representation of a mapping class group element ζ by the unitary operator $\mathcal{V}_\zeta$, which performs an *automorphism* that maps the code space back to itself, i.e., $\mathcal{V}_\zeta : \mathcal{H}_\Sigma \mapsto \mathcal{H}_\Sigma$. Therefore, $\mathcal{V}_\zeta$ is an element of the automorphism group of the code space $\mathcal{V}_\zeta \in \text{Aut}(\mathcal{H}_\Sigma)$, and is hence a logical gate.

B. Dehn twists

Here, we review a specific type of self-diffeomorphism called a Dehn twist. We first consider an annulus $A = S^1 \times [0, 1]$, which can be embedded in the (θ, r) plane as shown in Fig. 2(a). We define the twist map $D : A \mapsto A$ by the following formula:

$$D(\theta, t) = (\theta + 2\pi t, t). \quad (7)$$

Note that D is an orientation-preserving diffeomorphism fixing ∂A pointwise, and hence satisfies the definition of a mapping class. Thus, we call D (and the class of maps up to an additional diffeomorphism continuously connected to the identity) a left *Dehn twist*. The right Dehn twist is its inverse, i.e.,

$$D^{-1}(\theta, t) = (\theta - 2\pi t, t). \quad (8)$$

In order to visualize the transformation on the surface, we show two directed noncontractible lines/loops on the surface. The first one is a (blue) line connecting the inner and outer boundary, denoted by α , where the arrow represents the direction. The second one is a (red) noncontractible loop circulating the inner boundary, denoted by β . We can see the left Dehn twist performs the following map:

$$D : (\alpha, \beta) \mapsto (\alpha + \beta, \beta), \quad (9)$$

where $\alpha + \beta$ is the twisted line shown in the right panel in Fig. 2(a). The above map can also be used as an alternative definition of the Dehn twist. One can also see that the left Dehn twist is equivalent to a continuous counterclockwise 2π rotation of the outer boundary or a clockwise 2π rotation of the inner boundary. Similarly, the right Dehn twist performs the map

$$D^{-1} : (\alpha, \beta) \mapsto (\alpha - \beta, \beta), \quad (10)$$

where the $\alpha - \beta$ is twisted in the opposite direction. Since the annulus is equivalent (homotopic) to a cylinder, we hence have also defined the Dehn twist on a cylinder, as shown in Fig. 2(b).

For subsequent discussions, we also introduce the notion of a *half-twist* on an annulus as illustrated in Fig. 2(c). By itself, the half-twist is not an element of the mapping class group because it does not leave the boundary fixed. The left half-twist is defined as

$$\sqrt{D}(\theta, t) = (\theta + \pi + \pi t, t). \quad (11)$$

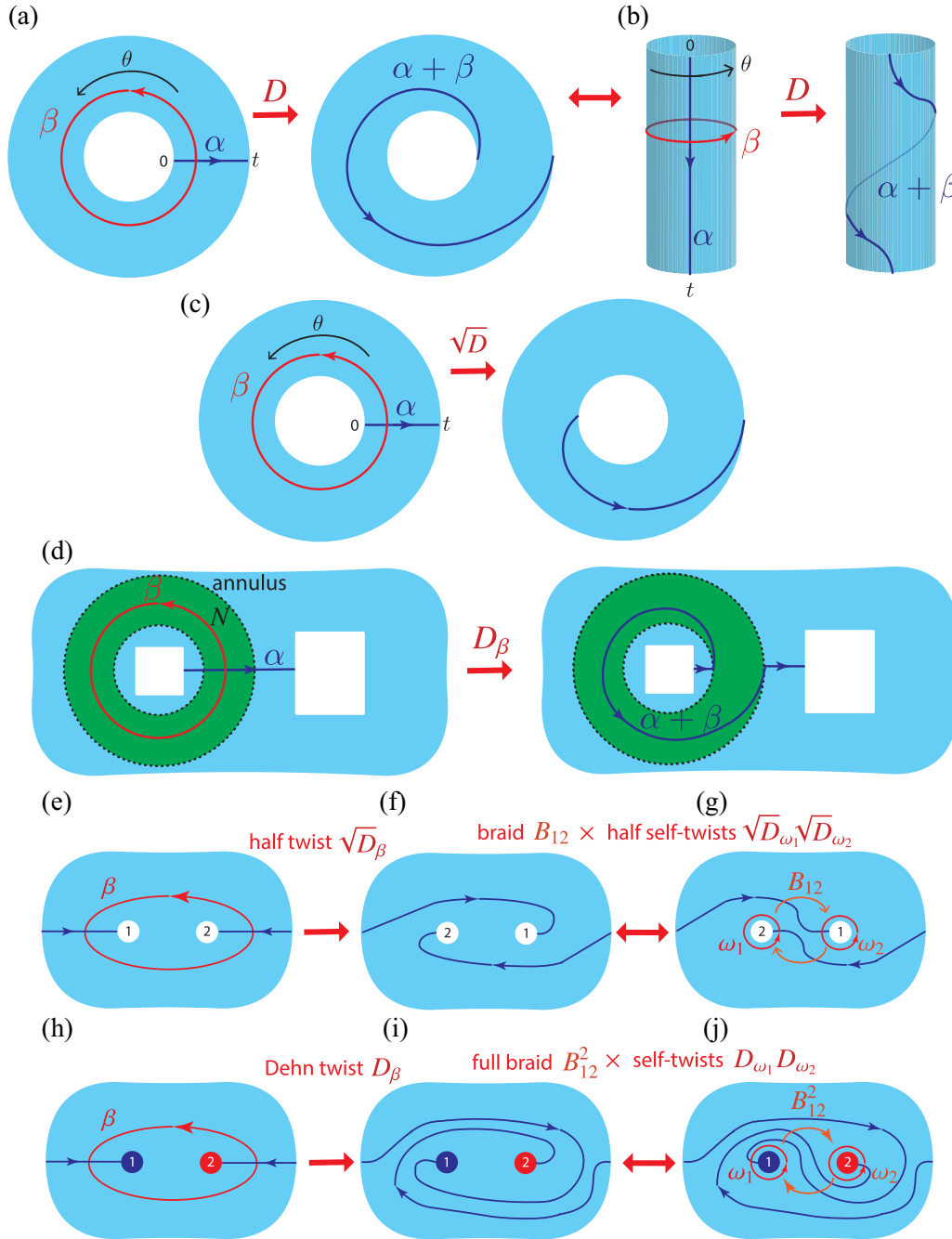


FIG. 2. Definition of Dehn twists, half-twists, and braids, as well as their relations.

Note that in our convention, the half-twist fixes the outer boundary and makes a clockwise π rotation on the inner boundary. Similarly, the right half-twist is defined as

$$\sqrt{D}^{-1}(\theta, t) = (\theta + \pi - \pi t, t), \quad (12)$$

which makes a counterclockwise π rotation on the inner boundary.

Now consider the generic surface $\Sigma_{g,p}$. One can perform a Dehn twist along the noncontractible loop β in $\Sigma_{g,p}$, as illustrated in Fig. 2(d). Note that the direction of the β loop (indicated by the arrow) determines whether it is a left or right Dehn twist. Let N be a regular neighborhood [shown

as a green belt in Fig. 2(d)] of β and f be an orientation-preserving diffeomorphism that maps the previously defined annulus to such a neighborhood: $f : A \mapsto N$. We can hence define a self-diffeomorphism $D_\beta : \Sigma_{g,p} \mapsto \Sigma_{g,p}$ as a Dehn twist about the β loop as follows:

$$D_\beta(\mathbf{x}) = \begin{cases} f D f^{-1} & \text{if } \mathbf{x} \in N, \\ \mathbf{x} & \text{otherwise.} \end{cases} \quad (13)$$

This definition says that D_β performs a Dehn twist D on the annulus N and fixes every point outside the annulus N . To visualize the change of the surface, we again use the noncontractible (blue) line α going across the inner and outer boundary of the neighborhood N , and the (red) loop

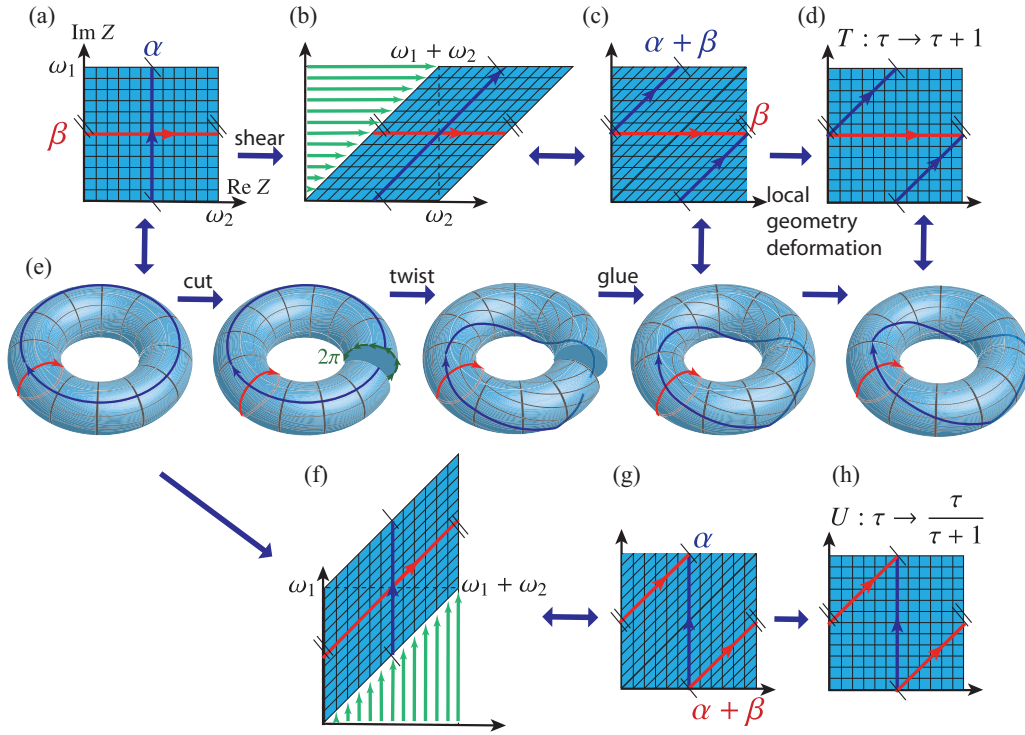


FIG. 3. The mathematical definition and implementation of Dehn twists T and U on a torus via Dehn surgery. The coordinate web represents the metric of the manifold in the continuous case or underlying lattice structure in the discrete case. The shear deformation (green arrows) in (b) implements a Dehn twist and is equivalent to Dehn surgery in (e): cut, twist, and glue. The local geometric deformation returns the metric/lattice to the original configuration.

β circulating around N for illustration. The Dehn twist D_β performs the following map on these two loops:

$$D_\beta : \alpha \mapsto \alpha + \beta, \quad D_\beta : \beta \mapsto \beta, \quad (14)$$

where the part of the twisted line $\alpha + \beta$ outside the annulus N remains fixed as illustrated by the right panel of Fig. 2(d). This map can also be used as the alternative definition of D_β . Similarly, one can also define the half-twist $\sqrt{D}_\beta$ on an arbitrary surface.

The representation of the Dehn twists on the topological ground-state subspace (i.e., the code subspace) is denoted by $\mathcal{D}_\beta$. Note that we have used different fonts to distinguish an MCG element and its representation. It induces the following transformations on the Wilson line/loop operators:

$$\mathcal{D}_\beta W_\alpha^a \mathcal{D}_\beta^\dagger = W_{\alpha+\beta}^a, \quad \mathcal{D}_\beta W_\beta^a \mathcal{D}_\beta^\dagger = W_\beta^a, \quad (15)$$

where W_α^a and W_β^a are Wilson line/loop on the α line and β cycle with topological charge a . The operator $W_{\alpha+\beta}^a$ represents the twisted Wilson loop.

C. Braids and braid group

Although braiding is often discussed in its own context (especially in physics), it is actually just a special type of mapping class. A particular example is the case of the braid group on p strands, which is equivalent to the mapping class group of a disk with p punctures: $B_p = \text{MCG}(D_p^2)$, where here D_p^2 denotes a disk with p punctures.

Braids between punctures can also be expressed in terms of Dehn twists, as follows. As shown in Figs. 2(e)–2(g), a right

half-twist along the β loop enclosing two punctures [Figs. 2(e) and 2(f)] is equivalent to braiding the two punctures, with additional half-self-twists around both punctures [Fig. 2(g)]. That is,

$$B_{1,2} = \sqrt{D}_\beta \sqrt{D_{\omega_1}^{-1}} \sqrt{D_{\omega_2}^{-1}}, \quad (16)$$

where $\sqrt{D}_\beta$, $\sqrt{D_{\omega_1}}$, $\sqrt{D_{\omega_2}}$ represent the half-twists around β , ω_1 , and ω_2 .

For a full braid, we thus have $B_{1,2}^2 = D_\beta D_{\omega_1}^{-1} D_{\omega_2}^{-1}$, as illustrated in Figs. 2(h)–2(j).

D. Mapping class group, Dehn surgery, and local geometry deformation on a torus

We now consider in more detail the case of the mapping class group of a torus T^2 . This will help provide the underlying mathematical intuition that forms the basis of our results.

The points of a torus can be specified by points z in the complex plane, modulo equivalences $z \sim z + \omega_1 \sim z + \omega_2$, for complex numbers ω_1 and ω_2 . The torus is thus parametrized by (ω_1, ω_2) or equivalently $(1, \tau)$ as shown in Fig. 3(a), where the modular parameter is defined to be $\tau = \omega_1/\omega_2$. The coordinate web indicates the local metric of a continuous manifold, or represents a lattice in the discrete case.

Arbitrary modular transformations belonging to the MCG of a torus can be achieved by the following transformation:

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \mapsto \begin{pmatrix} \omega'_1 \\ \omega'_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \quad \tau \mapsto \tau' = \frac{a\tau + b}{c\tau + d}, \quad (17)$$

satisfying $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. Therefore, the mapping class group of a torus is isomorphic to a special linear group, namely, $\text{MCG}(T^2) = \text{SL}(2, \mathbb{Z})$. The $\text{MCG}(T^2)$ is generated by two transformations $T : \tau \mapsto \tau + 1$ and $U : \tau \mapsto \frac{\tau}{\tau+1}$. Their matrix representations are

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \quad (18)$$

As shown in Fig. 3(b), a shear deformation induces a large diffeomorphism of the manifold and hence maps it back to a torus in Fig. 3(c), with an additional local metric/lattice deformation compared to the original torus in Fig. 3(a) indicated by the slanted coordinate web. Note that the original vertical geometric lines get stretched to diagonal lines. The above operation is equivalent to a Dehn surgery illustrated in Fig. 3(e), which cuts the torus along the β cycle into a cylinder, twist the cylinder by 2π along the β cycle (equivalent to the shearing) and then reglue the two edges of the cylinder back to a torus.

One can apply an additional *local geometry deformation* to transform the configuration in Fig. 3(c) to the one in Fig. 3(d) with the same metric/lattice structure as the original torus in Fig. 3(a). This local geometry deformation is a diffeomorphism isotopic to an identity MCG element $\mathbb{I}$, i.e., a trivial mapping class. In Sec. IV, we will see that this local geometry deformation can be interpreted as a retriangulation of the manifold.

Denoting the two noncontractible cycles of the torus as α (vertical) and β (horizontal), the combination of shear deformation (Dehn surgery) and local metric deformation achieves a self-diffeomorphism generating the following transformations on these two loops, respectively:

$$T : (\alpha, \beta) \mapsto (\alpha + \beta, \beta). \quad (19)$$

Similarly, one applies a combination of shear deformation (Dehn surgery) along the α loop and a local geometry deformation [Figs. 3(f)–3(h)] to induce the following transformation on the loops:

$$U : (\alpha, \beta) \mapsto (\alpha, \alpha + \beta). \quad (20)$$

Therefore, the two generators are Dehn twists along the two cycles, i.e., $T = D_\beta$ and $U = D_\alpha$.

The representation of the above two generators in the topologically ordered ground-state subspace (code subspace) are denoted by $\mathcal{T} = \mathcal{D}_\beta$ and $\mathcal{U} = \mathcal{D}_\alpha$. They induce the following transformations on the Wilson loop operators:

$$\mathcal{T} W_\alpha^a \mathcal{T}^\dagger = W_{\alpha+\beta}^a, \quad \mathcal{U} W_\beta^a \mathcal{U}^\dagger = W_{\alpha+\beta}^a, \quad (21)$$

where W_α^a and W_β^a are Wilson loops on the α cycle and β cycle with topological charge a . The operator $W_{\alpha+\beta}^a$ denotes the twisted Wilson loop.

We emphasize that, from the point of view of the mapping class group, the Dehn surgery (shear deformation) already performs a self-diffeomorphism which maps the topological manifold back to itself, i.e., $\Sigma \mapsto \Sigma$. However, such a map changes the local geometry, and in the discrete case the lattice structure, i.e., $\Lambda \mapsto \Lambda'$. For a topological state or code defined on the lattice Λ , the code space depends on the lattice structure and can be denoted by $\mathcal{H}_\Lambda$. Therefore, the Dehn surgery (shear deformation) itself changes the Hilbert space, i.e.,

$\mathcal{H}_\Lambda \rightarrow \mathcal{H}_{\Lambda'}$. Note that although $\mathcal{H}_\Lambda$ and $\mathcal{H}_{\Lambda'}$ are isomorphic, they are distinct subspaces of the full Hilbert space of the microscopic (physical) degrees of freedom (qubits). In order to realize a logical gate, which is an automorphism of the code space, one has to apply the additional trivial mapping class, i.e., the local geometry deformation in order to map the Hilbert space back to the original code space $\mathcal{H}_\Lambda$. Such a local geometry deformation can be implemented by a constant depth local quantum circuit as will be discussed in the later sections. As we will see in Sec. IV, this geometry dependence is related to the fact that the Hilbert space and wave function of a topological quantum field theory are not topological invariants and depend on the local geometry, in particular the triangulation.

Finally, we note that in this paper we focus on the situation that the topological states and codes are defined on discrete lattices. However, the notion of local geometry deformation also applies to the continuum case, and thus our results should be generalizable to topologically ordered states defined in the continuum.

III. THEORY FOR $\mathbb{Z}_N$ TORIC CODE

A. Local geometry deformation

We begin by defining the $\mathbb{Z}_N$ toric code and describing local quantum circuits that can be used to change the lattice geometry. We will subsequently use these geometrical transformations of the lattice structure to help implement our Dehn twist and braiding protocols.

1. $N = 2$

Let us begin with the $\mathbb{Z}_2$ toric code model with qubits located on the edges of a square lattice. We consider the case of periodic boundary conditions, so that the space is topologically a torus. The $\mathbb{Z}_2$ toric code has the following Hamiltonian:

$$H_{\mathbb{Z}_2} = - \sum_v X_{v,1} X_{v,2} X_{v,3} X_{v,4} - \sum_p Z_{p,1} Z_{p,2} Z_{p,3} Z_{p,4}, \quad (22)$$

where v and p specify the vertices and plaquettes of the lattice, X and Z are Pauli- X and Pauli- Z operators, and the numbers $1, \dots, 4$ index the four qubits associated with each vertex or plaquette. Violations of the vertex stabilizers are referred to as e particles, while plaquette violations are referred to as m particles.

Let us denote $|0\rangle$ and $|1\rangle$ to be the states associated with $+1$ and -1 eigenvalues for the Z operator, respectively. By taking the state $|0\rangle$ for each qubit to define the absence of a string and $|1\rangle$ to define the presence of a string, it is straightforward to see that the ground states of $H_{\mathbb{Z}_2}$ are associated with an equal weight superposition of all possible closed strings. On a torus, there are four topologically degenerate ground states, depending on whether there are an even or odd number of closed strings wrapping the two noncontractible cycles. In this basis, the ground state is a superposition of closed m strings. Alternatively, by working in the X basis, we can view the ground state to be a superposition of closed e strings.

The toric code model can in general be defined on any cellulation of a two-dimensional surface; a square lattice is

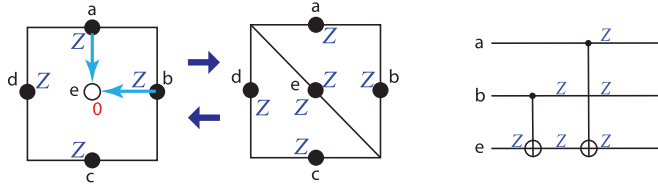


FIG. 4. The elementary move of adding (removing) a diagonal edge in the center of the plaquette, which splits (merges) the plaquette(s). The quantum circuit implementing such a move is composed of two CNOTs (blue arrows) targeting the qubits on the diagonal edge e , controlled by the other two qubits a and b on the same triangular plaquette Δ_{abe} .

just one particular simple choice. Given a toric code ground state on one cellulation, it is possible to convert it to a ground state on a different cellulation with a simple quantum circuit that effectively adds or removes extra qubits. This can be achieved with the basic moves shown in Fig. 4 [51,52]. We note that these moves have a close connection with notions of entanglement renormalization [52], as they can be used to rescale the wave function to coarser lattices.

The basic move in Fig. 4 adds an ancilla qubit (white circle in Fig. 4), which effectively adds an edge e and a triangular plaquette in the center of the square plaquette. The arrows represent the two-qubit CNOT gate, where the tail and head represent the control and target of the CNOTs. The corresponding quantum circuit is shown in the right panel. The Pauli operators in the Heisenberg picture are transformed by CNOTs as

$$\text{CNOT}(X \otimes I) \text{CNOT} = X \otimes X,$$

$$\text{CNOT}(I \otimes X) \text{CNOT} = I \otimes X,$$

$$\text{CNOT}(Z \otimes I) \text{CNOT} = Z \otimes I,$$

$$\text{CNOT}(I \otimes Z) \text{CNOT} = Z \otimes Z. \quad (23)$$

As indicated by the quantum circuit in Fig. 4, a single Z_e operator of the ancilla thus propagates through the two CNOTs into a 3-local stabilizer on the triangular plaquette:

$$Z_e \mapsto Z_a Z_b Z_e. \quad (24)$$

Since the ancilla qubit is initialized to $|0\rangle_e$, i.e., the eigenstate of Z_e with eigenvalue $+1$, the grown stabilizer $Z_a Z_b Z_e$ is also fixed at $+1$. According to Eq. (23), the original 4-local stabilizer $Z_a Z_b Z_c Z_d$ is untouched by the quantum circuit, therefore, its $+1$ eigenvalue is preserved. Since $Z_a Z_b Z_c Z_d = (Z_a Z_b Z_e)(Z_c Z_d Z_e)$, the other triangular stabilizer $Z_c Z_d Z_e$ is automatically fixed at $+1$. It is also straightforward to verify that the vertex Pauli- X stabilizers will also grow to include the new edge.

This procedure can be reversed to remove (disentangle) a qubit from the toric code ground state by applying the inverse of this procedure. In this case (but not for the generalization to $\mathbb{Z}_N$) the circuit is its own inverse. The above elementary move and its reverse is enough for all the local geometric deformation in our protocol of Dehn twist on a torus in Sec. III B and high two types of Dehn twists on high-genus surfaces in Sec. III E.

For other protocols discussed in the subsequent sections, we will use a number of other simple quantum circuits, which we refer to as gadgets, in order to implement other local geometric changes of the lattice structure.

Consider the moves shown in Fig. 5, which will be used for Dehn twists on an annulus in Sec. III C, and on high-genus surfaces in Sec. III E, as well as the braiding protocols in Sec. III D. Moves I and II achieve a fine graining (splitting one plaquette into two) and coarse graining (merging two

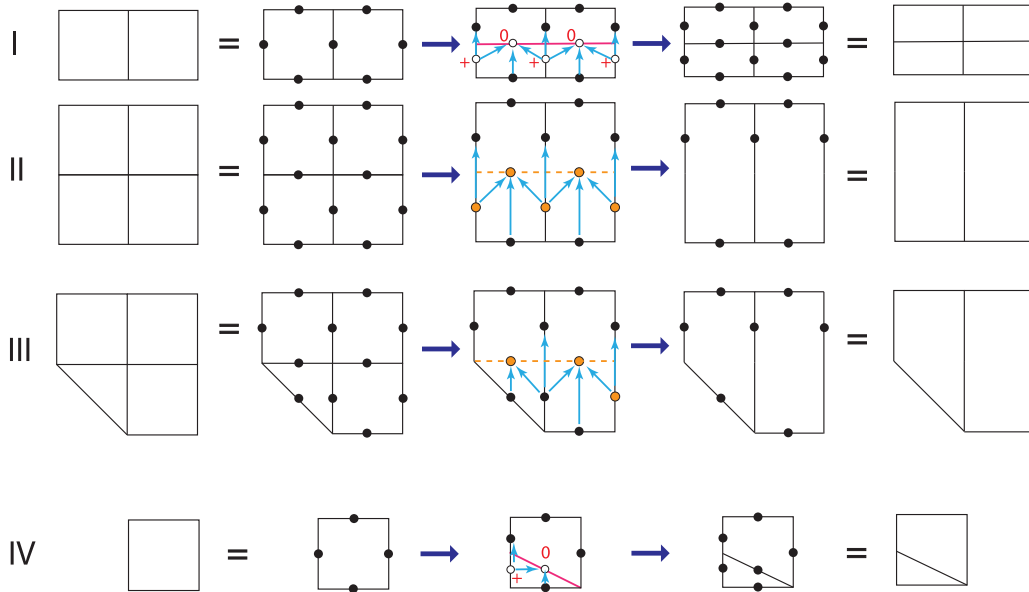


FIG. 5. Gadgets of local geometry deformation of the lattice, which will be used for braiding, and Dehn twists on an annulus and high-genus surfaces. The blue arrows represent the CNOT gates. The white (empty) circles represent the added ancilla qubits, which are initialized in $|0\rangle$ or $|+\rangle$. The yellow (filled) circles represent qubits to be removed from the code. Pink solid lines represent edges to be added to the code, while yellow dashed lines represent edges to be removed from the code.

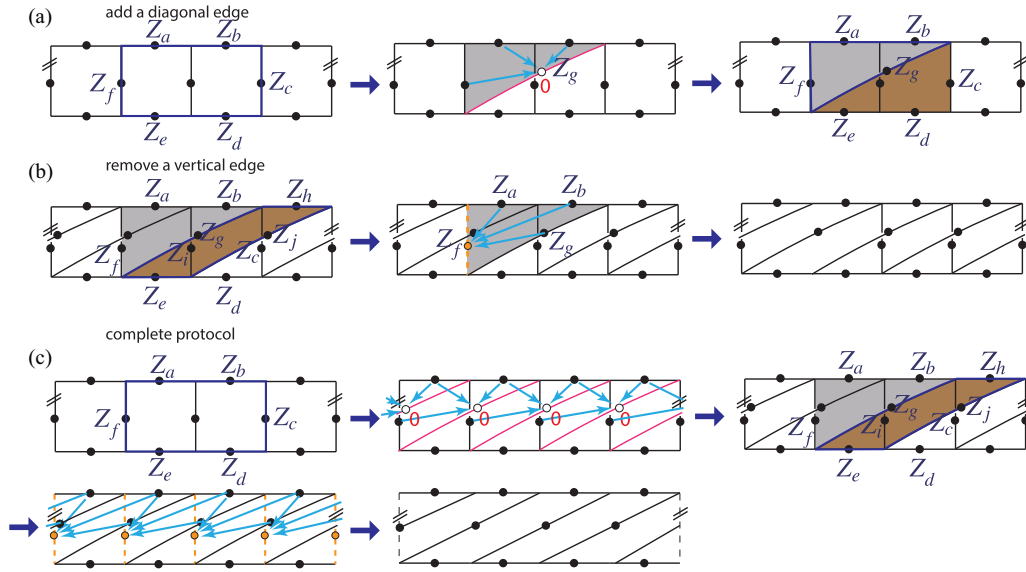


FIG. 6. Creating NNN slanted plaquettes through a constant depth circuit. The gray and brown shades represent stabilizers with triangular or parallelogram shape.

plaquettes into one), respectively, along the vertical directions. In move I, the ancilla qubits in white circles are initialized in the $|0\rangle$ state or the $|+\rangle$ state, depending on whether they are on the horizontal or vertical edges, as shown.

Move III implements a coarse graining that merges a square and a triangular plaquette together into a trapezoid, while merging the two neighboring square plaquettes into a single rectangular one. Finally, move IV splits a square plaquette into a triangular plaquette and a trapezoidal plaquette. Note that all of the required CNOT operations (indicated by blue arrows) commute with each other and thus can be implemented in any order.

The next set of moves will be used for alternative single-shot protocols for braiding, Dehn twists on an annulus, and high-genus surfaces, as well as multiple Dehn twists in a single shot in Sec. III F. They create slanted plaquettes with diagonal edges that can be of some arbitrary length ℓ . This can be achieved with a constant depth local circuit with a range ℓ . The protocol for the simplest case, namely, creating a slanted plaquette with a next-nearest-neighbor (NNN) diagonal edge [which we can label as the vector (2,1)] on a $\mathbb{Z}_2$ toric code is shown in Fig. 6. In Fig. 6(a), we apply three CNOTs targeting an ancilla g initialized at $|0\rangle$ (the $+1$ eigenstate of Z_g), conditioned by the qubits a , b , and f . According to Eq. (23), the operator Z_g is transformed as

$$Z_g \mapsto Z_a Z_b Z_g Z_f, \quad (25)$$

which effectively introduces the NNN diagonal edge overpassing a vertical edge and the triangular plaquette (gray shadow) associated with a four-body stabilizer $Z_a Z_b Z_g Z_f = +1$ coexisting with all the previous stabilizers, as shown in Fig. 6(a). Since the initial configuration in Fig. 6(a) has a two-plaquette stabilizer $Z_a Z_b Z_c Z_d Z_e Z_f = +1$, the stabilizer on the other triangle (brown shadow) divided by the diagonal edge, $Z_g Z_c Z_d Z_e$, is automatically set to $+1$, due to the decomposition

$Z_a Z_b Z_c Z_d Z_e Z_f = (Z_a Z_b Z_f Z_g)(Z_g Z_c Z_d Z_e)$. One can also easily verify the transformation of the vertex stabilizers.

One important fact is that the addition of different NNN diagonal edges (2,1) shown in Fig. 6(b) can be done in parallel. In this way, one creates the slanted plaquettes (one shown in brown shadow) with the NNN diagonal edges overpassing the vertical edges of the original square plaquettes. To see the stabilizers corresponding to the slanted plaquettes are fixed at $+1$, we can proceed as follows.

First, we note that from the argument above, the elongated triangular stabilizers are all one, e.g., $Z_b Z_h Z_j Z_i = +1$ and $Z_a Z_b Z_f Z_g = +1$. Next, we consider the trapezoid (union of the region in gray and brown shadow) which can be considered as a combination of a square plaquette associated with the stabilizer $Z_a Z_i Z_e Z_f = +1$ and the triangle on the right associated with the stabilizer $Z_b Z_h Z_j Z_i = +1$. Therefore, the multiplication of these two stabilizers gives rise to the trapezoid stabilizer $Z_a Z_b Z_h Z_j Z_e Z_f = (Z_b Z_h Z_j Z_i)(Z_i Z_a Z_e Z_f) = +1$. Now, the trapezoid can also be split into the triangle on the left (gray shadow) with stabilizer $Z_a Z_b Z_f Z_g = +1$ and the slanted plaquette with stabilizer $Z_h Z_j Z_e Z_g$, i.e., $Z_a Z_b Z_h Z_j Z_e Z_f = (Z_h Z_j Z_e Z_g)(Z_a Z_b Z_f Z_g) = +1$. It follows that the stabilizer associated with the slanted plaquette $Z_h Z_j Z_e Z_g$ is also fixed at $+1$.

Now, we can get rid of all the vertex edges [which we can label as the vector (0,1)] and also the triangular stabilizers, with the circuit shown in Fig. 6(c), where, for example, the qubit f is disentangled by the three CNOTs with the controls being the other three sites associated with the triangular stabilizer (gray shadow). Similar to the procedure of adding diagonal edges, the removal of the vertical edges can also be done in parallel as shown in Fig. 6(c). One hence ends up with a lattice with slanted plaquettes containing NNN diagonal edges in Fig. 6(c).

This protocol for creating slanted plaquettes spanned by the horizontal edge (1,0) and NNN diagonal edge (2,1) can

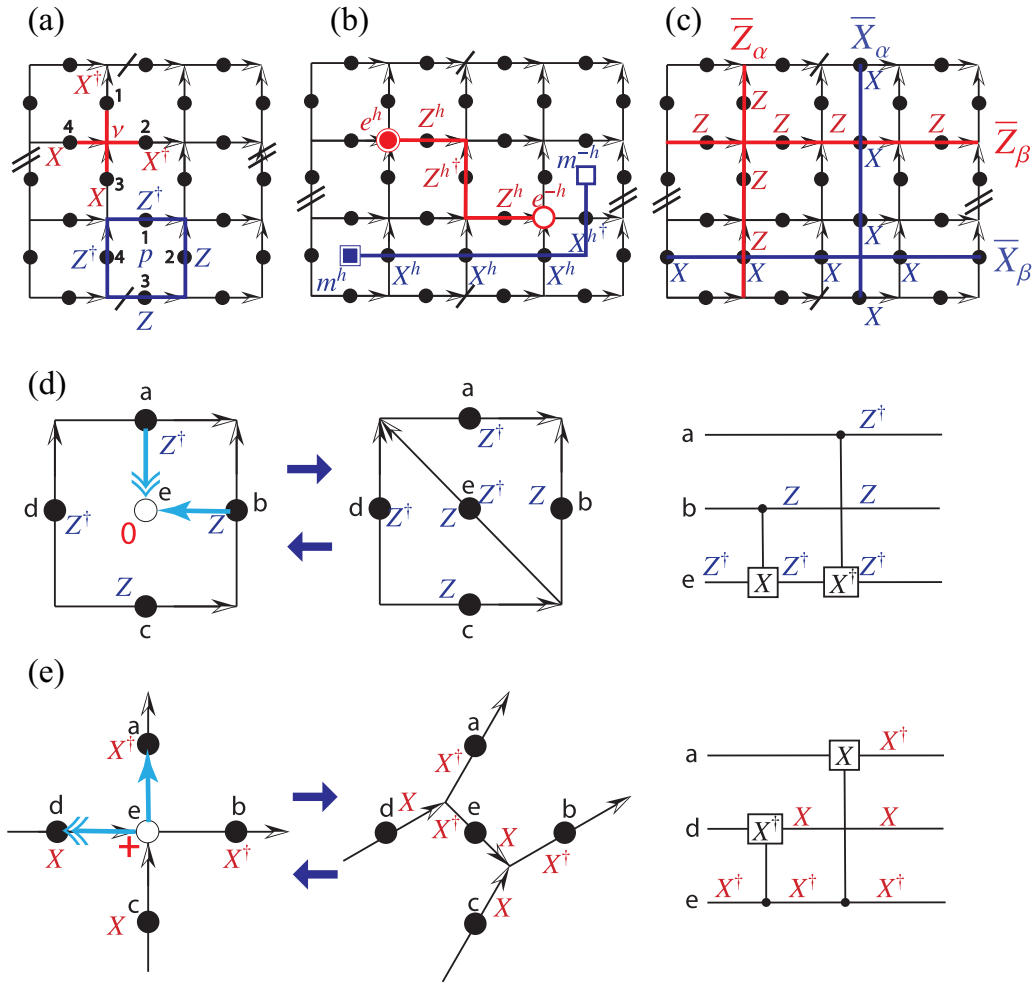


FIG. 7. Definition and elementary move for the $\mathbb{Z}_N$ toric code. The edges of the lattice are directed (with arrows) to represent the Hamiltonian. (a) Two types of stabilizers. (b) Two types of anyons strings with corresponding anyon pairs in the ends. (c) Two types of logical string operators along two directions on a torus. (d) Elementary move for adding an edge. The single and double arrows represent the CX and $CX^\dagger$ gates, respectively. (e) Elementary move for removing an edge.

be generalized to creating more slanted plaquettes spanned by $(1,0)$ and $(n,1)$, where n is an arbitrary integer. By symmetry, it can also be generalized to creating slanted plaquette spanned by the vertical edge $(0,1)$ and the diagonal edge $(1,n)$. Even more generally, we can create a parallelogram plaquette spanned by the edge (m,n) and (m',n') .

We emphasize that since the plaquettes can all be operated on in parallel, we can convert a lattice with square plaquettes to any type of slanted plaquettes by a constant depth circuit, independent of the size of the lattice.

2. General N

Now, we generalize the above results to the $\mathbb{Z}_N$ toric code. On each edge of the lattice we now have a qudit with N states. The Hamiltonian is

$$H_{\mathbb{Z}_N} = - \sum_v X_{v,1}^\dagger X_{v,2}^\dagger X_{v,3} X_{v,4} - \sum_p Z_{p,1}^\dagger Z_{p,2}^\dagger Z_{p,3}^\dagger Z_{p,4}, \quad (26)$$

where the vertex and plaquette operators are illustrated in Fig. 7. Here, the shift operators (generalized Pauli operators)

for the N -level qudits are defined by

$$\begin{aligned} X &= \sum_{n=0}^{N-1} |(n+1) \bmod N\rangle \langle n|, \\ X^\dagger &= \sum_{n=0}^{N-1} |(n-1) \bmod N\rangle \langle n|, \\ Z &= \sum_{n=0}^{N-1} \omega^n |n\rangle \langle n|, \end{aligned} \quad (27)$$

where $\omega = e^{2\pi i/N}$. The shift operators satisfy the Weyl algebra $ZXZ^\dagger X^\dagger = \omega$. A useful way of representing the Hamiltonian is to consider the edges to be directed, as the arrows on the edges indicate in Fig. 7.

The topological charges in this model are electric charge e^h , magnetic charge m^h , and the composites $e^h m^l$, where $h, l = (0, 1, 2, \dots, N-1) \bmod N$. As shown in Fig. 7, anyon e^h and its antiparticle e^{-h} can be created out of the vacuum by a string operator with Z^h on the horizontal edges and $Z^{h^\dagger}$ on the vertical edges. Similarly, m^h and m^{-h} can be created by a

string involving X^h and $X^{h\dagger}$. The logical qudit shift operators on a torus are $\bar{X}_\alpha$, $\bar{X}_\beta$, $\bar{Z}_\alpha$, and $\bar{Z}_\beta$ as shown in Fig. 7(c) up to local deformation.

The generalization of CNOT is the controlled- X gate defined as

$$CX = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} |n\rangle\langle n| \otimes |(m+n) \bmod N\rangle\langle m|, \quad (28)$$

i.e., the value of the target qudit undertakes a conditional addition of the value of the control qudit. The shift operators in the Heisenberg picture are transformed by CX or $CX^\dagger$ as

$$\begin{aligned} CX(X \otimes I)CX^\dagger &= X \otimes X, & CX(I \otimes X)CX^\dagger &= I \otimes X, \\ CX^\dagger(X \otimes I)CX &= X \otimes X^\dagger, & CX(Z \otimes I)CX^\dagger &= Z \otimes I, \\ CX(I \otimes Z)CX^\dagger &= Z^\dagger \otimes Z, & CX^\dagger(I \otimes Z)CX &= Z \otimes Z. \end{aligned} \quad (29)$$

All of the gadgets described in the previous section for locally changing the geometry of the lattice can be straightforwardly generalized to the case of the Z_N toric code by replacing CNOT with CX or $CX^\dagger$.

For example, consider the splitting of a plaquette shown in Fig. 7(d). A single $Z_e^\dagger$ operator of the ancilla propagates through the CX (single arrow) and $CX^\dagger$ (double arrow) into a 3-local stabilizer on the triangular plaquette, i.e.,

$$Z_e^\dagger \mapsto Z_a^\dagger Z_b Z_e^\dagger, \quad (30)$$

according to the Hermitian conjugate of the last line in Eqs. (29), namely,

$$CX(I \otimes Z^\dagger)CX^\dagger = Z \otimes Z^\dagger, \quad CX^\dagger(I \otimes Z^\dagger)CX = Z^\dagger \otimes Z^\dagger. \quad (31)$$

Note that the tail and head of the arrows represent the control and target of the CX and $CX^\dagger$ gates, as in the $N = 2$ case for CNOT gates.

Since the ancilla qubit is initialized in the state $|0\rangle_e$, i.e., the eigenstate of Z_e and $Z_e^\dagger$ with eigenvalue $+1$, the grown stabilizer $Z_a^\dagger Z_b Z_e^\dagger$ is also fixed at $+1$. According to Eqs. (29), the original 4-local plaquette stabilizer $Z_a^\dagger Z_b Z_c Z_d^\dagger$ is untouched by the quantum circuit, therefore, its $+1$ eigenvalue is preserved. Since $Z_a^\dagger Z_b Z_c Z_d^\dagger = (Z_a^\dagger Z_b Z_e^\dagger)(Z_e Z_c Z_d^\dagger)$, the other triangle stabilizer $Z_c Z_d^\dagger Z_e$ is automatically fixed at $+1$.

Similarly, one can verify that the vertex stabilizer terms are also grown appropriately. One can also add vertices to the lattice. To split a vertex with four edges into two vertices with three edges, we follow the procedure shown in Fig. 7(d). The ancilla qubit e is initialized in the $|+\rangle_e$ state, i.e. the eigenstate of X_e and $X_e^\dagger$ with eigenvalue $+1$. A single $X_e^\dagger$ operator of the ancilla propagates through the CX (single arrow) and $CX^\dagger$ (double arrow) into a 3-local vertex stabilizer, i.e.,

$$X_e^\dagger \mapsto X_a^\dagger X_d X_e^\dagger, \quad (32)$$

according to the Hermitian conjugate of the relations in Eqs. (29), namely,

$$\begin{aligned} CX(X^\dagger \otimes I)CX^\dagger &= X^\dagger \otimes X^\dagger, \\ CX^\dagger(X^\dagger \otimes I)CX &= X^\dagger \otimes X. \end{aligned} \quad (33)$$

Since the ancilla qubit is initialized in the $|+\rangle_e$ state, the eigenvalue of the grown stabilizer $X_a^\dagger X_d X_e^\dagger$ is also fixed at $+1$. Similar to the case with the plaquette stabilizers, the other three-term vertex stabilizer $X_b^\dagger X_c X_e$ is also automatically fixed at $+1$.

In the following sections, we describe in detail the protocols for implementing Dehn twists and braids for the $\mathbb{Z}_N$ toric code state. Most of our results will be presented for the case $N = 2$; the generalization to arbitrary N is straightforward.

B. Dehn twist on a torus

Using the basic moves described in the previous section to entangle/disentangle ancilla qubits in the toric code ground state, we can then implement the protocol for applying a Dehn twist on the torus. The complete protocol is shown in Fig. 8. For illustration purposes, we show the transformations of both small (contractible) loops and large (noncontractible) loops of e (red) and m (blue) anyons, respectively. These loops are created by applying Pauli string operators $\bar{Z} = \prod_j Z_j$ and $\bar{X} = \prod_j X_j$, respectively.

We start with the lattice configuration shown in Fig. 8(a). We then incorporate the ancilla qubits (white) in the center of each plaquette by applying the basic moves, as shown in Fig. 8(b). This can be done with two transversal CNOT operations, first with the qubits on vertical edges as the control qubits, and next with the qubits on horizontal edges as the control qubits. Next, as depicted in Fig. 8(c), we apply two transversal CNOT operations targeted on the vertical edges (yellow circles). This causes the qubits on the vertical edges to become disentangled, thus effectively removing them from participating in the toric code ground state. This results in the toric code ground state existing on the slanted lattice shown in Fig. 8(d); the ground state is then a superposition of slanted loops. The sequence of operations from Figs. 8(a)–8(d) is therefore a local, finite depth quantum circuit, which we label $\mathcal{LU}_\beta$. This circuit changes the local geometry of the topological wave function from a condensation of square-shaped loops to a condensation of slanted loops, which exactly corresponds to the local geometry deformation we introduced before in Sec. II (Fig. 3).

To complete the Dehn twist protocol, we perform a shear deformation by a qubit permutation $\mathcal{P}_\sigma$ shown in Fig. 8(e). The qubits in each row are cyclically permuted to the right by a number of spacings depicted by the green arrows. This leads to the configuration shown in Fig. 8(f), which we can see is equivalent to the starting lattice geometry. Therefore, the shear recovers the original lattice, and at the same time induces a Dehn twist of both the large e and m loops along β , namely,

$$\begin{aligned} \mathcal{P}_\sigma \mathcal{LU}_\beta : \bar{X}_\alpha &\longrightarrow \bar{X}_{\alpha+\beta}, \\ \mathcal{P}_\sigma \mathcal{LU}_\beta : \bar{Z}_\alpha &\longmapsto \bar{Z}_{\alpha+\beta}, \end{aligned} \quad (34)$$

and hence $\mathcal{P}_\sigma \mathcal{LU}_\beta = \mathcal{T} = \mathcal{D}_\beta$. The small loops are deformed but remain closed under this combined transformation. Therefore, the ground state continues to be an equal weight superposition of closed loops, and the topological ground-state (code) subspace is preserved under this operation.

By implementing the $\mathcal{LU}$ transformation to slant the lattice in the other direction and then a permutation corresponding to

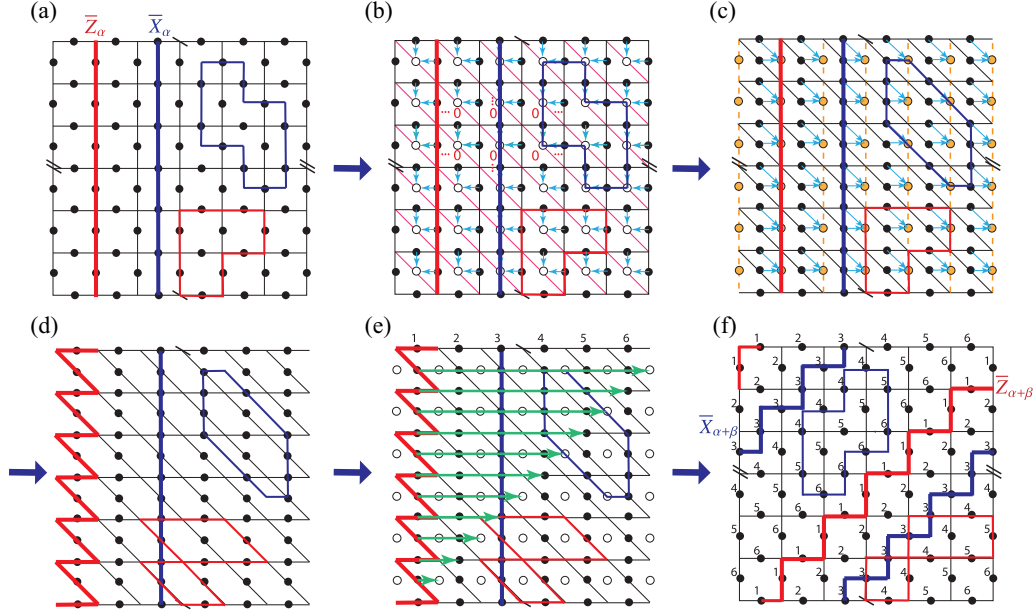


FIG. 8. Implementing Dehn twist $\mathcal{D}_\beta$ on a torus for the $\mathbb{Z}_2$ toric code. The thick lines represent the noncontractible loops of two types $\bar{Z}$ (red) and $\bar{X}$ (blue). The thin lines represent local contractible loops of two types. The blue arrows in (b) and (c) represent CNOTs, in order to add qubits (white circles, initialized at $|0\rangle$) and edges (pink solid lines), or to remove qubits (yellow circles) and edges (yellow dashed lines). The green thick arrows in (e) represent permutation of qubits in a shearing pattern.

a vertical shear, we can analogously obtain a Dehn twist along α . It is clear that the same operation can also be performed on a cylinder instead of a torus, where we perform a Dehn twist along the single closed noncontractible cycle of the cylinder.

We note that a different set of protocols for implementing a Dehn twist on a torus in the context of the $\mathbb{Z}_2$ toric code was proposed recently in Ref. [53]. Either one applies a sequence of long-range CNOTs with an $\mathcal{O}(d)$ overhead, or performs only CNOTs in parallel across the lattice with constant time overhead. In the latter protocol, the role of the long-range permutations was not discussed, which is crucial for consideration of multiple Dehn twists.

C. Dehn twist on an annulus

In order to set up the rest of the required protocols for arbitrary braids and Dehn twists, we now consider protocols which implement a Dehn twist along the noncontractible cycle β of an annulus. Again, here we restrict our attention to the case of the $\mathbb{Z}_2$ toric code. We consider two distinct protocols for implementing the Dehn twist $\mathcal{D}_\beta$ along β .

1. Protocol 1: Twist via shearing

The right Dehn twist $\mathcal{D}_\beta$ on a single annulus can be implemented according to the protocol shown in Fig. 9. The essence of this protocol is to implement a 2π rotation of the inner boundary by a sequence of operations combining local finite depth circuit and qubit permutation:

$$\mathcal{D}_\beta = \prod_{i=1}^9 \mathcal{P}_{\sigma,i} \mathcal{L} \mathcal{U}_i. \quad (35)$$

Since the 2π rotation of the square boundary defect can always be decomposed into a sequence of shears, we can just perform entangling and disentangling gates followed by a qubit permutation to realize each shearing process.

In Fig. 9(a) we illustrate a logical string operator $\bar{X}_\alpha$ along the α line connecting the inner and outer boundaries of the annulus. The first step of shearing the boundary defect is shown in Fig. 9(a), panels I–VI. In panel II, we add (entangle) qubits (white circles) and effectively add edges (red lines) to the code using one step of the elementary CNOT circuits from Fig. 4. In panel III, we then remove (disentangle) qubits and effectively remove edges (dashed yellow lines), which leads to the deformed lattice in panel IV. The lattice in panel IV is a deformed square lattice except the defect region. In particular, it is coarse grained in the region above the defect and fine grained in the region below, and is sheared on the two sides. In panel V, we perform a permutation of qubits, where some of them are moved to the unoccupied ancilla qubits (white circles). After that, we recover the regular square as shown in panel VI with the original square defect being sheared to a parallelogram.

The second step of shearing the defect is shown in Fig. 9(b), panels I–V. Now in panels I and II we add and remove qubits and edges to get the deformed lattice in panel III, where the shearing and coarse- and fine-graining procedures are opposite in the top and bottom parts. Now we again perform qubit permutation in opposite directions in the top and bottom parts as shown in panel IV, leading to the further sheared parallelogram defect in panel V where the vertex A is permuted to the upper right corner. We repeat this procedure of shearing the parallelogram defect as shown in Fig. 9(c) and finally shear it back to the square defect in Fig. 9(d). The square boundary defect has been rotated by a full 2π

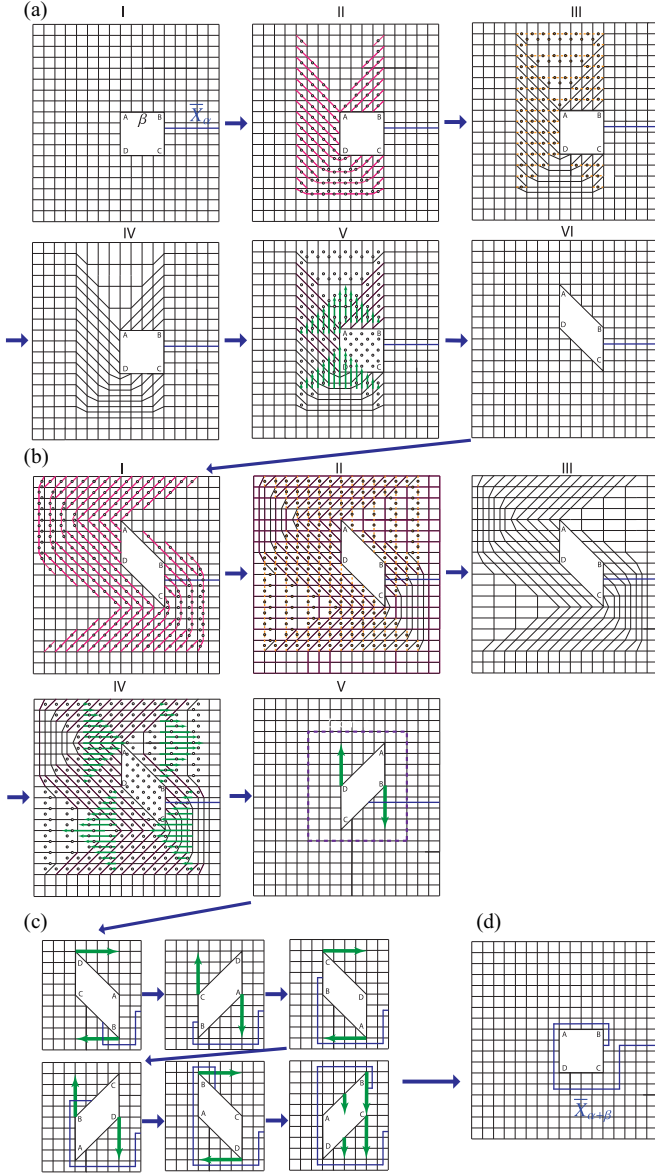


FIG. 9. Dehn twist along the boundary of the inner defect (the β loop) on an annulus via shearing of the defect. An overall 2π rotation (indicated by the letters on the four corners) is induced by nine steps of shearing.

cycle which leads to a right Dehn twist of the string $\bar{X}$ or $\bar{Z}$ connecting the inner and outer boundary of the annulus along the β loop, i.e.,

$$\mathcal{D}_\beta : \bar{X}_\alpha \mapsto \bar{X}_{\alpha+\beta}, \quad \mathcal{D}_\beta : \bar{Z}_\alpha \mapsto \bar{Z}_{\alpha+\beta}. \quad (36)$$

The transformation of the $\bar{X}$ string is illustrated in Fig. 9(d).

Note that in total we have passed through eight configurations of the parallelogram defect, and have used nine composite steps of local quantum circuit and qubit permutation in total. Finally, we note that if we start with an annulus geometry having a parallelogram defect in the middle as shown in Fig. 9(a), panel VI, and end up with the same shape, the total number of composite steps can be reduced to 8.

2. Protocol 2: Single-shot twist

Here, we described another procedure for implementing the Dehn twist around the β loop on the annulus. In contrast to the previous protocol, this protocol can be implemented in a single step, so $\mathcal{D}_\beta = \mathcal{P}_\sigma \mathcal{L} \mathcal{U}_\beta$, rather than the nine steps used in the previous protocol. However, the price to pay is that this protocol requires the local quantum circuit to have two-qubit gates with a range $r \approx 9$.

Consider a hole inside a square lattice with perimeter L . As an example, the case for $L = 12$ is shown in Fig. 10(a). Only a portion of the entire lattice has been shown in the figure. By the following procedure we can perform a Dehn twist around this hole defect.

(1) We start by labeling the sites as shown in Fig. 10(a). For the sake of clarity, only some of the labels are shown. We divide the lattice into rings. The sites that lie on the hole's boundary compose the 0th ring ($n = 0$), the next ring consists of sites that neighbor the 0th ring, and so on. In Fig. 10(a), the rings are plotted with thicker lines to be distinguished. We then number the sites in each ring, starting from the site on the upper left corner [$m = 0$, denoted by red color in Fig. 10(a)] and moving clockwise. Note that the n th ring has $L + 8n$ sites, so if (n, m) represents the m th site on the n th ring, m ranges between 0 to $L + 8n - 1$.

Given the labels, one can describe the lattice by the set of links between the sites. For example, the $(0,0)$ site is connected to $(1,1)$ and $(1,19)$ sites from the $n = 1$ ring, $(0,1)$ is connected to $(1,2)$, and so on.

(2) Now, we shift the labels on the n th ring by $9n$ sites clockwise along the ring. So, the site that was originally labeled as (n, m) will now be labeled as $(n, m - 9n)$. Figure 10(b) shows some of the shifted labels for the $L = 12$ case.

We will do this for $0 \leq n \leq L - 1$. Note that the sites on the L th ring would have been shifted by $9L$ sites. But, since the L th ring has $L + 8L = 9L$ sites in total, this shift is equivalent to doing nothing. So, for $n \geq L$ we leave the rest of the lattice unchanged.

(3) Now, we implement a local, finite depth quantum circuit to reconnect the new labels the same way that the original labels were connected. So, for example, in the $L = 12$ case, the new $(0,0)$ site should be connected to the sites that carry the labels $(1,1)$ and $(1,19)$ after the shift, as is shown in Fig. 10(c).

Note that although the sites that were connected to each other in the original lattice were nearest neighbors, after the shift we need to connect sites that are further apart. Nevertheless, the range of those links remains finite and independent of the code distance. Since the labels on the n th and $(n + 1)$ th rings have been shifted by $9n$ and $9(n + 1)$ sites, respectively, they have been moved only by nine sites *relative* to each other. So, the modified links' range would be at most 9.

The whole reconnecting procedure can be done in two steps using the protocol in Fig. 6. Figure 10(d) shows the resulting lattice after changing the links.

(4) Finally, we implement a permutation $\mathcal{P}_\sigma$ of qubits such that each site label is moved back to its original position.

Figure 11 illustrates how a string operator transforms under this procedure. Figure 11(a) shows the hole defect and a string that terminates on the hole's boundary. After reconnecting

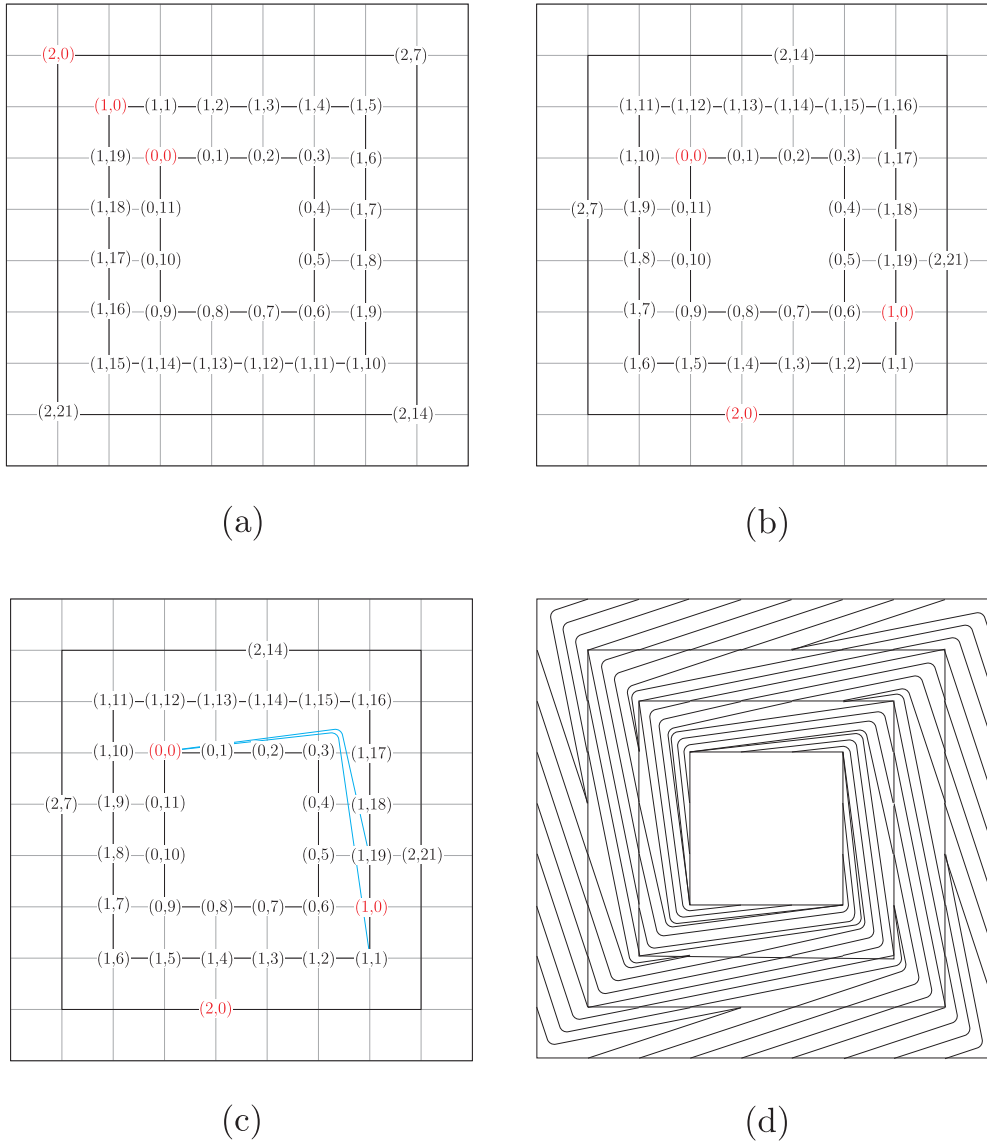


FIG. 10. (a) A hole with perimeter $L = 12$ in a square lattice. The qubits are on the edges but are not shown for simplicity. The sites are divided into rings surrounding the hole and are numbered clockwise. We start with the upper left corner, shown in red numbers, and move clockwise. (b) We shift the labels on the n th ring by $9n$ sites. The labels which originally corresponded to upper left corners are shown in red. (c) After shifting labels, we reconnect the sites as they were connected originally. For example, since the (0,0) site was connected to (1,1) and (1,19) in (a), it should be reconnected by the blue lines shown above. (d) The modified lattice after reconnecting the sites.

the sites, the lattice and the string will look as shown in Fig. 11(b). Finally, after permuting the qubits, we recover the original lattice with the string encircling the hole as is shown in Fig. 11(c).

D. Braiding

1. Protocol 1: Braiding in multiple steps

In this section we demonstrate a protocol for braiding which takes a finite number of steps: $\mathcal{B}_{12} = \prod_i \mathcal{P}_{\sigma_i} \mathcal{L} \mathcal{U}_i$. For concreteness, we will demonstrate how to implement a full braid between two hole defects in the $\mathbb{Z}_2$ toric code state. The protocol can be straightforwardly applied to the case of half-braids involving any type of holes, anyons, or twist defects. Furthermore, the protocol only includes nontrivial operations

that act in a given subregion of the system. Therefore, it can be applied to braids on any surface.

An intuitive way to understand the protocol is through the picture of entanglement renormalization and the multiscale entanglement renormalization ansatz (MERA). The essence of entanglement renormalization and the MERA circuit can be understood as a coarse-graining process that “merges” several qubits together, effectively removing (disentangling) several qubits, as illustrated in Fig. 12(a). In the context of topological order, one can think of this process as shrinking the manifold which supports the topological state. The reverse of such a process is “fine graining” which “splits” one qubit into several, effectively adding (entangling) qubits to the code. One can think of this process as stretching the manifold. We consider anyons or defects as punctures in the manifold as illustrated in Fig. 12(b) which are distance d apart. In order to

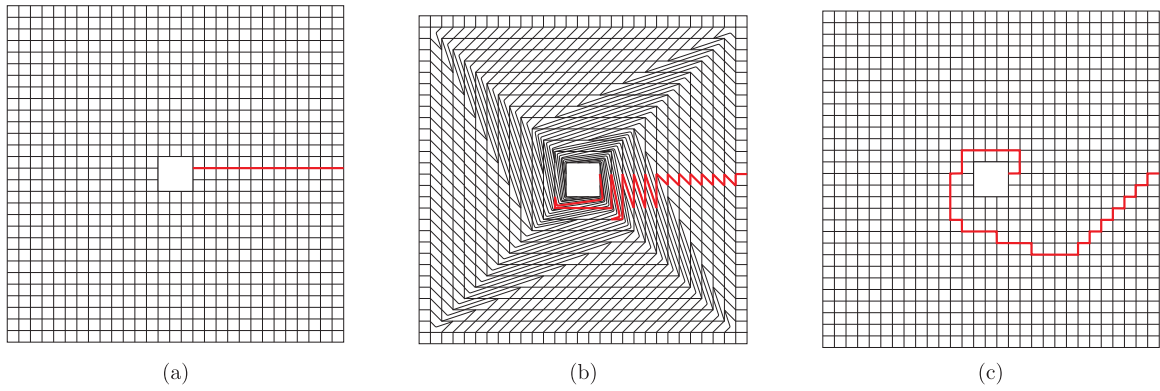


FIG. 11. Performing Dehn twist around the β loop in one shot. (a) A boundary defect with length $L = 12$ in a square lattice. A string operator which terminates on the hole's boundary is also shown in red color. (b) How the lattice and the string look like after reconnecting the sites. (c) Following the permutation we recover the original lattice, but now the string operator goes around the hole defect.

separate the two punctures further, one can perform one layer of the entanglement renormalization circuit (with constant depth) locally to stretch (fine graining) the region between the two punctures, which effectively adds qubits into the system. The distance between the two punctures has effectively been increased by a constant factor that is independent of the initial separation of the two punctures. Now, the manifold is effectively enlarged due to the addition of qubits. In order to preserve the shape of the manifold away from the region of the punctures, one can also perform one layer of the entanglement renormalization circuit locally to squeeze (coarse grain) the region on the top and bottom sides of the punctures, as shown in Fig. 12(c). Thus, one ends up with the same overall shape

of the manifold, with the two punctures being separated by a factor of 2, i.e., $d \rightarrow 2d$. One could also stretch the manifold only on one side as illustrated in Fig. 12(d), which effectively moves the top puncture upward in this case and keeps the other puncture fixed.

The concrete braiding circuit for the surface code (toric code on a planar geometry) is shown in Fig. 13. In Fig. 13(a), we show the setup under consideration, i.e., a pair of Z-cut defects with smooth boundary (red) and a pair of X-cut defects with rough boundary (blue) in a surface code. Each pair of defects form a “double-cut” logical qubit. The braiding of a Z-cut defect around the another X-cut defect implements a logical CNOT gate.

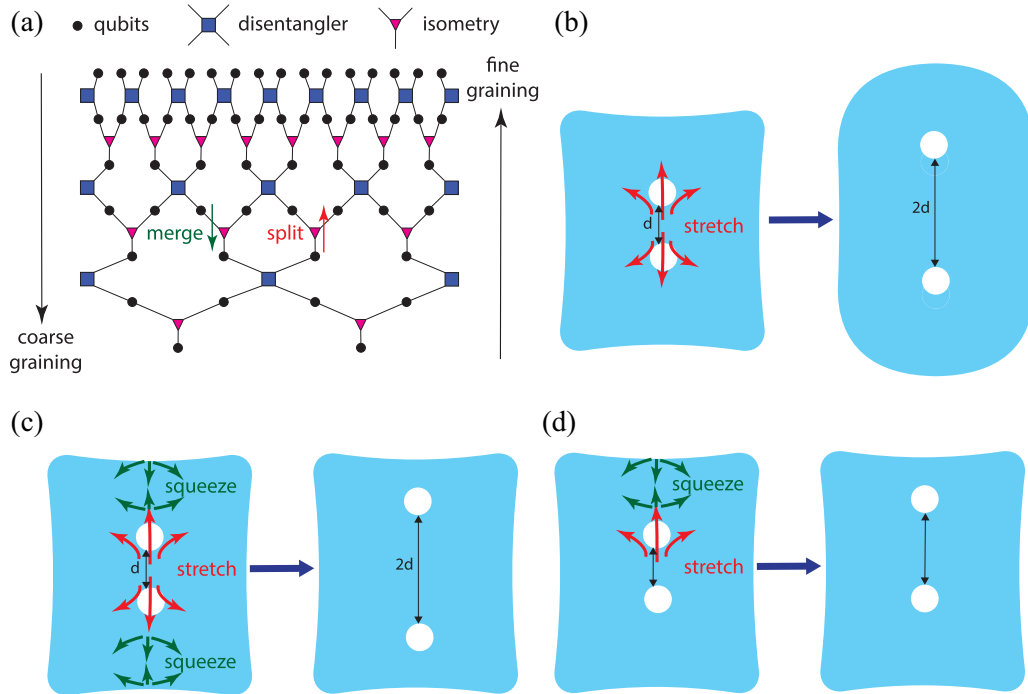


FIG. 12. Understanding the essence of braiding via the equivalence of local entanglement renormalization (its inverse) and manifold stretching (squeezing). (a) The global MERA circuit performing coarse graining (downward) or fine graining (upward). (b) Locally stretch the manifold in the region between the two punctures to enlarge their distance by a factor of 2. (c) Aside from stretching, also squeeze the upper and lower regions to preserve the shape of the manifold. (d) Stretch the puncture only in one direction to only move one puncture upward.

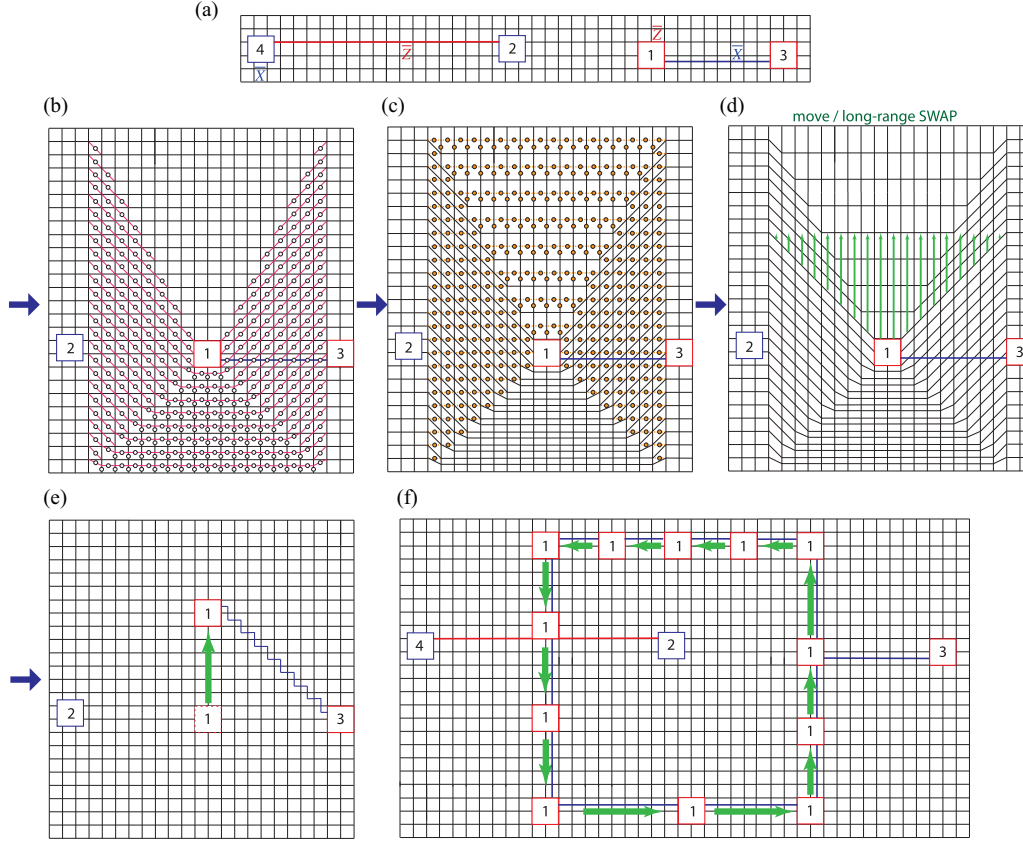


FIG. 13. Braiding protocol via single-shot long-distance moving of the boundary defects. (a) Two pairs of boundary defects (X cut and Z cut). (b) Adding qubits (white circles) and edges (pink solid lines). (c) Removing qubits (yellow circles) and edges (yellow dashed lines). (d) Permutation of qubits in a shearing pattern $\mathcal{P}_\sigma$. (e) Effectively move the defect 1 with distance $\mathcal{O}(d)$ after the permutation. (f) A full braiding is implemented by 12 steps of $\mathcal{LU}$ and $\mathcal{P}_\sigma$.

In Fig. 13(b), one adds qubits (white circles) and edges into the code in the region below defect 1, which effectively stretches the manifold. One also adds diagonal edges on the two sides which creates a shearing pattern. Now, in Fig. 13(c), one removes half of the qubits (yellow circles) and edges in the region above defect 1, in order to compensate for the added qubits. Thus, we preserve the total number of qubits participating in the topological state. These operations can be performed on different plaquettes in parallel, so we have a local finite depth quantum circuit that implements these transformations. After the transformation, one obtains a deformed square lattice, where the top part is squeezed, the lower part is stretched, and the side part is sheared.

We then permute the qubits as shown in Fig. 13(d), to revert back to the regular square lattice shown in Fig. 13(f). Note that the lattice in Figs. 13(d) and 13(e) are isomorphic to each other, and the qubit permutation is exactly the implementation of the isomorphism between the two lattices. As a consequence, defect 1 effectively moves $\mathcal{O}(d)$ sites upward, as shown in Fig. 13(e). We see that up to the overall permutation of qubits, this transformation occurs through a finite number of local two-qubit operations, independent of the code distance d . This is in sharp contrast to conventional schemes which need $\mathcal{O}(d)$ steps.

Crucially, the number of sites that a defect can move in our scheme is bounded by the distance to the closest defect

perpendicular to the moving direction. Therefore, to implement a braid, we need to break up the braid into a finite number of steps, where in each step the defect moves by an amount set by the distance to the closest defect.

We see that a single full braiding operation can be always performed with a constant number of steps, independent of system size and code distance. As summarized in Fig. 13(f), we have demonstrated how the full braid can be achieved with 12 steps:

$$\mathcal{B}_{1,2}^2 = \prod_{i=1}^{12} \mathcal{P}_{\sigma,i} \mathcal{LU}_i. \quad (37)$$

We note that there is freedom in our choice of steps to implement this protocol. The precise number of steps can be altered depending on the exact geometry and finite-size effects, but the main point is that the number of steps is finite and independent of code distance and system size. The precise protocol that we illustrated has a slightly asymmetric nature (four horizontal steps in the upper region and two horizontal steps in the lower region). This is due to the fact that the maximum distance a puncture can be moved is bounded by the separation of the defects 1 and 2, various finite-size effects, and some choices made to minimize the number of steps in the protocol.

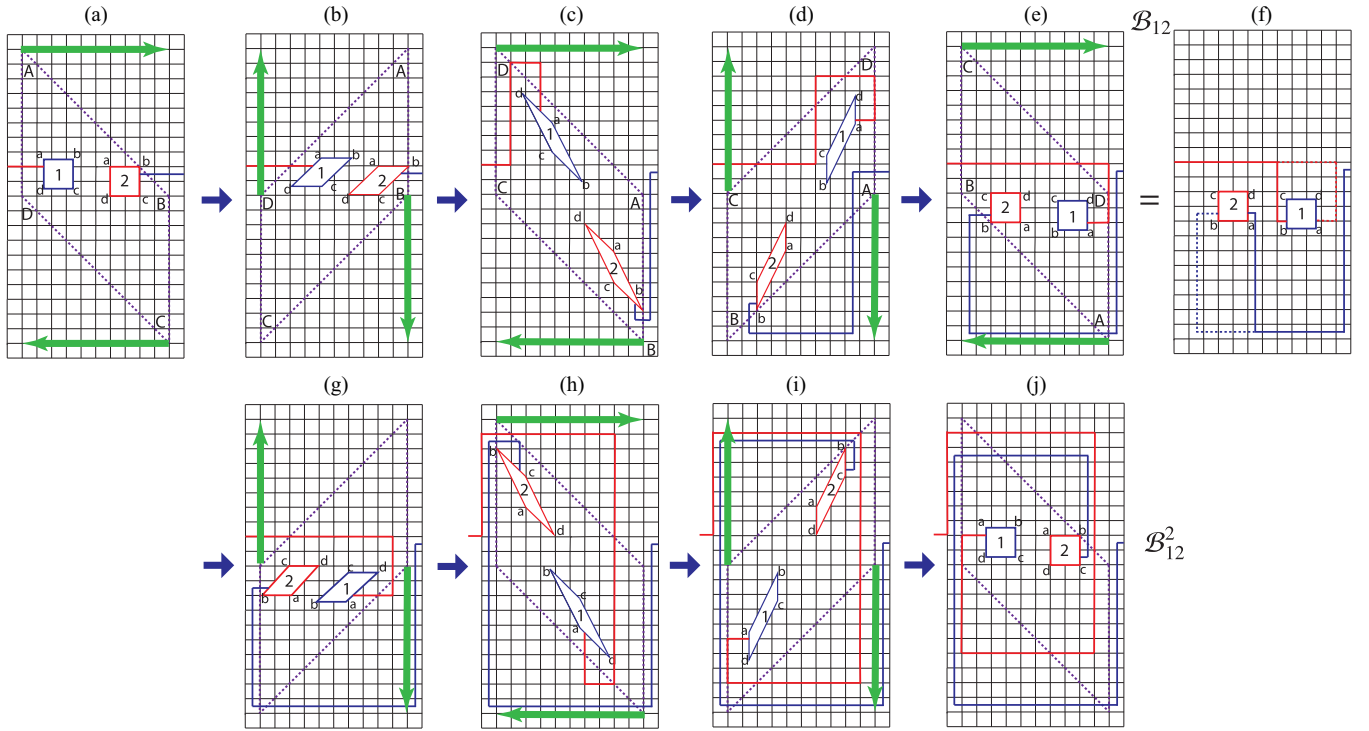


FIG. 14. Implementing braiding (full braiding) via half-twists (Dehn twists) through multiple steps of shearing operations. The parallelogram (purple dashed lines) surrounding both punctures is a guide to the eye. The red and blue dashed loops in (f) indicate gauge transformations, which shows that the half-self-twists around each puncture are trivial operations.

Note that in this example we considered a full braid because in the $\mathbb{Z}_2$ toric code, that is the only way to achieve a nontrivial logical operation by braiding hole defects. However, one can also consider half-braids (single exchange) which requires $\approx$ six steps; a half-braid on two twist defects also implements a nontrivial logical operation.

2. Protocol 2: Braiding via half-twist

The previous full braiding protocol in Sec. III D 1 needs 12 steps, independent of the code distance. One can consider a different protocol to further reduce the number of steps, by utilizing the equivalence between (half-) braiding of two punctures and a half-twist around the two punctures. The protocol of the latter can be readily adapted from the protocol for implementing a Dehn twist on an annulus.

As has been discussed in Sec. II C [Figs. 2(e)–2(g)], a half-twist along the β loop enclosing two punctures is equivalent to (half-) braiding the two punctures, with additional half-self-twists around each puncture. That is, $\sqrt{\mathcal{D}_\beta} = \mathcal{B}_{1,2} \sqrt{\mathcal{D}_{\omega_1}} \sqrt{\mathcal{D}_{\omega_2}}$, where here $\sqrt{\mathcal{D}_\beta}$, $\sqrt{\mathcal{D}_{\omega_1}}$, $\sqrt{\mathcal{D}_{\omega_2}}$ represent the half-twists around β , ω_1 , and ω_2 .

The half-twists $\sqrt{\mathcal{D}_{\omega_i}}$ for $i = 1, 2$ can be implemented entirely by a local finite depth quantum circuit. In general, the effect of the half-twists $\sqrt{\mathcal{D}_{\omega_i}}$ around each puncture is simply to ensure that the punctures return to exactly their original configuration. For example, a hole or an anyon can have some nontrivial geometric shape, such that a half-twist changes its orientation, so that the half-twists $\sqrt{\mathcal{D}_{\omega_i}}$ are necessary to recover the exact original configuration. In the case of anyons or twist defects, where the spatial extent is $\mathcal{O}(1)$,

independent of the code distance, it is clear that this can be accomplished entirely by a local finite depth quantum circuit. In the case of holes, even though the linear size of the hole is $\mathcal{O}(d)$, we have seen how the lattice geometry can be changed everywhere in parallel through a local finite depth quantum circuit. Alternatively, we can just implement the half-twists $\sqrt{\mathcal{D}_{\omega_i}}$ using the same protocol for implementing $\sqrt{\mathcal{D}_\beta}$, except by twisting around each puncture individually.

In the case that the two punctures are not identical, as illustrated in Figs. 2(h)–2(j), we have to perform a full braid to return the system to the original configuration and perform a logical gate. In this case, we can perform a full Dehn twist around the β loop, which is equivalent to a full braid with additional self-Dehn-twists around the two punctures, i.e., $\mathcal{D}_\beta = \mathcal{B}_{1,2} \mathcal{D}_{\omega_1} \mathcal{D}_{\omega_2}$. For anyons, the full twist $\mathcal{D}_{\omega_i}$ simply gives an overall phase (the topological twist of the anyon), and can therefore be ignored; in the case of hole defects, the overall phase is trivial. Therefore, in both cases the self-Dehn-twists can be dropped. For twist defects, the branch cut of the twist defect winds around by 2π locally around the twist defect, and can be undone by a local finite depth circuit.

We show the protocol for braiding holes in a surface code in Fig. 14, following the defect-shearing protocol for the Dehn twist on an annulus in Fig. 9. We consider a parallelogram with edges (dashed lines) equivalent to the β loop enclosing the two defects, as shown in Fig. 14(a). In order to apply a half or full Dehn twist along the β loop, we can rotate the parallelogram by π and 2π , respectively, which can be further decomposed into a sequence of shearing operations as discussed in Fig. 9. The sequence of shears spatially changes the location of the defects and also shears the defects themselves.

After four steps of shearing, we reach the configuration in Fig. 14(e), with the parallelogram and the two defects being rotated by π (see the labeling of the vertices). By tracking the configuration of reference Wilson loop operators, we can see that this procedure exactly leads to a half-twist, equivalent to braiding and additional half-self-twists around the defects [see Fig. 2(b)], i.e., $\sqrt{\mathcal{D}}_\beta = \mathcal{B}_{1,2} \sqrt{\mathcal{D}_{\omega_1}} \sqrt{\mathcal{D}_{\omega_2}}$.

As we can see from Fig. 14(f), we can perform a gauge transformation by applying a membrane of stabilizers to deform the Wilson loop configuration to undo the self-twist, which is a manifestation of the trivial self-twist of the holes. The result is a configuration that only corresponds to a single braid $\mathcal{B}_{1,2}$ [see Fig. 2(c)].

To get a nontrivial logical gate in the $\mathbb{Z}_2$ surface code, one has to consider the braiding of two different types of hole defects, i.e., those with smooth and rough boundaries, respectively. Therefore, we need to continue the protocol with an additional four steps to achieve a full braid [see Figs. 14(g)–14(j)]. In Fig. 14(j), the two defects come back to the original configuration in Fig. 14(a), and hence achieve a full braid $\mathcal{B}_{1,2}^2$.

The protocol described above still uses a finite number of steps, so that $\mathcal{B}_{1,2} = \prod_{i=1}^4 \mathcal{P}_{\sigma_i} \mathcal{LU}_i$, i.e., four steps for braiding. The full braiding needs 8 steps, which is less than the 12-step protocol with cyclic moving of one defect shown in Fig. 13.

We note that by using non-nearest-neighbor two-qubit gates, one can implement the Dehn twist protocol described in Fig. 10, which needs only one step. Therefore, we can have $\mathcal{B}_{1,2} = \mathcal{P}_{\sigma} \mathcal{LU}$, at the expense of using a longer-range local quantum circuit.

E. Dehn twists on high-genus surfaces

Now we consider generating the whole mapping class group of a high-genus surface, i.e., $\text{MCG}(\Sigma_g)$. It is well known that $\text{MCG}(\Sigma_g)$ can be generated by $3g - 1$ Dehn twists [45], which are of three types and denoted by α , β , and γ , respectively, as illustrated in Fig. 15(a) (the arrows indicate the convention of the twist direction in this paper). As we can see in Fig. 15(b), such a high-genus surface can be realized in terms of a bilayer system with holes, where the boundaries of holes in different layers are glued together appropriately.

The protocols we have developed so far for performing Dehn twists on a cylinder and an annulus can easily be adapted to performing Dehn twists along any of the α , β , and γ loops of a high-genus surface. In the following, we provide the details for implementing these protocols.

1. Dehn twist along α and γ loops

We first show the protocol for implementing a Dehn twist along the α loop in Fig. 16. As in the situation of Dehn twists on a torus as shown in Fig. 8, we perform the local geometry deformation to get a “solenoid” region with the slanted plaquette structure as shown in Figs. 16(a)–16(c). Here, we choose the length of the solenoid such that each diagonal line winds around the solenoid once. In Fig. 16(d), we perform the shear through the long-range permutation of the sites, similar to the situation in Fig. 8. Note that the two ends of the solenoid are fixed and no sites at these ends are

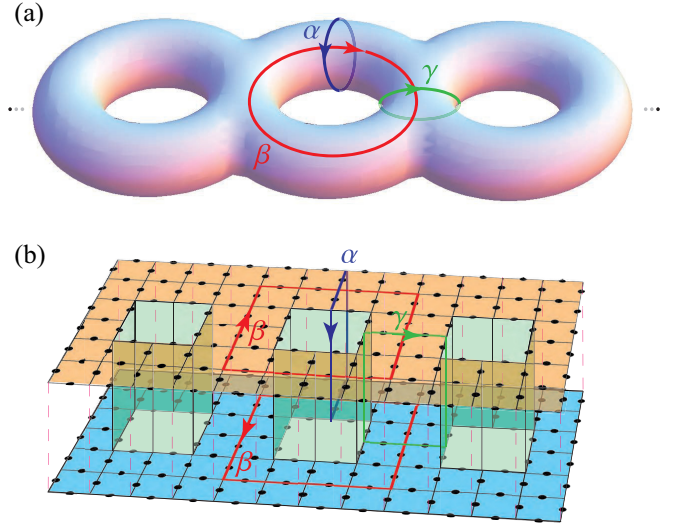


FIG. 15. (a) Three types of noncontractible loops on a high-genus surface: α , β , and γ . Dehn twists along these loops generate the whole mapping class group. (b) The equivalent loops in a bilayer topological system coupled via “wormholes.” The equivalent β loops on the upper and lower layers have opposite orientations.

permuted. Therefore, we get the following transformation:

$$\begin{aligned} \mathcal{P}_{\sigma} \mathcal{LU}_{\alpha} : \bar{X}_{\beta} &\mapsto \bar{X}_{\alpha+\beta}, \\ \mathcal{P}_{\sigma} \mathcal{LU}_{\alpha} : \bar{Z}_{\beta} &\mapsto \bar{Z}_{\alpha+\beta}, \end{aligned} \quad (38)$$

where only the loop $\bar{X}$ is illustrated in the figure. The transformation is exactly the Dehn twist $\mathcal{D}_{\alpha}$.

The Dehn twist along the γ loop, illustrated in Fig. 17, is almost identical. Here, the “solenoid” region containing the slanted plaquettes is located at the handle between the two pairs of punctures. We get the following transformation:

$$\begin{aligned} \mathcal{P}_{\sigma} \mathcal{LU}_{\gamma} : \bar{X}_{\gamma} &\mapsto \bar{X}_{\beta+\gamma}, \\ \mathcal{P}_{\sigma} \mathcal{LU}_{\gamma} : \bar{Z}_{\gamma} &\mapsto \bar{Z}_{\beta+\gamma}, \end{aligned} \quad (39)$$

which is exactly the Dehn twist $\mathcal{D}_{\gamma}$.

2. Dehn twist along β loops

As indicated by Fig. 15(b), one can perform the Dehn twist along the β loop in either of the two layers. As such, it becomes equivalent to performing a Dehn twist on an annulus with the inner boundary enclosed by the β loop. When both layers are viewed from the top as in Fig. 15(b), we see that the directionality of the Dehn twist depends on the layer, as shown.

Now, we can directly apply the protocol of the Dehn twist $\mathcal{D}_{\beta}$ on an annulus to the Dehn twist $\mathcal{D}_{\beta}$ on a high-genus surface. By applying the protocol discussed in Sec. III C 2, we can apply $\mathcal{D}_{\beta}$ by a single step $\mathcal{D}_{\beta} = \mathcal{P}_{\sigma} \mathcal{LU}_{\beta}$. Alternatively, we can apply the shearing protocol of Sec. III C 1. Assuming we choose the β loop located in the upper layer, we can just apply the nine (for square boundary defect) or eight (for parallelogram defect) composite steps of shearing punctures to twist the string in the upper layer as shown in Figs. 18(a)–18(c).

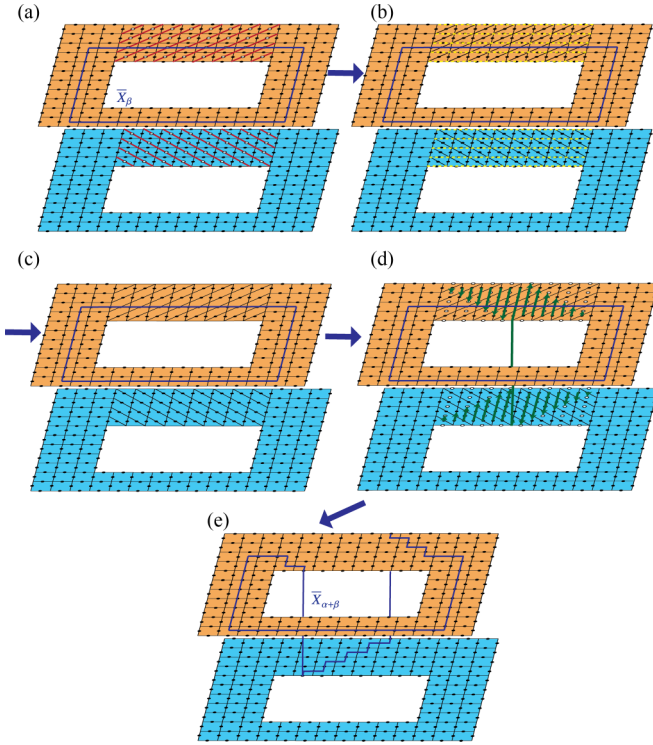


FIG. 16. Dehn twist along the α loop. (a) Adding qubits (white circles) and edges (red solid lines). (b) Removing qubits and edges (yellow dashed lines). (c) Obtain a lattice with a “gift-wrapping” (slanted) pattern in the solenoid region. (d) Shear permutation (green arrows) of the qubits in the solenoid region. (e) A Dehn twist is achieved and the lattice is mapped to the original geometry in (a).

Note that in the high-genus surface, the boundary of the inner hole of the top layer is “glued” to the inner hole of the bottom layer. In the case of passive topological quantum computation (TQC), during the Dehn twist protocol the Hamiltonian along this inner boundary should be turned off; alternatively, in the case of active QEC, the stabilizer measurements along the inner boundary that glue the two layers should be turned off. This is because during the Dehn twist protocol, the inner boundary of the one layer is twisting relative to the

inner boundary of the other layer, and therefore they cannot be glued together with local interactions until the Dehn twist protocol is completed.

This whole protocol gives the desired Dehn twist $\mathcal{D}_\beta$, which performs the map $\mathcal{D}_\beta : \bar{X}_\gamma \mapsto \bar{X}_{\beta+\gamma}$ in the illustrated case in Fig. 18(l). We can further reduce the number of composite steps by a variant of the protocol, shown in Fig. 19. Instead of doing the full twist on an annulus on one of the two layers, we can do half-twists on both layers simultaneously, but with opposite orientation. More concretely, we start with the same square defects on both layers, and perform the shearing of the defects in opposite directions on both layers, thus effectively performing a π ($-\pi$) rotation of the square defect in the upper (lower) layer. The half-twist is manifested by the fact the vertex A is permuted to the opposite (lower-right) corner at the end of the protocol. After gluing the two layers back, we achieve a Dehn twist along the β loop within total five defect-shearing steps, i.e., $\mathcal{D}_\beta = \prod_{i=1}^5 \mathcal{P}_{\sigma,i} \mathcal{LU}_i$. Note that the loop configuration of twisted Wilson loops ($\bar{X}$ or $\bar{Z}$) at the end of this protocol can be continuously deformed back to the loop configuration at the end of the previous protocol, i.e., equivalent to $\bar{X}_{\beta+\gamma}$ or $\bar{Z}_{\beta+\gamma}$.

F. Multiple Dehn twists in a single shot: Proof of Theorem 2

Here, we consider performing n Dehn twists around either the α , β , or γ loops. That is, we consider $\mathcal{D}_{n\omega} = \mathcal{D}_\omega^n$, for $\omega = \alpha_i$, β_i , or γ_i . We will show that we can perform $\mathcal{D}_{n\omega}$ through a quantum circuit $\mathcal{P}_\sigma \mathcal{LU}_{n\omega}$, where $\mathcal{LU}_{n\omega}$ is a local quantum circuit with finite depth independent of code distance d and system size.

In *Protocol 1* below, we will find that $\mathcal{LU}_{n\omega}$ has a depth that scales as $\mathcal{O}(\log n)$, but independent of code distance and system size. In *Protocol 2* below, $\mathcal{LU}_{n\omega}$ has a depth independent of n , but the range of two-qubit gates in $\mathcal{LU}_{n\omega}$ is $r = \mathcal{O}(n)$. While *Protocol 1* will be generalized to non-Abelian codes in Sec. IV, no such generalization exists for *Protocol 2*.

We note that for any topologically ordered phase of matter, $\mathcal{D}_\omega^k = 1$ for some finite k .² In the case of the $\mathbb{Z}_N$ toric code, $k = N$. Therefore, when considering n Dehn twists, we see that $n < k$, while the code distance d can be made arbitrarily large. Below, we will always assume we are in the limit $n < k \ll d$.

1. Protocol 1: Expanding the lattice

The idea of the first protocol is based on the observation that on an asymmetric torus elongated along one direction, as shown in Fig. 20, multiple Dehn twists can be applied in parallel along the same cycle. For example, in Fig. 20 the β cycle is twice the length of the α cycle. The protocol consisting of a finite depth local quantum circuit followed by long-range qubit permutations, as illustrated in Fig. 20, implements a double Dehn twist $\mathcal{D}_{2\alpha} = \mathcal{U}^2$ in “one shot” (through a constant depth circuit), leading to the transformation on the illustrated

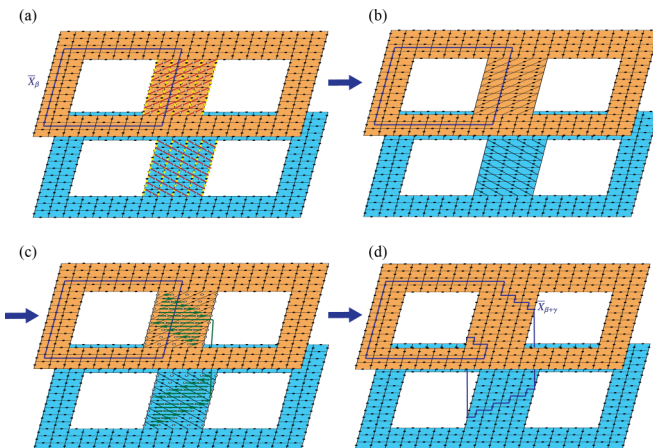
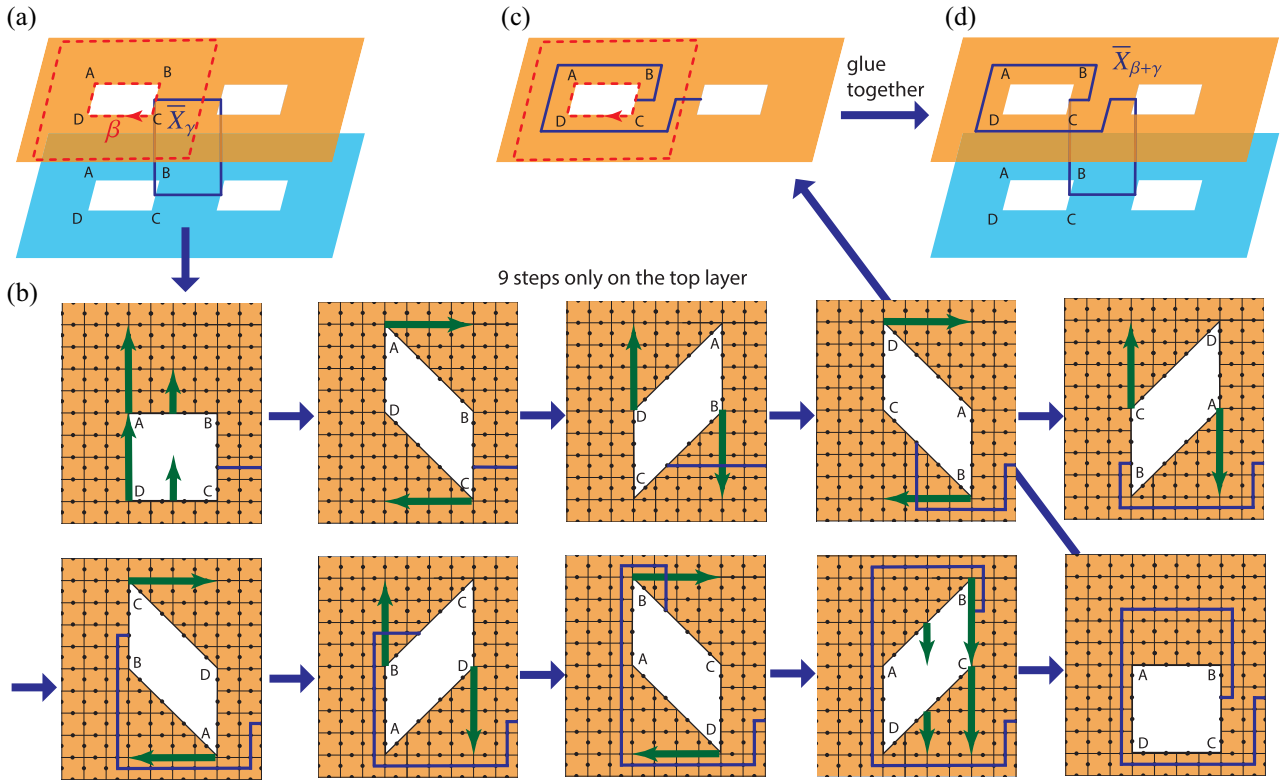


FIG. 17. Dehn twist along the γ loop.

²This follows from Vafa’s theorem [83], which states that the topological spins are always rational numbers.

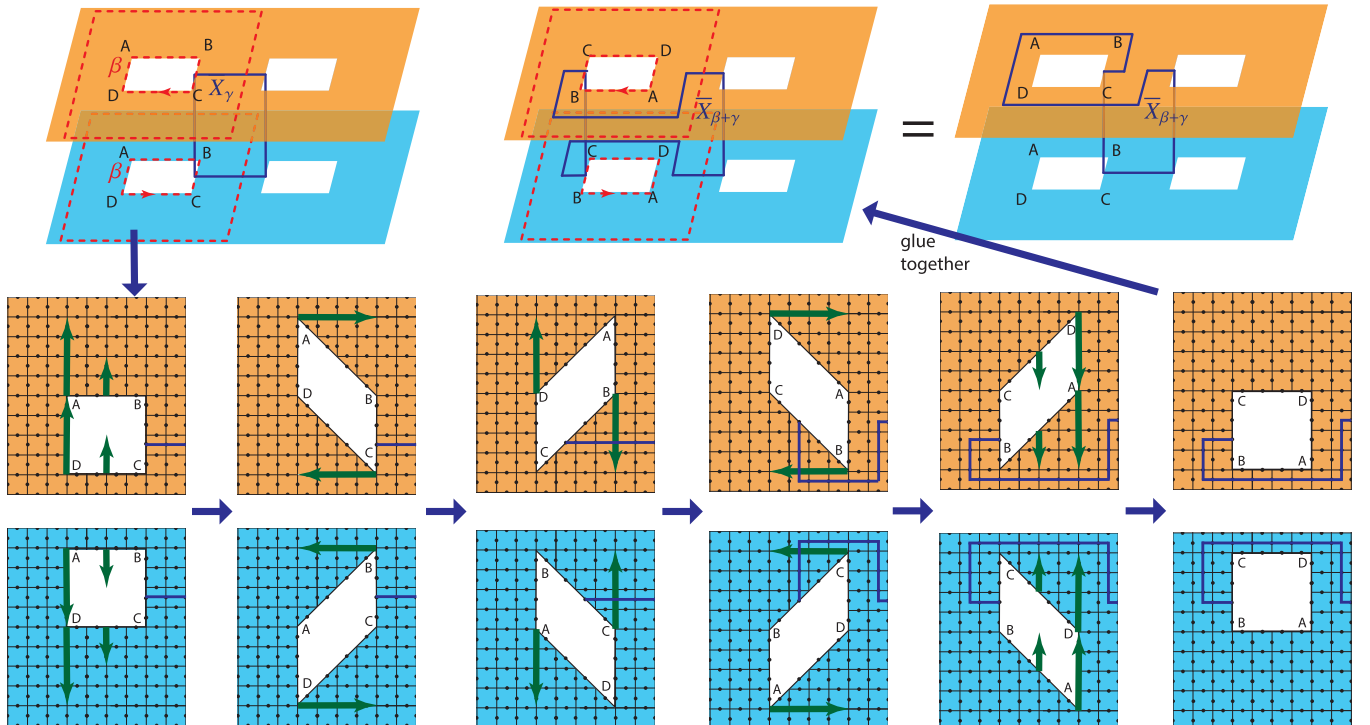
FIG. 18. Dehn twist along the β loop on a high-genus surface via nine steps of shearing on the top layers.

logical string operator

$$\mathcal{D}_{2\alpha} \bar{X}_\beta \mathcal{D}_{2\alpha}^\dagger = \bar{X}_{2\alpha+\beta}.$$

(40)

In general, for a fixed code geometry with the $\alpha(\beta)$ cycle n times the length of the $\beta(\alpha)$ cycle, one can implement the multi-Dehn twist $\mathcal{D}_{n\beta}(\mathcal{D}_{n\alpha})$ in a single shot. This is a

FIG. 19. Dehn twist along the β loop implemented by two half-twists of an annulus on both layers with five puncture-shearing steps in total.

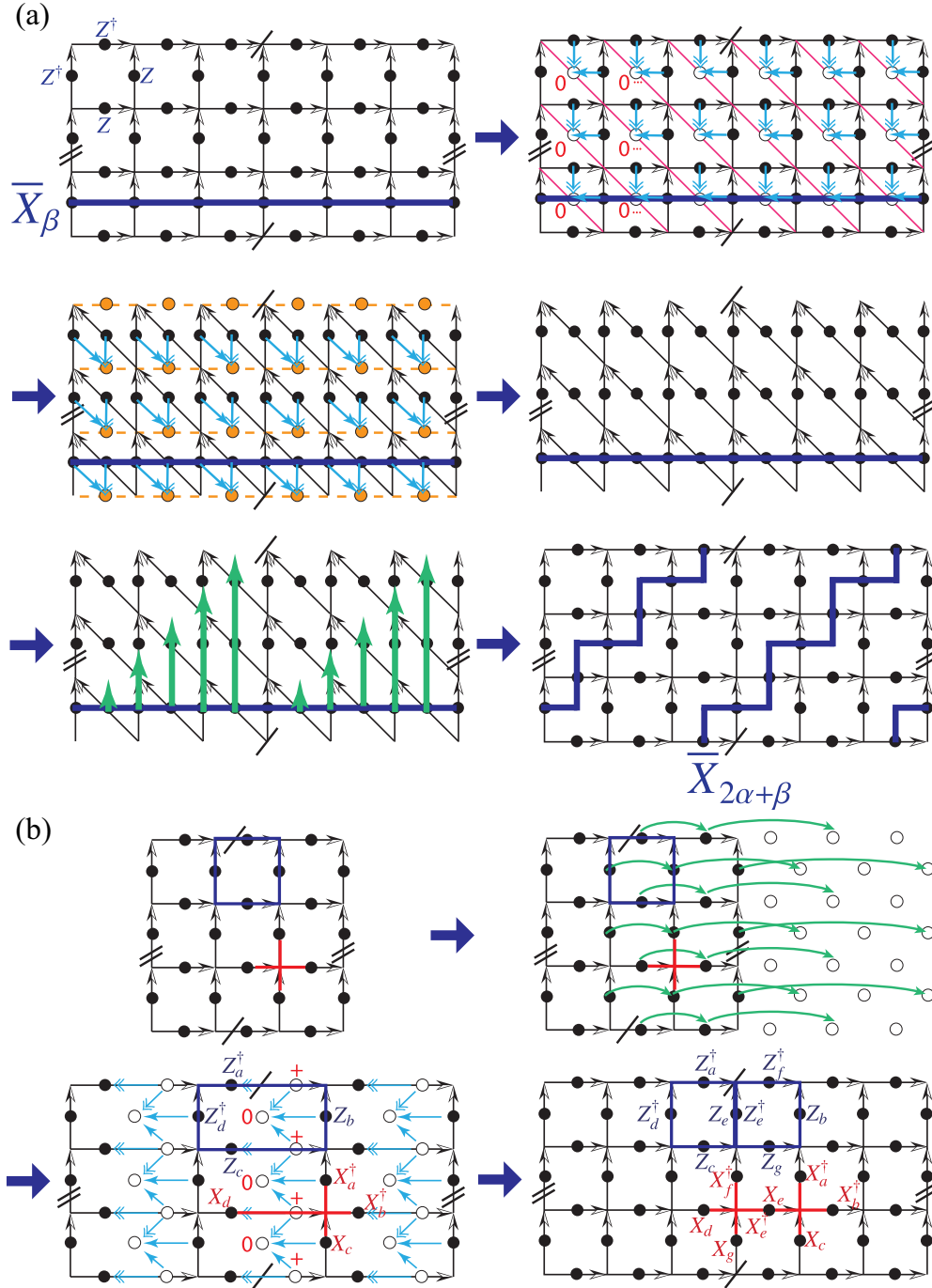


FIG. 20. Multiple Dehn twists implemented in a single shot on a $\mathbb{Z}_N$ toric code via changing the aspect ratio of the torus. The single and double arrows represent the CX and $CX^\dagger$ gates, respectively. The curved green arrows in (b) represent permutations.

remarkable result, which demonstrates that by increasing the system size (number of qubits) by n times and with fixed code distance d (determined by the shorter length of the torus), the time complexity of implementing a particular logical gate sequence can be decreased by n times, i.e., one can trade space for time. Nevertheless, the price to pay is that in such an asymmetric geometry, one can only implement the dual single Dehn twist $\mathcal{D}_\alpha(\mathcal{D}_\beta)$ in n shots.

In order to exploit the above observation, we consider the flexibility to adjust the aspect ratio of the torus during the computation, using entanglement renormalization. As shown

in Fig. 20(b), to be able to implement a double Dehn twist in one shot, we aim to increase the length of the torus along the β direction by a factor of 2. We consider ancilla qubits to the right side of the system. We next perform a qubit permutation $\mathcal{P}_{\sigma_1}$ to achieve an effective dilation of the system by increasing the horizontal size of each plaquette by a factor of 2. Now in order to also increase the number of qubits by a factor of 2, we add/entangle the ancilla qubits (initialized at $|0\rangle$ or $|+\rangle$) by the elementary moves composed of CX and $CX^\dagger$ gates. According to Eq. (31), the ancilla initialized at $|0\rangle$ is the eigenstate of $Z_e^\dagger$, and transformed by the two CX and

one $CX^\dagger$ as

$$Z_e^\dagger \mapsto Z_e^\dagger Z_f^\dagger Z_b Z_g, \quad (41)$$

which introduces a new plaquette stabilizer fixed to be +1. Meanwhile, the original plaquette stabilizer involving four qubits is transformed by the two $CX^\dagger$ as

$$Z_a^\dagger Z_b Z_c Z_d^\dagger \mapsto Z_a^\dagger Z_b Z_c Z_d^\dagger Z_f^\dagger Z_g, \quad (42)$$

according to Eqs. (29) and (31), which corresponds to a large stabilizer (fixed to be +1) involving two plaquettes and seven qubits. This large stabilizer can be decomposed into two stabilizers as $Z_a^\dagger Z_b Z_c Z_d^\dagger Z_f^\dagger Z_g = (Z_a^\dagger Z_c Z_d^\dagger Z_e)(Z_e^\dagger Z_f^\dagger Z_b Z_g)$, with the operator on e being canceled. This makes sure the other new plaquette operator $(Z_a^\dagger Z_c Z_d^\dagger Z_e)$ is automatically fixed at +1. Therefore, we see that the original plaquette stabilizer is split into two plaquette stabilizers. Similarly, we have the following transformation for the X stabilizers according to Eqs. (29) and (33), i.e.,

$$\begin{aligned} X_e^\dagger &\mapsto X_e^\dagger X_g X_d X_f^\dagger, \\ X_a^\dagger X_b^\dagger X_c X_d &\mapsto X_a^\dagger X_b^\dagger X_c X_d X_f^\dagger X_g, \end{aligned} \quad (43)$$

which effectively splits the original vertex stabilizers into two. This can be verified by the decomposition $X_a^\dagger X_b^\dagger X_c X_g X_d X_f^\dagger = (X_a^\dagger X_b^\dagger X_c X_e)(X_e^\dagger X_g X_d X_f^\dagger)$.

The above procedure increases the number of plaquette and vertices by a factor of 2. The whole elongating process is performed in one shot by a combination of a qubit permutation and a local finite depth quantum circuit. In general, in order to increase the length of the torus in a particular direction by a factor of n , one needs $\log_2 n$ shots of the above transformations, which is a well-known fact for state preparation with MERA [54]. This protocol provides an exponential improvement over implementing the Dehn twist sequentially n times.

It is straightforward to see that the above result can be extended to any of the α , β , and γ loops of a high-genus surface, or to braids. This proves the first statement [Eq. (3)] of Theorem 2 in the context of the $\mathbb{Z}_N$ toric code.

2. Protocol 2: Increasing the interaction range

In the previous scheme, we fixed the interaction range (nearest neighbor) of the local unitaries ($\mathcal{LU}$) and applied multiple Dehn twists in parallel by changing the aspect ratio of the torus. Here, we demonstrate a second protocol, where we apply a single step of $\mathcal{PLU}$. By increasing the range of $\mathcal{LU}$ to be $\mathcal{O}(n)$, this allows us to apply n Dehn twists, $\mathcal{D}_{no}$ in a single shot. This protocol shows how the interaction range can be turned into computational power.

We consider a square lattice in Fig. 21(a). Then, we add the NNN diagonal edge $(1, -2)$ and the triangular stabilizer plaquette (shadow) by applying a CX and two $CX^\dagger$ gates conditioned by qubits a , b , and c , and targeting qubit d . d is initialized in state $|0\rangle$. We apply this to other triangular stabilizer plaquettes in parallel as well, as illustrated in Figs. 21(b) and 21(c). According to the last two identities in Eq. (29), the entangling gates on the shadowed plaquette induce the following transformation:

$$Z_d \mapsto Z_a^\dagger Z_b Z_c Z_d. \quad (44)$$

Since $|0\rangle$ is the +1 eigenstate of Z_d , we fix the triangular stabilizer $Z_a^\dagger Z_b Z_c Z_d$ at +1. A similar result holds for all the other added plaquettes.

We now remove all the horizontal edges $(1,0)$ with the CX gates shown in Fig. 21(d). According to Eq. (31), the entangling gates on the shadowed plaquette induce the following transformation:

$$Z_a^\dagger Z_b Z_c Z_d \mapsto Z_a^\dagger, \quad (45)$$

which disentangles the qubit a (yellow circle). We then reach the double slanted lattice with NNN diagonal edges in Fig. 21(e), which is in contrast to the slanted lattice with NN diagonal edges in Fig. 20(a).

Now, we apply a qubit permutation P_σ shown by the green arrows in Fig. 21(e). This maps the state back to the original lattice in Fig. 20(a) with a double Dehn twist, leading to the following transformation on the illustrated logical string operator $\mathcal{D}_{2\alpha} \bar{X}_\beta \mathcal{D}_{2\alpha}^\dagger = \bar{X}_{2\alpha+\beta}$.

We see that the maximal range of the finite depth local quantum circuit $\mathcal{LU}$ has an increased range relative to the case of a single Dehn twist, as now there are two-qubit gates involving a qubit and its next-nearest diagonal neighbor. One can straightforwardly generalize the above protocol to apply n Dehn twists in a single shot, with the maximal range $r \sim \mathcal{O}(n)$ in the $\mathcal{LU}$ circuit.

It is straightforward also to adapt this protocol to the case of Dehn twists about any of the α , β , γ loops of a high-genus surface, or to braids. This proves the second statement [Eq. (4)] of Theorem 2 in the context of $\mathbb{Z}_N$ toric code.

IV. THEORY FOR GENERAL TOPOLOGICAL CODES

In this section we generalize the discussions presented in Sec. III to the case of arbitrary nonchiral topologically ordered states. In particular, this analysis applies to both general Abelian and non-Abelian topological states. When applied to certain non-Abelian codes, such as the Fibonacci surface code [13], our results imply that a universal, fault-tolerant set of logical gates can be performed with constant time overhead.

The class of topologically ordered states that we consider will be referred to as Turaev-Viro codes, which are based on Turaev-Viro-Barrett-Westbury (TVBW) topological quantum field theories [7,8,17]. Reference [44] contains a recent review aimed at physicists, and contains the conventions that we follow. These states are associated with exactly solvable models, such as the Levin-Wen model [10], which can realize all topologically ordered states that admit gapped boundaries. Topologically ordered states that can be obtained in this way are referred to as “nonchiral” topological states. Chiral topological states, such as fractional quantum Hall (FQH) states, have topologically protected gapless edge modes and cannot be obtained from such a construction, thus, they are not included in our analysis.

References [11,13] discussed utilizing these TVBW TQFTs as topological QECCs for quantum computation. As such, the code space corresponds to the ground-state subspace of the exactly solvable Levin-Wen Hamiltonians (and their generalizations).

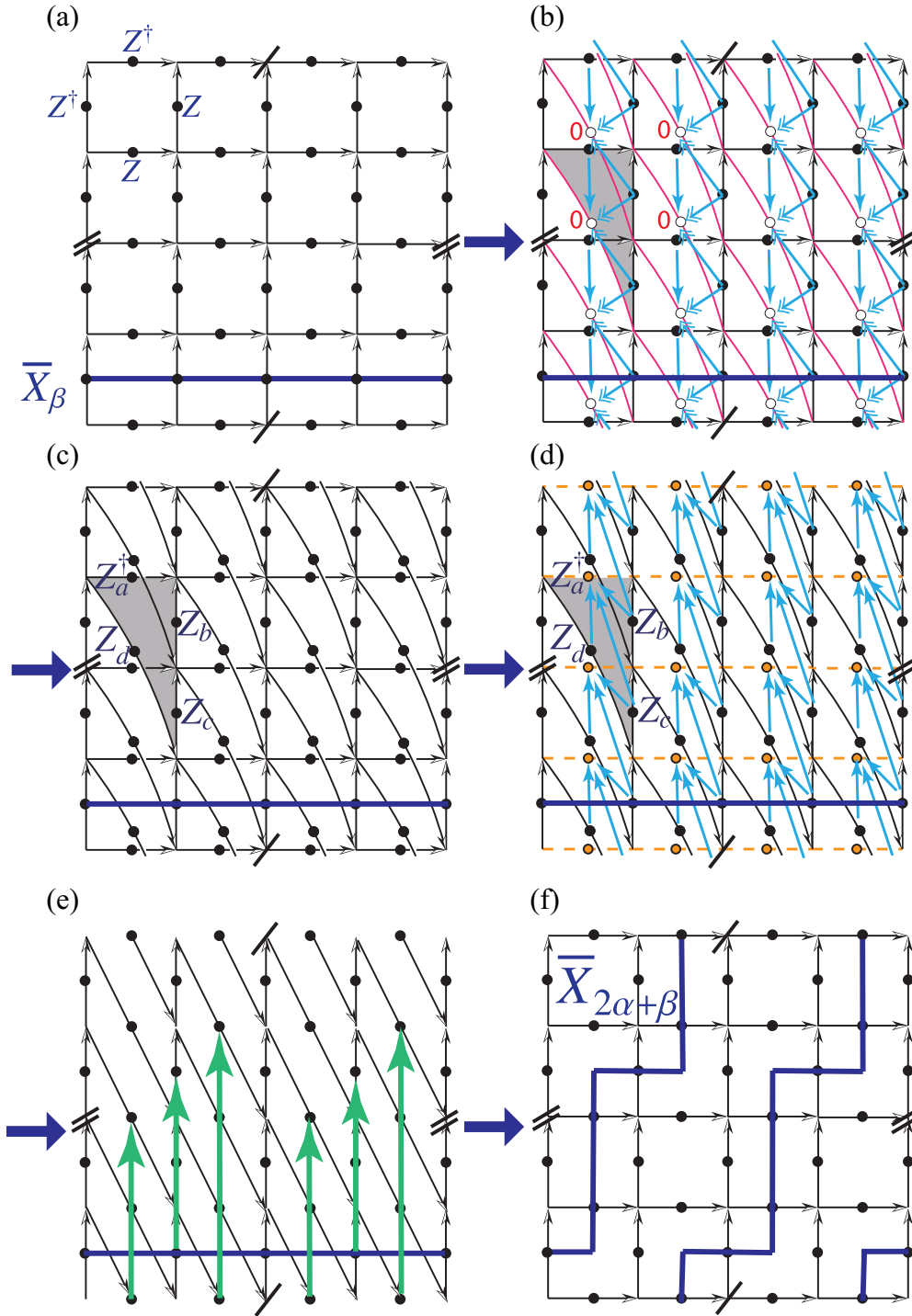


FIG. 21. Implementing multiple Dehn twists in a single shot via increasing the range of interaction on a $\mathbb{Z}_N$ toric code. In (b)–(d), the diagonal edges overpass the horizontal edges, i.e., equivalent to a bridge structure.

We note that some of the results of this section, in particular those of Secs. IV B and IV D 2, are also summarized in Ref. [43].

A. Turaev-Viro codes

The TVBW TQFT associates to a closed surface Σ a finite-dimensional Hilbert space $\mathcal{H}_\Sigma$. In the context of QECC, this space can be viewed as the code subspace of

a Turaev-Viro code. We use Λ to denote a triangulation of Σ , together with a local ordering of the vertices of the triangulation. This local ordering is referred to as a branching structure, and implies that each edge of Λ is directed. We further use $\hat{\Lambda}$ to denote the dual cellulation associated with Λ , which also defines a directed graph. For concreteness, we will first consider cases where Λ and $\hat{\Lambda}$ define triangular and honeycomb lattices, respectively.

The Turaev-Viro codes (alternatively, the TVBW TQFTs) take as input a unitary fusion category $\mathcal{C}$. The data of $\mathcal{C}$ are specified by the following. $\mathcal{C}$ contains a set of N “simple objects” $\{0, 1, 2, \dots, N-1\}$. Any triplet (a, b, c) of simple objects define a vector space V_{ab}^c . The dimension of this vector space defines the fusion rules N_{ab}^c :

$$N_{ab}^c = \dim V_{ab}^c,$$

where N_{ab}^c is a non-negative integer. The fusion rules N_{ab}^c can be summarized through the formal relation

$$a \times b = \sum_c N_{ab}^c c. \quad (46)$$

Given a vector space $V_d^{abc} \equiv \bigoplus_e V_{ab}^e \otimes V_{ec}^d \simeq \bigoplus_f V_{bc}^f \otimes V_{af}^d$, $\mathcal{C}$ defines a unitary map F_d^{abc} :

$$F_d^{abc} : \bigoplus_e V_{ab}^e \otimes V_{ec}^d \mapsto \bigoplus_f V_{bc}^f \otimes V_{af}^d. \quad (47)$$

In components, the F symbols are written as $[F_d^{abc}]_{(e,\alpha,\beta),(f,\mu,\nu)}$, where $\alpha = 0, \dots, N_{ab}^e - 1$, $\beta = 0, \dots, N_{ec}^d - 1$, $\mu = 0, \dots, N_{bc}^f - 1$, and $\nu = 0, \dots, N_{af}^d - 1$. When all $N_{ab}^c = 0, 1$, the F symbols can be written in components as F_{def}^{abc} . The F symbols satisfy a set of nontrivial self-consistency equations known as the pentagon equations.

In a unitary fusion category, the topological charge conjugate $\bar{a}$ is determined by the unique label $\bar{a}$ that satisfies $N_{a\bar{a}}^0 = 1$. Furthermore, the identity object 0 fuses trivially with all other objects: $N_{a0}^b = N_{0a}^b = \delta_{ab}$.

Below, for simplicity we will restrict to cases where $N_{ab}^c = 0, 1$, although this restriction is not necessary for the validity of our results. The TVBW TQFT provides an explicit wave function as follows. Each edge of Λ (equivalently, of $\hat{\Lambda}$) is associated with a local N -dimensional Hilbert space (qudit), where the states are labeled by the simple objects $\{0, 1, 2, \dots, N-1\}$. The wave-function amplitude for a particular state on Λ can be explicitly determined by evaluating a discrete path integral (state sum) over a triangulated 3-manifold M , whose boundary $\partial M = \Sigma$. The triangulation (and corresponding branching structure) of M restricts to Λ on ∂M . We will not review the state sum here; we refer the reader to Refs. [7–9, 11, 17, 44] for various presentations of the state sum.

An important property of the wave functions is that, for all states with nonzero amplitude, vertices of the dual graph satisfy the fusion rules, and as a result

$$\Psi \left(\begin{array}{c} b \\ \swarrow \quad \searrow \\ a \\ \swarrow \quad \searrow \\ c \end{array} \right) = 0 \quad \text{if} \quad N_{ab}^c = 0. \quad (48)$$

If the qudit on a particular edge is in the state $|a\rangle$, we say that there is a string of type a passing through that edge. The wave function can then be viewed as a superposition of closed string-net configurations consistent with these string fusion rules [10].

The original Turaev-Viro construction, together with the corresponding Levin-Wen Hamiltonian, assume a certain tetrahedral symmetry that imposes many relations among the

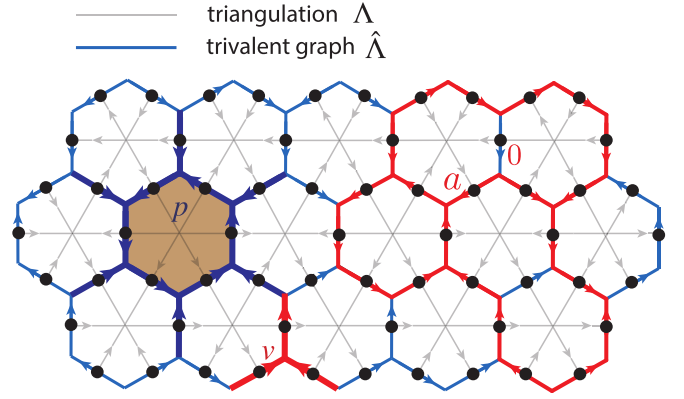


FIG. 22. Definition of the Levin-Wen Hamiltonian and Turaev-Viro codes on a triangulated manifold (light gray lines indicate the triangulation Λ) and the corresponding trivalent graph $\hat{\Lambda}$ (blue lines). The arrows on the lines specify the branching structure. The thin red lines represent the string nets. The thick blue and red lines illustrate the plaquette and vertex projectors, respectively.

F symbols; as such, not any unitary fusion category can be taken as input. The extension due to Barrett and Westbury relaxes the tetrahedral symmetry, at the cost of needing more careful consideration of the branching structure; the Barrett-Westbury generalization therefore applies to arbitrary unitary fusion categories.³ Our protocols in the following sections do not assume any tetrahedral symmetry, and thus apply to the full Barrett-Westbury generalization.

In the case where the F symbols satisfy additional relations due to tetrahedral symmetry (which we will not summarize here), the wave functions of the TVBW TQFT are exact ground states of a commuting projector Hamiltonian known as the Levin-Wen Hamiltonian [10].⁴ The Hamiltonian is

$$H_{\hat{\Lambda}} = - \sum_v Q_v - \sum_p B_p, \quad (49)$$

where v and p label the vertices and plaquettes of $\hat{\Lambda}$. The three-body vertex projection operator Q_v depends only on the three edges incident to the vertex v , and is defined by

$$Q_v \left| \begin{array}{c} b \\ \swarrow \quad \searrow \\ a \\ \swarrow \quad \searrow \\ c \end{array} \right\rangle = N_{ab}^c \left| \begin{array}{c} b \\ \swarrow \quad \searrow \\ a \\ \swarrow \quad \searrow \\ c \end{array} \right\rangle. \quad (50)$$

Recall for simplicity we have restricted to the case $N_{ab}^c = 0, 1$.

As an example, the Fibonacci Levin-Wen model has $N = 2$ and therefore each edge of the trivalent graph contains two types of strings, corresponding to the two states $(|0\rangle, |1\rangle)$ of

³Actually, the Barrett-Westbury construction applies to general “spherical” fusion categories, but from the perspective of topological quantum states of matter, the stricter condition of unitarity is required of the fusion category.

⁴In the case of Abelian topological states, Ref. [84] provided a generalization of the Levin-Wen models to relax the assumption of tetrahedral symmetry.

a qubit. The fusion rules are specified as

$$N_{ab}^c = \begin{cases} 1 & \text{if } abc = 000, 011, 101, 110, 111, \\ 0 & \text{otherwise.} \end{cases} \quad (51)$$

The corresponding string-net configuration satisfying the fusion rules can be illustrated on the right side of Fig. 22, where the edges with the red string correspond to an occupied site ($|1\rangle$) and the edges without a string correspond to an unoccupied site ($|0\rangle$) (in the Fibonacci case the arrows on the graph and string net can be ignored).

On a honeycomb lattice, as shown in Fig. 22, the plaquette operator B_p is a 12-body operator that depends on the six qudits on the hexagonal plaquette and also on the qudits on the six legs connecting to the hexagon. The operator can be written as $B_p = \sum_s d_s B_p^s / D^2$, where $d_s = |F_{s00}^{ss}|^{-1}$ is the quantum dimension of the string label s , $D = \sum_s \sqrt{d_s^2}$ is the total quantum dimension, and the operator B_p^s is defined via

$$B_p^s = \left| \begin{array}{c} b \\ \hline a \rightarrow \text{hexagon} \leftarrow d \\ \hline f \end{array} \right\rangle = \sum_{g'h'i'j'k'l'} B_{p,ghijkl}^{s,g'h'i'j'k'l'}(abcdef) \left| \begin{array}{c} b \\ \hline a \rightarrow \text{hexagon} \leftarrow d \\ \hline f \end{array} \right\rangle, \quad (52)$$

where the tensor coefficients are

$$B_{p,ghijkl}^{s,g'h'i'j'k'l'}(abcdef) = F_{\bar{s}h'g'}^{b\bar{g}h} F_{\bar{s}h'i'}^{c\bar{h}i} F_{\bar{s}j'f}^{d\bar{i}j} F_{\bar{s}k'e}^{e\bar{j}k} F_{\bar{s}l'd}^{f\bar{k}l} F_{\bar{s}g'a}^{a\bar{l}g}. \quad (53)$$

The plaquette term B_p flips the string-net configurations, such as the one shown in Fig. 22, to other configurations constrained by the fusion rules. The ground state of the Hamiltonian (49) in a particular topological sector is an equal-probability superposition of all the possible string-net configurations connected by the local action of the plaquette operators.

Just as in the case of the $\mathbb{Z}_2$ surface code (toric code), the Turaev-Viro codes can be implemented through an active error correction approach by repeated measurements of the commuting vertex and plaquette operators. Reference [13] presents details of the quantum circuits that can be used to measure these operators in the context of the Fibonacci surface code. Importantly, the quantum circuits for measuring these operators need only contain single- and two-qubit operations. Ongoing progress has been made on syndrome extraction, decoding, and error correction [55–58]. In particular, the decoder for a phenomenological Fibonacci code has been simulated numerically in Ref. [57].

The basic building block of the whole scheme is the F operation F_{def}^{abc} . The F operation can be viewed as a controlled-

F operation

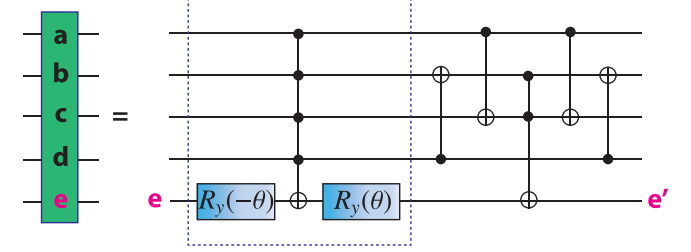


FIG. 23. The building block of the quantum computation scheme: the F operation quantum circuit for the Fibonacci surface code.

unitary operation, where the external a, b, c, d legs are the control qudits that determine the resulting unitary F_d^{abc} , whose matrix elements are $[F_d^{abc}]_{ef}$.

In the case of the Fibonacci surface code, the only nontrivial F matrix is

$$F_1^{111} = \begin{pmatrix} \phi^{-1} & \phi^{-\frac{1}{2}} \\ \phi^{-\frac{1}{2}} & -\phi^{-1} \end{pmatrix}, \quad (54)$$

where $\phi = \frac{\sqrt{5}+1}{2}$ is the golden ratio. All other F symbols are either 1 or 0, depending on whether they are consistent with the fusion rules.

A quantum circuit implementing the F operations in the Fibonacci surface code was presented in Ref. [13] and is shown in Fig. 23. The circuit inside the blue dashed box, consisting of a five-qubit Toffoli gate sandwiched by two single-qubit rotations, implements the F matrix in Eq. (54). Here, $R_y(\pm\theta) = e^{\pm i\theta\sigma_y/2}$ are single-qubit rotations about the y axis with angle $\theta = \tan^{-1}(\phi^{-\frac{1}{2}})$. All the other maps are taken care of by the rest of the quantum circuit.

B. Local geometry deformation

The key property of the wave functions of the TVBW TQFT that forms the basis of our approach is that wave functions associated to different graphs can be related to each other via a series of local moves. These moves are known as Pachner moves, and correspond to retriangulations of the manifold. In the (2+1)-dimensional [(2+1)D] path-integral state sum, the retriangulations of the 3-manifold can be performed by 2-3 and 1-4 Pachner moves. The 2-3 Pachner move replaces 2 tetrahedra by 3 and vice versa, while the 1-4 Pachner move replaces 1 tetrahedra by 4 and vice versa. At the 2D surface $\Sigma = \partial M$ of the 3-manifold, these moves restrict to 2-2 and 1-3 Pachner moves for retriangulation of a surface, illustrated in Figs. 24 and 25.

The path-integral state sum of the TVBW TQFT that determines the wave function thus also determines how to relate wave functions on different graphs that are related by these local Pachner moves. For example, we have the relations

$$\Psi_{\hat{\Lambda}'} \left(\begin{array}{c} b \quad c \\ \diagdown \quad \diagup \\ e \\ \diagup \quad \diagdown \\ a \quad d \end{array} \right) = \sum_f F_{def}^{abc} \Psi_{\hat{\Lambda}} \left(\begin{array}{c} b \quad c \\ \diagdown \quad \diagup \\ f \\ \diagup \quad \diagdown \\ a \quad d \end{array} \right), \quad (55)$$

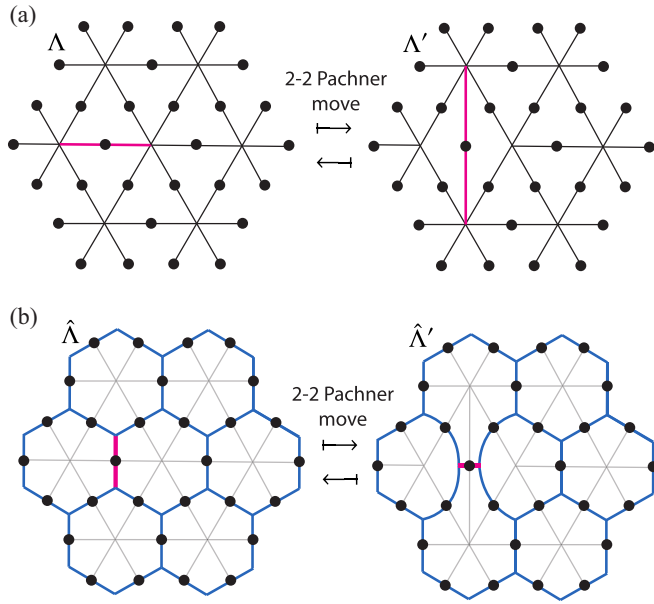


FIG. 24. Definition of the 2-2 Pachner move (F move) on the triangulation grid and the corresponding trivalent graph. The pink edges represent the edges being switched during the moves.

$$\Psi_{\hat{\Lambda}'} \left(\begin{array}{c} b \quad c \\ \diagdown \quad \diagup \\ e \\ \diagup \quad \diagdown \\ a \quad d \end{array} \right) = \sum_f G_{def}^{abc} \Psi_{\hat{\Lambda}} \left(\begin{array}{c} b \quad c \\ \diagdown \quad \diagup \\ f \\ \diagup \quad \diagdown \\ a \quad d \end{array} \right), \quad (56)$$

$$\Psi_{\hat{\Lambda}'} \left(\begin{array}{c} b \quad d \quad c \\ \diagdown \quad \diagup \quad \diagdown \\ e \quad f \\ \diagup \quad \diagdown \\ a \end{array} \right) = [F_{fce}^{abd}]^* \sqrt{\frac{d_d d_f}{d_c}} \Psi_{\hat{\Lambda}} \left(\begin{array}{c} b \quad c \\ \diagdown \quad \diagup \\ f \\ \diagup \quad \diagdown \\ a \end{array} \right). \quad (57)$$

We have defined

$$G_{def}^{abc} = F_{fac}^{bed} \sqrt{\frac{d_e d_f}{d_a d_c}}, \quad (58)$$

where G_d^{abc} is a unitary matrix:

$$G_d^{abc} [G_d^{abc}]^\dagger = \sum_f G_{def}^{abc} (G_{def}^{abc})^* = \delta_{ee'}. \quad (59)$$

These equations are to be interpreted as follows. We consider two different graphs, specified by $\hat{\Lambda}$ and $\hat{\Lambda}'$, whose duals Λ and Λ' can be viewed as two different triangulations of the same surface. $\hat{\Lambda}$ and $\hat{\Lambda}'$ only differ in the local patch that is illustrated. The equations show how to relate the wave-function amplitudes for the states associated with the different graphs.

We have chosen to illustrate the wave functions using the dual cellulations $\hat{\Lambda}$, but of course one could also use the triangulation Λ . Other similar relations exist as well; we do not list all of them here, but they can be easily derived from

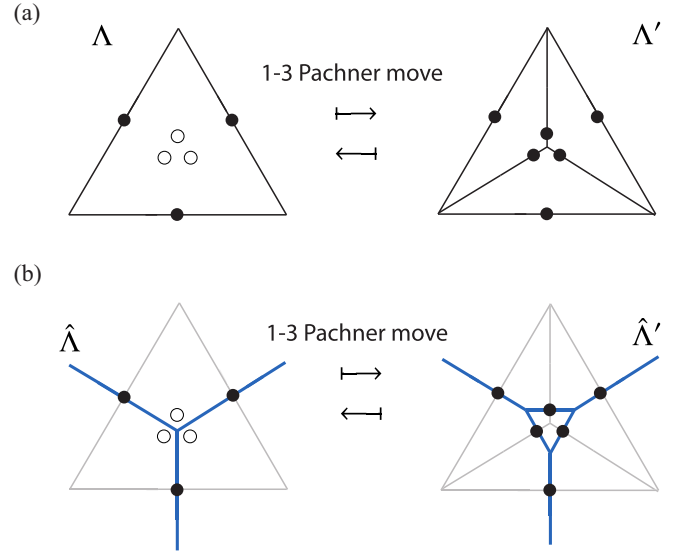


FIG. 25. Definition of the 1-3 Pachner move on the triangulation grid and the corresponding trivalent graph.

the path-integral state sum of the TVBW TQFT by considering the 2-2 and 1-3 Pachner moves with various choices of branching structures.

Since the external legs of the above diagrams are all fixed, F_d^{abc} and G_d^{abc} can be viewed as controlled-unitary gates, which effectively change the lattice geometry inside the plaquette defined by the edges a, b, c, d of the original triangulation Λ (see Fig. 24). This implies that these local moves can be performed in parallel over extensive regions of the lattice, a property which we will exploit in subsequent sections.

The 1-3 Pachner move, shown in Fig. 25 and Eq. (57), adds additional edges and thus additional qudits to the lattice. We can obtain the state on the new lattice from a state on the old lattice by appropriately initializing new qudits and applying a local unitary circuit shown in Fig. 26. We first consider three extra qudits, each initialized to the $|0\rangle$ state. Next, we consider applying a CX operation, which takes $|b\rangle|0\rangle \mapsto |b\rangle|b\rangle$, to the qudits shown in Fig. 26(a). At the same time, we apply S :

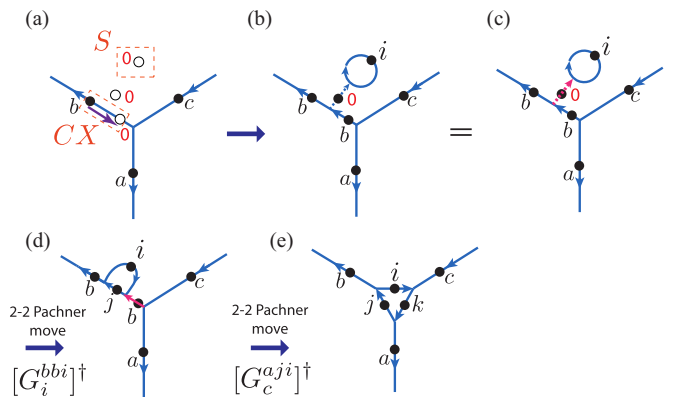


FIG. 26. Implementing the 1-3 Pachner move with a unitary circuit by attaching/detaching a tadpole diagram in the center of an arbitrary plaquette, followed/preceded by two 2-2 Pachner moves.

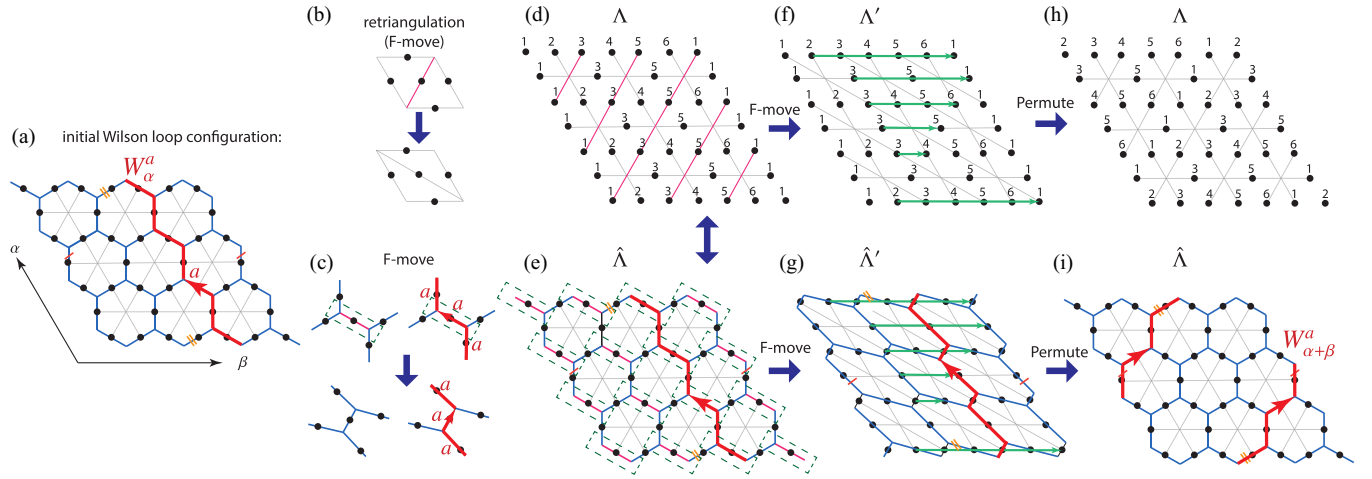


FIG. 27. Dehn twist on a torus for the Turev-Viro code. The upper rows [(b), (d), (f), (g)] show the protocol on the triangulation (light gray), while the lower rows [(c), (e), (g), (i)] illustrate the protocol on the dual trivalent graph (blue). The pink lines indicate the edges to be switched during the F moves. The green dashed box specifies the four external legs of the F moves on the trivalent graph. The green arrows represent qubit permutations, with the column numbers specified before and after the permutation.

$|0\rangle \mapsto \sum_i \frac{d_i}{D} |i\rangle$ to the topmost qudit, which effectively builds a “tadpole diagram” connected to the original graph through the edge with the remaining ancilla in the $|0\rangle$ state, as shown in Fig. 4(b). Note that the original edge labeled by b is split into two edges with the same label b . The equivalence to the tadpole diagram has implicitly used the concept of a smooth string net (see Refs. [10,11] for details). Next, we apply a sequence of two 2-2 Pachner moves by applying the unitary operations $[G_i^{bbi}]^\dagger$ and $[G_c^{aji}]^\dagger$, as shown in Fig. 26(e).

Therefore, we see that the 2-2 Pachner moves and the 1-3 Pachner moves, which locally change the lattice geometry, can be effectively implemented as unitary operations on the many-qudit quantum state. In what follows, we will demonstrate how to use these moves in parallel across large regions of the lattice, in order to build up nontrivial larger-scale geometry deformations through constant depth circuits. We will then subsequently use these large-scale geometry deformations for braiding and Dehn twists.

For simplicity, from now on we will call all the 2-2 Pachner moves as F moves, with the implication that depending on the branching structures they can correspond to different unitaries such as those in Eqs. (55) and (56), and others. All these unitaries will be proportional to the F symbols with additional normalization factor such as the case in Eqs. (56) and (58).

C. Dehn twist on a torus and a cylinder

Let us now consider the case where $\Sigma = T^2$ is a torus. Thus, we consider the case where the honeycomb lattice $\hat{\Lambda}$ and the triangulation Λ have periodic boundary conditions, as shown in Fig. 27(a). The initial lattice and triangulation correspond to a microscopic Hilbert space $\mathcal{H}_\Lambda$. To help demonstrate the resulting Dehn twist, we follow an initial Wilson loop W_α^a , i.e., a noncontractible string with string type a passing along the α cycle.

The elementary move in this protocol is the retriangulation obtained by the F move (2-2 Pachner move). As illustrated

in Fig. 27(b), this corresponds to flipping the edge (pink) between two neighboring triangles, and can be achieved by the controlled-unitary F operation discussed in the previous section. Figure 27(c) illustrates the effect on the dual lattice and on the string operator W_α^a .

We mark all the varying edges of the triangulation by pink lines in Fig. 27(d), with the corresponding moves in the dual graph indicated by the dashed box and pink edges in Fig. 27(e). The new triangulation Λ' and honeycomb lattice $\hat{\Lambda}'$ are shown in Figs. 27(f) and 27(g), corresponding to a different microscopic Hilbert space $\mathcal{H}_{\Lambda'}$. Since we can perform all of these moves by controlled-unitary operations in parallel, this retriangulation can be done via a local quantum circuit with $\mathcal{O}(1)$ depth.

To recover the original triangulation Λ and trivalent graph $\hat{\Lambda}$, we now perform a collective permutation of the qudits $\mathcal{P}_\sigma$, indicated by the arrows in Figs. 27(f) and 27(g). The qudits in each row are cyclically permuted by a spacing indicated by the length of the arrow in that row.

After the permutation, the column labels of the qudits are changed into the configuration shown in Fig. 27(j), which corresponds to the honeycomb lattice configuration in Fig. 27(i) where the Wilson loop now also winds around the β cycle. In the end of the protocol, we thus come back to the original triangulation Λ and honeycomb lattice $\hat{\Lambda}$ with permuted sites, and hence the same microscopic Hilbert space $\mathcal{H}_\Lambda$. This is exactly a self-diffeomorphism and corresponds to a Dehn twist $\mathcal{D}_\beta : W_\alpha^a \mapsto W_{\alpha+\beta}^a$. We have thus shown that the Dehn twist $\mathcal{D}_\beta$ can be implemented by a constant depth local unitary quantum circuit, followed by a permutation on the qudits.

We note that in the illustration depicted in Fig. 27, as well as all the illustrations for the following protocols, we have not drawn the branching structure in order to keep the illustrations simple. It can be verified that all the protocols can be adapted to the case where the branching structure is taken into account, so that the branching structure after the protocol is the same as before the protocol.

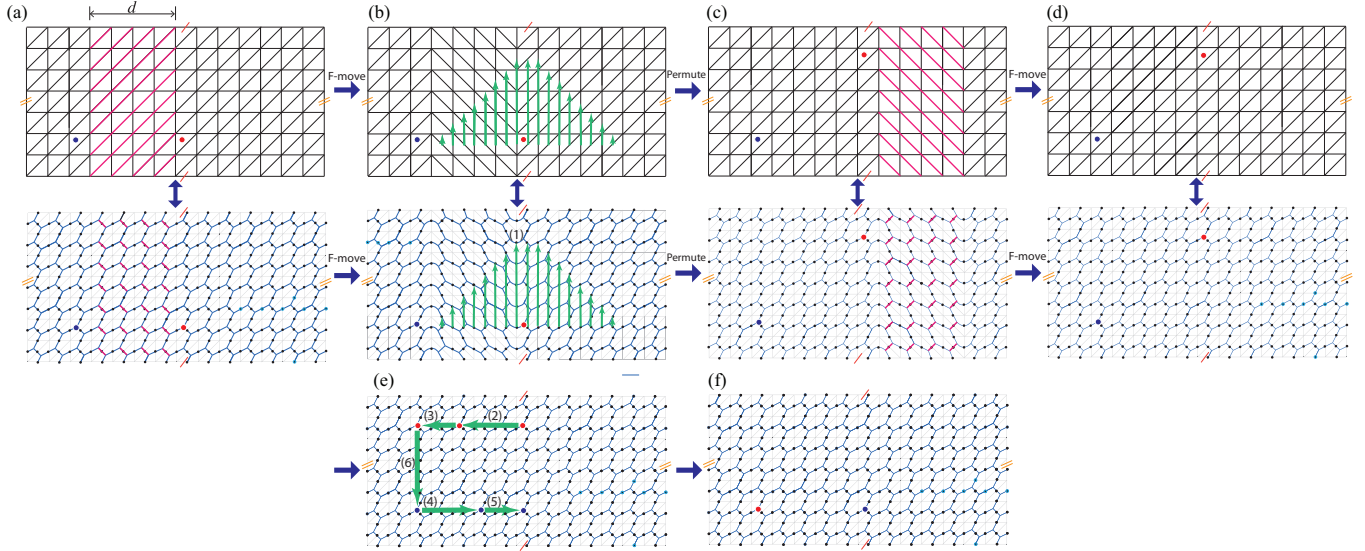


FIG. 28. Braiding of two punctures, e.g., non-Abelian anyons (blue and red circles) in Turaev-Viro codes, with constant depth circuit on a space with periodic boundary conditions (e.g., a genus $g > 0$ surface). The scheme can also be applied to punctures on a bilayer system with holes and appropriate boundary conditions, which is a way to effectively obtain a single-layer system on a high-genus surface.

D. Braiding

Now, we consider constant depth circuits for implementing braiding in Turaev-Viro codes. We will consider braiding of “punctures,” with the understanding that punctures can refer to anyons (including non-Abelian anyons), holes, or bulk twist defects. An important ingredient in the braiding protocols includes the ability to move the punctures by a distance of order the code distance d , by a local finite depth quantum circuit, followed by a permutation of qubits. When these protocols are applied to the case of the non-Abelian anyons of the Fibonacci surface code, they imply the possibility of a universal fault-tolerant gate set by a constant depth local quantum circuit, together with qubit permutations.

1. Braiding on a space with periodic boundary conditions

We first consider the simpler situation where the punctures are located on a space with periodic boundary conditions, such as a torus (Fig. 28). Physically this can be realized in a planar geometry by considering a bilayer system with holes and appropriate boundary conditions.

We start the protocol in Fig. 28(a) with two punctures separated by d plaquettes. The code distance $d = 4$ in Fig. 28(a). The upper panel of Fig. 28(a) shows the triangulation Λ and the lower panel shows the corresponding trivalent graph $\hat{\Lambda}$. As above, for simplicity we do not illustrate the branching structure.

The first step is to perform a retriangulation inside the cylindrical strip of width d between the two punctures. The retriangulation corresponds to the F moves (2-2 Pachner moves) discussed above; the edges to be flipped are indicated in pink thick lines, and the corresponding change of the dual graph $\hat{\Lambda}$ is indicated in the lower panel. The retriangulation can be performed via the controlled F operations discussed in the previous sections, and can be performed in parallel over

the entire strip. Therefore, this corresponds to a local quantum circuit with depth $\mathcal{O}(1)$.

The resulting triangulation and trivalent graph is shown in Fig. 28(b). We then permute the qudits in a shearing pattern (indicated by the green arrows) to reach the configuration in Fig. 28(c). Specifically, the qudits in each column are permuted cyclically, by a number of steps indicated by the length of the arrow in each column. This sequence of moves causes the puncture on the right (red) to be moved vertically by d spacings via a local constant depth quantum circuit, followed by a permutation. Note that due to the periodic boundary conditions, some qudits on top are permuted to the bottom.

Now, the triangulation pattern is modified in the strip of width d to the right of the (red) puncture, compared to the original configuration in Fig. 28(a). To recover the original triangulation, we perform another retriangulation, as shown in Fig. 28(d).

The overall effect of this cycle, therefore, is that we have moved the puncture vertically upward by d spacings, via a constant depth local quantum circuit, followed by a permutation of qudits, and followed finally by another constant depth local quantum circuit.

Another five steps complete the cycle of braiding, exchanging the red and blue punctures as shown in Figs. 28(e) and 28(f). Therefore, we have seen how we can braid two punctures in six steps, each of which contains a permutation of qudits sandwiched between two constant depth local quantum circuits. Note that in the above protocol, in each step the punctures can only be moved by a distance that is bounded by the distance to the closest puncture.

If we have n punctures, we can consider placing the punctures along a line, with a spacing d between each puncture. Then, neighboring punctures along the line can be braided using the protocol described above. Therefore, this method demonstrates how to perform elementary braids for any number of punctures on a space with periodic boundary conditions.

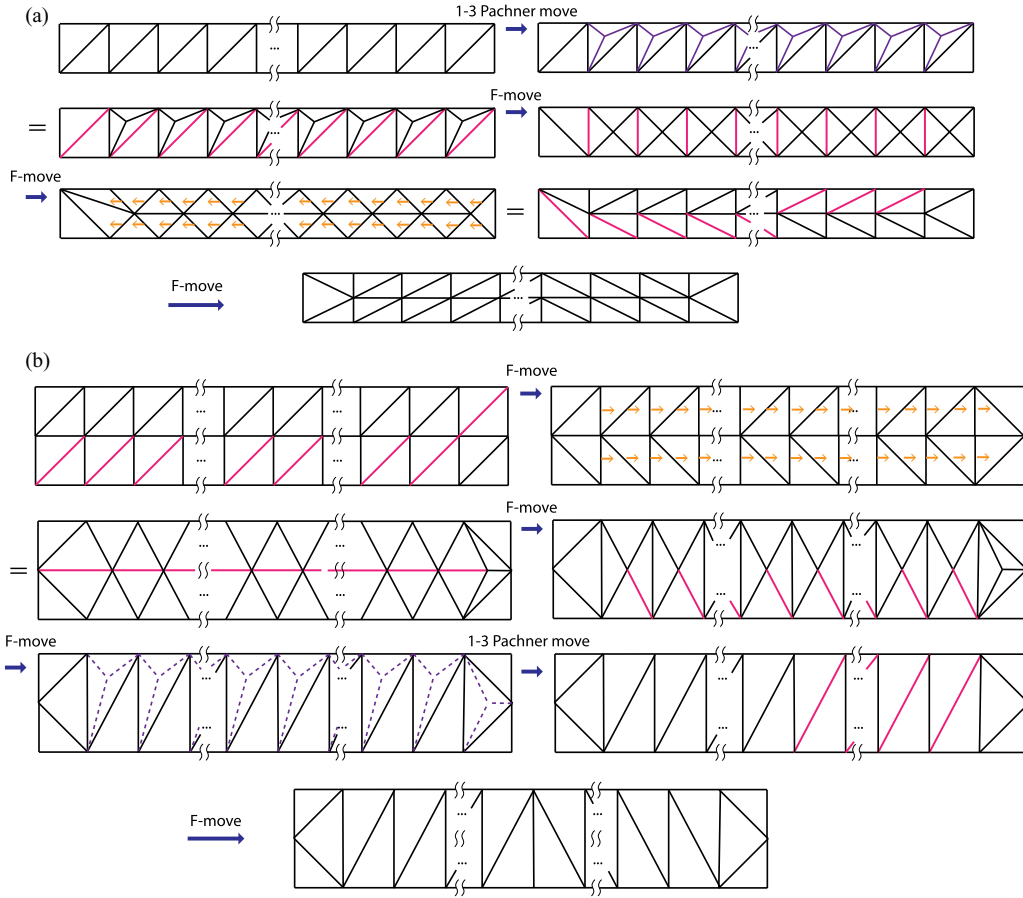


FIG. 29. Gadgets for local geometry deformation in Turaev-Viro codes. The solid (dashed) purple lines represent added (removed) edges during the 1-3 Pachner moves. The pink line represents the switched edges in F moves. The yellow arrow indicates the equivalence between two triangulations by locally shifting the positions of the edges, which can be physically implemented by local SWAPs.

2. Braiding on a disk subregion

Here, we demonstrate a protocol for braiding punctures where the whole protocol acts only within a disk subregion of the space. As such, it can be applied to braiding punctures on any space, and in a way which does not affect the system outside of the disk subregion.

In the case of braiding on a disk subregion, our protocol will require merging and splitting plaquettes in a manner similar to the case of the toric code in Fig. 13. Two important building blocks of our protocol are demonstrated in Fig. 29. In Fig. 29(a), we consider a single row of arbitrary length. By utilizing ancilla qudits, we can implement the 1-3 Pachner moves, which increase the number of vertices of the triangulation. By a finite sequence of F moves (2-2 Pachner moves) and local SWAPs, we can effectively split a single row of arbitrary length L into two rows, with a constant (independent of the length) number of steps (i.e., a constant depth local unitary circuit). In Fig. 29(b), we illustrate how two rows can be converted into a single row by a finite number of steps.

Note that in both of these protocols, the qudits on the outer boundary of the rows shown are completely unaffected, acting as control qudits for the unitary operations. This then allows the similar transformations to be applied to a large number of rows in parallel.

Using the above gadgets for splitting or combining rows, we can now demonstrate our braiding protocol, shown in Fig. 30. In the first step, in the region below the right puncture (red), we split rows of varying lengths in two rows, while combining rows in the region above the puncture, in a manner illustrated in Fig. 30(b). This can be performed by a local quantum circuit with a constant depth, independent of the spacing between punctures. Similar to the protocol for toric code (Fig. 13), we create a lattice with a shearing pattern on the left and right sides of the (red) puncture; the regions above the (red) puncture being coarse grained (effectively compressed) while the region below the (red) puncture is fine grained (effectively stretched).

The dual graph in the lower panel illustrates the positions of the qudits. The white and yellow circles represent added (entangled) and removed (disentangled) qudits, respectively. Now, via long-range permutation of qudits (indicated by green arrows), we reach the configuration in Fig. 30(c) where the (red) anyon is moved up. To recover the original triangulation and corresponding trivalent graph, we apply another step of retriangulation in the strip on the right of the (red) anyon (indicated by the pink thick lines), and hence map back to the original lattice in Fig. 30(d).

The above protocol, which uses a constant depth local unitary circuit and long-range qudit permutations, effectively

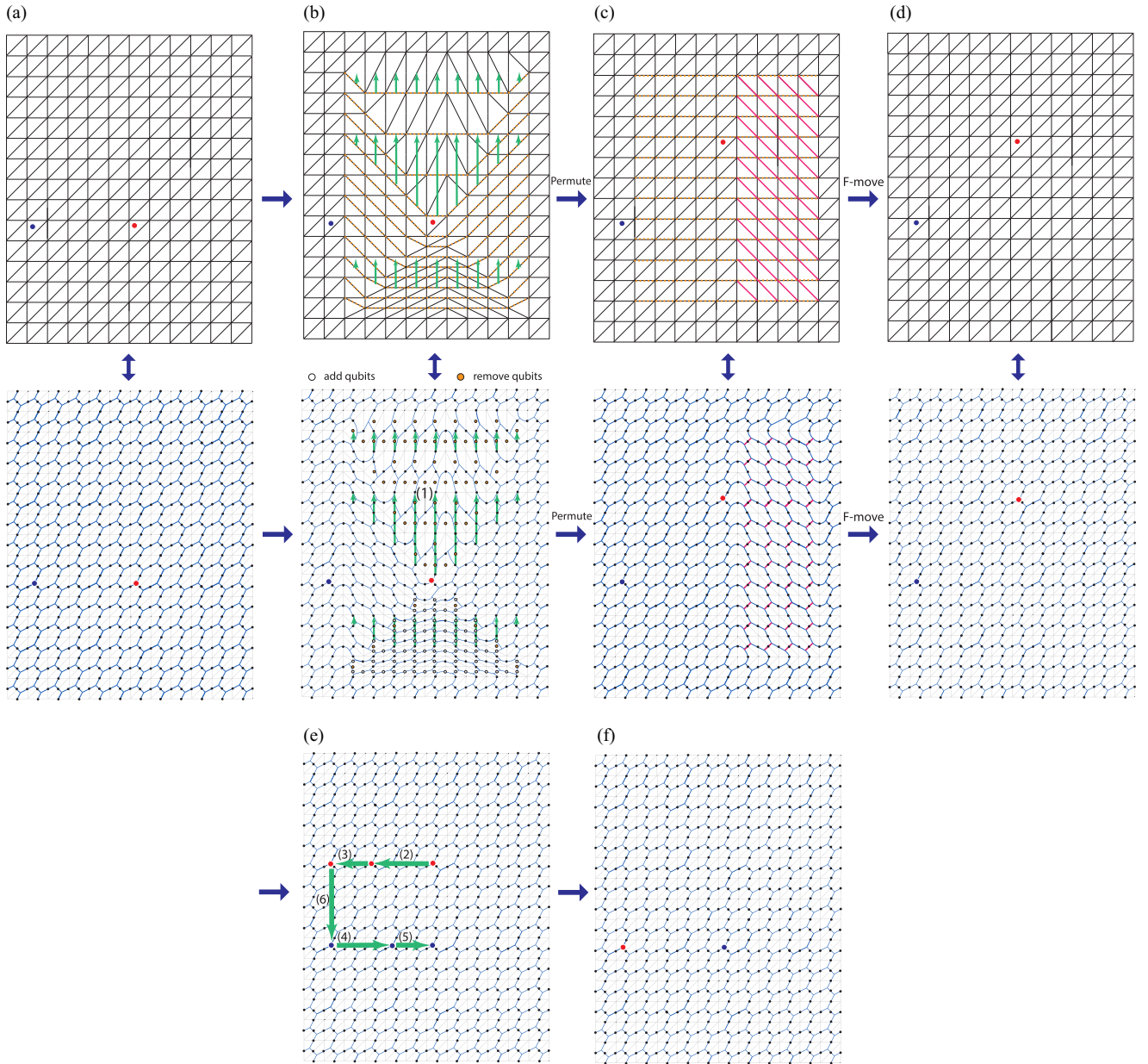


FIG. 30. Braiding of two non-Abelian anyons in Turaev-Viro codes with constant depth circuit on a disk subregion, keeping the boundary fixed. The yellow dashed lines in (b) and (c) show the equivalent edges before and after the permutation.

moves one puncture vertically by a distance of the order of the separation between the nearest puncture, which is on the order of the code distance. To complete a braiding cycle, we apply another five shots of a similar procedure, which then effectively braids the two punctures around each other as illustrated in Figs. 30(e) and 30(f).

To summarize, a single braiding operation, either in a single patch or utilizing periodic boundary conditions, can be performed in a constant number of steps, independent of the system size and code distance. Note that this is in contrast with the previous computation schemes of the Turaev-Viro code presented in Ref. [11], where braiding or Dehn twists are implemented by sequential F moves with circuit depth of $\mathcal{O}(d)$.

In this case we have demonstrated a six-step procedure:

$$\mathcal{B}_{1,2} = \prod_{i=1}^6 \mathcal{LU}'_i \mathcal{P}_{\sigma,i} \mathcal{LU}_i. \quad (60)$$

Note that each step is composed of a constant depth local quantum circuit $\mathcal{LU}_i$ corresponding to a retriangulation of the manifold, a permutation of qubits $\mathcal{P}_{\sigma,i}$ over a distance $\mathcal{O}(d)$, and another local circuit $\mathcal{LU}'_i$ in order to retriangulate the manifold back to the original triangulation. Here, we choose three smaller steps in each of the six steps of the move because of the symmetry of the intermediate retriangulation pattern shown in Figs. 28(c) and 30(c). Alternatively, one can also exchange the last retriangulation with the permutation and

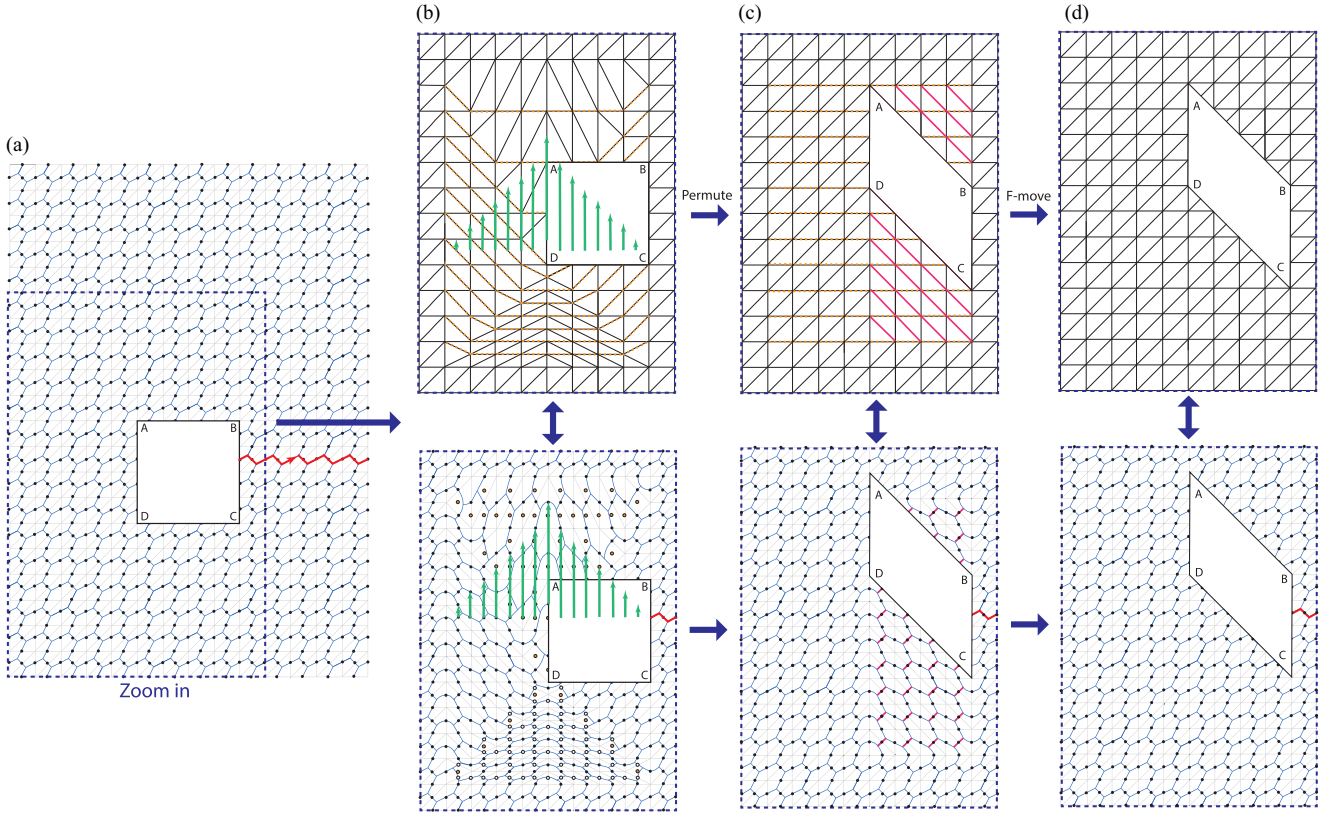


FIG. 31. Implementation of Dehn twist $\mathcal{D}_\beta$ in Turaev-Viro codes, on an annulus along the β loop encircling the boundary defect via a sequence of shearing operation. The red line with arrows represents Wilson-line operators connecting the inner and outer boundaries.

merge the two retriangulations into one, so that

$$\mathcal{B}_{1,2} = \prod_{i=1}^6 \tilde{\mathcal{P}}_{\sigma,i} \tilde{\mathcal{L}}\mathcal{U}_i. \quad (61)$$

This results in a less symmetric triangulation during intermediate steps of the protocol, and the local circuit $\tilde{\mathcal{L}}\mathcal{U}_i$ will have a somewhat longer (but still finite, independent of d) interaction range.

We note that, as discussed in the context of the $\mathbb{Z}_N$ toric code, another approach to braiding is to consider a half-Dehn-twist along a loop that surrounds the two punctures. In this case, the protocol to perform a Dehn twist on an annulus can then be easily adapted to perform a half-Dehn-twist along the loop surrounding two punctures. In the following section, we demonstrate protocols for a full Dehn twist on an annulus for the Turaev-Viro codes. The adaptation of this protocol to a half-Dehn-twist to perform braiding is straightforward and will thus not be presented explicitly here. Nevertheless, we note that it can be used to effectively perform the braiding in “one shot” by a single $\mathcal{L}\mathcal{U}$ followed by a permutation.

E. Dehn twists on an annulus

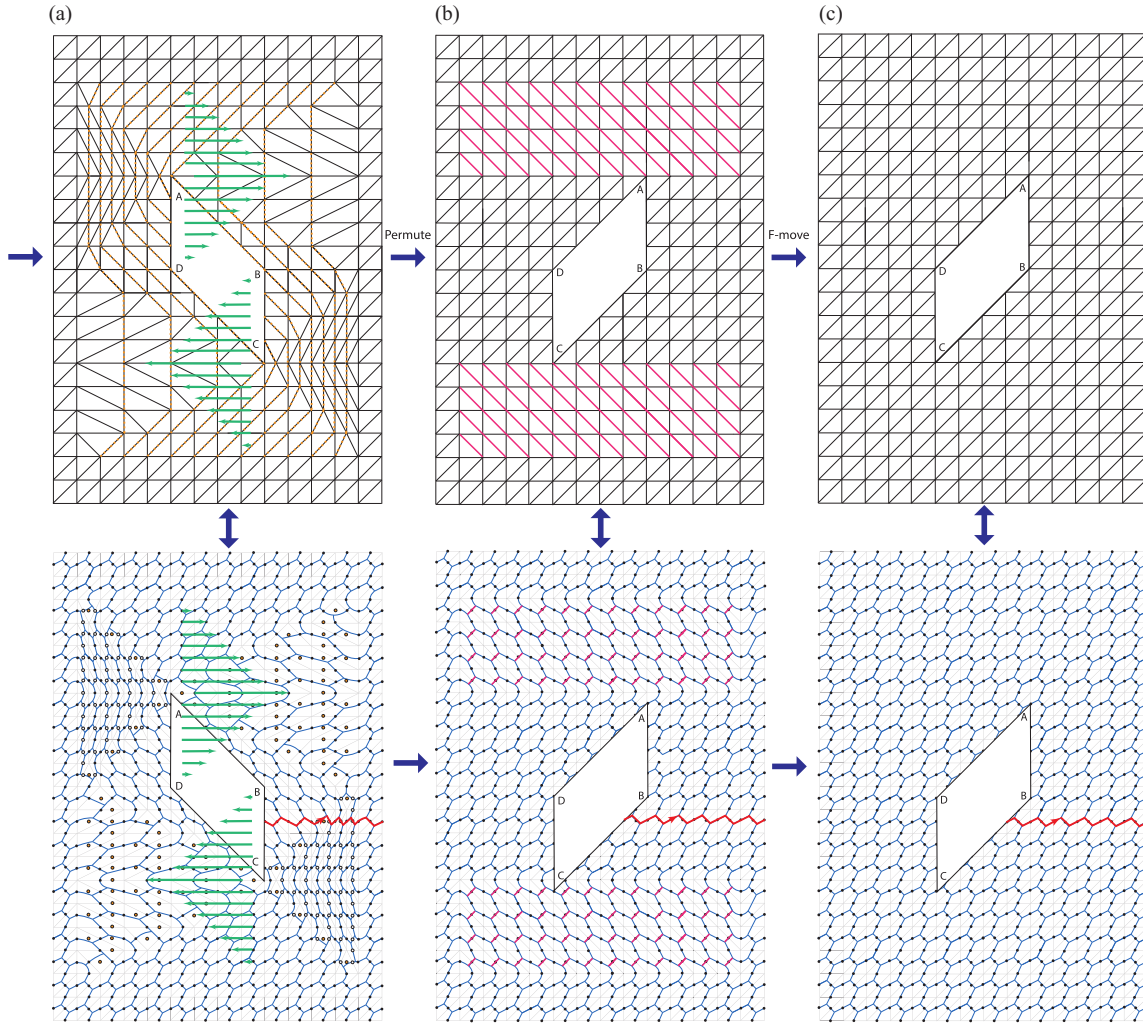
In this section, we demonstrate protocols for implementing a Dehn twist on an annulus in the Turaev-Viro codes. This operation can then be utilized for braiding (discussed above), or for performing Dehn twists along the β loops of a high-genus surface, discussed in the subsequent section.

1. Protocol 1: Twist via shearing

We begin with an annulus depicted in Fig. 31(a), where we have drawn a red string α from the inner to the outer boundary to track its motion during the course of the protocol and to verify when we have completed a Dehn twist. We note that the qudits on the inner and outer boundaries are unaffected, aside from undergoing the relevant permutations. Therefore, our protocol can be used for any choice of boundary conditions, or even when this annulus is viewed as a piece of a larger surface.

As we have demonstrated in the case of the toric code, we can perform an effective rotation of the inner boundary by 2π relative to the outer boundary through a sequence of shear deformations. The steps in this protocol are analogous to those used in the braiding protocol above. For the first shearing, we start with a constant depth local quantum circuit $\mathcal{L}\mathcal{U}$, which implements a retriangulation to reach the configuration shown in Fig. 31(b). The corresponding geometry deformation on the trivalent graph is shown in the lower panel. After a permutation P_σ , one reaches the pattern in Fig. 31(c). To revert to the original triangulation, we apply another constant depth local circuit $\mathcal{L}\mathcal{U}'$, to reach the configuration in Fig. 31(d). We see that the lattice has returned to the original form, while the inner hole has been effectively sheared upward.

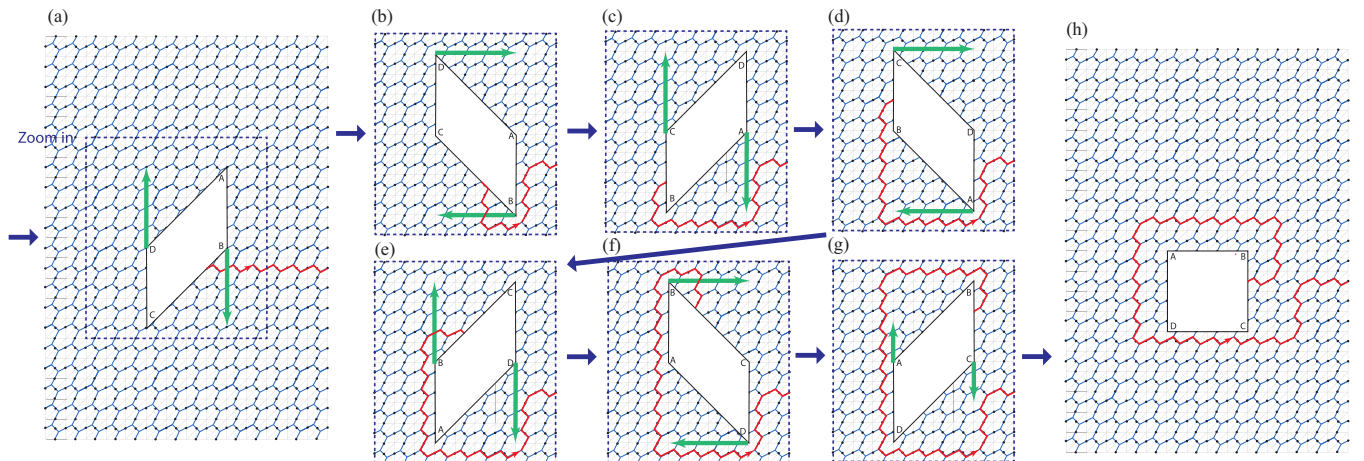
Now, we continue and perform the second shear. We first apply a constant depth local unitary to effectively induce the retriangulation shown in Fig. 32(a), followed by a permutation to reach the configuration in Fig. 32(b). A third constant depth local unitary effectively returns the system to the original

FIG. 32. Implementation of Dehn twist $\mathcal{D}_\beta$ in Turaev-Viro codes on an annulus (a continuation of Fig. 31).

triangulation, with the net effect being a horizontal shear that takes the vertex A to the upper right corner and the vertex C to the lower left corner.

We continue this basic procedure through another seven shears, as shown in Fig. 33. The net effect of the whole

procedure is thus to effectively rotate the inner boundary of the annulus by 2π . The system is returned to its original configuration. By tracking the red string α , we see that $\alpha \rightarrow \alpha + \beta$, as shown in Fig. 33(h), and thus we have performed a Dehn twist along the β curve of the annulus.

FIG. 33. Implementation of Dehn twist $\mathcal{D}_\beta$ in Turaev-Viro codes on an annulus (a continuation of Figs. 31 and 32).

In sum, our protocol uses nine steps of shearing, each of which is composed of three smaller steps: a constant depth local circuit, followed by a permutation, and then another constant depth local circuit. Thus, when acting in the code subspace, we have

$$\begin{aligned}\mathcal{D}_\beta &= \prod_{i=1}^9 \mathcal{LU}'_i \mathcal{P}_{\sigma_i} \mathcal{LU}_i \\ &= \prod_{i=1}^9 \tilde{\mathcal{P}}_{\sigma_i} \tilde{\mathcal{LU}}_i.\end{aligned}\quad (62)$$

As in the braiding discussion, in the second line we have commuted the last permutation through the last $\mathcal{LU}$ and combined neighboring local quantum circuits.

2. Protocol 2: Single-shot twist

As in the discussion for the toric code in Sec. III C 2, in the general Turaev-Viro codes it is also possible to perform the Dehn twist around the annulus in a “single shot,” in the sense that we need only one constant depth local quantum circuit $\mathcal{LU}$ followed by a permutation. The previous shearing protocol, in contrast, used nine such steps. However, in this single-shot procedure the local unitary circuit has longer range, although the range is still independent of code distance and system size.

The implementation of this protocol follows almost the same steps described in Sec. III C 2 for the case of the toric code. However, in contrast to the toric code case, the local unitary circuit required in this case to perform the analog of step 3 in Sec. III C 2 is somewhat more involved. To explain the protocol, we first focus on a single ring encircling the inner boundary and explain how one can utilize F moves (2-2 Pachner moves) to reconnect the vertices accordingly. After we have understood the case of a single ring, we can then use the same gadget in parallel across different rings to implement the full protocol.

We begin by grouping the vertices of the triangulation into rings of increasing size surrounding the inner boundary; we thus label the vertices as (n, m) where n labels the ring and m labels the vertices along the ring, with m increasing clockwise starting at the top left [see Fig. 34(a)]. We first perform an F move (2-2 Pachner move) as shown in Fig. 34(a) which facilitates reconnection of vertices.

Next, we relabel the vertices by shifting the labels $(n, m) \rightarrow (n, m - 9n)$ as shown in Fig. 34(c). Our goal now is to perform a constant depth local unitary, corresponding to a constant number of F moves, such that the edges of the final lattice connect vertices with the same labels as at the start of the protocol. To do this, consider first the two sequences of F moves shown in Figs. 34(c) and 34(d). After these F moves, we see that the number of edges coming out of each vertex matches that of the original lattice in Fig. 34(b), up to a cyclic permutation of the vertices.

Therefore, to get to the desired configuration, we perform a total of six steps, each of which consists of applying F moves in parallel around the ring, as shown in Figs. 34(e)–34(g). The exact number of steps varies among the rings, starting from nine for the innermost ring and decreasing as

one moves further away. After these steps, we see that the graph consisting of the vertices and edges is the same as it was before the relabelling $(n, m) \rightarrow (n, m - 9n)$. Finally, we apply the F moves shown in Fig. 34(h) to account for the first F moves done in Fig. 34(a).

Note that in the gadgets described in Fig. 34, the qudits at the boundaries of the rings (i.e., on the horizontal and vertical thick edges in Fig. 34) are used only as control qudits. This allows us to perform this transformation on all of the rings in parallel. For a hole with L qudits on its boundary, we have to modify L rings to perform the twist. Figure 35(b) shows the state of the whole annulus after performing the gadget on all 12 rings in parallel. We see that the configuration in Fig. 35(b) can be achieved by a constant depth local quantum circuit.

Finally, by a permutation we can recover the original lattice [Fig. 35(c)]. By following a red string α throughout the protocol, we see that a Dehn twist around the β loop has been performed. In sum, we thus see that a Dehn twist around a loop encircling the inner boundary of an annulus can be achieved by a constant depth local quantum circuit $\mathcal{LU}_\beta$, followed by a permutation $\mathcal{D}_\beta = \mathcal{P}_\sigma \mathcal{LU}_\beta$.

F. Dehn twists on a high-genus surface for Turaev-Viro codes

The results of the previous sections can be straightforwardly adapted to the case of braids and Dehn twists on a genus- g surface with p punctures, i.e., the generators of $\text{MCG}(\Sigma_{g,p})$. To see this explicitly, we can think of a genus- g surface in terms of two layers of a Turaev-Viro code, connected via wormholes along boundaries of disconnected holes, as shown in Figs. 36 and 37. In other words, a genus- g surface is equivalent to a bilayer version of the state, with g disconnected gapped boundaries. Note that we choose the triangulation and trivalent graph patterns on the two layers being symmetric up to a mirror reflection.

The α and γ Dehn twists, shown explicitly in Figs. 36 and 37, can be applied by straightforward generalizations of the Dehn twist protocols on a torus. The Dehn twist along the β loop can be applied by using the protocol for the Dehn twist on an annulus for the top or bottom layer, as illustrated in Fig. 18.

Finally, in the presence of p punctures, the additional generators needed for $\text{MCG}(\Sigma_{g,p})$ are the braids between punctures i and $i + 1$, i.e., $\mathcal{B}_{i,i+1}$. Here, we can straightforwardly use the protocols shown in Figs. 28–30 to implement these braids.

G. Multiple Dehn twists in a single shot: Proof of Theorem 2

In this section we prove Theorem 2 in the context of Turaev-Viro codes. We demonstrate how to implement n Dehn twists along any cycle of a genus- g surface using constant depth local circuits and permutations. Specifically, we show that in the code subspace, $\mathcal{D}_{n\omega} = \prod_{i=1}^k \mathcal{P}_{\sigma_i} \mathcal{LU}_i$, where the $\mathcal{LU}_i$ are local quantum circuits with depth independent of n , code distance d , and system size, while $k = \mathcal{O}(\log_2 n)$.

We start by considering a torus with its length in the β cycle being n times as much as its length along the α cycle, as illustrated in Fig. 38 (where $n = 2$ in this example). After applying the F moves [Fig. 38(b)] similar to the case in Fig. 27 and permutation of qubits along the β cycle [Fig. 38(b)], we

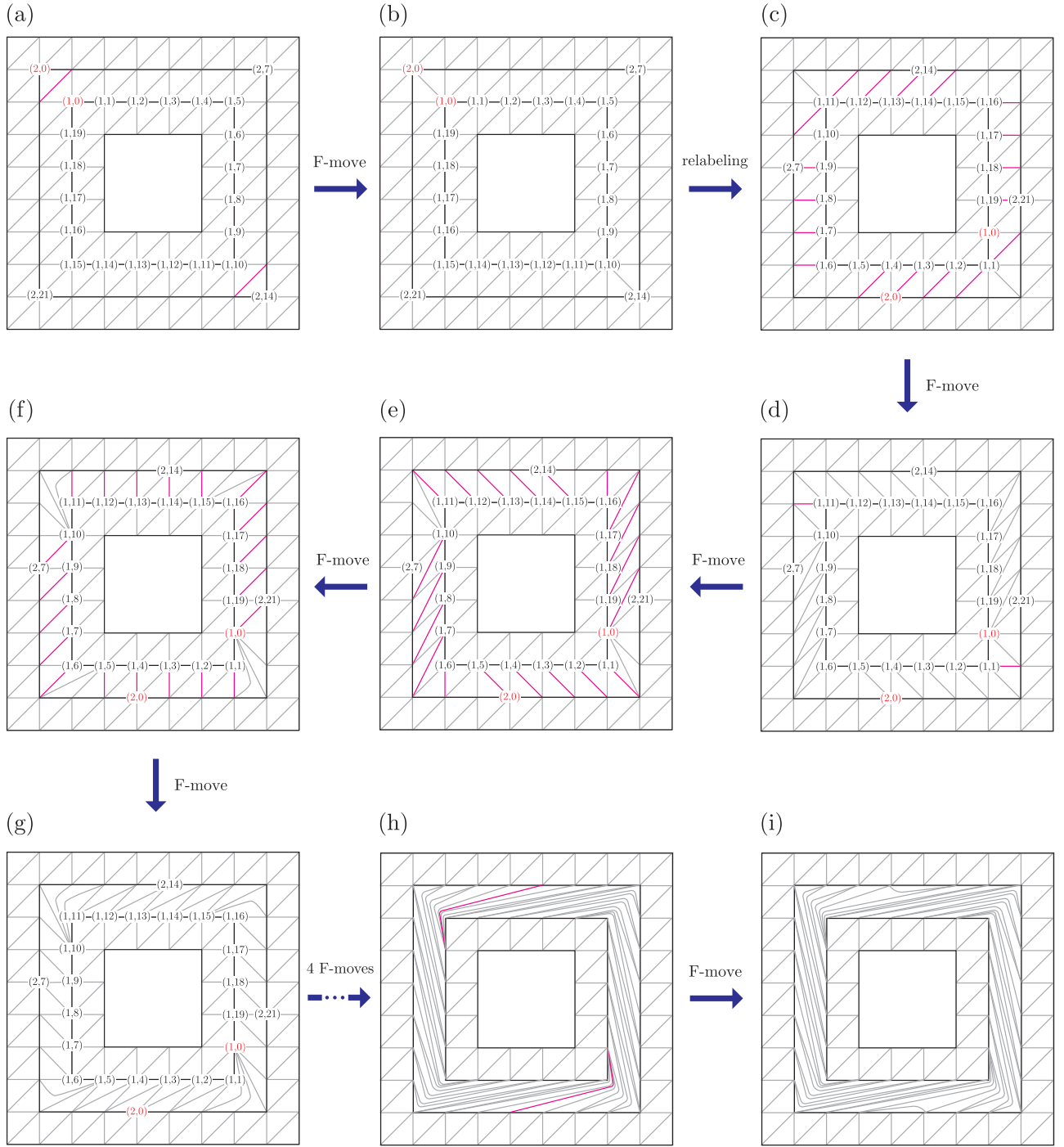


FIG. 34. The gadget used for the alternative protocol of Dehn twist on an annulus in Turaev-Viro codes.

achieve a Dehn twist $\mathcal{D}_{n\alpha}$ in a single shot, corresponding to the following transformation: $\mathcal{D}_{n\alpha} : W_{\beta}^a \mapsto W_{n\alpha+\beta}^a$ [with the $n = 2$ case illustrated in Fig. 38(c)]. By changing the opposite aspect ratio of the torus, we can also achieve the Dehn twist $\mathcal{D}_{n\beta}$ in a single shot. This again suggests that by increasing the number of qubits participating in the topological state by n times and with fixed code distance d , the time complexity of implementing a particular logical gate sequence can be decreased by n times, even in the case of the more computationally powerful non-Abelian code.

Now, to prove the theorem, we also need to be able to freely adjust the aspect ratio of the torus, as in the case of Abelian stabilizer codes in Sec. III F 1. To do this, we apply a plaquette-dividing protocol to grow the torus along a particular direction, allowing us to adjust its aspect ratio, as shown in Fig. 39. Here, by a sequence of 1-3 Pachner moves and F moves, we see that we can double the size of the code by a constant depth local unitary, followed by a permutation on the qubits. By repeating this procedure, one is able to stretch the torus along a particular direction by n

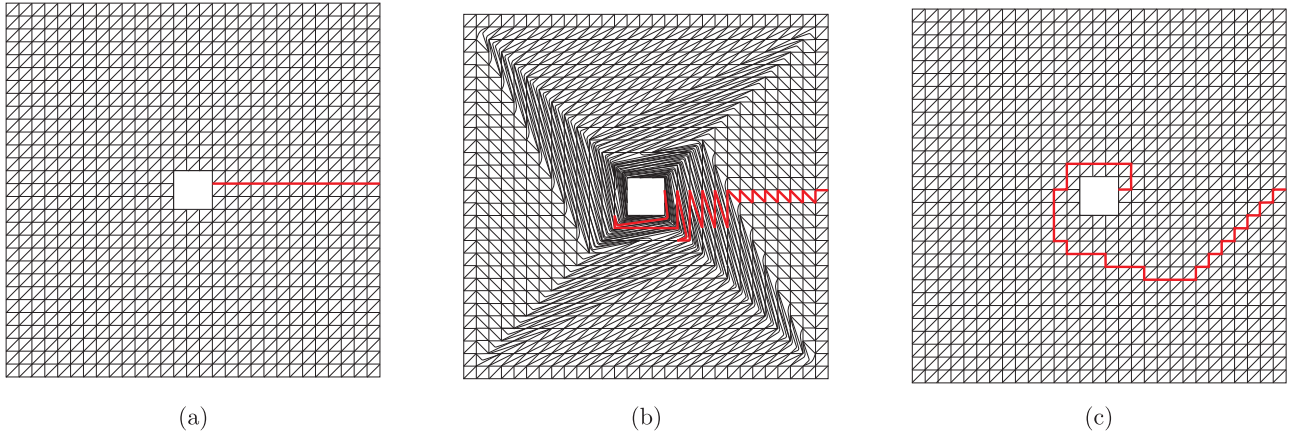


FIG. 35. Alternative protocol of Dehn twist on an annulus for Turaev-Viro codes.

times with $\log_2 n$ steps of the above procedure. Therefore, one gains an exponential speedup when merging n Dehn twists together, as compared with applying them sequentially. It is straightforward to generalize the above procedure to Dehn twists along any of the $3g - 1$ loops on the genus- g surface.

Finally, we note that, unlike the situation in Abelian stabilizer codes where multiple Dehn twists can be applied in a single shot by increasing the range of the local quantum circuit (second part of Theorem 2) (Sec. III F 2), we have not found such a general property for the Turaev-Viro codes. The key difference is that, in the Abelian stabilizer codes, it is always possible to create an arbitrarily long-range stabilizer in just a single shot (two small steps) with long-range entangling gates (e.g., CNOT). In contrast, in the non-Abelian codes, we create a plaquette involving an edge with length $\mathcal{O}(n)$ by applying $\mathcal{O}(n)$ steps of F moves.

V. ARBITRARY MCG ELEMENTS: PROOF OF THEOREM 3

So far we have demonstrated how one can implement all elementary Dehn twists and braids by a constant depth local circuit followed by a permutation. Here, we consider how to implement arbitrary elements of the MCG, proving Theorem 3.

Specifically, let $\zeta \in \text{MCG}(\Sigma_{g,p})$ be an arbitrary element of the mapping class group of $\Sigma_{g,p}$, and $\mathcal{V}_\zeta$ be its representation on the code subspace. ζ has a presentation in terms of k Dehn

twists and braids, as the latter generate the mapping class group. According to Theorem 1, each Dehn twist and braid can be implemented by a constant depth local quantum circuit $\mathcal{LU}_i$ followed by a permutation $\mathcal{P}_{\sigma_i}$. Therefore,

$$\mathcal{V}_\zeta \otimes I|\Psi\rangle = (\mathcal{V}_\zeta|\Phi\rangle) \otimes |\Pi\rangle_a = \prod_{i=1}^k \mathcal{P}_{\sigma_i} \mathcal{LU}_i |\Phi\rangle \otimes |\Pi\rangle_a. \quad (63)$$

Recall that $|\Phi\rangle$ is the many-qubit topological state, and $|\Pi\rangle_a$ is a product state on ancilla qubits.

Now, the idea is that since the $\mathcal{P}_{\sigma_i}$'s are just relabelings of the qubits, they can be deferred until after all of the circuits $\mathcal{LU}_i$. To get an intuition first, we set the general case aside and look at the braiding protocol described in Sec. III D 1. Note that the braiding protocol was also in the form of $\mathcal{B} = \prod_{i=1}^{12} \mathcal{P}_{\sigma_i} \mathcal{LU}_i$. After performing $\mathcal{LU}_1$, just before doing the long-range permutations shown in Fig. 13(d), one can see that the code is actually in square-lattice form and just stretched in some parts and squeezed in other parts. So, one can perform the circuit $\mathcal{LU}_2$ on the same lattice, skipping the long-range permutations of Fig. 13(d). The price to pay is that in the deformed square lattice, some plaquettes are elongated by a factor of 2, so if the $\mathcal{LU}_2$ gates involve qubits that are one lattice constant apart, now the range will increase to 2. By using the same idea in each step, one can defer all permutations until the end and perform all of them at once.

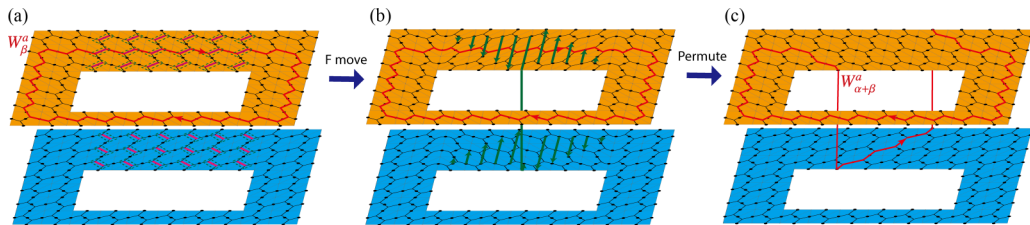
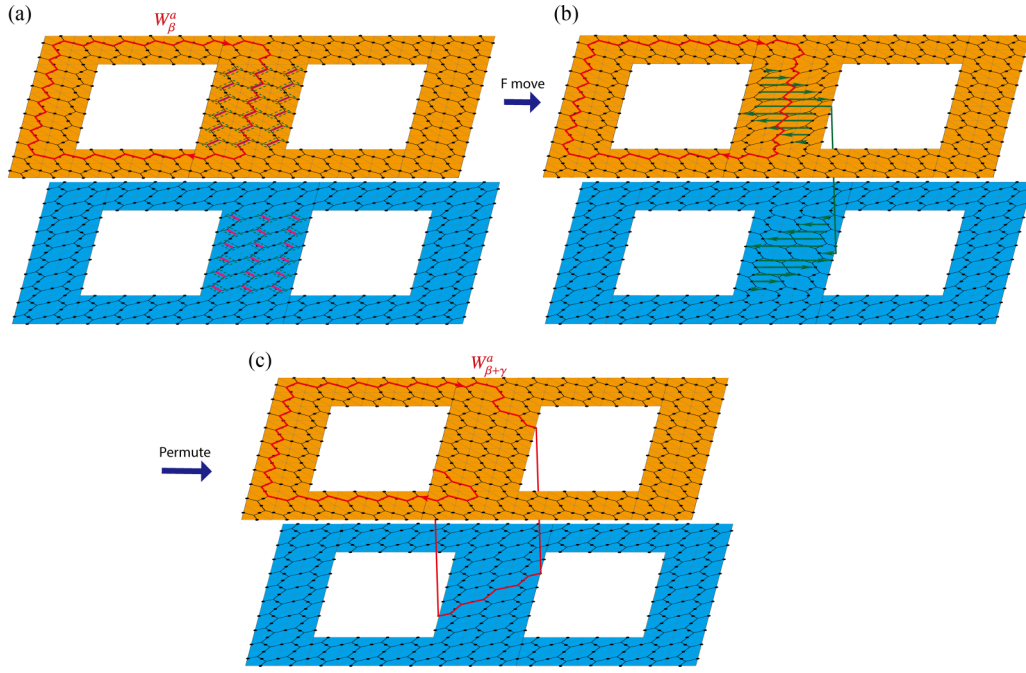


FIG. 36. Implementation of Dehn twist along the α loop in Turaev-Viro code. The red line with arrows represents the Wilson loop. The pink thick lines represent the edges being switched during the F moves, with the green dashed boxes specifying the four external legs. The dark green arrows represent qubit permutation with a shearing pattern.

FIG. 37. Implementation of Dehn twist along the γ loop in Turaev-Viro code.

Now, we return to the general case. By inserting appropriate factors of $\mathcal{P}_{\sigma_i} \mathcal{P}_{\sigma_i}^{-1}$, we can turn the right-hand side of Eq. (63) into

$$\left(\prod_{i=1}^k \mathcal{P}_{\sigma_i} \right) \prod_{i=1}^k (\mathcal{P}_{\sigma_1}^{-1} \mathcal{P}_{\sigma_2}^{-1} \dots \mathcal{P}_{\sigma_{i-1}}^{-1} \mathcal{LU}_i \mathcal{P}_{\sigma_{i-1}} \dots \mathcal{P}_{\sigma_2} \mathcal{P}_{\sigma_1}) |\Phi\rangle |\Pi\rangle. \quad (64)$$

Let $\tilde{\sigma}_i = \prod_{j=1}^{i-1} \sigma_j$. Then, Eq. (63) takes the simple form

$$\begin{aligned} \mathcal{V}_\zeta \otimes I |\Psi\rangle &= (\mathcal{V}_\zeta |\Phi\rangle) \otimes |\Pi\rangle_a \\ &= \mathcal{P}_{\tilde{\sigma}_{k+1}} \prod_{i=1}^k \mathcal{P}_{\tilde{\sigma}_i}^{-1} \mathcal{LU}_i \mathcal{P}_{\tilde{\sigma}_i} |\Phi\rangle \otimes |\Pi\rangle_a. \end{aligned} \quad (65)$$

One can interpret $\tilde{\mathcal{LU}}_i = \mathcal{P}_{\tilde{\sigma}_i}^{-1} \mathcal{LU}_i \mathcal{P}_{\tilde{\sigma}_i}$ as $\mathcal{LU}_i$ but performed over relabeled qubits. However, locality of $\mathcal{LU}_i$ alone does not guarantee locality of $\tilde{\mathcal{LU}}_i$ since the permutations are nonlocal operations. Nevertheless, as discussed in more detail in the subsequent section, the permutations that we use for the Dehn twist and braid protocols have a special structure (referred to in the subsequent section as a connectivity-preserving isomorphism), which keep local gates local, although they may increase the range of the gates by a constant factor. Thus, $\tilde{\mathcal{LU}}_i$ would still be a local quantum circuit.

Locality of each term in the product, together with the fact that k does not depend on code distance, ensures that the whole product in Eq. (65) is a local constant depth circuit, with depth independent of code distance and system size, which we denote by $\mathcal{LU}_\zeta$. Then,

$$\mathcal{V}_\zeta \otimes I |\Psi\rangle = (\mathcal{V}_\zeta |\Phi\rangle) \otimes |\Pi\rangle_a = \mathcal{P}_{\tilde{\sigma}_{k+1}} \mathcal{LU}_\zeta |\Phi\rangle \otimes |\Pi\rangle_a. \quad (66)$$

Since the permutations appearing in Theorem 1 change distance, the range of gates in $\mathcal{LU}_i$ and $\tilde{\mathcal{LU}}_i$ differ by a constant factor (independent of code distance). Let us consider a typical permutation σ which we use to implement one of braids or Dehn twists. If $D(m, n)$ denotes the distance between two qubits labeled by m and n , then,

$$\max_{m,n} \frac{D(m, n)}{D(\sigma(m), \sigma(n))} = c_\sigma \quad (67)$$

is independent of code distance and system size since otherwise σ would turn a local operator into a nonlocal one.

Let c be the maximum c_σ , and ρ be the maximum gate range used in the Dehn twists and braids that compose ζ . According to Eq. (67) the range of gates in $\mathcal{P}_{\sigma_i}^{-1} \mathcal{LU}_i \mathcal{P}_{\sigma_i}$ would be $c \rho$ at most. This in turn means the range of gates in $\tilde{\mathcal{LU}}_i$ would be at most $c^i \rho$. And since i can be k at most, the range

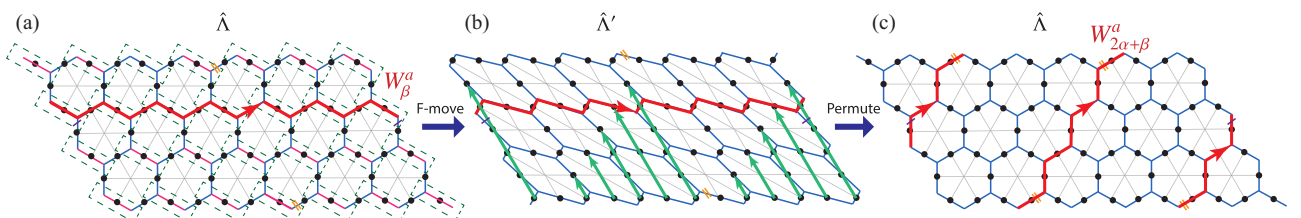


FIG. 38. Implementing multiple Dehn twists in Turaev-Viro code in a single shot by choosing a particular aspect ratio of the torus.

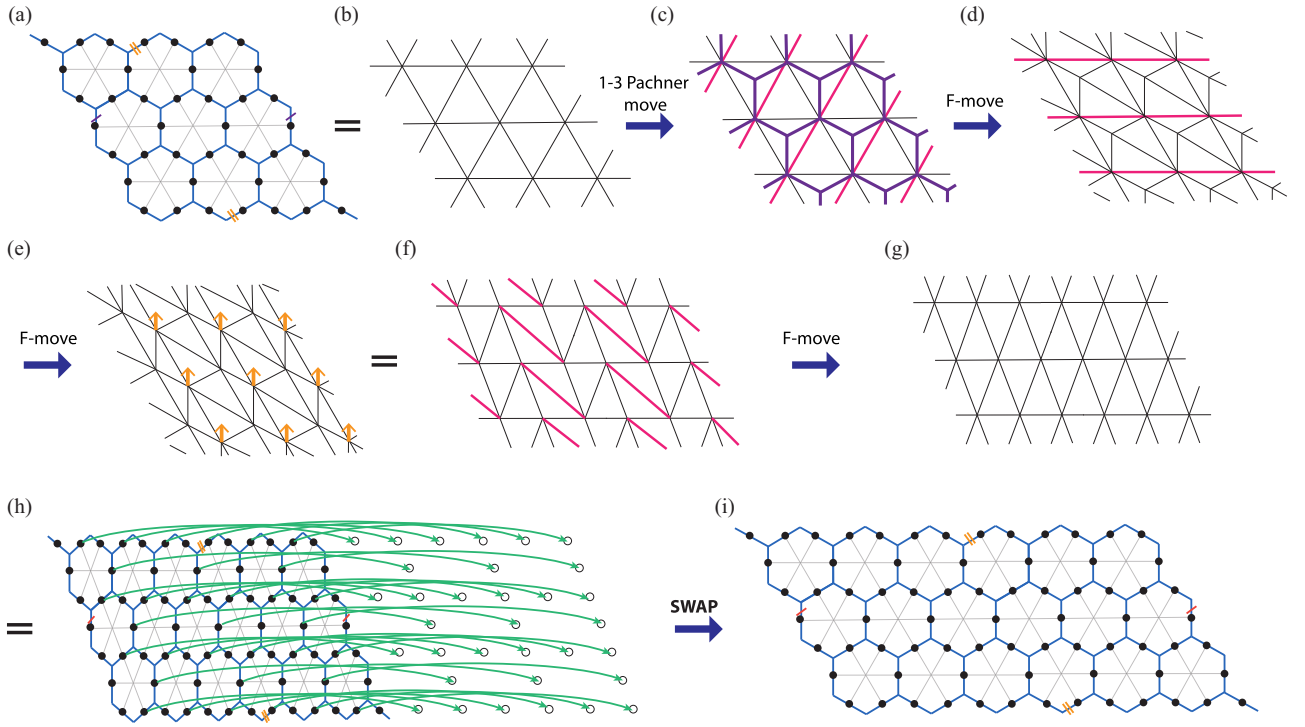


FIG. 39. Double the size of the torus along one cycle with by dividing a plaquette into two and long-range permutation. The yellow arrow in (e) indicates the equivalence of the triangulation in (e) and (f) by a local shift of the vertices. The curved green arrows in (h) represents long-range SWAPs.

of gates in $\mathcal{LU}_\zeta$ is $r = \mathcal{O}(c^k)$, which completes the proof of Theorem 3.

VI. TOPOLOGICAL PROTECTION, FAULT TOLERANCE, AND EXPERIMENTAL PLATFORMS

In this section, we argue that logical operations consisting of the $\mathcal{LU}$ and $\mathcal{P}_\sigma$ operations, which compose the Dehn twists and braids, are topologically protected and fault-tolerant logical operations, assuming appropriate error models.

A. Intrinsic protection and error propagation

A major consideration of fault tolerance is the extent to which logical gates propagate errors. For a given state with a preexisting local error on site j , i.e., $E_j|\psi\rangle$, one can apply a logical gate U to it. Even if the logical gate U is perfect, i.e., it does not generate new errors, the preexisting error may propagate under U , which can be seen from the following identity:

$$UE_j|\psi\rangle = UE_jU^\dagger(U|\psi\rangle). \quad (68)$$

This means that the ideal target state $U|\psi\rangle$ suffers from the propagated error $UE_jU^\dagger$.

We first consider the constant depth local quantum circuit $\mathcal{LU}$ implementing the local geometry deformation discussed in the previous sections, which consists only of a finite sequence of geometrically local gates supported on a region of radius $\mathcal{O}(1)$. These are also referred to as *locality-preserving unitaries* [29,30]. Any logical gate that can be performed in terms of a locality-preserving unitary is fault tolerant because the propagation of preexisting errors is restricted by causality:

a single-site error can only spread to an error supported within a Lieb-Robinson “light cone” [59] with radius $\mathcal{O}(1)$. Therefore, constant depth $\mathcal{LU}$, which composes the first part of the Dehn twist and braid protocols presented in this paper, also preserves the locality of errors, and are hence intrinsically fault tolerant.

Now, we consider the second part of our proposed logical gates, the qubit permutation $\mathcal{P}_\sigma$. While the Eastin-Knill theorem rules out the possibility of a universal transversal logical gate set, it was commented in Refs. [28,60] that qubit permutation could be a possible loophole to circumvent the theorem.⁵ It was also pointed out in Ref. [28] that such qubit permutations, taken in isolation, would be fault tolerant.

Below, we analyze the topological protection and fault-tolerance property of the long-range permutations $\mathcal{P}_\sigma$ that we use in some detail, and also specify the experimental platforms with which fault tolerance of such qubit permutations are expected.

We note that in this paper, we use a specific class of permutations $\mathcal{P}_\sigma$ that we can refer to as a *connectivity-preserving isomorphism* (CPI). While the CPI can permute qubits over long distances, it preserves the local connectivity of the underlying lattice structure of the codes/Hamiltonian. More concretely, for a pair of neighboring vertices v_1 and v_2 in the original lattice (triangulation) Λ , and the permuted

⁵Qubit permutation was also extensively studied in depth by Ref. [60] in the context of stabilizer codes, despite the fact that it was realized that universal gate sets cannot be achieved even with the addition of such permutations.

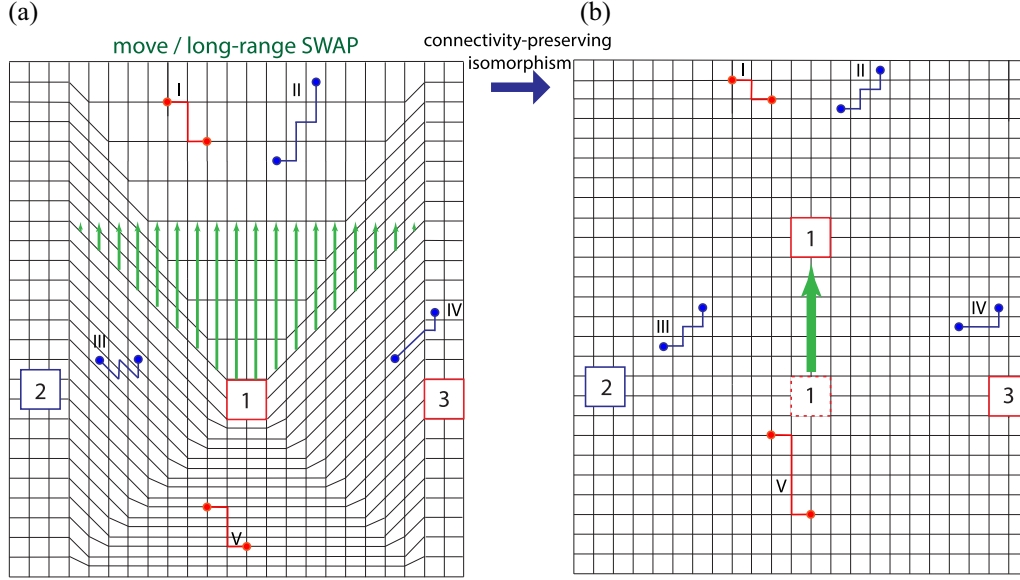


FIG. 40. Propagation of preexisting errors under the qubit permutation $\mathcal{P}_\sigma$ in the braiding scheme from Fig. 13. Five error strings before and after the permutation are illustrated.

vertices $\sigma(v_1), \sigma(v_2)$, the edge $e[v_1, v_2]$ connecting the original vertices v_1 and v_2 is exactly transformed to the edge connecting the new vertices, i.e.,

$$\mathcal{P}_\sigma : e[v_1, v_2] \mapsto e[\sigma(v_1), \sigma(v_2)]. \quad (69)$$

Moreover, the new edge $e[\sigma(v_1), \sigma(v_2)]$ also remains local, in the sense that it has length of $\mathcal{O}(1)$. More specifically, $\mathcal{P}_\sigma$ performs an isomorphism from the original lattice Λ to a new deformed lattice Λ' with completely the same connectivity. Thus, we see that CPI is clearly a more general type of transformation compared to a locality-preserving unitary $\mathcal{LU}$. As we will describe in more detail, a CPI is intrinsically protected because the many-qubit quantum state remains in the ground-state subspace of a local Hamiltonian; in other words, the many-body quantum state remains in the code subspace of a geometrically local topological QECC. A generic permutation does not have this property, as it typically moves the quantum state out of the ground-state subspace of a geometrically local Hamiltonian and thus necessarily generates errors (anyons). On the other hand, the qubit permutation considered in our paper, the CPI, is still more general than the type of qubit permutations that were considered in Ref. [60], which are automorphisms, i.e., isomorphisms that map the code space back to itself, namely, $AUT : \mathcal{H}_\Lambda \mapsto \mathcal{H}_\Lambda$. In our case, $\mathcal{P}_\sigma$ maps the code space on a triangulation Λ to that of a different triangulation Λ' : $\mathcal{P}_\sigma : \mathcal{H}_\Lambda \rightarrow \mathcal{H}_{\Lambda'}$. The $\mathcal{LU}$ is then needed to map back to the original code space $\mathcal{H}_\Lambda$.

To analyze the topological protection of a CPI, we first consider propagation of preexisting errors under a perfect (error-free) CPI. Let us consider an error string (the two end points of the string corresponding to anyons) with a length l much smaller than the code distance $l \ll d$, and which has support on sites $\{j_1, j_2, j_3, \dots, j_n\}$, as illustrated in Fig. 40. Our CPI permutation $\mathcal{P}_\sigma$ maps the string onto the new sites $\{\sigma(j_1), \sigma(j_2), \sigma(j_3), \dots, \sigma(j_n)\}$, which is a deformed error string with a length of

the same order as before, i.e., $\mathcal{O}(l) \ll d$. One can compare the configurations in Figs. 40(a) and 40(b), and see that the error strings I and II get squeezed, III and IV get deformed, and V gets stretched. Therefore, although our connectivity-preserving isomorphism does not preserve the location of errors, it only changes the length of the error string by a constant factor (independent of code distance) and hence preserves the characteristic length of the error strings. It is also worth emphasizing that the above properties suggest that the history of errors (anyons) can be completely tracked by the software (with the assistance of syndrome measurement) for the purpose of decoding in active error corrections. We note that the decoder for our scheme will differ from the standard ones, and thus requires further development in future works.

To summarize, the combined ($\mathcal{LU}$ and $\mathcal{P}_\sigma$) constant depth logical gate (denoted by U) maps a local operator O with support in a region $\mathcal{R}$ to another local operator $O' = U^\dagger O U$ with support in a region $\mathcal{R}'$, such that the area ratio between $\mathcal{R}'$ and $\mathcal{R}$ is bounded by an $\mathcal{O}(1)$ constant factor c , similar to the property of a locality-preserving unitary, namely,

$$\text{supp}(U^\dagger O U) \leq c \text{supp}(O). \quad (70)$$

Therefore, the constant depth logical gate is naturally topologically protected and can be made fault tolerant. Note that the difference from the definition of the locality-preserving unitary in Ref. [29] is that here the regions $\mathcal{R}$ and $\mathcal{R}'$ do not have to be in the vicinity of each other.

B. Additional error generation for the qubit permutation taking into account the hardware implementation

Now, we further consider additional errors that can be generated during the process of the qubit permutation. Just as with any discussion of fault tolerance, we have to assume a particular reasonable noise model. In our case, the permutations will yield fault-tolerant operations if errors in

any site-to-site permutation process, i.e., $j \mapsto \sigma(j) \equiv j'$, are independent. We consider two schemes and their experimental platforms for implementing the permutations: (1) moving qubits; (2) long-range SWAP gates utilizing ancillas.

(1) *Moving qubits.* For certain experimental systems, one can directly move the qubits to desired positions. Ion traps are currently the most promising platform for this purpose since high-fidelity fast shuttling of individual ion qubits has been realized experimentally [32–34]. Shuttling of ions has also been proposed and demonstrated for a scalable quantum computation architecture [46–48], including fault-tolerant quantum computation with surface codes on a 2D qubit array [47].

In practice, the shuttling process may suffer from noise, the major one being heating of ions [34], which is independent for individual ions. That is, there is no correlated noise between individual shuttling processes. One can model the shuttling process of a single qubit by the standard Pauli error channel (for simplicity here we only consider qubits, as opposed to qudits):

$$\mathcal{E}_{j'} = \left\{ \sqrt{1 - p_{j'}^x - p_{j'}^y - p_{j'}^z} I, \sqrt{p_{j'}^x} X_{j'}, \sqrt{p_{j'}^y} Y_{j'}, \sqrt{p_{j'}^z} Z_{j'} \right\}, \quad (71)$$

where $p_{j'}^x$, $p_{j'}^y$, and $p_{j'}^z$ are error probabilities of the Pauli- X , $-Y$, and $-Z$ errors to occur at site j' . For uncorrelated noise, the joint probability of a Pauli- i error ($i = X, Y, Z$) occurring at site j'_1 and a Pauli- k error occurring at site j'_2 is $p_{j'_1}^i p_{j'_2}^k$. Therefore, for a length- n error string of the form $\bigotimes_{j'=1}^n X_{j'}$, the probability of the error is $\prod_{j'=1}^n p_{j'}^x$. This result can also be easily generalized to the case where the operators on each site of the error string are of different types. For an error string of $\mathcal{O}(d/2)$ length, the error becomes uncorrectable and a logical error occurs. However, such an error string is exponentially suppressed in this uncorrelated noise model, which implies the existence of an error threshold.

(2) *Long-range SWAP.* For experimental systems with long-range connectivity, one can implement the permutation $\mathcal{P}_\sigma$ using long-range SWAP operations.

Here, we focus on the simplest implementation: we use two sets of qubits, a data register labeled by the set $\{j\}$ and storing the topological state $|\Psi(\{j\})\rangle$, and a temporary register labeled by the set $\{j'\}$ storing a product state of zeros, i.e., $\bigotimes_{j'} |0\rangle_{j'}$. We can take the two sets of qubits to form a superlattice structure in real space, such that corresponding qubits in the two registers are nearest neighbors if $j = j'$. We then SWAP all the information in the data register $\{j\}$ to the temporary register $\{j'\}$, according to the map $j' = \sigma(j)$. We can then just continue the computation in the new register $\{j'\}$, which now becomes the data register, and will later SWAP the data back to original data register $\{j\}$ if another permutation is needed. The qubits in the temporary register can also serve as ancilla qubits for syndrome measurement during the active error correction so extra resources may not be required.

Assuming the errors that occur on individual SWAP operations between the pair of sites j and j' are independent (uncorrelated), the errors that occur during this SWAP process can again be captured by the Pauli noise channel [Eq. (71)] on each site. Therefore, the noise property will be similar

to the case of moving qubits, and the operation is thus fault tolerant.

A more complicated implementation will be making an additional SWAP of the information in the temporary register $\{j'\}$ back to the original data register $\{j\}$. This two-step SWAP procedure makes the error analysis slightly more complicated, but does not change the essence of the fault tolerance.

Some experimental platforms that can be used for such long-range SWAP operations are listed below.

(I) *Long-range connectivity in ion traps* mediated by motional (phonon) modes of ions [61]. All the long-range SWAPs can be performed in parallel and have uncorrelated noise if there is a separate phonon mode mediating each individual SWAP.

(II) *Modular architecture of 3D superconducting cavities* [62–65]. The quantum information is stored in microwave cavity photons. This architecture has reconfigurable long-range connectivity between cavity nodes, routed by microwave circulators and superconducting cables [62]. One possible scheme is through direct quantum state transfer between remote cavity nodes in a network, which is equivalent to a long-range SWAP [63,64]. The noise is uncorrelated if different cables are used for individual SWAP processes. An alternative scheme is through remote entanglement generation and teleportation [62,65], which also has uncorrelated noise for individual teleportation channels.

(III) *Circuit QED with cavity buses* [66,67]. Here, long-range interaction between superconducting qubits or semiconductor spin qubits can be mediated by cavity array serving as quantum buses [68–74].

(IV) *Rydberg atoms.* Here, long-range gates can be realized via Rydberg-blockade mechanisms [75–78].

C. Additional time overhead to achieve full fault tolerance

So far, we have argued that our protocols for logical gates are topologically protected and can be made fault tolerant because they do not spread errors by more than a $\mathcal{O}(1)$ factor. Fault-tolerant quantum error correction requires syndrome measurements, decoders to deduce the error strings from the syndrome measurements, and finally an error recovery procedure.

Although the logical gate applied here is constant depth itself, dramatically differing from the situation of conventional braiding, extra time overhead might be introduced due to the error correction and decoding processes after the application of the gate in order to achieve full fault tolerance. In the absence of measurement error, the error syndrome can be decoded immediately after the application of the gate since the error string is only changed by an $\mathcal{O}(1)$ constant factor independent of the code distance d , and hence remains correctable.

However, in the presence of faulty measurement, one is expected to perform $\mathcal{O}(d)$ rounds of measurements to decode the error syndrome, similar to the case of surface code error correction. In the context of transversal gates or constant depth local quantum circuits [30], the error string either does not change or is stretched at most linearly, and one hence expects that only $\mathcal{O}(d)$ rounds of measurements are needed for every $\mathcal{O}(d)$ times of the application of such logical gates. The

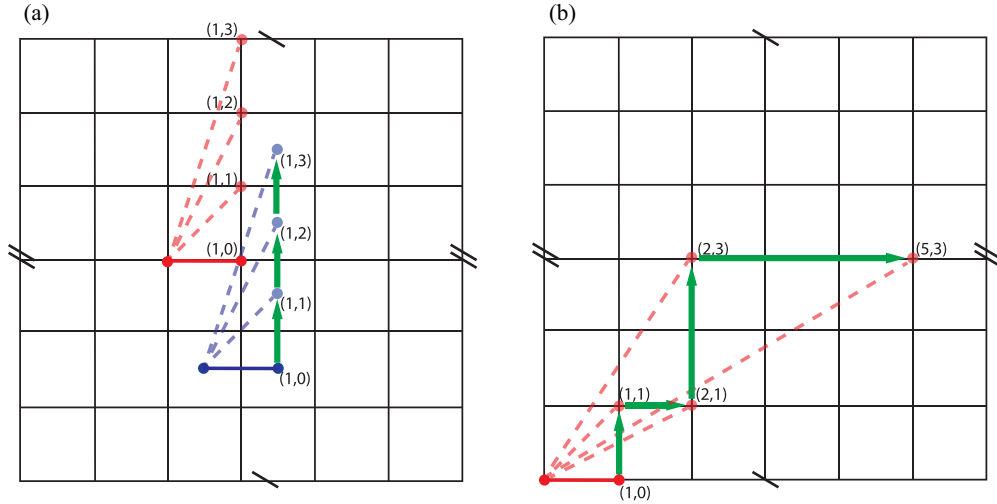


FIG. 41. Illustration of the propagation of error strings after a successive application of the instantaneous logical gates (Dehn twists) on a torus (for the $\mathbb{Z}_N$ toric code example). (a) A sequence of Dehn twists along the same α cycle (vertical) leads to a linear growing of error strings. (b) An alternating sequence of Dehn twists along both cycles leads to an exponential growing of error strings.

overall time overhead is hence $\mathcal{O}(1)$ in these situations. In the context of our logical gates, which are geometrically nonlocal, the overall time overhead depends on the detailed property of the logical circuits of the corresponding quantum algorithms. We note that our protocols do not introduce plaquette or vertex operators whose measurement outcomes are unknown (in contrast to, for example, lattice surgery methods for surface code). In other words, the eigenvalues of all new plaquette and vertex operators in our protocols are predetermined from the values of previously known plaquette and vertex operators. Thus, while not studied explicitly here, we expect that the space-time paths of the error syndromes can be fault tolerantly decoded.

In the cases when the error string is stretched linearly with time, the overall time complexity is still $\mathcal{O}(1)$. We can illustrate this situation with the $\mathbb{Z}_N$ toric code supported on a torus as shown in Fig. 41(a). Here, we apply a sequence of Dehn twists along the α cycle of the torus, i.e., $(\mathcal{D}_\alpha)^n$. We use (l_x, l_y) to denote the length of error string. We can see that for both types of error strings (e and m), the error string starting at the $(1,0)$ configuration grows with the following sequence $(1, 0) \rightarrow (1, 1) \rightarrow (1, 2) \rightarrow (1, 3) \dots \rightarrow (1, n-1) \rightarrow (1, n)$. We note that, in Fig. 41(a), we make the illustration in a way that the position of the left anyon is fixed, which occurs in the situation that the cut of the Dehn twist goes through the left anyon. In the more general situations, the left anyon will also be permuted, while the relative motion of the right anyon and hence length of the anyon string remains the same as illustrated in the figure. As one can easily conclude, the error string is stretched linearly, similar to the situation of constant depth local quantum circuit. Therefore, $\mathcal{O}(d)$ rounds of measurements are needed for applying every $\mathcal{O}(d)$ Dehn twists $\mathcal{D}_\alpha$. We note that the same conclusion applies exactly to the more general family of Turaev-Viro codes.

In the worst-case scenario, when certain sequence of braids are repetitively applied in the same region and stretch the error string exponentially with time which leads to it

growing to the code distance d in $\log d$ time steps. Therefore, we expect that $\mathcal{O}(d)$ rounds of syndrome measurements are needed for every $\mathcal{O}(\log d)$ logical gates, giving rise to an overall $\mathcal{O}(d/\log d)$ time overhead. This type of situation can be illustrated with Fig. 41(b). Here, we apply an alternating sequence of Dehn twists along both cycles of a torus, i.e., $\mathcal{D}_\alpha \mathcal{D}_\beta \mathcal{D}_\alpha \mathcal{D}_\beta \dots$. For an error string starting with the $(1,0)$ configuration, we end up with the following sequence of string growing: $(1, 0) \rightarrow (1, 1) \rightarrow (2, 1) \rightarrow (2, 3) \rightarrow (5, 3) \rightarrow (5, 8) \dots (l_x, l_y) \rightarrow (l_x + l_y, l_y) \rightarrow (l_x + l_y, l_x + 2l_y) \dots$. The property of adding the length component along both directions at the next round leads to an exponential growing the length of the error string. We note that the estimate $\mathcal{O}(d/\log d)$ is not rigorous since there may be some other more complex mechanisms which could potentially change the analysis of the time overhead. More detailed analysis needs to be done in the future works.

Finally, we remark that, since the logical gates in our cases have bounded support in space, i.e., around a single handle or pair of anyons, the logical gates applied in separate regions may not stretch the error strings consecutively. Therefore, for certain class of sequential logical circuits, we expect that the additional average time overhead for the decoding is still $\mathcal{O}(1)$.

Regarding the error recovery protocols, we note that these protocols are trivial for Abelian surface codes because any logical errors can be stored and corrected in software by “updating the Pauli frame” of the computation. However, for the non-Abelian Turaev-Viro codes, the error recovery schemes are more sophisticated [55,56,58] and need further exploration.

VII. DISCUSSION

We have demonstrated that braids and Dehn twists, which are the generators of $\text{MCG}(\Sigma_{g,p})$, can be achieved by a constant depth local unitary circuit, followed by a permutation on qubits. By utilizing long-range SWAP operations and

physical ancilla qubits, the permutations can also be achieved through a constant depth quantum circuit. These results thus imply that for topological codes with local interactions (local syndromes) in Euclidean space (excluding the class of hyperbolic codes), the space-time overhead for implementing a single logical operation on encoded qubits can be made optimal, which scales as $\mathcal{O}(d^2)$. Other proposed protocols for realizing universal fault-tolerant gate sets, such as those which use magic-state distillation, code switching, or other measurement-based schemes, require a space-time cost for a single logical gate that scales as $\mathcal{O}(d^3)$, including a time overhead that necessarily diverges with code distance when classical computational resources are included. We note that this estimate of the overhead cost is simply for implementing logical gates on logical qubits encoded using QECCs with code distance d . To estimate the total space-time overhead for *fault-tolerant* computation, we should also include the additional space-time overhead required for the error correction protocols, which include the syndrome measurements, decoder, and error recovery procedures. We estimate that, in the presence of measurement errors, an additional overhead lower-bounded by $\mathcal{O}(d/\log d)$ is introduced in the worst case of the logical circuit (gate sequence) in order to decode the space-time paths of the error syndromes, leading to an overall $\mathcal{O}(d^3/\log d)$ space-time overhead. We leave a more detailed study of this to future work.

In the previous section we discussed how the long-range permutations are natural to implement in a number of different experimental platforms. From a broader perspective, our scheme demonstrates at a fundamental level the significant advantage of long-range connectivity in quantum architectures for implementing fault-tolerant quantum computation. In addition, our study essentially provides a vision to bridge ideas from quantum communication, such as robust quantum state transfer and teleportation, and ideas from fault-tolerant quantum computation.

With respect to systems with purely local interactions between physical qubits, our results can still be of practical relevance. Long-range SWAP operations can be implemented through local operations together with a long-range entangled Bell pair. Thus, we can envision a quantum architecture where during the quantum computation, long-range Bell pairs are continuously being created, perhaps with entirely local operations, in parallel but independently of the quantum computation. The logical gates then utilize these long-range Bell pairs as a resource as part of the computation. This approach also shares the spirit with distributed quantum computation [79,80].

From a conceptual perspective, our results imply a different view on braiding and Dehn twists. Conventionally, braiding and Dehn twists are considered in terms of *adiabatic* processes. In terms of ground states of gapped Hamiltonians, braiding and Dehn twists are achieved by adiabatic evolution with a local Hamiltonian, which takes a time that diverges with system size or distance between anyons. This is manifested in active error correction approaches to quantum computation by requiring that braids of holes, twist defects, and non-Abelian anyons to require either (a) a local unitary quantum circuit whose depth grows with code distance, or (b) a

sequence of measurements, where fault tolerance requires that the measurements be performed d times.⁶ Our results, on the other hand, show that braiding and Dehn twists fundamentally need not be thought of as an adiabatic process.

Rather, the essence of Dehn twists can in some sense be thought of as actually an appropriate permutation on the qubits, which effectively implements the appropriate large diffeomorphism (diffeomorphism not continuously connected to the identity) on the space. The constant depth local unitary circuit can be thought of as a “trivial” diffeomorphism that, while it changes the geometry of the space, can be thought of in the continuum limit as being continuously connected to the identity operation. However, this trivial operation is required because we need to return the system after the permutation back to the original subspace.

Our braid protocols that utilize the movement of punctures (e.g., anyons) over large distances of the order of the code distance can instead be interpreted as follows. Rather than considering moving the anyon through the many-body state by applying a string operator of length d , we leave the anyon where it is and we “grow” the state to one side of the anyon by a factor of 2 and “shrink” it to the other side by a factor of 2, through a type of local entanglement renormalization protocol. This allows us to effectively change the distance between anyons by a factor of 2; if the anyons are separated by a distance of order d , this effectively allows motion of the anyon by d steps in essentially a single shot. A remarkable consequence of this is that the number of steps required to bring two anyons a distance ℓ apart to the same location goes like $\log \ell$ because our protocol only allows motion by a distance that is bounded by the distance to the nearest anyon.

Let us now compare our results to those of Ref. [31], where it was shown that certain mapping class group elements can be realized through a finite sequence of transversal SWAP operations in a multilayer topological state with appropriate boundary conditions and defects. The result of Ref. [31] can, in our context, be stated as follows. Certain mapping class group operations $\gamma \in \text{MCG}(\Sigma_{g,p})$ that are of finite order (i.e., $\gamma^k = 1$ for some k , called torsion elements) can be implemented purely in terms of a permutation on qubits. Furthermore, the structure of the permutation is such that it is an automorphism of the lattice (it keeps the lattice exactly invariant). This allows us to consider folding the system into a multilayer system (“quantum origami”) such that the permutation reduces to SWAP operations between layers. However, braids and Dehn twists are fundamentally different, as they have infinite order. The permutations they require are not automorphisms of the lattice, but rather more general connectivity-preserving isomorphisms. It is intriguing that we now have two classes of mapping class group operations that can be achieved with constant time overhead: (1) braids and Dehn twists, and (2) torsion elements.

⁶In certain cases in three-dimensional codes, fault tolerance with a single round of measurements may be possible, but the classical processing to determine the measurement outcome still grows with the code distance.

From the point of view of mathematics, our result demonstrates how mapping class group elements can be achieved as a map on the triangulation of a surface. Braids and Dehn twists correspond to permutations of vertices of the triangulation, followed by a finite sequence of Pachner moves to recover the original triangulation. Crucially, the length of the sequence of Pachner moves is independent of the length of any noncontractible cycle and number of vertices in the triangulation.

We have established our results for all nonchiral topologically ordered states, which can be captured by Turaev-Viro-Barrett-Westbury TQFTs. These include all of the Kitaev quantum double models and Levin-Wen string-net models as special cases. While we have not explicitly considered higher

dimensions, we expect analogous results to straightforwardly apply in higher dimensions as well, as the TVBW TQFTs have natural generalizations to higher dimensions in terms of higher categories, such as the Crane-Yetter-Walker-Wang TQFTs [9,81,82].

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- [1] X.-G. Wen, *Quantum Field Theory of Many-Body Systems* (Oxford University Press, Oxford, 2004).
 - [2] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Non-abelian anyons and topological quantum computation, *Rev. Mod. Phys.* **80**, 1083 (2008).
 - [3] Z. Wang, *Topological Quantum Computation* (American Mathematical Society, Providence, RI, 2010).
 - [4] A. Kitaev, Fault-tolerant quantum computation by anyons, *Ann. Phys.* **303**, 2 (2003).
 - [5] M. H. Freedman and D. A. Meyer, Projective plane and planar quantum codes, *Found. Comput. Math.* **1**, 325 (2001).
 - [6] E. T. Campbell, B. M. Terhal, and C. Vuillot, Roads towards fault-tolerant universal quantum computation, *Nature (London)* **549**, 172 (2017).
 - [7] V. Turaev and O. Viro, State sum invariants of 3-manifolds and quantum 6j-symbols, *Topology* **31**, 865 (1992).
 - [8] J. W. Barrett and B. W. Westbury, Invariants of piecewise-linear 3-manifolds, *Trans. Amer. Math. Soc.* **348**, 3997 (1996).
 - [9] K. Walker, Tqfts, <http://canyon23.net/math/tc.pdf>.
 - [10] M. A. Levin and X.-G. Wen, String-net condensation: A physical mechanism for topological phases, *Phys. Rev. B* **71**, 045110 (2005).
 - [11] R. Koenig, G. Kuperberg, and B. W. Reichardt, Quantum computation with Turaev-Viro codes, *Ann. Phys.* **325**, 2707 (2010).
 - [12] A. G. Fowler, M. Mariantoni, J. M. Martinis, and A. N. Cleland, Surface codes: Towards practical large-scale quantum computation, *Phys. Rev. A* **86**, 032324 (2012).
 - [13] N. E. Bonesteel and D. P. DiVincenzo, Quantum circuits for measuring Levin-Wen operators, *Phys. Rev. B* **86**, 165113 (2012).
 - [14] M. Barkeshli, P. Bonderson, M. Cheng, and Z. Wang, Symmetry fractionalization, defects, and gauging of topological phases, *Phys. Rev. B* **100**, 115147 (2019).
 - [15] I. Cong, M. Cheng, and Z. Wang, Topological quantum computation with gapped boundaries, [arXiv:1609.02037](https://arxiv.org/abs/1609.02037).
 - [16] E. Witten, Quantum field theory and the jones polynomial, *Commun. Math. Phys.* **121**, 351 (1989).
 - [17] V. Turaev, *Quantum Invariants of Knots and 3-manifolds* (Walter de Gruyter, Berlin, 1994).
 - [18] X. Wen, Topological orders in rigid states, *Int. J. Mod. Phys. B* **4**, 239 (1990).
 - [19] Y. You and M. Cheng, Measuring modular matrices by shearing lattices, [arXiv:1502.03192](https://arxiv.org/abs/1502.03192).
 - [20] M. Barkeshli, Charge $2e/3$ Superconductivity and Topological Degeneracies without Localized Zero Modes in Bilayer Fractional Quantum Hall States, *Phys. Rev. Lett.* **117**, 096803 (2016).
 - [21] P. Bonderson, M. Freedman, and C. Nayak, Measurement-only topological quantum computation via anyonic interferometry, *Ann. Phys.* **324**, 787 (2009).
 - [22] P. Bonderson, Measurement-only topological quantum computation via tunable interactions, *Phys. Rev. B* **87**, 035113 (2013).
 - [23] M. Barkeshli and M. Freedman, Modular transformations through sequences of topological charge projections, *Phys. Rev. B* **94**, 165108 (2016).
 - [24] A. Lavasani and M. Barkeshli, Low overhead Clifford gates from joint measurements in surface, color, and hyperbolic codes, *Phys. Rev. A* **98**, 052319 (2018).
 - [25] A. G. Fowler, Time-optimal quantum computation, [arXiv:1210.4626](https://arxiv.org/abs/1210.4626).
 - [26] M. Hastings (private communication).
 - [27] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, Cambridge, 2010).
 - [28] B. Eastin and E. Knill, Restrictions on Transversal Encoded Quantum Gate Sets, *Phys. Rev. Lett.* **102**, 110502 (2009).
 - [29] M. E. Beverland, O. Buerschaper, R. Koenig, F. Pastawski, J. Preskill, and S. Sijher, Protected gates for topological quantum field theories, *J. Math. Phys.* **57**, 022201 (2016).
 - [30] S. Bravyi and R. König, Classification of Topologically Protected Gates for Local Stabilizer Codes, *Phys. Rev. Lett.* **110**, 170503 (2013).
 - [31] G. Zhu, M. Hafezi, and M. Barkeshli, Quantum origami: Transversal gates for quantum computation and measurement of topological order, *Phys. Rev. Research* **2**, 013285 (2020).
 - [32] R. Bowler, J. Gaebler, Y. Lin, T. R. Tan, D. Hanneke, J. D. Jost, J. P. Home, D. Leibfried, and D. J. Wineland, Coherent Diabatic Ion Transport and Separation in a Multizone Trap Array, *Phys. Rev. Lett.* **109**, 080502 (2012).
 - [33] A. Walther, F. Ziesel, T. Ruster, S. T. Dawkins, K. Ott, M. Hettrich, K. Singer, F. Schmidt-Kaler, and U. Poschinger, Controlling Fast Transport of Cold Trapped Ions, *Phys. Rev. Lett.* **109**, 080501 (2012).

- [34] K. Wright, J. M. Amini, D. L. Faircloth, C. Volin, S. Charles Doret, H. Hayden, C.-S. Pai, D. W. Landgren, D. Denison, T. Killian, R. E. Slusher, and A. W. Harter, Reliable transport through a microfabricated X-junction surface-electrode ion trap, *New J. Phys.* **15**, 033004 (2013).
- [35] S. Bravyi and A. Kitaev, Universal quantum computation with ideal Clifford gates and noisy ancillas, *Phys. Rev. A* **71**, 022316 (2005).
- [36] A. Paetznick and B. W. Reichardt, Universal Fault-Tolerant Quantum Computation with Only Transversal Gates and Error Correction, *Phys. Rev. Lett.* **111**, 090505 (2013).
- [37] C. Jones, P. Brooks, and J. Harrington, Gauge color codes in two dimensions, *Phys. Rev. A* **93**, 052332 (2016).
- [38] S. Bravyi and A. Cross, Doubled color codes, [arXiv:1509.03239](https://arxiv.org/abs/1509.03239).
- [39] H. Bombin, Gauge color codes: optimal transversal gates and gauge fixing in topological stabilizer codes, *New J. Phys.* **17**, 083002 (2015).
- [40] H. Bombin, Dimensional jump in quantum error correction, *New J. Phys.* **18**, 043038 (2016).
- [41] T. Jochym-O'Connor and S. D. Bartlett, Stacked codes: Universal fault-tolerant quantum computation in a two-dimensional layout, *Phys. Rev. A* **93**, 022323 (2016).
- [42] T. J. Yoder, R. Takagi, and I. L. Chuang, Universal Fault-Tolerant Gates on Concatenated Stabilizer Codes, *Phys. Rev. X* **6**, 031039 (2016).
- [43] G. Zhu, A. Lavasani, and M. Barkeshli, Universal Fault-Tolerant Logical Gates with Constant Time Overhead for Quantum Computation, *Phys. Rev. Lett.* **125**, 050502 (2020).
- [44] M. Barkeshli, P. Bonderson, M. Cheng, C.-M. Jian, and K. Walker, Reflection and time reversal symmetry enriched topological phases of matter: path integrals, non-orientable manifolds, and anomalies, *Commun. Math. Phys.* **374**, 1021 (2020).
- [45] B. Farb and D. Margalit, *A Primer on Mapping Class Groups (PMS-49)* (Princeton University Press, Princeton, NJ, 2011).
- [46] J. P. Home, D. Hanneke, J. D. Jost, J. M. Amini, D. Liebfried, and D. J. Wineland, Complete methods set for scalable ion trap quantum information processing, *Science* **325**, 1224 (2009).
- [47] B. Lekitsch, S. Weidt, A. G. Fowler, K. Mølmer, S. J. Devitt, C. Wunderlich, and W. K. Hensinger, Blueprint for a microwave trapped ion quantum computer, *Sci. Adv.* **3**, e1601540 (2017).
- [48] H. Kaufmann, T. Ruster, C. T. Schmiegelow, M. A. Luda, V. Kaushal, J. Schulz, D. von Lindenfels, F. Schmidt-Kaler, and U. G. Poschinger, Scalable Creation of Long-Lived Multipartite Entanglement, *Phys. Rev. Lett.* **119**, 150503 (2017).
- [49] M. H. Freedman, M. Larsen, and Z. H. Wang, A modular functor which is universal for quantum computation, *Commun. Math. Phys.* **227**, 605 (2002).
- [50] N. E. Bonesteel, L. Hormozi, G. Zikos, and S. H. Simon, Braid Topologies for Quantum Computation, *Phys. Rev. Lett.* **95**, 140503 (2005).
- [51] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, Topological quantum memory, *J. Math. Phys.* **43**, 4452 (2002).
- [52] M. Aguado and G. Vidal, Entanglement Renormalization and Topological Order, *Phys. Rev. Lett.* **100**, 070404 (2008).
- [53] N. P. Breuckmann, C. Vuillot, E. Campbell, A. Krishna, and B. M. Terhal, Hyperbolic and semi-hyperbolic surface codes for quantum storage, *Quantum Sci. Technol.* **2**, 035007 (2017).
- [54] G. Vidal, Entanglement Renormalization, *Phys. Rev. Lett.* **99**, 220405 (2007).
- [55] W. Feng, Non-Abelian quantum error correction, Ph.D. thesis, Florida State University, 2015.
- [56] W. Feng, N. E. Bonesteel, and D. P. DiVincenzo (unpublished).
- [57] S. Burton, C. G. Brell, and S. T. Flammia, Classical simulation of quantum error correction in a Fibonacci anyon code, *Phys. Rev. A* **95**, 022309 (2017).
- [58] G. Dauphinais and D. Poulin, Fault-Tolerant Quantum Error Correction for non-Abelian Anyons, *Commun. Math. Phys.* **355**, 519 (2017).
- [59] E. H. Lieb and D. W. Robinson, *Statistical Mechanics* (Springer, Berlin, 1972), pp. 425–431.
- [60] B. Zeng, A. Cross, and I. L. Chuang, Transversality versus universality for additive quantum codes, *IEEE Trans. Inf. Theory* **57**, 6272 (2011).
- [61] N. M. Linke, D. Maslov, M. Roetteler, S. Debnath, C. Figgatt, K. A. Landsman, K. Wright, and C. Monroe, Experimental comparison of two quantum computing architectures, *Proc. Natl. Acad. Sci. USA* **114**, 3305 (2017).
- [62] P. Campagne-Ibarcq, E. Zalys-Geller, A. Narla, S. Shankar, P. Reinhold, L. D. Burkhardt, C. J. Axline, W. Pfaff, L. Frunzio, R. J. Schoelkopf, and M. H. Devoret, Deterministic Remote Entanglement of Superconducting Circuits Through Microwave Two-Photon Transitions, *Phys. Rev. Lett.* **120**, 200501 (2018).
- [63] P. Kurpiers, P. Magnard, T. Walter, B. Royer, M. Pechal, J. Heinsoo, Y. Salathé, A. Akin, S. Storz, J.-C. Besse, S. Gasparinetti, A. Blais, and A. Wallraff, Deterministic quantum state transfer and generation of remote entanglement using microwave photons, *Nature (London)* **558**, 264 (2018).
- [64] C. Axline, L. Burkhardt, W. Pfaff, M. Zhang, K. Chou, P. Campagne-Ibarcq, P. Reinhold, L. Frunzio, S. M. Girvin, L. Jiang, M. H. Devoret, and R. J. Schoelkopf, On-demand quantum state transfer and entanglement between remote microwave cavity memories, *Nat. Phys.* **14**, 705 (2018).
- [65] K. S. Chou, J. Z. Blumoff, C. S. Wang, P. C. Reinhold, C. J. Axline, Y. Y. Gao, L. Frunzio, M. H. Devoret, L. Jiang, and R. J. Schoelkopf, Deterministic teleportation of a quantum gate between two logical qubits, *Nature (London)* **561**, 368 (2018).
- [66] A. Blais, J. Gambetta, A. Wallraff, D. I. Schuster, S. M. Girvin, M. H. Devoret, and R. J. Schoelkopf, Quantum-information processing with circuit quantum electrodynamics, *Phys. Rev. A* **75**, 032329 (2007).
- [67] R. J. Schoelkopf and S. M. Girvin, Wiring up quantum systems, *Nature (London)* **451**, 664 (2008).
- [68] J. Majer, J. M. Chow, J. M. Gambetta, J. Koch, B. R. Johnson, J. A. Schreier, L. Frunzio, D. I. Schuster, A. A. Houck, A. Wallraff, A. Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, Coupling superconducting qubits via a cavity bus, *Nature (London)* **449**, 443 (2007).
- [69] F. Helmer, M. Mariantoni, A. G. Fowler, J. von Delft, E. Solano, and F. Marquardt, Cavity grid for scalable quantum computation with superconducting circuits, *Europhys. Lett.* **85**, 50007 (2009).
- [70] A. A. Houck, H. E. Tureci, and J. Koch, On-chip quantum simulation with superconducting circuits, *Nat. Phys.* **8**, 292 (2012).

- [71] G. Zhu, S. Schmidt, and J. Koch, Dispersive regime of the Jaynes–Cummings and Rabi lattice, *New J. Phys.* **15**, 115002 (2013).
- [72] M. Fitzpatrick, N. M. Sundaresan, A. C. Y. Li, J. Koch, and A. A. Houck, Observation of a Dissipative Phase Transition in a One-Dimensional Circuit QED Lattice, *Phys. Rev. X* **7**, 011016 (2017).
- [73] R. K. Naik, N. Leung, S. Chakram, P. Groszkowski, Y. Lu, N. Earnest, D. C. McKay, J. Koch, and D. I. Schuster, Random access quantum information processors using multi-mode circuit quantum electrodynamics, *Nat. Commun.* **8**, 1904 (2017).
- [74] G. Zhu, Y. Subaşı, J. D. Whitfield, and M. Hafezi, Hardware-efficient fermionic simulation with a cavity–QED system, *npj Quantum Inf.* **4**, 16 (2018).
- [75] M. Saffman, T. G. Walker, and K. Mølmer, Quantum information with Rydberg atoms, *Rev. Mod. Phys.* **82**, 2313 (2010).
- [76] D. Comparat and P. Pillet, Dipole blockade in a cold Rydberg atomic sample [Invited], *J. Opt. Soc. Am. B* **27**, A208 (2010).
- [77] K. M. Maller, M. T. Lichtman, T. Xia, Y. Sun, M. J. Piotrowicz, A. W. Carr, L. Isenhower, and M. Saffman, Rydberg-blockade controlled-NOT gate and entanglement in a two-dimensional array of neutral-atom qubits, *Phys. Rev. A* **92**, 022336 (2015).
- [78] H. Pichler, G. Zhu, A. Seif, P. Zoller, and M. Hafezi, Measurement Protocol for the Entanglement Spectrum of Cold Atoms, *Phys. Rev. X* **6**, 041033 (2016).
- [79] C. Cabrillo, J. I. Cirac, P. Garcia-Fernandez, and P. Zoller, Creation of entangled states of distant atoms by interference, *Phys. Rev. A* **59**, 1025 (1999).
- [80] S. Bose, P. L. Knight, M. B. Plenio, and V. Vedral, Proposal for Teleportation of an Atomic State via Cavity Decay, *Phys. Rev. Lett.* **83**, 5158 (1999).
- [81] L. Crane and D. Yetter, in *Quantum Topology*, edited by L. Kauffman and R. Baadhio (World Scientific, Singapore, 1993).
- [82] K. Walker and Z. Wang, (3+1)-TQFTs and topological insulators, *Front. Phys.* **7**, 150 (2012).
- [83] C. Vafa, Toward classification of conformal theories, *Phys. Lett. B* **206**, 421 (1988).
- [84] C.-H. Lin and M. Levin, Generalizations and limitations of string-net models, *Phys. Rev. B* **89**, 195130 (2014).