Localization of the first eigenfunction of a convex domain

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Abstract

We study the first Dirichlet eigenfunction of the Laplacian in a n-dimensional convex domain. For domains of a fixed inner radius, estimates of Chiti [11], [12], imply that the ratio of the L^2-norm and L^∞-norm of the eigenfunction is minimized when the domain is a ball. However, when the eccentricity of the domain is large the eigenfunction should spread out at a certain scale and this ratio should increase. We make this precise by obtaining a lower bound on the L^2-norm of the eigenfunction and show that the eigenfunction cannot localize to too small a subset of the domain. As a consequence, we settle a conjecture of van den Berg, [4], in the general n-dimensional case. The main feature of the proof is to obtain sufficiently sharp estimates on the first eigenvalue in order to estimate the first derivatives of the eigenfunction.

1 Introduction and statement of results

Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain and let λ be the first eigenvalue of the Dirichlet Laplacian on Ω. We denote the corresponding eigenfunction by u so that

$$\begin{cases} (\Delta + \lambda)u = 0 \text{ in } \Omega \\ \qquad\quad u = 0 \text{ on } \partial\Omega. \end{cases}$$

This first eigenfunction is of one sign, and we choose it so that $u(x) > 0$ in Ω. Our starting point for studying the behaviour of u and its level sets is that the convexity of Ω ensures that u is log-concave, [8]. In particular the superlevel sets

$$\{x \in \Omega : u(x) > c\}$$

are convex subsets of Ω. It is natural to study the shape of the level sets of u and how they depend on the geometry of Ω and the level under consideration. The quantity $|u(x)|^2$ can be interpreted as an (unnormalized) density for a free quantum particle in the domain Ω. The shape and location of the superlevel sets where u is comparable to its maximum value therefore correspond to the parts of Ω where the particle is most likely to be found. In this paper, we will obtain a lower bound on the $L^2(\Omega)$-norm of u in terms of its $L^\infty(\Omega)$-norm and length scales coming from the shape of Ω (see Theorem 1.1 below). The regions of Ω where Laplace eigenfunctions are of large magnitude relative to the rest of the domain, has received recent attention. For example, the torsion function has been used as a landscape function for predicting these regions of Ω, [13], [1], [22]. In [5], an upper bound on the efficiency ratio is given for a class of horn-shaped domains. This efficiency ratio is a weighted measure of the ratio of the L^1 and L^∞ norms of the first eigenfunction, and they then use this upper bound to provide sequences of domains Ω_m where the first eigenfunction u_m localizes to a small subset of the domain. That is, they demonstrate a sequence of domains Ω_m, and measurable sets $A_m \subset \Omega_m$, such that $\lim_{m\to\infty} |A_m|/|\Omega_m| = 0$, $\|u_m\|_{L^2(\Omega_m)} = 1$, and

$$\lim_{m\to\infty} \int_{A_m} |u_m|^2 = 1. \tag{1}$$

While in general the first eigenfunction can localize to a small subset of Ω relative to Ω itself, in the above sense, our result will place a restriction on how small this region can be. We will do this by studying lower bounds on the $L^2(\Omega)$-norm of the eigenfunction.

In [11], [12], Chiti provides a lower bound on the $L^2(\Omega)$-norm of u of the form

$$\|u\|_{L^2(\Omega)} \geq c_n^* \ \mathrm{inrad}(\Omega)^{n/2} \, \|u\|_{L^\infty(\Omega)} \tag{2}$$

Here $\mathrm{inrad}(\Omega)$ is the inner radius of Ω. In [10], a lower bound on the $L^n(\Omega)$-norm of the eigenfunction in its contact set is also given in terms of its $L^\infty(\Omega)$-norm, which is then used to show that the maximum of the eigenfunction is in the *heart* of the convex domain. See also [9] for further properties of the heart of a convex domain. The constant $c_n^* > 0$ in (2) depends only on the dimension, and is explicitly given in terms of Bessel functions (and their zeros). (In fact, this bound holds for any bounded, connected domain Ω.) Moreover, the constant c_n^* cannot be improved since equality in (2) holds when Ω is a ball. However, for Ω convex and when the diameter of Ω is large compared to its inner radius, one expects the eigenfunction to *spread out* along the diameter of Ω, and for the $L^2(\Omega)$-norm to increase relative to the $L^\infty(\Omega)$-norm. In terms of the estimate in (2), the question is then whether an estimate of the form

$$\|u\|_{L^2(\Omega)} \geq c_n \left(\mathrm{diam}(\Omega)/\mathrm{inrad}(\Omega)\right)^{\alpha} \mathrm{inrad}(\Omega)^{n/2} \, \|u\|_{L^\infty(\Omega)} \tag{3}$$

holds for all convex Ω, and some uniform $\alpha > 0$. Repeated applications of the Harnack inequality in overlapping balls is not sufficient to establish (3) for any $\alpha > 0$, and so any improvement of (2) must use the fact that u is an eigenfunction in a fundamental way. Kröger, [20], in two dimensions, and van den Berg, [4], in higher dimensions studied the first eigenfunction of a thin sector. Via a separation of variables in polar coordinates, and the properties of the resulting Bessel function in the radial variable, this example of the sector ensures that the maximal value of α for which (3) could hold is $\alpha = \frac{1}{6}$. Based on the intuition that the sector should be the convex domain for which the eigenfunction spreads out the least, van den Berg made the following conjecture:

Conjecture 1 ([4]) *There exists a constant $c_n > 0$, depending only on the dimension n, such that*

$$\|u\|_{L^2(\Omega)} \geq c_n \left(\mathrm{diam}(\Omega)/\mathrm{inrad}(\Omega)\right)^{1/6} \mathrm{inrad}(\Omega)^{n/2} \, \|u\|_{L^\infty(\Omega)} .$$

The two dimensional case of this conjecture has been established in [15]. Their proof uses an eigenvalue bound for the first eigenvalue of a class of one dimensional Schrödinger operators, and the work of Grieser and Jerison, [18], [17], on the first eigenfunction of a convex, planar domain.

In this paper, we bound $\|u\|_{L^2(\Omega)}$ from below in the general n-dimensional case. We call K a John ellipsoid associated to $\Omega \subset \mathbb{R}^n$ if K is an open ellipsoid contained within Ω and any other ellipsoid contained within Ω has volume at most that of K. John's lemma [19] ensures that such an ellipsoid K exists, is unique, and the dilation of K about its centre with scaling factor n contains Ω. We now fix the John ellipsoid K and define N_j to be the lengths of the axes of K with

$$N_1 \geq N_2 \geq \cdots \geq N_n .$$

Our main theorem provides a lower bound on the scale at which the eigenfunction can localize by establishing a lower bound on the $L^2(\Omega)$-norm of u in terms of its $L^\infty(\Omega)$-norm, and the length scales N_j.

Theorem 1.1 *There exists a constant $c_n > 0$, depending only on the dimension n, such that*

$$\|u\|_{L^2(\Omega)} \geq c_n N_n^{n/2} \prod_{j=1}^{n-1} \left(N_j/N_n\right)^{1/6} \|u\|_{L^\infty(\Omega)} .$$

In particular, $\prod_{j=1}^{n-1} \left(N_j/N_n\right)^{1/6} \geq \left(N_1/N_n\right)^{1/6}$, and by the properties of the John ellipsoid,

$$N_1 \leq \mathrm{diam}(\Omega) \leq nN_1, \qquad N_n \leq \mathrm{inrad}(\Omega) \leq nN_n .$$

Therefore, Theorem 1.1 settles Conjecture 1. In two dimensions, our methods can obtain an explicit constant for c_2 (see Theorem 1.2 below).

Remark 1.1 *Let $M_1 \geq M_2 \geq \cdots \geq M_n$ be the lengths of the axes of a John ellipsoid for the superlevel set $\{x \in \Omega : u(x) > \frac{1}{2}\max_\Omega u\}$. In the course of proving Theorem 1.1 we will show that $M_j \geq c_n(N_j/N_n)^{1/3}N_n$ for some constant $c_n > 0$. In terms of the localization statement in (1), this theorem places a restriction on how small the sets A_m where the eigenfunctions localize can be.*

To prove Theorem 1.1 we first obtain an upper bound on the directional derivatives of u in terms of the length scales N_j. After a rotation we will assume that the axes of K lie along the coordinate axes.

Theorem 1.2 *For each j, $1 \leq j \leq n$, there exists a constant $C_{j,n}$, depending only on j and the dimension n, such that the derivative $\partial_{x_j} u(x)$ satisfies*

$$\left\|\partial_{x_j} u\right\|_{L^2(\Omega)} \leq C_{j,n} N_n^{-1} \left(N_j/N_n\right)^{-1/3} \|u\|_{L^2(\Omega)}.$$

In two dimensions, the constants $C_{1,2}$, $C_{2,2}$ have the explicit upper bounds

$$C_{1,2} \leq (33)^{1/2}\pi, \qquad C_{2,2} \leq \alpha_{0,1},$$

where $\alpha_{0,1}$ is the first zero of the Bessel function $J_0(r)$. For $n = 2$, the constant c_2 appearing in Theorem 1.1 has an explicit lower bound in terms of $C_{1,2}$ and $C_{2,2}$ of

$$c_2 \geq \tfrac{1}{96} \cdot \tfrac{1}{16^2} C_{1,2}^{-1/4} C_{2,2}^{-1/4}.$$

Remark 1.2 *If we denote u_m to be the m-th Dirichlet eigenfunction of Ω, then the estimate in Theorem 1.2 continues to hold, with a constant $C_{j,n}$ replaced by a constant $C_{m,j,n}$ depending only on m, j, and n.*

Via a dilation we can also assume that $N_n = 1$ when proving these theorems, and by taking a constant multiple of u, we also assume that $\max_\Omega u = 1$. To prove Theorem 1.2 we will begin by using the eigenfunction equation to write

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}x = \lambda \int_\Omega |u|^2 \,\mathrm{d}x,$$

and we will also use the variational formulation of the first eigenvalue,

$$\lambda = \inf\left\{\frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x}{\int_\Omega |v|^2 \,\mathrm{d}x} : v \in H_0^1(\Omega), v \neq 0\right\}.$$

We combine these to prove an upper bound on the eigenvalue λ in terms of the eigenvalues of the cross-sections of Ω (see Proposition 2.2 for the precise statement). This then reduces the proof of Theorem 1.2 to obtaining sufficiently sharp upper bounds on the eigenvalues of $(n-j)$-dimensional cross-sections of Ω. We prove the desired eigenvalue bounds by induction on j, and will carry out the proof in Section 2. To prove Theorem 1.1, we will also use in a crucial way the log concavity of the eigenfunction u, [8]. In particular, this will allow us to reduce estimating the $L^2(\Omega)$-norm of u to estimating the lengths of the axes of a John ellipsoid associated to the superlevel set

$$\Omega_{1/2} = \left\{x \in \Omega : u(x) > \tfrac{1}{2}\right\}.$$

The desired estimate follows from using the derivative bounds in Theorem 1.2, and we will prove Theorem 1.1 in Section 3. Finally, in Section 4 we discuss known estimates in the two dimensional case, and future directions in higher dimensions. In [18], Jerison introduces a length scale L depending on the geometry of the convex, planar domain, and together with Grieser uses it to study the shape of the first (and second) eigenfunction, [17], [16]. In particular, their results imply comparable upper and lower bounds on $\|u\|_{L^2(\Omega)}$ in terms of this length scale L. It is natural to ask how to construct analogous length scales controlling the shape of the first eigenfunction in higher dimensions, and in Section 4 we discuss this in more detail.

Remark 1.3 *Throughout, constants which we will denote by C, C_1, c_1 etc, are constants which depend only on the dimension. We also say that two quantities are comparable (and write as $\sim$) if they can be bounded in terms of each other up to a constant depending only on n.*

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2 Gradient bounds for the eigenfunction

In this section we prove Theorem 1.2. The key step in the proof is to obtain appropriate upper bounds on the eigenvalue λ. In fact, we will carry out an inductive step, which will require estimates on the first Dirichlet eigenvalue of $(n-k)$-dimensional cross-sections of Ω for $0 \leq k \leq n-1$. To write down the eigenvalue bounds that we will establish, we first introduce the following notation: Given i, with $1 \leq i \leq n-k-1$, and a point $x \in \mathbb{R}^{n-k}$, we write

$$x = (x_1, x_2, \dots, x_{n-k}) = (X_i, X'_{n-k-i}) \in \mathbb{R}^{n-k},$$

with $X_i \in \mathbb{R}^i$, $X'_{n-k-i} \in \mathbb{R}^{n-k-i}$. Now let W be a $(n-k)$-dimensional convex domain. For each $Y_i \in \mathbb{R}^i$, we denote the $(n-k-i)$-dimensional cross-sections of W by

$$W(Y_i) = \left\{ x = (X_i, X'_{n-k-i}) \in W : X_i = Y_i \right\} \subset \mathbb{R}^{n-k}.$$

For us, W will either be the original convex domain Ω (with $k=0$) or a $(n-k)$-dimensional cross-section of Ω, for some $1 \leq k \leq n-1$. The sets $W(Y_i)$ can also be viewed as $(n-k-i)$-dimensional convex domains in $\mathbb{R}^{n-k-i}$, and this is how we will view them below in order to study the Dirichlet eigenvalue problem on $W(Y_i)$.

Definition 2.1 *For a $(n-k)$-dimensional convex domain W, let $\lambda(W)$ be its first Dirichlet eigenvalue. For i, with $1 \leq i \leq n-k-1$, and $Y_i \in \mathbb{R}^i$, let $\mu(Y_i; W)$ be the first Dirichlet eigenvalue of $W(Y_i)$, and define $\mu_i^*(W)$ by*

$$\mu_i^*(W) = \min_{Y_i} \mu(Y_i; W). \tag{4}$$

We also formally define $\mu_{n-k}^*(W) = 0$, and then for $1 \leq i \leq n-k$ set

$$\delta_i(W) = \lambda(W) - \mu_i^*(W).$$

We can obtain gradient bounds on the first Dirichlet eigenfunction of W in terms of $\delta_i(W)$ via the following proposition.

Proposition 2.2 *Let $u_W(x)$ be the first Dirichlet eigenfunction of W. Then, for each $1 \leq i \leq n-k$, with $\delta_i(W)$ as in Definition 2.1, the gradient bounds*

$$\sum_{\ell=1}^{i} \int_W \left|\partial_{x_\ell} u_W(x)\right|^2 \, \mathrm{d}x \leq \delta_i(W) \int_W \left|u_W(x)\right|^2 \, \mathrm{d}x$$

hold. In particular, $\delta_i(W) \geq 0$ for all i.

Proof of Proposition 2.2: Since u_W is a Dirichlet eigenfunction with eigenvalue $\lambda(W)$ we have

$$\int_W \left|\nabla u_W(x)\right|^2 \, \mathrm{d}x = \lambda(W) \int_W |u_W(x)|^2 \, \mathrm{d}x. \tag{5}$$

For $i = n-k$, we have $\delta_{n-k}(W) = \lambda(W)$ and then the estimate holds (with equality) immediately. We now fix i with $1 \leq i < n-k$. For each $X_i \in \mathbb{R}^i$ such that $W(X_i)$ is non-empty, the function $u_W(X_i, \cdot)$ is an admissible test function for the first eigenvalue on $W(X_i)$. Therefore,

$$\begin{aligned} \sum_{\ell=i+1}^{n-k} \int_{W(X_i)} \left|\partial_{x_\ell} u_W(X_i, X'_{n-k-i})\right|^2 \mathrm{d}X'_{n-k-i} &\geq \mu(X_i; W) \int_{W(X_i)} \left|u_W(X_i, X'_{n-k-i})\right|^2 \mathrm{d}X'_{n-k-i} \\ &\geq \mu_i^*(W) \int_{W(X_i)} \left|u_W(X_i, X'_{n-k-i})\right|^2 \mathrm{d}X'_{n-k-i}. \end{aligned}$$

Since this holds for each X_i, we integrate in X_i and then use it in (5) to get

$$\mu_i^*(W)\int_W |u_W(x)|^2\,\mathrm{d}x + \sum_{\ell=1}^{i}\int_W |\partial_{x_\ell}u_W(x)|^2\,\mathrm{d}x \le \lambda(W)\int_W |u_W(x)|^2\,\mathrm{d}x.$$

The estimate in the proposition then follows from the definition of $\delta_i(W)$. □

As before, we set $u(x) = u_\Omega(x)$, $\lambda = \lambda(\Omega)$, and for ease of notation, we write $\mu(Y_i;\Omega) = \mu(Y_i)$, $\mu_i^* = \mu_i^*(\Omega)$. Using Proposition 2.2, in order to prove Theorem 1.2 it is sufficient to establish the following eigenvalue bounds.

Proposition 2.3 *For all j, $1 \le j \le n$, there exists a constant $C_{j,n}$ such that*

$$\mu_j^* \le \lambda \le \mu_j^* + C_{j,n}^2 N_j^{-2/3}.$$

From Proposition 2.2 we have $\lambda - \mu_j^* \ge 0$, and so we only need to prove the upper bound. Since $\mu_n^* = 0$, and Ω has inner radius comparable to $N_n = 1$, the estimate in the proposition certainly holds for $j = n$. This is because the Dirichlet eigenvalues are monotonic with respect to inclusion. We will prove Proposition 2.3 by induction on j (starting with $j = n$ as the base case, and then decreasing j). To establish the inductive step we will use the variational formulation of the first Dirichlet eigenvalue. We will construct an appropriate test function involving the eigenfunctions corresponding to the minimal eigenvalue μ_j^* of the j-dimensional cross-sections of Ω. To demonstrate the method let us first use it to prove the proposition in the two dimensional case. In two dimensions, the estimate in Proposition 2.3 is also contained in the work of Jerison [18] and Grieser-Jerison [17]. The estimate in Proposition 2.3 for $j = n-1$ is also given in [6] (see the proof of Theorem 1.5 and in particular equation (4.8)) and [7] (see the proof of Theorem 1.1 and in particular equation (2.2)). There, explicit eigenvalue bounds are found using the monotonicity of Dirichlet eigenvalues with respect to inclusion.

Proof of Proposition 2.3 in two dimensions: In the two dimensional case, we just need to consider $j = 1$. After a translation along the x_1-axis, we may assume that the minimal value $\mu_1^* = \mu_1(Y_1)$ is attained at $Y_1 = 0$. (Note that this point is at a point where the height of the domain Ω in the x_2-direction is largest.) Let $\psi(x_2)$ be the corresponding $L^2(\Omega(0))$-normalized first Dirichlet eigenfunction of the interval $\Omega(0)$, extended to be zero outside of $\Omega(0)$. By the properties of the John ellipsoid of Ω, we can find a point $x = (x_1, x_2) \in \Omega$ with $|x_1| = N_1$, and so without loss of generality, we assume that $x^* = (N_1, x_2^*) \in \Omega$ for some x_2^*. By translating in the x_2-direction we may assume that $x_2^* = 0$, and after this translation there still exists a constant C such that $|x_2| \le C$ on the support of $\psi(x_2)$.

We now define a test function that we can use in the variational formulation of the first eigenvalue λ: We set $v(x_1, x_2)$ to be the function

$$v(x_1,x_2) = \chi(x_1)\psi\left(x_2 N_1/(N_1 - x_1)\right). \tag{6}$$

Here $\chi(x_1) \ge 0$ is a smooth cut-off function, such that

$$\begin{aligned}
&\chi(x_1) = 1 \text{ for } \tfrac{1}{2}N_1^{1/3} \le x_1 \le N_1^{1/3},\\
&\chi(x_1) = 0 \text{ for } x_1 \ge 2N_1^{1/3},\ x_1 \le \tfrac{1}{4}N_1^{1/3}.
\end{aligned}$$

The function $\chi(x_1)$ can in particular be chosen so that $|\chi'(x_1)| \le CN_1^{-1/3}$. The domain Ω contains the interval $\Omega(0)$ and the point $x^* = (N_1, 0)$, and so also contains the convex hull of these two sets. Therefore, for each $x_1 \in [0, N_1]$, the cross-section $\Omega(x_1)$ contains the interval $\frac{N_1 - x_1}{N_1}\Omega(0)$. In particular, this ensures that $v(x_1, x_2)$ is equal to zero on the complement of Ω, and we can use it in the variational formulation of the first eigenvalue λ. That is,

$$\lambda \le \frac{\int_\Omega |\nabla v(x)|^2\,\mathrm{d}x}{\int_\Omega |v(x)|^2\,\mathrm{d}x}. \tag{7}$$

We can write the right hand side of (7) as

$$\frac{\int_\Omega \chi(x_1)^2 \frac{N_1^2}{(N_1-x_1)^2} \left|\psi'\left(x_2 N_1/(N_1-x_1)\right)\right|^2 \, \mathrm{d}x}{\int_\Omega \chi(x_1)^2 \left|\psi\left(x_2 N_1/(N_1-x_1)\right)\right|^2 \, \mathrm{d}x} + \frac{\int_\Omega \left|\partial_{x_1} v(x)\right|^2 \, \mathrm{d}x}{\int_\Omega |v(x)|^2 \, \mathrm{d}x},$$

and on the support of $\chi(x_1)$ we have

$$\left|\frac{N_1}{N_1-x_1} - 1\right| \leq C N_1^{-2/3}. \tag{8}$$

Therefore, since $\psi(x_2)$ is an eigenfunction on $\Omega(0)$ with eigenvalue μ_1^*, we have

$$\lambda \leq \mu_1^* + C N_1^{-2/3} + \frac{\int_\Omega \left|\partial_{x_1} v(x)\right|^2 \, \mathrm{d}x}{\int_\Omega |v(x)|^2 \, \mathrm{d}x}. \tag{9}$$

The x_1-derivative of v is given by

$$\partial_{x_1} v(x_1, x_2) = \chi'(x_1)\psi\left(x_2 N_1/(N_1-x_1)\right) - \chi(x_1)\frac{x_2 N_1}{(N_1-x_1)^2}\psi'\left(x_2 N_1/(N_1-x_1)\right). \tag{10}$$

We have $|\chi'(x_1)| \leq C N_1^{-1/3}$, $|\psi'\left(x_2 N_1/(N_1-x_1)\right)| \leq C$, and $|x_2| \leq C$ on the support of ψ. Combining this with the estimate $N_1/(N_1-x_1)^2 \leq C N_1^{-1}$ on the support of $\chi(x_1)$, from (9) we obtain

$$\lambda \leq \mu_1^* + C N_1^{-2/3},$$

as required. □

Remark 2.1 *In the above proof, we could have chosen the cut-off function to have been adapted to an interval of length N_1^α for any $0 < \alpha < 1$. A larger value of α improves the estimate in (10), while a smaller value of α improves the estimate in (8). The exponent $\alpha = \frac{1}{3}$ is chosen to optimize the total overall error from these two estimates.*

Remark 2.2 *In the 2 dimensional case, it is straightforward to obtain an explicit estimate on the constants $C_{j,n}$ given in the statement of Theorem 1.2: For $j = 2$, since Ω contains a disc of radius 1, which has first eigenvalue given by the square of the first zero $\alpha_{0,1}$ of the Bessel function $J_0(r)$, we can set $C_{2,2} = \alpha_{0,1}$. To get an explicit estimate for $j = 1$, we first note that Ω contains an isosceles triangle with base of length $\pi\left(\mu_1^*\right)^{-1/2} \geq 1$ and height $\frac{1}{4}N_1$. Within this triangle is a rectangle of dimensions $N_1^{1/3}$ and $\pi\left(\mu_1^*\right)^{-1/2}\left(1 - 4N_1^{-2/3}\right)$, which has an explicit first Dirichlet eigenvalue leading to*

$$\begin{aligned}\lambda &\leq \mu_1^* \left(1 - 4N_1^{-2/3}\right)^{-2} + \pi^2 N_1^{-2/3} \\ &\leq \mu_1^* + \left(\frac{8\mu_1^*}{(1-4N_1^{-2/3})^2} + \pi^2\right) N_1^{-2/3}.\end{aligned}$$

Since $\mu_1^* \leq \pi^2$, this gives an explicit estimate for $C_{1,2}$ of

$$C_{1,2}^2 \leq \frac{8\pi^2}{(1/2)^2} + \pi^2 = 33\pi^2.$$

whenever $N_1^{2/3} \geq 8$. If $N_1^{2/3} < 8$, then we can use the estimate $\lambda \leq \alpha_{0,1}^2 \leq 33\pi^2 N_1^{-2/3}$ to obtain the same estimate for $C_{1,2}$.

We now prove the general case.

Proof of Proposition 2.3: We first recall that the estimate in the proposition holds for $j = n$, and that the lower bound holds for all j. We will prove the upper bound by induction on j, using $j = n$ as the base case. Our inductive hypothesis is that there exists constants C_j such that

$$\lambda \leq \mu_j^* + C_j N_j^{-2/3} \tag{11}$$

for $k+1 \leq j \leq n$, and we will prove that there exists a constant C_k such that (11) holds for $j = k$. Analogously to the two dimensional case, we will prove this estimate by using an appropriate test function in the variational formulation for λ. The minimal value μ_k^* is given by $\mu(Y_k)$ for some $Y_k \in \mathbb{R}^k$, and we let $\psi(X'_{n-k})$ be the $L^2(\Omega(Y_k))$-normalized first Dirichlet eigenfunction of the $(n-k)$-dimensional cross-section $\Omega(Y_k)$, and extended to be zero outside $\Omega(Y_k)$. (We recall that in our notation $X'_{n-k} = (x_{k+1}, x_{k+2}, \ldots, x_n)$.) Our test function will involve this eigenfunction, and we first use Proposition 2.2 to establish bounds on the components of the gradient of $\psi(X'_{n-k})$, under the inductive hypothesis.

Lemma 2.4 *Assuming that the estimate in (11) holds for j satisfying $k+1 \leq j \leq n$, there exists a constant C (depending on the constants C_j) so that for each such j in this range,*

$$\int_{\Omega(Y_k)} \left|\partial_{x_j}\psi(X'_{n-k})\right|^2 \, \mathrm{d}X'_{n-k} \leq C N_j^{-2/3} \int_{\Omega(Y_k)} \left|\psi(X'_{n-k})\right|^2 \, \mathrm{d}X'_{n-k} = C N_j^{-2/3}.$$

Proof of Lemma 2.4: The eigenfunction $\psi(X'_{n-k})$ on $\Omega(Y_k)$ has eigenvalue μ_k^*, and analogously to Definition 2.1, for $k+1 \leq j \leq n$, we define $\mu_{k,j}^*$ to be the minimum eigenvalue over all $(n-j)$-dimensional cross-sections of $\Omega(Y_k)$ in the X'_{n-j} variables. Since $\Omega(Y_k) \subset \Omega$, by the definitions of the minima $\mu_{k,j}^*$ and μ_j^* we automatically have

$$\mu_j^* \leq \mu_{k,j}^*.$$

Combining this with the inductive hypothesis in (11), for each $k+1 \leq j \leq n$ we obtain

$$\mu_k^* \leq \lambda \leq \mu_j^* + C_j N_j^{-2/3} \leq \mu_{k,j}^* + C_j N_j^{-2/3}. \tag{12}$$

Therefore, setting W to be the $(n-k)$-dimensional convex domain $\Omega(Y_k)$, and using the notation from Definition 2.1 we have

$$\delta_i(W) = \lambda(W) - \mu_i^*(W) = \mu_k^* - \mu_{k,i+k}^* \leq C_{i+k} N_{i+k}^{-2/3}.$$

for $1 \leq i \leq n-k$. The gradient bounds in the statement of the lemma then immediately follow from Proposition 2.2, using that $\psi(X'_{n-k})$ is $L^2(\Omega(Y_k))$-normalized. □

We now define the test function that we will use to bound λ. We first translate the domain Ω in the X_k-variables so that the point Y_k with $\mu(Y_k) = \mu_k^*$ is at the origin, which we denote by 0_k. Then, using the above notation, $\psi(X'_{n-k})$ is the first Dirichlet eigenfunction of the $(n-k)$-dimensional cross-section $\Omega(0_k)$. By the properties of the John ellipsoid of Ω, there exists a k-dimensional parallelepiped P of dimensions comparable to $N_1 \times N_2 \times \cdots \times N_k$ contained in the intersection of Ω with a k-dimensional plane $\{X'_{n-k} = \text{ constant}\}$. By translating Ω in the X'_{n-k} variables we will assume that this k-dimensional plane is $\{X'_{n-k} = 0'_{n-k}\}$. Note that after this translation, there exists a constant C such that

$$\operatorname{proj}_j(\Omega(0_k)) \subset \{|x_j| \leq C N_j\} \tag{13}$$

for $k+1 \leq j \leq n$. Here $\operatorname{proj}_j(\Omega(0_k))$ is the projection of $\Omega(0_k)$ onto the x_j-axis. Since Ω contains the above parallelepiped P, there exists a $(k-1)$-dimensional sphere contained in $\{X'_{n-k} = 0'_{n-k}\}$, centred at the origin 0_k in the X_k-variables, of radius R_1 with $R_1 \sim N_1$, and with the following property: There exists a direction $\mathbf{e}$ in the X_k-variables and number θ_k, with $\theta_k \sim N_k/N_1$, such that the subset, S_k, of the sphere making an angle at most θ_k with $\mathbf{e}$, is contained within Ω. (Note that in the case of $k = 1$, the sphere is 0-dimensional, and the above reduces to the existence of a point in Ω at a distance comparable to N_1 from the $(n-1)$-dimensional cross-section $\Omega(0_1)$.)

We now let Γ_k be the k-dimensional cone in the X_k-variables generated by the set S_k, with vertex at the origin 0_k. This cone Γ_k contains a k-dimensional cube of side length comparable to $N_k^{1/3}$, at a distance comparable to $N_1 N_k^{-2/3}$ from the origin. We can therefore define a cut-off function $\chi(X_k)$ adapted to this cube (so that $\chi(X_k) = 1$ in the middle half of the cube, and 0 outside the cube), with $|\nabla\chi(X_k)| \leq CN_k^{-1/3}$. Our test function is then

$$w(x) = w(X_k, X'_{n-k}) = \chi(X_k)\psi\left(X'_{n-k}R_1/(R_1 - r_k)\right). \tag{14}$$

Here $r_k = (x_1^2 + x_2^2 + \cdots + x_k^2)^{1/2}$ is the distance to the origin 0_k in the X_k-plane. Since Ω is convex, it contains the convex hull of the $(n-k)$-dimensional cross-section $\Omega(0_k)$ and the set S_k. Therefore, given $X_k \in S_k$, $s \in [0,1]$, the $(n-k)$-dimensional cross-section of Ω at $sX_k \in \Gamma_k$ contains the set

$$\left(\frac{R_1 - |sX_k|}{R_1}\right)\Omega(0_k) = (1-s)\Omega(0_k).$$

Thus, the test function $w(x)$ vanishes outside of Ω, and so can be used to obtain an upper bound on λ. We therefore have

$$\lambda \leq \frac{\int_\Omega |\nabla_{X_k} w(x)|^2 \,\mathrm{d}x}{\int_\Omega |w(x)|^2 \,\mathrm{d}x} + \frac{\int_\Omega \left|\nabla_{X'_{n-k}} w(x)\right|^2 \,\mathrm{d}x}{\int_\Omega |w(x)|^2 \,\mathrm{d}x}, \tag{15}$$

and we deal with each term separately. We can write the second term in (15) as

$$\frac{\int_\Omega \frac{R_1^2}{(R_1-r_k)^2}|\chi(X_k)|^2 \left|\left(\nabla_{X'_{n-k}}\psi\right)\left(X'_{n-k}R_1/(R_1-r_k)\right)\right|^2 \,\mathrm{d}x}{\int_\Omega |\chi(X_k)|^2 \left|\psi\left(X'_{n-k}R_1/(R_1-r_k)\right)\right|^2 \,\mathrm{d}x}, \tag{16}$$

and on the support of $\chi(X_k)$ we have

$$\left|\frac{R_1}{R_1 - r_k} - 1\right| \leq CN_k^{-2/3}. \tag{17}$$

Therefore, since $\psi(X'_{n-k})$ has eigenvalue μ_k^* on $\Omega(0)$, we can bound the quantity in (16) by $\mu_k^* + CN_k^{-2/3}$. We now turn to the first term in (15). We can bound the magnitude of $\nabla_{X_k} w(x)$ by

$$\left|(\nabla\chi(X_k))\,\psi\left(X'_{n-k}R_1/(R_1-r_k)\right)\right| + \left|\chi(X_k)\frac{R_1}{(R_1-r_k)^2}X'_{n-k}\cdot\left(\nabla_{X'_{n-k}}\psi\right)\left(X'_{n-k}R_1/(R_1-r_k)\right)\right|. \tag{18}$$

Since $|\nabla\chi(X_k)| \leq CN_k^{-1/3}$, the contribution from the first term in (18) leads to a contribution of size $CN_k^{-2/3}$ to (15). Using $|R_1 - r_k| \geq cN_1$, together with the lengths of the projections of $\Omega(0)$ onto each axis from (13), we can bound the second term in (18) by

$$CN_1^{-1}\sum_{j=k+1}^{n} N_j \left|\left(\partial_{x_j}\psi\right)\left(X'_{n-k}R_1/(R_1-r_k)\right)\right|.$$

Therefore, by Lemma 2.4, we can bound the contribution to (15) from the second term in (18) by

$$CN_1^{-2}\sum_{j=k+1}^{n} N_j^2 N_j^{-2/3} = CN_1^{-2}\sum_{j=k+1}^{n} N_j^{4/3}.$$

Since $N_1 \geq N_2 \geq \cdots \geq N_n$, this can be bounded by $CN_1^{-2}N_{k+1}^{4/3} \leq CN_k^{-2/3}$. Putting everything together, we obtain

$$\lambda \leq \mu_k^* + CN_k^{-2/3}.$$

This is precisely the inductive step, and so completes the proof of the proposition. □

Remark 2.3 *Denoting λ_m to be the m-th Dirichlet eigenvalue of Ω, a small modification of the proof of Proposition 2.3 ensures the existence of a constant $C_{j,m,n}$ such that*

$$\mu_j^* \leq \lambda_m \leq \mu_j^* + C_{j,m,n} N_j^{-2/3}. \tag{19}$$

The only change is that in place of $\chi(X_k)$, we require m functions $\chi_m(X_k)$, with $|\nabla\chi_m(X_k)| \leq C_m N_k^{-1/3}$, chosen such that

$$w_m(x) = \chi_m(X_k)\psi(X'_{n-k}R_1/(R_1 - r_k))$$

are orthogonal. The estimate in (19) in particular ensures that if u_m is the corresponding m-th eigenfunction, then it also satisfies the derivative estimates in Theorem 1.2 with a constant $C_{j,m,n}$.

3 A lower bound on the $L^2(\Omega)$-norm of the eigenfunction

In this section we prove Theorem 1.1 by combining the derivative estimates from Theorem 1.2 with the log concavity of the eigenfunction. Since u is log concave, the superlevel set $\Omega_{1/2}$ is a convex subset of Ω. In particular, we can associate the John ellipsoid $E_{1/2}$ to $\Omega_{1/2}$. Let v_j be the unit directions along which the axes of $E_{1/2}$ lie, and let M_j be the corresponding lengths of the axes. We also let e_j be the unit directions along the cartesian coordinate axes. The first step is to show that $\Omega_{1/2}$ determines the $L^2(\Omega)$-norm of u.

Lemma 3.1 *There exist constants C_1, $c_1 > 0$ such that*

$$c_1 \prod_{j=1}^{n} M_j \leq \int_\Omega |u(x)|^2 \,\mathrm{d}x \leq C_1 \prod_{j=1}^{n} M_j.$$

Proof of Lemma 3.1: The lower bound follows immediately from the definitions of M_j and the properties of the John ellipsoid. To obtain the upper bound we use the log concavity of u: The projection of the superlevel set $\Omega_{1/2}$ onto each v_j axis is comparable to M_j. The function $\log(u)$ is concave and attains a maximum of 0 in Ω. Therefore, the projection of the sets

$$\Omega_{2^{-m}} = \{x \in \Omega : u(x) \geq 2^{-m}\} = \{x \in \Omega : |\log(u(x))| \leq m\,|\log(1/2)|\}$$

onto each v_j axis is at most a constant multiplied by mM_j. Therefore,

$$\int_\Omega |u(x)|^2 \,\mathrm{d}x = \sum_{m=1}^{\infty} \int_{\Omega_{2^{-m}} \setminus \Omega_{2^{-m+1}}} |u(x)|^2 \,\mathrm{d}x \leq C \prod_{j=1}^{n} M_j \sum_{m=1}^{\infty} m^n 2^{-2(m-1)},$$

and this gives the desired upper bound in the lemma. □

Remark 3.1 *In the two dimensional case, the constants c_1 and C_1 can be given explicitly by*

$$c_1 = \frac{\pi}{16}, \qquad C_1 = 8\pi \sum_{m=1}^{\infty} m^2 2^{-2(m-1)} = 96\pi.$$

We now reorder the directions v_j to ensure that $M_1 \geq M_2 \geq \cdots \geq M_n$, and from the lower bound in Lemma 3.1 to prove Theorem 1.1 it is sufficient to prove the following lower bound on each M_j.

Proposition 3.2 *There exists a constant $c > 0$ such that for each j, $1 \leq j \leq n$, the axis length M_j satisfies the lower bound $M_j \geq cN_j^{1/3}$.*

Proof of Proposition 3.2: Since Ω has inner radius comparable to 1, the point where u attains its maximum is at a distance at least $c > 0$ from the boundary (see Theorem 1 in [21] in two dimensions, and Theorem 1.6 in [14] in higher dimensions). Therefore, by interior elliptic estimates, M_n is certainly comparable to

1. Given k, with $1 \le k \le n-1$, let w_k be a unit direction in $\mathbb{R}^n$ which lies in the projection of $\mathbb{R}^n$ onto the first k coordinates. That is, w_k is a linear combination of e_j for $1 \le j \le k$. We then consider the cross-sections of Ω

$$\Omega_{w_k}(t) = \{x \in \Omega : x \cdot w_k = t\},$$

which as t varies give the $(n-1)$-dimensional slices of Ω which are orthogonal to w_k. For each t, we can consider the $L^2(\Omega_{w_k}(t))$-norm squared of u,

$$\int_{\Omega_{w_k}(t)} |u(x)|^2 \, \mathrm{d}\sigma_{n-1}(x; w_k), \tag{20}$$

where $\mathrm{d}\sigma_{n-1}(x; w_k)$ is the flat $(n-1)$-dimensional surface measure on $\Omega_{w_k}(t)$. Suppose that the expression in (20) is maximized when $t = t^*$, and set

$$B_k^* = \int_{\Omega_{w_k}(t^*)} |u(x)|^2 \, \mathrm{d}\sigma_{n-1}(x; w_k).$$

We can now use Theorem 1.2 to obtain a lower bound on the L^2-norm of u in terms of B_k^*.

Lemma 3.3 *With $C_{k,n}$ as in the statement of Theorem 1.2, for each k, $1 \le k \le n$, and any such direction w_k,*

$$\int_\Omega |u(x)|^2 \, \mathrm{d}x \ge \tfrac{1}{4}(C_{k,n})^{-1/2} B_k^* N_k^{1/3}.$$

Proof of Lemma 3.3: Fix a point $x_{t^*} \in \Omega_{w_k}(t^*)$ and for each s choose x_s such that $(x_{t^*} - x_s) \cdot w_k = t^* - s$ and $|x_{t^*} - x_s| = |t^* - s|$. Then, extending u by zero outside Ω, for any t we can write

$$u(x_t) = u(x_{t^*}) + \int_{t^*}^t \partial_{w_k} u(x_s) \, \mathrm{d}s,$$

where $\partial_{w_k} u$ is the directional derivative $w_k \cdot \nabla u$. This implies that

$$\begin{aligned} |u(x_t)|^2 &\ge \tfrac{1}{2}|u(x_{t^*})|^2 - \left(\int_{t^*}^t \partial_{w_k} u(x_s) \, \mathrm{d}s \right)^2 \\ &\ge \tfrac{1}{2}|u(x_{t^*})|^2 - |t - t^*| \left| \int_{t^*}^t |\partial_{w_k} u(x_s)|^2 \, \mathrm{d}s \right|. \end{aligned}$$

We now integrate over the $(n-1)$ variables orthogonal to w_k. Since w_k lies in the projection of $\mathbb{R}^n$ onto the first k coordinates, we can use Theorem 1.2 with $j = k$ to bound $\partial_{w_k} u$. We therefore have

$$\int_{\Omega_{w_k}(t)} |u(x)|^2 \, \mathrm{d}\sigma_{n-1}(x; w_k) \ge \tfrac{1}{2} \int_{\Omega_{w_k}(t^*)} |u(x)|^2 \, \mathrm{d}\sigma_{n-1}(x; w_k) - C_{k,n} |t - t^*| N_k^{-2/3} \int_\Omega |u(x)|^2 \, \mathrm{d}x. \tag{21}$$

In particular, for

$$|t - t^*| \le \tfrac{1}{4} C_{k,n}^{-1} N_k^{2/3} \left(\int_\Omega |u(x)|^2 \, \mathrm{d}x \right)^{-1} \int_{\Omega_{w_k}(t^*)} |u(x)|^2 \, \mathrm{d}\sigma_{n-1}(x; w_k),$$

the estimate in (21) implies that

$$\int_{\Omega_{w_k}(t)} |u(x)|^2 \, \mathrm{d}\sigma_{n-1}(x; w_k) \ge \tfrac{1}{4} \int_{\Omega_{w_k}(t^*)} |u(x)|^2 \, \mathrm{d}\sigma_{n-1}(x; w_k) = \tfrac{1}{4} B_k^*.$$

Therefore,

$$\int_\Omega |u(x)|^2 \, \mathrm{d}x \ge \tfrac{1}{16} C_{k,n}^{-1} N_k^{2/3} \left(\int_\Omega |u(x)|^2 \, \mathrm{d}x \right)^{-1} (B_k^*)^2,$$

and rearranging implies the estimate in the lemma. □

The final step is to show that for each k, $1 \le k \le n$, we can choose such a unit direction w_k lying in k-dimensional space spanned by $e_1, e_2, \dots, e_k$, such that

$$B_k^* \ge c_3 \prod_{j=1, j\neq k}^{n} M_j. \tag{22}$$

Inserting this in Lemma 3.3 and using the upper bound in Lemma 3.1 implies that

$$C_1 \prod_{j=1}^{n} M_j \ge \int_\Omega |u(x)|^2 \, \mathrm{d}x \ge \tfrac{1}{4} c_3 C_{k,n}^{-1/2} N_k^{1/3} \prod_{j=1, j\neq k}^{n} M_j. \tag{23}$$

Thus (22) implies that M_k is bounded from below by a multiple of $N_k^{1/3}$.

To prove Proposition 3.2 (and hence also Theorem 1.1), we are thus left to prove (22), and we first consider $k = 1$: Consider the $(n-1)$-dimensional cross-sections of $\Omega_{1/2}$ perpendicular to $w_1 = e_1$. Since $\Omega_{1/2}$ has volume comparable to $\prod_{j=1}^{n} M_j$ and diameter comparable to M_1, the volume of one of these cross-sections must be at least comparable to $\prod_{j=2}^{n} M_j$. In particular, this ensures that $B_1^* \ge \frac{1}{4} c \prod_{j=2}^{n} M_j$.

For $k \ge 2$, we first choose a unit direction w_k in the intersection of the k-dimensional plane spanned by $e_1, e_2, \dots, e_k$ and the $(n-k+1)$-dimensional plane spanned by $v_k, v_{k+1}, \dots, v_n$. Taking the $(n-1)$-dimensional cross-sections of $\Omega_{1/2}$ perpendicular to w_k, the volume of one of these cross-sections must be at least $c \prod_{j=1, j\neq k}^{n} M_j$. To see this, we first note that there is a $(n-k+1)$-dimensional cross-section of $\Omega_{1/2}$ which is perpendicular to $v_1, v_2, \dots, v_{k-1}$ and contains a $(n-k+1)$-dimensional ellipsoid E with axes of lengths $M_k, M_{k+1}, \dots, M_n$. In particular, the volume of one of the $(n-k)$-dimensional cross-sections of E which is perpendicular to w_k must be at least $c \prod_{j=k+1}^{n} M_j$. But w_k is also perpendicular to $v_1, v_2, \dots, v_{k-1}$, and the projection of $\Omega_{1/2}$ onto the v_j-direction is comparable to M_j. Therefore, there exists a $(n-k) + (k-1) = (n-1)$-dimensional cross-section of $\Omega_{1/2}$ perpendicular to w_k of volume at least $c \left(\prod_{j=1}^{k-1} M_j \right) \left(\prod_{j=k+1}^{n} M_j \right)$. This ensures that $B_k^* \ge \frac{1}{4} c \prod_{j=1, j\neq k}^{n} M_j$, and (22) holds. □

Remark 3.2 *In the two dimensional case, the above argument can give an explicit lower bound on M_1 and M_2, and hence on the constant appearing in Theorem 1.1: When $k = 1$, a cross-section of $\Omega_{1/2}$ perpendicular to $w_1 = e_1$ must have length at least $\frac{\pi}{16} M_2$, and the argument above leads to the explicit lower bound on B_1^* of $B_1^* \ge \frac{\pi}{64} M_2$. Inserting this lower bound for B_1^* in the estimate in Lemma 3.3 as in (23), and using the bounds in Remark 3.1, we obtain*

$$96\pi M_1 M_2 \ge \tfrac{1}{4} C_{1,2}^{-1/2} \left(\tfrac{\pi}{64} M_2\right) N_1^{1/3},$$

and so

$$M_1 \ge \tfrac{1}{96} \cdot \tfrac{1}{256} C_{1,2}^{-1/2} N_1^{1/3}.$$

When $k = 2$, a cross-section of $\Omega_{1/2}$ perpendicular to $w_2 = v_2$ must have length at least M_1, and this leads to a lower bound on B_2^* of $B_2^* \ge \frac{1}{4} M_1$. This then leads to the lower bound on M_2 of

$$M_2 \ge \tfrac{1}{96\pi} \cdot \tfrac{1}{16} C_{2,2}^{-1/2} N_2^{1/3}.$$

From Lemma 3.1 and Remark 3.1, we have the lower bound

$$\|u\|_{L^2(\Omega)} \ge \tfrac{\pi^{1/2}}{4} M_1^{1/2} M_2^{1/2}.$$

Therefore, the above estimates on M_1 and M_2 lead to the explicit lower bound

$$c_2 \ge \tfrac{1}{96} \cdot \tfrac{1}{16^2} C_{1,2}^{-1/4} C_{2,2}^{-1/4}$$

given in the statement of Theorem 1.2. Inserting these lower bounds for B_1^* and B_2^* in the estimate in Lemma 3.3 as in (23), and using the bounds in Remark 3.1, then gives an explicit lower bound on M_1 and M_2 in terms of the constants $C_{j,n}$ appearing in Proposition 2.3 and Remark 2.2.

Remark 3.3 *The estimate in Theorem 1.1 will not hold for all non-convex domains, even for those with volume comparable to its diameter. For example, consider the dumbbell domain where $N_1 - 1$ unit balls are joined in a line to a ball of radius 2 by a series of thin necks. As the widths of the necks tends to 0, the first Dirichlet eigenfunction also tends to zero in all but the ball of radius 2. Therefore, an estimate on the L^2-norm as in Theorem 1.1 for these domains cannot hold uniformly as the width of the necks decrease.*

However, there are many non-convex domains for which the estimates in Theorem 1.2 will still hold. As seen in the proof of Proposition 2.3, the convexity of the domain Ω is only used at those parts of Ω within a distance $N_k^{1/3}$ of the slices Y_k leading to the minimal eigenvalues μ_k^*. In the two dimensional case, estimates on the location of these slices within the domain are known ([18], [17]). It would be very interesting to establish the analogous estimates in higher dimensions (see Section 4 for further details and questions). To convert the estimates in Theorem 1.2 to prove Theorem 1.1, we use in particular the convexity of the superlevel set $\Omega_{1/2}$ in order to estimate the volume of various k-dimensional slices of $\Omega_{1/2}$, which should also hold for domains which are suitably close to a convex set.

4 The two-dimensional case

Theorem 1.1 provides a lower bound on the $L^2(\Omega)$-norm of u. In two dimensions, Jerison and Grieser have given a precise characterization of the shape of u in terms of the geometry of Ω. To state this, we first rotate so that the projection of the planar domain onto the x_2-axis is the smallest and dilate so that this projection is of length 1. Then, we can write Ω as

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : a \leq x_1 \leq b, f_1(x_1) \leq x_2 \leq f_2(x_1)\}.$$

Here $b - a$ is comparable to N_1, f_1, f_2 are convex, concave functions respectively, and $0 \leq h(x) = f_2(x_1) - f_1(x_1)$ is a concave function, attaining a maximum of 1.

Definition 4.1 ([18]) *Define L to be the largest value such that $1 - L^{-2} \leq h(x_1) \leq 1$ on an interval I of length L.*

Since $h(x_1)$ is concave, the value of L satisfies $cN_1^{1/3} \leq L \leq CN_1$, and $L \sim N_1$, $L \sim N_1^{1/3}$ is attained when Ω is a rectangle, circular sector respectively. Any intermediate value of L can be obtained by, for example, forming the trapezoid of a rectangle of diameter L attached to a right angled triangle. In [18], [16], [17], Grieser and Jerison obtain estimates on the first and second Dirichlet eigenfunction in terms of this length scale L. Their approach is to perform an approximate separation of variables in Ω. Since the cross-section of Ω at x_1 has eigenvalue $\frac{\pi^2}{h(x_1)^2}$, a separation of variables leads to the ordinary differential operator

$$\mathcal{L} = -\frac{d^2}{dx_1^2} + \frac{\pi^2}{h(x_1)^2}$$

on the interval $[a, b]$. Grieser and Jerison approximate λ and u in terms of the first eigenvalue and eigenfunction of $\mathcal{L}$, and the approximation becomes stronger as the diameter of Ω increases. As a consequence of their work, the following $L^2(\Omega)$ bound holds in this planar case.

Theorem 4.2 (Grieser-Jerison, [17]) *There exists an absolute constant C such that the superlevel set $\{u > \frac{1}{2}\max_\Omega u\}$ has diameter bounded between $C^{-1}L$ and CL, and*

$$C^{-1}L^{1/2}\left\|u\right\|_{L^\infty(\Omega)} \leq \left\|u\right\|_{L^2(\Omega)} \leq CL^{1/2}\left\|u\right\|_{L^\infty(\Omega)}.$$

Using the definition of L from Definition 4.1 to compare the estimate in Theorem 4.2 with the lower bound in Theorem 1.1 in two dimensions, we note the following. When L is comparable to $N_1^{1/3}$, such as for a circular sector or right angled triangle, the bounds in the two theorems agree and in particular the lower bound in Theorem 1.1 is sharp. However, for $L \gg N_1^{1/3}$ Theorem 4.2 says that the eigenfunction

u has spread out by more than $N_1^{1/3}$ in the x_1-direction and so the $L^2(\Omega)$-norm of u is larger than that given in Theorem 1.1.

In higher dimensions, we can begin an analogous discussion. Consider the thin sector in $\mathbb{R}^n$ of the form

$$\{(r,\theta) : 0 < r < N_1, \theta \in D^{n-1}\},$$

where D^{n-1} is a geodesic disc of radius 1 in S^{n-1}. As shown in [4], for this domain, the lower bound given in Theorem 1.1 is sharp. If the domain Ω is instead a parallelepiped, then the superlevel set $\{u > \frac{1}{2}\max_\Omega u\}$ takes up a uniform portion of the whole domain. For a parallelepiped, this leads to the estimate

$$\|u\|_{L^2(\Omega)} \sim \text{ Volume}(\Omega)^{1/2} \|u\|_{L^\infty(\Omega)} \sim \prod_{j=1}^{n} N_j^{1/2} \|u\|_{L^\infty(\Omega)}.$$

Therefore, in dimensions higher than two it is natural to ask whether one can define analogous length scales to that of L from Definition 4.1 which govern the shape of the first eigenfunction.

Question 4.3 *Fix c, with $0 < c < 1$. Can we use the geometry of Ω to determine n length scales $M_1 \geq M_2 \geq \cdots \geq M_n$, and n directions $v_1, v_2, \ldots, v_n$ in $\mathbb{R}^n$ such that the John ellipsoid of*

$$\{x \in \Omega : u(x) > c \max_\Omega u\}$$

has axes along the directions v_j and of lengths comparable to M_j?

This question is open in any dimension higher than two. Let us normalize $\Omega \subset \mathbb{R}^n$ so that it has inner radius equal to 1, and its projection onto the x_n-axis is of length comparable to 1. Then, we can certainly choose v_n to point in the x_n-direction and take $M_n \sim 1$. The question is then to determine the remaining $n-1$ length scales and orientation. The results of this paper show that the lengths M_j must satisfy the lower bound $M_j \geq cN_j^{1/3}$. In [2], another preliminary step towards answering this question has been carried out: Consider the operator

$$-\Delta_{x_1,x_2} + \frac{\pi^2}{h(x_1,x_2)^2}, \tag{24}$$

with Dirichlet boundary conditions on a two dimensional convex domain D. Here $h(x_1, x_2)$ is a concave function on D, attaining a minimum of 1. The first eigenfunction of this operator still has convex superlevel sets and in [2], length scales L_1, L_2 and an orientation of the domain D are found in terms of D and h, which govern the intermediate level sets of this first eigenfunction. In particular, the $L^2(D)$-norm is comparable to $L_1^{1/2}L_2^{1/2}$ multiplied by the $L^\infty(D)$-norm of the eigenfunction.

The operator in (24) can be used to make progress of answering the question in the three dimensional case. For three dimensional domains of the form

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in D, 0 \leq x_3 \leq h(x_1, x_2)\},$$

an approximate separation of variables into (x_1, x_2) and x_3-variables leads to the operator in (24). It is shown in [3] that when L_1 and L_2 are sufficiently close in size ($L_1 \leq L_2^{3/2-\delta}$ for any fixed $\delta > 0$), this separation of variables provides a good approximation to the first eigenfunction of Ω. In particular, referring back to Question 4.3, in this case we can set $M_1 = L_1$, $M_2 = L_2$, $M_3 = 1$, and the orientation of D also governs the behaviour of the first Dirichlet eigenfunction of Ω. To make further progress towards fully answering Question 4.3, even in the three dimensional case, a key step is to determine the orientation of the superlevel sets of u, as in general it will not be the same as that of Ω itself. Especially as the dimension of Ω increases, it is unclear how to determine this orientation.

References

[1] D. Arnold, G. David, D. Jerison, S. Mayboroda, and M. Filoche, *The effective confining potential of quantum states in disordered media*, Phys. Rev. Lett. 116 (2016), Article Number: 0566

[2] T. Beck, *The shape of the level sets of the first eigenfunction of a class of two dimensional Schrödinger operators*, Trans. Amer. Math. Soc. 370 (2018), 3197–3244.

[3] T. Beck, *Level set shape for ground state eigenfunctions on convex domains*, PhD thesis.

[4] M. van den Berg, *On the L^∞-Norm of the First Eigenfunction of the Dirichlet Laplacian*, Potential Analysis 13 (2000), 361–366.

[5] M. van den Berg, F. Della Pietra, G. Di Blasio, and N. Gavitone, *Efficiency and localisation for the first Dirichlet eigenfunction*, arXiv:1905.06591v4, (2020).

[6] M. van den Berg, V. Ferone, C. Nitsch, and C. Trombetti, *On Pólya's inequality for torsional rigidity and first Dirichlet eigenvalue*, Integral Equations and Operator Theory 86, (2016), 579–600.

[7] M. van den Berg, V. Ferone, C. Nitsch, and C. Trombetti, *On a Pólya functional for rhombi, isosceles triangles, and thinning convex sets*, Revista Matemática Iberoamericana, 36 (2020), 2091–2105.

[8] H. J. Brascamp and E. H. Lieb, *On extensions of the Brunn-Minkowski and Prekopa Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Funct. Anal. 22 (1976), 366–389.

[9] L. Brasco and R. Magnanini, *The heart of a convex body*, in Geometric properties for parabolic and elliptic PDEs, 49–66, Springer, Milan, 2013.

[10] L. Brasco, R. Magnanini, and P. Salani, *The location of the hot spot in a grounded convex conductor*, Indiana Univ. Math. J. 60 no. 2 (2011), 633–659.

[11] G. Chiti, *A reverse Hölder inequality for the eigenfunctions of linear second order elliptic operators*, J. Applied Mathematics and Physics 33 (1982), 143–148.

[12] G. Chiti, *An isoperimetric inequality for the eigenfunctions of linear second order elliptic operators*, Boll. Un. Mat. Ital. A 1 (1982), 145–151.

[13] M. Filoche and S. Mayboroda, *Universal mechanism for Anderson and weak localization*, Proc. Natl. Acad. Sci. USA 109 (2012), 14761–14766.

[14] B. Georgiev and M. Mukherjee, *Nodal Geometry, Heat Diffusion and Brownian Motion*, Anal. PDE 11 no. 1 (2018), 133–148.

[15] B. Georgiev, M. Mukherjee, and S. Steinerberger, *A Spectral Gap Estimate and Applications*, Potential Analysis, 49, (2018) 635–645.

[16] D. Grieser and D. Jerison, *Asymptotics of the first nodal line of a convex domain*, Invent. Math. 125 no. 2 (1996), 197–219.

[17] D. Grieser and D. Jerison, *The size of the first eigenfunction of a convex planar domain*, J. Amer. Math. Soc. 11 no. 1 (1998), 41–72.

[18] D. Jerison, *The diameter of the first nodal line of a convex domain*, Ann. of Math. 141 (1995), 1–33.

[19] F. John, *Extremum problems with inequalities as subsidiary conditions*, Studies and Essays Presented to R. Courant on his 60th Birthday. January 8 (1948), pp.187–204.

[20] P. Kröger, *On the ground state eigenfunction of a convex domain in Euclidean space*, Potential Analysis 5 (1996), 103–108.

[21] M. Rachh and S. Steinerberger, *On the location of maxima of solutions of Schrödingers equation*, Comm. Pure. Appl. Math. 71, (2018) 525–537.

[22] S. Steinerberger, *Localization of Quantum States and Landscape Functions*, Proc. Amer. Math. Soc., 145 (2017), 2895–2907.

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