

IMPLICITIZATION OF TENSOR PRODUCT SURFACES VIA VIRTUAL PROJECTIVE RESOLUTIONS

ELIANA DUARTE AND ALEXANDRA SECELEANU

ABSTRACT. We derive the implicit equations for certain parametric surfaces in three-dimensional projective space termed tensor product surfaces. Our method computes the implicit equation for such a surface based on the knowledge of the syzygies of the base point locus of the parametrization by means of constructing an explicit virtual projective resolution.

1. INTRODUCTION

The *residual resultant* of a system of polynomial equations is a polynomial on the coefficients of the system that vanishes if and only if the system has a solution outside the zero set of another prescribed system of polynomial equations. Residual resultants for projective space were introduced in [BEM01] and further developed in [Bus01] for the case of \mathbb{P}^2 . In this article we consider residual resultants over $\mathbb{P}_k^1 \times \mathbb{P}_k^1$.

For projective space, the computation of the residual resultant relies on producing a free resolution of an ideal having the same vanishing locus as the residual (colon) ideal of the two systems of polynomial equations. In this article we formulate a similar approach to compute a residual resultant over $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ where we replace the free resolution of the residual ideal with a *virtual resolution*. This allows the derivation of the residual resultant from smaller, more manageable complexes than the more standard free resolutions. Besides being shorter than their free resolution counterparts, virtual resolutions also exhibit a closer relationship with Castelnuovo–Mumford regularity than minimal free resolutions. We exploit this relationship and present Algorithm 4.14 to compute residual resultants over $\mathbb{P}_k^1 \times \mathbb{P}_k^1$.

Our motivation to study residual resultants over $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ comes from implicitization in geometric modeling. In this context, a *tensor product surface* is the closure of the image Λ of a rational map $\lambda : \mathbb{P}_k^1 \times \mathbb{P}_k^1 \dashrightarrow \mathbb{P}_k^3$ defined by four bihomogeneous polynomials $p_0, p_1, p_2, p_3 \in k[\mathbb{P}_k^1 \times \mathbb{P}_k^1] = k[s, t, u, v]$ as

$$\lambda([s : t], [u : v]) = [p_0(s, t, u, v) : p_1(s, t, u, v) : p_2(s, t, u, v) : p_3(s, t, u, v)].$$

The *base points* of λ are the common zeros of the polynomials p_0, p_1, p_2, p_3 . The *implicitization problem* for tensor product surfaces consists in finding the equation whose vanishing defines the surface Λ in \mathbb{P}^3 . This problem has its origins in the

Received by the editor August 6, 2019, and, in revised form, March 1, 2020.

2010 *Mathematics Subject Classification.* Primary 13P15; Secondary 13D02, 14Q10.

The first author was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)-314838170, GRK2297 MathCoRe.

The second author was supported by NSF grant DMS-1601024 and EpSCOR award OIA-1557417.

seminal papers [SC95, CGZ00] and has been considered further in [KSZ92, D'A02, Bot11].

Three methods can be used to solve the implicitization problem for tensor product surfaces: Gröbner bases, resultants, and Rees algebras. Gröbner basis methods are the least satisfactory since they tend to be computationally intensive. Thus, it is primarily the latter two techniques which are used. Since classical resultants fail in the presence of base points, following the work of Busé [Bus01], we propose the use of residual resultants over $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ to solve the implicitization problem for tensor product surfaces in this case. We present this approach in Algorithm 5.5.

The structure of this paper is as follows: in section 2 we give the necessary background on residual resultants, with special attention to the case of biprojective space. In section 4 we derive effective methods to compute the residual resultant based on a virtual projective resolution for certain ideals of minors. In section 5 we show how this theory can be applied to the implicitization problem for tensor product surfaces. Finally, section 6 contains many completed examples that illustrate our results.

Conventions. Throughout the paper \mathbb{N} denotes the set of nonnegative integers.

A rational map $\lambda : \mathbb{P}_k^1 \times \mathbb{P}_k^1 \dashrightarrow \mathbb{P}_k^3$ corresponds to a ring map $S = k[\mathbb{P}_k^3] \rightarrow k[\mathbb{P}_k^1 \times \mathbb{P}_k^1] = R$. The coordinate ring $R = k[\mathbb{P}_k^1 \times \mathbb{P}_k^1] = k[s, t, u, v]$ is \mathbb{Z}^2 graded by $\deg(s, t) = (1, 0)$ and $\deg(u, v) = (0, 1)$. In R the bihomogeneous irrelevant ideal is $B = \langle s, t \rangle \cap \langle u, v \rangle$, and we fix an ideal G defining the base point locus of the rational map λ . The corresponding ideal sheaf is denoted \mathcal{G} . We assume throughout that the base locus is supported at a finite set of points. The coordinate functions of the rational map λ are elements of G with indeterminate coefficients from a polynomial ring C . These functions generate an ideal $P = (p_0, p_1, p_2, p_3)$ in the ring $C[s, t, u, v]$. Implicitization purposes lead to considering the ideal $\mathbb{F} = (p_0W - p_3X, p_1W - p_3Y, p_2W - p_3Z)$ of the ring $T = C[s, t, u, v, X, Y, Z, W]$. The extension of the bihomogeneous irrelevant ideal to this ring is denoted $\mathbb{B} = BT$, and the implicit equation for the image of λ is the polynomial $H \in S = k[X, Y, Z, W]$ such that $\mathbb{F} \cap S = (H)$. Throughout, $V(-)$ denotes the scheme-theoretic vanishing locus of an ideal.

2. A RESIDUAL RESULTANT FOR $\mathbb{P}_k^1 \times \mathbb{P}_k^1$

In this section we give an overview of the theory and construction for a residual resultant over a biprojective space. We closely follow the exposition in [BEM01] and [Bus01], adapting the statements for the case of the variety $Q = \mathbb{P}_k^1 \times \mathbb{P}_k^1$.

Algebraically, classical resultant computations can be phrased as follows: given commutative rings $A = k[x_0, \dots, x_m] = k[\mathbb{P}_k^m]$ and $C = k[C_{ij} : 0 \leq i \leq n, 1 \leq j \leq \dim_k(A_{d_i})]$, where the latter is viewed as a ring of indeterminate coefficients, form the polynomial ring $T = C[x_0, \dots, x_m] = C \otimes_k A$ and define a set of homogeneous polynomials $F_0, \dots, F_m \in T$

$$F_i(C_{ij}, x_0, \dots, x_n) = \sum_{m_j \in A_{d_i}} C_{ij} m_j.$$

One is interested in finding a generator for the principal ideal $I = (F_0, \dots, F_m) \cap C$, which is called the resultant of F_0, \dots, F_m . The resultant is a unique (up to scaling by constants) irreducible polynomial in C [GKZ08, Chapter 12]. The ring C is a standard graded ring with $\deg(C_{ij}) = 1$. Denote by C_1 the degree one generators

of C . For a point $c = (c_{ij}) \in \mathbb{P}_k(C_1)$ define the evaluation map at c to be the A -module homomorphism $e_c : T \rightarrow A$, $e_c(C_{ij}) = c_{ij}$ induced by the analogous k -linear map $C \rightarrow k(c)$. The zero locus of the ideal I ,

$$V(I) = \{c \in \mathbb{P}_k(C_1) : V(e_c(F_0), \dots, e_c(F_m)) \neq \emptyset\},$$

consists of the coefficients $c = (c_{ij})$ for which the equations $e_c(F_0), \dots, e_c(F_m)$ have common solutions in \mathbb{P}_k^m .

We proceed to describe a modified version of this classical resultant termed the residual resultant. If A is the coordinate ring of a variety Q and C, T are as above, consider two sets of homogeneous polynomials $g_0, \dots, g_n \in A$ and $F_0, \dots, F_m \in (g_0, \dots, g_n)T$. The residual resultant is a generator for the principal ideal $I = (F : G) \cap C$, where $F = (F_0, \dots, F_m)$ and $G = (g_0, \dots, g_n)$. The zero locus of this ideal,

$$V(I) = \{c \in \mathbb{P}_k(C_1) : V(e_c(F_0), \dots, e_c(F_m)) \setminus V(g_0, \dots, g_n) \neq \emptyset\},$$

consists of the coefficients c_{ij} for which the equations $e_c(F_0), \dots, e_c(F_m)$ have common solutions outside the common zero locus of g_0, \dots, g_n in Q .

We now rephrase the problem in the language of algebraic geometry. The classical resultant is interpreted in this language in [Jou91, Jou95] and [GKZ08, Propositions 3.1 and 3.3]. Following the exposition in [BEM01], let Q be an irreducible projective variety of dimension $\dim(Q) = m$ over the algebraically closed field k . Consider $m + 1$ invertible sheaves $\mathcal{L}_0, \dots, \mathcal{L}_m$ on Q , and let $V_i = H^0(Q, \mathcal{L}_i)$ be the vector space spanned by the global sections of the sheaf \mathcal{L}_i . Proposition 2.1 sets up the residual resultant as a polynomial that captures the condition for a set of global sections f_0, \dots, f_m ($f_i \in V_i$) to vanish on the variety Q . This resultant is a polynomial in the coefficients of each f_i with respect to the basis of the vector space V_i .

More precisely, given a set of polynomials $F_i = \sum_{b_j \in B_i} C_{ij} b_j \in T$ expressed in terms of fixed bases B_i for each vector space V_i , their resultant is a polynomial $\text{Res}_{V_0, \dots, V_m} \in C$. For any $c \in \mathbb{P}_k(C_1)$, if $f_i = e_c(F_i)$, then $\text{Res}_{V_0, \dots, V_m}(f_0, \dots, f_m)$ denotes the polynomial

$$\text{Res}_{V_0, \dots, V_m}(f_0, \dots, f_m) = e_c(\text{Res}_{V_0, \dots, V_m}).$$

From this point onward, we use the notation F_0, \dots, F_m for elements of T and f_0, \dots, f_m for specializations $f_i = e_c(F_i)$ at some $c \in \mathbb{P}_k(C_1)$.

Proposition 2.1 ([BEM01, Proposition 1]). *Suppose that each V_i generates the sheaf \mathcal{L}_i on Q and that V_i is very ample on a nonempty open subset U of Q . Then there exists an irreducible polynomial on $\prod_{i=0}^m V_i$, denoted by $\text{Res}_{V_0, \dots, V_m}$ and called the (V_0, \dots, V_m) -resultant, which satisfies*

$$\text{Res}_{V_0, \dots, V_m}(f_0, \dots, f_m) = 0 \iff \exists x \in Q : f_0(x) = \dots = f_m(x) = 0.$$

Moreover, $\text{Res}_{V_0, \dots, V_m}$ is homogeneous in the coefficients of each f_i , and of degree $\int_Q \prod_{j \neq i} c_1(\mathcal{L}_j)$.

We will follow the aforementioned result to define a *residual resultant* for $Q = \mathbb{P}_k^1 \times \mathbb{P}_k^1$. This follows readily using the methods of [BEM01], but it is important for our purposes to establish the notation in terms of sheaves on $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ instead of \mathbb{P}_k^n . For this reason we include a discussion of the setup below.

From this point on let $R = k[s, t, u, v]$ denote the bigraded coordinate ring of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ over an algebraically closed field k , with $\deg(s) = \deg(t) = (1, 0)$ and $\deg(u) = \deg(v) = (0, 1)$. Let $R_{(a,b)}$ denote the set of elements in R of bidegree

(a, b) . Recall that the smallest geometrically irrelevant ideal of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ is $B = (s, t) \cap (u, v)$. This yields a family of geometrically irrelevant ideals for $\mathbb{P}_k^1 \times \mathbb{P}_k^1$, i.e., $\mathcal{B} = \{\mathfrak{p} \in \text{Spec}(R) : B \subseteq \mathfrak{p}\}$.

Definition 2.2. The B -saturation of an ideal $I \subset R$ is the ideal $I^{\text{sat}} = \bigcup_{i=0}^{\infty} I : B^i$, where $I : B^i = \{f \in R : fB^i \in I\}$. The geometric importance of the B -saturation stems from the fact that for bihomogeneous ideals $I \subseteq R$, the following varieties agree that $V(I) = V(I^{\text{sat}})$. Analogously, one defines the B -saturation of an R -module M to be $M^{\text{sat}} = \bigoplus_{\mathbf{d} \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}} H^0(\widetilde{M}(\mathbf{d}), \mathbb{P}_k^1 \times \mathbb{P}_k^1)$, where \widetilde{M} denotes the sheafification of the module M .

Let $Q = \mathbb{P}_k^1 \times \mathbb{P}_k^1 = \text{Proj}(R)$ and consider a bihomogeneous ideal $G = (g_1, \dots, g_n) \subseteq R$, where $\deg g_j = (k_j, l_j)$. Let \mathcal{G} be the coherent sheaf of ideals associated to G . Consider pairs of nonnegative integers (a_i, b_i) , $0 \leq i \leq 2$, such that $(a_i, b_i) \geq (k_j, l_j)$ entrywise for all i, j , which yield the sheaves $\mathcal{G}(a_i, b_i) = \mathcal{G} \otimes_{\mathcal{O}_Q} \mathcal{O}_Q(a_i, b_i)$ for $0 \leq i \leq 2$. The vector space $V_i = H^0(Q, \mathcal{G}(a_i, b_i))$ is the set of polynomials of degree (a_i, b_i) which belong to the saturation of the ideal G . We denote by $\pi : \widetilde{Q} \rightarrow Q$ the blow-up of Q along the sheaf of ideals \mathcal{G} . The inverse image of the sheaf $\mathcal{G} = \pi^{-1}\mathcal{G} \cdot \mathcal{O}_Q$ is an invertible sheaf on \widetilde{Q} . The sheaf $\widetilde{\mathcal{G}} \otimes \pi^*(\mathcal{O}_Q(a_i, b_i))$ is denoted by $\widetilde{\mathcal{G}}(a_i, b_i)$.

Proposition 2.4 establishes the existence of a residual resultant polynomial, which cuts out the locus of those polynomials $f_0, f_1, f_2 \in V_0 \times V_1 \times V_2$ for which the common vanishing of f_0, f_1, f_2 contains a point not in $V(G)$. It also gives an algebraic criterion for this geometric condition, namely that the saturations of the two ideals $G = (g_1, \dots, g_n)$ and $F = (f_0, f_1, f_2)$ with respect to B are distinct. In order to establish this fact we need the following definition.

Definition 2.3. An ideal $I \subseteq R$ is said to be *locally a complete intersection* if $I_{\mathfrak{p}}$ can be generated by a regular sequence for every prime ideal $\mathfrak{p} \in \text{Spec}(R) \setminus \mathcal{B}$.

Proposition 2.4 ([BEM01, Proposition 3]). Let $G = (g_1, \dots, g_n) \subseteq R$ be a codimension two locally complete intersection ideal, with $\deg(g_j) = (k_j, l_j)$. Choose bihomogeneous polynomials $f_i \in V_i = H^0(Q, \mathcal{G}(a_i, b_i))$ for $i = 0, 1, 2$ such that $F = (f_0, f_1, f_2)$ and the following condition holds:

$$(a_i, b_i) \geq (k_{j_1} + 1, l_{j_1}) \text{ for some } j_1 \text{ and } (a_i, b_i) \geq (k_{j_2}, l_{j_2} + 1) \text{ for some } j_2.$$

Then there exists a polynomial in $C = \prod_{i=0}^2 k[V_i]$, denoted $\text{Res}_{\mathcal{G}, \{(a_i, b_i)\}_{i=0}^2}$, which satisfies

(1)

$$\text{Res}_{\mathcal{G}, \{(a_i, b_i)\}_{i=0}^2}(f_0, f_1, f_2) = 0 \Leftrightarrow \exists x \in \widetilde{Q} : \pi^*(f_0)(x) = \pi^*(f_1)(x) = \pi^*(f_2)(x) = 0$$

(2)

$$\Leftrightarrow \exists y \in \mathbb{P}_k^1 \times \mathbb{P}_k^1 \text{ such that } y \in V(F) \setminus V(G)$$

(3)

$$\Leftrightarrow F^{\text{sat}} \neq G^{\text{sat}}.$$

Proof. Let $i \in \{0, 1, 2\}$ and consider the vector space of global sections $V_i = H^0(Q, \mathcal{G}(a_i, b_i))$. The sections $s \in V_i$ generate the invertible sheaf $\mathcal{G}(a_i, b_i)$ on an open subset of Q , namely $Q \setminus Z$, where Z is the zero-dimensional scheme defined by G . Following [Har77, Ch.II.7.17.3] we blow-up $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ at the subscheme defined by \mathcal{G} . Then $\widetilde{\mathcal{G}}(a_i, b_i)$ is globally generated by the pullbacks $\pi^*(s)$

for $s \in H^0(Q, \mathcal{G}(a_i, b_i))$. Thus for all $i \in \{0, 1, 2\}$, if we let \tilde{V}_i be the vector subspace generated by the pullbacks $\pi^*(s)$, $s \in V_i$ then \tilde{V}_i generates $\tilde{\mathcal{G}}(a_i, b_i)$ on \tilde{Q} .

Next we show that each $\tilde{\mathcal{G}}(a_i, b_i)$ is very ample on an open subset U of \tilde{Q} . Suppose (a_i, b_i) satisfies the inequality conditions in the statement of the proposition. Let S_k be the subvariety of \tilde{Q} defined by the vanishing of $\pi^*(g_{j_k})$, and let $U_k = \tilde{Q} \setminus S_k$ for $k = 1, 2$. Set $U = U_1 \cap U_2$. We show that the map $\Gamma_i : U \rightarrow \mathbb{P}(\tilde{V}_i)$, $x \mapsto \{\pi^*(f) \mid f \in V_i, \pi^*(f)(x) = 0\}$ is an embedding. Since a point in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ is a pair (p_1, p_2) where p_i are points in the i th factor, there is a form L_1 of bidegree $(1, 0)$ or L_2 of bidegree $(0, 1)$ that vanishes at the given point but not at another point $(q_1, q_2) \in \mathbb{P}_k^1 \times \mathbb{P}_k^1$ according to whether $p_1 \neq q_1$ or $p_2 \neq q_2$. We say that such a form separates $(p_1, p_2), (q_1, q_2)$. In the former case there is a global section in \tilde{V}_i which is a multiple of $L_1 g_{j_1}$ and which separates $\pi^*(p_1, p_2)$ and $\pi^*(q_1, q_2)$ in U . Analogously, if $(p_1, p_2), (q_1, q_2)$ are separated by a form of bidegree $(0, 1)$, there is a global section in \tilde{V}_i which is a multiple of $L_2 g_{j_2}$ and which separates $\pi^*(p_1, p_2)$ and $\pi^*(q_1, q_2)$ in U . A proof that the differential condition for very ampleness holds follows in a similar fashion to [BEM01, Proposition 3] by the use of the appropriate separating form in each case. We conclude that each $\tilde{\mathcal{G}}(a_i, b_i)$ is very ample on the nonempty open subset U .

The first equivalence of the conclusion follows by applying Proposition 2.1 to the invertible sheaves $\tilde{\mathcal{G}}(a_i, b_i)$ on \tilde{Q} . For $(2) \Rightarrow (1)$ notice that if $y \notin V(G)$ and $f_0(y) = f_1(y) = f_2(y) = 0$, then, for the unique $x \in \tilde{Q}$ such that $\pi(x) = y$, we have $\pi^*(f_0)(x) = \pi^*(f_1)(x) = \pi^*(f_2)(x) = 0$. The equivalence $(2) \Leftrightarrow (3)$ follows from the identities $V(F) = V(F^{\text{sat}})$ and $V(G) = V(G^{\text{sat}})$. It remains to show that $(3) \Rightarrow (1)$, equivalently, if $\pi^*(f_0), \pi^*(f_1), \pi^*(f_2)$ do not vanish simultaneously on \tilde{Q} , then $F^{\text{sat}} = G^{\text{sat}}$. Since G is locally a complete intersection, the sheaf $\mathcal{G}/\mathcal{G}^2$ is locally free of rank 2. Hence, setting \mathcal{F} to be the ideal sheaf corresponding to F , one sees that the inclusion $\mathcal{F} \hookrightarrow \mathcal{G}$ is a surjection locally at $\mathfrak{p} \in X$. Thus $\mathcal{F} = \mathcal{G}$, and hence $F^{\text{sat}} = G^{\text{sat}}$ holds true. \square

More restrictive conditions on the generators of G allow us to define residual resultants with respect to sections of additional degrees. Below are two sample results of this flavor.

Proposition 2.5. *Let $G = (g_1, \dots, g_n) \subseteq R$ be a codimension two locally complete intersection ideal, with $\deg(g_j) = (a, b)$. Choose bihomogeneous polynomials $f_i \in V_i = H^0(Q, \mathcal{G}(a+1, b))$ for $i = 0, 1, 2$ such that $F = (f_0, f_1, f_2)$. If there exists a form $q \in R_{(a, b-1)}$ so that $\langle u, v \rangle q \subseteq G$, then $\text{Res}_{\mathcal{G}, (a+1, b)}$ exists and satisfies the conclusion of Proposition 2.4. Likewise if $f_i \in V_i = H^0(Q, \mathcal{G}(a, b+1))$ and there exists a form $p \in R_{(a-1, b)}$ so that $\langle s, t \rangle p \subseteq G$ then $\text{Res}_{\mathcal{G}, (a, b+1)}$ exists and satisfies the conclusion of Proposition 2.4*

Proof. As in the proof of Proposition 2.4 it suffices to prove that $\tilde{\mathcal{G}}(a+1, b)$ and $\tilde{\mathcal{G}}(a, b+1)$ are very ample on an open subset. We prove this for $\tilde{\mathcal{G}}(a+1, b)$; a similar argument works for $\tilde{\mathcal{G}}(a, b+1)$ by interchanging the roles of q and p . Set $g_{j_1}, g_{j_2} \in G$, and let U be the open set described in the proof of Proposition 2.4. It suffices to show that sections in $\mathcal{G}(a+1, b)$ separate points $x, y \in \mathbb{P}_k^1 \times \mathbb{P}_k^1 \setminus (V(g_{j_1}) \cup V(g_{j_2}))$. Suppose $I(x) = (\ell_1, h_1)$ and $I(y) = (\ell_2, h_2)$ with $\ell_i \in R_{(1, 0)}$ and $h_i \in R_{(0, 1)}$. If ℓ_1 is not a multiple of ℓ_2 , then the form $\ell_1 g_{j_1} \in G_{(a+1, b)}$ vanishes at x and not at y . If ℓ_1 is a multiple of ℓ_2 , then h_1 is not a multiple of h_2 . Pick a form $\ell \in R_{(1, 0)}$

which is not a multiple of ℓ_1 and ℓ_2 . Then the form $\ell h_1 q \in G_{(a+1,b)}$ separates x and y . A similar argument as in the proof of Proposition 2.4 shows that sections in $\tilde{\mathcal{G}}(a+1, b)$ also separate tangents. Thus $\tilde{\mathcal{G}}(a+1, b)$ is very ample. \square

Proposition 2.6. *Let $G = \langle l_1 h_2, l_2 h_1 \rangle$ with $l_j \in R_{(1,0)}$ and $h_j \in R_{(0,1)}$, $j \in \{1, 2\}$. Choose bihomogeneous polynomials $f_i \in V_i = H^0(Q, \mathcal{G}(2, 1))$ for $i = 0, 1, 2$ such that $F = (f_0, f_1, f_2)$. Then $\text{Res}_{\mathcal{G}, (2,1)}$ exists and satisfies the conclusion of Proposition 2.4. Analogously, if $f_i \in V_i = H^0(Q, \mathcal{G}(1, 2))$, then $\text{Res}_{\mathcal{G}, (1,2)}$ exists and satisfies the conclusion of Proposition 2.4.*

Proof. Following the proof of Proposition 2.4, it is only necessary to prove that $\tilde{\mathcal{G}}(2, 1)$ and $\tilde{\mathcal{G}}(1, 2)$ are very ample on an open subset. We do the proof for $\tilde{\mathcal{G}}(2, 1)$; a similar argument works for $\tilde{\mathcal{G}}(1, 2)$ by interchanging the roles of the lines l_i and h_i . Set U to be the open set in the proof of Proposition 2.4 with $g_{j_1} = l_1 h_2$ and $g_{j_2} = l_2 h_1$. It suffices to show that sections in $\mathcal{G}(2, 1)$ separate points $x, y \in \mathbb{P}_k^1 \times \mathbb{P}_k^1 \setminus (V(g_{j_1}) \cup V(g_{j_2}))$. Note that x, y do not lie on any of the lines defined by the forms l_1, l_2, h_1, h_2 . Suppose $I(x) = (L_1, H_1)$ and $I(y) = (L_2, H_2)$ with $L_i \in R_{(1,0)}$ and $H_i \in R_{(0,1)}$. If L_1 is not a multiple of L_2 , then the form $g_{j_1} L_1 \in G_{(2,1)}$ vanishes at x and not at y . If L_1 is a multiple of L_2 , then H_1 is not a multiple of H_2 . Write $H_1 = ch_1 + dh_2$; then the form $l_1 l_2 H_1 = cl_2 g_{j_2} + dl_2 g_{j_1} \in G_{(2,1)}$ separates x, y . A similar argument shows that sections in $\tilde{\mathcal{G}}(2, 1)$ also separate tangents. Thus $\tilde{\mathcal{G}}(2, 1)$ is very ample on U . \square

Remark 2.7. By the assumption on the codimension of G , the ideal sheaf \mathcal{G} in Propositions 2.4, 2.5 defines a zero-dimensional scheme. Propositions 2.4, 2.5 apply when G defines a reduced set of points in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$, since such an ideal is locally a complete intersection by [CFG⁺16, Lemma 4.1]. However, not all ideals G that fit the hypotheses of Proposition 2.4 define reduced sets of points in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. For example, $G = \langle s^2 t^2, u^2 v^2 \rangle$ is a (global) complete intersection, hence this ideal is also locally a complete intersection, which is not reduced.

Suppose that the ideal sheaf \mathcal{G} defines a zero-dimensional scheme Z composed of p points P_1, \dots, P_p . We denote by e_i the multiplicity of the point P_i in Z . We have

$$e_i = \dim_k(\mathcal{O}_{Z, P_i}), \text{ where } \mathcal{O}_Z = \mathcal{O}_{\mathbb{P}_k^1 \times \mathbb{P}_k^1} / \mathcal{G},$$

and hence $\sum_{i=1}^p e_i = \dim_k H^0(Z, \mathcal{O}_Z)$.

Remark 2.8. One important aspect to recall from the proof of Proposition 2.1 [BEM01, Proposition 1] is that the incidence variety defined by

$$\widetilde{W} = \left\{ (x, f_0, \dots, f_m) \in \tilde{Q} \times \prod_{i=0}^m \mathbb{P}(V_i) : f_0(x) = \dots = f_m(x) = 0 \right\} \subseteq \tilde{Q} \times \prod_{i=0}^m \mathbb{P}(V_i)$$

has codimension $m+1$. In the context of Proposition 2.4, $m = \dim \tilde{Q} = 2$ because \tilde{Q} is the blowup of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ at the scheme Z defined by \mathcal{G} . Therefore the incidence variety \widetilde{W} in this case is contained in $\tilde{Q} \times \prod_{i=0}^2 \mathbb{P}(V_i)$ and it is of codimension 3. Let E denote the exceptional locus of the blow-up of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ at Z . Then $\tilde{Q} \setminus E$ is isomorphic to $Q \setminus Z$. The open set

$$U = \left\{ (x, f_0, f_1, f_2) \in \tilde{Q} \setminus E \times \prod_{i=0}^2 \mathbb{P}(V_i) : \pi^*(f_0)(x) = \pi^*(f_1)(x) = \pi^*(f_2)(x) = 0 \right\}$$

is dense in \widetilde{W} and isomorphic to

$$W = \left\{ (x, f_0, f_1, f_2) \in (Q \setminus Z) \times \prod_{i=0}^2 \mathbb{P}(V_i) : f_0(x) = f_1(x) = f_2(x) = 0 \right\}.$$

Thus W is of codimension three in $(Q \setminus Z) \times \prod_{i=0}^2 \mathbb{P}(V_i)$.

In the next proposition we compute the degree of the residual resultant in the coefficients of each polynomial f_i . A general formula for this degree is given in Proposition 2.1 [BEM01, Proposition 1], and the case for \mathbb{P}^2 is treated in [Bus01]. We will now deduce this degree for the residual resultant in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$; the proof follows the same lines as for \mathbb{P}^2 , except that the computation of the intersection product is now performed on the blow-up of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ at Z .

Proposition 2.9. *The polynomial $\text{Res}_{\mathcal{G}, \{(a_i, b_i)\}_{i=0}^2}$ is multihomogeneous in the coefficients of each V_i , of degree N_i for $i = 0, 1, 2$ with*

$$N_0 = a_1 b_2 + b_1 a_2 - \sum_{i=1}^p e_i, \quad N_1 = a_0 b_2 + b_0 a_2 - \sum_{i=1}^p e_i,$$

$$\text{and } N_2 = a_0 b_1 + b_0 a_1 - \sum_{i=1}^p e_i.$$

Proof. We compute the integer N_0 ; the computation of N_1, N_2 is carried out in a similar fashion. Fix $i = 0$. By Propositions 2.1, N_0 equals

$$\int_{\widetilde{Q}} c_1(\widetilde{\mathcal{G}}(a_1, b_1)) c_1(\widetilde{\mathcal{G}}(a_2, b_2)),$$

where $c_1(\mathcal{F})$ denotes the first Chern class of the sheaf \mathcal{F} over \widetilde{Q} and $\int_{\widetilde{Q}}$ denotes the degree map on \widetilde{Q} . Denote by $H = \pi^*(h)$ and $L = \pi^*(l)$ the pullbacks of generic hyperplanes in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ that generate the divisor class group $\text{Cl}(\mathbb{P}_k^1 \times \mathbb{P}_k^1) \cong \mathbb{Z}^2$. Each E_i , $i = 1, \dots, p$, denotes the exceptional divisor of the blow-up π above each point P_i defined by \mathcal{G} , and E_i^{red} the reduced scheme of E_i . Following [Ful84], $c_1(\widetilde{\mathcal{G}}(a_i, b_i)) = a_i H + b_i L - \sum_{i=1}^p E_i$. Since $E_i \cdot E_j = 0$ if $i \neq j$, $H \cdot E_i = L \cdot E_i = 0$, and $L^2 = H^2 = 0$, we obtain

$$\begin{aligned} \int_{\widetilde{Q}} c_1(\widetilde{\mathcal{G}}(a_1, b_1)) c_1(\widetilde{\mathcal{G}}(a_2, b_2)) &= \int_{\widetilde{Q}} (a_1 H + b_1 L - \sum_{i=1}^p E_i) (a_2 H + b_2 L - \sum_{i=1}^p E_i) \\ &= \int_{\widetilde{Q}} a_1 b_2 H \cdot L + a_2 b_1 H \cdot L + \sum_{i=1}^p E_i^2. \end{aligned}$$

Now let f_1 (resp., f_2) be generic global sections of $\mathcal{G}(a_1, b_1)$ (resp., $\mathcal{G}(a_2, b_2)$), and let $D_{f_1} := V(f_1)$ (resp., $D_{f_2} := V(f_2)$) be the divisor corresponding to the vanishing of the section f_1 (resp., f_2) in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. We have

$$\begin{aligned} \pi^* D_{f_1} &= \widetilde{D}_{f_1} + \sum_{i=1}^p E_i = \widetilde{D}_{f_1} + \sum_{i=1}^p m_i E_i^{\text{red}} \text{ and} \\ \pi^* D_{f_2} &= \widetilde{D}_{f_2} + \sum_{i=1}^p E_i = \widetilde{D}_{f_2} + \sum_{i=1}^p n_i E_i^{\text{red}}, \end{aligned}$$

where \tilde{D}_{f_1} (resp., \tilde{D}_{f_2}) is the strict transform of D_{f_1} (resp., D_{f_2}) and where m_i (resp., n_i) is the multiplicity of f_1 (resp., f_2) at the point P_i [Ful84, Section 4.3]. Now $\tilde{D}_{f_1} \cdot \tilde{D}_{f_2} = 0$, and since \mathcal{G} is a local complete intersection, for each point $P_i \in Z$ we have $m_i n_i = e_i$ [Ful84, Section 12.4]. We deduce that

$$\sum_{i=1}^p E_i^2 = \sum_{i=1}^p m_i n_i E_i^{\text{red}^2} = \sum_{i=1}^p e_i E_i^{\text{red}^2}.$$

By the projection formulae, we know that $\int_{\tilde{Q}} H \cdot L = 1$ and $\int_{\tilde{Q}} E_i^{\text{red}^2} = -1$. Therefore

$$N_0 = a_1 b_2 + b_1 a_2 - \sum_{i=1}^p e_i.$$

□

We shall give a method for the effective computation of the residual resultant on $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ in section 4 after reviewing the notion of virtual complexes on $\mathbb{P}_k^1 \times \mathbb{P}_k^1$, which will prove useful in computing the residual resultants.

3. VIRTUAL RESOLUTIONS IN $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ AND MULTIGRADED REGULARITY

Free resolutions have played an important role in the effective computation of resultants. It is shown in [GKZ08] that the classic projective resultant in \mathbb{P}^n can be computed via a Koszul complex. In a similar manner, [Bus01] and [BFM01] use the Eagon–Northcott complex and variants of it to compute residual resultants with respect to locally complete intersection ideals over \mathbb{P}_k^2 and complete intersection ideals on \mathbb{P}_k^n , respectively. The Castelnuovo–Mumford regularity of the ideal resolved by this complex is a crucial ingredient for the computation of the residual resultant, and the ability to explicitly exhibit a free resolution has the advantage of giving a straightforward way to calculate the regularity. For $\mathbb{P}_k^1 \times \mathbb{P}_k^1$, general recipes for the free resolutions of the analogous ideal are not available, even under the above-mentioned assumptions. We overcome this obstacle by showing that virtual resolutions in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ have the same good properties that free resolutions have for the computation of resultants and residual resultants in \mathbb{P}^n , and we give an explicit description for a virtual resolution of certain determinantal ideals.

Two bigraded rings are of central importance for the purpose of residual implicitization on $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. The first is the coordinate ring $R = k[s, t, u, v]$ of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$, equipped with a natural \mathbb{Z}^2 grading obtained from viewing \mathbb{Z}^2 as the Picard group of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. For simplicity, we call rings graded by \mathbb{Z}^2 bigraded. For a finitely generated R -module M and a bidegree $\nu \in \mathbb{Z}^2$, the Hilbert function of M at ν is $H_M(\nu) = \dim_k M_\nu$.

The second ring of interest is $T = R \otimes_k C$, where $C = k[C_{ij}]$ is a ring of indeterminate coefficients as in section 2. Note that T is the coordinate ring of the variety $(\mathbb{P}_k^1 \times \mathbb{P}_k^1) \times \prod_{i=0}^m V_i$ and, moreover, $\mathbb{B} = BT$ is the irrelevant ideal for this variety. We equip the ring T with a \mathbb{Z}^2 grading given by $\deg_T(c \otimes r) = \deg_R(r)$ for any $r \in R, c \in C$. Thus T is a finitely generated C -algebra with $R_{(0,0)} = C$. For any bidegree $\nu \in \mathbb{Z}^2$ the bigraded component of T in bidegree ν , $T_\nu = R_\nu \otimes_k C$, is a free C -module minimally generated by a basis of R_ν .

3.1. Virtual resolutions in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. Virtual resolutions for $\mathbb{P}_k^1 \times \mathbb{P}_k^1$, also known as B -torsion complexes, have been discussed in the literature in [MS04] and [CDS07], among others. Our interest in these complexes was sparked by [BES20].

An R -module M is B -torsion if $B^i M = 0$ for some i .

Definition 3.1. A bigraded complex of free R -modules $P_i = \bigoplus_j R(-a_{ij}, -b_{ij})$ of the form

$$\mathbf{F} : 0 \longrightarrow P_m \xrightarrow{\varphi_m} \cdots \longrightarrow P_1 \xrightarrow{\varphi_1} P_0$$

is called a *virtual resolution* of a module M if $(H_0(\mathbf{F}))^{\text{sat}} \cong M^{\text{sat}}$ and all the homology modules $H_i(\mathbf{F})$ with $i > 0$ are B -torsion. Note that every free resolution is automatically a virtual resolution.

Virtual resolutions were introduced in [BES20], where it is pointed out how these resolutions capture the geometry of subvarieties of products of projective spaces in an optimal manner. For example, saturated ideals defining finite sets of points in \mathbb{P}^2 have a Hilbert–Burch resolution. This is not the case for ideals of sets of points in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$; however, there is a virtual version of this theorem for points in biprojective space.

Proposition 3.2 ([BES20, Corollary 4.2]). *Every zero-dimensional subscheme Z of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ has a virtual Hilbert–Burch resolution, i.e., there exists an $(m+1) \times m$ matrix φ such that the complex $0 \longrightarrow R^{m+1} \xrightarrow{\varphi} R^m$ is a resolution for $I_m(\varphi)$ and $V(I_m(\varphi)) = Z$.*

Corollary 3.3. *If $G \subseteq R$ is an ideal defining a not necessarily reduced set of points in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$, there exists an ideal G' such that $G^{\text{sat}} = G'^{\text{sat}}$, and G' has a Hilbert–Burch resolution. Moreover, G is locally a complete intersection if and only if G' is locally a complete intersection.*

Proof. The first statement is an algebraic reformulation of Proposition 3.2, while the second follows since $G^{\text{sat}} = G'^{\text{sat}}$ implies that $G_{\mathfrak{p}} = G'_{\mathfrak{p}}$ for $\mathfrak{p} \in \text{Spec}(R) \setminus \mathcal{B}$. \square

From a practical standpoint, there are two methods of producing virtual projective resolutions as in the above corollary. The first method, presented in [BES20, Theorem 4.1], is to set $G' = G \cap \langle s, t \rangle^a$ or $G' = G \cap \langle u, v \rangle^a$ for $a \gg 0$. It is an interesting and currently open problem to estimate the magnitude of the integer a required to obtain an ideal G' having a Hilbert–Burch resolution in this manner.

The second method, termed the virtual resolution of the pair $(R/G, \nu)$ in [BES20, Theorem 3.1], produces a virtual projective resolution of R/G by fixing a degree ν in the regularity region of R/G (see Definition 3.7) and considering the subcomplex of the minimal free resolution of R/G consisting of the free modules of degrees $\mu \leq \nu + (1, 1)$. While this approach always produces a virtual projective resolution, it does not guarantee that the virtual resolution of the pair $(R/G, \nu)$ is a Hilbert–Burch resolution. If successful, this method produces an ideal G' generated by the generators of G of degrees $\leq \nu + (1, 1)$ which has fewer generators than G and of lower degrees. This is a desirable feature from a computational perspective.

Example 3.4. Consider the ideal $G = \langle s, u \rangle \cap \langle t, v \rangle = \langle st, sv, tu, uv \rangle$ of a set of two points in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. A free resolution and a virtual resolution of G with $G' = \langle sv, tu \rangle$ are shown below. Note how the virtual resolution is much simpler than the free resolution and the ideal $G' = \langle sv, tu \rangle$ defines the same variety as G .

$$\begin{array}{ccccccc}
0 & \longrightarrow & R & \xrightarrow{\begin{pmatrix} v \\ -u \\ -t \\ s \end{pmatrix}} & R^4 & \xrightarrow{\begin{pmatrix} -u & -v & 0 & 0 \\ s & 0 & 0 & -v \\ 0 & t & -u & 0 \\ 0 & 0 & s & t \end{pmatrix}} & R^4 & \xrightarrow{\begin{pmatrix} st & tu & sv & uv \end{pmatrix}} & G & \longrightarrow & 0 \\
\\
0 & \longrightarrow & R & \xrightarrow{\begin{pmatrix} -sv \\ tu \end{pmatrix}} & R^2 & \xrightarrow{\begin{pmatrix} tu & sv \end{pmatrix}} & G' & \longrightarrow & 0.
\end{array}$$

This example is an instance of a more general phenomenon.

Example 3.5. If G defines a set Z of r general points in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$, from [BES20, Example 5.10] it follows that G has a virtual resolution

$$\begin{array}{ll}
0 \longrightarrow R(-2, -2p) \longrightarrow R(-1, -p)^2 \longrightarrow R & \text{if } r = 2p \text{ and} \\
0 \longrightarrow R(-2, -2p - 1) \longrightarrow \begin{array}{c} R(-1, -p) \\ \oplus \\ R(-1, -p - 1) \end{array} \longrightarrow R & \text{if } r = 2p + 1.
\end{array}$$

In particular, any set of general points in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ is virtually a complete intersection. Further details on which sets of points in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ are virtual complete intersections appear in [GLM19].

The notion of virtual resolution can be extended to modules over the ring T , where the meaning of the word virtual is understood to be with respect to the irrelevant ideal $\mathbb{B} = BT$. To see why this is a natural extension we start by defining a T -module M to be \mathbb{B} -torsion if $(\mathbb{B})^i M = 0$ for some $i \geq 0$. The following lemma shows that this notion is equivalent to the notion of B -torsion for R -modules.

Lemma 3.6. *A T -module M is \mathbb{B} -torsion if and only if M is B -torsion as an R -module.*

Proof. Denote by M_R the structure of M as an R -module induced by the restriction of scalars. The claim follows from the identity $(\mathbb{B}^i M)_R = B^i M_R$. \square

By analogy with Definition 3.1 we say that a bigraded complex of free T -modules $P_i = \bigoplus_j T(-a_{ij}, -b_{ij})$ of the form $\mathbf{F} : 0 \rightarrow P_m \rightarrow \cdots \rightarrow P_1 \rightarrow P_0$ is a *virtual resolution* of a T -module M if $(H_0(\mathbf{F}))^{\text{sat}} \cong M^{\text{sat}}$ and for $i > 0$ the homology modules $H_i(\mathbf{F})$ are \mathbb{B} -torsion. In view of Lemma 3.6, \mathbf{F} is a virtual resolution of the T -module M if and only if it is a virtual resolution for the R -module M_R .

3.2. Multigraded regularity: Strong and weak forms. In this paper we make use of a notion of (weak) regularity developed in [MS04]. Although this applies to modules over a polynomial ring graded by a finitely generated abelian group, we are primarily interested in modules over the rings R and T introduced at the beginning of this section, graded by the group $\text{Pic}(\mathbb{P}_k^1 \times \mathbb{P}_k^1) = \mathbb{Z}^2 = \mathbb{Z}\mathbf{c}_1 \oplus \mathbb{Z}\mathbf{c}_2$, where $\mathbf{c}_1, \mathbf{c}_2$ are the standard basis vectors. To explain the notion of bigraded regularity define

the sets

$$\begin{aligned} \text{St}_i &= \begin{cases} \{(r, s) \in \mathbb{Z}^2 : r + s = -i - 1, r < 0, s < 0\} & \text{for } i > 0, \\ \{(r, s) \in \mathbb{Z}^2 : r + s = -i, r \geq 0, s \geq 0\} & \text{for } i \leq 0, \end{cases} \\ &= \begin{cases} \{(-i, -1), (-i + 1, -2), \dots, (-2, -i + 1), (-1, -i)\} & \text{for } i > 0, \\ \{(-i, 0), (-i - 1, 1), \dots, (1, -i - 1), (0, -i)\} & \text{for } i \leq 0. \end{cases} \end{aligned}$$

Definition 3.7. A module M over a bigraded ring is said to be *weakly ν -regular* with respect to the irrelevant ideal B of that ring if $H_B^i(M)_\mu = 0$ for all $i \geq 0$ and $\mu \in \text{St}_i + \nu + \mathbb{N}^2$. We denote by $\text{reg}(M)$ the set of all elements $\mu \in \mathbb{Z}^2$ such that M is weakly μ -regular; we call this set the *regularity region* of M .

As before, the notion of regularity for T -modules and R -modules are closely related.

Lemma 3.8. For a T -module M and a bidegree $\mu \in \mathbb{Z}^2$, M is weakly μ -regular with respect to \mathbb{B} if and only if M is weakly μ -regular as an R -module with respect to B .

Proof. By independence of basis for local cohomology $H_B^i(M) \cong H_{\mathbb{B}}^i(M)$ as T -modules, whence $H_B^i(M)_\mu = 0$ if and only if $H_{\mathbb{B}}^i(M)_\mu = 0$. \square

One of the main applications of (multigraded) regularity consists of controlling the growth of Hilbert functions. Specifically, if M is a μ -regular bigraded module, then the Hilbert function $H_M(\nu)$ agrees with a polynomial $P_M(\nu)$, termed the Hilbert polynomial of M , for all values $\nu \in (\mu + \mathbb{N}^2) \setminus \mu$; see [MS05, Corollary 2.15.]. Furthermore, [MS04, Proposition 6.7] shows that if I is a B -saturated ideal defining a finite set of points in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$, then $\text{reg}(R/I)$ is exactly the set of elements $\mu \in \mathbb{Z}^2$ for which the Hilbert function $H_{R/I}(\mu)$ is equal to the Hilbert polynomial $P_{R/I}(\mu)$.

An important observation from [MS04] is that the regularity region of a module M can be estimated from any virtual projective resolution of M . We give a version of this result adapted to our setup.

Proposition 3.9 ([MS04, Theorem 1.5]). Let M be a finitely generated bigraded module. If $0 \rightarrow P_3 \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ is a virtual projective resolution for M , then

$$\mathcal{R} = \bigcup_{\sigma: [3] \rightarrow [2]} \left(\bigcap_{1 \leq i \leq 3} -\mathbf{c}_{\sigma(1)} - \dots - \mathbf{c}_{\sigma(i)} + \text{reg}(P_i) \right) \subseteq \text{reg}(M).$$

Unlike the case where the grading group is \mathbb{Z} , the minimal free resolution of a bigraded module M does not completely determine its regularity region. This shortcoming is overcome by introducing a related notion of strong regularity developed in [HW04].

Definition 3.10. A bigraded module M is said to be *strongly (a, b) -regular* if

$$\begin{aligned} H_{(s,t)}^i(M)_{(k,k')} &= 0 \quad \forall k \geq a + 1 - i \quad \forall k', \\ H_{(u,v)}^i(M)_{(k,k')} &= 0 \quad \forall k' \geq b + 1 - i \quad \forall k, \text{ and} \\ H_{(s,t,u,v)}^i(M)_{(k+k')} &= 0 \quad \forall k + k' \geq a + b + 1 - i. \end{aligned}$$

We denote by $\text{reg}^s(M)$ the set of all pairs $(a, b) \in \mathbb{Z}^2$ such that M is strongly (a, b) -regular.

It is shown in [HW04, Corollary 4.5] that $\nu \in \text{reg}^s(M)$ implies $\nu \in \text{reg}(M)$. The advantage of strong regularity is that it can be read from the minimal free resolution for the module M . Indeed, [HW04, Theorem 4.10] shows that if for all i the bigraded shifts in the i th homological degree of the minimal free resolution of a module M belong to

$$\text{DReg}_i(a, b) = \mathbb{Z}_-^2 + \text{St}_{-i} + \mu, \text{ where } \mathbb{Z}_- = \{n \in \mathbb{Z} : n \leq 0\},$$

then M is strongly μ -regular and thus also weakly μ -regular.

3.3. Eagon–Northcott complex and bigraded regularity. We follow the notation from the original paper by Eagon and Northcott [EN62]. Let R be a noetherian commutative ring, and let

$$\alpha : \bigoplus_{i=1}^r R(-c_i, -d_i) \rightarrow \bigoplus_{i=1}^q R(-e_i, -f_i)$$

be a bihomogeneous map where q, r are positive integers with $q \leq r$. Let $I_q(\alpha)$ denote the ideal generated by the maximal minors of any matrix ϕ_α representing α with respect to a choice of bases X_1, X_2, \dots, X_r for the domain of α and Y_1, Y_2, \dots, Y_q for the target of α . Consider the free graded R -modules

$$K = \bigwedge \left(\bigoplus_{i=1}^r R(-c_i, -d_i) \right) = \bigwedge (X_1, X_2, \dots, X_r),$$

where $\deg(X_i) = (-c_i, -d_i)$, and

$$S = \text{Sym} \left(\bigoplus_{i=1}^q R(-e_i, -f_i) \right) = \text{Sym}(Y_1, Y_2, \dots, Y_q),$$

where $\deg(Y_i) = (e_i, f_i)$. Set $K_i = \bigwedge^i (X_1, X_2, \dots, X_r)$ and $S_j = \text{Sym}_j(Y_1, Y_2, \dots, Y_q)$. Let $(-e, -f) = \sum_{i=1}^q (-e_i, -f_i)$. The k th row of the matrix $\phi_\alpha = (a_{ij})$ determines a Koszul differential Δ_k on K given by

$$\Delta_k(X_{i_1} \wedge \cdots \wedge X_{i_n}) = \sum_{p=1}^n (-1)^{p+1} a_{k i_p} X_{i_1} \wedge \cdots \widehat{X_{i_p}} \cdots \wedge X_{i_n}.$$

The Eagon–Northcott complex associated to the map α is the complex given by

$$0 \rightarrow K_r \otimes_R S_{r-q} \rightarrow \cdots \rightarrow K_{q-i} \otimes_R S_i \rightarrow K_{q+1} \otimes_R S_1 \rightarrow K_q \rightarrow R(-e, -f),$$

where the first map $\bigwedge^q \alpha : K_q \rightarrow \bigwedge^q (\bigoplus_{i=1}^q R(-e_i, -f_i)) = R(-e, -f)$ maps $X_{i_1} \wedge \cdots \wedge X_{i_q}$ to the maximal minor Δ_{i_1, \dots, i_q} of ϕ_α determined by the columns i_1, \dots, i_q . The rest of the differentials are specified on the basis elements of $K_{q-i} \otimes_R S_i$ as follows:

$$d(X_{i_1} \wedge \cdots \wedge X_{i_{q-i}} \otimes Y_1^{\nu_1} \cdots Y_q^{\nu_q}) = \sum_j \Delta_j(X_{i_1} \wedge \cdots \wedge X_{i_{q-i}}) \otimes Y_1^{\nu_1} \cdots Y_j^{\nu_j-1} \cdots Y_q^{\nu_q},$$

where $\nu_1 + \cdots + \nu_s = i$ and the sum is over those indices j for which $\nu_j > 0$. With the degree conventions in place this is a complex of free bigraded modules and bidegree $(0, 0)$ maps. It is convenient to shift the complex above so that the homological degree 0 component is generated in bidegree $(0, 0)$. Henceforth we refer to the shifted version below as the Eagon–Northcott complex $EN(\alpha)$:

$$(4) \quad 0 \rightarrow (K_r \otimes_R S_{r-q})(e, f) \rightarrow \cdots \rightarrow (K_{q+1} \otimes_R S_1)(e, f) \rightarrow K_q(e, f) \rightarrow R.$$

The principal application of the Eagon–Northcott complex is in resolving the ideal of minors of matrices $I_q(\phi_\alpha)$ when these ideals have maximum possible height, i.e., $\text{ht}(I_q(\phi_\alpha)) = r - q + 1$. The following lemmas are important in establishing the exactness and computing the homology of the Eagon–Northcott complex in our case of interest.

Lemma 3.11. *Using the notation of subsection 3.3, suppose $r = q + h$. Then*

- (1) *if $\text{ht}(I_q(\alpha)) = h$, then $EN(\alpha)$ has $H_i(EN(\alpha)) = 0$ for $i \geq 2$,*
- (2) *if $\text{ht}(I_q(\alpha)) = h + 1$, then $EN(\alpha)$ is a resolution for $I_q(\alpha)$.*

Proof. By [EN62, Theorem 1 Section 5], the homology of the complex $EN(\alpha)$ satisfies

$$\begin{aligned} \max\{i : H_i(EN(\alpha)) \neq 0\} &= r - q + 1 - \text{ht}(I_q(\alpha)) \\ &= \begin{cases} q + h - q + 1 - h = 1 & \text{in case (1),} \\ q + h - q + 1 - h - 1 = 0 & \text{in case (2).} \end{cases} \end{aligned}$$

□

Remark 3.12. Suppose $q \leq r - 1$ and consider a restriction

$$\alpha' : \bigoplus_{i=1, i \neq i_0}^r R(-c_i, -d_i) \rightarrow \bigoplus_{i=1}^q R(-e_i, -f_i)$$

of the map α defined above, which gives rise to the module

$$K' = \bigwedge(X_1, \dots, \widehat{X_{i_0}}, \dots, X_r).$$

Since K' is naturally a submodule of K , it follows from (4) that $EN(\alpha')$ is a subcomplex of $EN(\alpha)$. In particular, if the degrees of the generators of the free module $EN(\alpha)_i$ belong to $\text{DReg}_i(a, b)$, then so do the degrees of the generators of the free module $EN(\alpha')_i$ implying that $\text{reg}^s(I_q(\phi_\alpha)) \subseteq \text{reg}^s(I_q(\phi'_\alpha))$. In a similar fashion, if $\mathcal{R}, \mathcal{R}'$ are the weak regularity regions of $R/I_q(\alpha)$ and $R/I_q(\alpha')$ specified by Proposition 3.9, then $\mathcal{R} \subseteq \mathcal{R}'$.

Example 3.13. We illustrate by showing the Eagon–Northcott complex when G is a complete intersection. Assume $q = 2, r = 4$, and $(e_i, f_i) = (k_i, l_i)$, while $(c_1, d_1) = (k_1 + k_2, l_1 + l_2)$ and $(c_i, d_i) = (a, b)$ for $2 \leq i \leq 4$. The bigraded shifts in the Eagon–Northcott complex are illustrated below, based on the degrees of the

standard bases of the free modules in the complex (4), where $(e, f) = (k_1 + k_2, l_1 + l_2)$:

$$\begin{array}{ccccc}
 & & & & R(-2a + k_1, -2b + l_1)^3 \\
 & & & & \oplus \\
 & & & & R(-2a + k_2, -2b + l_2)^3 \\
 & & & & \oplus \\
 & & & & R(-3a + 2k_1 + k_2, -3b + 2l_1 + l_2) \\
 & & & & \oplus \\
 & & & & R(-3a + k_1 + 2k_2, -3b + l_1 + 2l_2) \\
 \\
 0 \longrightarrow & R(-3a + 2k_1, -3b + 2l_1) & \longrightarrow & & \\
 & \oplus & & & \\
 & R(-3a + 2k_2, -3b + 2l_2) & & & \\
 & \oplus & & & \\
 & R(-3a + k_1 + k_2, -3b + l_1 + l_2) & & & \\
 \\
 & & & R(-a, -b)^3 & \\
 & & & \oplus & \\
 & & & R(-2a + k_1 + k_2, -2b + l_1 + l_2)^3 & \\
 & & & \longrightarrow & R.
 \end{array}$$

The following result generalizes Example 3.13.

Proposition 3.14. *Let $\alpha : \bigoplus_{i=1}^{n-1} R(-c_i, -d_i) \oplus R(-a, -b)^3 \rightarrow \bigoplus_{j=1}^n R(-e_j, -f_j)$ be a bidegree preserving map, and set $(c, d) = \sum_{i=1}^{n-1} (c_i, d_i)$ and $(e, f) = \sum_{j=1}^n (e_j, f_j)$. Then the degrees of the minimal generators for the free R -modules in the complex $EN(\alpha)$, listed by homological degree, are as follows:*

degree	shifts
0	$(0, 0)$
1	$(a + c - e, b + d - f), (2a + c - e - c_i, 2b + d - f - d_i)$ $(3a + c - e - c_i - c_j, 3b + d - f - d_i - d_j), i \neq j$
2	$(2a + c - e - e_j, 2b + d - f - f_j), (3a + c - e - c_i - e_j, 2b + d - f - d_i - f_j)$
3	$(3a + c - e - e_i - e_j, 3b + d - f - f_i - f_j)$

In particular, if $a \geq e_j$ for some $1 \leq j \leq n$ and $b \geq f_j$ for some $1 \leq j \leq n$ and $EN(\alpha)$ is a virtual resolution for a module $R/I_q(\phi_\alpha)$, then the bigraded regularity of $R/I_q(\phi_\alpha)$ can be estimated by

$$\begin{aligned}
 \mathcal{R}(\alpha) &= \left(3a + c - e - \min_{1 \leq i \leq j \leq n} (e_i + e_j), 3b + d - f - \min_{1 \leq i \leq j \leq n} (f_i + f_j) \right) + \text{St}_{-3} + \mathbb{N}^2 \\
 &\subseteq \text{reg}(R/I_q(\phi_\alpha)).
 \end{aligned}$$

Proof. The shifts listed in the table follow from the graded structure of the complex (4).

Denoting by P_i the free module in the i th homological degree in $EN(\alpha)$, we claim that $\text{reg}(P_{i+1}) \subseteq \text{reg}(P_i) + (1, 1)$ for $0 \leq i \leq 4$. Using the fact that for any two modules U, V $\text{reg}(U \oplus V) = \text{reg}(U) \cap \text{reg}(V)$ ([MS04, Lemma 7.1]) one can easily compute the regularity of a graded free R -module $\bigoplus_{i=1}^q R(-m_i, -n_i) = (\max_{1 \leq i \leq q} m_i, \max_{1 \leq i \leq q} n_i) + \mathbb{N}^2$. Thus, to establish the claim it is sufficient to show that the maximum of the first components of the degrees listed in row i of the table above is strictly smaller than the maximum of the first components of the degrees listed in row $i + 1$ of the table and the analogous statement for the second components. For $i = 0$ this is clear, so we assume $i > 0$. Notice that $a \geq e_j$ for some $1 \leq j \leq n$ and $b \geq f_j$ for some $1 \leq j \leq n$ ensures that each component of the degrees listed in row i of the table above is strictly smaller than some component of the degrees listed in row $i + 1$, which establishes the claim.

In particular, $\text{reg}(P_{i+1}) \subseteq \text{reg}(P_i) + (1, 1)$ implies that

$$-\mathbf{c}_{\sigma(1)} - \cdots - \mathbf{c}_{\sigma(i)} + \text{reg}(P_i) \subseteq -\mathbf{c}_{\sigma(1)} - \cdots - \mathbf{c}_{\sigma(i)} - \mathbf{c}_{\sigma(i+1)} + \text{reg}(P_{i+1}) \text{ for } 0 \leq i \leq 2.$$

The statement of Proposition 3.9 can now be simplified to say

$$\bigcup_{\sigma:[3] \rightarrow [2]} (-\mathbf{c}_{\sigma(1)} - \mathbf{c}_{\sigma(2)} - \mathbf{c}_{\sigma(3)} + \operatorname{reg}(P_3)) \subseteq \operatorname{reg}(M), \text{ i.e., } \operatorname{reg}(P_3) + \operatorname{St}_{-3} \subseteq \operatorname{reg}(M).$$

Using the explicit formula for the regularity of a free module deduced above yields the desired estimate

$$\left(3a + c - e - \min_{1 \leq i \leq j \leq n} (e_i + e_j), 3b + d - f - \min_{1 \leq i \leq j \leq n} (f_i + f_j) \right) + \operatorname{St}_{-3} + \mathbb{N}^2 \subseteq \operatorname{reg}(R/I_q(\phi_\alpha)).$$

□

For a depiction of the regularity regions obtained by applying this proposition in various examples, see Examples 6.1, 6.2, 6.3, 6.5

Remark 3.15. All the results of this section continue to hold verbatim for T -modules. In particular, if $\alpha : \bigoplus_{i=1}^{n-1} T(-c_i, -d_i) \oplus T(-a, -b)^3 \rightarrow \bigoplus_{j=1}^n T(-e_j, -f_j)$ is a bidegree preserving map, $(c, d) = \sum_{i=1}^{n-1} (c_i, d_i)$, and $(e, f) = \sum_{j=1}^n (e_j, f_j)$, then the region $\mathcal{R}(\alpha)$ of Proposition 3.14 is contained in the regularity region of $I_q(T/I_q(\phi_\alpha))$, provided that the Eagon–Northcott complex is a virtual projective resolution for this module.

Note that the region $\mathcal{R}(\alpha)$ only depends on the numerical information regarding the degrees in which the domain and target of the map α are generated and not on the rule defining α . In particular, applying an evaluation map to the source and target of α induces an R -linear map $e_c(\alpha) : \bigoplus_{i=1}^{n-1} R(-c_i, -d_i) \oplus R(-a, -b)^3 \rightarrow \bigoplus_{j=1}^n R(-e_j, -f_j)$ such that $\mathcal{R}(e_c(\alpha)) = \mathcal{R}(\alpha)$.

4. EFFECTIVE COMPUTATION OF THE RESIDUAL RESULTANT

4.1. Virtual resolutions for effective computations. Let $G = (g_1, \dots, g_n)$ be a bihomogeneous ideal. To emphasize the universality of the resultant in terms of the generators of G we fix a presentation matrix φ for G having entries in $R = C'[s, t, u, v]$, where C' is a coefficient ring for the generators of G . This is equivalent to allowing the basepoints of the parameterization to have indeterminate coordinates from the ring C' . We shall refer to the matrix φ which fits in the exact sequence $R^\ell \xrightarrow{\varphi} R^n \rightarrow G \rightarrow 0$ as the syzygy matrix of G . For $0 \leq i \leq m$, let $(a_i, b_i) \in \mathbb{N}^2$ and set $C = k[C_{ij}^\alpha : 0 \leq i \leq m, 1 \leq j \leq n]$, where for each pair i, j , the index α enumerates the elements m_α of a monomial basis of $R_{(a_i - k_j, b_i - l_j)}$. Define $H_{ji} = \sum_\alpha C_{ij}^\alpha m_\alpha$, $F_i = \sum_{j=1}^n H_{ji} g_j$, so $H_{ji} \in T_{(a_i - k_j, b_i - l_j)}$ and $F_i \in T_{(a_i, b_i)}$. This can be written concisely as

$$(5) \quad \begin{bmatrix} F_0 & \cdots & F_m \end{bmatrix} = \begin{bmatrix} g_1 & \cdots & g_n \end{bmatrix} \Psi, \text{ where } \Psi = [H_{ji}]_{1 \leq j \leq n, 0 \leq i \leq m} \in \mathcal{M}_{n \times (m+1)}(T).$$

Lastly, set $\mathbb{F} = (F_0, \dots, F_m)$ and notice that the previous equation gives the containment $\mathbb{F} \subseteq G$. We study the algebraic counterpart of the residual resultant developed in Section 2. As mentioned previously, we denote by h_{ji} and ψ the images of H_{ji} and Ψ under any evaluation homomorphism $e_c : T \rightarrow R$.

We aim to express the residual resultant for the pair of ideals \mathbb{F}, G in terms of the minimal free resolution for the residual ideal $I = \mathbb{F} :_T G$. In turn, we will approximate this resolution by a virtual projective resolution based on the structure matrix Ψ defined above as well as the syzygy matrix φ for G . We start

with a lemma that relates the ideal $\mathbb{F} :_T G$ to the ideal of maximal minors of the matrix $\varphi \oplus \Psi \in \mathcal{M}_{n \times (\ell+m+1)}(T)$.

Lemma 4.1. *Let $\mathbb{F} = (F_0, \dots, F_m) \subseteq G = (g_1, \dots, g_n)$ be homogeneous ideals in T with the sets of generators of the two ideals related by*

$$[F_0 \quad \cdots \quad F_m] = [g_1 \quad \cdots \quad g_n] \Psi.$$

Let φ denote the $n \times \ell$ matrix of syzygies of G . Then the following hold:

- (1) $I_n(\varphi \oplus \Psi) \subseteq \text{Ann}(\text{coker}(\varphi \oplus \Psi)) = \mathbb{F} :_T G$.
- (2) *If $\text{ht}(\mathbb{F} :_T G) = m - n + \ell + 2$, then equality holds in the above containment.*

Proof. Computing ranks along the exact sequence $R^\ell \xrightarrow{\varphi} R^n \rightarrow R \rightarrow R/G \rightarrow 0$ gives $\ell \geq n - 1$ and thus $\ell + m \geq n$; hence $I_n(\varphi \oplus \Psi)$ is the ideal of maximal minors of $\varphi \oplus \Psi$. Note that $\varphi \oplus \Psi$ appears in the following bigraded presentation for G/\mathbb{F} :

$$\bigoplus_{i=0}^{m+\ell+1} T(-e_i, -d_i) \xrightarrow{\varphi \oplus \Psi} \bigoplus_{i=1}^n T(-k_i, -l_i) \rightarrow G/\mathbb{F} \rightarrow 0.$$

A theorem of Buchsbaum–Eisenbud [BE77] on Fitting ideals, applied to the presentation above, gives the containment below, with equality instead of the rightmost containment when $\text{ht}(I_n(\varphi \oplus \Psi)) = m - n + \ell + 2$:

$$(6) \quad \text{Ann}(\text{coker}(\varphi \oplus \Psi))^n \subseteq I_n(\varphi \oplus \Psi) \subseteq \text{Ann}(\text{coker}(\varphi \oplus \Psi)).$$

Combining the containment above and the identity

$$\text{Ann}(\text{coker}(\varphi \oplus \Psi)) = \text{Ann}(G/\mathbb{F}) = (\mathbb{F} :_T G)$$

gives the first statement of the lemma. Furthermore, if $\text{ht}(\mathbb{F} :_T G) = m - n + \ell + 2$, the containment (6) and the generalized principal ideal theorem (see [Eis95, Exercise 10.9]) $\text{ht}(I_n(\varphi \oplus \Psi)) \leq m - n + \ell + 2$ yield $\text{ht}(I_n(\varphi \oplus \Psi)) = m - n + \ell + 2$, which gives the second statement of the lemma. \square

Note that the identity $I_n(\varphi \oplus \Psi) = \mathbb{F} :_T G$ can hold even if the hypothesis of statement (2) above is not met, as illustrated in Example 6.2.

Corollary 4.2. *The statement of the lemma holds over the ring R whenever F_0, \dots, F_m and Ψ are specialized via evaluation to R .*

We exploit the close relation between $I_n(\varphi \oplus \Psi)$ and $\mathbb{F} :_T G$ established in Lemma 4.1 to obtain a virtual resolution of $\mathbb{F} :_T G$. First, due to Proposition 3.2 we may assume that G is an ideal with a Hilbert–Burch resolution provided the degrees of the generators of \mathbb{F} are high enough. The exact meaning of this reduction is made precise in the following proposition.

Lemma 4.3. *Suppose that G defines a zero-dimensional subscheme of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ and \mathbb{F} is an arbitrary ideal of T . Then there exists an ideal G' of R that has the following properties:*

- (1) $V(G) = V(G')$,
- (2) G' has a Hilbert–Burch resolution,
- (3) $\text{Res}_{\mathcal{G}, \{(a_i, b_i)\}_{i=0}^2} = \text{Res}_{\mathcal{G}', \{(a_i, b_i)\}_{i=0}^2}$ for (a_i, b_i) satisfying the condition in Proposition 2.4,
- (4) $(\mathbb{F} : G)^{\text{sat}} = (\mathbb{F} : G')^{\text{sat}}$, where saturation is taken with respect to the ideal \mathbb{B} of T ,

- (5) a complex \mathbf{F} of free T -modules is a virtual projective resolution for $\mathbb{F} : G$ if and only if \mathbf{F} is a virtual projective resolution for $\mathbb{F} : G'$ as well.

Moreover, if the reduced subscheme of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ defined by G consists of r general points and $\mathbb{F} \subseteq G$ is an ideal of T such that the generators of \mathbb{F} have bidegrees lying in the interior of the region \mathcal{S} shown in Figure 1, then $\mathbb{F} \subseteq G'$.

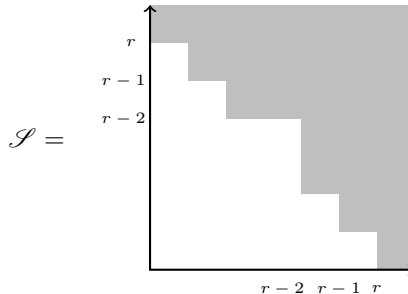


FIGURE 1. Region \mathcal{S} referred to in Lemma 4.3

Proof. Let G' be the ideal given by Corollary 3.3, which establishes that it satisfies properties (1) and (2) listed above. Note that property (1) is equivalent to $G^{\text{sat}} = G'^{\text{sat}}$ and therefore $\mathcal{G} = \mathcal{G}'$, which tautologically yields property (3).

For (4), consider $\mathfrak{p} \in \text{Spec}(T)$. If $B \not\subseteq \mathfrak{p}$ (equivalently $\mathbb{B} \not\subseteq \mathfrak{p}$), then the equality $G^{\text{sat}} = G'^{\text{sat}}$ implies that $G_{\mathfrak{p}} = G'_{\mathfrak{p}}$, and therefore we have $(\mathbb{F} : G)_{\mathfrak{p}} = \mathbb{F}_{\mathfrak{p}} : G_{\mathfrak{p}} = \mathbb{F}_{\mathfrak{p}} : G'_{\mathfrak{p}} = (\mathbb{F} : G')_{\mathfrak{p}}$, which is equivalent to $(\mathbb{F} : G)^{\text{sat}} = (\mathbb{F} : G')^{\text{sat}}$. For (5), recall that \mathbf{F} is a virtual resolution of $\mathbb{F} : G$ if and only if $(H_0(\mathbf{F}))^{\text{sat}} = (\mathbb{F} : G)^{\text{sat}} = (\mathbb{F} : G')^{\text{sat}}$ and $H_i(\mathbf{F})$ is \mathbb{B} -torsion for $i > 0$.

When the reduced locus of G consists of r general points, then G' can be taken to have one of the two types of resolutions presented in Example 3.5 or those obtained from the ones presented by interchanging the two coordinates of each bidegree. By Proposition 3.9 the region \mathcal{S} is contained in the union of the regularity regions of the two possible cyclic modules R/G' afforded by the value of r . Note that \mathcal{S} is also contained in the regularity region of R/G because the resolutions in Example 3.5 are virtual projective resolutions for R/G . Because $G_{\nu} = G_{\nu}^{\text{sat}} = G'_{\nu}^{\text{sat}} = G'_{\nu}$ for any $\nu = \mu + (i, j)$ with $\mu \in \mathcal{S}$ and $i, j \in \mathbb{N}$ with $i + j > 0$ (denote this by $\nu \in \mathcal{S}^0$), we have that $F \subseteq \bigcup_{\nu \in \mathcal{S}^0} G_{\nu}$ implies $F \subseteq \bigcup_{\nu \in \mathcal{S}^0} G'_{\nu} \subseteq G'$. \square

The existence of a Hilbert–Burch resolution for G is a key ingredient in our results, and from this point on we assume that G satisfies this property. We further assume that $m = 2$ since this is the setup for a residual resultant over $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. Under these conditions $\varphi \in M_{n \times (n-1)}(T)$ and the matrix $\varphi \oplus \Psi$ in Lemma 4.1 is an $n \times (n + 2)$ matrix.

Proposition 4.4. *Assume that $G \subseteq R$ has a Hilbert–Burch resolution, $\mathbb{F} = (F_0, F_1, F_2)$, and suppose that for every $\mathfrak{p} \in \text{Spec}(R) \setminus \mathcal{B}$ with $\text{ht}(\mathfrak{p}) = 2$ there is an equality $\mathbb{F}_{\mathfrak{p}} = G_{\mathfrak{p}}$. Then the Eagon–Northcott complex $EN(\varphi \oplus \Psi)$ is a virtual resolution for the module $T/I_n(\varphi \oplus \psi)$.*

Proof. Throughout this proof, let $\text{Min}(I)$ denote the set of minimal primes of an ideal I .

Recall from Remark 2.8 that the incidence variety $W \subset (Q \setminus Z) \times \prod_{i=0}^2 V_i$ has codimension three. Since $V(\mathbb{F} :_T G) \subseteq W \cup Z$, it follows that there is a containment

$$\text{Min}_T(\mathbb{F} :_T G) \subseteq \text{Min}_T(I_W) \cup \mathcal{B}_T \cup \text{Min}_T(G),$$

where \mathcal{B}_T and $\text{Min}_T(G)$ are the set of primes in \mathcal{B} and $\text{Min}_R(G)$, respectively, extended to T . Since $\mathbb{F}_{\mathfrak{p}} = G_{\mathfrak{p}}$ holds for any $\mathfrak{p} \in \text{Spec}(R) \setminus \mathcal{B}$ with height of \mathfrak{p} equal to two, it follows that in fact any prime of $\text{Min}_T(G)$ that is also an associated prime of $\mathbb{F} :_T G$ is in \mathcal{B}_T , thus the containment above reduces to

$$(7) \quad \text{Min}_T(\mathbb{F} :_T G) \subseteq \text{Min}_T(I_W) \cup \mathcal{B}_T.$$

Consider the T -module $H = \bigoplus_{i=0}^{n+2} H_i(EN(\varphi \oplus \Psi))$. From [Eis95, Theorem A2.59] it follows that $I_n(\varphi \oplus \psi)$ annihilates H , therefore there is a containment

$$(8) \quad \text{Min}_T(H) \subseteq \text{Min}_T(I_n(\varphi \oplus \Psi)).$$

The containments $(\mathbb{F} :_T G)^n \subseteq I_n(\varphi \oplus \Psi) \subseteq (\mathbb{F} :_T G)$ noted in the proof of Lemma 4.1 imply $\sqrt{\mathbb{F} :_T G} = \sqrt{I_n(\varphi \oplus \Psi)}$, and hence $\text{Min}_T(I_n(\varphi \oplus \Psi)) = \text{Min}_T(\mathbb{F} :_T G)$. Therefore, from equations (7), (8) we deduce

$$(9) \quad \text{Min}_T(H) \subseteq \text{Min}_T(I_n(\varphi \oplus \Psi)) = \text{Min}_T(\mathbb{F} :_T G) \subseteq \text{Min}_T(I_W) \cup \mathcal{B}_T.$$

Let P be any ideal of T of height at least 3; in particular, this applies to any $P \in \text{Min}_T(I_W)$ since the codimension of W is 3 by Remark 2.8. Then the complex $EN(\varphi \oplus \Psi) \otimes_T T_P = EN_{T_P}(\varphi \oplus \Psi)$ is exact by Lemma 3.11(2) because $\text{ht}(I_n(\varphi \oplus \Psi)_P) = \text{ht}(P) \geq 3$. It follows that $H_P = 0$ and therefore P is not in the support of H , so $P \notin \text{Ass}_T(H)$. This shows that the associated primes of H have height 2 and further reduce equation (9) to

$$(10) \quad \text{Min}_T(H) \subseteq \text{Min}_T(I_n(\varphi \oplus \Psi)) \subseteq \mathcal{B}_T.$$

Therefore $\text{Ann}_T(H) = Q_1 \cap Q_2$, where Q_1 is $\langle s, t \rangle$ -primary and Q_2 is $\langle u, v \rangle$ -primary. Now $\langle s, t \rangle^a \subseteq Q_1$ for some $a \geq 0$, and similarly $\langle u, v \rangle^b \subseteq Q_2$ for some $b \geq 0$, hence for $m \geq \max\{a, b\}$ we have the desired conclusion

$$B^m = \langle s, t \rangle^m \cap \langle u, v \rangle^m \subseteq Q_1 \cap Q_2 \subseteq \text{Ann}_T(H).$$

□

Remark 4.5. Example 6.2 illustrates the fact that it is possible for the Eagon–Northcott complex in Corollary 4.7 to be a virtual projective resolution while not being a resolution, i.e., not being exact.

Remark 4.6. In the setup of this section, where $F_i = \sum_{j=1}^n H_{ji} g_j$ for $0 \leq i \leq m$, $H_{ji} = \sum_{\alpha} C_{ij}^{\alpha} m_{\alpha}$, and α runs over the elements of a monomial basis of $R_{(a_i - k_j, b_i - l_j)}$, the hypothesis that there is an equality $\mathbb{F}_{\mathfrak{p}} = G_{\mathfrak{p}}$ for every $\mathfrak{p} \in \text{Spec}(R) \setminus \mathcal{B}$ with $\text{ht}(\mathfrak{p}) = 2$ holds true whenever G is locally a complete intersection (see Lemma 4.8). However, we prefer to state Proposition 4.4 including this hypothesis, since we shall use it in a slightly more general context in section 5 and also draw a closer analogy with the following corollary.

Corollary 4.7. *Suppose that G has a Hilbert–Burch resolution and the ideal $F = (f_0, f_1, f_2)$ arising by specializing the coefficients of F_0, F_1, F_2 to values in k satisfies $F^{\text{sat}} = G^{\text{sat}}$. Denote by ψ the corresponding specialization of the matrix Ψ in the setup at the beginning of this section. Then the Eagon–Northcott complex $EN(\varphi \oplus \psi)$ over R is a virtual resolution for the module $R/I_n(\varphi \oplus \psi)$.*

Proof. By Corollary 4.2, the conclusion of Lemma 4.1 still holds for f_0, f_1, f_2 . The hypothesis $F^{\text{sat}} = G^{\text{sat}}$ implies that $F_{\mathfrak{p}} = G_{\mathfrak{p}}$ for all $\mathfrak{p} \in \text{Spec}(R) \setminus \mathcal{B}$ and thus $\text{Min}(F :_R G) \subseteq \mathcal{B}$. Therefore the proof of Proposition 4.4 starting at equation (10) applies to show that $EN(\varphi \oplus \psi)$ is B -torsion as a complex over R . \square

Lemma 4.8. *Assume that G is a locally complete intersection ideal and $F_i = \sum_{j=1}^n H_{ji} g_j$ for $0 \leq j \leq m$, where $H_{ji} = \sum_{\alpha} C_{ij}^{\alpha} m_j^{\alpha}$ and m_j^{α} runs over the elements of a monomial basis of $R_{(a-k_j, b-l_j)}$ for some $(a, b) \in \mathbb{N}^2$. Then there is an equality $\mathbb{F}_{\mathfrak{p}} = G_{\mathfrak{p}}$ for every $\mathfrak{p} \in \text{Spec}(R) \setminus \mathcal{B}$ with $\text{ht}(\mathfrak{p}) = 2$.*

Proof. Let $\mathfrak{p} \in \text{Spec}(R) \setminus \mathcal{B}$ be an ideal with $\text{ht}(\mathfrak{p}) = 2$. We show that for any pair i, j we have $H_{ji} \notin \mathfrak{p}$. Assume the contrary, fix α_0 in the indexing set of monomials in $R_{(a-k_j, b-l_j)}$, and consider the prime ideal $\mathfrak{q} = \mathfrak{p} + (C_{ij}^{\alpha} : \alpha \neq \alpha_0)$. Then $H_{ji} \in \mathfrak{p}$ implies that $C_{ij}^{\alpha_0} m_j^{\alpha_0} \in \mathfrak{q}$, and since $C_{ij}^{\alpha_0} \notin \mathfrak{q}$ this yields $m_j^{\alpha_0} \in \mathfrak{q}$, which in turn implies that $m_j^{\alpha_0} \in \mathfrak{q} \cap R = \mathfrak{p}$ for any α_0 . We deduce that $R_{(a-k_j, b-l_j)} = \langle s, t \rangle^{a-k_j} \cap \langle u, v \rangle^{b-l_j} \subseteq \mathfrak{p}$, and consequently $\mathfrak{p} \in \mathcal{B}$, a contradiction. Therefore the elements H_{ij} become units in $T_{\mathfrak{p}}$. Since $G_{\mathfrak{p}}$ is a complete intersection with $\dim_{k(\mathfrak{p})} G_{\mathfrak{p}}/G_{\mathfrak{p}}^2 = 2$ and $\mathbb{F}_{\mathfrak{p}} \subseteq G_{\mathfrak{p}}$ is generated by 3 elements which are pairwise independent in $G_{\mathfrak{p}}/G_{\mathfrak{p}}^2$, the equality $\mathbb{F}_{\mathfrak{p}} = G_{\mathfrak{p}}$ follows. \square

4.2. A matrix representation for the residual resultant. The computation of the residual resultant hinges on the following proposition, which identifies a matrix whose rank drops when evaluated at any point of the residual resultant. In an alternate terminology, the following proposition gives a matrix representation for the residual resultant.

Proposition 4.9. *Let g_1, \dots, g_n and f_0, f_1, f_2 be polynomials in R with $f_i \in R_{(a_i, b_i)}$ related by the identities $f_i = \sum_{j=1}^n h_{ji} g_j$. Set $G = (g_1 \dots, g_n)$, $\psi = [h_{ji}]$, and assume that G has a Hilbert–Burch syzygy matrix φ . Let θ be a presentation map for the cyclic module $R/I_n(\varphi \oplus \psi)$. The following statements are equivalent:*

- (1) $\text{Res}_{\mathcal{G}, \{(a_i, b_i)\}_{i=0}^2}(f_0, f_1, f_2) \neq 0$,
- (2) $V(I_n(\varphi \oplus \psi)) = \emptyset$,
- (3) *the restriction of the map θ to degree ν is surjective for all degrees $\nu = \mu + (p, p')$ such that $\mu \in \mathcal{R}(\varphi \oplus \psi)$, $(p, p') \in \mathbb{N}^2$ and $p + p' > 0$.*

Proof.

(1) \Leftrightarrow (2) : By Proposition 2.4, the condition $\text{Res}_{G, \{(a_i, b_i)\}_{i=1}^s}(f_0, f_1, f_2) \neq 0$ is equivalent to $F^{\text{sat}} = G^{\text{sat}}$, which is equivalent to $F^{\text{sat}} :_R G^{\text{sat}} = R$. In view of Corollary 4.2, this translates to $\sqrt{I_n(\varphi \oplus \psi)^{\text{sat}}} = \sqrt{(F :_R G)^{\text{sat}}} = \sqrt{(F^{\text{sat}} :_R G^{\text{sat}})} = R$, that is, $V(I_n(\varphi \oplus \psi)) = \emptyset$.

(1) \Rightarrow (3) : By Proposition 2.4 $\text{Res}_{G, \{(a_i, b_i)\}_{i=0}^2}(f_0, f_1, f_2) \neq 0$ implies that $F^{\text{sat}} = G^{\text{sat}}$, whence Corollary 4.7 implies that the Eagon–Northcott complex is a virtual projective resolution for $R/I_n(\varphi \oplus \psi)$ and this module is μ -regular for $\mu \in \mathcal{R}(\varphi \oplus \psi)$. Since $V(I_n(\varphi \oplus \psi)) = V(F :_R G) = \emptyset$ by hypothesis and (1) \Rightarrow (2), we deduce from [MS05, Corollary 2.15] that $H_{R/I_n(\varphi \oplus \psi)}(\nu) = 0$ for bidegrees $\nu = \mu + (p, p')$ such that $(p, p') \in \mathbb{N}^2$ and $p + p' > 0$. Since the cokernel of the restriction of the map θ to degree ν is $R/I_n(\varphi \oplus \psi)_{\nu}$, and by the previous considerations $R/I_n(\varphi \oplus \psi)_{\nu} = 0$, we deduce that this map is surjective.

For (3) \Rightarrow (2) we prove the contrapositive. Suppose that $V(I_n(\varphi \oplus \psi))$ is not empty. Due to the equality $V(I_n(\varphi \oplus \psi)) = V(F :_R G)$, there exists a point $\xi \in$

$V(F) \setminus V(G)$. Evaluating the following identity encompassing the expressions $f_i = \sum_{j=1}^n h_{ji} g_j$ and the fact that φ is a syzygy matrix for G at ξ ,

$$\begin{bmatrix} 0 & \cdots & 0 & f_0 & f_1 & f_2 \end{bmatrix} = \begin{bmatrix} g_1 & \cdots & g_n \end{bmatrix} [\varphi \oplus \psi]$$

shows that the rank of the matrix $\varphi \oplus \psi$ evaluated at ξ is not maximal ($< n$). Hence all the maximal minors of $\varphi \oplus \psi$ vanish at ξ . Since all these minors generate $I_n(\varphi \oplus \psi)$ we deduce that for arbitrary $\nu \in \mathbb{N}^2$ any polynomial in the image of the map θ_ν vanishes at ξ . Since for any point ξ there exist polynomials in R_ν that do not vanish at ξ , it follows that the map θ_ν is not surjective. \square

Remark 4.10. Proposition 4.9 relates the nonvanishing of $\text{Res}_{\mathcal{G}, \{(a_i, b_i)\}_{i=0}^2}(f_0, f_1, f_2)$ to the presentation of the module $R/I_n(\varphi \oplus \psi)$ restricted to any bidegree in the interior of the region $\mathcal{R}(\varphi \oplus \psi)$ described in Proposition 3.14. Note that by Remark 3.15 this region is stable under specialization, that is, $\mathcal{R}(\varphi \oplus \psi) = \mathcal{R}(\varphi \oplus \Psi)$.

We now proceed to convert Proposition 4.9 into an effective computational tool.

In order to make the matrix representation for the residual resultant explicit we recall the first map of the Eagon–Northcott complex (4) associated to the matrix $\varphi \oplus \Psi$ over T ,

$$d = \bigwedge^q (\varphi \oplus \Psi) : \bigoplus_{\{i_1, \dots, i_q\} \subset [r]} TX_{i_1} \wedge \cdots \wedge X_{i_q} \rightarrow T, \quad X_{i_1} \wedge \cdots \wedge X_{i_q} \mapsto \Delta_{i_1, \dots, i_q}.$$

Here Δ_{i_1, \dots, i_q} is the maximal minor of $\varphi \oplus \Psi$ corresponding to the columns i_1, \dots, i_q , and the T -module generated by $X_{i_1} \wedge \cdots \wedge X_{i_q}$ is generated in degree $\deg(\Delta_{i_1, \dots, i_q})$. For $\nu \in \mathbb{Z}^2$, let d_ν denote the map d restricted to bidegree ν . Since for any bidegree ν , $T_\nu = C \otimes_k R_\nu$ is a free C -module, we obtain a map of finitely generated free C -modules:

$$d_\nu : \bigoplus_{\{i_1, \dots, i_q\} \subset [r]} T_{\nu - \deg(\Delta_{i_1, \dots, i_q})} \rightarrow T_\nu.$$

An explicit matrix representing the map d_ν can be obtained in four steps:

- (1) fix a basis for the vector space $\bigoplus_{\{i_1, \dots, i_q\} \subset [r]} T_{\nu - \deg(\Delta_{i_1, \dots, i_q})}$,
- (2) apply the map d_ν to this basis,
- (3) fix a basis for R_ν and express the result of step (2) in terms of this basis as vectors with entries in C ,
- (4) form a matrix with entries in C denoted Θ_ν having these vectors as columns.

Note that for step (1), a standard basis of this vector space consists of elements $mX_{i_1} \wedge \cdots \wedge X_{i_q}$ such that m is a monomial in R with $\deg(m) = \nu - \deg(\Delta_{i_1, \dots, i_q})$ for some $\{i_1, \dots, i_q\} \subseteq [n]$. Then in step (2) one obtains $d_\nu(m) = m \cdot \Delta_{i_1, \dots, i_q}$.

For any bidegree $\nu \in \mathbb{Z}^2$ we denote by θ_ν the image of the matrix Θ_ν defined above under an evaluation homomorphism. According to part (3) of Proposition 4.9, from this point onward we let ν be a bidegree such that $\nu = \mu + (p, p')$ with $\mu \in \mathcal{R}(\varphi \oplus \Psi)$ and $(p, p') \in \mathbb{N}^2, p + p' > 0$. When this holds we say that ν is in the interior of $\mathcal{R}(\varphi \oplus \Psi)$.

Proposition 4.11. *If ν is in the interior of $\mathcal{R}(\varphi \oplus \Psi)$, then any nonzero minor of size $\dim_k(R_\nu)$ of the matrix Θ_ν is a multihomogeneous polynomial in the coefficients C_{ij}^α of F_0, F_1, F_2 and a multiple of $\text{Res}_{\mathcal{G}, \{(a_i, b_i)\}_{i=0}^2}$.*

In light of Proposition 4.9 this proof follows along the lines of the argument in [Bus01]. We include the details for completeness.

Proof. First observe that any minor ρ of the matrix Θ_ν is multihomogeneous in the coefficients of each F_i for $i = 0, 1, 2$. Indeed, if F_i is multiplied by a scalar $\lambda \in k$, then the same is true for the column in $\varphi \oplus \Psi$ that corresponds to the coefficients of λF_i . Consequently, any column in Θ_ν containing the coefficients of an element $m \cdot \Delta_{i_1, \dots, i_s}$ such that Δ_{i_1, \dots, i_s} involves a column corresponding to the coefficients of λF_i is multiplied by a factor of λ^n . This implies that ρ is homogeneous of degree $n \cdot \dim_k R_\nu \cdot d$ in the coefficients of F_i , where d is the number of columns that appear in the submatrix of Θ_ν that have a factor of λ^n .

Next, fix ρ to be a maximal minor of Θ_ν . We want to show that ρ vanishes at every point where the resultant vanishes, for this implies that ρ is a multiple of $\text{Res}_{\mathcal{G}, \{(a_i, b_i)\}_{i=0}^2}$. Let $Q = \mathbb{P}_k^1 \times \mathbb{P}_k^1$, and let \tilde{Q} be the blow-up of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ along the sheaf of ideals associated to G . Define $\tilde{Q}^0 = \tilde{Q} \setminus E$ where E is the exceptional divisor in \tilde{Q} . Let

$$Z^0 = V\left(\text{Res}_{\mathcal{G}, \{(a_i, b_i)\}_{i=0}^2}\right) = \{c = (c_{ij}) : \exists x \in \tilde{Q}^0, \pi^*(f_0) = \pi^*(f_1) = \pi^*(f_2) = 0\},$$

i.e., Z^0 is the set of coefficients such that the pullbacks of the sections f_0, f_1, f_2 have a common root outside the exceptional divisor E . Suppose there is a choice of coefficients $c \in Z^0$ such that $e_c(\rho) \neq 0$. This implies that θ_ν is surjective because ρ is a maximal nonvanishing minor of size $\dim_k(R_\nu)$. However, since $c \in Z^0$, the specialized sections f_0, f_1, f_2 have a common root in $V(F) \setminus V(G)$ by Proposition 2.4. Using the equivalence (1) \Leftrightarrow (3) of Proposition 4.9 implies that θ_ν cannot be surjective, a contradiction. Therefore $e_c(\rho) = 0$, and since $c \in Z^0$ was arbitrary, ρ vanishes on Z^0 . As \tilde{Q}^0 is dense in \tilde{Q} , Z^0 is also dense in $Z = \{c = (c_{ij}) : \exists x \in \tilde{Q}, \pi^*(f_0) = \pi^*(f_1) = \pi^*(f_2) = 0\}$. Consequently, ρ vanishes on Z , i.e., ρ vanishes at all the points where $\text{Res}_{G, \{(a_i, b_i)\}_{i=0}^2}$ vanishes. \square

Proposition 4.12. *For ν in the interior of $\mathcal{R}(\varphi \oplus \Psi)$ and $0 \leq i \leq 2$ there exists a nonzero maximal minor of Θ_ν of degree N_i in the coefficients of F_i , where N_i is given in Proposition 2.9.*

Proof. Without loss of generality we assume $i = 0$. Choose a specialization $F = (f_0, f_1, f_2)$ such that $F^{\text{sat}} = G^{\text{sat}}$ and such that the ideal $F' = (f_1, f_2)$ has height two. In this case the variety $V(F' :_R G)$ has degree

$$\deg(F' :_R G) = \deg(F') - \deg(G) = a_1 b_2 + b_1 a_2 - \sum_{i=1}^p e_i = N_0.$$

Denote by ψ_{12} the submatrix of ψ consisting of the columns corresponding to the coefficients of f_1, f_2 . Since $F' \subseteq F' :_R G$ we deduce $\text{ht}(F' :_R G) \geq \text{ht}(F') = 2$. In view of Lemma 4.1 and Lemma 3.11 we conclude that $F' :_R G = I_2(\varphi \oplus \psi_{12})$ and $EN(\psi \oplus \psi_{12})$ is a resolution of $R/(F' :_R G)$. Moreover, by Corollary 4.7, since $F^{\text{sat}} = G^{\text{sat}}$ it follows that $EN(\varphi \oplus \psi)$ is a virtual projective resolution for $R/I_2(\varphi \oplus \psi)$.

Let $\mathcal{R}' = \mathcal{R}(\varphi \oplus \psi_{12})$ denote the region specified by Proposition 3.9, which is contained in the weak regularity region of $R/I_n(\varphi \oplus \psi_{12})$, and let $\mathcal{R} = \mathcal{R}(\varphi \oplus \psi)$ be the corresponding region for $R/I_n(\varphi \oplus \psi)$. Using Remark 3.12, since $\nu \in \mathcal{R}$ it follows that $\nu \in \mathcal{R}'$, hence $R/I_n(\varphi \oplus \psi_{12})$ is also ν -regular. By [MS05, Corollary 2.15] we deduce that $H_{R/(F' :_R G)}(\nu) = N_0$. Therefore

$$\dim_k(I_n(\varphi \oplus \psi_{12}))_\nu = \dim_k(F' :_R G)_\nu = \dim_k R_\nu - N_0.$$

Denote by θ_{12} the matrix corresponding to the Eagon–Northcott complex of $\varphi \oplus \psi_{12}$. Following the discussion before Proposition 4.11, the image of this matrix, $(I_n(\varphi \oplus \psi_{12}))_\nu$, is the vector space

$$\text{Span}_k \{m \cdot \Delta_{i_1, \dots, i_q} : \text{none of the columns } i_1, \dots, i_q \text{ involve the column of coefficients of } F_0\}.$$

Hence we can choose exactly $\dim_k R_\nu - N_0$ columns in the matrix θ_ν that are independent and do not involve the coefficients of F_0 ; therefore the same is true for the matrix Θ_ν . Denote the submatrix consisting of these columns by Θ_{ν, F_1, F_2} . Next, by Proposition 4.9 it follows that the map θ_ν is surjective and thus its image has dimension $\dim_k R_\nu$. Thus the vector space

$$\text{Span}_k \{m \cdot \Delta_{i_1, \dots, i_q} : \text{one column in } i_1, \dots, i_q \text{ is a coefficient column of } F_0\}$$

has dimension N_0 . Therefore there exists N_0 linearly independent columns in θ_ν that only involve the coefficients of F_0 , and the same is true for Θ_ν . Denote the submatrix given by these columns matrix by Θ_{ν, F_0} . The columns of Θ_{ν, F_1, F_2} together with the columns of Θ_{ν, F_0} span a vector space of dimension $\dim_k R_\nu$, hence the maximal minor corresponding to these columns is a maximal nonvanishing minor of Θ_ν . Furthermore, since the entries of Θ_ν are linear in the coefficients of F_0 , the determinant of this minor has degree N_0 in the coefficients of F_0 , as desired. \square

Proposition 4.13. *The greatest common divisor of the maximal minors of the matrix Θ_ν is exactly $\text{Res}_{G, \{(a_i, b_i)\}_{i=0}^2}$.*

Proof. Let d be the greatest common divisor of the maximal minors of Θ_ν . Proposition 4.11 implies that d is a multiple of $\text{Res}_{G, \{(a_i, b_i)\}_{i=0}^2}$. However, Proposition 4.12 states that the degree of d in the coefficients of F_0 is less than or equal to N_0 , and on the other hand Proposition 2.9 implies that $\text{Res}_{G, \{(a_i, b_i)\}_{i=0}^2}$ has degree N_0 in the coefficients of F_0 . Therefore the degree of d in the coefficients of F_0 is equal to N_0 . The same argument for $i = 1, 2$ allows us to conclude that $d = \text{Res}_{G, \{(a_i, b_i)\}_{i=0}^2}$ since they have the same degree with respect to all sets of coefficients. \square

Proposition 4.13 gives a practical method to compute the residual resultant. Note that Lemma 4.8 yields that the Eagon–Northcott complex gives a virtual projective resolution in this context.

Algorithm 4.14 (Computation of the residual resultant).

Input: G a locally complete intersection ideal with syzygy matrix φ , Ψ as in equation (5).

- (1) Pick ν in the interior of the regularity region $\mathcal{R}(\varphi \oplus \Psi)$.
- (2) Compute the matrix Θ_ν as explained before Proposition 4.11.
- (3) Compute a maximal minor δ_i of degree N_i in the coefficients of F_i for $0 \leq i \leq 2$.
- (4) Return $\gcd(\det(\delta_0), \det(\delta_1), \det(\delta_2))$.

Examples illustrating this algorithm can be found in section 6.1.

Remark 4.15. Steps (3) and (4) in the above algorithm are computationally expensive. However, we can replace these two steps by the computation of the determinant of the bidegree ν strand of the complex $EN(\varphi \oplus \Psi)$. Briefly, the determinant

of a complex is an alternating product of minors of the matrices of the differentials in the complex. Theorem 34 in [GKZ08, Appendix A] establishes an equality between the gcd of the maximal minors of the first differential of a complex and the determinant of a complex under certain hypotheses. Such hypotheses are satisfied for the complex $EN(\varphi \oplus \Psi)$, and therefore we can use determinants of complexes in this setting. We refer the reader to Appendix A in [GKZ08] for a detailed construction of the determinant of a complex. Although computing the determinant of a complex can also be computationally expensive, by comparison it is faster than computing the gcd of the maximal minors.

5. IMPLICITIZATION OF TENSOR PRODUCT SURFACES

We now describe the specific setting of interest for our paper. First we establish the relation between the residual resultant and the implicit equation of Λ , and immediately after we give explicit steps for its computation. Setting the coordinate ring of \mathbb{P}_k^3 to be $S = k[X, Y, Z, W]$, our goal is to find the equation $H \in S$ defining the algebraic variety

$$\Lambda = \overline{\text{im}(\lambda)} = \{[x : y : z : w] \in \mathbb{P}_k^3 : p_0w - p_3x = p_1w - p_3y = p_2w - p_3z = 0\} = V(H),$$

where $\lambda : \mathbb{P}_k^1 \times \mathbb{P}_k^1 \dashrightarrow \mathbb{P}_k^3$ is a rational map as described in the introduction.

Let $P = \langle p_0, p_1, p_2, p_3 \rangle$ be the ideal of R generated by the polynomials that define the parameterization λ , and set $T = R \otimes_k S$. We assume that the p_i have no common factors and that P is a height two ideal in R that defines a local complete intersection set of points. Let $G = P^{\text{sat}}$ denote the B -saturated ideal that defines the set of points in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$, and set \mathcal{G} to be the sheaf of ideals on $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ associated to G . Since $P^{\text{sat}} = G$, the sheaf $\mathcal{G}(a, b)$ is generated by its global sections p_0, p_1, p_2, p_3 on $\mathbb{P}_k^1 \times \mathbb{P}_k^1 \setminus V(G)$. We denote by $\pi : \tilde{Q} \rightarrow \mathbb{P}_k^1 \times \mathbb{P}_k^1$ the blow-up of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ along \mathcal{G} , and by \tilde{p}_i the global section $\pi^*(p_i)$ of the sheaf $\tilde{\mathcal{G}}_{(a,b)}$ for $i = 0, 1, 2, 3$. Since $\tilde{\mathcal{G}}_{(a,b)}$ is an invertible sheaf on \tilde{Q} and $\tilde{p}_0, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3$ are global sections that generate it, we deduce that there is a morphism

$$\tilde{\lambda} : \tilde{Q} \rightarrow \mathbb{P}^3$$

such that $\tilde{\lambda}^*\mathcal{O}(1) \cong \tilde{\mathcal{G}}_{(a,b)}$ and $\tilde{\lambda}^*(x) = \tilde{p}_0$, $\tilde{\lambda}^*(y) = \tilde{p}_1$, $\tilde{\lambda}^*(z) = \tilde{p}_2$, $\tilde{\lambda}^*(w) = \tilde{p}_3$ ([Har77, Ch.II.7]). As \tilde{Q} is projective and irreducible, we have $\tilde{\lambda}_*(\tilde{Q}) = \deg(\tilde{Q}/\Lambda)\Lambda$, where Λ is the rational surface in \mathbb{P}^3 and $\deg(\tilde{Q}/\Lambda)$ is the degree of the surjective map $\tilde{\lambda} : \tilde{Q} \rightarrow \Lambda$.

Let β be the following regular map:

$$\begin{aligned} \beta : U = \mathbb{P}_k^1 \times \mathbb{P}_k^1 \setminus V(G) &\longrightarrow \mathbb{P}^3 \\ [s : t] \times [u, v] &\mapsto (p_0 : p_1 : p_2 : p_3). \end{aligned}$$

Proposition 5.1. *The degree of Λ divides*

$$2ab - \sum_{i=1}^p e_i,$$

where e_i is defined before Remark 2.8, and it is equal to this number when β is birational.

Proof. We have $\deg(\tilde{\lambda}_*(\tilde{Q})) = \deg(\tilde{Q}/\Lambda) \cdot \deg(\Lambda)$. Next, we compute $\deg(\tilde{\lambda}_*(\tilde{Q}))$ by

$$\deg(\tilde{\lambda}_*(\tilde{Q})) = \int_{\tilde{Q}} c_1(\tilde{\lambda}^* \mathcal{O}(1))^2 = \int_{\tilde{Q}} c_1(\tilde{\mathcal{G}}_{(a,b)})^2 = 2ab - \sum_{i=1}^p e_i.$$

The last equality above follows from the same computation as in the proof of Proposition 2.9. Thus $\deg(\tilde{\lambda}_*(\tilde{Q})) = 2ab - \sum_{i=1}^p e_i$, which proves the first part of the statement.

Now we consider the following diagram, where E denotes the exceptional divisor of the blow-up π :

$$\begin{array}{ccc} \tilde{Q} \setminus E & \xrightarrow{\tilde{\lambda}|_{\tilde{Q} \setminus E}} & \mathbb{P}^3 \\ \pi \downarrow & & \\ U = \mathbb{P}_k^1 \times \mathbb{P}_k^1 \setminus V(G) & \xrightarrow{\beta} & \mathbb{P}^3 \end{array}$$

Since by construction $\tilde{\lambda}$ is unique and since the vertical map is an isomorphism outside the exceptional divisor, we deduce that $\tilde{\lambda}|_{\tilde{Q} \setminus E} = \beta \circ \pi$ and hence $\deg(\tilde{Q}/\Lambda) = \deg(U/\beta(U))$, which is one if π is birational. \square

The next proposition establishes the relation between residual resultants in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ and the implicitization problem for tensor product surfaces with basepoints.

Proposition 5.2. *Suppose that $(a, b) \geq (k_i, l_i)$ for all i , $(a, b) \geq (k_{j_1} + 1, l_{j_1})$ for some j_1 , and $(a, b) \geq (k_{j_2}, l_{j_2} + 1)$ for some j_2 . Then*

$$(11) \quad \text{Res}_{\mathcal{G}, (a,b)}(p_0 - Xp_3, p_1 - Yp_3, p_2 - Zp_3) = H(X, Y, Z, 1)^{\deg(U/\beta(U))}$$

with $\deg(U/\beta(U)) = 1$ if β is birational.

Proof. The residual resultant is defined as a general resultant over the blow-up of $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ along \mathcal{G} . Let ξ denote a point in $\tilde{Q} \setminus V(\tilde{p}_3)$, and let \tilde{W} denote the variety

$$\{\xi \times (x, y, z) \mid \tilde{p}_0(\xi) - x\tilde{p}_3(\xi) = \tilde{p}_1(\xi) - y\tilde{p}_3(\xi) = \tilde{p}_2(\xi) - z\tilde{p}_3(\xi) = 0\}.$$

Note that considering only points in $\tilde{Q} \setminus V(\tilde{p}_3)$ for the incidence variety is not a restriction. Indeed, if ξ is such that $\tilde{p}_3(\xi) = 0$, then for some $i \in \{0, 1, 2\}$ we must have $\tilde{p}_i(\xi) \neq 0$ because $\tilde{p}_0, \tilde{p}_1, \tilde{p}_2, \tilde{p}_3$ generate the sheaf $\tilde{\mathcal{G}}(a, b)$ on \tilde{Q} . Thus ξ cannot be a solution of the system $\tilde{p}_0(\xi) - x\tilde{p}_3(\xi) = \tilde{p}_1(\xi) - y\tilde{p}_3(\xi) = \tilde{p}_2(\xi) - z\tilde{p}_3(\xi) = 0$. Consider the following diagram:

$$\begin{array}{ccc} \tilde{W} & \xrightarrow{\pi_2} & \mathbb{P}^3 \setminus V(W) \\ \pi_1 \downarrow & & \uparrow \beta \\ \tilde{Q} \setminus V(\tilde{p}_3) & \xrightarrow{\pi} & \mathbb{P}_k^1 \times \mathbb{P}_k^1 \setminus V(G) \end{array}$$

The cycle in \mathbb{P}^3 that represents the residual resultant is exactly $\pi_{2*}(\tilde{W})$, i.e., $\pi_{2*}(\tilde{W}) = \deg(\tilde{W}/\pi_2(\tilde{W}))\Lambda$ (in the generic case we have $\deg(\tilde{W}/\pi_2(\tilde{W})) = 1$). As the blow-up π is an isomorphism outside the exceptional divisor, the equation that defines $\pi_{2*}(\tilde{W})$ vanishes if and only if the point $(x, y, z, 1) \in \mathbb{P}^3$ is in Λ . We deduce that

$$\text{Res}_{\mathcal{G}, (a,b)}(p_0 - Xp_3, p_1 - Yp_3, p_2 - Zp_3) = H(X, Y, Z, 1)^{\deg(\tilde{W}/\pi_2(\tilde{W}))}.$$

Now the map $\beta|_{\mathbb{P}_k^1 \times \mathbb{P}_k^1 \setminus V(p_3)}$ makes the above diagram commute, and since π is birational, we deduce that $\deg(\widetilde{W}/\pi_2(\widetilde{W})) = \deg(U/\beta(U))$. \square

Remark 5.3. It follows from Proposition 2.9 that in the case where $\deg(f_0) = \deg(f_1) = \deg(f_2) = (a, b)$, then $\text{Res}_{\mathcal{G}, \{(a,b)\}}$ has degree $2ab - \sum_{i=1}^p e_i$ in the coefficients of f_0, f_1 , and f_2 . Looking at the degrees of the polynomials in equation (11) from Proposition 5.2, we deduce that $2ab - \sum_{i=1}^p e_i = \deg(U/\beta(U)) \cdot \deg(H)$. This yields an alternate proof of the first assertion in Proposition 5.1.

Proposition 5.2 establishes that the residual resultant of $\mathbb{F} = (F_0, F_1, F_2) = (p_0 - Xp_3, p_1 - Yp_3, p_2 - Zp_3)$ with respect to \mathcal{G} computes the implicit equation $H = 0$. To use the methods presented in section 4 to compute the implicit equation of a tensor product surface via residual resultants, we assume that the given parameterization has a special form. To set up a parametrization λ we start with a locally complete intersection ideal $G = \langle g_1, \dots, g_n \rangle \subseteq R$ of height two with a Hilbert–Burch resolution and four bihomogenous polynomials $p_0, p_1, p_2, p_3 \in R_{(a,b)}$ related by

$$(12) \quad \begin{bmatrix} p_0 & p_1 & p_2 & p_3 \end{bmatrix} = \begin{bmatrix} g_1 & \cdots & g_n \end{bmatrix} \begin{bmatrix} h_{ji} \end{bmatrix}, \quad h_{ji} \in R_{(a-k_i, b-l_i)}.$$

Second, we assume $P^{\text{sat}} = G^{\text{sat}}$. The importance of this assumption is clarified in the following Lemma 5.4 and guarantees that we can use the Eagon–Northcott complex of $I_n(\varphi \oplus \Psi)$ to find suitable degrees in the regularity region of $T/I_n(\varphi \oplus \Psi)$.

Lemma 5.4. *Suppose that G is a locally complete intersection ideal and $P^{\text{sat}} = G^{\text{sat}}$. Then the ideal $\mathbb{F} = (p_0 - Xp_3, p_1 - Yp_3, p_2 - Zp_3)$ has the property that $\mathbb{F}_{\mathfrak{p}} = G_{\mathfrak{p}}$ for any ideal $\mathfrak{p} \in \text{Spec } R \setminus \mathcal{B}$ with $\text{ht}(\mathfrak{p}) = 2$.*

Proof. Let $\mathfrak{p} \in \text{Spec}(R) \setminus \mathcal{B}$ be an ideal with $\text{ht}(\mathfrak{p}) = 2$. Since $P^{\text{sat}} = G^{\text{sat}}$ it follows that $P_{\mathfrak{p}} = G_{\mathfrak{p}}$, and since $G_{\mathfrak{p}}$ is a complete intersection it is furthermore the case that $\dim_{k(\mathfrak{p})} P_{\mathfrak{p}}/P_{\mathfrak{p}}^2 = 2$. Now $\mathbb{F}_{\mathfrak{p}} \subseteq P_{\mathfrak{p}}$ is generated by 3 elements which are pairwise independent in $P_{\mathfrak{p}}/P_{\mathfrak{p}}^2$, thus the equality $\mathbb{F}_{\mathfrak{p}} = G_{\mathfrak{p}}$ follows. \square

Using the relation from equation (12), we can write the polynomials $F_0 = p_0 - Xp_3, F_1 = p_1 - Yp_3, F_2 = p_2 - Zp_3$ from Proposition 5.2 as

$$(13) \quad \begin{bmatrix} F_0 & F_1 & F_2 \end{bmatrix} = \begin{bmatrix} p_0 - Xp_3 & p_1 - Yp_3 & p_2 - Zp_3 \end{bmatrix} = \begin{bmatrix} g_1 & \cdots & g_n \end{bmatrix} \Psi.$$

Based on Algorithm 4.14 we derive a version that is tailored to the implicitization problem.

Algorithm 5.5 (Implicitization algorithm).

Input: G a locally complete intersection ideal, P as in equation (12) such that $P^{\text{sat}} = G^{\text{sat}}$.

- (1) Set $\Psi = [h_{i0} - Xh_{i3} \quad h_{i0} - Yh_{i3} \quad h_{i2} - Zh_{i3}]_{1 \leq i \leq n}$ as in equation (13).
- (2) Pick ν in the interior of the regularity region $\mathcal{R}(\varphi \oplus \Psi)$.
- (3) Compute the matrix Θ_{ν} as explained before Proposition 4.11.
- (4) Compute a maximal minor δ_i of degree N_i in the coefficients of F_i for $0 \leq i \leq 2$.
- (5) Return $\gcd(\det(\delta_0), \det(\delta_1), \det(\delta_2))$.

Examples illustrating this algorithm can be found in section 6.2.

For practical applications it is important to estimate the size of the matrix Θ_{ν} which governs the complexity of this algorithm. We estimate the size of this

matrix for the case when G is a complete intersection ideal with generators of degree (k, l) . This applies to general base points as illustrated in Example 3.5. Assuming as in Example 3.13 that the entries of Ψ have degree $(a - k, b - l)$, we see that the matrix $\varphi \oplus \Psi$ has three minors of degree (a, b) and three of degree $(2a - 2k, 2b - 2l)$. One may choose $\nu = (\nu_1, \nu_2) = (3a - 2k + i, 3b - 2l + j) \in \mathcal{R}(\varphi \oplus \Psi)$ for $(i, j) \in \text{St}_{-3} = \{(3, 0), (2, 1), (1, 2), (0, 3)\}$ based on Proposition 3.14. The remarks preceding Proposition 4.11 give that Θ_ν has size

$$\begin{aligned} & (3(\nu_1 - a + 1)(\nu_2 - b + 1) + 3(\nu_1 - 2a + 2k + 1)(\nu_2 - 2b + 2l + 1)) \times (\nu_1 + 1)(\nu_2 + 1) \\ &= 3((2a - 2k + i + 1)(2b - 2l + j + 1) + (a + i + 1)(b + j + 1)) \\ &\quad \times (3a - 2k + i + 1)(3b - 2l + j + 1). \end{aligned}$$

We wish to write this in terms of the degree of the tensor product surface, (a, b) . The smallest size is attained when Ψ is a matrix of degree $(1, 1)$ forms, that is, $(a, b) = (k + 1, l + 1)$. This yields the best case estimate,

$$3((i + 3)(j + 3) + (a + i + 1)(b + j + 1)) \times (a + i + 3)(b + j + 3) = O(3ab \times ab).$$

The largest size is attained when (k, l) is assumed negligible in comparison with (a, b) , yielding a worst-case estimate of order $O(15ab \times 9ab)$.

Comparing our methods with the results in [Bot11] where implicitization of tensor product surfaces of bidegree (a, b) is realized based on matrices of size on the order of $2ab \times 2ab$, we see that our method can in the best case scenario be more efficient, but as the degree of the tensor product surface grows farther apart from the degree of G , our matrices become progressively larger than those proposed by [Bot11].

Remark 5.6. If the hypothesis $P^{\text{sat}} = G^{\text{sat}}$ is not satisfied, Algorithm 5.5 no longer applies since the presentation map Θ for $R/I_n(\varphi \oplus \Psi)$ described in Proposition 4.9 is no longer surjective when restricted to any bidegree. However, given a bidegree $\nu \in \mathbb{N}^2$, if the dimension of the cokernel of Θ_ν is c , then the proof of Proposition 4.11 shows that the resultant divides the greatest common divisor of the generators of the c th Fitting ideal of Θ_ν , i.e., the minors of size $(\dim_k R_\nu - c) \times (\dim_k R_\nu - c)$ for Θ_ν . This is illustrated in Example 6.4.

Remark 5.7. As highlighted in Remark 4.15, steps (3) and (4) can be replaced by the computation of the determinant of a complex. Determinants of complexes are also used in syzygy approach methods for implicitization of triangular and tensor product surfaces; see for instance [Cha06, Bus01, Bot11]. More importantly, in the context of implicitization it is sufficient to compute Θ_ν . The matrix Θ_ν is known as an implicit matrix representation of the surface. Matrix representations are a useful alternative to implicit equations to represent a surface. A detailed account of their use in geometric modeling is outlined by Busé [Bus14].

6. EXAMPLES

6.1. Examples of computing residual resultants.

Example 6.1 (Residual resultant of one reduced point). We compute the residual resultant $\text{Res}_{G, (1,1)}$, where $G = \langle s, v \rangle$ is the defining ideal of the reduced point

$[0 : 1] \times [1 : 0]$ in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. Consider the system

$$\begin{aligned} F_0 &= (uc_{00} + vc_{01})s + (sc_{02} + tc_{03})v, \\ F_1 &= (uc_{10} + vc_{11})s + (sc_{12} + tc_{13})v, \\ F_2 &= (uc_{20} + vc_{21})s + (sc_{22} + tc_{23})v, \end{aligned}$$

and let $T = C \otimes R$, where $C = k[c_{ij}]$ is the ring of generic coefficients. The ideal G is a complete intersection and the matrix $\varphi \oplus \Psi$ is

$$\varphi \oplus \Psi = \begin{pmatrix} -v & uc_{00} + vc_{01} & uc_{10} + vc_{11} & uc_{20} + vc_{21} \\ s & sc_{02} + tc_{03} & sc_{12} + tc_{13} & sc_{22} + tc_{23} \end{pmatrix}.$$

To calculate $\text{Res}_{G,(1,1)}$, we find a bidegree ν as in Remark 4.10 and compute the matrix Θ_ν . Let J denote the ideal $I_2(\varphi \oplus \Psi)$. From Proposition 3.14, since the numerical parameters for this example are $(a, b) = (c, d) = (e, f) = (1, 1)$ and $(e_1, f_1) = (1, 0)$, $(e_2, f_2) = (0, 1)$, we obtain the estimate

$$\begin{aligned} \mathcal{R} &= \left(3a + c - e - \min_{1 \leq i \leq j \leq n} (e_i + e_j), 3b + d - f - \min_{1 \leq i \leq j \leq n} (f_i + f_j) \right) + \text{St}_{-3} + \mathbb{N}^2 \\ &= (3, 3) + \text{St}_{-3} + \mathbb{N}^2 \\ &= ((3, 0) + \mathbb{N}^2) \cup ((2, 1) + \mathbb{N}^2) \cup ((1, 2) + \mathbb{N}^2) \cup ((0, 3) + \mathbb{N}^2) \subseteq \text{reg}(R/J). \end{aligned}$$

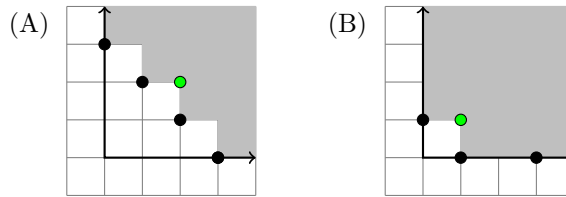


FIGURE 2. Example 6.1: (A) regularity region $\mathcal{R}(\varphi \oplus \Psi)$; (B) strong regularity region.

We can choose any ν in the interior of the regularity region to set up Θ_ν . For $\nu = (2, 2)$, $\Theta_{(2,2)}$ is a 9×24 matrix. We can alternatively use the notion of strong regularity to find bidegrees such that T/J is ν -regular. The two regularity regions are shown in Figure 2. Computing the minimal free resolution for J with *Macaulay2* [GS02] yields

$$\begin{aligned} 0 \rightarrow T(-1, -2) \oplus T(-2, -1) \oplus T(-2, -2) \rightarrow T(-1, -1)^2 \oplus T(-1, -2)^3 \oplus T(-2, -1)^3 \\ \rightarrow T(-1, -1)^6 \rightarrow T \rightarrow T/J \rightarrow 0, \end{aligned}$$

and hence

$$((1, 0) + \mathbb{N}^2) \cup ((0, 1) + \mathbb{N}^2) = \text{reg}^s(T/J) \subseteq \text{reg}(T/J).$$

This means we can compute the determinant of the EN complex restricted to bidegree $(1, 1)$ to find the residual resultant of the system. This yields the matrix $\Theta_{(1,1)}$ of size 6×12 . The residual resultant is

$$\begin{aligned} \text{Res}_{\mathcal{G},(1,1)} = & -c_{03}c_{11}c_{20} - c_{03}c_{12}c_{20} + c_{01}c_{13}c_{20} + c_{02}c_{13}c_{20} + c_{03}c_{10}c_{21} - c_{00}c_{13}c_{21} \\ & + c_{03}c_{10}c_{22} - c_{00}c_{1,3}c_{22} - c_{01}c_{10}c_{23} - c_{02}c_{10}c_{23} + c_{00}c_{11}c_{23} + c_{00}a_{12}c_{23}. \end{aligned}$$

For this example we can compute $\text{Res}_{\mathcal{G},(1,1)}$ in a much simpler way. Indeed, we can rewrite the system above as a linear system having three unknowns su, sv, tv . This system has the coefficient matrix

$$\rho = \begin{pmatrix} c_{00} & c_{01} + c_{02} & c_{03} \\ c_{10} & c_{11} + c_{12} & c_{13} \\ c_{20} & c_{21} + c_{22} & c_{23} \end{pmatrix},$$

hence the system has a solution whenever this determinant is zero. Indeed, one can check that the displayed equation above gives $\text{Res}_{G,(1,1)} = \det(\rho)$.

Example 6.2 (Residual resultant of two complete intersection points). We compute the residual resultant $\text{Res}_{\mathcal{G},(1,2)}$, where $G = \langle uv, s \rangle$ is a complete intersection defining a set of two reduced complete points in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ that lie on the same line in one of the rulings. Consider the system

$$\begin{aligned} F_0 &= (sc_{00} + tc_{01})uv + (u^2c_{02} + uvc_{03} + v^2c_{04})s, \\ F_1 &= (sc_{10} + tc_{11})uv + (u^2c_{12} + uvc_{13} + v^2c_{14})s, \\ F_2 &= (sc_{20} + tc_{21})uv + (u^2c_{22} + uvc_{23} + v^2c_{24})s. \end{aligned}$$

According to Proposition 2.9, $\text{Res}_{\mathcal{G},(1,2)}$ is of degree 2 in the coefficients of each F_i . We set up the matrix

$$\begin{aligned} &\varphi \oplus \Psi \\ &= \begin{pmatrix} -s & sc_{00} + tc_{01} & sc_{10} + tc_{11} & sc_{20} + tc_{21} \\ uv & u^2c_{02} + uvc_{03} + v^2c_{04} & u^2c_{12} + uvc_{13} + v^2c_{14} & u^2c_{22} + uvc_{23} + v^2c_{24} \end{pmatrix}. \end{aligned}$$

Let J denote the ideal $I_2(\varphi \oplus \Psi)$. In a similar fashion as in Example 6.1, we compute the regularity region for R/J specified in Proposition 3.14 as illustrated in Figure 3. From this region it follows that we may use $\nu = (1, 6)$. The matrix $\Theta_{(1,6)}$ is of size 14×30 . The strong regularity region in this case is depicted in Figure 3 (right)

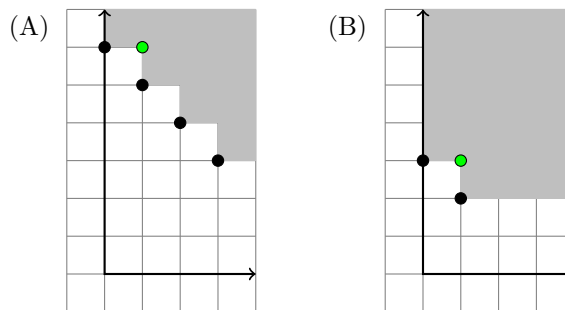


FIGURE 3. Example 6.2: (A) Regularity region $\mathcal{R}(\varphi \oplus \Psi)$; (B) strong regularity region.

and is given by

$$\text{reg}^s(T/J) = ((1, 2) + \mathbb{Z}_+^2) \cup ((0, 3) + \mathbb{Z}_+^2).$$

Estimating the regularity of T/J using the strong regularity region allows the use of the bidegree $\nu = (1, 3)$, for which the matrix $\Theta_{(1,3)}$ is an 8×12 matrix. The polynomial $\text{Res}_{\mathcal{G},(1,2)}$ contains 141 terms.

In this example, it is true that $\mathbb{F} : G = I_2(\varphi \oplus \Psi)$, but not for the reason given in the hypothesis of Lemma 4.1(2). The Eagon–Northcott complex obtained from

the matrix $\varphi \oplus \Psi$ is a virtual projective resolution for $I_2(\varphi \oplus \Psi)$, but it is not a resolution for this ideal since it is not exact. This is to be expected considering the proof of Lemma 3.11 because $\text{ht}(I_2(\varphi \oplus \Psi)) = 2$. However, this allows us to estimate the regularity of $T/I_2(\varphi \oplus \Psi)$ using Proposition 3.14 as pictured in Figure 3.

6.2. Examples of implicitization. In this section we illustrate the techniques we developed in the previous sections to compute the implicit equation of a map $\mathbb{P}_k^1 \times \mathbb{P}_k^1 \rightarrow \mathbb{P}^3$ defined by four bihomogeneous polynomials of bidegree (a, b) .

Example 6.3. Let $I = \langle s, u \rangle \cap \langle t, v \rangle$ be the ideal from Example 3.4 which defines two noncollinear points in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. This set is pictured below together with its Hilbert function:

$$\begin{array}{c} \begin{array}{cc} s & t \\ \hline \bullet & \bullet \\ \hline \bullet & \bullet \\ \hline \end{array} \begin{array}{l} u \\ v \end{array} \quad H_X = \begin{array}{c|c|c|c|c} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 2 & 2 & 2 \\ \hline 1 & 2 & 2 & 2 & 2 \\ \hline 2 & 2 & 2 & 2 & 2 \\ \hline 3 & 2 & 2 & 2 & 2 \end{array}$$

Let $G = \langle sv, tu \rangle$, and denote by g_1, g_2 the two generators of G . Here G is a complete intersection with resolution

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -sv \\ tu \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} tu & sv \end{pmatrix}} G \longrightarrow 0.$$

Note that $G^{\text{sat}} = I$, so, while G is not saturated, $V(G) = V(I)$, and therefore the complex displayed above is a Hilbert–Burch virtual resolution for I . Next we consider the ideal $P = \langle p_0, p_1, p_2, p_3 \rangle$, where $[p_0 \ p_1 \ p_2 \ p_3] = [g_1 \ g_2] h$ and h is the 2×4 matrix

$$h = \begin{bmatrix} s & t & 0 & 0 \\ 0 & 0 & s & t \end{bmatrix}.$$

The bihomogeneous polynomials $p_0 = s^2v, p_1 = stv, p_2 = stu, p_3 = t^2u$ define a parameterization of a tensor product surface of bidegree $(2, 1)$ with two basepoints given by $V(P)$. Note that the homogeneous implicit equation for this surface is easily obtained and equal to $YZ - XW = 0$. Since the primary decomposition of the ideal P is $P = (s^2, st, t^2) \cap (s, u) \cap (t, v) \cap (u, v)$, it follows that $P^{\text{sat}} = G^{\text{sat}}$. We obtain the matrix Ψ by writing

$$\begin{aligned} & [p_0 - Xp_3 \quad p_1 - Yp_3 \quad p_2 - Zp_3] \\ &= [g_1 \quad g_2] \underbrace{\begin{bmatrix} h_{01} - h_{31}X & h_{11} - h_{31}Y & h_{21} - h_{31}Z \\ h_{02} - h_{32}X & h_{12} - h_{32}Y & h_{22} - h_{32}Z \end{bmatrix}}_{\Psi}. \end{aligned}$$

The ideal G satisfies the conditions in Proposition 2.6, hence $\text{Res}_{\mathcal{G}, (2,1)}$ exists. To obtain the implicit equation using a residual resultant we set up the matrix Θ_ν for a bidegree ν according to Remark 4.10. On one hand we compute the regularity region of $EN(\phi \oplus \Psi)$ following Proposition 3.14. On the other hand we compute the strong regularity region determined by a minimal free resolution of T/J . The regions found by these two methods and the shifts in the minimal free resolution of R/J are displayed in Figure 4.

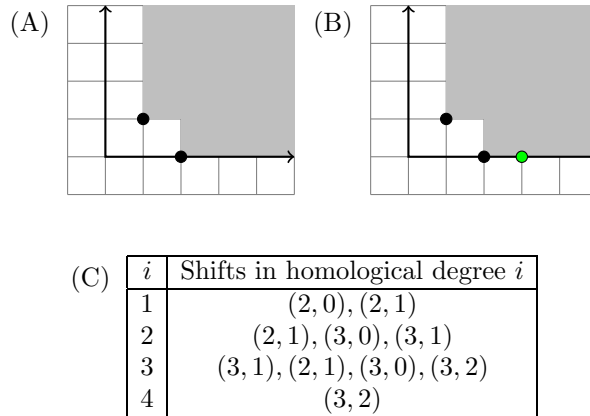


FIGURE 4. Example 6.3: (A) regularity region from Proposition 3.14; (B) strong regularity; and (C) bigraded shifts of a minimal free resolution of T/J .

Note that in this example, the two methods of estimating the regularity region for the module $T/I_n(\varphi \oplus \Psi)$ agree as shown in Figure 4. Now for $\nu = (3, 0)$ one has

$$\Theta_\nu = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -Y & 0 & -Z & 1 & 1 & 0 \\ X & -Y & 0 & -Z & -Z & 1 \\ 0 & X & 0 & 0 & 0 & -Z \end{bmatrix},$$

whence $I_4(\Theta_\nu) = \langle YZ - X \rangle$ gives the implicit equation restricted to the affine set $W = 1$. The method in [Bot11] for $\nu = (2a - 1, b - 1) = (3, 0)$ also produces a matrix representation of the implicit equation of size 4×6 .

Example 6.4. Using the same setup as in Example 6.3, we change the entries of the matrix h that determines the parametrization ideal P . Set

$$h = \begin{bmatrix} su & sv & 0 & tu + sv \\ 0 & tu & su & tv \end{bmatrix},$$

so $P = \langle s^2uv, t^2u^2 + s^2v^2, st u^2, st uv + t^2uv + s^2v^2 \rangle$. The generators of P define a tensor product surface of bidegree $(2, 2)$ with two basepoints. The support of P and G is the same; however, the primary decomposition of P reveals that the point corresponding to (s, u) has multiplicity 2 in the scheme defined by P . However, the primary component corresponding to this point in P is (s, u^2) , therefore this ideal is still locally a complete intersection and our results apply.

In this case we cannot use the Eagon–Northcott complex $EN(\varphi \oplus \Psi)$ to compute bidegrees in the regularity region of $T/I_2(\varphi \oplus \Psi)$ because the first homology module of $EN(\varphi \oplus \Psi)$ is not B -torsion. In fact, the first homology is a torsion module supported at the point with multiplicity 2 i.e., $\langle s, u \rangle$. This shows the necessity of the hypothesis of Proposition 4.4.

The free resolution of $T/I_2(\varphi \oplus \Psi)$ is $0 \rightarrow T^5 \rightarrow T^{23} \rightarrow T^{32} \rightarrow T^{19} \rightarrow T^6 \rightarrow C$, and the strong regularity region $\text{reg}^s(T/I_n(\varphi \oplus \Psi)) = ((2, 4) + \mathbb{Z}_+^2) \cup ((3, 3) + \mathbb{Z}_+^2) \cup ((4, 2) + \mathbb{Z}_+^2)$ is depicted in Figure 5. Therefore for $\nu = (5, 2)$ the

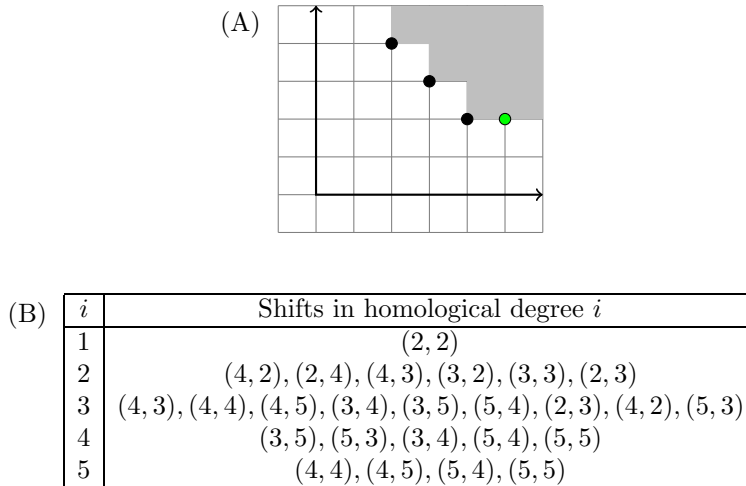


FIGURE 5. Example 6.4 (A) strong regularity and (B) shifts in the resolution for $T/I_n(\varphi \oplus \Psi)$.

18×24 matrix $\Theta_{(5,2)}$ provides the implicit equation. The method in [Bot11] for $\nu = (2a - 1, b - 1) = (3, 1)$ also produces a matrix representation of size 8×11 .

Although $(3, 2)$ is not inside the strong regularity region, we can use this bidegree to set up a 12×12 matrix Θ_ν whose determinant vanishes, but that has an 11×11 maximal minor whose determinant is a multiple of the implicit equation of the tensor product surface:

$$\Theta_{(3,2),11 \times 11} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -X & 0 & -Y+1 \\ X & 0 & Y-1 & 0 & 0 & 0 & Z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -X & 1 \\ X & -1 & Y & 0 & -X-Y & 0 & Z & 0 & -Z & -X & Z \\ 0 & X & 0 & Y-1 & X & 0 & 0 & Z & 0 & 0 & -Z \\ 0 & 0 & -1 & 0 & -X & 1 & 0 & -1 & 0 & -X & Z \\ X & X & Y & Y & 0 & -X-Y & Z & Z & 0 & -Z & 0 \\ 0 & 0 & 0 & 0 & 0 & X & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -X & 0 & 0 & 0 & 0 & 0 \\ 0 & X & 0 & Y & 0 & 0 & 0 & Z & 0 & 0 & 0 \end{pmatrix}.$$

The implicit equation is the degree 5 factor of

$$X \cdot (X^4 Y + X^3 Y Z + X^2 Y Z^2 + X Y^2 Z^2 + X Y Z^3 - X^4 - 2 X^2 Z^2 - Z^4).$$

The 11×11 minors of Θ_ν yield, in addition to the above minor, the polynomials

$$(Y-1) \cdot (X^4 Y + X^3 Y Z + X^2 Y Z^2 + X Y^2 Z^2 + X Y Z^3 - X^4 - 2 X^2 Z^2 - Z^4) \text{ and}$$

$$Z \cdot (X^4 Y + X^3 Y Z + X^2 Y Z^2 + X Y^2 Z^2 + X Y Z^3 - X^4 - 2 X^2 Z^2 - Z^4),$$

thus the greatest common divisor of these minors is the implicit equation.

In this example the cokernel of $\Theta_{(3,2)}$ is 1-dimensional, and the sum of the multiplicities of the basepoints is three, but there are two basepoints. This illustrates the observation made in Remark 5.6 that the residual resultant can be recovered as a divisor of the submaximal minors of $\Theta_{(3,2)}$ even if the base points in P have higher multiplicity than the points in G .

Example 6.5. We continue with the setup from Example 6.3 and change h to

$$h^\top = \begin{pmatrix} \frac{4}{9}su + tu + sv + \frac{9}{5}tv & 10su + \frac{1}{2}tu + \frac{2}{3}sv + \frac{2}{3}tv \\ \frac{1}{3}su + \frac{10}{7}tu + \frac{9}{4}sv + \frac{2}{9}tv & \frac{8}{5}su + \frac{1}{2}tu + \frac{5}{7}sv + \frac{2}{3}tv \\ su + \frac{4}{5}tu + sv + \frac{5}{8}tv & 2su + \frac{7}{3}tu + sv + \frac{9}{5}tv \\ \frac{3}{5}su + \frac{7}{3}tu + sv + 8tv & \frac{4}{5}su + \frac{7}{3}tu + \frac{3}{10}sv + \frac{7}{9}tv \end{pmatrix}.$$

This choice of h determines the ideal P and a tensor product surface of bidegree $(2, 2)$ with two basepoints $V(P)$ and $P^{\text{sat}} = G^{\text{sat}}$. We use Proposition 3.14 to obtain the regularity region of $EN(\varphi \oplus \Psi)$ depicted in Figure 6. The resolution of $T/I_2(\varphi \oplus \Psi)$ is

$$0 \rightarrow T^{119} \rightarrow T^{171} \rightarrow T^{71} \rightarrow T^{24} \rightarrow T^6 \rightarrow T.$$

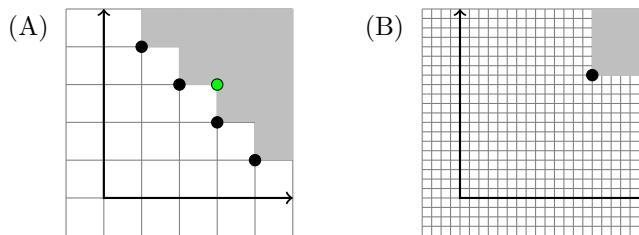


FIGURE 6. Example 6.5: (A) regularity region of $EN(\varphi \oplus \Psi)$ and (B) strong regularity region with corner $(14, 11)$.

The strong regularity region for this example is considerably worse than the regularity provided by Proposition 3.14. For $\nu = (3, 3)$, Θ_ν is a matrix of size 16×24 . Although the point $\nu = (2, 3)$ is not in the interior of the regularity regions in Figure 6, the matrix Θ_ν provides a 12×12 determinantal representation for the implicit equation of the surface. The method in [Bot11] for $\nu = (2a-1, b-1) = (3, 1)$ produces a matrix representation of size 8×10 . Botbol's method gives the implicit equation as a quotient of two determinants, whereas the residual resultant gives a representation as a single determinant.

7. OPEN PROBLEMS

The results we present open the avenue for a more detailed study of virtual resolutions from a computational point of view and for their use in elimination theory. We formulate some questions that we encountered in our investigations and of possible further generalizations of residual resultants.

Question 7.1. The computation of the residual resultant $\text{Res}_{\mathcal{G}, \{(a_i, b_i)\}_{i=0}^2}$ relies on the availability of Hilbert–Burch resolutions of points in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$. As highlighted after Corollary 3.3, it is always possible to find Hilbert–Burch for any ideal of points. However, there could be several different choices for this type of complex. Given a set of points in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$, what are all possible virtual Hilbert–Burch resolutions of the defining ideal of the points? Is it possible to formulate an algorithm to construct them? How does the choice of virtual Hilbert–Burch resolution affect Algorithms 4.14 and 5.5?

Question 7.2. The regularity region $\mathcal{R}(\alpha)$ that we present in Proposition 3.14 makes the computation of the residual resultant feasible. However, as we observed in Examples 6.2, 6.4, and 6.5 we can choose a bidegree $\nu \notin \mathcal{R}(\alpha)$ for which the computation still holds. Are there better bounds on the regularity region of the module $T/I_n(\varphi \oplus \Psi)$ to compute the residual resultant?

Question 7.3. We generalized residual resultants in \mathbb{P}^2 [Bus01] to residual resultants in $\mathbb{P}_k^1 \times \mathbb{P}_k^1$ and computed them by using virtual resolutions. Is it possible to generalize residual resultants in \mathbb{P}^n from [BEM01] to residual resultants in multi-projective spaces and use virtual resolutions to compute them?

ACKNOWLEDGMENTS

Computations using the software system Macaulay2 [GS02] were crucial for the development of this paper. We thank Laurent Busé for suggesting that we explore residual resultants on $\mathbb{P}_k^1 \times \mathbb{P}_k^1$.

REFERENCES

- [BES20] C. Berkesch, D. Erman, and G. G. Smith, *Virtual resolutions for a product of projective spaces*, *Algebr. Geom.* **7** (2020), 460–481, DOI 10.14231/AG-2020-013.
- [Bot11] N. Botbol, *The implicit equation of a multigraded hypersurface*, *J. Algebra* **348** (2011), 381–401, DOI 10.1016/j.jalgebra.2011.09.019. MR2852248
- [BE77] D. A. Buchsbaum and D. Eisenbud, *What annihilates a module?*, *J. Algebra* **47** (1977), no. 2, 231–243, DOI 10.1016/0021-8693(77)90223-X. MR476736
- [BEM01] L. Busé, M. Elkadi, and B. Mourrain, *Resultant over the residual of a complete intersection: Effective methods in algebraic geometry (Bath, 2000)*, *J. Pure Appl. Algebra* **164** (2001), no. 1-2, 35–57, DOI 10.1016/S0022-4049(00)00144-4. MR1854329
- [Bus01] L. Busé, *Residual resultant over the projective plane and the implicitization problem*, *Proceedings of the 2001 International Symposium on Symbolic and Algebraic Computation*, ACM, New York, 2001, pp. 48–55, DOI 10.1145/384101.384109. MR2049730
- [Bus14] L. Busé, *Implicit matrix representations of rational Bézier curves and surfaces*, *Comput.-Aided Des.* **46** (2014), 14–24, DOI 10.1016/j.cad.2013.08.014. MR3124172
- [Cha06] M. Chardin, *Implicitization using approximation complexes*, *Algebraic geometry and geometric modeling*, *Math. Vis.*, Springer, Berlin, 2006, pp. 23–35, DOI 10.1007/978-3-540-33275-6_2. MR2279841
- [CFG⁺16] S. Cooper, G. Fatabbi, E. Guardo, A. Lorenzini, J. Migliore, U. Nagel, A. Seceleanu, J. Szpond, and A. Van Tuyl, *Symbolic powers of codimension two cohen-macaulay ideals*, *arXiv preprint arXiv:1606.00935* (2016).
- [CDS07] D. Cox, A. Dickenstein, and H. Schenck, *A case study in bigraded commutative algebra, Syzygies and Hilbert functions*, *Lect. Notes Pure Appl. Math.*, vol. 254, Chapman & Hall/CRC, Boca Raton, FL, 2007, pp. 67–111, DOI 10.1201/9781420050912.ch3. MR2309927
- [CGZ00] D. Cox, R. Goldman, and M. Zhang, *On the validity of implicitization by moving quadrics of rational surfaces with no base points*, *J. Symbolic Comput.* **29** (2000), no. 3, 419–440, DOI 10.1006/jsco.1999.0325. MR1751389
- [D’A02] C. D’Andrea, *Macaulay style formulas for sparse resultants*, *Trans. Amer. Math. Soc.* **354** (2002), no. 7, 2595–2629, DOI 10.1090/S0002-9947-02-02910-0. MR1895195
- [EN62] J. A. Eagon and D. G. Northcott, *Ideals defined by matrices and a certain complex associated with them*, *Proc. Roy. Soc. London Ser. A* **269** (1962), 188–204, DOI 10.1098/rspa.1962.0170. MR142592
- [Eis95] D. Eisenbud, *Commutative algebra: With a view toward algebraic geometry*, *Graduate Texts in Mathematics*, vol. 150, Springer-Verlag, New York, 1995. MR1322960
- [Ful84] W. Fulton, *Intersection theory*, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*, vol. 2, Springer-Verlag, Berlin, 1984. MR732620

- [GLLM19] J. Gao, Y. Li, M. Loper, and A. Matoo, *Virtual complete intersection in $\mathbb{P}^1 \times \mathbb{P}^1$* , arXiv preprint [arXiv:1905.0999](https://arxiv.org/abs/1905.0999) (2019).
- [GKZ08] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2008. Reprint of the 1994 edition. MR2394437
- [GS02] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, 2002.
- [Har77] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977. MR0463157
- [HW04] J. W. Hoffman and H. H. Wang, *Castelnuovo-Mumford regularity in biprojective spaces*, Adv. Geom. **4** (2004), no. 4, 513–536, DOI 10.1515/adv.2004.4.4.513. MR2096526
- [Jou91] J.-P. Jouanolou, *Le formalisme du résultant* (French), Adv. Math. **90** (1991), no. 2, 117–263, DOI 10.1016/0001-8708(91)90031-2. MR1142904
- [Jou95] J.-P. Jouanolou, *Aspects invariants de l'élimination* (French), Adv. Math. **114** (1995), no. 1, 1–174, DOI 10.1006/aima.1995.1042. MR1344713
- [KSZ92] M. M. Kapranov, B. Sturmfels, and A. V. Zelevinsky, *Chow polytopes and general resultants*, Duke Math. J. **67** (1992), no. 1, 189–218, DOI 10.1215/S0012-7094-92-06707-X. MR1174606
- [MS04] D. Maclagan and G. G. Smith, *Multigraded Castelnuovo-Mumford regularity*, J. Reine Angew. Math. **571** (2004), 179–212, DOI 10.1515/crll.2004.040. MR2070149
- [MS05] D. Maclagan and G. G. Smith, *Uniform bounds on multigraded regularity*, J. Algebraic Geom. **14** (2005), no. 1, 137–164, DOI 10.1090/S1056-3911-04-00385-6. MR2092129
- [SC95] T. Sederberg and F. Chen, *Residual resultant over the projective plane and the implicitization problem*, Siggraph 1995 conference proceedings, 1995, pp. 301–308.

MAX-PLANCK-INSTITUTE FOR MATHEMATICS IN THE SCIENCES, LEIPZIG; OTTO-VON-GUERICKE UNIVERSITÄT, MAGDEBURG

Email address: eliana.duarte@ovgu.de

MATHEMATICS DEPARTMENT, UNIVERSITY OF NEBRASKA–LINCOLN, LINCOLN, NEBRASKA 68588

Email address: aseceleanu@unl.edu