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Stability of a Subcritical Fluid Model for Fair Bandwidth Sharing with General File Size Distributions

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Abstract

This work concerns the asymptotic behavior of solutions to a (strictly) subcritical fluid model for a data communication network, where file sizes are generally distributed and the network operates under a fair bandwidth sharing policy. Here we consider fair bandwidth sharing policies that are a slight generalization of the α-fair policies of Mo and Walrand (2000). It has been a standing problem to prove stability of the data communications network model of Massoulié and Roberts (2000), operating under fair bandwidth sharing policies, when the offered load is less than capacity (subcritical conditions). A crucial step in an approach to this problem is to prove stability of subcritical fluid model solutions. Paganini et al. (2012) introduced a Lyapunov function for this purpose and gave an argument, assuming that fluid model solutions are sufficiently smooth in time and space that they are strong solutions of a partial differential equation and assuming that no fluid level on any route touches zero before all route levels reach zero. The aim of the current paper is to prove stability of the subcritical fluid model without these strong assumptions.

Starting with a slight generalization of the Lyapunov function proposed by Paganini et al. (2012), assuming that each component of the initial state of a measure-valued fluid model solution, as well as the file size distributions, have no atoms and have finite first moments, we prove absolute continuity in time of the composition of the Lyapunov function with any subcritical fluid model solution and describe the associated density. We use this to prove that the Lyapunov function composed with such a subcritical fluid model solution converges to zero as time goes to infinity. This implies that each component of the measure-valued fluid model solution converges vaguely on $(0, \infty)$ to the zero measure as time goes to infinity. Under the further assumption that the file size distributions have finite p-th moments for some $p > 1$ and that each component of the initial state of the fluid model solution has finite p-th moment, it is proved that the fluid model solution reaches the measure with all components equal to the zero measure in finite time and that the time to reach this zero state has a uniform bound for all fluid model solutions having a uniform bound on the initial total mass and the p-th moment of each component of the initial state. In contrast to the analysis in Paganini et al. (2012), we do not need their strong smoothness assumptions on fluid model solutions and we rigorously treat the realistic, but singular situation, where the fluid level on some routes becomes zero while other route levels remain positive.

Contents

1. Introduction

The design and analysis of congestion control mechanisms for modern data networks such as the Internet is a challenging problem. Mathematical models at various levels have been introduced in an effort to provide insight into some aspects of this problem. In particular, Massoulié and Roberts (2000) introduced a stochastic model called a flow level model that aimed to capture the connection level dynamics of file arrivals and departures in a network where bandwidth is dynamically shared amongst flows which correspond to continuous transfers of individual elastic files. A natural family of "fair" bandwidth sharing policies was introduced by Mo and Walrand (2000) around the same time. These policies are often referred to as (weighted) α-fair policies, since a parameter $\alpha \in (0,\infty)$ (and optional weight parameters) is associated with the family. The cases $\alpha = 1$ (proportional fairness) and $\alpha \to \infty$ (max-min fairness) have received particular attention.

One of the first natural questions to ask about the flow level model operating under an α-fair bandwidth sharing policy is "when is it stable?". Here we take stability to mean that a Markov process describing the model is positive Harris recurrent. Assuming Poisson arrivals and exponential file sizes, this is a solved problem. Indeed, under these assumptions, Lyapunov functions constructed by De Veciana et al. (2001) for weighted max-min fair and proportionally fair policies, and by Bonald and Massoulié (2001) for weighted α-fair policies ($\alpha \in (0,\infty)$), can be used to establish positive recurrence of the Markov chain that tracks the number of flows on each route, provided the network is subcritically loaded, i.e., the average load on each link is less than its capacity. Kelly and Williams (2004) proved that subcriticality is necessary for positive recurrence of the Markov chain. Ye et al. (2005) generalized the stability result to where the arrival processes are stationary renewal processes, but the file sizes are still exponentially distributed, and the bandwidth sharing policies come from a class of utility based policies that include the weighted α-fair policies.

When the interarrival time and file sizes are generally distributed, the process that records the number of flows on each route is usually not Markovian and a more complicated Markovian state descriptor is needed to track the dynamics of the model. Much less is known concerning stability in this general situation, although a few cases have been treated. Massoulié (2007) showed stability of subcritical networks under the proportionally fair policy with Poisson arrivals and phase-type distributions for file sizes. Bramson (2010) proved that subcritical networks operating under weighted max-min fair policies and having general interarrival and file size distributions are stable, provided the file size distributions have finite p-th moments for some $p > 2$.

One general approach to exploring stability of stochastic networks uses fluid models, solutions of which are obtained as functional law of large numbers limits from the original stochastic network. The idea of this approach is to first prove that the fluid model for a subcritical network is stable (i.e., all fluid model solutions converge towards the zero state) and then to use this to infer stability of the original stochastic model. This methodology has been successfully used to obtain sufficient conditions for stability of a variety of multiclass queueing networks (see Bramson (2008), Dai (1995) and the references therein) and was the approach used in the work by Massoulié (2007) mentioned above.

Gromoll and Williams (2009) used a measure-valued process to track the dynamics of the flow level model with general interarrival and file size distributions when operating under a member of a family of fairly general bandwidth sharing policies that includes the weighted α-fair policies of Mo and Walrand (2000). They showed that, under law of large numbers scaling, the measure-valued processes corresponding to a sequence of flow level models are tight and any weak limit point of the sequence is almost surely a continuous solution of a measure-valued fluid model. In Gromoll and Williams (2008), the same authors also established stability of the fluid model for weighted α-fair bandwidth sharing policies ($\alpha \in (0,\infty)$), for linear networks and simple tree networks under subcritical loading. In this context, the zero state is the measure with each component equal to the zero measure on $[0,\infty)$.

Chiang et al. (2006) obtained the same fluid model as Gromoll and Williams (2009) (but with a zero initial condition) from the flow level model via a different law of large numbers scaling limit in which the arrival rate and bandwidth capacity are allowed to grow to infinity proportionally, but the bandwidth per flow stays uniformly bounded. They used the fluid model to derive some conclusions concerning rate stability for the flow level model when file sizes have general distributions with compact support, and for bandwidth sharing policies that are a slight generalization of the α-fair policies of Mo and Walrand (2000), in which the parameter $\alpha \in (0,\infty)$ is allowed to vary with the route. For their stability result, their α parameters need to be sufficiently small.

Paganini et al. (2012) developed a Lyapunov function to study the stability of the fluid model introduced by Gromoll and Williams (2009) for all weighted α-fair policies ($\alpha \in (0,\infty)$). Using this function, under the assumptions that fluid model solutions are sufficiently smooth that they have densities that are strong solutions of a nonlinear parabolic partial differential equation, and that no fluid level on any route touches zero before all route levels reach zero, Paganini et al. (2012) proved stability of the subcritical fluid model. The aim of the current paper is to prove stability of the subcritical fluid model without the strong assumptions of Paganini et al. (2012).

In this paper, starting with a slight generalization of the Lyapunov function proposed by Paganini et al. (2012) (to accommodate the generalization of α-fair policies introduced by Chiang et al. (2006)), assuming that each component of the initial state of a fluid model solution and the file size distributions have no atoms and have finite first moments, we prove absolute continuity in time of the composition of the Lyapunov function with any subcritical fluid model solution and describe the associated density. We use this to prove that the Lyapunov function composed with such a subcritical fluid model solution converges to zero as time goes to infinity. This implies that each component of the fluid model solution converges vaguely on $(0,\infty)$ to the zero measure as time goes to infinity. Under the further assumption that the file size distributions have finite p-th moments for some $p>1$ and that each component of the initial state of the fluid model solution has finite p-th moments, it is proved that the fluid model solution reaches the zero state in finite time and that the time to reach the zero state has a uniform bound for all fluid model solutions having a uniform bound on the initial total mass and the p-th moment of the initial state. In contrast to the analysis of Paganini et al. (2012), we do not need their strong smoothness assumptions on fluid model solutions and we rigorously treat the realistic, but singular situation, where the fluid level on some routes becomes zero while other route levels remain positive.

In terms of stability of the original flow level model of Massoulié and Roberts (2000), the thesis of Nam Lee (2008) provides sufficient conditions for positive recurrence of an age-based Markovian state descriptor for the flow level model, in terms of stability of the associated fluid model. In a similar manner to that described by Paganini et al. (2012), this result can be used to establish such positive recurrence under the nominal underloaded condition, provided a number of additional technical assumptions are satisfied, including a light-tailed assumption on the file size distributions and a commonly used spread-out assumption on the interarrival times. Since the formal description of such a result involves considerable setup and description of the flow level model and of the substantial technical assumptions of Lee (2008), we leave the formal description of such a result to a subsequent article coauthored with Lee, where we also hope to simplify the technical assumptions of the thesis of Lee (2008).

The structure of the paper is as follows. In Section 2 we recall the fluid model of Gromoll and Williams (2009). In Section 3, we introduce assumptions on the parameters under which our results will be proved and define the Lyapunov function H (as a function of measures); this is a slight variant of the function proposed by Paganini et al. (2012). We also introduce the composition $\mathcal{H}^\zeta$ of H with a fluid model solution ζ, and a function $\mathcal{K}^\zeta$ which is used to describe the density in time of $\mathcal{H}^\zeta$. Our main results, as outlined in the paragraph above, are stated in Section 4. The proofs of these main results are given in Sections 6 and 7. Some preliminary lemmas needed for our proofs are given in Section 5.

Our proofs have benefitted from prior works of others. In particular, our starting point is the clever Lyapunov function posited by Paganini et al. (2012). The preliminary results in Section 5 include three lemmas taken from the work of Paganini et al. (2012) and four lemmas and a corollary giving some basic properties of fluid model solutions. The proofs of the fluid model solution results extend some techniques developed by Gromoll et al. (2002) for a critical fluid model of a single class processor sharing queue. The latter is a special case of a bandwidth sharing model with one route and one link. The final result in Section 5 is our proof of the continuity of the function $\mathcal{H}^\zeta(\cdot)$, which is a critical precursor to our proof of absolute continuity of this function. The key new results, Theorem 4.1 and Corollary 4.1, are proved in Section 6. These rely on a result proved in Section 6.1, where we show that smoothed versions of each component of a fluid model solution satisfy certain parabolic partial differential equations on intervals of time where the fluid level for the component is not zero. This provides a rigorous formulation of a partial differential equation assumed to hold by Paganini et al. (2012). A similar smoothing technique was also used by Puha and Williams (2016), in the study of the asymptotic behavior of critical fluid model solutions for a single class processor sharing queue. Our method is a little different from that of Puha and Williams (2016) in that we smooth the entire fluid model solution, not just the initial condition. Theorems 4.2 and 4.3 are proved in Section 7. Having Theorem 4.1 and Corollary 4.1 in place, these proofs follow a similar line of argument to that of Paganini et al. (2012). However, we do generalize from having a common parameter α for all routes to the case where there is a separate α_i for each route $i\in\mathcal{I}$, and we also establish uniformity of the convergence to the zero state under suitable conditions. Throughout, our proofs need to deal with the more complex bandwidth sharing model and especially to deal with the singular situation where the fluid level for some routes can reach zero while other route levels remain positive.

1.1. Notation

Let $\mathbb{R}=(-\infty,\infty)$ and $\mathbb{R}_+=[0,\infty)$. Define $\mathbf{C}_b^1(\mathbb{R})$ (resp. $\mathbf{C}_b^1(\mathbb{R}_+)$) to be the set of once continuously differentiable functions $f:\mathbb{R}\to\mathbb{R}$ (resp. $f:\mathbb{R}_+\to\mathbb{R}$) that together with their first derivatives are continuous and bounded on $\mathbb{R}$ (resp. $\mathbb{R}_+$). Let $\mathbf{C}_c^\infty(\mathbb{R})$ be the set of infinitely differentiable functions defined on the real line that have compact support. Let $\mathbb{1}_A$ denote the indicator function of a set A.

Let $\mathbf{M}$ be the set of finite nonnegative Borel measures on $\mathbb{R}_+$, endowed with the topology of weak convergence. Given $\xi\in\mathbf{M}$, let $\mathbf{L}^1(\xi)$ denote the set of Borel measurable functions from $\mathbb{R}_+$ into $\mathbb{R}$ that are integrable with respect to ξ. For $f\in\mathbf{L}^1(\xi)$, let $\langle f,\xi\rangle=\int_{\mathbb{R}_+} f\,d\xi$. Also for any non-negative Borel measurable function $f\notin\mathbf{L}^1(\xi)$, let $\langle f,\xi\rangle=+\infty$. For $x\in\mathbb{R}_+$, let $\chi(x)=x$. Define $\mathbf{M}_1=\{\xi\in\mathbf{M}:\langle\chi,\xi\rangle<\infty\}$. Let $\mathbf{K}=\{\xi\in\mathbf{M}:\xi(\{x\})=0 \text{ for all } x\in\mathbb{R}_+\}$, the set of continuous measures in $\mathbf{M}$, and let $\mathbf{K}_1=\mathbf{M}_1\cap\mathbf{K}$. Let $\mathbf{A}$ denote the elements of $\mathbf{M}$ that are absolutely continuous (with respect to Lebesgue measure).

For $\mathbf{I}\in\mathbb{N}$, let $\mathcal{I}=\{1,\dots,\mathbf{I}\}$ and define

$$\mathbf{M}^{\mathbf{I}}=\{(\xi_1,\dots,\xi_{\mathbf{I}}):\xi_i\in\mathbf{M} \text{ for all } i\in\mathcal{I}\},$$

$$\mathbf{K}^{\mathbf{I}}=\{(\xi_1,\dots,\xi_{\mathbf{I}}):\xi_i\in\mathbf{K} \text{ for all } i\in\mathcal{I}\},$$

$$\mathbf{K}_1^{\mathbf{I}} = \{(\xi_1, \ldots, \xi_{\mathbf{I}}) : \xi_i \in \mathbf{K}_1 \text{ for all } i \in \mathcal{I}\},$$

$$\mathbf{A}^{\mathbf{I}} = \{(\xi_1, \ldots, \xi_{\mathbf{I}}) : \xi_i \in \mathbf{A} \text{ for all } i \in \mathcal{I}\}.$$

Here $\mathbf{M}^{\mathbf{I}}$ has its product topology and the other sets have the induced topologies as subsets of $\mathbf{M}^{\mathbf{I}}$. Fluid model solutions will take values in $\mathbf{M}^{\mathbf{I}}$ and we shall refer to the measure $\xi \in \mathbf{M}^{\mathbf{I}}$ that has ξ_i equal to the zero measure on $\mathbb{R}_+$ for all $i \in \mathcal{I}$, as the zero measure (in $\mathbf{M}^{\mathbf{I}}$) or the zero state (for the fluid model).

2. Fluid Model

Here we recall the fluid model developed by Gromoll and Williams (2009) as a functional law of large numbers approximation to the flow level model of Massoulié and Roberts (2000) operating under a bandwidth sharing policy such as one of the α-fair policies of Mo and Walrand (2000). This fluid model (with a zero initial condition) was also obtained by Chiang et al. (2006) from the flow level model operating under a slight generalization of the α-fair policies of Mo and Walrand (2000). This used a different law of large numbers scaling limit from Gromoll and Williams (2009); in particular, in the work of Chiang et al. (2006), the arrival rate and bandwidth capacity were allowed to grow to infinity proportionally. We begin by introducing the fluid model parameters.

2.1. Parameters

Consider finitely many resources (e.g., links in a communication network) labelled by $j \in \mathcal{J} \equiv \{1, \ldots, \mathbf{J}\}$, and a finite set of routes labeled by $i \in \mathcal{I} \equiv \{1, \ldots, \mathbf{I}\}$. A route $i \in \mathcal{I}$ is simply a non-empty subset of $\mathcal{J}$ and is interpreted as the set of resources used by the route. Let R be the $\mathbf{J} \times \mathbf{I}$ incidence matrix satisfying $R_{ji} = 1$ if resource j is used by route i, and $R_{ji} = 0$ otherwise. Each resource $j \in \mathcal{J}$ has a fixed (bandwidth) capacity $C_j > 0$.

Fix a vector $\nu = (\nu_1, \ldots, \nu_{\mathbf{I}})$ where $\nu_i > 0$ for each $i \in \mathcal{I}$, and a vector $\vartheta = (\vartheta_1, \ldots, \vartheta_{\mathbf{I}})$ where for each $i \in \mathcal{I}$, ϑ_i is a Borel probability measure on $\mathbb{R}_+$ that does not charge the origin and has finite mean, i.e., $\langle \chi, \vartheta_i \rangle < \infty$. For $i \in \mathcal{I}$, the constant ν_i represents the mean arrival rate of files to route i and ϑ_i represents the distribution for the sizes of files arriving to route i.

For each $i \in \mathcal{I}$, $\mu_i \equiv \frac{1}{\langle \chi, \vartheta_i \rangle}$ is the reciprocal of the mean of the distribution ϑ_i and $\rho_i \equiv \frac{\nu_i}{\mu_i}$ is interpreted as the *nominal load* (average bandwidth needed) on route i. For each $i \in \mathcal{I}$, let ϑ_i^e be the *excess lifetime distribution* associated with ϑ_i. The probability measure ϑ_i^e is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}_+$ and has density

$$p_i^e(x) = \mu_i \langle \mathbb{1}_{(x,\infty)}, \vartheta_i \rangle \quad \text{for all } x \in \mathbb{R}_+. \tag{2.1}$$

For each $i \in \mathcal{I}$, we define $N_i(x) = \langle \mathbb{1}_{[0,x]}, \vartheta_i \rangle$, $\overline{N}_i(x) = 1 - N_i(x)$, $N_i^e(x) = \langle \mathbb{1}_{[0,x]}, \vartheta_i^e \rangle$, and $\overline{N}_i^e(x) = 1 - N_i^e(x)$ for each $x \in \mathbb{R}_+$. Note that $\mu_i^{-1} = \int_0^\infty \overline{N}_i(x)dx$ and $p_i^e(x) = \mu_i \overline{N}_i(x)$ for all $x \in \mathbb{R}_+$. For $\xi \in \mathbf{M}^{\mathbf{I}}$, for each $i \in \mathcal{I}$, define $\overline{M}_\xi^i(x) = \langle \mathbb{1}_{(x,\infty)}, \xi_i \rangle$ for each $x \in \mathbb{R}_+$.

2.2. Bandwidth Sharing Policy

We will consider a family of bandwidth sharing policies that were studied by Chiang et al. (2006) and that are a slight generalization of the α-fair policies of Mo and Walrand (2000).

The bandwidth allocations in the fluid model change dynamically as a function of the amount of fluid on each route. We will need the following notation to describe them. For each $z \in \mathbb{R}_+^{\mathbf{I}}$, let $\mathcal{I}_+(z) = \{i \in \mathcal{I} : z_i > 0\}$ and $\mathcal{O}(z) = \{\psi \in \mathbb{R}_+^{\mathbf{I}} : \psi_i = 0 \text{ for all } i \notin \mathcal{I}_+(z)\}$.

Fix parameters $\alpha_i > 0, \kappa_i > 0$, for each $i \in \mathcal{I}$. Let $\alpha = (\alpha_1, \ldots, \alpha_{\mathbf{I}})$ and $\kappa = (\kappa_1, \ldots, \kappa_{\mathbf{I}})$. The following optimization problem will be used to define the bandwidth sharing policy associated with the pair of vector parameters (α, κ). Given $z \in \mathbb{R}_+^{\mathbf{I}}$, the vector of bandwidth allocations $\phi(z)$ associated with z is the unique value of $\psi \in \mathcal{O}(z)$ that solves the following *utility maximization problem:*

$$\text{maximize} \sum_{i \in \mathcal{I}_+(z)} \kappa_i z_i U_i\left(\frac{\psi_i}{z_i}\right) \text{ subject to } \sum_{i \in \mathcal{I}} R_{ji}\psi_i \le C_j \text{ for all } j \in \mathcal{J},\ \psi \in \mathcal{O}(z), \tag{2.2}$$

where for each i, $U_i : [0, \infty) \to [-\infty, \infty)$ is a utility function of the form

$$U_i(x_i) = \begin{cases} \frac{1}{1-\alpha_i} x_i^{1-\alpha_i} & \text{if } \alpha_i \ne 1, \\ \log(x_i) & \text{if } \alpha_i = 1. \end{cases}$$

Remark 2.1 *For $i \in \mathcal{I}_+(z)$, we have $\phi_i(z) > 0$ because, either $U_i(0) = -\infty$ if $\alpha_i \ge 1$, or $U_i(0) = 0$ and $U_i'(x_i) \to +\infty$ as $x_i \to 0$ if $\alpha_i \in (0,1)$. Let $\mathcal{S}(z) = \{\psi \in \mathbb{R}_+^{\mathbf{I}} : \psi_i > 0$ for all $i \in \mathcal{I}_+(z), \psi_i = 0$ for all $i \notin \mathcal{I}_+(z)\}$. Then one can restrict the choice of ψ to the set $\mathcal{S}(z)$ for the utility maximization problem. The uniqueness of the maximizer follows from the strict concavity of the utility functions U_i, $i \in \mathcal{I}_+(z)$. Furthermore, for $z \in \mathbb{R}_+^{\mathbf{I}}$, $\phi_i(\cdot)$ is continuous at z for each $i \in \mathcal{I}_+(z)$. If $\alpha_i = \alpha \in (0, \infty)$ for all $i \in \mathcal{I}$, this last statement was proved by Kelly and Williams (2004). When $\alpha_i \in (0,1)$ for all $i \in \mathcal{I}$, it was noted by Chiang et al. (2006) that a similar proof to that of Kelly and Williams (2004) can be used to establish this result. Similar ideas can be used to give a proof for all $\alpha_i \in (0, \infty), i \in \mathcal{I}$. For completeness, in Lemma A.1 in the Appendix, we give such a proof.*

2.3. Definition of Fluid Model Solutions

The fluid model of Gromoll and Williams (2009), with the bandwidth sharing policy described in the previous subsection, is described below. For the remainder of the paper, the parameters $(R, C, \alpha, \kappa, \nu, \vartheta)$ are fixed and the bandwidth allocation function ϕ is as specified in the previous section.

Definition 2.1 *Given a continuous function $\zeta : [0, \infty) \to \mathbf{M}^{\mathbf{I}}$, define the auxiliary functions (z, Λ, τ, u, w) by the following for all $t \ge 0$:*

$$z(t) = \langle 1, \zeta(t) \rangle,$$

$$\begin{aligned}
\Lambda(t) &= \phi(z(t)), \\
\tau_i(t) &= \int_0^t \Big(\Lambda_i(s)\mathbb{1}_{(0,\infty)}\big(z_i(s)\big) + \rho_i \mathbb{1}_{\{0\}}\big(z_i(s)\big)\Big)ds, \quad i \in \mathcal{I}, \\
u(t) &= Ct - R\tau(t), \\
w(t) &= \langle \chi, \zeta(t) \rangle.
\end{aligned}$$

In Definition 2.1 the i-th component of $w(\cdot)$ represents the fluid workload for route i, $w_i(t) = \langle \chi, \zeta_i(t) \rangle$, $t \geq 0$. The fluid workload per link is given by $\widetilde{w}_j(t) = \sum_{i\in\mathcal{I}} R_{ji} w_i(t)$, $t \geq 0$.

A fluid model solution is defined through projections against test functions in the class

$$\mathcal{C} = \{f \in \mathbf{C}_b^1(\mathbb{R}_+) : f(0) = f'(0) = 0\}.$$

Definition 2.2 *A fluid model solution associated with the parameters $(R, C, \alpha, \kappa, \nu, \vartheta)$ is a continuous function $\zeta : [0,\infty) \to \mathbf{M}^{\mathbf{I}}$ that, together with its auxiliary functions (z, Λ, τ, u), satisfies:*

(i) $\langle \mathbb{1}_{\{0\}}, \zeta(t) \rangle = 0$ *for all* $t \geq 0$*.*

(ii) u_j *is nondecreasing for all* $j \in \mathcal{J}$*,*

(iii) for each $f \in \mathcal{C}$*,* $i \in \mathcal{I}$*, and* $t \geq 0$*,*

$$\langle f, \zeta_i(t) \rangle = \langle f, \zeta_i(0) \rangle - \int_0^t \langle f', \zeta_i(s) \rangle \frac{\Lambda_i(s)}{z_i(s)} \mathbb{1}_{(0,\infty)}(z_i(s))ds + \nu_i \langle f, \vartheta_i \rangle \int_0^t \mathbb{1}_{(0,\infty)}\big(z_i(s)\big)ds. \quad (2.3)$$

Remark 2.2 *The auxiliary function w associated with ζ satisfies the following for all $t \geq 0$ for those i for which $w_i(0) < \infty$:*

$$w_i(t) = w_i(0) + \int_0^t \Big(\rho_i - \Lambda_i\big(z(s)\big)\Big)\mathbb{1}_{(0,\infty)}\big(z_i(s)\big)ds; \quad (2.4)$$

see Lemma 3.3 of Gromoll and Williams (2009) and Lemma 4 of Chiang et al. (2006) for the method of proof.

Remark 2.3 *The third property in Definition 2.2 can be extended to hold for all functions $f \in \tilde{\mathcal{C}} = \{f \in \mathbf{C}_b^1(\mathbb{R}_+) : f(0) = 0\}$. A proof of this is given in Lemma A.2 in Appendix A.*

Remark 2.4 *The fluid limit result proved by Gromoll and Williams (2009) yields fluid model solutions which have initial states that are continuous measures and which have finite workload, i.e., for which $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$. Indeed, in order for fluid model solutions to be continuous functions of time, the initial condition cannot have any atoms. For our main results, we will be assuming that $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$, see §4.*

2.4. Additional Notation for Fluid Model Solutions

Suppose that $\zeta(\cdot)$ is a fluid model solution. We shall often use $\overline{M}_t^i(x)$ in place of $\overline{M}_{\zeta(t)}^i(x)$ to simplify notation. Let (z,Λ) be auxiliary functions associated with ζ, as in Definition 2.1. For each $i\in\mathcal{I}$ and $0\le s<t<\infty$, let

$$S_{s,t}^i=\int_s^t \frac{\Lambda_i(r)}{z_i(r)}\mathbb{1}_{(0,\infty)}\big(z_i(r)\big)dr. \tag{2.5}$$

Note that this may take the value $+\infty$. However, if $z_i(r)>0$ for all $r\in[s,t]$, then $S_{s,t}^i<\infty$, since $\Lambda_i(\cdot)$ is bounded and $z_i(\cdot)$ is continuous (hence it is bounded away from zero on the interval $[s,t]$). Indeed, $r\to S_{r,t}^i$ is continuously differentiable on $[s,t]$ because $\Lambda_i(\cdot)=\phi_i(z(\cdot))$ is continuous on $[s,t]$, since $z\to\phi_i(z)$ is continuous at points z where $z_i>0$ (see Remark 2.1) and $r\to z_i(r)$ is continuous, and furthermore $r\to z_i(r)$ is continuous and bounded away from zero on $[s,t]$. We interpret $S_{s,t}^i$ as the cumulative amount of bandwidth per unit of fluid allocated to route i over the time interval $[s,t]$.

3. Lyapunov Function

3.1. Assumptions

3.1.1. Subcritical Parameters Henceforth in this paper, we shall assume that the fluid model is subcritical, that is, the parameters (R,ρ,C) satisfy the following assumption.

Assumption 1

$$\sum_{i\in\mathcal{I}} R_{ji}\rho_i<C_j \quad \text{for all } j\in\mathcal{J}. \tag{3.1}$$

This condition means that the average load on each link is strictly less than its capacity. Under this condition, we can choose a sufficiently small $\delta>0$ such that

$$\tilde\rho_i\equiv(1+\delta)\rho_i,\ \text{for all } i\in\mathcal{I}, \tag{3.2}$$

satisfies

$$\sum_{i\in\mathcal{I}} R_{ji}\tilde\rho_i<C_j \text{ for all } j\in\mathcal{J} \quad\text{and}\quad (1-\delta)(1+\delta)^{\alpha_i+1}>1 \ \text{ for all } i\in\mathcal{I}. \tag{3.3}$$

We fix such a sufficiently small $\delta>0$ henceforth and define

$$\theta_i(x)=\left(1-\frac{m_i\mu_i}{\alpha_i}\int_0^x \overline{N}_i(u)du\right)^{-\alpha_i} \quad \text{for all } x\in[0,\infty),\ i\in\mathcal{I}, \tag{3.4}$$

where $m_i\in(0,\alpha_i)$ is defined so that $\left(\frac{\alpha_i}{m_i}\right)^{\alpha_i}=(1-\delta)(1+\delta)^{\alpha_i+1}$ holds for all $i\in\mathcal{I}$. Since $\mu_i\int_0^\infty \overline{N}_i(u)du=1$, we have

$$\begin{aligned}1\ge 1-\frac{m_i\mu_i}{\alpha_i}\int_0^x\overline{N}_i(u)du&=\frac{m_i}{\alpha_i}\left(1-\mu_i\int_0^x\overline{N}_i(u)du\right)+1-\frac{m_i}{\alpha_i}\\&=\frac{m_i\overline{N}_i^e(x)}{\alpha_i}+1-\frac{m_i}{\alpha_i}\\&\ge 1-\frac{m_i}{\alpha_i}>0,\end{aligned}$$

and so $\theta_i(\cdot)$ is positive and bounded above and below on $[0,\infty)$ for all $i\in\mathcal{I}$.

3.1.2. File Size Distributions The following assumption will be used in Lemma 5.8 to prove continuity in time of $\mathcal{H}^\zeta$, the composition of the Lyapunov function H (defined below) with a suitable fluid model solution ζ. This continuity property ultimately features in our proof of the absolute continuity of $\mathcal{H}^\zeta$ as a function of time and the convergence of fluid model solutions to the zero state.

Assumption 2 *For each $i \in \mathcal{I}$, the probability measure ϑ_i is in $\mathbf{K}_1$, i.e., it has no atoms and has finite first moment.*

Remark 3.1 *We already assumed that ϑ_i has finite first moment, so the additional assumption here is that it has no atoms.*

The additional assumption below, will be used in showing that under suitable constraints on the initial conditions, fluid model solutions reach the zero state in finite time. Indeed, we will prove that the time can be chosen uniformly provided there is a uniform bound on the initial workload vector and on the p-th moments of the components of the initial state of the fluid model solutions.

Assumption 3 *There is $p \in (1,\infty)$ such that $B_{\vartheta,p} \equiv \max_{i\in\mathcal{I}} \langle \chi^p, \vartheta_i \rangle < \infty$.*

3.2. The Functions H and $\mathcal{H}^\zeta$

Definition 3.1 *Given $\xi \in \mathbf{M}^{\mathbf{I}}$, for each $i \in \mathcal{I}$, define*

$$H_i(\xi) = \frac{\kappa_i}{\tilde{\rho}_i^{\alpha_i}} \int_0^\infty \left(\langle 1_{(x,\infty)}, \xi_i \rangle \right)^{\alpha_i+1} \theta_i(x) dx, \tag{3.5}$$

and define

$$H(\xi) = \sum_{i\in\mathcal{I}} \frac{H_i(\xi)}{\alpha_i + 1}. \tag{3.6}$$

The function H will be our Lyapunov function. It is a slight generalization of the one used by Paganini et al. (2012), where we have made adjustments to allow for the fact that our α_i can depend on i. Note that for $\xi \in \mathbf{M}^{\mathbf{I}}$, $H(\xi) \in [0,\infty]$ and $H(\xi) = 0$ if and only if $\xi_i((0,\infty)) = 0$ for all $i \in \mathcal{I}$. We shall ultimately be applying H to $\xi \in \mathbf{K}_1^{\mathbf{I}}$. For such ξ, $H(\xi)$ is finite and such that $H(\xi) = 0$ if and only if ξ_i is the zero measure on $\mathbb{R}_+ = [0,\infty)$ for each $i \in \mathcal{I}$.

Definition 3.2 *Given a fluid model solution $\zeta(\cdot)$, for each $t \ge 0$ and $i \in \mathcal{I}$, define $\overline{M}_t^i(x) = \langle 1_{(x,\infty)}, \zeta_i(t) \rangle$ and*

$$\mathcal{H}_i^\zeta(t) = H_i(\zeta(t)) = \frac{\kappa_i}{\tilde{\rho}_i^{\alpha_i}} \int_0^\infty \left(\overline{M}_t^i(x) \right)^{\alpha_i+1} \theta_i(x) dx \quad \textit{for all} \quad i \in \mathcal{I}, \tag{3.7}$$

and let

$$\mathcal{H}^\zeta(t) = H(\zeta(t)) = \sum_{i\in\mathcal{I}} \frac{\mathcal{H}_i^\zeta(t)}{\alpha_i + 1}. \tag{3.8}$$

The following provides a sufficient condition for $\mathcal{H}^\zeta(t)$ to be finite-valued for all $t \in [0,\infty)$.

Proposition 3.1 *Let $\zeta(\cdot)$ be a fluid model solution. Suppose that $i \in \mathcal{I}$ such that $w_i(0) = \langle \chi, \zeta_i(0)\rangle < \infty$. Then $\mathcal{H}_i^\zeta(t)$ is finite for all $t \geq 0$.*

Proof. Fix $t \geq 0$. By (2.4) we have that $w_i(t)$ is finite. Also, since $\overline{M}^i_t(x) \leq z_i(t)$ and $w_i(t) = \int_0^\infty \overline{M}^i_t(x)dx$, we have

$$\begin{aligned}\mathcal{H}_i^\zeta(t) &\leq \frac{\kappa_i \|\theta_i\|_\infty}{\tilde{\rho}_i^{\alpha_i}} (z_i(t))^{\alpha_i} \int_0^\infty \overline{M}^i_t(x)dx \\ &= \frac{\kappa_i \|\theta_i\|_\infty}{\tilde{\rho}_i^{\alpha_i}} (z_i(t))^{\alpha_i} w_i(t) < \infty,\end{aligned} \tag{3.9}$$

where $\|\theta_i\|_\infty = \sup_{x\in[0,\infty)} |\theta_i(x)|$. $\square$

3.3. The Function $\mathcal{K}^\zeta$

In this section, we introduce the function $\mathcal{K}^\zeta$, which arises in taking the derivative of the function $t \to \mathcal{H}^\zeta(t)$.

Definition 3.3 *Suppose that $\zeta(\cdot)$ is a fluid model solution. Define for each $i \in \mathcal{I}$ and $t \geq 0$,*

$$\begin{aligned}\mathcal{K}_i^\zeta(t) = \tilde{\rho}_i^{-\alpha_i}\Bigg(&-\kappa_i \Lambda_i(t)\big(z_i(t)\big)^{\alpha_i} \\ &-\kappa_i \int_0^\infty \big(\overline{M}^i_t(x)\big)^{\alpha_i} \left(\frac{\Lambda_i(t)}{z_i(t)} \mathbb{1}_{(0,\infty)}\big(z_i(t)\big)\right) \overline{M}^i_t(x)\theta_i'(x)dx \\ &+\kappa_i(\alpha_i+1)\nu_i \int_0^\infty \big(\overline{M}^i_t(x)\big)^{\alpha_i} \overline{N}_i(x)\theta_i(x)dx \Bigg)\end{aligned} \tag{3.10}$$

and let

$$\mathcal{K}^\zeta(t) = \sum_{i\in\mathcal{I}_+(z(t))} \frac{\mathcal{K}_i^\zeta(t)}{\alpha_i+1} \text{ for all } t \geq 0. \tag{3.11}$$

Remark 3.2 *In (3.10), if $z_i(t) = 0$, we interpret the right member of the equality to be zero and so $\mathcal{K}_i^\zeta(t) = 0$ in this case.*

Proposition 3.2 *Suppose that $\zeta(\cdot)$ is a fluid model solution. Then, for each $i \in \mathcal{I}$ and $t \geq 0$, $\sup_{s\in[0,t]} |\mathcal{K}_i^\zeta(s)| < \infty$.*

Proof. Fix $i \in \mathcal{I}$ and $t \geq 0$. For each $s \in [0,t]$, let

$$\begin{aligned}k_1(s) &= \tilde{\rho}_i^{-\alpha_i}\kappa_i\Lambda_i(s)\left(z_i(s)\right)^{\alpha_i}, \\ k_2(s) &= \tilde{\rho}_i^{-\alpha_i}\kappa_i \int_0^\infty \left(\overline{M}^i_s(x)\right)^{\alpha_i} \left(\frac{\Lambda_i(s)}{z_i(s)} \mathbb{1}_{(0,\infty)}\big(z_i(s)\big)\right) \overline{M}^i_s(x)\theta_i'(x)dx, \\ k_3(s) &= \tilde{\rho}_i^{-\alpha_i}\kappa_i(\alpha_i+1)\nu_i \int_0^\infty \left(\overline{M}^i_s(x)\right)^{\alpha_i} \overline{N}_i(x)\theta_i(x)dx.\end{aligned}$$

Noting that $\theta_i'(x) = m_i\mu_i(\theta_i(x))^{\frac{\alpha_i+1}{\alpha_i}}\overline{N}_i(x)$ for all $x \in \mathbb{R}_+$, $\|\theta_i\|_\infty < \infty$, $\frac{\overline{M}^i_s(\cdot)}{z_i(s)}\mathbb{1}_{(0,\infty)}\big(z_i(s)\big) \le 1$, $|\Lambda_i(\cdot)| \le \max_j C_j$, $\overline{M}^i_s(\cdot) \le z_i(s) < \infty$, and $\int_0^\infty \overline{N}_i(x)dx = \langle \chi, \vartheta_i\rangle = \mu_i^{-1} < \infty$, we have that $k_1(s), k_2(s), k_3(s)$ are well defined, non-negative and finite for each $s \in [0,t]$. Indeed,

$$\sup_{s\in[0,t]} k_1(s) \le \tilde{\rho}_i^{-\alpha_i}\kappa_i(\max_j C_j)\Big(\sup_{s\in[0,t]} z_i(s)\Big)^{\alpha_i} < \infty, \tag{3.12}$$

$$\sup_{s\in[0,t]} k_2(s) \le \tilde{\rho}_i^{-\alpha_i}\kappa_i\Big(\sup_{s\in[0,t]} z_i(s)\Big)^{\alpha_i}(\max_j C_j)\, m_i\, \|\theta_i\|_\infty^{\frac{\alpha_i+1}{\alpha_i}} < \infty, \tag{3.13}$$

$$\sup_{s\in[0,t]} k_3(s) \le \tilde{\rho}_i^{-\alpha_i}\kappa_i(\alpha_i+1)\nu_i\Big(\sup_{s\in[0,t]} z_i(s)\Big)^{\alpha_i}\langle\chi,\vartheta_i\rangle\|\theta_i\|_\infty < \infty. \tag{3.14}$$

Noting that

$$\mathcal{K}_i^\zeta(s) = -k_1(s) - k_2(s) + k_3(s), \quad \text{for all } s \in [0,t], \tag{3.15}$$

the result follows. □

4. Main Results

Theorem 4.1 *Suppose that Assumptions 1 and 2 hold. Further suppose that $\zeta(\cdot)$ is a fluid model solution with $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$. For each $i \in \mathcal{I}$, the function $\mathcal{H}_i^\zeta(\cdot)$ is absolutely continuous with respect to Lebesgue measure on $[0,\infty)$, and $\mathcal{K}_i^\zeta(\cdot)$ is a density for $\mathcal{H}_i^\zeta(\cdot)$, i.e., for each $t \ge 0$,*

$$\mathcal{H}_i^\zeta(t) - \mathcal{H}_i^\zeta(0) = \int_0^t \mathcal{K}_i^\zeta(s)ds. \tag{4.1}$$

Furthermore, for each $t \ge 0$,

$$\mathcal{K}_i^\zeta(t) \le \kappa_i\big(z_i(t)\big)^{\alpha_i}\left(\frac{-\Lambda_i(t)}{\tilde{\rho}_i^{\alpha_i}} + \frac{\tilde{\rho}_i(1-\delta)}{\Lambda_i(t)^{\alpha_i}}\right)\mathbb{1}_{(0,\infty)}\big(z_i(t)\big), \tag{4.2}$$

where the right member is interpreted to be zero if $z_i(t) = 0$.

Corollary 4.1 *Under the assumptions of Theorem 4.1, $\mathcal{H}^\zeta(\cdot)$ is absolutely continuous with respect to Lebesgue measure on $[0,\infty)$ and $\mathcal{K}^\zeta(\cdot)$ is a density for $\mathcal{H}^\zeta(\cdot)$. In addition, for all $t \ge 0$ we have*

$$\mathcal{K}^\zeta(t) \le -\delta \sum_{i\in\mathcal{I}_+(z(t))} \frac{\kappa_i\tilde{\rho}_i}{\alpha_i+1}\left(\frac{z_i(t)}{\Lambda_i(t)}\right)^{\alpha_i}. \tag{4.3}$$

The proofs of Theorem 4.1 and Corollary 4.1 are presented in Section 6.

Theorem 4.2 *Suppose that Assumptions 1 and 2 hold. For any fluid model solution $\zeta(\cdot)$ with $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$, $\mathcal{H}^\zeta(t)$ decreases monotonically towards zero as $t \to \infty$. Furthermore, for any $W > 0$,*

$$\lim_{t\to\infty}\sup\{\mathcal{H}^\zeta(t) : \zeta \text{ is a fluid model solution, } \zeta(0) \in \mathbf{K}_1^{\mathbf{I}}, \max_{i\in\mathcal{I}}(\langle 1, \zeta_i(0)\rangle, \langle\chi,\zeta_i(0)\rangle) \le W\} = 0.$$

Consequently, $\zeta_i(t)$ as a measure on $(0,\infty)$ converges vaguely[1] *to the zero measure on $(0,\infty)$ as $t \to \infty$ for each $i \in \mathcal{I}$.*

[1] That is, $\langle f, \zeta_i(t)\rangle \to 0$ as $t \to \infty$ for each continuous function f with compact support in $(0,\infty)$.

The following theorem shows that with the addition of Assumption 3 (with $p \in (1,\infty)$) to the assumptions of Corollary 4.1, and assuming the components of the initial fluid state have finite p-th moments, we have that the fluid model solution reaches the zero state in finite time, and the hitting time of the zero state is uniformly bounded for fluid model solutions starting in $\{\xi \in \mathbf{K}_1^{\mathbf{I}} : \max_{i\in\mathcal{I}}(\langle \mathbb{1}, \xi_i\rangle,\ \langle \chi^p, \xi_i\rangle) \le W\}$ for any fixed $W > 0$.

Theorem 4.3 *Suppose that Assumptions 1, 2 and 3 hold and let $p \in (1,\infty)$ be as in Assumption 3. For each $W \ge 1$ there exists $T_W > 0$ such that for all fluid model solutions $\zeta(\cdot)$ satisfying $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$ and $\max_{i\in\mathcal{I}}(\langle \mathbb{1}, \zeta_i(0)\rangle, \langle \chi^p, \zeta_i(0)\rangle) \le W$, we have $\zeta(t) = \mathbf{0}$, the zero measure in $\mathbf{M}^{\mathbf{I}}$, for all $t \ge T_W$.*

The proofs of Theorems 4.2 and 4.3 are given in Section 7.

5. Preliminary Lemmas

The following three lemmas are similar to Lemma 1, a result in Section III.C, and Lemma 5 in Paganini et al. (2012). For the first lemma, Paganini et al. (2012) indicated the idea for a proof. Here we provide more details, for completeness. For the other two lemmas, we provide the statements and the short proofs as a convenience to the reader.

Lemma 5.1 *Fix $z \in \mathbb{R}_+^{\mathbf{I}}$. Recall that $\phi(z)$ solves the maximization problem (2.2). Let ψ be a vector in $\mathbb{R}_+^{\mathbf{I}}$ such that $\psi_i > 0$ for all $i \in \mathcal{I}_+(z)$ and $\sum_{i\in\mathcal{I}} R_{ji}\psi_i \le C_j$ for all $j \in \mathcal{J}$. Then*

$$\sum_{i\in\mathcal{I}_+(z)} \kappa_i U_i'\left(\frac{\phi_i(z)}{z_i}\right)\big(\psi_i - \phi_i(z)\big) \le 0, \tag{5.1}$$

where, for each $i \in \mathcal{I}_+(z)$, $U_i'(x)$ is the derivative of $U_i(x)$ when $x > 0$.

Proof. Since (5.1) holds trivially for $z = 0$, we may assume that $z \ne 0$. Let $\tilde{\phi}(z) = \big(\phi_i(z) : i \in \mathcal{I}_+(z)\big)$ and $\widetilde{\Psi} = \{\tilde{\psi} = (\tilde{\psi}_i : i \in \mathcal{I}_+(z)), \tilde{\psi}_i > 0$ for all $i \in \mathcal{I}_+(z)\}$. For each $i \in \mathcal{I}_+(z)$, U_i is a concave, continuously differentiable function on $(0,\infty)$. Then the following function is concave and continuously differentiable on $\widetilde{\Psi}$:

$$f(\tilde{\psi}) = \sum_{i\in\mathcal{I}_+(z)} \kappa_i z_i U_i\left(\frac{\tilde{\psi}_i}{z_i}\right),\ \tilde{\psi} \in \widetilde{\Psi}.$$

Consider the set

$$\mathcal{F}(z) = \left\{\tilde{\psi} \in \widetilde{\Psi} : \sum_{i\in\mathcal{I}_+(z)} R_{ji}\tilde{\psi}_i \le C_j \quad \text{for all } j \in \mathcal{J}\right\}.$$

Then f achieves its maximum value on $\mathcal{F}(z)$ at $\tilde{\phi}(z)$. We claim that $\nabla f\big(\tilde{\phi}(z)\big)\cdot\big(\tilde{\psi} - \tilde{\phi}(z)\big) \le 0$, for any $\tilde{\psi} \in \mathcal{F}(z)$. For a proof by contradiction, suppose there is $\tilde{\psi} \in \mathcal{F}(z)$ such that $\nabla f\big(\tilde{\phi}(z)\big)\cdot\big(\tilde{\psi} - \tilde{\phi}(z)\big) > 0$. Then for any $t \in [0,1]$, $\gamma(t) = t\tilde{\psi} + (1-t)\tilde{\phi}(z)$ is in $\mathcal{F}(z)$, since this set is convex, and

$\frac{d}{dt} f\big(\gamma(t)\big)\big|_{t=0} = \nabla f\big(\tilde{\phi}(z)\big) \cdot \big(\tilde{\psi} - \tilde{\phi}(z)\big) > 0$, by our assumption. It follows that for all sufficiently small $t > 0$, we have $f\big(\gamma(t)\big) > f\big(\tilde{\phi}(z)\big)$, which contradicts the fact that $\tilde{\phi}(z)$ is the optimal solution of the maximization problem. Thus, $\nabla f\big(\tilde{\phi}(z)\big) \cdot \big(\tilde{\psi} - \tilde{\phi}(z)\big) \leq 0$. Computing the gradient of f, and using the fact that $\phi_i(z) = \tilde{\phi}_i(z)$ for all $i \in \mathcal{I}_+(z)$, it follows that

$$\sum_{i \in \mathcal{I}_+(z)} \kappa_i U_i' \left(\frac{\phi_i(z)}{z_i} \right) \big(\tilde{\psi}_i - \phi_i(z)\big) \leq 0 \text{ for all } \tilde{\psi} \in \mathcal{F}(z).$$

For a ψ satisfying the hypotheses of the lemma, $\tilde{\psi} = (\psi_i : i \in \mathcal{I}_+(z))$ is in $\mathcal{F}(z)$ and so the above inequality holds for it. Since the sum in the above does not involve $(\psi_i : i \notin \mathcal{I}_+(z))$, it follows that (5.1) holds for ψ. $\square$

Lemma 5.2 *Let $g(s) = s^a\big((a+1)q - bs\big)$ for $s \geq 0$ where a, b, q are fixed strictly positive real numbers. Then g has a maximum of $\left(\frac{aq}{b}\right)^a q$ at $s = \frac{aq}{b}$.*

Proof. Differentiating g with respect to $s > 0$, we have:

$$g'(s) = (a+1)s^{a-1}\big(aq - bs\big),$$

which is zero on $(0, \infty)$ only when $s = \frac{aq}{b}$ and noting the sign of g' on either side of this value, we see that g has a local maximum at $s = \frac{aq}{b}$ with value $\left(\frac{aq}{b}\right)^a q$. Further noting that g is continuous on $[0, \infty)$ and is zero at $s = 0$ and tends to $-\infty$ as $s \to \infty$, we see that the local maximum is the global maximum. $\square$

Lemma 5.3 *For any strictly positive real numbers, a, b, q, we have*

$$-\frac{b}{q^a} + \frac{q}{b^a} \leq (a+1)\frac{q-b}{b^a}.$$

Proof. Let $f(x) = x^{a+1}$ for $x \geq 0$. Then f is a convex function. The tangent line at $x = q$ is a support line and so $b^{a+1} \geq q^{a+1} + (a+1)q^a(b-q)$. Dividing both sides by $q^a b^a$ yields the desired result. $\square$

The remaining lemmas in this section contain various results for fluid model solutions that will be used in later sections. The proof of Lemma 5.4 is the same as that of Proposition 4.2 in Gromoll et al. (2002), so we omit it. The proofs of Lemmas 5.5 and 5.6, are similar to those of Lemmas 4.1 and 4.3, respectively, from the work of Gromoll et al. (2002). Since some details are a bit different, we provide the proofs for our context as a convenience to the reader.

Recall the third property in Definition 2.2. We now state a version of this property that holds for a class of functions of both time and space. Let $\mathbf{C}_b([0,\infty) \times \mathbb{R}_+)$ denote the set of continuous, bounded functions on $[0,\infty) \times \mathbb{R}_+$, and let $\mathbf{C}_b^1([0,\infty) \times \mathbb{R}_+)$ denote the set of once continuously differentiable functions defined on $[0,\infty)\times\mathbb{R}_+$ which, together with their first partial derivatives are bounded on $[0,\infty)\times\mathbb{R}_+$. That is, $f(s,x)$, $f_s(s,x)=\frac{\partial}{\partial s}f(s,x)$ and $f_x(s,x)=\frac{\partial}{\partial x}f(s,x)$ are continuous and bounded by a constant for all $(s,x)\in[0,\infty)\times\mathbb{R}_+$.

Lemma 5.4 *Let $\zeta:[0,\infty)\to\mathbf{M}^{\mathbf{I}}$ be continuous. Then for each $f\in\mathbf{C}_b([0,\infty)\times\mathbb{R}_+)$ and $i\in\mathcal{I}$,*

$$t\to\langle f(t,\cdot),\zeta_i(t)\rangle$$

is a continuous function of $t\in[0,\infty)$.

Proof. The proof is the same as that of Proposition 4.2 in Gromoll et al. (2002). □

The proofs of the next two lemmas are similar to those of Lemmas 4.1 and 4.3 in Gromoll et al. (2002). However, since bandwidth sharing is more general than the processor sharing treated in Gromoll et al. (2002), and since special care is needed in our setting to treat the fact that $z_i(\cdot)$ can be zero at times, we give the full proofs here. We note that the special case where $x=0$ in (5.6) follows from Appendix A in Borst et al. (2014). A dynamic equation for $z(t)$ for all $t\geq 0$ is also derived there.

Lemma 5.5 *Suppose that $\zeta(\cdot)$ is a fluid model solution, $i\in\mathcal{I}$, and $0\leq s<t<\infty$ are such that $\zeta_i(r)\neq 0$ for all $s<r<t$. Then for each $f\in\mathbf{C}_b^1([0,\infty)\times\mathbb{R}_+)$ such that $f(\cdot,0)\equiv 0$, we have that the following holds.*

$$\begin{aligned}\langle f(t,\cdot),\ \zeta_i(t)\rangle &= \langle f(s,\cdot),\zeta_i(s)\rangle+\int_s^t\langle f_r(r,\cdot),\zeta_i(r)\rangle dr\\ &\quad-\int_s^t\langle f_x(r,\cdot),\zeta_i(r)\rangle\frac{\Lambda_i(r)}{z_i(r)}\mathbb{1}_{(0,\infty)}\big(z_i(r)\big)dr+\nu_i\int_s^t\langle f(r,\cdot),\vartheta_i\rangle\mathbb{1}_{(0,\infty)}\big(z_i(r)\big)dr.\end{aligned}\tag{5.2}$$

Proof. Suppose that ζ, s,t, $i\in\mathcal{I}$ and f are as in the statement of the lemma. Then for $r,r+h\in(s,t)$ we have

$$\begin{aligned}\langle f(r+h,\cdot),\zeta_i(r+h)\rangle-\langle f(r,\cdot),\zeta_i(r)\rangle &= \langle f(r+h,\cdot),\zeta_i(r+h)\rangle-\langle f(r,\cdot),\zeta_i(r+h)\rangle\\ &\quad+\langle f(r,\cdot),\zeta_i(r+h)\rangle-\langle f(r,\cdot),\zeta_i(r)\rangle.\end{aligned}\tag{5.3}$$

In the following, for clarity, we write f_1 for the first partial derivative of f with respect to its first variable, and f_2 for its first partial derivative with respect to its second variable. The first difference on the right hand side of the equation (5.3) equals

$$\begin{aligned}\Big\langle\int_r^{r+h}f_1(u,\cdot)du,\zeta_i(r+h)\Big\rangle &= \Big\langle\int_0^1 f_1(r+hv,\cdot)hdv,\zeta_i(r+h)\Big\rangle\\ &= h\int_0^1\langle f_1(r+hv,\cdot),\zeta_i(r+h)\rangle dv,\end{aligned}$$

where we have used Fubini's theorem to change the order of integration to obtain the last equality. For each $v \in [0,1]$ define a function $f^v : [0,\infty) \times \mathbb{R}_+ \to \mathbb{R}$ by $f^v(y,x) = f_1(r+(y-r)v, x)$ for $(y,x) \in [0,\infty) \times \mathbb{R}_+$. Then $f^v \in \mathbf{C}_b([0,\infty) \times \mathbb{R}_+)$, and so by Lemma 5.4, $y \to \langle f^v(y,\cdot), \zeta_i(y)\rangle$ is a continuous function of $y \in [0,\infty)$. Noting that $f^v(r+h,\cdot) = f_1(r+hv,\cdot)$, it follows that for each $v \in [0,1]$,

$$\lim_{h\to 0}\langle f_1(r+hv,\cdot), \zeta_i(r+h)\rangle = \lim_{h\to 0}\langle f^v(r+h,\cdot), \zeta_i(r+h)\rangle = \langle f^v(r,\cdot), \zeta_i(r)\rangle = \langle f_1(r,\cdot), \zeta_i(r)\rangle.$$

Combining the above and since $f_1(\cdot,\cdot)$ is bounded by a constant and $\sup_{0\le u\le t} z_i(u)$ is finite by the continuity of $z_i(\cdot)$, using the bounded convergence theorem we have

$$\lim_{h\to 0} \frac{\langle f(r+h,\cdot), \zeta_i(r+h)\rangle - \langle f(r,\cdot), \zeta_i(r+h)\rangle}{h} = \int_0^1 \langle f_1(r,\cdot), \zeta_i(r)\rangle dv = \langle f_1(r,\cdot), \zeta_i(r)\rangle.$$

Now consider the last difference on the right hand side of (5.3). For fixed $r \in (s,t)$, we can use Lemma A.2 with $f(r,\cdot)$ in place of $f(\cdot)$ there, to conclude that

$$\begin{aligned}\langle f(r,\cdot), \zeta_i(r+h)\rangle - \langle f(r,\cdot), \zeta_i(r)\rangle = &-\int_r^{r+h} \langle f_2(r,\cdot), \zeta_i(u)\rangle \frac{\Lambda_i(u)}{z_i(u)} \mathbb{1}_{(0,\infty)}\big(z_i(u)\big) du \\ &+ \nu_i \langle f(r,\cdot), \vartheta_i\rangle \int_r^{r+h} \mathbb{1}_{(0,\infty)}\big(z_i(u)\big) du.\end{aligned}$$

Now, since $z_i(u) > 0$ for $u \in (s,t)$, we have that

$$u \to \langle f_2(r,\cdot), \zeta_i(u)\rangle \frac{\Lambda_i(u)}{z_i(u)} \mathbb{1}_{(0,\infty)}\big(z_i(u)\big),$$

is continuous on (s,t). Consequently, by the fundamental theorem of calculus,

$$\lim_{h\to 0} \frac{1}{h} \int_r^{r+h} \langle f_2(r,\cdot), \zeta_i(u)\rangle \frac{\Lambda_i(u)}{z_i(u)} \mathbb{1}_{(0,\infty)}\big(z_i(u)\big) du = \langle f_2(r,\cdot), \zeta_i(r)\rangle \frac{\Lambda_i(r)}{z_i(r)} \mathbb{1}_{(0,\infty)}\big(z_i(r)\big). \tag{5.4}$$

Finally, we note that $r \to \langle f(r,\cdot), \vartheta_i\rangle$ is continuous on $[0,\infty)$ by the bounded convergence theorem. Combining all of the above, and replacing $f_1(r,x), f_2(r,x)$ with $f_r(r,x), f_x(r,x)$, we can conclude that $r \to \langle f(r,\cdot), \zeta_i(r)\rangle$ is once continuously differentiable on (s,t), with continuous derivative given by

$$\langle f_r(r,\cdot), \zeta_i(r)\rangle - \langle f_x(r,\cdot), \zeta_i(r)\rangle \frac{\Lambda_i(r)}{z_i(r)} + \nu_i \langle f(r,\cdot), \vartheta_i\rangle, \quad r \in (s,t). \tag{5.5}$$

Integrating over any closed interval $[s_1,t_1]$ contained in (s,t), we obtain that (5.2) holds with s_1, t_1 in place of s,t, respectively. Then invoking Lemma 5.4 again for the continuity of $r \to \langle f(r,\cdot),\ \zeta_i(r)\rangle$ from the right at s and the left at t, and noting the boundedness of the integrands in the integrals in (5.2), we see that we can let $s_1 \downarrow s$ and $t_1 \uparrow t$ to obtain the desired result. $\square$

Lemma 5.6 *Suppose that $\zeta(\cdot)$ is a fluid model solution, $i \in \mathcal{I}$ and $0 \le s < t < \infty$ such that $\zeta_i(r) \ne 0$ for all $r \in [s,t]$. Then*

$$\overline{M}^i_t(x) = \overline{M}^i_s(x + S^i_{s,t}) + \nu_i \int_s^t \overline{N}_i(x + S^i_{u,t}) du \quad \textit{for all } x \in \mathbb{R}_+. \tag{5.6}$$

Proof. Because $\zeta_i(\cdot) \neq 0$ on $[s,t]$, $z_i(\cdot)$ is strictly positive on $[s,t]$ and since it is continuous, there is $s_1 \in [0,s]$, where $s_1 < s$ if $s \neq 0$ and $s_1 = 0$ if $s = 0$, such that $z_i(\cdot)$ is still strictly positive on $[s_1,t]$. Then $u \to S^i_{u,t}$ is continuously differentiable on $[s_1,t]$, with $\frac{dS^i_{u,t}}{du} = -\frac{\Lambda_i(u)}{z_i(u)}$ for $u \in [s_1,t]$. Consider $g \in \mathbf{C}^1_b(\mathbb{R})$ with $g(x) = 0$ for all $x \le 0$. Note that by the continuous differentiability of g, we must have $g'(x) = 0$ for all $x \le 0$. Let

$$f(u,x) = g(x - S^i_{u,t}), \quad u \in [s_1,t],\ x \in \mathbb{R}_+.$$

Then, $f \in \mathbf{C}^1_b([s_1,t] \times \mathbb{R}_+)$ where for $u \in [s_1,t]$ and $x \in \mathbb{R}_+$,

$$f_u(u,x) = \frac{g'(x - S^i_{u,t})\Lambda_i(u)}{z_i(u)} \quad \text{and} \quad f_x(u,x) = g'(x - S^i_{u,t}).$$

Because $g(x) = 0$ and $g'(x) = 0$ for all $x \le 0$, we have for $u \in [s_1,t]$, $f(u,0) = 0$ and $f_x(u,0) = 0$. Let $\epsilon \in (0,(t-s)/2)$. We wish to construct a function f^ϵ that satisfies the conditions in Lemma 5.5 and that equals f on $[s,t-\epsilon] \times \mathbb{R}_+$. Let $h^\epsilon \in \mathbf{C}^1_b([0,\infty))$ be such that

$$h^\epsilon(u) = \begin{cases} 1, & u \in [s,t-\epsilon], \\ 0, & u \in [0,s_1) \cup [t,\infty). \end{cases}$$

Note that if $s = 0$, then $[0,s_1) = \emptyset$. When $[0,s_1) \neq \emptyset$, we have by continuity (from the left) of h_ϵ and h'_ϵ that $h_\epsilon(s_1) = 0$ and $h'_\epsilon(s_1) = 0$. Extend f to be identically equal to zero on $([0,s_1) \cup (t,\infty)) \times \mathbb{R}_+$ and define

$$f^\epsilon(u,x) = f(u,x)h^\epsilon(u), \quad u \in [0,\infty), x \in \mathbb{R}_+.$$

Then, $f^\epsilon \in \mathbf{C}^1_b([0,\infty) \times \mathbb{R}_+)$ with $f^\epsilon(\cdot,0) \equiv 0$ and $f^\epsilon = f$ on $[s,t-\epsilon] \times \mathbb{R}_+ \subset [s_1,t] \times \mathbb{R}_+$. Upon replacing f,t in (5.2) with $f^\epsilon, t-\epsilon$, respectively, we obtain

$$\begin{aligned}
\langle f(t-\epsilon,\cdot), \zeta_i(t-\epsilon)\rangle =& \langle f(s,\cdot),\zeta_i(s)\rangle + \int_s^{t-\epsilon} \langle g'(\cdot - S^i_{u,t}), \zeta_i(u)\rangle \frac{\Lambda_i(u)}{z_i(u)} du \\
& - \int_s^{t-\epsilon} \langle g'(\cdot - S^i_{u,t}), \zeta_i(u)\rangle \frac{\Lambda_i(u)}{z_i(u)} du \\
& + \nu_i \int_s^{t-\epsilon} \langle g(\cdot - S^i_{u,t}), \vartheta_i\rangle du \\
=& \langle g(\cdot - S^i_{s,t}), \zeta_i(s)\rangle + \nu_i \int_s^{t-\epsilon} \langle g(\cdot - S^i_{u,t}), \vartheta_i\rangle du.
\end{aligned} \tag{5.7}$$

Similar to Lemma 5.4, $u \to \langle f(u,\cdot), \zeta(u)\rangle$ is continuous on $[s,t]$ and so we can let $\epsilon \to 0$ in the above to obtain

$$\langle g(\cdot), \zeta_i(t)\rangle = \langle f(t,\cdot), \zeta_i(t)\rangle = \langle g(\cdot - S^i_{s,t}), \zeta_i(s)\rangle + \nu_i \int_s^t \langle g(\cdot - S^i_{u,t}), \vartheta_i\rangle du. \tag{5.8}$$

For $x \in \mathbb{R}_+$, to obtain (5.6) from (5.8), consider a sequence of nonnegative functions $\{g_n\}_{n=0}^\infty \subset \mathbf{C}^1_b(\mathbb{R})$ satisfying $g_n(x) = 0$ for all $x \le 0$ and all n, and such that g_n increases to $\mathbb{1}_{(x,\infty)}$ pointwise on $\mathbb{R}$ and apply the monotone convergence theorem. $\square$

Lemma 5.7 *Suppose that Assumption 2 holds and let $\zeta(\cdot)$ be a fluid model solution. Suppose that $0 \le s < t < \infty$ and $i \in \mathcal{I}$ such that $\zeta_i(s) \in \mathbf{K}$ and $z_i(r) > 0$ for all $r \in (s,t)$. Then $\zeta_i(r) \in \mathbf{K}$ for all $r \in (s,t)$.*

Proof. Fix $r \in (s,t)$. It suffices to show that $x \to \overline{M}^i_r(x)$ is continuous.

We first consider the case where $z_i(s) > 0$. From (5.6), we have for all $x \in \mathbb{R}_+$,

$$\overline{M}^i_r(x) = \overline{M}^i_s(x + S^i_{s,r}) + \nu_i \int_s^r \overline{N}_i(x + S^i_{u,r})du. \tag{5.9}$$

Because $\zeta_i(s) \in \mathbf{K}$, $y \to \overline{M}^i_s(y)$ is continuous and it follows that the first term on the right hand side above is continuous as a function of x. From the assumption that $\vartheta_i \in \mathbf{K}$, we have that $y \to \overline{N}_i(y)$ is continuous (and bounded). It follows from the dominated convergence theorem that the second term on the right hand side above is continuous as a function of x. This completes the proof when $z_i(s) > 0$.

Now suppose that $z_i(s) = 0$. Then for $s < s_0 < r < t_0 < t$, we have $z_i(\cdot) > 0$ on $[s_0, t_0]$, and so by (5.6) we have for all $x \in \mathbb{R}_+$,

$$\overline{M}^i_r(x) = \overline{M}^i_{s_0}(x + S^i_{s_0,r}) + \nu_i \int_{s_0}^r \overline{N}_i(x + S^i_{u,r})du. \tag{5.10}$$

Fix $x_0 \in \mathbb{R}_+$ and let $\epsilon > 0$. Because $z_i(\cdot)$ is continuous and $z_i(s) = 0$, we can choose s_0 close enough to s so that $\overline{M}^i_{s_0}(\cdot) \le z_i(s_0) < \epsilon/4$. It follows that the difference of two evaluations of the first term in the right hand side of (5.10), where the evaluations are at x_0 and $x \in \mathbb{R}_+$, has magnitude less than $\epsilon/2$. For this fixed value of s_0, the last term in (5.10) is continuous as a function of x (because $\vartheta_i \in \mathbf{K}$). Combining the properties of the first and last terms in the right hand side of (5.10), it follows that there is $\delta > 0$ such that whenever $|x - x_0| < \delta$, we have

$$|\overline{M}^i_r(x) - \overline{M}^i_r(x_0)| \le \epsilon.$$

Since $\epsilon > 0$ and $x_0 \in \mathbb{R}_+$ were arbitrary, it follows that $x \to \overline{M}^i_r(x)$ is continuous when $z_i(s) = 0$. □

Corollary 5.1 *Suppose that Assumption 2 holds and $\zeta(\cdot)$ is a fluid model solution with $\zeta(0) \in \mathbf{K}^{\mathbf{I}}$. Then $\zeta(t) \in \mathbf{K}^{\mathbf{I}}$ for all $t > 0$.*

Proof. Fix $i \in \mathcal{I}$ and $t > 0$. If $z_i(t) = 0$, then $\zeta_i(t) = 0$ is in $\mathbf{K}$. On the other hand, if $z_i(t) > 0$, then by the continuity of $z_i(\cdot)$, there is an open interval $V = (a,b)$ containing t such that $z_i(s) > 0$ on V and either $a = 0$ or $z_i(a) = 0$. In either case, $\zeta_i(a) \in \mathbf{K}$, and it follows from Lemma 5.7 that $\zeta_i(s) \in \mathbf{K}$ for all $s \in V$ and in particular, $\zeta_i(t) \in \mathbf{K}$. □

Lemma 5.8 *Suppose that Assumption 2 holds. Let $\zeta(\cdot)$ be a fluid model solution with $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$. Then for each $i \in \mathcal{I}$, $\mathcal{H}_i^{\zeta}(\cdot)$ as defined in (3.7) is continuous on $[0,\infty)$.*

Proof. Fix $i \in \mathcal{I}$ and $t_0 \in [0,\infty)$.

We first consider the case where $z_i(t_0) = 0$. Then $\mathcal{H}_i^{\zeta}(t_0) = 0$. Note that $z_i(\cdot)$ is continuous. Also, because $w_i(0) = \langle \chi, \zeta_i(0) \rangle < \infty$ by assumption, it follows from (2.4), that $w_i(\cdot)$ is continuous. Then, because $z_i(t_0) = w_i(t_0) = 0$, it follows from the continuity of $z_i(\cdot)$, $w_i(\cdot)$ and (3.9), that $\mathcal{H}_i^{\zeta}(s)$ tends to zero as $s \to t_0$. So $\mathcal{H}_i^{\zeta}(\cdot)$ is continuous at t_0.

We now turn to the case where $z_i(t_0) > 0$. By the continuity of $z_i(\cdot)$, there is a neighborhood $[s,t]$ of t_0 on which $z_i(r) \neq 0$ for all $r \in [s,t]$, where we may choose $s < t_0 < t$ if $t_0 \neq 0$ and $s = t_0 < t$ if $t_0 = 0$. Since $\zeta(0) \in \mathbf{K}^{\mathbf{I}}$, it follows from Corollary 5.1 that for all $r \in [0,\infty)$, $x \to \overline{M}_r^i(x)$ is continuous. From Lemma 5.6, with r in place of t there, we have for each $r \in [s,t]$ that for each $x \in \mathbb{R}_+$,

$$\overline{M}_r^i(x) = \overline{M}_s^i(x + S_{s,r}^i) + \int_s^r \nu_i \overline{N}_i(x + S_{u,r}^i) du.$$

It then follows from the continuity of $\overline{M}_s^i(\cdot)$ and $\overline{N}_i(\cdot)$ (because ϑ_i is continuous) on $\mathbb{R}_+$, and the continuity of $r \to S_{u,r}^i$ for $r \in [s,t]$, for each fixed $u \in [s,t]$, that $r \to \overline{M}_r^i(x)$ is continuous for each $x \in \mathbb{R}_+$. Now, for $r \in [s,t]$,

$$\mathcal{H}_i^{\zeta}(r) = \frac{\kappa_i}{\rho_i^{\alpha_i}} \int_0^{\infty} \left(\overline{M}_r^i(x)\right)^{\alpha_i} \theta_i(x) \overline{M}_r^i(x) dx,$$

where the integrand is dominated by $\|\theta_i\|_\infty (\sup_{u \in [s,t]} z_i(u))^{\alpha_i} \overline{M}_r^i(\cdot)$. By the generalized Lebesgue dominated convergence theorem and the fact that $w_i(r) = \int_0^\infty \overline{M}_r^i(x) dx$ is continuous as a function of r, we have that $\mathcal{H}_i^{\zeta}(r) \to \mathcal{H}_i^{\zeta}(t_0)$ as $r \to t_0$. □

6. Proofs of Theorem 4.1 and Corollary 4.1

6.1. Smooth Approximation of Measures

We use an approximation argument to prove Theorem 4.1. To prepare for this, for each positive integer n, let $\varphi_n \in \mathbf{C}_c^\infty(\mathbb{R})$ be such that $\varphi_n \geq 0, \varphi_n(x) = 0$ for all $x \in (-\infty, -\frac{1}{n}] \cup [\frac{1}{n}, \infty)$, $\varphi_n(x) = \varphi_n(-x)$ for all $x > 0$, and $\int_{\mathbb{R}} \varphi_n(x) dx = 1$. Given $\xi \in \mathbf{M}$ and $n \in \mathbb{N}$, let ξ^n be the nonnegative, absolutely continuous Borel measure on $\mathbb{R}_+$ whose density is given by $d_n(x) = \int_{\mathbb{R}_+} \varphi_n(x-y) \xi(dy) = \int_{\mathbb{R}_+} \varphi_n(y-x) \xi(dy)$ for $x \in \mathbb{R}_+$, where we have used the symmetry of φ_n for the last equality. Note that $d_n(\cdot)$ is in $\mathbf{C}_b^\infty(\mathbb{R}_+)$, since φ_n is infinitely differentiable with compact support and ξ is a finite measure on $\mathbb{R}_+$. For any bounded, Borel measurable function f defined on $\mathbb{R}_+$, let $(f * \varphi_n)(y) = \int_{\mathbb{R}_+} \varphi_n(y-x) f(x) dx$ for $y \in \mathbb{R}_+$. Then, by Fubini's theorem,

$$\langle f, \xi^n \rangle = \int_{\mathbb{R}_+} f(x) \int_{\mathbb{R}_+} \varphi_n(y-x) \xi(dy) dx = \langle f * \varphi_n, \xi \rangle. \tag{6.1}$$

The following lemma can be proved in the same manner as Lemma 7.12 of Puha and Williams (2016), so we omit the proof.

Lemma 6.1 *Let $\xi \in \mathbf{K}_1$. For each $n \in \mathbb{N}$ and $x \in \mathbb{R}_+$, we have*

$$\left\langle \mathbb{1}_{(x+\frac{1}{n},\infty)}, \xi \right\rangle \le \langle \mathbb{1}_{(x,\infty)}, \xi^n \rangle \le \left\langle \mathbb{1}_{\left((x-\frac{1}{n})^+,\infty\right)}, \xi \right\rangle, \tag{6.2}$$

$$\langle \chi, \xi \rangle - \frac{\langle \mathbb{1}, \xi \rangle}{n} \le \langle \chi, \xi^n \rangle \le \langle \chi, \xi \rangle + \frac{\langle \mathbb{1}, \xi \rangle}{n}. \tag{6.3}$$

Furthermore, we have $\xi^n \in \mathbf{A}$ for each $n \in \mathbb{N}$ and as $n \to \infty$,

$$\xi^n \xrightarrow{w} \xi \quad \textit{and} \quad \langle \chi, \xi^n \rangle \to \langle \chi, \xi \rangle. \tag{6.4}$$

Given a fluid model solution $\zeta(\cdot)$, for each $t \ge 0$ and $i \in \mathcal{I}$, let $\{\zeta_i^n(t)\}_{n=1}^\infty$ be the approximating sequence of measures for $\zeta_i(t)$, as defined above with $\zeta_i(t)$ in place of ξ. For any positive integer ℓ, let $\mathcal{C}_{0,\ell} = \{g \in \mathbf{C}_b^1(\mathbb{R}_+) : g = 0 \text{ on } [0, \frac{1}{\ell}]\}$. For $g \in \mathcal{C}_{0,\ell}$ and all $n > \ell$, we have $(g * \varphi_n)(0) = 0$ and $(g * \varphi_n)'(0) = 0$. It follows that $g * \varphi_n \in \mathcal{C}$. By (2.3), with $g * \varphi_n$ replacing f and noting that $(g * \varphi_n)'(\cdot) = (g' * \varphi_n)(\cdot)$, we have for any $t \ge 0$,

$$\begin{aligned} \langle g * \varphi_n, \zeta_i(t) \rangle = \langle g * \varphi_n, \zeta_i(0) \rangle &- \int_0^t \langle g' * \varphi_n, \zeta_i(s) \rangle \frac{\Lambda_i(s)}{z_i(s)} \mathbb{1}_{(0,\infty)}(z_i(s)) ds \\ &+ \nu_i \langle g * \varphi_n, \vartheta_i \rangle \int_0^t \mathbb{1}_{(0,\infty)}\big(z_i(s)\big) ds. \end{aligned} \tag{6.5}$$

Then, using (6.1), we can rewrite the above as

$$\langle g, \zeta_i^n(t) \rangle = \langle g, \zeta_i^n(0) \rangle - \int_0^t \langle g', \zeta_i^n(s) \rangle \frac{\Lambda_i(s)}{z_i(s)} \mathbb{1}_{(0,\infty)}(z_i(s)) ds + \nu_i \langle g, \vartheta_i^n \rangle \int_0^t \mathbb{1}_{(0,\infty)}\big(z_i(s)\big) ds. \tag{6.6}$$

For each positive integer n, $i \in \mathcal{I}$, $t \ge 0$ and $x \in \mathbb{R}_+$, let

$$\overline{M}_t^{i,n}(x) = \langle \mathbb{1}_{(x,\infty)}, \zeta_i^n(t) \rangle, \qquad \overline{N}^{i,n}(x) = \langle \mathbb{1}_{(x,\infty)}, \vartheta_i^n \rangle. \tag{6.7}$$

The following lemma is key to our proof of Theorem 4.1. It provides a rigorous formulation of the partial differential equation result assumed in Paganini et al. (2012).

Lemma 6.2 *Assume that $\zeta(\cdot)$ is a fluid model solution. Suppose that $i \in \mathcal{I}$ and $0 \le a < b < \infty$ are such that $z_i(t) \neq 0$ for all $t \in [a, b]$. Then, for each positive integer ℓ and all $n > \ell$, $t \to \overline{M}_t^{i,n}(x)$ is continuously differentiable on $[a, b]$ for each fixed $x \in \mathbb{R}_+$, and $x \to \overline{M}_t^{i,n}(x)$ is continuously differentiable on $[\frac{1}{\ell}, \infty)$ for each fixed $t \in [a, b]$, and furthermore,*

$$\frac{\partial \overline{M}_t^{i,n}(x)}{\partial t} = \frac{\Lambda_i(t)}{z_i(t)} \frac{\partial \overline{M}_t^{i,n}(x)}{\partial x} + \nu_i \overline{N}^{i,n}(x), \tag{6.8}$$

for $t \in [a, b]$, $x \ge \frac{1}{\ell}$, where the partial derivatives with respect to time at $t = a, b$ are from the right and left, respectively, and the partial derivative with respect to x at $x = 1/\ell$ is from the right.

Proof. For each $s \in [0,\infty)$, $i \in \mathcal{I}$ and fixed n, by the definition of $\zeta_i^n(s)$, $m_s^{i,n}(\cdot) = \int_{\mathbb{R}_+} \varphi_n(y - \cdot)\zeta_i(s)(dy)$ is the $\mathbf{C}_b^\infty$ density function for the measure $\zeta_i^n(s)$. Thus, $x \to \overline{M}_s^{i,n}(x)$ is continuously differentiable on $[0,\infty)$ with derivative function $-m_s^{i,n}(\cdot)$. By the finiteness of $\zeta_i(s)$, $\lim_{x\to\infty} m_s^{i,n}(x) = 0$. Using integration by parts, for any $g \in \mathcal{C}_{0,\ell}$ that has compact support in $\mathbb{R}_+$, we have for each $n > \ell$, using the facts that g is bounded, $g(\frac{1}{\ell}) = 0$, and g is zero outside some compact set, we have

$$\langle g', \zeta_i^n(s)\rangle = \int_{\frac{1}{\ell}}^\infty g'(x) m_s^{i,n}(x)dx = -\int_{\frac{1}{\ell}}^\infty g(x)\frac{dm_s^{i,n}(x)}{dx}dx. \tag{6.9}$$

Now suppose, as in the statement of the lemma, that $i \in \mathcal{I}$ and $0 \le a < b < \infty$ such that $z_i(s) \neq 0$ for $s \in [a,b]$. Then we have from (6.6) that for any $t \in [a,b]$,

$$\langle g, \zeta_i^n(t)\rangle - \langle g, \zeta_i^n(a)\rangle = -\int_a^t \langle g', \zeta_i^n(s)\rangle \frac{\Lambda_i(s)}{z_i(s)}ds + \nu_i\langle g, \vartheta_i^n\rangle(t-a). \tag{6.10}$$

Fix $\ell, n > \ell$, $x_0 \ge \frac{1}{\ell}$ and $z > x_0$. Combining the above and considering a sequence $\{g_m\}_{m=1}^\infty$ of non-negative functions in $\mathcal{C}_{0,\ell}$ that have compact support and that converge monotonically upwards to $\mathbb{1}_{(x_0,z)}$, we obtain using monotone and dominated convergence (noting that $\frac{dm_s^{i,n}(x)}{dx} = -\langle \varphi_n'(\cdot - x), \zeta_i(s)\rangle$ is uniformly bounded for all $s \in [a,b]$ and $x \in \mathbb{R}_+$), that for all $t \in [a,b]$ and $z > x_0$,

$$\begin{aligned}\langle \mathbb{1}_{(x_0,z)}, \zeta_i^n(t)\rangle - \langle \mathbb{1}_{(x_0,z)}, \zeta_i^n(a)\rangle &= \int_a^t \left\langle \mathbb{1}_{(x_0,z)}, \frac{dm_s^{i,n}}{dx}\right\rangle \frac{\Lambda_i(s)}{z_i(s)}ds + \nu_i\langle \mathbb{1}_{(x_0,z)}, \vartheta_i^n\rangle(t-a)\\ &= \int_a^t (m_s^{i,n}(z) - m_s^{i,n}(x_0))\frac{\Lambda_i(s)}{z_i(s)}ds + \nu_i\langle \mathbb{1}_{(x_0,z)}, \vartheta_i^n\rangle(t-a).\end{aligned}$$

We can let $z \to \infty$, using monotone and bounded convergence, plus the fact that $\lim_{z\to\infty} m_s^{i,n}(z) = 0$ for each $s \in [a,t]$, to conclude that for each $t \in [a,b]$ and $x_0 \ge \frac{1}{\ell}$,

$$\langle \mathbb{1}_{(x_0,\infty)}, \zeta_i^n(t)\rangle - \langle \mathbb{1}_{(x_0,\infty)}, \zeta_i^n(a)\rangle = -\int_a^t m_s^{i,n}(x_0)\frac{\Lambda_i(s)}{z_i(s)}ds + \nu_i\langle \mathbb{1}_{(x_0,\infty)}, \vartheta_i^n\rangle(t-a). \tag{6.11}$$

Rewriting, we have for all $t \in [a,b]$ and $x \ge \frac{1}{\ell}$,

$$\overline{M}_t^{i,n}(x) - \overline{M}_a^{i,n}(x) = \int_a^t \frac{\Lambda_i(s)}{z_i(s)}\frac{\partial \overline{M}_s^{i,n}(x)}{\partial x}ds + \nu_i \overline{N}^{i,n}(x)(t-a). \tag{6.12}$$

For fixed $x \ge \frac{1}{\ell}$, $s \to \frac{\partial \overline{M}_s^{i,n}(x)}{\partial x} = -m_s^{i,n}(x)$ is continuous, because the fluid model solution ζ_i is continuous as a function of time, and also that $s \to \frac{\Lambda_i(s)}{z_i(s)}$ is continuous on $[a,b]$, because $z_i(\cdot)$ is strictly positive there, we see that $t \to \overline{M}_t^{i,n}(x)$ is continuously differentiable on $[a,b]$, and by differentiating (6.12), we obtain (6.8). Since all of the other properties have been verified, this completes the proof. □

6.2. Proof of Theorem 4.1

Proof of Theorem 4.1. Assume that the hypotheses of Theorem 4.1 hold. Since $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$, we have by Corollary 5.1 and (2.4), that for each $t \geq 0$, $\zeta(t) \in \mathbf{K}_1^{\mathbf{I}}$. It follows that for each $i \in \mathcal{I}$ and $t \geq 0$, $x \to \overline{M}_t^i(x)$ is continuous and integrable with respect to Lebesgue measure (with integral equal to $\langle \chi, \zeta_i(t) \rangle < \infty$) on $[0,\infty)$. Also, $x \to \overline{N}_i(x)$ is continuous and integrable with respect to Lebesgue measure (with integral equal to $\langle \chi, \vartheta_i \rangle < \infty$) over $[0,\infty)$.

Fix $i \in \mathcal{I}$. Because $\mathcal{K}_i^\zeta(\cdot)$ is bounded and measurable on $[0,t]$ for each $t \geq 0$, to prove the absolute continuity of $\mathcal{H}_i^\zeta(\cdot)$, it suffices to prove that (4.1) holds for each $t \geq 0$. We first prove that if $0 \leq a < b < \infty$ such that $z_i(s) \neq 0$ for all $s \in [a,b]$, then

$$\mathcal{H}_i^\zeta(b) - \mathcal{H}_i^\zeta(a) = \int_a^b \mathcal{K}_i^\zeta(s) ds. \tag{6.13}$$

Assume that $0 \leq a < b < \infty$ such that $z_i(s) \neq 0$ for all $s \in [a,b]$. For (6.14), we shall use the definition of $\mathcal{K}_i^\zeta(\cdot)$, the facts that $\Lambda_i(\cdot) \leq \max_j C_j$, $z_i(\cdot)$ is bounded on $[a,b]$, being continuous there, $\overline{M}_s^i(x) \leq z_i(s)$ for all $x \in \mathbb{R}_+$ and $s \in [a,b]$, $\left|\frac{\Lambda_i(\cdot)}{z_i(\cdot)}\right|$ is bounded on $[a,b]$ because $z_i(\cdot)$ is continuous and strictly positive there, $\theta_i'(x) = m_i \mu_i (\theta_i(x))^{\frac{\alpha_i+1}{\alpha_i}} \overline{N}_i(x)$ for all $x \in \mathbb{R}_+$, $\|\theta_i\|_\infty < \infty$, and $\int_0^\infty \overline{N}_i(x) dx = \langle \chi, \vartheta_i \rangle = \mu_i^{-1} < \infty$. With these, we see that by dominated convergence,

$$\begin{aligned} \int_a^b \mathcal{K}_i^\zeta(s) ds = {} & -\tilde{\rho}_i^{-\alpha_i} \kappa_i \int_a^b \Lambda_i(s) (z_i(s))^{\alpha_i} ds \\ & + \lim_{\ell \to \infty} \tilde{\rho}_i^{-\alpha_i} \kappa_i \int_a^b \int_{\frac{1}{\ell}}^{\ell} (\overline{M}_s^i(x))^{\alpha_i} \left(\frac{-\Lambda_i(s)}{z_i(s)} \overline{M}_s^i(x) \theta_i'(x) + (\alpha_i+1) \nu_i \overline{N}_i(x) \theta_i(x) \right) dx ds. \end{aligned} \tag{6.14}$$

Now, for positive integers ℓ and $n > \ell$, because $\vartheta_i \in \mathbf{K}_1$ and $\zeta_i(s) \in \mathbf{K}_1$ for all $s \in [a,b]$, by Lemma 6.1, we have that as $n \to \infty$, $\overline{N}_i^n(x) \to \overline{N}_i(x)$ for each $x \in (0,\infty)$ and $\overline{M}_s^{i,n}(x) \to \overline{M}_s^i(x)$ for each $x \in (0,\infty)$, $s \in [a,b]$. Moreover, $\overline{N}_i^n(x) \leq \overline{N}_i((x-1)^+)$ and $\overline{M}_s^{i,n}(x) \leq \overline{M}_s^i((x-1)^+) \leq z_i(s)$ for all $x \in \mathbb{R}_+$ and $s \in [a,b]$, where $x \to \overline{N}_i((x-1)^+)$ has integral on $(0,\infty)$ bounded by $\int_0^\infty \overline{N}_i(x) dx + 1 < \infty$, and there is a uniform bound on $z_i(\cdot)$ for all $s \in [a,b]$. It then follows by the dominated convergence theorem (using the boundedness of θ_i' and θ_i from above) that for each fixed positive integer ℓ,

$$\begin{aligned} & \int_a^b \int_{\frac{1}{\ell}}^{\ell} (\overline{M}_s^i(x))^{\alpha_i} \left(\frac{-\Lambda_i(s)}{z_i(s)} \overline{M}_s^i(x) \theta_i'(x) + (\alpha_i+1) \nu_i \overline{N}_i(x) \theta_i(x) \right) dx ds \\ & = \lim_{n \to \infty} \int_a^b \int_{\frac{1}{\ell}}^{\ell} (\overline{M}_s^{i,n}(x))^{\alpha_i} \left(\frac{-\Lambda_i(s)}{z_i(s)} \overline{M}_s^{i,n}(x) \theta_i'(x) + (\alpha_i+1) \nu_i \overline{N}_i^n(x) \theta_i(x) \right) dx ds. \end{aligned} \tag{6.15}$$

Using integration by parts on the first term, the expression above is equal to

$$\begin{aligned} \lim_{n \to \infty} \Bigg(& \int_a^b \left(\frac{\Lambda_i(s)}{z_i(s)} \right) [-(\overline{M}_s^{i,n}(\cdot))^{\alpha_i+1} \theta_i(\cdot)]_{\frac{1}{\ell}}^{\ell} ds \\ & + (\alpha_i+1) \int_a^b \int_{\frac{1}{\ell}}^{\ell} (\overline{M}_s^{i,n}(x))^{\alpha_i} \left(\frac{\Lambda_i(s)}{z_i(s)} \frac{\partial \overline{M}_s^{i,n}(x)}{\partial x} + \nu_i \overline{N}_i^n(x) \right) \theta_i(x) dx ds \Bigg) \end{aligned}$$

$$= \lim_{n\to\infty} \left(\int_a^b \left(\frac{\Lambda_i(s)}{z_i(s)} \right) [-(\overline{M}_s^{i,n}(\cdot))^{\alpha_i+1}\theta_i(\cdot)]_{\frac{1}{\ell}}^{\ell} ds \right. \tag{6.16}$$
$$\left. +(\alpha_i+1)\int_a^b \int_{\frac{1}{\ell}}^{\ell} (\overline{M}_s^{i,n}(x))^{\alpha_i} \left(\frac{\partial \overline{M}_s^{i,n}(x)}{\partial s} \right) \theta_i(x) dx ds \right),$$

where we have used Lemma 6.2 for the last equality. By Fubini's theorem (where the joint measurability of the integrand follows from (6.8) and the fact that the partial derivative with respect to x there is given by $-m_s^{i,n}(x)$), the quantity above is equal to

$$\lim_{n\to\infty} \left(\int_a^b \left(\frac{\Lambda_i(s)}{z_i(s)} \right) [-(\overline{M}_s^{i,n}(\cdot))^{\alpha_i+1}\theta_i(\cdot)]_{\frac{1}{\ell}}^{\ell} ds + (\alpha_i+1)\int_{\frac{1}{\ell}}^{\ell} \int_a^b (\overline{M}_s^{i,n}(x))^{\alpha_i} \left(\frac{\partial \overline{M}_s^{i,n}(x)}{\partial s} \right) \theta_i(x) ds dx \right)$$
$$= \lim_{n\to\infty} \left(\int_a^b \left(\frac{\Lambda_i(s)}{z_i(s)} \right) \left[-(\overline{M}_s^{i,n}(\ell))^{\alpha_i+1}\theta_i(\ell) + \left(\overline{M}_s^{i,n}\left(\frac{1}{\ell}\right)\right)^{\alpha_i+1} \theta_i\left(\frac{1}{\ell}\right) \right] ds \right.$$
$$\left. + \int_{\frac{1}{\ell}}^{\ell} \left((\overline{M}_b^{i,n}(x))^{\alpha_i+1} - (\overline{M}_a^{i,n}(x))^{\alpha_i+1} \right) \theta_i(x) dx \right)$$
$$= \int_a^b \left(\frac{\Lambda_i(s)}{z_i(s)} \right) \left[-(\overline{M}_s^{i}(\ell))^{\alpha_i+1}\theta_i(\ell) + \left(\overline{M}_s^{i}\left(\frac{1}{\ell}\right)\right)^{\alpha_i+1} \theta_i\left(\frac{1}{\ell}\right) \right] ds \tag{6.17}$$
$$+ \int_{\frac{1}{\ell}}^{\ell} \left((\overline{M}_b^{i}(x))^{\alpha_i+1} - (\overline{M}_a^{i}(x))^{\alpha_i+1} \right) \theta_i(x) dx,$$

where we have used bounded convergence to pass to the limit for the last equality. Observe that as $\ell \to \infty$, we have $\overline{M}_s^i(\ell) \to 0$, $\overline{M}_s^i\left(\frac{1}{\ell}\right) \to z_i(s)$, $\theta_i\left(\frac{1}{\ell}\right) \to 1$ and there is a uniform bound for $(s,x) \to \overline{M}_s^i(x)$ and $x \to \theta_i(x)$ for all $s \in [a,b], x \in \mathbb{R}_+$. Combining this with the fact that $(\overline{M}_s^i(x))^{\alpha_i+1} \le (z_i(s))^{\alpha_i}\overline{M}_s^i(x)$, which is integrable on $\mathbb{R}_+$ for $s = a, b$, we see that as $\ell \to \infty$, the above expression converges to

$$\int_a^b \left(\frac{\Lambda_i(s)}{z_i(s)} \right) (z_i(s))^{\alpha_i+1} ds + \int_0^\infty \left((\overline{M}_b^{i}(x))^{\alpha_i+1} - (\overline{M}_a^{i}(x))^{\alpha_i+1} \right) \theta_i(x) dx. \tag{6.18}$$

On substituting the above into (6.14), we obtain

$$\int_a^b \mathcal{K}_i^\zeta(s) ds = -\tilde{\rho}_i^{-\alpha_i}\kappa_i \int_a^b \Lambda_i(s)(z_i(s))^{\alpha_i} ds$$
$$+\tilde{\rho}_i^{-\alpha_i}\kappa_i \int_a^b \left(\frac{\Lambda_i(s)}{z_i(s)} \right) (z_i(s))^{\alpha_i+1} ds$$
$$+\tilde{\rho}_i^{-\alpha_i}\kappa_i \int_0^\infty \left((\overline{M}_b^{i}(x))^{\alpha_i+1} - (\overline{M}_a^{i}(x))^{\alpha_i+1} \right) \theta_i(x) dx$$
$$= \mathcal{H}_i^\zeta(b) - \mathcal{H}_i^\zeta(a), \tag{6.19}$$

as desired.

We now turn to proving (4.1) for each $t \ge 0$. It clearly holds for $t = 0$, so we consider $t > 0$ fixed. If $z_i(s) \ne 0$ for all $s \in [0,t]$, then the result follows immediately from (6.13) with $a = 0$ and $b = t$. So we only need to treat the case where $z_i(s) = 0$ for some $s \in [0,t]$. Assuming this, let $s^* = \inf\{s \in$

$[0,t]: z_i(s)=0\}$ and $t^* = \sup\{s \in [0,t] : z_i(s) = 0\}$. Then, $0 \le s^* \le t^* \le t$, $z_i(s^*) = z_i(t^*) = 0$ and $z_i(s) > 0$ for $s \in (0,s^*) \cup (t^*,t)$. (Note that the interval $(0,s^*)$ is empty if $z_i(0)=0$ and (t^*,t) is empty if $z_i(t)=0$.) In any event, we can write the open set $\mathcal{T}_t^i = \{s \in (0,t) : z_i(s) > 0\}$ as a (finite or countable) union of disjoint open intervals:

$$\mathcal{T}_t^i = (0,s^*) \cup \left(\bigcup_n (s_n,t_n) \right) \cup (t^*,t),$$

where $\bigcup_n (s_n,t_n) \subset (s^*,t^*)$ and $z_i(s_n) = z_i(t_n) = 0$ for each n.

For each fixed n, for $s_n < a < b < t_n$, we have that (6.13) holds. Then using the continuity of $\mathcal{H}_i^\zeta(\cdot)$ (see Lemma 5.8) and the boundedness of $\mathcal{K}_i^\zeta$ on $[s_n,t_n]$ that we can let $a \downarrow s_n$ and $b \uparrow t_n$ in the last equation, to obtain

$$\mathcal{H}_i^\zeta(t_n) - \mathcal{H}_i^\zeta(s_n) = \int_{(s_n,t_n)} \mathcal{K}_i^\zeta(s)ds.$$

Moreover, since $z_i(s_n) = z_i(t_n) = 0$, $\mathcal{H}_i^\zeta(s_n) = \mathcal{H}_i^\zeta(t_n) = 0$ in the above. Thus we have

$$\int_{(s_n,t_n)} \mathcal{K}_i^\zeta(s)ds = 0. \tag{6.20}$$

In a similar manner, we can obtain

$$\mathcal{H}_i^\zeta(s^*) - \mathcal{H}_i^\zeta(0) = \int_{(0,s^*)} \mathcal{K}_i^\zeta(s)ds, \tag{6.21}$$

where $\mathcal{H}_i^\zeta(s^*) = 0$, and

$$\mathcal{H}_i^\zeta(t) - \mathcal{H}_i^\zeta(t^*) = \int_{(t^*,t)} \mathcal{K}_i^\zeta(s)ds, \tag{6.22}$$

where $\mathcal{H}_i^\zeta(t^*) = 0$. Combining all of the above, and using the integrability of $\mathcal{K}_i^\zeta$ on $[0,t]$, the fact that $\mathcal{K}_i^\zeta(\cdot)$ is zero on $(0,t) \setminus \mathcal{T}_t^i$, and the disjointness of the intervals $\{(s_n,t_n)\}$, we have

$$\begin{aligned}\int_0^t \mathcal{K}_i^\zeta(s)ds &= \int_{(0,s^*)} \mathcal{K}_i^\zeta(s)ds + \sum_n \int_{(s_n,t_n)} \mathcal{K}_i^\zeta(s)ds + \int_{(t^*,t)} \mathcal{K}_i^\zeta(s)ds \\ &= -\mathcal{H}_i^\zeta(0) + 0 + \mathcal{H}_i^\zeta(t),\end{aligned}$$

which is the desired result (4.1).

We now prove (4.2). Because both sides are zero when $z_i(t) = 0$, it suffices to consider the case where $z_i(t) > 0$. In this case,

$$\begin{aligned}\tilde{\rho}_i^{\alpha_i} \mathcal{K}_i^\zeta(t) &= -\kappa_i \Lambda_i(t) \big(z_i(t)\big)^{\alpha_i} \\ &\quad + \kappa_i \int_0^\infty \big(\overline{M}_t^i(x)\big)^{\alpha_i} \left(-\frac{\Lambda_i(t)}{z_i(t)} \overline{M}_t^i(x) m_i \theta_i(x)^{\frac{\alpha_i+1}{\alpha_i}} \mu_i \overline{N}_i(x) + (\alpha_i+1)\nu_i \overline{N}_i(x)\theta_i(x) \right) dx \\ &= -\kappa_i \Lambda_i(t) \big(z_i(t)\big)^{\alpha_i}\end{aligned}$$

$$
\begin{aligned}
&+\kappa_i \int_0^\infty \big(\overline{M}^i_t(x)\big)^{\alpha_i}\Big(-\frac{\Lambda_i(t)}{z_i(t)}\overline{M}^i_t(x) m_i \theta_i(x)^{\frac{1}{\alpha_i}} + (\alpha_i+1)\rho_i\Big)\mu_i\theta_i(x)\overline{N}_i(x)dx \\
&\le -\kappa_i\Lambda_i(t)(z_i(t))^{\alpha_i} + \kappa_i\int_0^\infty \Big(\frac{\alpha_i}{m_i}\Big)^{\alpha_i}\rho_i^{\alpha_i+1}\Big(\frac{z_i(t)}{\Lambda_i(t)}\Big)^{\alpha_i}\mu_i\overline{N}_i(x)dx \\
&= -\kappa_i\Lambda_i(t)(z_i(t))^{\alpha_i} + \frac{\kappa_i\alpha_i^{\alpha_i}\rho_i^{\alpha_i+1}(z_i(t))^{\alpha_i}}{m_i^{\alpha_i}(\Lambda_i(t))^{\alpha_i}},
\end{aligned}
$$

where we used Lemma 5.2, with $a=\alpha_i$, $q=\rho_i$, $b=\frac{\Lambda_i(t)}{z_i(t)}\theta_i(x)^{\frac{1}{\alpha_i}}m_i$ with $z_i(t)>0$, for the inequality, and the fact that $\int_0^\infty \mu_i\overline{N}_i(x)dx=1$ for the last equality. Recall that $\delta>0$ and $m_i\in(0,\alpha_i)$ were chosen so that $\big(\frac{\alpha_i}{m_i}\big)^{\alpha_i}=(1-\delta)(1+\delta)^{\alpha_i+1}>1$ and $\tilde\rho_i=(1+\delta)\rho_i$ satisfies (3.3). Using that in the above expression, we obtain when $z_i(t)>0$,

$$
\begin{aligned}
\mathcal{K}_i^\zeta(t) &\le \kappa_i(z_i(t))^{\alpha_i}\Big(-\frac{\Lambda_i(t)}{\tilde\rho_i^{\,\alpha_i}} + \frac{\tilde\rho_i(1-\delta)}{(\Lambda_i(t))^{\alpha_i}}\Big) \\
&= \kappa_i(z_i(t))^{\alpha_i}\Big(-\frac{\Lambda_i(t)}{\tilde\rho_i^{\,\alpha_i}} + \frac{\tilde\rho_i}{(\Lambda_i(t))^{\alpha_i}} - \delta\frac{\tilde\rho_i}{(\Lambda_i(t))^{\alpha_i}}\Big) \\
&\le \kappa_i(z_i(t))^{\alpha_i}\Big((\alpha_i+1)\frac{\tilde\rho_i-\Lambda_i(t)}{(\Lambda_i(t))^{\alpha_i}} - \delta\frac{\tilde\rho_i}{(\Lambda_i(t))^{\alpha_i}}\Big), \qquad (6.23)
\end{aligned}
$$

where the last step follows by Lemma 5.3 with $a=\alpha_i$, $b=\Lambda_i(t)$ and $q=\tilde\rho_i$. The first inequality yields (4.2). We shall use the last inequality to prove Corollary 4.1. □

6.3. Proof of Corollary 4.1

Proof of Corollary 4.1. Given the results of Theorem 4.1, all that requires proof is the inequality. For fixed $t\ge 0$ and $i\in\mathcal{I}_+(z(t))$, $U_i'\big(\frac{\Lambda_i(t)}{z_i(t)}\big)=\big(\frac{z_i(t)}{\Lambda_i(t)}\big)^{\alpha_i}$. Furthermore, $\tilde\rho$ has positive components and satisfies $\sum_{i\in\mathcal{I}}R_{ji}\tilde\rho_i<C_j$ for all $j\in\mathcal{J}$. Then, by (6.23) and replacing $z,\psi,\phi(z)$ by $z(t),\tilde\rho,\Lambda(t)$, respectively, in Lemma 5.1, we obtain

$$
\begin{aligned}
\mathcal{K}^\zeta(t) &= \sum_{i\in\mathcal{I}_+(z(t))}\frac{\mathcal{K}_i^\zeta(t)}{\alpha_i+1} \\
&\le \sum_{i\in\mathcal{I}_+(z(t))}\kappa_i\Big(\frac{z_i(t)}{\Lambda_i(t)}\Big)^{\alpha_i}\big(\tilde\rho_i-\Lambda_i(t)\big) - \delta\sum_{i\in\mathcal{I}_+(z(t))}\frac{\kappa_i\tilde\rho_i}{\alpha_i+1}\Big(\frac{z_i(t)}{\Lambda_i(t)}\Big)^{\alpha_i} \\
&\le -\delta\sum_{i\in\mathcal{I}_+(z(t))}\frac{\kappa_i\tilde\rho_i}{\alpha_i+1}\Big(\frac{z_i(t)}{\Lambda_i(t)}\Big)^{\alpha_i}.
\end{aligned}
$$

7. Proofs of Theorems 4.2 and 4.3

Theorem 4.1 and Corollary 4.1 are the main new results of this paper. In particular, these results are given proofs that, in contrast to Paganini et al. (2012), do not make strong smoothness assumptions on fluid model solutions and deal with the singular situation where some components of a fluid model solution may touch zero before all components reach zero. With these results in place, Theorems 4.2 and 4.3 follow in a similar manner to the arguments presented in Paganini et al.

(2012). However, we do generalize from having a common parameter α for all routes to the case where there is a separate α_i for each route $i\in\mathcal{I}$. We also establish the uniformity of the convergence to the zero state under suitable conditions.

Proof of Theorem 4.2. Let $\zeta(\cdot)$ be a fluid model solution with $\zeta(0)\in\mathbf{K}_1^{\mathbf{I}}$ and suppose that

$$\max_{i\in\mathcal{I}}(\langle \mathbb{1},\zeta_i(0)\rangle,\langle\chi,\zeta_i(0)\rangle)\le W, \tag{7.1}$$

for some finite, positive constant W. By (3.9) and the fact that $w_i(t)\le w_i(0)+\rho_i t$, we have

$$\mathcal{H}_i^\zeta(t)\le \big(A_W+Bt\big)(z_i(t))^{\alpha_i}\quad \text{for all } t\ge 0,\ i\in\mathcal{I}, \tag{7.2}$$

where

$$A_W=W\cdot\max_{i\in\mathcal{I}}\left(\frac{\kappa_i\|\theta_i(\cdot)\|_\infty}{\tilde{\rho}_i^{\alpha_i}}\right)\qquad\text{and}\qquad B=\max_{i\in\mathcal{I}}\left(\frac{\kappa_i\|\theta_i(\cdot)\|_\infty\rho_i}{\tilde{\rho}_i^{\alpha_i}}\right).$$

Let $\tilde{\rho}_\perp=\min_{i\in\mathcal{I}}\tilde{\rho}_i$ and $C=\max_{j\in\mathcal{J}}C_j$. It follows from (7.2) that

$$\kappa_i\tilde{\rho}_i\left(\frac{z_i(t)}{\Lambda_i(t)}\right)^{\alpha_i}\ge\frac{\kappa_i\mathcal{H}_i^\zeta(t)\tilde{\rho}_\perp}{C^{\alpha_i}(A_W+Bt)}\qquad\text{for all } t\ge 0,\ i\in\mathcal{I}. \tag{7.3}$$

Combining this with Corollary 4.1, we have for all $t\ge 0$,

$$\mathcal{K}^\zeta(t)\le-\delta\sum_{i\in\mathcal{I}_+(z(t))}\frac{\kappa_i\tilde{\rho}_i}{\alpha_i+1}\left(\frac{z_i(t)}{\Lambda_i(t)}\right)^{\alpha_i}\le-\frac{\delta D\tilde{\rho}_\perp}{A_W+Bt}\mathcal{H}^\zeta(t)$$

where $D=\min_{i\in\mathcal{I}}\frac{\kappa_i}{C^{\alpha_i}}$, and we have used the definition of $\mathcal{H}^\zeta(t)$ given in (3.8), as well as the fact that $\mathcal{H}_i^\zeta(t)=0$ for $i\notin\mathcal{I}_+(z(t))$.

Recall that $\mathcal{H}^\zeta(t)\ge 0$ for all $t\ge 0$. Because $\mathcal{K}^\zeta(\cdot)$ is the density (in time) for the absolutely continuous function $\mathcal{H}^\zeta(\cdot)$, we see from the above that $\mathcal{H}^\zeta(\cdot)$ is monotone decreasing with time and it is strictly decreasing on $\{s\ge 0:\mathcal{H}^\zeta(s)>0\}$. Let $\eta=\inf\{t\ge 0:\mathcal{H}^\zeta(t)=0\}$. Then for $0\le t<\eta$, we have

$$\begin{aligned}\log\mathcal{H}^\zeta(t)&=\log\mathcal{H}^\zeta(0)+\int_0^t\frac{\mathcal{K}^\zeta(s)}{\mathcal{H}^\zeta(s)}\,ds\\&\le\log\mathcal{H}^\zeta(0)-\int_0^t\frac{\delta D\tilde{\rho}_\perp}{A_W+Bs}ds.\end{aligned}$$

We observe that this holds for $t\ge\eta$ as well, since $\log\mathcal{H}^\zeta(t)=-\infty$ for such t. The last integral in the above diverges as $t\to\infty$. From this it follows that $\log\mathcal{H}^\zeta(t)\to-\infty$ as $t\to\infty$, and so whether η is finite or infinite, we have that $\lim_{t\to\infty}\mathcal{H}^\zeta(t)=0$. Moreover, this convergence is uniform for all fluid model solutions satisfying $\zeta(0)\in\mathbf{K}_1^{\mathbf{I}}$ and (7.1). (Note that $\mathcal{H}_i^\zeta(0)$ is bounded by $A_W W^{\alpha_i}$ for this.)

In a similar manner to that in Remark 3 in Paganini et al. (2012), for each $i\in\mathcal{I}$, because the weight function $\theta_i(\cdot)$ is bounded above and below on $[0,\infty)$, the convergence of $\mathcal{H}^\zeta(t)$ to zero as

$t \to \infty$ implies that $\overline{M}^i_t(\cdot)$ converges to zero in $\mathbf{L}^{\alpha_i+1}$ (with Lebesgue measure) as $t \to \infty$, and since $\overline{M}^i_t(x)$ is monotone decreasing as a function of $x \in (0,\infty)$, it follows that $\overline{M}^i_t(x)$ converges to zero as $t \to \infty$ for each $x \in (0,\infty)$. Consequently, $\zeta_i(t)$ as a measure on $(0,\infty)$ converges vaguely to zero as $t \to \infty$ for each $i \in \mathcal{I}$. □

We shall next prove Theorem 4.3. For the remainder of the section we shall assume that Assumptions 1–3 hold and that $W \geq 1$ is fixed. Let $p \in (1,\infty)$ be such that $B_{\vartheta,p} < \infty$, as in Assumption 3. We shall need the following supporting propositions.

Proposition 7.1 *Suppose that $\zeta(\cdot)$ is a fluid model solution such that $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$ and for each $i \in \mathcal{I}$, $\langle \chi^p, \zeta_i(0)\rangle \leq W$. Then for each $i \in \mathcal{I}$ and $t \geq 0$,*

$$\langle \chi^p, \zeta_i(t)\rangle \leq W + \nu_i t B_{\vartheta,p}. \tag{7.4}$$

Proof. By Remark 2.3, the fluid model equation (2.3) holds for ζ for all $f \in \tilde{\mathcal{C}} = \{f \in \mathbf{C}_b^1(\mathbb{R}_+) : f(0) = 0\}$. Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions in $\tilde{\mathcal{C}}$ such that $f_n(0) = 0$, $f_n' \geq 0$ on $[0,\infty)$ for all n and $0 \leq f_n \uparrow \chi^p$ on $[0,\infty)$ as $n \to \infty$. Equation (2.3) holds with f replaced by f_n and discarding the first integral term which is a non-negative integral since $f_n' \geq 0$, we obtain for each $i \in \mathcal{I}$ and $t \geq 0$,

$$\begin{aligned}\langle f_n, \zeta_i(t)\rangle &\leq \langle f_n, \zeta_i(0)\rangle + \nu_i \langle f_n, \vartheta_i\rangle \int_0^t \mathbb{1}_{(0,\infty)}\big(z_i(s)\big)ds \\ &\leq \langle \chi^p, \zeta_i(0)\rangle + \nu_i \langle \chi^p, \vartheta_i\rangle t \\ &\leq W + \nu_i t B_{\vartheta,p}.\end{aligned}$$

Letting $n \to \infty$ and using monotone convergence, we obtain

$$\langle \chi^p, \zeta_i(t)\rangle \leq W + \nu_i t B_{\vartheta,p},$$

as desired. □

Proposition 7.2 *Under the conditions of Proposition 7.1, for each $i \in \mathcal{I}$ and $t \geq 0$,*

$$w_i(t) \leq (W + \nu_i t B_{\vartheta,p})^{\frac{1}{p}} (z_i(t))^{\frac{1}{q}}, \tag{7.5}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Hölder's inequality and Proposition 7.1, we have

$$\begin{aligned}w_i(t) &= \langle \chi, \zeta_i(t)\rangle \\ &\leq \Big(\langle \chi^p, \zeta_i(t)\rangle\Big)^{\frac{1}{p}} \Big(\langle 1, \zeta_i(t)\rangle\Big)^{\frac{1}{q}} \\ &\leq (W + \nu_i t B_{\vartheta,p})^{\frac{1}{p}} (z_i(t))^{\frac{1}{q}}.\end{aligned}$$

□

Proof of Theorem 4.3. Applying Proposition 7.2 to (3.9), we have for all fluid model solutions $\zeta(\cdot)$ satisfying $\zeta(0)\in\mathbf{K}_1^{\mathbf{I}}$ and $\langle\chi^p,\zeta_i(0)\rangle\le W$ for all $i\in\mathcal{I}$,

$$\mathcal{H}_i^\zeta(t)\le C^\dagger(A^\dagger+B^\dagger t)^{1-\beta}(z_i(t))^{\alpha_i+\beta}, \tag{7.6}$$

where $\beta=\frac{1}{q}=1-\frac{1}{p}\in(0,1)$, $A^\dagger=W\ge 1$, $B^\dagger=(\max_{i\in\mathcal{I}}\nu_i)B_{\vartheta,p}$ and $C^\dagger=\max_{i\in\mathcal{I}}\left(\frac{\kappa_i\|\theta_i(\cdot)\|_\infty}{\tilde{\rho_i}^{\alpha_i}}\right)$. Using this, for $i\in\mathcal{I}_+(z(t))$, we have

$$\frac{\kappa_i\tilde{\rho}_i}{\alpha_i+1}\left(\frac{z_i(t)}{\Lambda_i(t)}\right)^{\alpha_i}\ge\frac{\kappa_i\tilde{\rho}_i}{(\alpha_i+1)C^{\alpha_i}(C^\dagger)^{\frac{\alpha_i}{\alpha_i+\beta}}}\left(\frac{\mathcal{H}_i^\zeta(t)}{(A^\dagger+B^\dagger t)^{1-\beta}}\right)^{\frac{\alpha_i}{\alpha_i+\beta}},$$

where $C=\max_j C_j$.

By Proposition 7.2, if $z_i(0)=\langle\mathbb{1},\zeta_i(0)\rangle\le W$ and $\langle\chi^p,\zeta_i(0)\rangle\le W$ for all $i\in\mathcal{I}$, then $w_i(0)=\langle\chi,\zeta_i(0)\rangle\le W$ for all $i\in\mathcal{I}$. Then, by Theorem 4.2, there is $T_1<\infty$ such that for all fluid model solutions ζ satisfying $\zeta(0)\in\mathbf{K}_1^{\mathbf{I}}$ and $\max_{i\in\mathcal{I}}(\langle\mathbb{1},\zeta_i(0)\rangle,\langle\chi^p,\zeta_i(0)\rangle)\le W$, we have $\mathcal{H}^\zeta(t)\le 1$ for all $t\ge T_1$. Then by the definition of $\mathcal{H}^\zeta(\cdot)$ and the fact that $A^\dagger\ge 1$, we have $\frac{\mathcal{H}_i^\zeta(t)}{(A^\dagger+B^\dagger t)^{1-\beta}}\le 1$ for all $i\in\mathcal{I}$ and $t\ge T_1$. On setting

$$\gamma=\min_{i\in\mathcal{I}}\left\{\frac{\kappa_i\tilde{\rho}_i}{(\alpha_i+1)C^{\alpha_i}(C^\dagger)^{\frac{\alpha_i}{\alpha_i+\beta}}}\right\},\qquad \alpha^\dagger=\max_{i\in\mathcal{I}}\alpha_i\quad\text{ and }\quad\gamma^\dagger=\frac{\gamma}{\mathbf{I}^{\frac{\alpha^\dagger}{\alpha^\dagger+\beta}}},$$

and noting that $x^{\frac{\alpha_i}{\alpha_i+\beta}}\ge x^{\frac{\alpha^\dagger}{\alpha^\dagger+\beta}}$ for all $i\in\mathcal{I}$ when $0<x\le 1$, it follows from the above that for all $t\ge T_1$,

$$\begin{aligned}\sum_{i\in\mathcal{I}_+(z(t))}\frac{\kappa_i\tilde{\rho}_i}{\alpha_i+1}\left(\frac{z_i(t)}{\Lambda_i(t)}\right)^{\alpha_i}&\ge\gamma\sum_{i\in\mathcal{I}_+(z(t))}\left(\frac{\mathcal{H}_i^\zeta(t)}{(A^\dagger+B^\dagger t)^{1-\beta}}\right)^{\frac{\alpha^\dagger}{\alpha^\dagger+\beta}}\\&\ge\frac{\gamma}{(A^\dagger+B^\dagger t)^{\frac{\alpha^\dagger(1-\beta)}{\alpha^\dagger+\beta}}}\max_{i\in\mathcal{I}}(\mathcal{H}_i^\zeta(t))^{\frac{\alpha^\dagger}{\alpha^\dagger+\beta}}\\&\ge\frac{\gamma}{(A^\dagger+B^\dagger t)^{\frac{\alpha^\dagger(1-\beta)}{\alpha^\dagger+\beta}}}\max_{i\in\mathcal{I}}\left\{\left(\frac{\mathcal{H}_i^\zeta(t)}{\alpha_i+1}\right)^{\frac{\alpha^\dagger}{\alpha^\dagger+\beta}}\right\}\\&\ge\frac{\gamma}{(A^\dagger+B^\dagger t)^{\frac{\alpha^\dagger(1-\beta)}{\alpha^\dagger+\beta}}}\left(\frac{\sum_{i\in\mathcal{I}}\frac{\mathcal{H}_i^\zeta(t)}{\alpha_i+1}}{\mathbf{I}}\right)^{\frac{\alpha^\dagger}{\alpha^\dagger+\beta}}\\&=\frac{\gamma^\dagger\,(\mathcal{H}^\zeta(t))^{\frac{\alpha^\dagger}{\alpha^\dagger+\beta}}}{(A^\dagger+B^\dagger t)^{\frac{\alpha^\dagger(1-\beta)}{\alpha^\dagger+\beta}}}.\end{aligned}\tag{7.7}$$

Then, by Corollary 4.1, the density $\mathcal{K}^\zeta(\cdot)$ for $\mathcal{H}^\zeta(\cdot)$ satisfies for all $t\ge T_1$,

$$\mathcal{K}^\zeta(t)\le-\delta\frac{\gamma^\dagger\,(\mathcal{H}^\zeta(t))^{\frac{\alpha^\dagger}{\alpha^\dagger+\beta}}}{(A^\dagger+B^\dagger t)^{\frac{\alpha^\dagger(1-\beta)}{\alpha^\dagger+\beta}}}. \tag{7.8}$$

Let $\eta = \inf\{t \geq 0 : \mathcal{H}^\zeta(t) = 0\}$. On $[0, \eta)$, $(\mathcal{H}^\zeta(\cdot))^{\frac{\beta}{\alpha^\dagger+\beta}}$ is absolutely continuous with density given by the left hand member of the following string of (in)equalities, which hold for all $T_1 \leq t < \eta$:

$$\frac{\beta}{\alpha^\dagger+\beta}\mathcal{H}^\zeta(t)^{\frac{-\alpha^\dagger}{\alpha^\dagger+\beta}}\mathcal{K}^\zeta(t) \leq \frac{\beta}{\alpha^\dagger+\beta}\left(\frac{-\delta\gamma^\dagger}{(A^\dagger+B^\dagger t)^{\frac{\alpha^\dagger(1-\beta)}{\alpha^\dagger+\beta}}}\right) = -\gamma^\ddagger\frac{d(A^\dagger+B^\dagger t)^{\frac{\beta(1+\alpha^\dagger)}{\alpha^\dagger+\beta}}}{dt} \tag{7.9}$$

where $\gamma^\ddagger = \frac{\delta\gamma^\dagger}{B^\dagger(1+\alpha^\dagger)} > 0$. Integrating in time, we obtain for $T_1 \leq t < \eta$,

$$(\mathcal{H}^\zeta(t))^{\frac{\beta}{\alpha^\dagger+\beta}} \leq (\mathcal{H}^\zeta(T_1))^{\frac{\beta}{\alpha^\dagger+\beta}} - \gamma^\ddagger(A^\dagger+B^\dagger t)^{\frac{\beta(1+\alpha^\dagger)}{\alpha^\dagger+\beta}} + \gamma^\ddagger(A^\dagger+B^\dagger T_1)^{\frac{\beta(1+\alpha^\dagger)}{\alpha^\dagger+\beta}}. \tag{7.10}$$

The right hand side of (7.10) goes to $-\infty$ as $t \to \infty$. Because $\mathcal{H}^\zeta(\cdot)$ is non-negative, it follows that $\mathcal{H}^\zeta(\cdot)$ reaches zero in finite time and stays there forever after. Assuming $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$, by Corollary 5.1 and (2.4), $\zeta(t) \in \mathbf{K}_1^{\mathbf{I}}$ for all $t \geq 0$, and it follows that $\zeta(t) = \mathbf{0}$ for all t such that $\mathcal{H}^\zeta(t) = 0$. Moreover, because $\mathcal{H}^\zeta(T_1)$ is bounded by 1, and T_1 was chosen to be the same for all fluid model solutions ζ satisfying $\zeta(0) \in \mathbf{K}_1^{\mathbf{I}}$ and $\max_{i\in\mathcal{I}}(\langle \mathbb{1}, \zeta_i(0)\rangle, \langle \chi^p, \zeta_i(0)\rangle) \leq W$, it follows that there is a uniform bound $T_W < \infty$ for the time for these fluid model solutions to reach the zero state and stay there forever after. $\square$

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Appendix A: Supplementary Lemmas

Lemma A.1 *For each $z^* \in \mathbb{R}_+^{\mathbf{I}}$, $\phi_i(\cdot)$ is continuous at z^* for each $i \in \mathcal{I}_+(z^*)$.*

Proof. We first observe that $\kappa_i z_i U_i(\psi_i/z_i) = \kappa_i z_i^{\alpha_i} U_i(\psi_i)$ for all $\psi_i \geq 0, i \in \mathcal{I}_+(z), z \in \mathbb{R}_+^{\mathbf{I}}$. So the objective function in the utility maximization problem (2.2) is the same as $\sum_{i\in\mathcal{I}_+(z)} \kappa_i z_i^{\alpha_i} U_i(\psi_i)$.

Fix $z^* \in \mathbb{R}_+^{\mathbf{I}}$. We want to show that for each $i \in \mathcal{I}_+(z^*)$, $z \to \phi_i(z)$ is continuous at $z = z^*$, where $\phi(z) = (\phi_1(z), \ldots, \phi_{\mathbf{I}}(z))$ is the optimal solution of (2.2). For convenience, for $z \in \mathbb{R}_+^{\mathbf{I}}$, let

$$G_z(\psi^+) = \sum_{i\in\mathcal{I}_+(z)} \kappa_i z_i^{\alpha_i} U_i(\psi_i),$$

where $\psi^+ = (\psi_i : i \in \mathcal{I}_+(z))$ will be regarded as a vector in $\mathbb{R}_+^{|\mathcal{I}_+(z)|}$ (this vector contains all of the positive entries of any feasible vector $\psi \in \mathbb{R}_+^{\mathbf{I}}$ for the optimization problem (2.2)).

Let $\epsilon > 0$ be sufficiently small that the open ball B_ϵ in $\mathbb{R}_+^{|\mathcal{I}_+(z^*)|}$, that is centered at $\phi^*(z^*) = \{\phi_i(z^*) : i \in \mathcal{I}_+(z^*)\}$ and has radius $\epsilon > 0$, is a strictly positive distance from the boundary of the orthant $\mathbb{R}_+^{|\mathcal{I}_+(z^*)|}$. Let D_ϵ denote the compact set of $\psi^\dagger = (\psi_i : i \in \mathcal{I}_+(z^*))$ in $\mathbb{R}_+^{|\mathcal{I}_+(z^*)|}$ that satisfy the constraints:

$$\sum_{i\in\mathcal{I}_+(z^*)} R_{ji}\psi_i \leq C_j \text{ for all } j \in \mathcal{J}, \quad \psi_i \geq 0 \text{ for all } i \in \mathcal{I}_+(z^*), \quad \psi^\dagger \notin B_\epsilon.$$

We claim that there is $\eta > 0$ and $\delta_1 > 0$ such that for all $z \in \mathbb{R}_+^{\mathbf{I}}$ satisfying $|z - z^*| < \delta_1$ and $\psi^\dagger = (\psi_i : i \in \mathcal{I}_+(z^*))$ in D_ϵ, we have

$$\sum_{i \in \mathcal{I}_+(z^*)} \kappa_i z_i^{\alpha_i} U_i(\psi_i) < G_{z^*}(\phi^*(z^*)) - \eta. \tag{A.1}$$

Here $|\cdot|$ denotes the usual Euclidean norm. Note that the sum in (A.1) is only over $i \in \mathcal{I}_+(z^*)$, even though the functions being summed have z_i not z_i^* in them. The claim can be proved using an argument by contradiction as follows. Suppose that for each positive integer n there is $z^n \in \mathbb{R}_+^{\mathbf{I}}$ such that $|z^n - z^*| < 1/n$ and $\psi^{\dagger,n} = (\psi_i^n, i \in \mathcal{I}_+(z^*)) \in D_\epsilon$ such that

$$\sum_{i \in \mathcal{I}_+(z^*)} \kappa_i (z_i^n)^{\alpha_i} U_i(\psi_i^n) \geq G_{z^*}(\phi^*(z^*)) - \frac{1}{n}. \tag{A.2}$$

Then $z^n \to z^*$ as $n \to \infty$ and, since D_ϵ is compact, by passing to a suitable subsequence, denoted by $\{n_k\}_{k=1}^\infty$, we may assume that $\psi^{\dagger,n_k} \to \psi^*$ for some $\psi^* \in D_\epsilon$ as $k \to \infty$. For any $i \in \mathcal{I}_+(z^*)$ such that $\alpha_i \in (0,1)$, the term $\kappa_i (z_i^n)^{\alpha_i} U_i(\psi_i^n)$ in the left member of (A.2) is jointly continuous in z_i^n and ψ_i^n and so with n replaced by n_k, this term tends to the finite value $\kappa_i (z_i^*)^{\alpha_i} U_i(\psi_i^*)$ as $k \to \infty$. For any $i \in \mathcal{I}_+(z^*)$ such that $\alpha_i \in [1,\infty)$, if $\psi_i^* > 0$, then $U_i(\psi_i^{n_k})$ tends to $U_i(\psi_i^*)$ as $k \to \infty$; on the other hand, if $\psi_i^* = 0$ then $U_i(\psi_i^{n_k})$ tends to $-\infty$. In fact, the latter cannot occur; because, if it did, taking the lim inf as $k \to \infty$ in (A.2), with n_k in place of n, and using the fact that $z_i^{n_k} \to z_i^* > 0$ as $k \to \infty$ for $i \in \mathcal{I}_+(z^*)$, would yield a contradiction to the finiteness of the right member of (A.2). Consequently, we can pass to the limit as $k \to \infty$ in (A.2), with n_k in place of n, to conclude that $\psi^* \in D_\epsilon$ and

$$\sum_{i \in \mathcal{I}_+(z^*)} \kappa_i (z_i^*)^{\alpha_i} U_i(\psi_i^*) \geq G_{z^*}(\phi^*(z^*)). \tag{A.3}$$

Recognizing the left member above as $G_{z^*}(\psi^*)$, this implies that $\psi^* \in D_\epsilon$ and $\phi^*(z^*) \in B_\epsilon$ are two distinct maximizers for the optimization of $G_{z^*}(\cdot)$ over the set

$$\left\{ \psi^\dagger \in \mathbb{R}_+^{|\mathcal{I}_+(z^*)|} : \sum_{i \in \mathcal{I}_+(z^*)} R_{ji} \psi_i \leq C_j \text{ for all } j \in \mathcal{J} \right\}.$$

This contradicts the uniqueness of such a maximizer (see Remark 2.1). This last contradiction implies that the claim associated with (A.1) is true.

Without loss of generality, we can assume that the $\delta_1 > 0$ in the claim proved above is small enough that for all $z \in \mathbb{R}_+^{\mathbf{I}}$ such that $|z - z^*| < \delta_1$ we have $z_i > 0$ for all $i \in \mathcal{I}_+(z^*)$, which implies that $\mathcal{I}_+(z^*) \subset \mathcal{I}_+(z)$ and for $\psi^+ = (\psi_i : i \in \mathcal{I}_+(z)) \in \mathbb{R}_+^{|\mathcal{I}_+(z)|}$,

$$G_z(\psi^+) = \sum_{i \in \mathcal{I}_+(z^*)} \kappa_i z_i^{\alpha_i} U_i(\psi_i) + \sum_{i \in \mathcal{I}_+(z) \setminus \mathcal{I}_+(z^*)} \kappa_i z_i^{\alpha_i} U_i(\psi_i). \tag{A.4}$$

Note $z_i^* = 0$ for all $i \in \mathcal{I}_+(z) \setminus \mathcal{I}_+(z^*)$, and for all ψ^+ satisfying

$$\sum_{i \in \mathcal{I}_+(z)} R_{ji} \psi_i \leq C_j, \quad \text{for all } j \in \mathcal{J}, \tag{A.5}$$

we have $U_i(\psi_i) \leq U_i(C^*)$ for all $i \in \mathcal{I}_+(z)$ where $C^* = \max_{j \in \mathcal{J}} C_j$. It follows that there is $\delta_2 \in (0, \delta_1)$ such that the last sum in (A.4) is less than $\eta/4$ for all $z \in \mathbb{R}_+^{\mathbf{I}}$ satisfying $|z - z^*| < \delta_2$ and $\psi^+ \in \mathbb{R}_+^{|\mathcal{I}_+(z)|}$ satisfying (A.5).

Combining this with (A.1) and (A.4), we see that for such z and ψ^+, if in addition, $\psi^\dagger = (\psi_i : i \in \mathcal{I}_+(z^*)) \notin B_\epsilon$, then

$$G_z(\psi^+) \leq G_{z^*}(\phi^*(z^*)) - \frac{3\eta}{4}. \tag{A.6}$$

On the other hand, consider the first sum on the right side of (A.4). By the continuity of the $U_i(\cdot)$ on $(0,\infty)$ and since $\phi_i(z^*) > 0$ for all $i \in \mathcal{I}_+(z^*)$, there is $\delta_3 \in (0, \delta_2)$ such that for all $z \in \mathbb{R}_+^{\mathbf{I}}$ satisfying $|z - z^*| < \delta_3$ and all $\psi^\dagger = (\psi_i : i \in \mathcal{I}_+(z^*))$ satisfying $|\psi^\dagger - \phi^*(z^*)| < \delta_3$, we have

$$\psi_i > 0 \text{ for all } i \in \mathcal{I}_+(z^*) \quad \text{ and } \quad \sum_{i \in \mathcal{I}_+(z^*)} \kappa_i z_i^{\alpha_i} U_i(\psi_i) \geq G_{z^*}(\phi^*(z^*)) - \frac{\eta}{4}. \tag{A.7}$$

In particular, an allowed value of such a $\psi^\dagger$ is $\psi^\ddagger = (\phi_i^*(z^*) - \frac{\delta_3}{2\mathbf{I}} : i \in \mathcal{I}_+(z^*))$. For this $\psi^\ddagger$, if $j \in \mathcal{J}$ such that $\sum_{i \in \mathcal{I}_+(z^*)} R_{ji} \phi_i^*(z^*) = C_j$, we must have that $R_{ji} = 1$ for some $i \in \mathcal{I}_+(z^*)$ and then $\sum_{i \in \mathcal{I}_+(z^*)} R_{ji} \psi_i \leq C_j - \frac{\delta_3}{2\mathbf{I}}$. Furthermore, there is $\delta_4 \in (0, \delta_3/2\mathbf{I})$ such that for those $j \in \mathcal{J}$ satisfying $\sum_{i \in \mathcal{I}_+(z^*)} R_{ji} \phi_i^*(z^*) < C_j$, we have $\sum_{i \in \mathcal{I}_+(z^*)} R_{ji} \phi^*(z^*) < C_j - \delta_4$. Then, $\psi^\ddagger$ satisfies $\sum_{i \in \mathcal{I}_+(z^*)} R_{ji} \psi_i < C_j - \delta_4$ for all $j \in \mathcal{J}$. Then, for any $z \in \mathbb{R}_+^{\mathbf{I}}$ satisfying $|z - z^*| < \delta_4$, we can define a vector $\psi^\flat(z)$ in $\mathbb{R}_+^{|\mathcal{I}_+(z)|}$ such that $\psi_i^\flat(z) = \psi_i^\ddagger$ for $i \in \mathcal{I}_+(z^*)$ and $\psi_i^\flat(z) = \frac{\delta_4}{\mathbf{I}}$ for all $i \in \mathcal{I}_+(z) \setminus \mathcal{I}_+(z^*)$. Then $\sum_{i \in \mathcal{I}_+(z)} R_{ji} \psi_i^\flat(z) \leq C_j$ for all $j \in \mathcal{J}$, and by (A.4) and (A.7),

$$G_z(\psi^\flat(z)) \geq G_{z^*}(\phi^*(z^*)) - \frac{\eta}{4} + \sum_{i \in \mathcal{I}_+(z) \setminus \mathcal{I}_+(z^*)} \kappa_i z_i^{\alpha_i} U_i\left(\frac{\delta_4}{\mathbf{I}}\right). \tag{A.8}$$

For $i \in \mathcal{I}_+(z) \setminus \mathcal{I}_+(z^*)$, we have that $z_i^* = 0$, and so there is $\delta_5 \in (0, \delta_4)$ such that the sum that is the last term in the above is smaller in magnitude than $\eta/4$ for all $z \in \mathbb{R}_+^{\mathbf{I}}$ satisfying $|z - z^*| < \delta_5$. Then, for all $z \in \mathbb{R}_+^{\mathbf{I}}$ such that $|z - z^*| < \delta_5$, we have that $\psi^\flat(z)$ (expanded to a vector in $\mathbb{R}_+^{\mathbf{I}}$ that has zeros for the components indexed by $i \notin \mathcal{I}_+(z)$) is feasible for the optimization problem (2.2) and by (A.8) and (A.6),

$$\begin{aligned} G_z(\psi^\flat(z)) &\geq G_{z^*}(\phi^*(z^*)) - \frac{\eta}{2} \\ &\geq G_z(\psi^+) + \frac{\eta}{4} \end{aligned}$$

for all $\psi^+ = (\psi_i : i \in \mathcal{I}_+(z)) \in \mathbb{R}_+^{|\mathcal{I}_+(z)|}$ that satisfy (A.5) and are such that $\psi^\dagger = (\psi_i : i \in \mathcal{I}_+(z^*))$ is not in B_ϵ. It follows that the optimal solution $\phi(z)$ must be such that $(\phi_i(z) : i \in \mathcal{I}_+(z^*))$ is in B_ϵ. Hence $\sum_{i \in \mathcal{I}_+(z^*)} |\phi_i(z) - \phi_i(z^*)|^2 < \epsilon^2$ whenever $|z - z^*| < \delta_5$. This proves the desired continuity. $\square$

Lemma A.2 *Let $\tilde{\mathcal{C}} = \{f \in \mathbf{C}_b^1(\mathbb{R}_+) : f(0) = 0\}$. If $\zeta(\cdot)$ is a solution for the fluid model, then for each $f \in \tilde{\mathcal{C}}$, property (iii) in Definition 2.2 still holds.*

Proof. Fix $f \in \tilde{\mathcal{C}}$. We first consider the case where f has compact support contained in $[0, M]$ for some $M > 1$. Let $\{g_n\}_{n=0}^\infty$ be a uniformly bounded sequence of continuous functions on $\mathbb{R}_+$ such that each g_n has support in $[0, M]$, $g_n(0) = 0$, and $g_n(x) \to f'(x)$ pointwise for each $x \in (0, \infty)$ as $n \to \infty$. For each n, let $f_n(x) = \int_0^x g_n(t) dt$, $x \in [0, \infty)$. Then $f_n \in \mathcal{C}$ for each n, $f_n' = g_n$ converges to f' pointwise and boundedly on $(0, \infty)$, and by bounded convergence, f_n converges pointwise to f on $[0, M]$ and also on $[M, \infty)$ since $f_n(x) = f_n(M) \to f(M) = f(x)$ for all $x \geq M$.

The property (2.3) holds with f_n, g_n in place of f, f', respectively. Hence,

$$\langle f_n, \zeta_i(t) \rangle = \langle f_n, \zeta_i(0) \rangle - \int_0^t \langle g_n, \zeta_i(s) \rangle \frac{\Lambda_i(s)}{z_i(s)} \mathbb{1}_{(0,\infty)}(z_i(s)) ds + \nu_i \langle f_n, \vartheta_i \rangle \int_0^t \mathbb{1}_{(0,\infty)}(z_i(s)) ds. \tag{A.9}$$

By the bounded convergence theorem, since $\zeta_i(t)$, $\zeta_i(0)$, $\zeta_i(s)$, ϑ_i are finite measures on $\mathbb{R}_+$ that do not charge the origin, as $n\to\infty$, we have $\langle f_n,\zeta_i(t)\rangle \to \langle f,\zeta_i(t)\rangle$, $\langle f_n,\zeta_i(0)\rangle \to \langle f,\zeta_i(0)\rangle$, $\langle g_n,\zeta_i(s)\rangle \to \langle f',\zeta_i(s)\rangle$ for each $s\geq 0$, and $\langle f_n,\vartheta_i\rangle \to \langle f,\vartheta_i\rangle$. Furthermore,

$$\sup_{s\in[0,t]}\left|\langle g_n,\zeta_i(s)\rangle \frac{\Lambda_i(s)}{z_i(s)}\right| \leq \sup_n \|g_n\|_\infty (\max_{j\in\mathcal{J}} C_j) < \infty.$$

Combining the above and using bounded convergence again for the second term in the right side of (A.9), we can let $n\to\infty$ in (A.9) to show that (2.3) holds for f. Thus, (2.3) holds for $f\in\tilde{\mathcal{C}}$ that has compact support in $[0,M]$ for any $M>1$. In particular, for an arbitrary $f\in\tilde{\mathcal{C}}$, it holds with $f\chi_M$ in place of f and $(f\chi_M)' = f'\chi_M + f\chi_M'$ in place of f', where χ_M is a function in $\mathbf{C}_b^1(\mathbb{R}_+)$ that equals 1 on $[0,M-1]$, is zero on $[M,\infty)$, and is monotonically decreasing on $[M-1,M]$ with first derivative bounded in absolute value by 2. Then using the facts that $f\chi_M$ and $(f\chi_M)'$ converge pointwise and boundedly on $\mathbb{R}_+$ to f and f', respectively, as $M\to\infty$, using bounded convergence again, we conclude that (2.3) holds for all $f\in\tilde{\mathcal{C}}$. □

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