European Journal of Mathematics
https://doi.org/10.1007/s40879-021-00462-z

RESEARCH/REVIEW ARTICLE

On Novodvorskii's theorem and the Oka principle

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Received: 21 October 2020 / Revised: 28 January 2021 / Accepted: 19 February 2021

Abstract
We give an exposition of Novodvorskii's theorem in Banach algebra K-theory, asserting that the Gelfand transform for a commutative Banach algebra induces an isomorphism in topological K-theory.

Keywords K-theory · Commutative Banach algebras · Oka principle

Mathematics Subject Classification 46L80 · 46J10

1 Introduction

The purpose of this paper is to give an expository account of the following theorem of Novodvorskii [18], which was one of the most striking discoveries from the early days of K-theory for Banach algebras:

Theorem *If A is a commutative and unital Banach algebra A with Gelfand spectrum X, then the Gelfand transform induces an isomorphism*

$$K_*(A) \xrightarrow{\cong} K_*(C(X))$$

in (topological) K-theory.

There are several precursors to Novodvorskii's theorem. In fact essentially the same result was proved by Arens [3], in the same way, although K-theory was not mentioned. We refer the reader to [19,20] for early surveys, as well as to the introduction of [6]

Nigel Higson's research was partially supported by the NSF grant DMS-1952669.

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Published online: 31 March 2021

for further remarks. But the clean and simple statement of Novodvorskii's theorem causes it to stand out, especially now that K-theory has become a familiar topic in Banach algebra theory.

The theorem is a K-theoretic version of the Oka principle from several complex variables, in the form established by Grauert [11–13], and it may be proved by a reduction to Grauert's work. This was Novodvorskii's approach. In contrast, the proof we shall present here takes full advantage of the computational framework that K-theory provides, and minimizes prerequisites from several complex variables.

The K-theory approach makes plain a striking analogy between the proof of the Oka principle and the proof of a classic result in topology, namely the Jordan–Brouwer separation theorem:

Theorem *The complement of an embedded $(n-1)$-sphere in an n-sphere has precisely two connected components.*

The separation theorem quickly reduces to the following assertion, which is the heart of the matter:

Theorem *The complement of an embedded k-cube in an n-sphere has the singular homology of a point.*

This is proved by induction on the dimension of the cube. First, the dimension zero case is trivial. Next, by writing a k-cube C as a union of two closed half-cubes that intersect along a midplane, by assuming the result for this midplane, which is a lower-dimensional cube, and by invoking the Mayer–Vietoris sequence, we find that the theorem is true for a given embedding of C if and only if it is true for the embeddings of the two half-cubes.

Each of these half-cubes we may in turn cut in two, along hyperplanes parallel to the first cut; then we may do the same to the four resulting quarter cubes; and so on. A simple diagram chase and an application of the continuity property of singular homology show that the theorem holds for the embedding of C if and only if it holds for the embeddings of all $\bigcap_j C_j$, where $C_0 = C$ and where C_j is one of the halves that is obtained by bisecting C_{j-1}, as above. But $\bigcap_j C_j$ is a cube of lower dimension, so the induction hypothesis applies, and the proof is complete.

Novodvorskii's theorem may be proved in essentially the same way, using a combination of Mayer–Vietoris and continuity in K-theory (meaning compatibility with direct limits).[1]

To begin, the key instance of the theorem, to which all others may be reduced, is that of the Banach algebra $B(X)$ that is associated to a polynomially convex compact set $X \subseteq \mathbb{C}^n$, and is constructed by completing the algebra of polynomial functions on X in the uniform norm. The Gelfand transform in this case is the inclusion of $B(X)$ into the algebra $C(X)$ of all continuous functions.

[1] See also [9, Section 7.5], where essentially the same point about the role of the Mayer–Vietoris property in the proof of the Oka principle is made in more categorical language.

The main task is to establish a K-theory Mayer–Vietoris sequence for the $B(X)$-algebras, leading to commuting diagrams of the form

$$\begin{array}{ccccccccc}
\cdots & \to & K_0(B(X_1 \cap X_2)) & \to & K_1(B(X_1 \cup X_2)) & \to & K_1(B(X_1) \oplus B(X_2)) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & K_0(C(X_1 \cap X_2)) & \to & K_1(C(X_1 \cup X_2)) & \to & K_1(C(X_1) \oplus C(X_2)) & \to & \cdots
\end{array}$$

Complex analysis enters here. Then, with the Mayer–Vietoris sequence in hand, the proof of Novodvorskii's theorem is by induction on the dimension of X, using repeated divisions of X by hyperplanes and an eventual direct limit argument.

More than once the Oka principle has been suggested as a possible starting point for new approaches to the Baum–Connes conjecture in the K-theory of group C^*-algebras [5]; see for example [2]. Those ideas have not yet advanced very far, but it was with possible noncommutative generalizations in mind that we have written this exposition.

2 The Mayer–Vietoris property

In this section we shall gather some results about Mayer–Vietoris sequences in Banach algebra K-theory. Most are easily derived from foundational properties of K-theory such as the six-term exact sequence, and so on. Generally we shall simply state these results. An exception is Theorem 2.7, which is more substantial and less well known.

We begin by recalling some facts about mapping cones of Banach algebra morphisms and double mapping cones of pairs of Banach algebra morphisms.

Definition 2.1 Let $\varphi : A \to B$ be a morphism of Banach algebras.[2] The *mapping cone* $\mathsf{MC}(\varphi)$ is the Banach algebra

$$\mathsf{MC}(\varphi) = \big\{(a, f) \in A \oplus C([0, 1], B) : \varphi(a) = f(0) \text{ and } f(1) = 0\big\}.$$

If we define the *suspension* of the Banach algebra B by

$$\mathsf{S}(B) = \{f \in C([0, 1], B) : f(0) = 0 = f(1)\},$$

then there are Banach algebra morphisms

$$\mathsf{S}(B) \xrightarrow{\iota} \mathsf{MC}(\varphi) \xrightarrow{\pi} A$$

[2] Throughout the paper, our morphisms are required to be continuous, but not contractive unless otherwise advertised. In particular, for us, an isomorphism of Banach algebras is not required to be isometric.

given by inclusion and projection, and they induce a six-term mapping cone exact sequence in K-theory

$$\begin{array}{ccccc} K_0(\mathsf{MC}(\varphi)) & \xrightarrow{\pi_*} & K_0(A) & \xrightarrow{\varphi_*} & K_0(B) \\ {\scriptstyle \iota_*}\uparrow & & & & \downarrow{\scriptstyle \iota_*} \\ K_1(B) & \xleftarrow[\varphi_*]{} & K_1(A) & \xleftarrow[\pi_*]{} & K_1(\mathsf{MC}(\varphi)). \end{array} \tag{2.1}$$

This uses the suspension and periodicity isomorphisms

$$K_0(\mathsf{S}(B)) \cong K_1(B) \quad \text{and} \quad K_1(\mathsf{S}(B)) \cong K_0(B).$$

Definition 2.2 Given a diagram of Banach algebras and Banach algebra morphisms of the form

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle \varphi} \\ C & \xrightarrow[\psi]{} & D \end{array} \tag{2.2}$$

we define the *double mapping cone* $\mathsf{DMC}(\varphi, \psi)$ to be the Banach algebra

$$\begin{aligned} \mathsf{DMC}(\varphi, \psi) = \big\{ (b, f, c) \in B \oplus C([0,1], D) \oplus C \\ : \varphi(b) = f(0) \text{ and } \psi(c) = f(1) \big\}. \end{aligned}$$

There is an obvious surjective morphism from $\mathsf{DMC}(\varphi, \psi)$ to $B \oplus C$, and its kernel is $\mathsf{S}(D)$. Associated to the surjection is therefore a K-theory six-term exact sequence

$$\begin{array}{ccccc} K_0(\mathsf{DMC}(\varphi, \psi)) & \longrightarrow & K_0(B) \oplus K_0(C) & \longrightarrow & K_0(D) \\ \uparrow & & & & \downarrow \\ K_1(D) & \longleftarrow & K_1(B) \oplus K_1(C) & \longleftarrow & K_1(\mathsf{DMC}(\varphi, \psi)) \end{array} \tag{2.3}$$

that is analogous to (2.1).

Definition 2.3 Suppose the diagram (2.2) is part of a commuting square of Banach algebras and Banach algebra morphisms

$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \varphi} \\ C & \xrightarrow[\psi]{} & D. \end{array} \tag{2.4}$$

The associated *canonical morphism* is

$$\begin{aligned} A &\to \mathsf{DMC}(\varphi, \psi) \\ a &\mapsto (\beta(a), f_a, \gamma(a)), \end{aligned}$$

where $f_a\colon [0, 1] \to B$ is the constant function with value $\varphi(\beta(a)) = \psi(\gamma(a))$. We shall say that (2.4) has the *Mayer–Vietoris property* if the canonical morphism induces an isomorphism in K-theory.

If the diagram (2.4) has the Mayer–Vietoris property, then by substituting $K_*(A)$ for $K_*(\mathsf{DMC}(\varphi, \psi))$ in the six-term sequence (2.3) we obtain the six-term Mayer–Vietoris exact sequence

$$\begin{array}{ccccc} K_0(A) & \longrightarrow & K_0(B) \oplus K_0(C) & \longrightarrow & K_0(D) \\ \uparrow & & & & \downarrow \\ K_1(D) & \longleftarrow & K_1(B) \oplus K_1(C) & \longleftarrow & K_1(A). \end{array}$$

This explains the terminology in Definition 2.3.

We shall need two simple facts about the Mayer–Vietoris property related to a commuting cube of Banach algebras and Banach algebra morphisms

$$\begin{array}{ccccc} & & A_1 & \longrightarrow & B_1 \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ A_2 & & \longrightarrow & B_2 & \\ \downarrow & & \downarrow & & \downarrow \\ & & C_1 & \longrightarrow & D_1 \\ & \swarrow & & \swarrow & \\ C_2 & & \longrightarrow & D_2 & \end{array} \tag{2.5}$$

Lemma 2.4 *If either the front face or the back face in* (2.5) *has the Mayer–Vietoris property, and if all the morphisms from the back face to the front face induce isomorphisms in K-theory, then the opposite face has the Mayer–Vietoris property, too.*

Lemma 2.5 *If both the front face and the back face in* (2.5) *have the Mayer–Vietoris property, then the square composed of the mapping cone algebras for the morphisms from the back face to the front face has the Mayer–Vietoris property, too.*

Definition 2.6 The diagram (2.4) has the *pullback property*, or is a *pullback square*, if the morphism of Banach algebras

$$A \to \{(b, c) \in B \oplus C : \varphi(b) = \psi(c)\}$$

defined by the formula $a \mapsto (\beta(a), \gamma(a))$ is an isomorphism.

Our objective for the remainder of this section is to prove a Mayer–Vietoris theorem due to Bost, following Cartan [7] and Douady [8]. The formulation below has not appeared in print, but compare [6, Section 5.4], where the essential ideas are presented.

It is simple to prove that if the commuting square (2.4) is a pullback square, and if $\varphi[B] = D$, or if $\psi[C] = D$, then the square has the Mayer–Vietoris property (for instance, the algebraic argument in [17, Sections 1–3] is easily adapted to topological K-theory). The following theorem is a generalization:

Theorem 2.7 *If the commuting square of Banach algebras* (2.4) *is a pullback square, if*

$$\varphi[B] + \psi[C] = D,$$

and if one or both of the morphisms φ or ψ has dense image, then the square has the Mayer–Vietoris property.

The proof of Theorem 2.7 involves the definition of K-theory in terms of homotopy groups, some elementary ideas from homotopy theory, and an implicit function theorem in the Banach space context.

To begin, recall that the K-theory groups of A may be defined by $K_j(A) = \varinjlim \pi_{j-1}(\mathrm{GL}_n(A))$ for $j > 0$, using the standard embeddings on $\mathrm{GL}_n(A)$ into $\mathrm{GL}_{n+1}(A)$ [17, Section 3].[3]

Next, recall that a continuous map $f : E \to B$ between two topological spaces is a (*Serre*) *fibration* if f satisfies the homotopy lifting property [21, Chapter 1, Section 7] for maps from a cube of any finite dimension into B. If we fix a basepoint $e \in E$, set $b = f(e)$, and form the fiber $F = f^{-1}[b]$, then associated to a Serre fibration there is an exact sequence of homotopy groups

$$\cdots \to \pi_j(F, e) \to \pi_j(E, e) \to \pi_j(B, b) \to \pi_{j-1}(F, e) \to \pi_{j-1}(E, e) \to \cdots$$

If $A \to A/J$ is a surjective morphism of Banach algebras, then the induced morphism

$$\mathrm{GL}_n(A) \to \mathrm{GL}_n(A/J)$$

is a Serre fibration for every n, with fiber $\mathrm{GL}_n(J)$. The K-theory exact sequence

$$\cdots \to K_{j+1}(J) \to K_{j+1}(A) \to K_{j+1}(A/J) \to K_j(J) \to K_j(A) \to \cdots$$

is the associated exact sequence of homotopy groups (or rather the direct limit over n of these). The sequence is 2-periodic in j, leading to the six-term sequences mentioned earlier.

The above leads immediately to the following sufficient condition for a commuting square (2.4) with the pullback property to have the Mayer–Vietoris property (the pullback property allows us to identify $\mathrm{GL}_n(A)$ with the fiber of the map in the lemma):

[3] As usual, when a Banach algebra A does not have a multiplicative identity element, we embed A as an ideal in any Banach algebra $\widetilde{A}$ that does have one, and then define $\mathrm{GL}_n(A)$ to be the kernel of the morphism $\mathrm{GL}_n(\widetilde{A}) \to \mathrm{GL}_n(\widetilde{A}/A)$.

Lemma 2.8 *Suppose given a commuting square of Banach algebras* (2.4) *with the pullback property. If for every n the continuous map*

$$\pi : \mathrm{GL}_n(B) \times \mathrm{GL}_n(C) \to \mathrm{GL}_n(D) \tag{2.6}$$

given by the formula

$$\pi(S, T) \mapsto \varphi(S)^{-1}\psi(T)$$

is a Serre fibration, then the square has the Mayer–Vietoris property.

Thanks to this, to prove Theorem 2.7 it suffices to prove the following result:

Theorem 2.9 *Under the hypotheses of Theorem* 2.7*, the continuous map* π *in* (2.6) *is a Serre fibration.*

The main step in the proof of Theorem 2.9 will be as follows:

Lemma 2.10 *Under the hypotheses of Theorem* 2.7*, the continuous map* π *in* (2.6) *has a local right inverse, defined in a neighborhood* U *of the identity in* $\mathrm{GL}_n(D)$*. That is, there is a continuous map*

$$\sigma : U \to \mathrm{GL}_n(B) \times \mathrm{GL}_n(C)$$

for which the composition $\pi \circ \sigma$ *is the inclusion of* U *into* $\mathrm{GL}_n(D)$*.*

Proof of Theorem 2.9, assuming Lemma 2.10 The group

$$G = \mathrm{GL}_n(B) \times \mathrm{GL}_n(C)$$

acts on itself by right multiplication, and on $\mathrm{GL}_n(D)$ by the formula

$$R \cdot (S, T) = \varphi(S)^{-1} R \psi(T).$$

The map π in (2.6) is G-equivariant, so its image is a union of G-orbits. This, together with the assumption that one or both of φ or ψ has dense image and a computation using the exponential map, shows that each G-orbit in $\mathrm{GL}_n(D)$ is dense in each component of $\mathrm{GL}_n(D)$ that it intersects.

The existence of the local section σ implies that the image of π is open. Now, the complement of the image is also a union of G-orbits, and so from the above it follows that the image is a union of components in $\mathrm{GL}_n(D)$.

The fiber of π over the identity element of $\mathrm{GL}_n(D)$ is

$$\mathrm{GL}_n(A) \cong \big\{(\beta(Q), \gamma(Q)) \in \mathrm{GL}_n(B) \times \mathrm{GL}_n(C) : Q \in \mathrm{GL}_n(A)\big\}.$$

If $U \subseteq \mathrm{GL}_n(D)$ is the open set on which the section σ is defined, and if for $(S, T) \in \pi^{-1}[U]$ we set

$$\theta(S, T) = \sigma(\pi(S, T)) \cdot (S^{-1}, T^{-1}),$$

using multiplication in $\mathrm{GL}_n(B) \times \mathrm{GL}_n(C)$, then in fact $\theta(S, T) \in \mathrm{GL}_n(A)$ and the formula

$$(S, T) \mapsto (\theta(S, T), \pi(S, T))$$

defines a $\mathrm{GL}_n(A)$-equivariant homeomorphism from $\pi^{-1}[U]$ to $\mathrm{GL}_n(A) \times U$ (for the obvious left $\mathrm{GL}_n(A)$-actions). So $\pi^{-1}[U]$ is a trivial principal $\mathrm{GL}_n(A)$-bundle over U. Using right G-equivariance, we find that the map

$$\pi : \mathrm{GL}_n(B) \times \mathrm{GL}_n(C) \to \mathrm{GL}_n(D)$$

is a principal $\mathrm{GL}_n(A)$-bundle over its image, an open and closed subset of $\mathrm{GL}_n(D)$. Since principal bundles are fibrations the proof is complete. □

We shall construct the local right inverse in Lemma 2.10 using a version of the implicit function theorem, the essential part of which is the following simple nonlinear version of the open mapping theorem (we have simplified the original statement by specializing it).

Theorem 2.11 ([14, Theorem 1]) *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let F be a continuous function from a neighborhood of $0 \in \mathcal{X}$ into $\mathcal{Y}$ such that $F(0) = 0$. If there exists a surjective continuous linear operator $L : \mathcal{X} \to \mathcal{Y}$ such that*

$$\big\| F(x_1) - F(x_2) - L(x_1 - x_2) \big\| = \mathcal{O}\big((\|x_1\| + \|x_2\|)\|x_1 - x_2\|\big),$$

then the element $0 \in \mathcal{Y}$ lies in the interior of the image of F.

In order to construct a local section from this result we use the well-known Bartle–Graves theorem:

Theorem 2.12 (See [4, Theorem 4]) *Every continuous and surjective linear operator between Banach spaces has a continuous (but not necessarily linear) right inverse, which may be chosen so that it maps bounded subsets of $\mathcal{Y}$ to bounded subsets of $\mathcal{X}$.*

Corollary 2.13 *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let F be a continuous function from a neighborhood of $0 \in \mathcal{X}$ into $\mathcal{Y}$ such that $F(0) = 0$. If there exists a surjective continuous linear operator $L : \mathcal{X} \to \mathcal{Y}$ such that*

$$\big\| F(x_1) - F(x_2) - L(x_1 - x_2) \big\| = \mathcal{O}\big((\|x_1\| + \|x_2\|)\|x_1 - x_2\|\big),$$

then F has a continuous local right-inverse, defined on some neighborhood of $0 \in \mathcal{Y}$.

Proof Denote by $C_{\mathrm{b}}(\mathcal{Y}, \mathcal{X})$ the Banach space of bounded, continuous functions from $\mathcal{Y}$ to $\mathcal{X}$ (the norm is the supremum norm associated to the norm on $\mathcal{X}$) and define $C_{\mathrm{b}}(\mathcal{Y}, \mathcal{Y})$ similarly. Apply Theorem 2.11 to the continuous map

$$\boldsymbol{F} : C_{\mathrm{b}}(\mathcal{Y}, \mathcal{X}) \to C_{\mathrm{b}}(\mathcal{Y}, \mathcal{Y})$$

given by composition with F (it is defined on the open subset of $C_{\mathrm{b}}(\mathcal{Y}, \mathcal{X})$ consisting of functions from $\mathcal{Y}$ to the open subset of $\mathcal{X}$ on which F is defined) and the continuous linear operator

$$\boldsymbol{L}\colon C_{\mathrm{b}}(\mathcal{Y}, \mathcal{X}) \to C_{\mathrm{b}}(\mathcal{Y}, \mathcal{Y})$$

given by composition with L (it follows from the Bartle–Graves theorem that $\boldsymbol{L}$ is surjective). □

Proof of Lemma 2.10 Define a continuous map from a neighborhood of zero in the Banach space $M_n(B) \oplus M_n(C)$ into the Banach space $M_n(D)$ by the formula

$$F(S, T) = \log\bigl(\exp(-\varphi(S))\exp(\psi(T))\bigr).$$

Define a bounded linear map from $M_n(B) \oplus M_n(C)$ into $M_n(D)$ by

$$L(S, T) = \psi(T) - \varphi(S).$$

It is surjective, thanks to the hypotheses of Theorem 2.7. Now apply Corollary 2.13. Compose the local right inverse that it provides with the logarithm and exponential maps on $\mathrm{GL}_n(D)$ and $M_n(B) \oplus M_n(C)$ respectively, to obtain a local right inverse to the map (2.6). □

3 The Gelfand theorem and continuity

A famous theorem of Gelfand asserts that an element of a commutative Banach algebra with unit is invertible if and only if its Gelfand transform is invertible [10, Chapter I, Section 4]. This has the following consequence for K-theory, which we shall use several times:

Theorem 3.1 *If a morphism $\alpha\colon A \to B$ of commutative unital Banach algebras has dense image and induces a homeomorphism*

$$\alpha^*\colon \mathrm{Spec}(B) \xrightarrow{\cong} \mathrm{Spec}(A),$$

then it induces an isomorphism on K-theory.

Of course this is an immediate consequence of Novodvorskii's theorem, but the dense image assumption allows for a simple, direct proof, based on the following well-known result:

Theorem 3.2 (Karoubi density [16, Exercise II.6.15]) *Let $\alpha\colon A \to B$ be an injective morphism of Banach algebras whose image is dense in B. If for every n*

$$\alpha[\mathrm{GL}_n(A)] = M_n(\alpha[A]) \cap \mathrm{GL}_n(B),$$

then the group homomorphism $\alpha\colon \mathrm{GL}_n(A) \to \mathrm{GL}_n(B)$ *induces isomorphisms* $\pi_j(\mathrm{GL}_n(A)) \to \pi_j(\mathrm{GL}_n(B))$ *for every* j *and every* n*, and so in particular* α *induces an isomorphism in* K*-theory.*

Proof of Theorem 3.1 By adjoining units, if necessary, we may assume that A and B are unital. The hypothesis in the theorem implies that every multiplicative linear functional on A vanishes on kernel(α), since it factors (uniquely) through α. It follows that the induced morphism

$$\bar{\alpha}\colon A/\mathrm{kernel}(\alpha) \to B$$

also satisfies the hypothesis of the theorem.

If $f \in A/\mathrm{kernel}(\alpha)$ and if the element $\bar{\alpha}(f) \in B$ is invertible, then by this hypothesis and by Gelfand's theorem, f is invertible in $A/\mathrm{kernel}(\alpha)$. Using determinants, we find that if $[f_{ij}]$ is an $n \times n$ matrix over $A/\mathrm{kernel}(\alpha)$, and if for which $[\bar{\alpha}(f_{ij})]$ is an invertible matrix over B, then $[f_{ij}]$ is itself invertible. So the Karoubi density theorem applies to $\bar{\alpha}$.

Now if $a \in \mathrm{kernel}(\alpha)$, then Gelfand's theorem implies that $1 + a$ is invertible in A. Using

$$(1 + a)(1 + a') = 1 + a + a' + aa'$$

we find that the inverse has the form $1 + a'$ with $a' \in \mathrm{kernel}(\alpha)$. Using determinants, again, we find that

$$\mathrm{GL}_n(\mathrm{kernel}(\alpha)) = I + M_n(\mathrm{kernel}(\alpha)).$$

So $\mathrm{GL}_n(\mathrm{kernel}(\alpha))$ is contractible for all n, and hence

$$K_*(\mathrm{kernel}(\alpha)) = 0.$$

The theorem follows from this, the six-term exact sequence in K-theory, and the fact that $\bar{\alpha}$ induces an isomorphism in K-theory. □

Next, suppose given a diagram of Banach algebras and Banach algebra morphisms

$$A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \cdots$$

in which the the norm of each morphism α_r is 1, or less. The algebraic vector space direct limit is in fact an associative algebra, and the formula

$$\|a\| = \lim_{s\to\infty} \|\alpha_{s+r,r}(a)\|_{A_{s+r}},$$

where $a \in A_r$ and where $\alpha_{s+r,r}$ is the composition of the morphisms in the diagram from A_r to A_{s+r}, defines a submultiplicative seminorm. Forming the quotient by the

ideal of elements with seminorm 0, and then completing, we obtain the *Banach algebra direct limit*, which we shall write as $\varinjlim A_r$.

There are canonical morphisms from each A_r into the direct limit. We shall make frequent use of the following well-known continuity property of K-theory, which may be proved using a variation of the Karoubi density theorem.

Theorem 3.3 *The induced morphism*

$$\varinjlim K_*(A_r) \to K_*(\varinjlim A_r)$$

is an isomorphism of abelian groups.

A modest first application is the reduction of Novodvorskii's theorem to the case of (topologically) finitely generated Banach algebras.

Corollary 3.4 *If the Gelfand transform induces an isomorphism in K-theory for every unital and finitely generated commutative Banach algebra, then it induces an isomorphism in K-theory for every separable* (*in the sense of general topology*) *unital commutative Banach algebra.*

Proof A general unital commutative Banach algebra is the direct limit of its finitely generated subalgebras, and the Gelfand spectrum is the inverse limit of the Gelfand spectra of those subalgebras. Since the canonical morphism

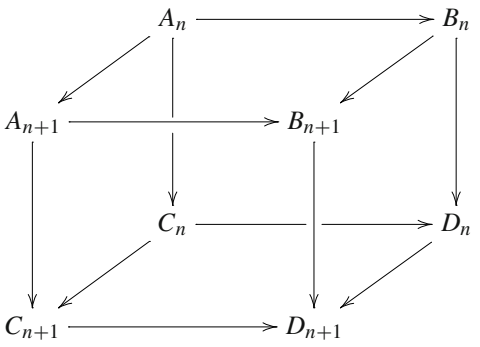
$$\varinjlim C(X_\alpha) \to C(\varprojlim X_\alpha)$$

is an isomorphism of Banach algebras, the continuity property of K-theory completes the proof. □

The obvious modification of the results of this section to handle general directed systems reduces Novodvorskii's theorem for not-necessarily-separable Banach algebras to the finitely generated case, too.

Finally, we shall take advantage of the compatibility of the Mayer–Vietoris property with direct limits.

Lemma 3.5 *Suppose given a sequence of commuting cubes*



with the morphisms from the back to front faces of norm 1 *or less, giving rise to a commuting square*

$$\begin{array}{ccc} \varinjlim A_n & \longrightarrow & \varinjlim B_n \\ \downarrow & & \downarrow \\ \varinjlim C_n & \longrightarrow & \varinjlim D_n. \end{array}$$

If all the back faces of the cubes have the Mayer–Vietoris property, then so does the direct limit square.

4 The Banach algebra of a polynomially convex compact set

Definition 4.1 A compact subset X of $\mathbb{C}^n$ is *polynomially convex* if it includes all points $w \in \mathbb{C}^n$ with the property that

$$|p(w)| \leqslant \max\{|p(z)| : z \in X\}$$

for every complex (holomorphic) polynomial function p on $\mathbb{C}^n$.

See for example Hormander's monograph [15]. Every compact convex set is polynomially convex. The polynomially convex compact subsets of $\mathbb{C}$ are the compact subsets with connected complements, but in higher dimensions there is a much richer set of possibilities. For instance every compact subset of $\mathbb{R}^n$ is polynomially convex in $\mathbb{C}^n$.

Definition 4.2 Let X be a compact subset of $\mathbb{C}^n$. We shall denote by $B(X)$ the closure in the supremum norm on X of the algebra of complex polynomial functions on X.

Obviously $B(X)$ is a commutative Banach algebra under pointwise multiplication. Each point $z \in X$ determines a multiplicative linear functional $\varepsilon_z \colon B(X) \to \mathbb{C}$ by evaluation at z, and it is easily checked that if X is polynomially convex, then these are the *only* multiplicative linear functionals. Hence:

Lemma 4.3 *Let X be a polynomially convex compact subset of $\mathbb{C}^n$. The map $z \mapsto \varepsilon_z$ is a homeomorphism from X onto the Gelfand spectrum of $B(X)$.*

Remark 4.4 In general, the Gelfand spectrum of $B(X)$ is the polynomial convex hull of X—the smallest polynomially convex set containing X.

Thanks to Lemma 4.3, if X is polynomially convex, then the Gelfand transform for $B(X)$ may be identified with the inclusion of $B(X)$ into the algebra $C(X)$ of all continuous complex-valued functions on X, and hence the Novodvorskii theorem for $B(X)$ states the following:

Theorem 4.5 *Let X be a polynomially convex compact subset of $\mathbb{C}^n$. The inclusion of $B(X)$ into $C(X)$ induces an isomorphism in K-theory.*

We shall prove this result in the next two sections. Actually the full Novodvorskii theorem follows easily from Theorem 4.5:

Theorem 4.6 *If the Gelfand transform for $B(X)$ induces an isomorphism in K-theory for every compact and polynomially convex subset $X \subseteq \mathbb{C}^n$ and every n, then the Gelfand transform induces an isomorphism in K-theory for every commutative unital Banach algebra.*

Proof If A is finitely generated by $a_1, \ldots, a_n$, so that the polynomials in these elements are dense in A, then the morphism $\varphi \mapsto (\varphi(a_1), \ldots, \varphi(a_n))$ is a homeomorphism from the Gelfand spectrum of A onto a polynomially convex compact subset X of $\mathbb{C}^n$.

The Gelfand transform maps A to a dense subalgebra of $B(X)$, and it follows from Theorem 3.1 that the induced map on K-theory is an isomorphism. So assuming that the Gelfand transform induces an isomorphism in K-theory for $B(X)$, it does so for A as well. The result now follows from Theorem 3.4. □

Remark 4.7 Novodvorskii actually formulated his theorem in [18] only for semi-simple algebras—those for which the Gelfand transform is injective. Theorem 4.6 shows that the more general result that we stated in Sect. 1 easily reduces to this case.

5 The Mayer–Vietoris property for polynomially convex compact sets

The purpose of this section is to prove the following result:

Theorem 5.1 *Let $X \subseteq \mathbb{C}^n$ be a polynomially convex compact set. Let α be an $\mathbb{R}$-linear functional on $\mathbb{C}^n$ and let $c \in \mathbb{R}$. If*

$$X' = \{z \in X : \alpha(z) \leqslant c\} \quad \text{and} \quad X'' = \{z \in X : \alpha(z) \geqslant c\},$$

then the commuting square

$$\begin{array}{ccc} B(X) & \longrightarrow & B(X') \\ \downarrow & & \downarrow \\ B(X'') & \longrightarrow & B(X' \cap X'') \end{array}$$

of restriction morphisms has the Mayer–Vietoris property.

We shall prove the theorem through a sequence of reductions and approximations, most of them involving the following concept:

Definition 5.2 A compact set $X \subseteq \mathbb{C}^n$ is a *polynomial polyhedron* if there are complex polynomial functions $p_1, \ldots, p_k$ on $\mathbb{C}^n$ such that

$$X = \{z \in \mathbb{C}^n : |p_1(z)|, \ldots, |p_k(z)| \leqslant 1\}.$$

Lemma 5.3 (See for example [15, Lemma 2.7.4]) *Let $X \subseteq \mathbb{C}^n$ be a polynomially convex compact set and let $U \subseteq \mathbb{C}^n$ be an open set containing X. There is a compact polynomial polyhedron L such that $X \subseteq L \subseteq U$.*

Corollary 5.4 *Every polynomially convex compact set is the intersection of a nested sequence of compact polynomial polyhedra.*

It follows from the corollary and from Lemma 3.5 that the special case of Theorem 5.1 in which X is a compact polynomial polyhedron implies the general case. So *from now on we shall assume that X is a compact polynomial polyhedron, as in Definition 5.2.*

Choose a positive number R so that

$$(z_1, \ldots, z_n) \in X \implies |z_1|, \ldots, |z_n| \leqslant R$$

and form the compact polydisk

$$W = \big\{(z, w) \in \mathbb{C}^{n+k} : |z_1|, \ldots, |z_n| \leqslant R \text{ and } |w_1|, \ldots, |w_k| \leqslant 1\big\}.$$

Let

$$Z = \big\{(z, w) \in W : p_1(z) = w_1, \ldots, p_k(z) = w_k\big\}$$

and define a polynomial map $\mu\colon \mathbb{C}^n \to \mathbb{C}^{n+k}$ by

$$\mu(z) = (z, p_1(z), \ldots, p_k(z)).$$

Observe that μ restricts to a homeomorphism from X to Z. The inverse of this homeomorphism is given by the coordinate projection from $\mathbb{C}^{n+k}$ to $\mathbb{C}^n$. Since μ and this inverse are both polynomial maps, it is evident that composition with μ gives an isomorphism of Banach algebras

$$\mu^*\colon B(Z) \xrightarrow{\cong} B(X).$$

As we shall see, the advantage of doing so is that Z is defined, as a subset of the polydisk W, by polynomial equations, namely the equations $q_j(z, w) = 0$, where

$$q_j(z, w) = p_j(z) - w_j, \quad j = 1, \ldots, k.$$

We shall need sets W_m, X_m and Z_m, $m = 1, 2, \ldots$, that approximate W, X and Z from the outside. For $m \geqslant 1$ we define

$$W_m = \big\{(z, w) \in \mathbb{C}^{n+k} : |z_1|, \ldots, |z_n| \leqslant R + 1/m \text{ and } |w_1|, \ldots, |w_k| \leqslant 1 + 1/m\big\}$$

and then define

$$Z_m = \big\{(z, w) \in W_m : q_1(z, w) = \cdots = q_k(z, w) = 0\big\}$$

and

$$X_m = \{z \in \mathbb{C}^n : \mu(z) \in Z_m\}.$$

Each X_{m+1} lies in the interior of X_m, and X is the intersection of all X_m. Similarly for W_m and Z_m. We also define

$$X'_m = \{z \in X_m : \alpha(z) \leqslant c + 1/m\}$$

and

$$X''_m = \{z \in X_m : \alpha(z) \geqslant c - 1/m\},$$

and similarly for W'_m, W''_m, Z'_m and Z''_m, using the same bounds on $\alpha(z)$. Once more we obtain nested decreasing sequences of compact sets, with each set contained in the interior of its predecessor.

The following result is relevant to the Mayer–Vietoris property in view of Theorem 2.7.

Proposition 5.5 *For every m,*

$$\begin{aligned}\operatorname{Image}\big[B(W'_m) \to, B(W'_m \cap W''_m)\big] \\ + \operatorname{Image}\big[B(W''_m) \to B(W'_m \cap W''_m)\big] = B(W'_m \cap W''_m),\end{aligned}$$

where the morphisms are restrictions.

In the proof we shall use the fact that $B(W'_m \cap W''_m)$ includes a dense family of functions that extend holomorphically to a neighborhood of $W'_m \cap W''_m$ (namely the polynomial functions, for instance), and that every function that is holomorphic in a neighborhood of W'_m or W''_m restricts to a function in $B(W'_m)$ or $B(W''_m)$, respectively. The latter is a general fact about polynomially compact convex sets (the Oka–Weil theorem) but it may be proved directly in the present simple case.

Proof It suffices to prove the following approximation result. There is a constant $C > 0$ such that if f is holomorphic in a neighborhood of $W'_m \cap W''_m$, then there are functions $f' \in B(W'_m)$ and $f'' \in B(W''_m)$ such that

$$f|_{W'_m \cap W''_m} = f'|_{W'_m \cap W''_m} + f''|_{W'_m \cap W''_m} \tag{5.1}$$

and

$$\|f'\|_{W'_m}, \|f''\|_{W''_m} \leqslant C\|f\|_{W'_m \cap W''_m}. \tag{5.2}$$

Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\varphi \equiv 0$ in a neighborhood of $(-\infty, c - 1/m]$ and $\varphi \equiv 1$ in a neighborhood of $[c + 1/m, \infty)$. Then define $\varphi : \mathbb{C}^{n+k} \to \mathbb{R}$ by $\varphi(z) = \sigma(\alpha(z))$.

Choose linear coordinates $v_1, \ldots, v_{n+k}$ on $\mathbb{C}^{n+k}$ with $\alpha(z) = \mathrm{Re}(v_1)$, and given f as above, let V be a compact polydisk that includes W_m within its interior, that is small enough that f is defined in a neighborhood of $V' \cap V'' = V \cap \{c - 1/m \leqslant \alpha(z) \leqslant c + 1/m\}$, and that is small enough that

$$\left\| \frac{\partial \varphi}{\partial \bar{v}_1} \cdot f \right\|_V \leqslant 2 \left\| \frac{\partial \varphi}{\partial \bar{v}_1} \cdot f \right\|_{W_m}. \tag{5.3}$$

Note that even though f is only defined near $V' \cap V''$, its product with $\partial\varphi/\partial\bar{v}_1$ extends by zero to a smooth function in a neighborhood of V. Next, let D be the projection of V onto the first complex coordinate (it is a compact disk in $\mathbb{C}$). Then define

$$g(v_1, \ldots, v_{n+k}) = \frac{1}{\pi} \int_D \left(\frac{\partial \varphi}{\partial \bar{v}_1} \cdot f \right) (v, v_2, \ldots, v_{n+k})(v - v_1)^{-1} \, d\lambda(v),$$

where the integration is with respect to the Lebesgue measure on the disk. The function g is defined and smooth on the interior of V, it is holomorphic there in the variables $v_2, \ldots, v_{n+k}$, and

$$\frac{\partial g}{\partial \bar{v}_1} = \frac{\partial \varphi}{\partial \bar{v}_1} \cdot f$$

(see for example [15, Theorem 1.2.2]). Moreover since $(v - v_1)^{-1}$ is uniformly integrable as a function of $v \in D$ as v_1 ranges over D,

$$\|g\|_{W_m} \leqslant \text{constant}_1 \cdot \left\| \frac{\partial \varphi}{\partial \bar{v}_1} \cdot f \right\|_V \leqslant \text{constant}_2 \cdot \left\| \frac{\partial \varphi}{\partial \bar{v}_1} \right\|_V \cdot \|f\|_{W'_m \cap W''_m},$$

where the second inequality uses (5.3) and both constants are independent of f.

Now set

$$f' = \varphi f - g \quad \text{and} \quad f'' = (1 - \varphi) f + g.$$

These are holomorphic in neighborhoods of W'_m and W''_m respectively (here the functions φf and $(1 - \varphi) f$ are extended by zero to W'_m and W''_m, respectively) and they satisfy (5.1) and (5.2), so the proof is complete. □

Unfortunately it is not easy to prove the counterpart of Proposition 5.5 for the spaces X_m or Z_m. Instead we shall replace the Banach algebras $B(X_m)$, etc, with others that are easier to handle.

Definition 5.6 Let V be a polynomially convex compact subset of $\mathbb{C}^{n+k}$, and let Y be the subset of V on which all the polynomials $q_1, \ldots q_k$ vanish. We shall write

$$I(V, Y) = \{f \in B(V) : f|_Y = 0\}$$

and

$$A(V, Y) = B(V)/I(V, Y).$$

The ideal $I(V, Y)$ in $B(V)$ is the kernel of the obvious restriction morphism $\mathrm{res}\colon B(V) \to B(Y)$.

Lemma 5.7 *The associated morphism of Banach algebras*

$$\overline{\mathrm{res}}\colon A(V, Y) \to B(Y)$$

induces an isomorphism in K-theory.

Proof The Gelfand spectrum of $A(V, Y)$ is mapped to the Gelfand spectrum of $B(V)$ by composition with the quotient morphism

$$B(V) \to A(V, Y).$$

This map of spectra is injective and continuous, and so it is a homeomorphism onto its image.

Now the image of the spectrum of $A(V, Y)$ in the spectrum of $B(V)$ is precisely the set of multiplicative linear functionals on $B(V)$ that vanish on $I(V, Y)$. The evaluation functionals $\varepsilon_y\colon B(V) \to \mathbb{C}$ associated to points $y \in Y$ obviously vanish on $I(V, Y)$. On the other hand, the evaluations at points $v \in V \setminus Y$ *do not* vanish on $I(V, Y)$, since for every such, at least one of the polynomials q_j vanishes on Y but not on v. It follows that the image in V of the Gelfand spectrum of $A(V, Y)$ is Y.

The morphism $\overline{\mathrm{res}}\colon A(V, Y) \to B(Y)$ is therefore spectrum-preserving. Since it also has dense range (consider polynomials), Theorem 3.1 applies. □

This leads us to analyze the commuting the square

$$\begin{array}{ccc} A(W_m, Z_m) & \longrightarrow & A(W'_m, Z'_m) \\ \downarrow & & \downarrow \\ A(W''_m, Z''_m) & \longrightarrow & A(W'_m \cap W''_m, Z'_m \cap Z''_m). \end{array}$$

Since the notation used above is a bit cumbersome, we shall simplify it and write the square as

$$\begin{array}{ccc} A(Z_m) & \longrightarrow & A(Z'_m) \\ \downarrow & & \downarrow \\ A(Z''_m) & \longrightarrow & A(Z'_m \cap Z''_m), \end{array}$$

which should not cause any confusion.

Lemma 5.8

$$\begin{aligned}\text{Image}\big[A(Z'_m) \to A(Z'_m \cap Z''_m)\big] \\ + \text{Image}\big[A(Z''_m) \to A(Z'_m \cap Z''_m)\big] = A(Z'_m \cap Z''_m).\end{aligned}$$

Proof This is an immediate consequence of Proposition 5.5, since the A-algebras are quotients of their B-algebra counterparts. □

Now form the Banach algebra

$$A(Z'_m, Z''_m) = \big\{(f', f'') \in A(Z'_m) \times A(Z''_m) : f'|_{Z'_m \cap Z''_m} = f''_{Z'_m \cap Z''_m}\big\}.$$

Proposition 5.9 *The commuting square*

$$\begin{array}{ccc} A(Z'_m, Z''_m) & \longrightarrow & A(Z'_m) \\ \downarrow & & \downarrow \\ A(Z''_m) & \longrightarrow & A(Z'_m \cap Z''_m) \end{array}$$

has the Mayer–Vietoris property.

Proof By construction, this is a pullback square. So the result follows from Lemma 5.8 and Theorem 2.7. □

To move from this to a proof of Theorem 5.1, we shall need to address the difference between $A(Z'_m, Z''_m)$ and $A(Z_m)$, and for this purpose we shall use the following fact. The proof will be evident to those familiar with coherent sheaves; we shall give an alternative proof from [1, Lemma 2] in an appendix.

Lemma 5.10 *Let Y be the interior of a compact polynomial polyhedron in $\mathbb{C}^n$ and let*

$$V = \big\{(z, w) \in Y \times \mathbb{C}^k : |w_1|, \ldots, |w_k| \leqslant 1\big\}$$

Let $p_1, \ldots, p_k$ be polynomial functions on $\mathbb{C}^n$, and define polynomial functions $q_1, \ldots, q_k$ on $\mathbb{C}^{n+k}$ by $q_j(z, w) = p_j(z) - w_j$. If f is a holomorphic function defined in a neighborhood of V, and if f vanishes on the common zero set of $q_1, \ldots, q_k$ in that neighborhood, then there are holomorphic functions $h_1, \ldots, h_k$ defined near V such that

$$f = q_1 h_1 + \cdots + q_k h_k.$$

There is a natural morphism from $A(Z_m)$ to $A(Z'_m, Z''_m)$, induced from restriction of functions on W_m to W'_m and W''_m. The following proposition supplies a near-inverse.

Lemma 5.11 *For each m there is a Banach algebra morphism*

$$\varphi_m \colon A(Z'_m, Z''_m) \to A(Z_{m+1})$$

for which the diagram

$$\begin{array}{ccc} A(Z'_m, Z''_m) & \longleftarrow & A(Z_m) \\ \downarrow & \searrow^{\varphi_m} & \downarrow \\ A(Z'_{m+1}, Z''_{m+1}) & \longleftarrow & A(Z_{m+1}) \end{array}$$

is commutative (*the unlabeled morphisms are all induced from restriction of functions*).

Proof Let $([f'], [f''])$ be an element of $A(Z'_m, Z''_m)$, where the square brackets denote equivalence classes of functions on W'_m and W''_m. It follows from the definitions that the difference $f' - f''$, which is defined on $W'_m \cap W''_m$, vanishes on the mutual zero-set of the polynomials $q_1, \ldots, q_k$.

Let us apply Lemma 5.10. By fitting a suitable compact polynomial polyhedron into the interior of $W'_m \cap W''_m$ whose interior, in turn, includes $W'_{m+1} \cap W''_{m+1}$, we find that there are holomorphic functions $h_1, \ldots, h_k$ defined on a neighborhood of $W'_{m+1} \cap W''_{m+1}$ such that

$$f' - f'' = q_1 h_1 + \cdots + q_k h_k$$

on this neighborhood. In particular, the identity holds on $W'_{m+1} \cap W''_{m+1}$.

By applying Proposition 5.5, to a polydisk that is slightly larger than W_{m+1}, we find that for each j there are functions $h'_j \in B(W'_{m+1})$ and $h''_j \in B(W''_{m+1})$, holomorphic in neighborhoods of W'_{m+1} and W''_{m+1}, respectively, such that

$$h_j = h'_j - h''_j$$

in a neighborhood of $W'_{m+1} \cap W''_{m+1}$.

Now define functions $g' \in B(W'_{m+1})$ and $g'' \in B(W''_{m+1})$ by

$$g' = f' - (q_1 h'_1 + \cdots + q_k h'_k) \quad \text{and} \quad g'' = f'' - (q_1 h''_1 + \cdots + q_k h''_k).$$

These are actually defined and holomorphic in neighborhoods of W'_{m+1} and W''_{m+1}, respectively, and they agree on a neighborhood of the intersection. So they determine a function g that is holomorphic in a neighborhood of W_{m+1}, and hence a function g in $B(W_{m+1})$.

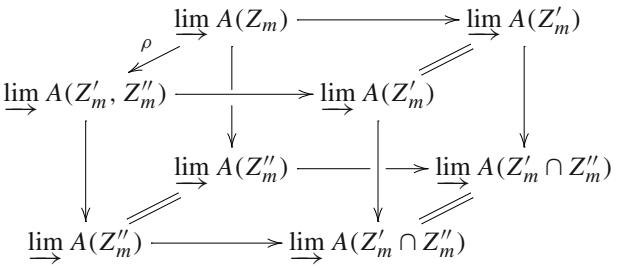
The associated element $[g] \in A(Z_{m+1})$ is characterized by the fact that g is equal to f' on Z'_{m+1}, and is equal to the f'' on Z''_{m+1}. This is because the morphism from $A(Z_{m+1})$ into $B(Z_{m+1})$ is injective. So $[g]$ depends only on $([f'], [f''])$.

To complete the proof, the formula

$$\varphi_m : ([f'], [f'']) \longmapsto [g]$$

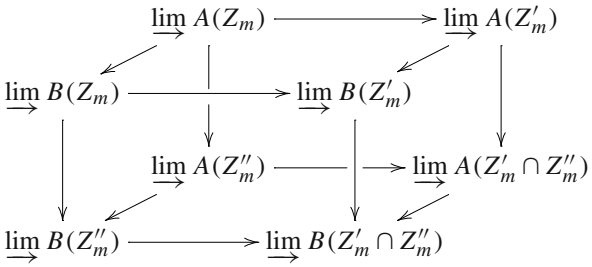
defines an algebra homomorphism from $A(Z'_m, Z''_m)$ to $A(Z_{m+1})$ that fits into the commuting diagram in the statement of the proposition. It follows from the closed graph theorem that φ_m is in addition continuous, and hence is a Banach algebra morphism, as required. □

Proof of Theorem 5.1 Form the commuting cube



with all morphisms identities or induced from restrictions. It follows from Theorem 3.3 and Lemma 5.11 that the morphism labelled ρ induces an isomorphism in K-theory. In addition, it follows from Lemma 3.5 and Proposition 5.9 that the front face has the Mayer–Vietoris property. Therefore it follows from Lemma 2.4 that the back face does too.

Now consider a second commuting cube:



It follows from Theorem 3.3 and Lemma 5.7 that the morphisms from the back face to the front induce isomorphisms in K-theory. So the front face has the Mayer–Vietoris property. But the front face is isomorphic to the commuting square in Theorem 5.1.□

6 Proof of the Novodvorskii theorem for $B(X)$

In this section we shall prove Theorem 4.5, that if X is a compact and polynomially convex subset of $\mathbb{C}^n$, then the inclusion of $B(X)$ into $C(X)$ induces an isomorphism in

K-theory. As we already observed, this leads to a complete proof of the Novodvorskii theorem.

Definition 6.1 If X is a polynomially convex compact subset of $\mathbb{C}^n$, then we shall denote by $D(X)$ the mapping cone from Definition 2.1 for the inclusion of $B(X)$ into $C(X)$.

We shall prove Theorem 4.5 by showing that $K_*(D(X)) = 0$.

Lemma 6.2 *Let $X \subseteq \mathbb{C}^n$ be a polynomially convex compact set. Let α be an $\mathbb{R}$-linear functional on $\mathbb{C}^n$ and let $c \in \mathbb{R}$. If*

$$X' = \{z \in X : \alpha(z) \leqslant c\} \quad \text{and} \quad X'' = \{z \in X : \alpha(z) \geqslant c\},$$

then the commuting square

$$\begin{array}{ccc} D(X) & \longrightarrow & D(X') \\ \downarrow & & \downarrow \\ D(X'') & \longrightarrow & D(X' \cap X'') \end{array}$$

of restriction morphisms has the Mayer–Vietoris property.

Proof The same diagram with either $B(X)$-type algebras or $C(X)$-type algebras has the Mayer–Vietoris property. So the lemma follows immediately from Lemma 2.5. □

Corollary 6.3 *Let $X = X' \cup X''$, as above. If $K_*(D(X' \cap X'')) = 0$, then the morphism*

$$K_*(D(X)) \to K_*(D(X')) \oplus K_*(D(X''))$$

induced from restriction is injective.

Lemma 6.4 *If $\{X_r : r = 1, 2, \dots\}$ is a decreasing sequence of polynomially convex compact subsets of $\mathbb{C}^n$, then the morphism*

$$\varinjlim K_*(D(X_r)) \to K_*\big(D\big(\textstyle\bigcap_r X_r\big)\big)$$

induced from restriction to the intersection is an isomorphism.

Proof This follows from the mapping cone six-term exact sequence, the same property for the $B(X)$-algebras and the $C(X)$-algebras, and the five lemma. □

Theorem 6.5 *If X is any compact and polynomially convex subset of $\mathbb{C}^n$, then $K_*(D(X)) = 0$.*

Proof First, the case when X is a point is trivial. Starting from this, we prove the theorem by induction, as follows. Suppose that $K_*(D(X)) = 0$ for all compact polynomially convex sets $X \subseteq \mathbb{C}^n$ that may be included within an $\mathbb{R}$-affine subspace of $\mathbb{C}^n$

of dimension at most $k-1$, where $k > 0$. Now suppose that X may be included in an $\mathbb{R}$-affine subspace $S \subseteq \mathbb{C}^n$ of dimension k, and suppose for the sake of a contradiction that $K_*(D(X)) \neq 0$.

Choose an $\mathbb{R}$-linear functional α on $\mathbb{C}^n$ that is non-constant on S, let a and b be the minimal and maximal values of α on X, and let $c = (a+b)/2$. Form the corresponding decomposition $X = X' \cup X''$ as in the statement of Lemma 6.2 above.

Let $x \in K_*(D(X))$ be a nonzero element. By Corollary 6.3, we may choose one of X' or X'', call it X_1, so that the image of x in $K_*(D(X_1))$ is nonzero.

Note that the difference between the maximum and minimum values of α on X_1 is $(b-a)/2$. By repeating the argument starting from X_1, we obtain $X_2 \subseteq X_1$ such that the image of x in $K_*(D(X_2))$ is nonzero, and such that the difference between the maximum and minimum values of α on X_2 is $(b-a)/4$.

Continuing, there is a decreasing sequence $\{X_r\}$ of compact polynomially convex subsets of X such that the image of x in $K_*(D(X_r))$ is nonzero and such that the difference between the maximum and minimum values of α on X_r is $(b-a)/2^r$.

But now let Y be the intersection of the X_r. Then α is constant on Y, so that Y may be included in an $\mathbb{R}$-affine subspace of dimension $k-1$, while by Lemma 6.4 the image of x in $K_*(D(Y))$ is nonzero. A contradiction. □

7 Appendix: A proof of Lemma 5.10

We start from the following result of Oka. The proof may be found in many texts; see for instance [15, Section 2.7].

Theorem 7.1 *Let p be a polynomial function on $\mathbb{C}^n$, and define*

$$\mu : \mathbb{C}^n \to \mathbb{C}^{n+1}$$

by $\mu(z) = (z, p(z))$. Let Y be a compact polynomial polyhedron in $\mathbb{C}^n$, let

$$V = \{(z, w) \in Y \times \mathbb{C} : |w| \leqslant 1\},$$

and let

$$X = \{z \in Y : |p(z)| \leqslant 1\}.$$

Each function that is holomorphic in a neighborhood of X may be written in the form

$$f(z) = h(\mu(z))$$

near X, where h is a function that is holomorphic in a neighborhood of V.

Proof of Lemma 5.10 Suppose first that $k = 1$. The function $q = q_1$ has nowhere vanishing gradient. So in a neighborhood of any point it may be chosen as the first coordinate in a local coordinate system. In that neighborhood there certainly exists a

holomorphic solution to the equation $f = qh$. These local solutions patch together to define a solution throughout a neighborhood of V.

Now suppose that the $k-1$ case of the lemma has been proved (for all Y). Let f be a holomorphic function near V that vanishes on the common zero set of $q_1, \ldots, q_k$, as in the statement of the lemma. Define

$$V' = \big\{(z, w) \in Y' \times \mathbb{C}^{k-1} : |w_1|, \ldots, |w_{k-1}| \leqslant 1\big\},$$

where

$$Y' = \{z \in Y : |p_k(z)| \leqslant 1\},$$

and define f' in a neighborhood of $V' \subseteq \mathbb{C}^{n+k-1}$ by

$$f'(z, w_1, \ldots, w_{k-1}) = f(z, w_1, \ldots, w_{k-1}, p_k(z)).$$

It vanishes on the common zero set of the polynomials

$$q_j'(z, w_1, \ldots, w_{k-1}) = p_j(z) - w_j, \quad j = 1, \ldots, k-1.$$

So by the $k-1$ case of the lemma it may be written as

$$f' = q_1' h_1' + \cdots + q_{k-1}' h_{k-1}',$$

for some $h_1', \ldots h_{k-1}'$ that are holomorphic near V'. By Oka's theorem there are $h_1, \ldots, h_{k-1}$ holomorphic near V such that

$$h_j(z, w_1, \ldots, w_{k-1}, p_k(z)) = h_j'(z, w_1, \ldots, w_{k-1})$$

for $j = 1, \ldots, k-1$ and for $(z, w_1, \ldots, w_{k-1})$ near V'. But then the function

$$f - (q_1 h_1 + \cdots + q_{k-1} h_{k-1})$$

is holomorphic in a neighborhood of V and vanishes on the zero set of q_k. So by the $k = 1$ case of the lemma, there is a function h_k that is holomorphic in a neighborhood of V such that

$$f - (q_1 h_1 + \cdots + q_{k-1} h_{k-1}) = q_k h_k,$$

near V, as required. □

Acknowledgements It is a pleasure to thank Jonathan Block and Jean-Benoît Bost for valuable discussions on the topic of this paper.

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