



Rigidity results for von Neumann algebras arising from mixing extensions of profinite actions of groups on probability spaces

Ionut Chifan¹ · Sayan Das¹

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Abstract

Motivated by Popa’s seminal work Popa (Invent Math 165:409–45, 2006), in this paper, we provide a fairly large class of examples of group actions $\Gamma \curvearrowright X$ satisfying the extended Neshveyev–Størmer rigidity phenomenon Neshveyev and Størmer (J Funct Anal 195(2):239–261, 2002): whenever $\Lambda \curvearrowright Y$ is a free ergodic pmp action and there is a $*$ -isomorphism $\Theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$ such that $\Theta(L(\Gamma)) = L(\Lambda)$ then the actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are conjugate (in a way compatible with Θ). We also obtain a complete description of the intermediate subalgebras of all (possibly non-free) compact extensions of group actions in the same spirit as the recent results of Suzuki (Complete descriptions of intermediate operator algebras by intermediate extensions of dynamical systems, To appear in Comm Math Phys. ArXiv Preprint: [arXiv:1805.02077](https://arxiv.org/abs/1805.02077), 2020). This yields new consequences to the study of rigidity for crossed product von Neumann algebras and to the classification of subfactors of finite Jones index.

1 Introduction

In the mid thirties Murray and von Neumann found a natural way to associate a von Neumann algebra to any measure preserving action $\Gamma \curvearrowright X$ of a countable group Γ on a probability space X . This is called the group measure space von Neumann algebra, denoted by $L^\infty(X) \rtimes \Gamma$. The most interesting case for study is when the initial action $\Gamma \curvearrowright X$ is free and ergodic, in which case the group measure space

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✉ Ionut Chifan
ionut-chifan@uiowa.edu

Sayan Das
sayan-das@uiowa.edu

¹ Department of Mathematics, The University of Iowa, 14 MacLean Hall, Iowa City, IA 52242, USA

construction is in fact a type II_1 factor. When X is a singleton the group measure space construction yields just the group von Neumann algebra that will be denoted by $L(\Gamma)$. The latter is a II_1 factor specifically when all nontrivial conjugacy classes of Γ are infinite (henceforth abbreviated as the icc property).

A problem of central importance in von Neumann algebras is to determine how much information about the action $\Gamma \curvearrowright X$ can be recovered from the isomorphism class of $L^\infty(X) \rtimes \Gamma$. An unprecedented progress in this direction emerged over the last decade from Popa's influential deformation/rigidity theory [68]. A remarkable achievement of this theory was the discovery of first classes of examples of actions that are entirely remembered by their von Neumann algebras; for some examples see [6, 11, 12, 14, 17, 18, 25, 29, 32, 35, 37–39, 58, 68, 69, 71–73, 78]. We refer the reader to the surveys [41, 77] for an overview of the recent developments.

There are two distinguished subalgebras of $L^\infty(X) \rtimes \Gamma$: the coefficient (or Cartan) subalgebra $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$ and the group von Neumann subalgebra $L(\Gamma) \subset L^\infty(X) \rtimes \Gamma$. The classification of group measure space von Neumann algebra is closely related to the study of these two inclusions of von Neumann algebras. For instance, in [74] Singer observed that the study of the inclusion $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$ amounts to the study of the equivalence relation induced by the orbits of $\Gamma \curvearrowright X$. Thus reconstructing the action $\Gamma \curvearrowright X$ from the inclusion $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$ relies upon the reconstruction from its orbits. This theme in contemporary ergodic theory is known as orbit equivalence rigidity. The study of orbit equivalence rigidity has received a lot of attention over the last couple of decades and has major consequences to the classification of von Neumann algebras in general, and the structure of the crossed product algebras in particular; for instance see [11, 27, 30, 32, 37, 49, 51].

Deriving information about the action $\Gamma \curvearrowright X$ from the other inclusion $L(\Gamma) \subset L^\infty(X) \rtimes \Gamma$ is another topic which is implicit in many core rigidity results in von Neumann algebras [52, 55, 66, 67]. When Γ is abelian $L^\infty(X) \rtimes \Gamma = \mathcal{R}$ is the hyperfinite II_1 factor and each of $L^\infty(X)$ and $L(\Gamma)$ is a maximal abelian subalgebra of \mathcal{R} (henceforth abbreviated as MASA). In their study on structural aspects of these MASAs in [52] Neshveyev and Størmer discovered that the positions of these two MASAs inside \mathcal{R} completely determines the action. More precisely, they showed the following: *Let Γ be an infinite abelian group, $\Gamma \curvearrowright X$ be a weak mixing action and $\Lambda \curvearrowright Y$ be any action. If there is a $*$ -isomorphism $\Theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$ satisfying $\Theta(L(\Gamma)) = L(\Lambda)$ and $\Theta(L^\infty(X))$ is inner conjugate to $L^\infty(Y)$ then $\Gamma \curvearrowright X$ is conjugate with $\Lambda \curvearrowright Y$ (in a way compatible to Θ).* They also conjectured the same statement holds without the inner conjugacy of the Cartan subalgebras condition. In other words the inclusion $L(\Gamma) \subset L^\infty(X) \rtimes \Gamma$ alone completely captures the entire crossed product structure of $L^\infty(X) \rtimes \Gamma$.

The first examples of actions satisfying the full statement of Neshveyev–Størmer conjecture emerged from the impressive work of Popa on the classification of von Neumann algebras associated with Bernoulli actions, [66, 67]. Specifically, using his influential deformation/rigidity theory Popa was able to show that this is the case for all clustering (e.g. Bernoulli) actions $\Gamma \curvearrowright X$ [67, Theorem 0.7]. Remarkably, this holds even when Γ is nonabelian. These significant initial advances strongly suggest that the Neshveyev–Størmer conjecture could hold in a much larger generality that supersedes the amenable regime (e.g. Γ is abelian). Motivated by this and the implicit

relevance to the study of rigidity aspects for crossed products it is natural to investigate the following extended version of the Neshveyev–Størmer rigidity question:

Question 1.1 (Extended Neshveyev–Størmer rigidity question). *Let Γ and Λ be icc countable discrete groups and let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be free, ergodic, pmp actions. Assume that there is a $*$ -isomorphism $\Theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$ such that $\Theta(L(\Gamma)) = L(\Lambda)$. Under what conditions on $\Gamma \curvearrowright X$ are the actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ conjugate?*

Besides Popa's examples at this time there are several other families of specific actions $\Gamma \curvearrowright X$ for which Question 1.1 has a solution. These arise mostly from decade-long developments in the classification of von Neumann algebras via Popa's deformation/rigidity program. For instance, this is the case for all W^* -superrigid actions (see [41] for a survey on W^* superrigidity and the references therein). Also, using [67, Theorem 5.2] one can easily see that the rigidity phenomenon in Question 1.1 is also satisfied by any weak mixing action $\Gamma \curvearrowright X$ for which, up to unitary conjugacy, $L^\infty(X)$ is the unique group measure space Cartan subalgebra of $L^\infty(X) \rtimes \Gamma$. This way one can get more examples using the recent results on uniqueness of Cartan subalgebras, see [11, 12, 39, 55, 72, 73] for example. However not much was known beyond these classes of examples and it remained open to find a more intrinsic approach to Question 1.1 which does not rely on uniqueness of Cartan subalgebras results from deformation/rigidity theory.

In this article we develop new technical aspects that enables us to partially answer Question 1.1. In particular we are able to describe a fairly large family of actions which covers many new examples beyond all the aforementioned classes, e.g. all nontrivial mixing extensions of free compact actions, satisfying the extended Neshveyev–Størmer rigidity phenomenon. More generally, we have the following result.

Theorem 1.2 *Let Γ be an icc group and let $\Gamma \curvearrowright^\sigma X$ be an action whose distal quotient $\Gamma \curvearrowright X_d$ is free and the extension $\pi : X \rightarrow X_d$ is (nontrivial) mixing. Let $\Lambda \curvearrowright^\alpha Y$ be any action. Assume that $\Theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$ is a $*$ -isomorphism such that $\Theta(L(\Gamma)) = L(\Lambda)$. Then there exist a unitary $x \in L(\Lambda)$, a character $\omega : \Gamma \rightarrow \mathbb{T}$, and a group isomorphism $\delta : \Gamma \rightarrow \Lambda$ such that $x\Theta(L^\infty(X))x^* = L^\infty(Y)$ and for all $a \in L^\infty(X)$, $\gamma \in \Gamma$ we have*

$$\Theta(au_\gamma) = \omega(\gamma)\Theta(a)x^*v_{\delta(\gamma)}x.$$

In particular, we have $x\Theta(\sigma_\gamma(a))x^ = \alpha_{\delta(\gamma)}(x\Theta(a)x^*)$ and hence $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are conjugate.*

Here $\{u_\gamma\}_{\gamma \in \Gamma}$ and $\{v_\lambda\}_{\lambda \in \Lambda}$ are the canonical group unitaries implementing the actions in $L^\infty(X) \rtimes \Gamma$ and $L^\infty(Y) \rtimes \Lambda$, respectively.

In particular the theorem implies that if Γ is any icc group then any action $\Gamma \curvearrowright X$ which admits a free profinite quotient $\Gamma \curvearrowright X_d$ with (nontrivial) mixing extension $\pi : X \rightarrow X_d$ satisfies the extended Neshveyev–Størmer rigidity question. As a concrete example let Γ be any icc residually finite group and let $\cdots \triangleleft \Gamma_n \triangleleft \cdots \triangleleft \Gamma_2 \triangleleft \Gamma_1 \triangleleft \Gamma$ be a resolution of finite index normal subgroups satisfying $\bigcap_n \Gamma_n = 1$. Consider the action

$\Gamma \curvearrowright (\Gamma/\Gamma_n, c_n)$ by left multiplication of Γ on the left cosets Γ/Γ_n seen as a finite probability space with the counting measure c_n and let $\Gamma \curvearrowright (Z, \mu) = \varprojlim (\Gamma/\Gamma_n, c_n)$ be the inverse limit of these actions. In addition let $\pi : \Gamma \curvearrowright \mathcal{O}(\mathcal{H})$ be any mixing orthogonal representation and let $\Gamma \curvearrowright (Y^\pi, \nu^\pi)$ be the corresponding Gaussian action. Then the diagonal action $\Gamma \curvearrowright (Y^\pi \times Z, \nu^\pi \times \mu)$ is profinite-by-(nontrivial) mixing, and hence by Theorem 1.2 the rigidity Question 1.1 has a positive solution in this case.

Theorem 1.2 is obtained by heavily exploiting, at the von Neumann algebraic level, the natural tension that occurs between mixing and compactness properties for actions. Briefly, let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be actions as in Theorem 1.2 so that $L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda$ with $L(\Gamma) = L(\Lambda)$. First we use the description of compactness via quasnormalizers from [15,36] to identify the von Neumann algebras of their distal parts, i.e. $L^\infty(X_d) \rtimes \Gamma = L^\infty(Y_d) \rtimes \Lambda$. In turn this is used to show that the mixing property of the extension $L^\infty(X_d) \subseteq L^\infty(X)$ is transferred through von Neumann equivalence to the extension $L^\infty(Y_d) \subseteq L^\infty(Y)$ (Theorem 2.10). Once these are established, some basic adaptations of Popa's intertwining techniques from [66] further show that the Cartan subalgebras $L^\infty(X)$ and $L^\infty(Y)$ are in fact unitarily conjugate. Then the desired result is derived from a general principle which states that for any free ergodic actions $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ of icc groups Γ and Λ , inner conjugacy of $L^\infty(X)$ and $L^\infty(Y)$ together with $L(\Gamma) = L(\Lambda)$ imply conjugacy of $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ (Theorem 4.5). This criterion for conjugacy of group actions generalizes the earlier works [52,67] and is obtained using the notion of height of elements with respect to groups from [42]. Specifically, using Dye's theorem and an averaging argument we show that Γ has large height with respect to Λ inside $L(\Lambda)$ (Theorem 4.4). By [42, Theorem 3.1] this further implies Γ is unitarily conjugate to Λ . Further exploiting the icc condition we deduce conjugacy of the actions (Theorem 4.5).

While Theorem 1.2 settles the extended Neshveyev-Størmer rigidity question for nontrivial extensions, two natural extreme situations, namely, when $\Gamma \curvearrowright X$ is either mixing or compact (even profinite) remain open. We believe that in both of these cases one should still get a positive answer and we formulate a few sub problems in this direction; see for instance Problem 4.12. However, in order to successfully tackle these questions, significant new technical advancements are needed. Specifically, if one pursues an approach similar to Theorem 1.2 the key step is to establish the inner conjugacy of $L^\infty(X)$ and $L^\infty(Y)$. In the presence of mixing this would follow if one can show there exist free factors $\Gamma \curvearrowright X_0$ of $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y_0$ of $\Lambda \curvearrowright Y$ whose von Neumann algebras coincide, i.e. $L^\infty(X_0) \rtimes \Gamma = L^\infty(Y_0) \rtimes \Lambda$; see Corollary 4.9. In turn this highlights the importance of studying intermediate subalgebras in the inclusion $L(\Gamma) \subset L^\infty(X) \rtimes \Gamma$. In addition this seems relevant even to the study of Question 1.1 for profinite actions.

Note that when Γ is icc, and $\Gamma \curvearrowright X$ is free, ergodic and pmp, the inclusion $L(\Gamma) \subset L^\infty(X) \rtimes \Gamma$ is an irreducible inclusion of II_1 factors. In his seminal paper [44] Jones pioneered the study of inclusions of type II_1 factors, or *subfactors*. Subfactor theory has had a number of striking applications over the years in various diverse branches of mathematics and mathematical physics, including Knot theory and Conformal Field theory, [45–47]. A major motivating question in Subfactor theory is the classification of all intermediate subalgebras. Pursuing this perspective, we were able to classify all the

intermediate subalgebras in compact extensions in the same spirit as Suzuki's recent results from [75]. To properly introduce our result we briefly recall some terminology. Given two actions $\Gamma \curvearrowright^\beta X_0$ and $\Gamma \curvearrowright^\alpha X$ we say that α is an extension of β if there is a Γ -equivariant factor map $\pi : X \rightarrow X_0$. At the von Neumann algebra level this induces an inclusion $L^\infty(X_0) \subseteq L^\infty(X)$ on which Γ acts naturally via $\alpha_\gamma(f) = f \circ \alpha_{\gamma^{-1}}$ when $f \in L^\infty(X)$. An *intermediate extension for π* (or *between $\Gamma \curvearrowright X_0$ and $\Gamma \curvearrowright X$*) is an action $\Gamma \curvearrowright Z$ for which there exist Γ -equivariant factor maps $\pi_1 : X \rightarrow Z$ and $\pi_2 : Z \rightarrow X_0$ such that $\pi_2 \circ \pi_1 = \pi$. Note that the intermediate extensions of π are in bijective correspondence with the Γ -invariant intermediate subalgebras of $L^\infty(X_0) \subseteq L^\infty(X)$. We show that there is a bijective correspondence between intermediate von Neumann algebras in crossed products and intermediate extensions of dynamical systems. More precisely, we have the following

Theorem 1.3 *Let Γ be an icc group and let $\Gamma \curvearrowright^\beta X_0$ be a pmp action. Let $\Gamma \curvearrowright X$ be an ergodic compact extension of β , [28]. Consider the corresponding group measure space von Neumann algebras and note that we have the following inclusion $L^\infty(X_0) \rtimes \Gamma \subseteq L^\infty(X) \rtimes \Gamma$. Then for any intermediate von Neumann subalgebra $L^\infty(X_0) \rtimes \Gamma \subseteq N \subseteq L^\infty(X) \rtimes \Gamma$ there exists an intermediate extension $\Gamma \curvearrowright Z$ between $\Gamma \curvearrowright X$ and $\Gamma \curvearrowright X_0$ satisfying $N = L^\infty(Z) \rtimes \Gamma$.*

In many respects this theorem complements the results from [75]; for instance, it covers various examples of non-free extensions, most notably, when X_0 is a singleton. In this situation our result provides a complete description of all intermediate von Neumann subalgebras in the inclusion $L(\Gamma) \subseteq L^\infty(X) \rtimes \Gamma$ for any compact ergodic action $\Gamma \curvearrowright X$ of any icc group Γ . This in turn yields new interesting consequences towards the classification of finite index subfactors. For example, combining Theorem 1.3 with the characterization of compactness via quasnormalizers from [36, Theorem 6.10], for any icc group Γ and any ergodic action $\Gamma \curvearrowright X$, we are able to classify all the intermediate subfactors $L(\Gamma) \subseteq N \subseteq L^\infty(X) \rtimes \Gamma$ with finite Jones index $[N : L(\Gamma)] < \infty$. Specifically we show that all such N could arise only from the transitive finite factors of $\Gamma \curvearrowright X$ (see part 2. in Corollary 1.4); in particular, this entails that the Jones index $[N : L(\Gamma)]$ is always a positive integer. This should be compared with the similar statement [62, Corollary 2.4] for the intermediate subfactors of the Cartan inclusion $L^\infty(X) \subset N \subseteq L^\infty(X) \rtimes \Gamma$ with $[L^\infty(X) \rtimes \Gamma : N] < \infty$.

Corollary 1.4 *Let Γ be an icc group and let $\Gamma \curvearrowright X$ be an ergodic pmp action. If $M = L^\infty(X) \rtimes \Gamma$ is the corresponding group measure space construction then the following hold:*

1. *For any intermediate von Neumann algebra $L(\Gamma) \subseteq N \subseteq L^\infty(X) \rtimes \Gamma$ satisfying $N \subseteq \mathcal{QN}_M(L(\Gamma))''$ there exists a factor $\Gamma \curvearrowright X_0$ of $\Gamma \curvearrowright X$ such that $N = L^\infty(X_0) \rtimes \Gamma$.*
2. *If $L(\Gamma) \subseteq N \subseteq L^\infty(X) \rtimes \Gamma$ is an intermediate subfactor with $[N : L(\Gamma)] < \infty$ then there is a finite, transitive factor $\Gamma \curvearrowright X_0$ of $\Gamma \curvearrowright X$ such that $N = L^\infty(X_0) \rtimes \Gamma$; in particular, $[N : L(\Gamma)] \in \mathbb{N}$. Thus for any subfactors $L(\Gamma) \subseteq N_1 \subseteq N_2 \subseteq L^\infty(X) \rtimes \Gamma$, with either $[N_1 : L(\Gamma)] < \infty$ or $\Gamma \curvearrowright X$ compact, we have $[N_2 : N_1] \in \mathbb{N} \cup \{\infty\}$.*

In particular, part 2. implies that for any icc group Γ with no proper finite index subgroups and any free ergodic action $\Gamma \curvearrowright X$ there are no nontrivial intermediate subfactors $L(\Gamma) \subseteq N \subseteq L^\infty(X) \rtimes \Gamma$ of finite index $[N : L(\Gamma)] < \infty$. For example this is the case for all Γ infinite simple groups, e.g. Tarski's monsters, Burger-Mozes groups [7], Camm's groups [8], or Bhattacharjee's groups [3], just to enumerate a few.

We point out in passing that Theorem 1.3 actually holds in a more general setting, namely, for actions of groups on compact extensions of possibly non-abelian von Neumann algebras; this notion is highlighted in Definition 3.9. In this generality our result yields a twisted version of Ge's splitting theorem for tensor products (see Corollary 3.13) in the same spirit as [75, Example 4.14].

The classification of the intermediate subalgebras in Theorem 1.3 is achieved through a new mix of analytic and algebraic techniques that combines factoriality arguments together with a general algebraic criterion outlined in Theorem 3.2. We also note the same criterion can be used in conjunction with various soft analytical arguments to successfully recover, in the finite von Neumann algebra case, several well-known results such as Ge's tensor splitting theorem [31, Theorem 3.1] or the Galois correspondence for group actions [21]. These applications are presented in Corollary 3.3 and Theorems 3.4 and 3.7.

Finally, Theorem 1.3 in combination with methods from Popa's deformation/rigidity theory and Jones' finite index subfactor theory provide new insight towards rigidity aspects for II_1 factors arising from profinite actions $\Gamma \curvearrowright X$ of icc property (T) groups Γ . While Ioana has already established in [37] that such actions are completely reconstructible from their orbits, significantly less is known about their rigid behavior at the von Neumann algebraic level. When Γ is in addition properly proximal, Boutonnet, Ioana and Peterson showed in [5] using boundary techniques [4] that all compact Cartan subalgebras in $L^\infty(X) \rtimes \Gamma$ are unitarily conjugate to $L^\infty(X)$. (For Γ direct products of nonamenable biexact groups this already follows from the earlier works [17,18].) Consequently, this combined with [37] yields that for any non-commensurable groups Γ and Λ and any free ergodic profinite actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ the von Neumann algebras $L^\infty(X) \rtimes \Gamma$ and $L^\infty(Y) \rtimes \Lambda$ are not isomorphic; remarkably, this is the case for lattices $\Gamma = \text{PSL}_n(\mathbb{Z})$ and $\Lambda = \text{PSL}_m(\mathbb{Z})$ for all $n \neq m$. However, without these additional assumption on Γ , the study of von Neumann algebraic rigidity aspects for profinite (or compact) actions $\Gamma \curvearrowright X$ remains an wide open problem. For example, even establishing strong rigidity results similar to the ones obtained in [67] by Popa for Bernoulli actions of rigid groups seems elusive at this time. While it is very plausible that such results should hold true, we only have the following partial result at this time in this direction.

Theorem 1.5 *Let Γ and Λ be icc property (T) groups. Let $\Gamma \curvearrowright X = \varprojlim X_n$ be a free ergodic profinite action and let $\Lambda \curvearrowright Y$ be a free ergodic compact action. Assume that $\Theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$ is a $*$ -isomorphism. Then $\Lambda \curvearrowright Y = \varprojlim Y_n$ is also a profinite action. Moreover, there exist $l \in \mathbb{N}$ and a unitary $w \in L^\infty(Y) \rtimes \Lambda$ such that $\Theta(L^\infty(X_{k+l}) \rtimes \Gamma) = w(L^\infty(Y_{k+1}) \rtimes \Lambda)w^*$ for every integer $k \geq 0$.*

This should be compared with Popa's work on inductive limits of II_1 factors [70]. Finally, the same strategy used in the proof of Theorem 1.2 can be successfully used in combination with Theorem 1.5 to provide a purely von Neumann algebraic approach

to a version of Ioana's orbit equivalence superrigidity theorem from [37]; see the proof of Theorem 5.3.

2 Some preliminaries and technical results

2.1 Popa's intertwining techniques

Over a decade ago, Popa introduced in [66, Theorem 2.1 and Corollary 2.3] a powerful analytic criterion for identifying intertwiners between arbitrary subalgebras of tracial von Neumann algebras. This is now termed *Popa's intertwining-by-bimodules technique*.

Theorem 2.1 [66] *Let (M, τ) be a separable tracial von Neumann algebra and let $P, Q \subseteq M$ be (not necessarily unital) von Neumann subalgebras. Then the following are equivalent:*

1. *There exist $p \in \mathcal{P}(P)$, $q \in \mathcal{P}(Q)$, a $*$ -homomorphism $\theta : pPp \rightarrow qQq$ and a partial isometry $0 \neq v \in qMp$ such that $\theta(x)v = vx$, for all $x \in pPp$.*
2. *For any group $\mathcal{G} \subset \mathcal{U}(P)$ such that $\mathcal{G}'' = P$ there is no sequence $(u_n)_n \subset \mathcal{G}$ satisfying $\|E_Q(xu_n y)\|_2 \rightarrow 0$, for all $x, y \in M$.*

If one of the two equivalent conditions from Theorem 2.1 holds then we say that *a corner of P embeds into Q inside M* , and write $P \prec_M Q$.

For further use, we record a result which states that whenever an irreducible subfactor $P \subseteq M$ intertwines into a II_1 subfactor $N \subseteq M$, then we can choose the image of the intertwining to be an irreducible subfactor itself inside a corner of N . Notice that our theorem is inspired by a similar result in the context of maximal abelian subalgebras [40, Lemma 1.5].

Theorem 2.2 *Let $P, N \subseteq M$ be II_1 factors such that $P' \cap M = \mathbb{C}$. If $P \prec_M N$, then one can find nonzero projections $p \in \mathcal{P}(P)$, $q_0 \in \mathcal{P}(N)$, a nonzero partial isometry $w_0 \in q_0Mp$, and a $*$ -isomorphism on its image $\psi_0 : pPp \rightarrow q_0Nq_0$ such that*

- i) $\psi_0(x)w_0 = w_0x$ for all $x \in pPp$, and
- ii) $\psi_0(pPp)' \cap q_0Nq_0 = \mathbb{C}q_0$.

Proof Since $P \prec_M N$, there exist nonzero projections $p \in \mathcal{P}(P)$, $q \in \mathcal{P}(N)$, a nonzero partial isometry $v \in qMp$, and a $*$ -isomorphism on its image $\psi : pPp \rightarrow qNq$ such that

$$\psi(x)v = vx \text{ for all } x \in pPp. \quad (2.1.1)$$

Since $P' \cap M = \mathbb{C}$, equation (2.1.1) implies that $v^*v = p$. Also, we get that $q' := vv^* \in \psi(pPp)' \cap qMq$. Using the Borel functional calculus, after replacing q with a nonzero subprojection if necessary, we may assume that

$$q = \text{support}(E_N(q')) \text{ and } c_0q \leq E_N(q') \leq c_1q \text{ for some scalars } c_1 \geq c_0 > 0. \quad (2.1.2)$$

Claim 2.3 *The von Neumann algebra $\psi(pPp)' \cap qNq$ is completely atomic.*

Proof of Claim 2.3. Notice that Eq. (2.1.1) yields $\psi(pPp)q' = vpPpv^*$. Since $P \subseteq M$ is irreducible, we have

$$q'(\psi(pPp)' \cap qMq)q' = \mathbb{C}q'. \quad (2.1.3)$$

By construction we have that

$$\psi(pPp)' \cap qNq \subseteq \psi(pPp)' \cap qMq. \quad (2.1.4)$$

Fix $0 \neq e \in \mathcal{P}(\psi(pPp)' \cap qNq)$. Using Eqs. (2.1.3)–(2.1.4), we have $q'eq' = \tau_{q'}(e)q'$, where $\tau_{q'}(e) = \frac{\tau(q'eq')}{\tau(q')}$. Therefore, using equation (2.1.2) in the one hand we get

$$\|eq'e\|_2 \geq \|E_N(eq'e)\|_2 = \|eE_N(q')e\|_2 \geq c_0\|e\|_2. \quad (2.1.5)$$

On the other hand, we have $\|eq'e\|_2^2 = \tau(eq'eq'e) = \tau_{q'}(e)\tau(eq'e) = \tau_{q'}^2(e)\tau(q')$. Hence,

$$\|eq'e\|_2 = \tau_{q'}(e)\|q'\|_2. \quad (2.1.6)$$

Combining (2.1.5), (2.1.6) and (2.1.2) we see that

$$\begin{aligned} c_0\|e\|_2 &\leq \tau_{q'}(e)\|q'\|_2 = \frac{\tau(q'eq')}{\tau(q')} \|q'\|_2 = \frac{\tau(eq')}{\tau(q')} \|q'\|_2 = \frac{\tau(eE_N(q'))}{\tau(q')} \|q'\|_2 \\ &\leq \frac{c_1\tau(e)}{\tau(q')} \|q'\|_2 = \frac{c_1\|e\|_2^2}{\|q'\|_2} \end{aligned}$$

Thus, $\|e\|_2 \geq \frac{c_0\|q'\|_2}{c_1}$ and hence $\tau(e) \geq \frac{c_0^2\tau(q')}{c_1^2}$. Since this holds for all $0 \neq e \in \mathcal{P}(\psi(pPp)' \cap qNq)$, we conclude that $\psi(pPp)' \cap qNq$ is completely atomic, thereby establishing the claim. \square

The Claim 2.3 implies that there is $0 \neq q_0 \in \psi(pPp)' \cap qNq$ such that $(\psi(pPp)q_0)' \cap q_0Nq_0 = \mathbb{C}q_0$. Let $w = q_0v$ and note that $w \neq 0$. Indeed, otherwise $0 = w = q_0v$ and hence $0 = q_0vv^*$. This would further imply that $0 = q_0E_N(vv^*)$ and hence $0 = q_0\text{support}(E_N(vv^*))$. Using relation (2.1.2) it would give $0 = q_0q = q_0$, which is a contradiction. Moreover, if we let $\psi_0(x) = \psi(x)q_0$, then we see that $\psi_0(x)w = wx$ for all $x \in pPp$. In particular, this equation entails that $w^*w \in pPp' \cap pMp = \mathbb{C}p$ and hence $w^*w = dp$, for a scalar $d > 0$. Therefore $w_0 = d^{-1/2}w$ is a partial isometry satisfying $\psi_0(x)w_0 = w_0x$, for all $x \in pPp$.

In conclusion, $\psi_0 : pPp \rightarrow \psi_0(pPp) = \psi(pPp)q_0 \subseteq q_0Nq_0$ is a $*$ -isomorphism onto its image, satisfying $\psi_0(pPp)' \cap q_0Nq_0 = \mathbb{C}q_0$. \square

2.2 Quasnormalizers of von Neumann subalgebras

Given an inclusion $N \subseteq M$, the quasi-normalizer $\mathcal{QN}_M(N)$ is the $*$ -subalgebra of M consisting of all elements $x \in M$ such that there exist $x_1, x_2, \dots, x_k \in M$ satisfying $Nx \subseteq \sum_i x_i N$ and $xN \subseteq \sum_i Nx_i$, [63]. The von Neumann algebra $\mathcal{QN}_M(N)''$ is

called the *quasi-normalizing algebra of N inside M* . This is an extension of normalization and it is precisely the von Neumann algebraic counterpart of the notion of commensurator in group theory. As usual, $\mathcal{N}_M(N) = \{u \in \mathcal{U}(M) : uNu^* = N\}$ denotes the *normalizing group* and $\mathcal{N}_M(N)''$ denotes the *normalizing algebra of N in M* . We obviously have $N \subseteq N \vee N' \cap M \subseteq \mathcal{N}_M(N)'' \subseteq \mathcal{QN}_M(N)'' \subseteq M$. In general the quasinormalizing algebra is (much) larger than the normalizer but there are natural instances when they coincide; e.g. when $N \subseteq M$ is a MASA it was shown in [64] that $\mathcal{QN}_M(A)'' = \mathcal{N}_M(A)''$. Quasinormalizers play an important role in the classification of von Neumann algebras and over the last decade there have been a sustained effort towards computing these algebras in various situations [64].

In this subsection we highlight some new computations of quasinormalizers of subalgebras in crossed products from [15] that are essential to deriving our main results from Sect. 4. If $\Gamma \curvearrowright^\sigma X$ is a free ergodic action and $M = L^\infty(X) \rtimes \Gamma$ then $\mathcal{QN}_M(L(\Gamma))''$ was computed in the following situations. When Γ is infinite abelian and σ is weak mixing Nielsen observed that $L(\Gamma)$ is a singular MASA in M [53]. Later Packer was able to show that the normalizer (and hence the quasinormalizer) depends only on the discrete spectrum of σ ; more precisely one has $\mathcal{QN}_M(L(\Gamma))'' = L^\infty(X_c) \rtimes \Gamma$, where $\Gamma \curvearrowright X_c$ is the maximal compact factor of $\Gamma \curvearrowright X$ [57]. More recently Ioana obtained a far-reaching generalization of Packer's result by showing that the same holds for every Γ and any ergodic action σ , [36, Sect. 6]. In [15] this analysis was completed at the entire level of the distal tower of $\Gamma \curvearrowright X$ using iterated quasinormalizers.

An action $\Gamma \curvearrowright X$ is called *distal* if it is the last element of an increasing finite or transfinite sequence $\Gamma \curvearrowright X_\beta$ of factors $\beta \leq \alpha$, such that $\Gamma \curvearrowright X_o$ is the trivial factor, each extension $\pi : X_{\beta+1} \rightarrow X_\beta$ is maximal compact, and for every limit ordinal $\beta \leq \alpha$ the action $\Gamma \curvearrowright X_\beta$ is the inverse limit of the preceding factors. The sequence $\{\Gamma \curvearrowright X_\beta\}_{\beta \leq \alpha}$ of factors is also called the Furstenberg-Zimmer tower of $\Gamma \curvearrowright X$. Furstenberg [28] and Zimmer [79] independently obtained the following structure theorem

Theorem 2.4 *Let $\Gamma \curvearrowright X$ be any action. Then there exists an ordinal α and a unique distal tower $\{\Gamma \curvearrowright X_\beta\}_{\beta \leq \alpha}$ such that the extension $\pi : X \rightarrow X_\alpha$ is weak mixing.*

In [15] Peterson and the first author obtained a purely von Neumann algebraic way of describing Furstenberg-Zimmer distal tower of factors for an action, namely as towers of quasinormalizers.

Theorem 2.5 *Let $\Gamma \curvearrowright^\sigma X$ be an ergodic action and let $\{\Gamma \curvearrowright X_\beta\}_{\beta \leq \alpha}$ be the corresponding Furstenberg-Zimmer tower. Let $M = L^\infty(X) \rtimes \Gamma$ and for all $\beta \leq \alpha$ let $M_\beta = L^\infty(X_\beta) \rtimes \Gamma$ be the corresponding cross-products von Neumann algebras. Then the following hold:*

1. *for all $\beta \leq \beta' \leq \alpha$ we have the following inclusions of von Neumann algebras $L(\Gamma) = M_o \subseteq M_\beta \subseteq M_{\beta'} \subseteq M_\alpha \subseteq M$;*
2. *for all $\beta \leq \alpha$ we have $\mathcal{QN}_M(M_\beta)'' = M_{\beta+1}$;*
3. *for every limit ordinal $\beta \leq \alpha$ we have $\overline{\bigcup_{\gamma < \beta} L^\infty(X_\gamma)}^{\text{WOT}} = L^\infty(X_\beta)$ and also $\overline{\bigcup_{\gamma < \beta} M_\gamma}^{\text{WOT}} = M_\beta$;*

4. There exists an infinite sequence $(\gamma_n)_n \subset \Gamma$ such that for every $x, y \in L^\infty(X) \ominus L^\infty(X_\alpha)$ we have that $\lim_{n \rightarrow \infty} \|E_{L^\infty(X_\alpha)}(x\sigma_{\gamma_n}(y))\|_2 = 0$.

2.3 Finite index inclusions of II_1 factors

A trace-preserving action $\Gamma \curvearrowright A$ on a finite von Neumann algebra is called *transitive* if A is abelian and there exist finitely many minimal projections $\mathcal{F} \subset \mathcal{P}(A)$ such that $\text{span} \mathcal{F} = A$ and for every $p, q \in \mathcal{F}$ there is $\gamma \in \Gamma$ such that $\sigma_\gamma(p) = q$. Throughout the paper the set \mathcal{F} will be denoted by $\text{At}(A)$ and will be called the atoms of A . In particular all atoms of A have same trace, i.e. $\dim(A)^{-1}$.

Lemma 2.6 *Let A be an abelian von Neumann algebra and let $\Gamma \curvearrowright^\sigma A$ be a trace preserving action. Assume that the inclusion $L(\Gamma) \subseteq A \rtimes \Gamma$ admits a finite Pimsner-Popa basis. Then A is completely atomic. Moreover, if $\Gamma \curvearrowright A$ is ergodic then $\Gamma \curvearrowright A$ is transitive.*

Proof By assumption there exist $m_1, \dots, m_k \in A \rtimes \Gamma = M$, with $E_{L(\Gamma)}(m_i m_j^*) = \delta_{i,j} p_i$ where $p_i \in \mathcal{P}(M)$, such that for all $x \in M$ we have $x = \sum_{i=1}^k E_{L(\Gamma)}(x m_i^*) m_i$. Thus, for all $x \in M$ we have $\|x\|_2^2 = \sum_{i=1}^k \|E_{L(\Gamma)}(x m_i^*)\|_2^2$. Approximating $m_i \in M$ using their Fourier decompositions and doing some basic calculations this further implies the following: for every $\varepsilon > 0$ one can find $a_j \in A$ with $1 \leq j \leq l$ and $c > 0$ so that for all $x \in (M)_1$ we have

$$\|x\|_2^2 \leq \varepsilon + c \sum_{i=1}^l \|E_{L(\Gamma)}(x a_i)\|_2^2. \quad (2.3.1)$$

Assume for the sake of contradiction that A has a diffuse corner, i.e. there is $0 \neq p \in A$ so that Ap is diffuse. Hence one can find a sequence of unitaries $u_n \in \mathcal{U}(Ap)$ so that for all $x \in Ap$ we have $\tau(u_n x) \rightarrow 0$, as $n \rightarrow \infty$. Since $a_i \in A$ we have $E_{L(\Gamma)}(u_n a_i) = \tau(u_n a_i) = \tau(u_n a_i p)$. Thus using (2.3.1) we get that $\tau(p) = \|u_n\|_2^2 \leq \varepsilon + c \sum_{i=1}^l \|E_{L(\Gamma)}(u_n a_i)\|_2^2 = \varepsilon + c \sum_{i=1}^l |\tau(u_n a_i p)|^2$ and since $\lim_{n \rightarrow \infty} \sum_{i=1}^l |\tau(u_n a_i p)|^2 = 0$ we get that $\tau(p) \leq \varepsilon$. Letting $\varepsilon \searrow 0$ we get $p = 0$, a contradiction.

To see the moreover part let $0 \neq q \in A$ be a minimal projection of maximal trace. Thus for all $\gamma \in \Gamma$ either $q\sigma_\gamma(q) = 0$ or $q = \sigma_\gamma(q)$. Thus the orbit $\mathcal{F} = \{\sigma_\gamma(q) \mid \gamma \in \Gamma\}$ is necessarily a finite set of (orthogonal) minimal projections of A . Let $t = \sum_{q \in \mathcal{F}} q$ and notice that $0 \neq t \in A$ is a projection satisfying $\sigma_\gamma(t) = t$ for all $\gamma \in \Gamma$. Since $\Gamma \curvearrowright A$ is ergodic it follows that $t = 1$. Since A is completely atomic this entails that $A = \text{span} \mathcal{F}$. Thus $\Gamma \curvearrowright A$ is transitive. \square

Proposition 2.7 *Let Γ, Λ be icc groups and let $\Gamma \curvearrowright A, \Lambda \curvearrowright B$ be transitive actions so that $A \rtimes \Gamma$ and $B \rtimes \Lambda$ are II_1 factors. Assume that $\theta : A \rtimes \Gamma \rightarrow B \rtimes \Lambda$ is a $*$ -isomorphism such that $\theta(L(\Gamma)) = L(\Lambda)$. Then $\dim(A) = \dim(B)$ and for every $a \in \text{At}(A)$, $b \in \text{At}(B)$ there is a unitary $u \in L(\Lambda)$ so that $\theta(L(\text{Stab}_\Gamma(a))) = u^* L(\text{Stab}_\Lambda(b)) u$. In addition, if there exists $a \in \text{At}(A)$ such that $\text{Stab}_\Gamma(a)$ is normal in Γ then for*

every $b \in \text{At}(B)$ then $\text{Stab}_\Lambda(b)$ is also normal in Λ ; moreover, $\Gamma/\text{Stab}_\Gamma(a) \cong \Lambda/\text{Stab}_\Lambda(b)$.

Proof To simplify the presentation, we assume that $A \rtimes \Gamma = B \rtimes \Lambda$ and $L(\Gamma) = L(\Lambda)$. Let $n = \dim(A)$ and fix $a \in \text{At}(A)$. Notice $\tau(a) = 1/n$ and hence $E_{L(\Gamma)}(a) = \tau(a)1 = 1/n$. Also for each $x \in L(\Gamma)$, using its Fourier decomposition, we have $axa = \sum_{\gamma \in \Gamma} \tau(xu_\gamma^{-1})au_\gamma a = \sum_{\gamma \in \Gamma} \tau(xu_\gamma^{-1})a\sigma_\gamma(a)u_\gamma = \sum_{\gamma \in \text{Stab}_\Gamma(a)} \tau(xu_\gamma^{-1})u_\gamma a = E_{L(\text{Stab}_\Gamma(a))}(x)a$. Since clearly $*\text{-alg}\{a, L(\Gamma)\} = A \rtimes \Gamma$ then, altogether, the above relations show that $A \rtimes \Gamma$ is the basic construction of the inclusion $L(\text{Stab}_\Gamma(a)) \subseteq L(\Gamma)$ and also $[\Gamma : \text{Stab}_\Gamma(a)] = [A \rtimes \Gamma : L(\Gamma)] = n$. A similar statement holds for $L(\Lambda) \subseteq B \rtimes \Lambda$. Since by assumption $[A \rtimes \Gamma : L(\Gamma)] = [B \rtimes \Lambda : L(\Lambda)]$ it follows that $\dim(A) = \dim(B) = n$. To show the remaining part of the statement fix $b \in \text{At}(B)$. By the factoriality assumption, since $\tau(a) = \tau(b) = 1/n$, there is a unitary $u \in A \rtimes \Gamma$ so that

$$b = uau^*. \quad (2.3.2)$$

Since $a \in A \rtimes \Gamma$ is the Jones projection for inclusion $L(\text{Stab}_\Gamma(a)) \subseteq L(\Gamma)$, by pull-down lemma there exists $m \in L(\Gamma)$ such that $b = uau^* = mam^*$. Thus one can check that $1/n = \tau(b) = E_{L(\Lambda)}(b) = E_{L(\Gamma)}(b) = E_{L(\Gamma)}(mam^*) = mE_{L(\Gamma)}(a)m^* = \tau(a)mm^* = (1/n)mm^*$. Hence $mm^* = 1$ which implies that $m \in L(\Lambda)$ is a unitary. Thus in equation (2.3.2) we can assume wlog that the unitary u belongs to $L(\Gamma)$. Hence using (2.3.2) we further have that $L(\text{Stab}_\Lambda(b)) = \{b\}' \cap L(\Lambda) = \{uau^*\}' \cap L(\Gamma) = \{uau^*\}' \cap uL(\Gamma)u^* = uL(\text{Stab}_\Gamma(a))u^*$, as desired. Since $\text{Stab}_\Gamma(a)$ is normal in Γ it follows from the above relation that $uu_\gamma u^* \in \mathcal{N}_{L(\Lambda)}(L(\text{Stab}_\Lambda(b)))$ for every $\gamma \in \Gamma$. Since Λ is icc and $[\Lambda : \text{Stab}_\Lambda(b)] < \infty$ then $L(\text{Stab}_\Lambda(b)) \subseteq L(\Lambda)$ is a irreducible inclusion of II_1 factors. Thus using [76, Corollary 5.3] we have that for every $\gamma \in \Gamma$ there exist a unitary $x \in L(\text{Stab}_\gamma(b))$ and $\lambda \in \Lambda$ such that $uu_\gamma u^* = xv_\lambda$. In particular this implies that $\text{Stab}_\Lambda(b)$ is normal in Λ and also $\Gamma/\text{Stab}_\Gamma(a) \cong \Lambda/\text{Stab}_\Lambda(b)$. \square

2.4 Mixing extensions

Let $B \subseteq A$ be an inclusion of von Neumann algebras and assume that $\Gamma \curvearrowright^\sigma A$ is an action that leaves the subalgebra B invariant. Throughout the paper we call such a system *an extension* and we denote it by $\Gamma \curvearrowright (B \subseteq A)$. When A is endowed with a state ϕ preserved by σ the extension is said to be ϕ -preserving and will be denoted by $\Gamma \curvearrowright (B \subseteq A, \phi)$. When A is a finite von Neumann algebra and ϕ is a faithful normal trace then $\Gamma \curvearrowright (B \subseteq A, \phi)$ is called a trace-preserving extension.

Definition 2.8 A trace-preserving extension $\Gamma \curvearrowright (B \subseteq A, \tau)$ is called mixing if for every $t, z \in A \ominus B$ we have $\lim_{\gamma \rightarrow \infty} \|E_B(t\sigma_\gamma(z))\|_2 = 0$.

Lemma 2.9 Let $\Gamma \curvearrowright (B \subseteq A, \tau)$ be a trace-preserving mixing extension. Then for every $t, z \in (A \rtimes \Gamma) \ominus (B \rtimes \Gamma)$ and every sequence $(x_n)_n \subset (L(\Gamma))_1$ that converges to 0 weakly, we have $\lim_{n \rightarrow \infty} \|E_{B \rtimes \Gamma}(tx_n z)\|_2 = 0$.

Proof Fix $t, z \in (A \rtimes \Gamma) \ominus (B \rtimes \Gamma)$. Consider the Fourier decompositions $t = \sum_{\gamma} E_A(tu_{\gamma^{-1}})u_{\gamma}$ and $z = \sum_{\gamma} u_{\gamma^{-1}}E_A(u_{\gamma}z)$ and notice that $E_B(tu_{\gamma}) = E_B(u_{\gamma}z) = 0$ for all $\gamma \in \Gamma$. Fix $\varepsilon > 0$. Using these decompositions and basic $\|\cdot\|_2$ -estimates one can find finite subsets $F, G \subset \Gamma$ such that

$$\|E_{B \rtimes \Gamma}(tx_n z)\|_2 \leq \frac{\varepsilon}{2} + \sum_{\delta \in F, \lambda \in G} \|E_{B \rtimes \Gamma}(E_A(tu_{\delta^{-1}})u_{\delta}x_n u_{\lambda^{-1}}E_A(u_{\lambda}z))\|_2. \quad (2.4.1)$$

Also fix $a, b \in A$ and $x \in L(\Gamma)$. Using the Fourier decomposition of $x \in L(\Gamma)$ we see that $E_{B \rtimes \Gamma}(axb) = \sum_{\gamma} \tau(xu_{\gamma^{-1}})E_{B \rtimes \Gamma}(au_{\gamma}b) = \sum_{\gamma} \tau(xu_{\gamma^{-1}})E_B(a\sigma_{\gamma}(b))u_{\gamma}$. Thus we have the formula

$$\|E_{B \rtimes \Gamma}(axb)\|_2^2 = \sum_{\gamma} |\tau(xu_{\gamma^{-1}})|^2 \|E_B(a\sigma_{\gamma}(b))\|_2^2. \quad (2.4.2)$$

Since $\Gamma \curvearrowright (B \subseteq A, \tau)$ is mixing and F, G are finite one can find a finite subset $H \subset \Gamma$ so that $\|E_B(E_A(tu_{\delta^{-1}})\sigma_{\gamma}(E_A(u_{\lambda}z)))\|_2 \leq \varepsilon/(\sqrt{8}|F||G|)$ for all $\gamma \in \Gamma \setminus H$, $\delta \in F$ and $\lambda \in G$. Also since $x_n \rightarrow 0$ weakly, and F, G, H are finite there is an integer n_0 such that $|\tau(x_n u_{\gamma^{-1}})| \leq \varepsilon/(\sqrt{8}|H||G||F||t\|_{\infty}\|z\|_{\infty})$ for all $\gamma \in G^{-1}H^{-1}F$ and $n \geq n_0$. Using these basic estimates in combination with formula (2.4.2) we see that for all $n \geq n_0$ we have

$$\begin{aligned} & \sum_{\delta \in F, \lambda \in G} \|E_{B \rtimes \Gamma}(E_A(tu_{\delta^{-1}})u_{\delta}x_n u_{\lambda^{-1}}E_A(u_{\lambda}z))\|_2 \\ &= \sum_{\delta \in F, \lambda \in G} \left(\sum_{\gamma \in H} |\tau(x_n u_{\lambda^{-1}\gamma^{-1}\delta})|^2 \|E_B(E_A(tu_{\delta^{-1}})\sigma_{\gamma}(E_A(u_{\lambda}z)))\|_2^2 \right. \\ & \quad \left. + \sum_{\gamma \in \Gamma \setminus H} |\tau(x_n u_{\lambda^{-1}\gamma^{-1}\delta})|^2 \|E_B(E_A(tu_{\delta^{-1}})\sigma_{\gamma}(E_A(u_{\lambda}z)))\|_2^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{\delta \in F, \lambda \in G} \left(\sum_{\gamma \in H} \frac{\varepsilon^2}{8|F|^2|G|^2|H||t\|_{\infty}^2\|z\|_{\infty}^2} \|E_B(t\sigma_{\gamma}(z))\|_2^2 + \frac{\varepsilon^2}{8|F|^2|G|^2} \|x_n\|_2^2 \right)^{\frac{1}{2}} \\ &\leq \left(\frac{\varepsilon^2}{8} + \frac{\varepsilon^2}{8} \right)^{\frac{1}{2}} = \frac{\varepsilon}{2}. \end{aligned}$$

This combined with (2.4.1) show that for every $\varepsilon > 0$ there exists n_0 such that for all $n \geq n_0$ we have $\|E_B(tx_n z)\|_2 \leq \varepsilon$, as desired. \square

Theorem 2.10 Let $\Gamma \curvearrowright (B \subseteq A, \tau)$ be a trace-preserving mixing extension. Also let $\Lambda \curvearrowright^{\alpha} (D \subseteq C, \tau)$ be a trace-preserving extension for which there exists a $*$ -isomorphism $\theta : A \rtimes \Gamma \rightarrow C \rtimes \Lambda$ satisfying $\theta(B \rtimes \Gamma) = D \rtimes \Lambda$ and $\theta(L(\Gamma)) = L(\Lambda)$. Then $\Lambda \curvearrowright (D \subseteq C, \tau)$ is a mixing extension.

Proof Suppressing θ from the notation we assume that $A \rtimes \Gamma = C \rtimes \Lambda$, $B \rtimes \Gamma = D \rtimes \Lambda$ and $L(\Gamma) = L(\Lambda)$. Fix $t, z \in C \ominus D$. We now show that for any infinite sequence $(\lambda_n)_n \subseteq \Lambda$ we have that

$$\lim_{n \rightarrow \infty} \|E_D(t\alpha_{\lambda_n}(z))\|_2 = 0. \quad (2.4.3)$$

Since $B \rtimes \Gamma = D \rtimes \Lambda$ we note that

$$\begin{aligned} E_{B \rtimes \Gamma}(z) &= E_{D \rtimes \Lambda}(z) = \sum_{\lambda \in \Lambda} E_D(zv_{\lambda^{-1}})v_\lambda = \sum_{\lambda \in \Lambda} E_D(E_C(zv_{\lambda^{-1}}))v_\lambda \\ &= \sum_{\lambda \in \Lambda} E_D(zE_C(v_{\lambda^{-1}}))v_\lambda = E_D(z) = 0. \end{aligned}$$

Similarly we have $E_{B \rtimes \Gamma}(t) = 0$. Since $t, z \in C$ and $D \rtimes \Lambda = B \rtimes \Gamma$ we see that

$$\begin{aligned} \|E_D(t\alpha_{\lambda_n}(z))\|_2 &= \|E_{D \rtimes \Lambda}(t\alpha_{\lambda_n}(z))\|_2 = \|E_{D \rtimes \Lambda}(tv_{\lambda_n}z)v_{\lambda_n^{-1}}\|_2 \\ &= \|E_{D \rtimes \Lambda}(tv_{\lambda_n}z)\|_2 = \|E_{B \rtimes \Gamma}(tv_{\lambda_n}z)\|_2. \end{aligned} \quad (2.4.4)$$

Since $(\lambda_n)_n$ is infinite the sequence $(v_{\lambda_n})_n \subset L(\Lambda) = L(\Gamma)$ converges weakly to 0. Thus applying Lemma 2.9 we get $\lim_{n \rightarrow \infty} \|E_{B \rtimes \Gamma}(tv_{\lambda_n}z)\|_2 = 0$ and hence (2.4.3) follows from (2.4.4). \square

For further use we recall the following technical variation of [66, Theorem 3.1]. The proof is essentially the same with the one presented in [66] and will be left to the reader.

Theorem 2.11 *Let $\Gamma \curvearrowright (B \subseteq A, \tau)$ be a trace-preserving mixing extension. Denote by $M = A \rtimes \Gamma \supset B \rtimes \Gamma = N$ the corresponding inclusion of crossed product von Neumann algebras. Then for every von Neumann subalgebra $C \subseteq N$ satisfying $C \not\prec_N B$ we have $\mathcal{QN}_M(C)'' \subseteq N$.*

3 Extensions satisfying the intermediate subalgebra property

Let $\Gamma \curvearrowright (P_0 \subseteq P)$ be an extension of tracial von Neumann algebras and consider the corresponding inclusion $P_0 \rtimes \Gamma \subseteq P \rtimes \Gamma$ of von Neumann algebras. Suzuki discovered in [75] that if P_0, P are abelian and $\Gamma \curvearrowright P_0$ is free then the extension $\Gamma \curvearrowright (P_0 \subseteq P)$ satisfies the *intermediate subalgebra property*, i.e. every intermediate subalgebra $P_0 \rtimes \Gamma \subseteq N \subseteq P \rtimes \Gamma$ arises as $N = Q \rtimes \Gamma$ for some Γ -invariant intermediate subalgebra $P_0 \subseteq Q \subseteq P$. In this section we establish the intermediate subalgebra property for new classes of extensions (e.g. compact) for icc groups Γ (see Theorem 3.10). In many respects these results complement Suzuki's as they cover several examples of non free extensions, for instance when $P_0 = \mathbb{C}1$. As a consequence, for all free ergodic pmp actions on probability spaces $\Gamma \curvearrowright X$ of icc groups Γ , we are able to completely describe all intermediate subfactors $L(\Gamma) \subseteq N \subseteq L^\infty(X) \rtimes \Gamma$ with finite index $[N : L(\Gamma)] < \infty$ (see Theorem 3.14). Our strategy also enables us to recover some well-known older results on intermediate subalgebras (see Corollary 3.3 and Theorems 3.4, 3.7).

We briefly introduce a few preliminaries. The first result describes the algebraic structure of fixed point subspaces associated with u.c.p. maps and it is essentially [2, Lemma 3.4]. For reader's convenience we also include a short proof.

Lemma 3.1 *Let M be a von Neumann algebra, and let φ be a faithful, normal state on M . Let $\Psi : M \rightarrow M$ be a normal, u.c.p. map. Define $\text{Har}(\Psi) = \{m \in M : \Psi(m) = m\}$ to be the fixed points of Ψ . If $\varphi \circ \Psi = \varphi$ then $\text{Har}(\Psi)$ is a von Neumann subalgebra of M .*

Proof From the definition it is clear that $\text{Har}(\Psi)$ is closed under sum and taking adjoint. Also since Ψ is normal, $\text{Har}(\Psi)$ is closed in the weak-operator topology. Thus, to finish the proof we only need to show that $\text{Har}(\Psi)$ is closed under product. Using the polarization identity, it suffices to show that whenever $x \in \text{Har}(\Psi)$ we have that $x^*x \in \text{Har}(\Psi)$ as well. By Kadison–Schwarz inequality we have that $\Psi(x^*x) \geq \Psi(x)^*\Psi(x) = x^*x$, where the last equality follows because $x \in \text{Har}(\Psi)$; thus $\Psi(x^*x) - x^*x \geq 0$. Since $\varphi \circ \Psi = \varphi$ we also have $\varphi(\Psi(x^*x) - x^*x) = 0$. Since φ is faithful, we get that $\Psi(x^*x) = x^*x$, thereby proving that $\text{Har}(\Psi)$ is an algebra. \square

Theorem 3.2 *Let $\Gamma \curvearrowright (P, \tau)$ be a trace preserving action on a finite von Neumann algebra P and consider the corresponding crossed product von Neumann algebra $P \rtimes \Gamma$. Let $P_0 \subseteq P$ be a Γ -invariant subalgebra. Assume that $P_0 \rtimes \Gamma \subseteq N \subseteq P \rtimes \Gamma$ is an intermediate von Neumann subalgebra. Then there is a Γ -invariant subalgebra $P_0 \subseteq Q \subseteq P$ so that $N = Q \rtimes \Gamma$ if and only if $E_N(P) \subseteq P$.*

Proof Denote by $M = P \rtimes \Gamma$ and let $E_P : M \rightarrow P$ and $E_N : M \rightarrow N$ be the canonical conditional expectations onto P and N , respectively. To see the direct implication, fix $a \in P$. Since $N = Q \rtimes \Gamma$ and $L(\Gamma) \subseteq N$ we have

$$\begin{aligned} E_N(a) &= \sum_{\gamma} E_Q(E_N(a)u_{\gamma^{-1}})u_{\gamma} = \sum_{\gamma} E_Q(E_N(au_{\gamma^{-1}}))u_{\gamma} \\ &= \sum_{\gamma} E_Q(au_{\gamma^{-1}})u_{\gamma} = \sum_{\gamma} E_Q(E_P(au_{\gamma^{-1}}))u_{\gamma} \\ &= \sum_{\gamma} E_Q(aE_P(u_{\gamma^{-1}}))u_{\gamma} = E_Q(a) \in P. \end{aligned}$$

Next we show the reverse implication. Let $e_P : L^2(M) \rightarrow L^2(P)$ and $e_N : L^2(M) \rightarrow L^2(N)$ be the canonical orthogonal projections. Since $E_N(P) \subseteq P$ then $E_P(E_N(a)) = E_N(a)$ for all $a \in P$. Therefore $E_P \circ E_N = E_N \circ E_P$ and hence $e_P e_N e_P = e_N e_P$. Taking adjoints we obtain $e_N e_P = e_P e_N$ and since $(e_P e_N e_P)^n$ converges to $e_N \wedge e_P$ in the strong-operator topology, as n tends to infinity, we conclude that $e_P e_N = e_N e_P = e_N \wedge e_P$. This also entails that $e_N \wedge e_P = e_{N \cap P}$ and thus

$$E_N \circ E_P = E_P \circ E_N = E_{N \cap P}. \quad (3.0.5)$$

Alternatively, one can show (3.0.5) just by using Lemma 3.1. Indeed since $E_N(P) \subseteq P$ then from assumptions $E_{N|_P} : P \rightarrow P$ is a u.c.p. map which preserves τ , a normal,

faithful, tracial state. Letting $\Psi = E_{N|P}$ we can easily see that $N \cap P \subseteq \text{Har}(\Psi) \subseteq E_N(P)$. Since we canonically have $E_N(P) \subseteq N \cap P$ we conclude that $\text{Har}(\Psi) = E_N(P) = P \cap N$. The last equality gives (3.0.5).

Notice that from assumptions $Q := N \cap P \subseteq P$ is a Γ -invariant von Neumann subalgebra of P containing P_0 . So to finish the proof of our implication we only need to show that $N = Q \rtimes \Gamma$. Since $Q \rtimes \Gamma \subseteq N$ canonically, we will only argue for the reverse inclusion. To see this fix $x \in N$ and consider its Fourier decomposition (in M) $x = \sum_{\gamma} x_{\gamma} u_{\gamma}$ where $x_{\gamma} \in P$. Since $L(\Gamma) \subseteq N$ we have $\sum_{\gamma} x_{\gamma} u_{\gamma} = x = E_N(x) = E_N(\sum_{\gamma} x_{\gamma} u_{\gamma}) = \sum_{\gamma} E_N(x_{\gamma}) u_{\gamma}$. By (3.0.5) we have $E_N(x_{\gamma}) = E_Q(x_{\gamma}) \in Q$ and hence $x_{\gamma} = E_Q(x_{\gamma}) \in Q$ for all $\gamma \in \Gamma$. Thus $x = \sum_{\gamma} E_Q(x_{\gamma}) u_{\gamma} \in Q \rtimes \Gamma$, as desired. \square

The conditional expectation property presented in the previous theorem can be used effectively to describe all the intermediate subalgebras for many inclusions arising from canonical constructions in von Neumann algebras. In the remaining part of the section we highlight several situations when this is indeed the case. For instance it provides a very fast approach to Ge's well known tensor-splitting theorem [31, Theorem 3.1] for finite von Neumann algebras.

Corollary 3.3 ([Ge, Theorem 3.1]) *Let P_1 be a factor and let P_2, N be von Neumann algebras such that $P_1 \otimes 1 \subseteq N \subseteq P_1 \bar{\otimes} P_2$. Assume there exist faithful normal states φ_1 on P_1 and φ_2 on P_2 , and a faithful, normal conditional expectation $E_N : P_1 \bar{\otimes} P_2 \rightarrow N$ preserving $\varphi := \varphi_1 \otimes \varphi_2$. Then $N = P_1 \bar{\otimes} Q$ for some (von Neumann) subalgebra $Q \subseteq P_2$.*

Proof We first claim that $E_N(1 \otimes P_2) \subseteq 1 \otimes P_2$. To see this fix $p_2 \in P_2$ and $p_1 \in P_1$. Since $N \supset P_1 \otimes 1$ we have $(p_1 \otimes 1)E_N(1 \otimes p_2) = E_N(p_1 \otimes p_2) = E_N((1 \otimes p_2)(p_1 \otimes 1)) = E_N(1 \otimes p_2)(p_1 \otimes 1)$. This implies that $E_N(1 \otimes p_2) \in (P_1 \otimes 1)' \cap (P_1 \bar{\otimes} P_2) = 1 \otimes P_2$, thereby proving the claim. So we have that $E_N : 1 \otimes P_2 \rightarrow 1 \otimes P_2$ is a u.c.p. map, preserving φ , a faithful, normal state. So by Lemma 3.1, $E_N(1 \otimes P_2)$ is a subalgebra of $1 \otimes P_2$, which we can identify as a von Neumann subalgebra $Q \subseteq P_2$. Under this identification we have that $1 \otimes Q = E_N(1 \otimes P_2) = N \cap (1 \otimes P_2)$. Hence $P_1 \bar{\otimes} Q \subseteq N$.

To show the reverse containment, we first claim that $(P_1 \otimes_{\text{alg}} P_2) \cap N$ is WOT-dense in N . Let $n \in N$. By Kaplansky's density theorem, we can find a bounded net $(x_{\lambda}) \subset P_1 \otimes_{\text{alg}} P_2$ such that $x_{\lambda} \rightarrow n$ in WOT. Since E_N is normal, we get that $E_N(x_{\lambda}) \rightarrow n$. If $x_{\lambda} = \sum_i p_i \otimes q_i$, with $p_i \in P_1$ and $q_i \in P_2$ then $E_N(x_{\lambda}) = E_N(\sum_i p_i \otimes q_i) = \sum_i (p_i \otimes 1)E_N(1 \otimes q_i) \in (P_1 \otimes_{\text{alg}} P_2) \cap N$, thereby establishing the claim. Now fix $n \in (P_1 \otimes_{\text{alg}} P_2) \cap N$. Then, there exist $p_1, \dots, p_k \in P_1$ and $q_1, \dots, q_k \in P_2$ so that $n = \sum_i p_i \otimes q_i$. Now, since $n \in N$ and $P_1 \otimes 1 \subseteq N$ we have

$$n = E_N(n) = E_N\left(\sum_{i=1}^k p_i \otimes q_i\right) = \sum_{i=1}^k E_N(p_i \otimes q_i) = \sum_{i=1}^k (p_i \otimes 1)E_N(1 \otimes q_i). \quad (3.0.6)$$

Since $E_N(1 \otimes q_i) \in Q$ then (3.0.6) implies that $(P_1 \otimes_{\text{alg}} P_2) \cap N \subseteq P_1 \otimes_{\text{alg}} Q$. As $(P_1 \otimes_{\text{alg}} P_2) \cap N$ is WOT-dense in N we get $N = P_1 \bar{\otimes} Q$. \square

We also record a twisted version of the above theorem.

Theorem 3.4 *Let P be a II_1 factor and let Q be a finite separable von Neumann algebra, equipped with a trace preserving action of Γ . Assume that $\Gamma \curvearrowright^\sigma P$ is an outer action. Then for any intermediate von Neumann subalgebra $P \rtimes \Gamma \subseteq N \subseteq (P \bar{\otimes} Q) \rtimes \Gamma$ there is a von Neumann subalgebra $Q_0 \subseteq Q$ such that $N = (P \bar{\otimes} Q_0) \rtimes \Gamma$.*

Proof Using Theorem 3.2 we only need to show that $E_N(Q) \subseteq Q$. Naturally, we have that $E_N(Q) \subseteq P' \cap (P \bar{\otimes} Q) \rtimes \Gamma$. We shall now briefly argue that $P' \cap (P \bar{\otimes} Q) \rtimes \Gamma \subseteq Q$, which will prove our claim. To see this fix $\sum_\gamma a_\gamma u_\gamma \in P' \cap (P \bar{\otimes} Q) \rtimes \Gamma$, where $a_\gamma \in P \bar{\otimes} Q$. Thus for every $\gamma \in \Gamma$ and $p \in P$ we have that $pa_\gamma = a_\gamma \sigma_\gamma(p)$. Fix $e \neq \gamma \in \Gamma$. Let $a_\gamma = \sum_i p_i \otimes q_i$, with $p_i \in P$ and $q_i \in Q$. We may assume that q_i are orthogonal with respect to τ_Q (by using the Gram-Schmidt process, and using the separability of Q). Thus we have $\sum_i (pp_i) \otimes q_i = p(\sum_i p_i \otimes q_i) = (\sum_i p_i \otimes q_i) \sigma_\gamma(p) = \sum_i (p_i \sigma_\gamma(p)) \otimes q_i$. As q_i 's are orthogonal we further get $pp_i = p_i \sigma_\gamma(p)$ for all i and $p \in P$. Since $\Gamma \curvearrowright P$ is outer, this implies $p_i = 0$ for all i and hence $a_\gamma = 0$. Thus, $P' \cap (P \bar{\otimes} Q) \rtimes \Gamma \subseteq P' \cap (P \bar{\otimes} Q) = Q$. Hence $N = (P \bar{\otimes} Q_0) \rtimes \Gamma$ where $Q_0 = E_N(Q)$. \square

If P is a II_1 factor then an action $\Gamma \curvearrowright P$ is called *centrally free* if the induced action $\Gamma \curvearrowright P' \cap P^\omega$ is properly outer (see [75, Definition 4.3]). Theorem 3.4 was first obtained by Y. Suzuki under the assumption that the $\Gamma \curvearrowright P$ is centrally free, [75, Example 4.14]. In general the centrally freeness assumption introduces certain limitations. For instance, if $P = L(\mathbb{F}_2)$ then $P' \cap P^\omega = \mathbb{C}$ and hence no nontrivial group admits a centrally free action on P . However, when P is the hyperfinite II_1 factor, then requiring the $\Gamma \curvearrowright P$ to be outer is the same as requiring the $\Gamma \curvearrowright P$ to be centrally free. This surprising result is a consequence of Ocneanu's central freedom lemma ([26, Lemma 15.25]). The reader may also consult [9] for another recent application of the central freedom lemma.

Theorem 3.5 *Let \mathcal{R} denote the hyperfinite type II_1 factor and let Γ be a discrete group acting on \mathcal{R} . Then $\Gamma \curvearrowright^\sigma \mathcal{R}$ is outer if and only if $\Gamma \curvearrowright^\sigma \mathcal{R}$ is centrally free.*

Proof Let $\Gamma \curvearrowright^\sigma \mathcal{R}$ be an outer action. Let $a \in \mathcal{R}' \cap \mathcal{R}^\omega$ and $\gamma \in \Gamma$ be such that $\sigma_{g^{-1}}(x)a = ax$ for all $x \in \mathcal{R}' \cap \mathcal{R}^\omega$. This clearly implies that $u_\gamma a \in (\mathcal{R}' \cap \mathcal{R}^\omega)' \cap (\mathcal{R} \rtimes \Gamma)^\omega$. Now, by Ocneanu's central freedom lemma we get that $(\mathcal{R}' \cap \mathcal{R}^\omega)' \cap (\mathcal{R} \rtimes \Gamma)^\omega = \mathcal{R} \vee (\mathcal{R}' \cap \mathcal{R} \rtimes \Gamma)^\omega = \mathcal{R}$ (where the last equality holds because $\Gamma \curvearrowright \mathcal{R}$ is outer). Thus $u_\gamma \in \mathcal{R}$ which implies that $\gamma = e$. Hence $\Gamma \curvearrowright^\sigma \mathcal{R}' \cap \mathcal{R}^\omega$ is outer.

Conversely, assume that $\Gamma \curvearrowright^\sigma \mathcal{R}' \cap \mathcal{R}^\omega$ is outer. We will show that $\mathcal{R}' \cap \mathcal{R} \rtimes \Gamma = \mathbb{C}$, which shall establish that $\Gamma \curvearrowright^\sigma \mathcal{R}$ is outer. Let $x \in \mathcal{R}' \cap \mathcal{R} \rtimes \Gamma$, and consider its Fourier decomposition $x = \sum_\gamma x_\gamma u_\gamma$, where $x_\gamma \in \mathcal{R}$. Now $x \in \mathcal{R}' \cap \mathcal{R} \rtimes \Gamma$ implies that $x_\gamma u_\gamma \in \mathcal{R}' \cap \mathcal{R} \rtimes \Gamma$ for all $\gamma \in \Gamma$. Hence $x_\gamma u_\gamma \in \mathcal{R} \vee (\mathcal{R}' \cap \mathcal{R} \rtimes \Gamma)^\omega = (\mathcal{R}' \cap \mathcal{R}^\omega)' \cap (\mathcal{R} \rtimes \Gamma)^\omega$ (where the last equality follows from Ocneanu's central freedom lemma). Thus we get $x_\gamma u_\gamma x = x x_\gamma u_\gamma$ for all $x \in \mathcal{R}' \cap \mathcal{R}^\omega$ which gives that $x_\gamma \sigma_\gamma(x) = x x_\gamma$, implying $\sigma_\gamma(x) x_\gamma x_\gamma^* = x_\gamma x_\gamma^* x$, which implies $E_{\mathcal{R}' \cap \mathcal{R}^\omega}(x_\gamma x_\gamma^*) x = x E_{\mathcal{R}' \cap \mathcal{R}^\omega}(x_\gamma x_\gamma^*)$, for all $x \in \mathcal{R}' \cap \mathcal{R}^\omega$.

Since $\Gamma \curvearrowright \mathcal{R}' \cap \mathcal{R}^\omega$ is outer, we get that $E_{\mathcal{R}' \cap \mathcal{R}^\omega}(x_\gamma x_\gamma^*) = 0$ for all $\gamma \neq e$. Since $E_{\mathcal{R}' \cap \mathcal{R}^\omega}$ is faithful, this further implies that $x_\gamma = 0$ for all $\gamma \neq e$. This implies that

$x \in \mathcal{R}' \cap \mathcal{R} = \mathbb{C}$, thereby establishing that $\mathcal{R}' \cap \mathcal{R} \rtimes \Gamma = \mathbb{C}$, which implies that $\Gamma \curvearrowright^\sigma \mathcal{R}$ is outer. \square

Theorem 3.4 leads to new examples of subalgebras in $(P \bar{\otimes} Q) \rtimes \Gamma$ that are amenable relative to $P \rtimes \Gamma$, [55, Definition 2.2]. Note that for von Neumann algebra inclusions $N \subseteq M$, the existence of a maximal amenable subalgebra P in M relative to N follows from [24, Lemma 2.7]. We remark that very similar methods were used in [48, Theorem 3.4] to provide examples of maximal Haagerup subalgebras arising from *extremely rigid actions* of an icc group.

Corollary 3.6 *Let P be a type II_1 factor, let Q be a finite von Neumann algebra, and let Γ be an amenable group acting outerly on P, Q (the actions are assumed to be trace preserving). Let $Q_0 \subseteq Q$ be a maximal amenable subalgebra. Then $(P \bar{\otimes} Q_0) \rtimes \Gamma$ is a maximal amenable subalgebra in $(P \bar{\otimes} Q) \rtimes \Gamma$ relative to $P \rtimes \Gamma$. In particular, if \mathcal{R} is the hyperfinite II_1 factor, and $\Gamma \curvearrowright \mathcal{R}$ is an outer trace preserving action, then $(\mathcal{R} \bar{\otimes} Q_0) \rtimes \Gamma$ is maximal amenable in $(\mathcal{R} \bar{\otimes} Q) \rtimes \Gamma$.*

Proof Let $(P \bar{\otimes} Q_0) \rtimes \Gamma \subseteq N \subseteq (P \bar{\otimes} Q) \rtimes \Gamma$. Then by Theorem 3.4 $N = (P \bar{\otimes} Q_1) \rtimes \Gamma$, with $Q_0 \subseteq Q_1 \subseteq Q$. If N is amenable relative to $P \rtimes \Gamma$, then Q_1 is amenable. By maximal amenability of Q_0 we obtain that $Q_1 = Q_0$ thereby establishing the result. \square

The next theorem re-establishes a well known Galois correspondence for group actions.

Theorem 3.7 *Let Γ be a group, let $\Lambda \triangleleft \Gamma$ be a normal subgroup, and let (P, τ) be a tracial von Neumann algebra. Assume that Γ acts on P via trace preserving automorphisms such that $(P \rtimes \Lambda)' \cap (P \rtimes \Gamma) = \mathbb{C}$. Then for any intermediate subfactor $P \rtimes \Lambda \subseteq N \subseteq P \rtimes \Gamma$ there exists an intermediate subgroup $\Lambda \leq K \leq \Gamma$ such that $N = P \rtimes K$.*

Proof Let $K = \{\gamma \in \Gamma : u_\gamma \in N\}$. Clearly, K is a group satisfying $\Lambda \leq K \leq \Gamma$. Also $P \subseteq P \rtimes \Lambda \subseteq N$ and hence $P \rtimes K \subseteq N \subseteq P \rtimes \Gamma$. Next we show that $N \subseteq P \rtimes K$.

First we claim for every $\gamma \in \Gamma$ there is $c_\gamma \in \mathbb{C}$ so that $E_N(u_\gamma) = c_\gamma u_\gamma$. Fix $\gamma \in \Gamma$ and let $\psi(x) = u_\gamma x u_\gamma^*$, for all $x \in L(\Gamma)$. Since Λ is normal in Γ , ψ restricts to an automorphism of $P \rtimes \Lambda$. Thus for all $x \in P \rtimes \Lambda$ we have $\psi(x) u_\gamma = u_\gamma x$ and hence $\psi(x) E_N(u_\gamma) = E_N(u_\gamma) x$. This implies that $E_N(u_\gamma)^* \psi(x) = x E_N(u_\gamma)^*$ and hence $E_N(u_\gamma) E_N(u_\gamma)^* \in (P \rtimes \Lambda)' \cap (P \rtimes \Gamma) = \mathbb{C}$. Let $d = E_N(u_\gamma) E_N(u_\gamma)^*$. Note that $0 \leq d \leq 1$. If $d \neq 0$, we get that $(d^{-1/2} E_N(u_\gamma)) (d^{-1/2} E_N(u_\gamma))^* = 1$, implying that $d^{-1/2} E_N(u_\gamma) \in \mathcal{U}(N)$. Next consider $u = u_\gamma^* E_N(d^{-1/2} u_\gamma) \in \mathcal{U}(M)$. For every $x \in P \rtimes \Lambda$ we can check that

$$uxu^* = d^{-1} u_\gamma^* E_N(u_\gamma) x E_N(u_\gamma)^* u_\gamma = d^{-1} u_\gamma^* E_N(u_\gamma) E_N(u_\gamma)^* \psi(x) u_\gamma = u_\gamma^* \psi(x) u_\gamma = x.$$

Hence $d^{-1/2} u_\gamma^* E_N(u_\gamma) = u \in (P \rtimes \Lambda)' \cap (P \rtimes \Gamma) = \mathbb{C}$. Thus $E_N(u_\gamma) = c_\gamma u_\gamma$ for some $c_\gamma \in \mathbb{C}$.

The claim shows that for any $\gamma \in \Gamma$, either $E_N(u_\gamma) = 0$ or $u_\gamma \in N$. Finally, if $N \ni n = \sum_{\gamma \in \Gamma} n_\gamma u_\gamma$ is its Fourier decomposition in $P \rtimes \Gamma$, then applying E_N , we see that $n = \sum_{\gamma \in \Gamma} n_\gamma E_N(u_\gamma) \in P \rtimes K$, as desired. \square

Below we highlight a few special cases of the above theorem, which are well known in the literature.

Corollary 3.8 1. ([21], [43, Theorem 3.13]) *Let M be a II_1 factor, and let Γ be a discrete group with an outer action on M . Let N be an intermediate subalgebra, i.e. $M \subseteq N \subseteq M \rtimes \Gamma$. Then there exists a subgroup K of Γ such that $N = M \rtimes K$.*
 2. *Let Γ be an icc group, and let $\Lambda \triangleleft \Gamma$ be a normal subgroup such that $L(\Lambda)' \cap L(\Gamma) = \mathbb{C}$. Then for any intermediate subfactor $L(\Lambda) \subseteq N \subseteq L(\Gamma)$ there exists an intermediate subgroup $\Lambda \leq K \leq \Gamma$ such that $N = L(K)$.*

Proof Since $\Gamma \curvearrowright M$ is outer, $M' \cap M \rtimes \Gamma = \mathbb{C}$. Taking $\Lambda = \{e\}$, and appealing to Theorem 3.7 yields the first statement.

Taking $P = \mathbb{C}$ in Theorem 3.7 yields the second statement. \square

In the remaining part of the section we show that the strategy presented in Theorem 3.2 can be successfully used to classify all intermediate subalgebras for inclusion of von Neumann algebras arising from compact extensions. This covers a new situation which complements the case of free extensions discovered in [75, Main Theorem]. To be able to properly introduce our result we first recall the following notion of compact extension of actions on von Neumann algebras:

Definition 3.9 Let $\Gamma \curvearrowright (P_0 \subseteq P)$ be an extension of tracial von Neumann algebras. One says that $\Gamma \curvearrowright (P_0 \subseteq P)$ is a *compact extension* if there exists $\mathcal{F} \subseteq P$ satisfying the following properties:

1. $\overline{\text{span} \mathcal{F}}^{\|\cdot\|_2} = L^2(P)$;
2. for every $f \in \mathcal{F}$ and $\varepsilon > 0$ there exist $\xi_1, \xi_2, \dots, \xi_n \in L^2(P)$ such that for every $\gamma \in \Gamma$ one can find $\kappa_i(\gamma) \in P_0$, with $i = \overline{1, n}$ satisfying $\sup_{1 \leq i \leq n, \gamma \in \Gamma} \|\kappa_i(\gamma)\|_\infty < \infty$ and

$$\|\sigma_\gamma(f) - \sum_{i=1}^n \kappa_i(\gamma) \xi_i\|_2 \leq \varepsilon.$$

When $P_0 = \mathbb{C}1$ we simply say that the action $\Gamma \curvearrowright P$ is compact.

Examples Assume that $\Gamma \curvearrowright X$ is an ergodic pmp action on a probability space X and let $\Gamma \curvearrowright X_0$ be a factor such that the extension $\pi : X \rightarrow X_0$ is compact in the usual sense [28, 79]. Then it is a routine exercise to show that the corresponding von Neumann algebraic extension $\Gamma \curvearrowright (L^\infty(X_0) \subseteq L^\infty(X))$ automatically satisfies the definition above. In particular whenever $\Gamma \curvearrowright X$ is an ergodic compact pmp action then $\Gamma \curvearrowright L^\infty(X)$ is compact in the above sense.

With this definition at hand we can now introduce the main result of this section.

Theorem 3.10 *Let Γ be an icc group and let $\Gamma \curvearrowright (P_0 \subseteq P)$ be a compact extension of tracial von Neumann algebras as in Definition 3.9. Let $P_0 \rtimes \Gamma \subseteq P \rtimes \Gamma$ be the corresponding inclusion of crossed product von Neumann algebras. Then for any intermediate von Neumann subalgebra $P_0 \rtimes \Gamma \subseteq N \subseteq P \rtimes \Gamma$ there exists an intermediate von Neumann subalgebra $P_0 \subseteq Q \subseteq P$ such that $N = Q \rtimes \Gamma$.*

Proof Let $M = P \rtimes \Gamma$. Denote by $E_N : M \rightarrow N$ the canonical trace preserving conditional expectation and note that it extends to a map from $L^2(M) \rightarrow L^2(M)$ by $E_N(\hat{m}) = \widehat{E_N(m)}$. Similarly, let $E : M \rightarrow P$ be the trace preserving conditional expectation. E also extends to a map $E : L^2(M) \rightarrow L^2(M)$. For every $\xi \in L^2(M)$ let $\tilde{\xi} = E_N(\xi) - E \circ E_N(\xi)$. With these notations at hand we prove the following

Claim 3.11 *for every $\xi \in \mathcal{F}$ and every $\varepsilon > 0$ there exists a finite set $K \subset \Gamma \setminus \{e\}$ and $\eta_1, \eta_2, \dots, \eta_n \in \text{span}PK$ such that for every $\gamma \in \Gamma$ there exist $\kappa_i(\gamma) \in P_0$ with $\sup_{\gamma \in \Gamma} \|\kappa_i(\gamma)\|_\infty < \infty$ such that*

$$\|\sigma_\gamma(\tilde{\xi}) - \sum_i \kappa_i(\gamma) \eta_i\|_2 \leq \varepsilon. \quad (3.0.7)$$

Proof of Claim 3.11. First notice that since $L(\Gamma) \subseteq N$ and P is Γ -invariant then for all $\xi \in L^2(M)$ and $\gamma \in \Gamma$ we have

$$E_N(u_\gamma \xi u_\gamma^*) = u_\gamma E_N(\xi) u_\gamma^*, \text{ and } E(u_\gamma \xi u_\gamma^*) = u_\gamma E(\xi) u_\gamma^*. \quad (3.0.8)$$

Fix $\xi \in \mathcal{F}$ and $\varepsilon > 0$. Since $\Gamma \curvearrowright^\sigma P_0 \subseteq P$ is a compact extension there is a finite set $\xi_1, \xi_2, \dots, \xi_n \in L^2(P)$ such that for every $\gamma \in \Gamma$ there exist $\kappa_i(\gamma) \in P_0$ with $\sup_{\gamma \in \Gamma} \|\kappa_i(\gamma)\|_\infty < \infty$ so that

$$\|\sigma_\gamma(\xi) - \sum_i \kappa_i(\gamma) \xi_i\|_2 \leq \frac{\varepsilon}{3}. \quad (3.0.9)$$

Using (3.0.8) in combination with (3.0.9) and the basic inequalities $\|E_N(m)\|_2, \|E \circ E_N(m)\|_2 \leq \|m\|_2$, for all $m \in L^2(M)$ we get that

$$\begin{aligned} \|\sigma_\gamma(E_N(\xi)) - \sum_i \kappa_i(\gamma) E_N(\xi_i)\|_2 &\leq \frac{\varepsilon}{3}, \text{ and } \|\sigma_\gamma(E \circ E_N(\xi)) - \sum_i \kappa_i(\gamma) E \circ E_N(\xi_i)\|_2 \\ &\leq \frac{\varepsilon}{3} \end{aligned}$$

Subtracting these relations and using the triangle inequality we conclude that

$$\|\sigma_\gamma(\tilde{\xi}) - \sum_i \kappa_i(\gamma) \tilde{\xi}_i\|_2 \leq \frac{2\varepsilon}{3}. \quad (3.0.10)$$

Approximating the ξ_i 's one can find a finite set $F \subset \Gamma \setminus \{e\}$ so that $\|\tilde{\xi}_i - \eta_i\|_2 \leq \varepsilon/(3n \sup_{\gamma \in \Gamma} \|\kappa_i(\gamma)\|_\infty)$ for all $1 \leq i \leq n$. Thus $\|\sum_i \kappa_i(\gamma) \tilde{\xi}_i - \sum_i \kappa_i(\gamma) \eta_i\|_2 \leq \varepsilon/3$ and combining it with (3.0.10) we get the desired conclusion. \square

Next we prove the following

Claim 3.12 *For every $\xi \in \mathcal{F}$ we have $\tilde{\xi} = 0$.*

Proof of the Claim 3.12. Fix $\varepsilon > 0$ and $\xi \in \mathcal{F}$. Approximating $\tilde{\xi}$ there exists a finite set $K \subseteq \Gamma \setminus \{e\}$ and $r \in \text{span} PK$ such that

$$\|\tilde{\xi} - r\|_2 \leq \varepsilon. \quad (3.0.11)$$

Also by Claim 3.11 there exists a finite set $G \subset \Gamma \setminus \{e\}$ and $\eta_1, \eta_2, \dots, \eta_n \in \text{span} PG$ such that for every $\gamma \in \Gamma$ there exist $\kappa_i(\gamma) \in P_0$ with $\sup_{\gamma \in \Gamma} \|\kappa_i(\gamma)\|_\infty < \infty$ such that

$$\|\sigma_\gamma(\tilde{\xi}) - \sum_i \kappa_i(\gamma) \eta_i\|_2 \leq \varepsilon. \quad (3.0.12)$$

Since Γ is icc and $G, K \subset \Gamma \setminus \{e\}$ are finite by [19, Proposition 3.4] there exists $\lambda \in \Gamma$ such that $\lambda K \lambda^{-1} \cap G = \emptyset$; in particular, we have

$$\langle u_\lambda r u_{\lambda^{-1}}, \sum_i \kappa_i(\lambda) \eta_i \rangle = 0. \quad (3.0.13)$$

Using (3.0.11) in combination with Cauchy–Schwarz inequality, (3.0.12), and (3.0.13) we see that

$$\begin{aligned} \|\tilde{\xi}\|_2^2 &= \|u_\lambda \tilde{\xi} u_\lambda^*\|_2^2 = |\langle u_\lambda \tilde{\xi} u_{\lambda^{-1}}, u_\lambda \tilde{\xi} u_{\lambda^{-1}} \rangle| \\ &\leq \varepsilon \|\tilde{\xi}\|_2 + |\langle u_\lambda r u_{\lambda^{-1}}, u_\lambda \tilde{\xi} u_{\lambda^{-1}} \rangle| \leq \varepsilon \|\tilde{\xi}\|_2 + \varepsilon \|r\|_2 + \left| \left\langle u_\lambda r u_{\lambda^{-1}}, \sum_i \kappa_i(\lambda) \eta_i \right\rangle \right| \\ &\leq \varepsilon \|\tilde{\xi}\|_2 + \varepsilon (\|\tilde{\xi}\|_2 + \varepsilon). \end{aligned}$$

Letting $\varepsilon \searrow 0$ we get $\tilde{\xi} = 0$, as desired. \square

Claim 3.12 implies that $E_N(\xi) = E \circ E_N(\xi)$ for all $\xi \in \mathcal{F}$. Since $\text{span} \mathcal{F}$ is dense in $L^2(P)$, these two maps agree on $L^2(P) \supseteq P$. Appealing to Theorem 3.2 we conclude that $N = Q \rtimes \Gamma$, for some subalgebra $P_0 \subseteq Q \subseteq P$. \square

Remarks After the first draft of the paper appeared on the ArXiv, we were kindly informed by Y. Jiang and A. Skalski that they had subsequently obtained a characterization of intermediate subfactors N satisfying $L(\Gamma) \subseteq N \subseteq L^\infty(X) \rtimes \Gamma$, with $\Gamma \curvearrowright X$ a profinite action, in an independent manner (see [48, Cor 3.11]).

Corollary 3.13 *Let Γ be an icc group and let $\Gamma \curvearrowright A$ and $\Gamma \curvearrowright B$ be trace preserving actions with $\Gamma \curvearrowright B$ compact, and where A is a II_1 factor. Consider the diagonal action $\Gamma \curvearrowright A \bar{\otimes} B$ and let $(A \bar{\otimes} B) \rtimes \Gamma$ be the corresponding crossed product von Neumann algebra. Then for any von Neumann subalgebra $A \rtimes \Gamma \subseteq N \subseteq (A \bar{\otimes} B) \rtimes \Gamma$ one can find a Γ -invariant von Neumann subalgebra $C \subseteq B$ so that $N = (A \bar{\otimes} C) \rtimes \Gamma$.*

Proof Since $\Gamma \curvearrowright B$ is compact one can see that $\Gamma \curvearrowright (A \subseteq A \rtimes B)$ is a compact extension and hence the conclusion follows from Theorem 3.10. \square

We end this section with an immediate application of Theorem 3.10 to the study of finite index subfactors. More specifically, we show that Theorem 3.10 can be used

effectively to completely describe all intermediate subfactors $L(\Gamma) \subseteq N \subseteq L^\infty(X) \rtimes \Gamma$ with $[N : L(\Gamma)] < \infty$ for any ergodic action $\Gamma \curvearrowright X$ of any icc group Γ .

Corollary 3.14 *Let Γ be an icc group and let $\Gamma \curvearrowright X$ be an ergodic action. Let $M = L^\infty(X) \rtimes \Gamma$ denote the corresponding group measure space von Neumann algebra. Then the following hold:*

1. *Suppose $L(\Gamma) \subseteq N \subseteq L^\infty(X) \rtimes \Gamma$ is an intermediate von Neumann subalgebra so that $N \subseteq \mathcal{QN}_M(L(\Gamma))''$. Then there exists a factor $\Gamma \curvearrowright X_0$ of $\Gamma \curvearrowright X$ such that $N = L^\infty(X_0) \rtimes \Gamma$.*
2. *For any intermediate subfactor $L(\Gamma) \subseteq N \subseteq L^\infty(X) \rtimes \Gamma$ with $[N : L(\Gamma)] < \infty$ there is a finite, transitive factor $\Gamma \curvearrowright X_0$ of $\Gamma \curvearrowright X$ such that $N = L^\infty(X_0) \rtimes \Gamma$; in particular $[N : L(\Gamma)] \in \mathbb{N}$. Thus for any subfactors $L(\Gamma) \subseteq N_1 \subseteq N_2 \subseteq L^\infty(X) \rtimes \Gamma$, with either $[N_1 : L(\Gamma)] < \infty$ or $\Gamma \curvearrowright X$ compact, we have $[N_2 : N_1] \in \mathbb{N} \cup \{\infty\}$.*
3. *If Γ has no proper finite index subgroups (e.g. Γ is simple) then there are no nontrivial intermediate subfactors $L(\Gamma) \subseteq N \subseteq L^\infty(X) \rtimes \Gamma$ with $[N : L(\Gamma)] < \infty$.*

Proof 1. Let $\Gamma \curvearrowright X_c$ be a maximal compact factor of $\Gamma \curvearrowright X$ and using [36, Theorem 6.9] we have that $\mathcal{QN}_M(L(\Gamma))'' = L^\infty(X_c) \rtimes \Gamma$. Altogether these show that $L(\Gamma) \subseteq N \subseteq L^\infty(X_c) \rtimes \Gamma$. Then the desired conclusion follows directly from Theorem 3.10.

2. Since $[N : L(\Gamma)] < \infty$ then N admits a finite left (and also a finite right) Pimsner-Popa basis over $L(\Gamma)$ and hence $N \subseteq \mathcal{QN}_M(L(\Gamma))''$. By part 1. there is a factor $\Gamma \curvearrowright X_0$ of $\Gamma \curvearrowright X$ such that $N = L^\infty(X_0) \rtimes \Gamma$. As Γ is icc and N is a factor we also have that $\Gamma \curvearrowright L^\infty(X_0)$ is ergodic. Since N admits a finite Pimsner-Popa basis over $L(\Gamma)$ then by Proposition 2.6 it follows that $\Gamma \curvearrowright L^\infty(X_0)$ is a transitive action. In particular X_0 is a finite probability space and $\Gamma \curvearrowright X_0$ is transitive. If $\Gamma_x \leq \Gamma$ is the stabilizer of an $x \in X_0$ one can also check that $[N : L(\Gamma)] = |X_0| = |\Gamma/\Gamma_x| \in \mathbb{N}$. The rest of the statement follows easily.
3. Assume that $[N : L(\Gamma)] < \infty$. From the proof of part 2. we have $N = L^\infty(X_0) \rtimes \Gamma$ where $\Gamma \curvearrowright X_0$ is an action on a finite set X_0 and also $[N : L(\Gamma)] = |X_0| = |\Gamma/\Gamma_x|$ where Γ_x is the stabilizer of $x \in X_0$. Since Γ has no nontrivial finite index subgroups then $\Gamma = \Gamma_x$ and hence $N = L(\Gamma)$.

□

Final remarks. The previous corollary also holds for intermediate subalgebras $L^\infty(X) \rtimes \Gamma \subseteq N \subset L^\infty(Y) \rtimes \Gamma$ with $[N : L^\infty(X) \rtimes \Gamma] < \infty$ for von Neumann algebras arising from extensions $\Gamma \curvearrowright L^\infty(X) \subseteq L^\infty(Y)$ of icc groups Γ . The proof is essentially the same as the one presented in Corollary 3.14 with the only difference that we use Theorem 2.5 instead of [36, Theorem 6.9]. Also, parts 1. and 2. hold for any von Neumann algebra N which admits a finite Pimsner-Popa basis over $L(\Gamma)$, if we use the Pimsner-Popa index [61] instead of Jones index [44].

In connection with the previous problems one may attempt to describe the subfactors of group von Neumann algebras $N \subseteq L(\Gamma)$ that are normalized by the Γ itself, i.e. $\Gamma \subset \mathcal{N}_{L(\Gamma)}(N)$. Very recently this problem was considered in [1] where a

complete description was obtained for Γ lattices in higher rank simple Lie groups via a noncommutative version of Margulis' normal subgroup theorem; in turn this was obtained using character rigidity techniques introduced [20,59]. In this work we make further progress on this question for many new families of groups Γ complementary to the ones from [1]. In particular, we show that under additional conditions on the relative commutant $N' \cap L(\Gamma)$ (e.g. finite dimensional) these subfactors are always "commensurable" with von Neumann algebras arising from the normal subgroups of Γ (Theorem 3.15). Moreover, in the case of all exact acylindrically hyperbolic groups [23,54], all nonamenable groups with positive first L^2 -Betti number, and all lattices in product of trees the same holds without any a priori assumptions on $N' \cap L(\Gamma)$ (see Theorem 3.16, Corollary 3.17, and part 3 in Theorem 3.15).

Theorem 3.15 *Let Γ be a countable discrete group and let $N \subset L(\Gamma)$ be a subfactor such that $\Gamma \subset \mathcal{N}_{L(\Gamma)}(N)$. Then there exists a normal subgroup $\Lambda \triangleleft \Gamma$ such that $N \subseteq L(\Lambda) \subseteq N \vee (N' \cap L(\Gamma))$. Moreover, we have the following*

- 1) *If $N' \cap L(\Gamma)$ is finite dimensional then the inclusions $N \subseteq L(\Lambda) \subseteq N \vee N' \cap L(\Gamma)$ have finite index; in particular, when $N' \cap L(\Gamma) = \mathbb{C}1$ then $N = L(\Lambda)$.*
- 2) *If $L(\Gamma)$ is solid¹ then either N is an amenable factor or the inclusions $N \subseteq L(\Lambda) \subseteq N \vee N' \cap L(\Gamma)$ have finite index. Moreover if $L(\Gamma)$ is strongly solid² then either N is finite dimensional or the inclusions $N \subseteq L(\Lambda) \subseteq N \vee N' \cap L(\Gamma)$ have finite index.*
- 3) *Γ be a simple group such that $L(\Gamma)$ is a prime factor, e.g. Burger-Mozes group [7], Camm's group [8] or Bhattacharjee's group [3] (see [10]). Then N is either finite dimensional or $[L(\Gamma) : N] < \infty$.*

Proof Denote by Σ the set of all $\gamma \in \Gamma$ for which there is $y \in \mathcal{U}(N)$ such that $\tau(yu_\gamma) \neq 0$. Note that Σ coincides with the set of all $\gamma \in \Gamma$ such that $E_N(u_\gamma) \neq 0$.

Fix $\gamma \in \Sigma$ and denote by $\phi_\gamma : N \rightarrow N$ the automorphism given by $\phi_\gamma(x) = u_\gamma x u_{\gamma^{-1}}$ for all $x \in N$. Thus $\phi_\gamma(x)u_\gamma = u_\gamma x$ and applying the expectation E_N we also have $\phi_\gamma(x)E_N(u_\gamma) = E_N(u_\gamma)x$ for all $x \in N$. These two relations give that $\phi_\gamma(x)E_N(u_\gamma)u_{\gamma^{-1}} = E_N(u_\gamma)xu_{\gamma^{-1}} = E_N(u_\gamma)u_{\gamma^{-1}}\phi_\gamma(x)$ for all $x \in N$; in particular, $a_\gamma := E_N(u_\gamma)u_{\gamma^{-1}} \in N' \cap L(\Gamma)$. Thus

$$E_N(u_\gamma) = a_\gamma u_\gamma. \quad (3.0.14)$$

Thus $E_N(u_\gamma)E_N(u_{\gamma^{-1}}) = a_\gamma a_\gamma^*$. Applying the expectation E_N and using $E_N \circ E_{N' \cap L(\Gamma)} = \tau$ (since N is a factor) we get $E_N(u_\gamma)E_N(u_{\gamma^{-1}}) = \tau(a_\gamma a_\gamma^*)1$. As $a_\gamma \neq 0$ one can find a unitary $b_\gamma \in N$ so that $E_N(u_\gamma) = \|a_\gamma\|_2 b_\gamma$. Combining with (3.0.14) we get $\|a_\gamma\|_2 b_\gamma = a_\gamma u_\gamma$ and hence $u_\gamma = \|a_\gamma\|_2 a_\gamma^* b_\gamma$. In particular we have $\|a_\gamma\|_2 a_\gamma^* \in \mathcal{U}(N' \cap L(\Gamma))$ and hence $u_\gamma \in \mathcal{U}(N)\mathcal{U}(N' \cap L(\Gamma)) \subseteq N \vee (N' \cap L(\Gamma))$. Let Λ be the set of all $\gamma \in \Gamma$ such that $u_\gamma = x_\gamma y_\gamma$, where $x_\gamma \in \mathcal{U}(N)$ and $y_\gamma \in \mathcal{U}(N' \cap L(\Gamma))$. Observe that $\Lambda \triangleleft \Gamma$ is in fact a normal subgroup. The previous relations show that $\Sigma \subseteq \Lambda$ and by the definition of Σ we have that $N \subseteq L(\Lambda)$. Since $L(\Lambda) \subseteq N \vee (N' \cap L(\Gamma))$ canonically, the first part of the conclusion follows.

¹ For every diffuse $A \subseteq L(\Gamma)$ the relative commutant $A' \cap L(\Gamma)$ is amenable

² For every diffuse amenable $A \subseteq L(\Gamma)$ the normalizer $\mathcal{N}_{L(\Gamma)}(A)''$ is amenable

Since $N' \cap L(\Gamma)$ is finite dimensional then $N \vee N' \cap L(\Gamma)$ admits left (and right) finite Pimsner-Popa basis over N and 1) follows.

If N is nonamenable, then $N' \cap L(\Gamma)$ is finite dimensional, as $L(\Gamma)$ is solid. The rest of 2) follows easily from 1).

If Γ is simple, then $\Lambda = \Gamma$, as Λ is a normal subgroup of Γ ; hence, $N \vee N' \cap L(\Gamma) = L(\Gamma)$. Since $L(\Gamma)$ is prime, this further implies that either N or $N' \cap L(\Gamma)$ is finite dimensional, and thus 3) follows from 1). \square

Next we show that whenever Γ is a “negatively curved” group then all subfactors $N \subseteq L(\Gamma)$ normalized by Γ are commensurable to subalgebras $L(\Lambda)$ arising from normal subgroups $\Lambda \triangleleft \Gamma$. Our proof relies heavily on the deformation/rigidity techniques for array/quasi-cocycles on groups that were introduced and studied in [13, 17–19]. We advise the reader to consult these references beforehand.

Let $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{H})$ be an orthogonal representation. Let $\mathcal{QH}_{as}^1(\Gamma, \pi)$ be the set of all unbounded *quasicocycles* into π , i.e. unbounded maps $q : \Gamma \rightarrow \mathcal{H}$ so that $d(q) := \sup_{\gamma, \lambda \in \Gamma} \|q(\gamma\lambda) - q(\gamma) - \pi_\gamma(q(\lambda))\| < \infty$. When the defect $d(q) = 0$ the set $\mathcal{QH}_{as}^1(\Gamma, \pi)$ is nothing but the first cohomology group $H^1(\Gamma, \pi)$.

Theorem 3.16 *Let $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{H})$ be an orthogonal mixing representation that is weakly contained in the left regular representation of Γ . Assume one of the following holds: a) Γ is exact and $\mathcal{QH}_{as}^1(\Gamma, \pi) \neq \emptyset$, or b) $H^1(\Gamma, \pi) \neq 0$. Let $N \subseteq L(\Gamma)$ be a subfactor satisfying $\Gamma \subset \mathcal{N}_{L(\Gamma)}(N)$. Then there is a normal subgroup $\Lambda \triangleleft \Gamma$ so that $N \subseteq L(\Lambda) \subseteq N \vee N' \cap L(\Lambda)$ and one of the following holds:*

1. N is finite dimensional, or
2. Λ is infinite amenable, or
3. $[L(\Lambda) : N] < \infty$.

Proof Let $M = L(\Gamma)$. By Theorem 3.15 there is $\Lambda \triangleleft \Gamma$, such that $N \subseteq L(\Lambda) \subseteq N \vee (N' \cap M)$ and moreover from its proof it follows that for every $\gamma \in \Lambda$ there are unitaries $a_\gamma \in N$ and $b_\gamma \in N' \cap L(\Lambda)$ so that

$$u_\gamma = a_\gamma b_\gamma. \quad (3.0.15)$$

Also since N is a factor, using Ge’s tensor splitting result (Theorem 3.3) we also get that

$$L(\Lambda) = N \vee (N' \cap L(\Lambda)). \quad (3.0.16)$$

Assume that Λ is nonamenable. Let $q \in \mathcal{QH}_{as}^1(\Gamma, \pi)$ and consider the restriction $q|_\Lambda$. One can easily see that the representation $\pi|_\Lambda^{\oplus \infty}$ is still mixing and is weakly contained in $\ell^2(\Lambda)$. Moreover since $\Lambda \triangleleft \Gamma$ is normal and the representation is mixing it follows that $q|_\Lambda$ is unbounded and hence $q|_\Lambda \in \mathcal{QH}_{as}^1(\Lambda, \pi|_\Lambda^{\oplus \infty})$. Thus by [13, Corollary 7.2] it follows that the finite conjugacy radical $FC(\Lambda)$ of Λ is finite and hence $\mathcal{Z}(L(\Lambda))$ is finite dimensional.

Assume that $N' \cap L(\Lambda)$ is amenable. If it is finite dimensional then (3.0.16) already implies 3. If not then there is a projection $0 \neq z \in \mathcal{Z}(N' \cap L(\Lambda)) = \mathcal{Z}(\Lambda)$ such that $(N' \cap L(\Lambda))z$ is isomorphic to the hyperfinite factor. Since Λ is nonamenable N

is also nonamenable. Thus $L(\Lambda)z$ has property (Gamma) and there is a sequence of $(u_n)_n$ of unitaries in $(N' \cap L(\Lambda))z$ such that $u^\omega := (u_n)_n \in (L(\Lambda)' \cap L(\Lambda)^\omega)z$ and $u^\omega \perp L(\Lambda)$; here ω is a free ultrafilter on \mathbb{N} . On the other hand using [19, Theorem 4.1] we get that $L(\Lambda)' \cap L(\Lambda)^\omega \subseteq L(\Lambda)$. Thus $u \perp u$, which is a contradiction.

Now assume that $N' \cap L(\Lambda)$ is nonamenable. If N is amenable then a similar argument as before shows that N is finite dimensional leading to I . Thus for the rest of the proof we assume that N and $N' \cap L(\Lambda)$ are nonamenable and we will show this leads to a contradiction.

Let $P = L(\Lambda)$. Following [17, Sect. 2.3] consider $V_t : L^2(\tilde{P}) \rightarrow L^2(\tilde{P})$ be the Gaussian deformation corresponding to the quasicocycle $q|_\Lambda \in \mathcal{QH}_{as}^1(\Lambda, \pi|_\Lambda^{\oplus \infty})$ where the supralgebra $P \subset \tilde{P}$ is the Gaussian dilation. Let $e_P : \tilde{P} \rightarrow P$ denote the orthogonal projection. Since $N' \cap L(\Lambda)$ is nonamenable there exists a nonzero projection $0 \neq p \in N' \cap L(\Lambda)$ such that $(N' \cap L(\Lambda))p$ has no amenable direct summand. Thus applying a spectral gap argument a la Popa (see for instance [17, Theorem 3.2]), we obtain that

$$\lim_{t \rightarrow 0} \left(\sup_{x \in (N)_1 p} \|e_P^\perp V_t(x)\|_2 \right) = 0, \quad \text{and} \quad \lim_{t \rightarrow 0} \left(\sup_{x \in (N' \cap L(\Lambda))_1} \|e_P^\perp V_t(x)\|_2 \right) = 0. \quad (3.0.17)$$

Fix $\varepsilon > 0$. Thus, using the transversality property from [17, Lemma 2.8], relations (3.0.17) and a simple calculation show that there exist $C, D > 0$ satisfying

$$\|P_{B_C}(x) - x\|_2 < \varepsilon \text{ for all } x \in \mathcal{U}(N)p, \quad \|P_{B_D}(y) - y\|_2 < \varepsilon \text{ for all } y \in \mathcal{U}(N' \cap P). \quad (3.0.18)$$

Here for every constant $C \geq 0$ we denoted by $B_C = \{\lambda \in \Lambda : \|q(\lambda)\|_{\mathcal{H}} \leq C\}$ and by P_{B_C} the orthogonal projection onto the Hilbert subspace of $L^2(\Lambda)$ spanned by B_C . Since by (3.0.15) we have $u_\gamma = a_\gamma b_\gamma$ then (3.0.18) imply

$$\|P_{B_C}(u_\gamma b_\gamma^* p) - u_\gamma b_\gamma^* p\|_2 < \varepsilon \quad \text{and} \quad \|P_{B_D}(b_\gamma^* p) - b_\gamma^* p\|_2 < \varepsilon \text{ for all } \gamma \in \Lambda. \quad (3.0.19)$$

Thus using triangle inequality, for all $\gamma \in \Lambda$, we also have

$$\|P_{B_C}(u_\gamma b_\gamma^* p) - u_\gamma P_{B_D}(b_\gamma^* p)\|_2 \leq \|P_{B_C}(u_\gamma p b_\gamma^*) - u_\gamma p b_\gamma^*\|_2 + \|P_{B_D}(b_\gamma^* p) - b_\gamma^* p\|_2 < 2\varepsilon. \quad (3.0.20)$$

Since $q|_\Lambda$ is unbounded, there exists $\gamma_0 \notin B_{C+D+3d(q|_\Lambda)}$. Also the quasicocycle relation and the triangle inequality show that $B_C B_D^{-1} \subseteq B_{C+D+3d(q|_\Lambda)}$ and thus $\gamma_0 \notin B_C B_D^{-1}$. Hence $\langle P_{B_C}(\xi), u_{\gamma_0} P_{B_D}(\eta) \rangle = 0$ for all $\xi, \eta \in L^2(\Lambda)$. Thus using inequalities 3.0.20 for $\gamma = \gamma_0$ and (3.0.19) we see that $4\varepsilon^2 \geq \|P_{B_C}(u_{\gamma_0} b_{\gamma_0}^* p) - u_{\gamma_0} P_{B_D}(b_{\gamma_0}^* p)\|_2^2 = \|P_{B_C}(u_{\gamma_0} b_{\gamma_0}^* p)\|_2^2 + \|u_{\gamma_0} P_{B_D}(b_{\gamma_0}^* p)\|_2^2 \geq \|u_{\gamma_0} b_{\gamma_0}^* p\|_2^2 + \|b_{\gamma_0}^* p\|_2^2 - 2\varepsilon^2 = 2\|p\|_2^2 - 2\varepsilon^2$. Thus $\|p\|_2^2 \leq 3\varepsilon^2$, which contradicts $p \neq 0$ when $\varepsilon \rightarrow 0$. This completes the proof of the first part of the theorem in the the case when q is a quasicocycle with $d(q) \neq 0$. When $d(q) = 0$ i.e. q is a cocycle the same proof works with the only difference that to derive the convergence 3.0.18, instead of using [17, Theorem 3.2] (which requires exactness of Γ) one can use the spectral gap arguments as in [58] or [78]. \square

When combined with results in geometric group theory the previous result leads to the following

Corollary 3.17 *Let Γ be a nonamenable group that is either exact and acylindrically hyperbolic or has positive first L^2 -Betti number. Let $N \subseteq L(\Gamma)$ be a subfactor such that $\Gamma \subset \mathcal{N}_{L(\Gamma)}(N)$. Then there is a nonamenable normal subgroup $\Lambda \triangleleft \Gamma$ so that $N \subseteq L(\Lambda) \subseteq N \vee N' \cap L(\Lambda)$ and one of the following holds:*

1. N is finite dimensional, or
2. $[L(\Lambda) : N] < \infty$.

Proof From [60] and [34] it follows that these families always have $\mathcal{QH}_{\text{as}}^1(\Gamma, \ell^2(\Gamma)) \neq \emptyset$. Hence the result follows directly from the previous theorem as both classes of nonamenable acylindrically hyperbolic groups and nonamenable groups with positive first L^2 -Betti number have finite amenable radical. \square

4 Actions that satisfy Neshveyev–Størmer rigidity

If Γ, Λ are abelian (or more generally amenable) groups, and $\Gamma \curvearrowright X, \Lambda \curvearrowright Y$ are free, ergodic, pmp actions, then $L^\infty(X) \rtimes \Gamma$ and $L^\infty(Y) \rtimes \Lambda$ are isomorphic to the hyperfinite II_1 factor \mathcal{R} . However, Neshveyev and Størmer proved that if we assume that $\Theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$ is an $*$ -isomorphism such that $\Theta(L^\infty(X))$ is unitarily conjugate to $L^\infty(Y)$ and $\Theta(L(\Gamma)) = L(\Lambda)$ then the actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are conjugate [52, Theorem 4.1]. Motivated by this group action conjugacy criterion, they further conjectured the following: if Γ, Λ are abelian groups, $\Gamma \curvearrowright X, \Lambda \curvearrowright Y$ are free, weak mixing, pmp actions and $\Theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$ is a $*$ -isomorphism satisfying $\Theta(L(\Gamma)) = L(\Lambda)$ then $\Gamma \curvearrowright X$ is conjugate to $\Lambda \curvearrowright Y$ [52, Conjecture]. Shortly after, using his influential deformation/rigidity theory Popa was able to prove the following striking result: if Γ, Λ are any countable groups, $\Gamma \curvearrowright^\sigma X, \Lambda \curvearrowright^\rho Y$ are free, ergodic actions, with σ Bernoulli (or more generally clustering), and $\Theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$ is an $*$ -isomorphism such that $\Theta(L(\Gamma)) = L(\Gamma)$ then $\Gamma \curvearrowright X$ is conjugate to $\Lambda \curvearrowright Y$, [67, Theorem 5.2]. In particular, this settled Neshveyev–Størmer conjecture for Bernoulli actions. Popa also showed that the study of the Neshveyev–Størmer rigidity question in the context of icc property (T) groups eventually leads to his remarkable proof of the *group measure space* version of Connes’ rigidity conjecture, [67, Theorem 0.1]. All these results motivate the study of the following *generalized Neshveyev–Størmer rigidity question*.

Question 4.1 *Let Γ and Λ be icc countable discrete groups and let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be free, ergodic, pmp actions. Assume that there is a $*$ -isomorphism $\Theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$ so that $\Theta(L(\Gamma)) = L(\Lambda)$. Under what conditions on $\Gamma \curvearrowright X$ can we conclude that $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are conjugate?*

Informally, the generalized Neshveyev–Størmer rigidity question asks, under what conditions can $\Gamma \curvearrowright X$ be completely recovered from the irreducible subfactor inclusion $L(\Gamma) \subset L^\infty(X) \rtimes \Gamma$.

Using existing literature, one can see that the generalized Neshveyev–Størmer rigidity phenomenon holds for the following classes of actions: all Bernoulli actions of icc groups, [67]; all W^* -superrigid actions, [6, 11, 12, 14, 17, 18, 25, 35, 38, 58, 71–73, 78]; all weak mixing \mathcal{C}_{gms} -superrigid actions, [67, Theorem 5.1]; and all mixing Gaussian actions [6, Corollary 3.9].

In this section we provide new classes of actions satisfying the generalized Neshveyev–Størmer question, most notably, all actions that appear as (nontrivial) mixing extension of free distal actions (see Theorem 4.8).

4.1 A criterion for conjugacy of group actions

Within the class of icc groups, we further generalize Neshveyev–Størmer’s aforementioned criterion for conjugacy of group actions on probability spaces by completely removing the weak mixing assumption of $\Gamma \curvearrowright X$ (see Theorem 4.5). In this context our result also generalizes [67, Theorem 0.7] as it covers many new actions (e.g. compact) that were not previously analyzed in this context. Our proof relies on the usage of the notion of height of elements in group von Neumann algebras introduced in [42]. In order to prove our result we need to establish first a few preliminary technical results on height of elements in group von Neumann algebras, [42, Definition 3.1].

Definition 4.2 A trace preserving action $\Gamma \curvearrowright^\sigma A$ on a finite von Neumann algebra A is called properly outer over the center of A if for every $\gamma \neq 1$ and every $0 \neq z \in \mathcal{Z}(A)$ such that $\sigma_\gamma(z) = z$ the automorphism $\sigma_\gamma : Az \rightarrow Az$ is not inner. When A is abelian this amounts to the usual freeness of the action $\Gamma \curvearrowright A$.

The following lemma is a basic generalization of Dye’s famous result in the case of group measure space von Neumann algebras. For readers’ convenience we include a short proof.

Lemma 4.3 Let $\Gamma \curvearrowright^\sigma A$ and $\Lambda \curvearrowright^\alpha B$ be properly outer actions. Also let $\Theta : A \rtimes \Gamma \rightarrow B \rtimes \Lambda$ be a $*$ -isomorphism such that $\Theta(A) = B$. Fix $\gamma \in \Gamma$ and let $\Theta(u_\gamma) = \sum_{\lambda \in \Lambda} a_\lambda v_\lambda$ be the Fourier decomposition of $\Theta(u_\gamma)$ in $B \rtimes \Lambda$. Then there are mutually orthogonal projections $\{e_\lambda\}_{\lambda \in \Lambda} \subset \mathcal{Z}(B)$ and unitaries $\{x_\lambda\}_{\lambda \in \Lambda} \subset B$ so that $a_\lambda = e_\lambda x_\lambda$ for all $\lambda \in \Lambda$. Also, $\sum_{\lambda \in \Lambda} e_\lambda = 1$.

Proof To ease our presentation we assume that $A = B$. Thus, $M := A \rtimes \Gamma = A \rtimes \Lambda$. Fix $\gamma \in \Gamma$ and let $u_\gamma = \sum_{\lambda \in \Lambda} a_\lambda v_\lambda$. Since $\sigma_\gamma(a)u_\gamma = u_\gamma a$ for all $a \in A$ then $\sigma_\gamma(a) \sum_{\lambda \in \Lambda} a_\lambda v_\lambda = \sum_{\lambda \in \Lambda} a_\lambda v_\lambda a = \sum_{\lambda \in \Lambda} a_\lambda \alpha_\lambda(a) v_\lambda$. Thus $\sigma_\gamma(a) a_\lambda = a_\lambda \alpha_\lambda(a)$ for all $a \in A$ and $\lambda \in \Lambda$. If $a_\lambda = w_\lambda |a_\lambda|$ is the polar decomposition of a_λ this further implies that for all $a \in A$ and $\lambda \in \Lambda$ we have

$$\sigma_\gamma(a) w_\lambda = w_\lambda \alpha_\lambda(a). \quad (4.1.1)$$

Hence $e_\lambda = w_\lambda w_\lambda^* \in \mathcal{Z}(B)$. Let $x_\lambda \in \mathcal{U}(A)$ such that $w_\lambda = x_\lambda e_\lambda$. Fix $\lambda \neq \mu \in \Lambda$. Using (4.1.1), for all $a \in A$ we have $\sigma_\gamma(a) e_\lambda = x_\lambda \alpha_\lambda(a) x_\lambda^* e_\lambda$ and $\sigma_\gamma(a) e_\mu = x_\mu \alpha_\mu(a) x_\mu^* e_\mu$. Thus $\sigma_\gamma(a) e_\lambda e_\mu = x_\lambda \alpha_\lambda(a) x_\lambda^* e_\lambda e_\mu = x_\mu \alpha_\mu(a) x_\mu^* e_\lambda e_\mu$. Letting $a = \alpha_{\mu^{-1}}(b)$ we get $\alpha_{\lambda\mu^{-1}}(b) e_\lambda e_\mu = x_\lambda^* x_\mu b e_\lambda e_\mu x_\mu^* x_\lambda$ for all $b \in A$. Also one can easily

check that $\alpha_{\lambda\mu^{-1}}(e_\lambda e_\mu) = e_\lambda e_\mu$. Since $\Lambda \curvearrowright^\alpha A$ is properly outer and $\lambda\mu^{-1} \neq 1$, we get $e_\lambda e_\mu = 0$; thus for all $\lambda \neq \mu$ we have $e_\lambda e_\mu = 0$.

As $u_\gamma \in \mathcal{U}(M)$ we have $1 = \sum_\lambda a_\lambda^* a_\lambda = |a_\lambda|^2 \leq \sum_\lambda e_\lambda \leq 1$. Thus $|a_\lambda|^2 = e_\lambda$ and hence $|a_\lambda| = e_\lambda$ for all $\lambda \in \Lambda$; moreover $\sum_\lambda e_\lambda = 1$. \square

With this result at hand we are now ready to prove the first technical result needed in the proof of Theorem 4.5.

Theorem 4.4 *Let $\Gamma \curvearrowright^\sigma A$ and $\Lambda \curvearrowright^\alpha B$ be properly outer actions. Assume that $\Theta : A \rtimes \Gamma \rightarrow B \rtimes \Lambda$ is a $*$ -isomorphism satisfying the following conditions:*

- i) $\Theta(A) = B$, and
- ii) *there exist $1 > \varepsilon > 0$ and a finite subset $K \subseteq B \rtimes \Lambda$ such that for every $\gamma \in \Gamma$ we have*

$$\|\Theta(u_\gamma) - \sum_{a,b,c,d \in K} a E_{L(\Lambda)}(b(\Theta(u_\gamma))c)d\|_2 \leq \varepsilon. \quad (4.1.2)$$

Then one can find $D > 0$ and finite subset $F \subseteq B$ such that for every $\gamma \in \Gamma$ there exists $\lambda \in \Lambda$ satisfying $\max_{b,c \in F} |\tau(b^ \Theta(u_\gamma) c v_\lambda)| \geq D > 0$.*

Proof As before assume that $A = B$ and notice that $M = A \rtimes \Gamma = A \rtimes \Lambda$. Let $1 > \varepsilon > 0$ and $K \subseteq M$ a finite subset such that for all $g \in \Gamma$ we have

$$\|u_\gamma - \sum_{z,t,w,r \in K} z E_{L(\Lambda)}(t u_\gamma w) r\|_2 \leq \varepsilon. \quad (4.1.3)$$

Approximating the elements of K via Kaplansky's density theorem we can assume there are finite subsets $F \subseteq A$, $G \subseteq \Lambda$ (some elements could be repeated finitely many times!) so that for all $\gamma \in \Gamma$ we have

$$\varepsilon \geq \|u_\gamma - \sum_{\substack{a,b,c,d \in F \\ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in G}} a v_{\lambda_1} E_{L(\Lambda)}(v_{\lambda_2}^{-1} b^* u_\gamma c v_{\lambda_3}) v_{\lambda_4}^{-1} d^*\|_2^2.$$

For the simplicity of writing we convene for the rest of the proof that $\sum_{\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in G} a, b, c, d \in F = \sum_{F, G}$. Thus for all $\gamma \in \Gamma$ we have

$$\begin{aligned} \varepsilon &\geq \|u_\gamma - \sum_{\lambda \in \Lambda} \sum_{F, G} a \tau(v_{\lambda_1 \lambda_2}^{-1} b^* u_\gamma c v_{\lambda_3 \lambda_4}^{-1} v_\lambda d^*)\|_2^2 \\ &\geq \sum_{\lambda \in \Lambda} (\|E_A(u_\gamma v_{\lambda}^{-1}) - \sum_{F, G} \tau(v_{\lambda_1 \lambda_2}^{-1} b^* u_\gamma c v_{\lambda_3 \lambda_4}^{-1} v_\lambda) a \alpha_\lambda(d^*)\|_2^2). \end{aligned} \quad (4.1.4)$$

By the previous lemma 4.3 we have that $E_A(u_\gamma v_\lambda) = e_\lambda x_\lambda$ with $x_\lambda \in \mathcal{U}(A)$ and $e_\lambda \in \mathcal{Z}(A)$. Then using $\| |f| - |g| \|_2 \leq \|f - g\|_2$ for $f, g \in A$ we see that the last

quantity in (4.1.4) is larger than

$$\begin{aligned}
 &\geq \sum_{\lambda \in \Lambda} \left(\left\| e_\lambda - \left| \sum_{F,G} \tau(v_{\lambda_1 \lambda_2^{-1}} b^* u_\gamma c v_{\lambda_3 \lambda_4^{-1} \lambda^{-1}}) a \alpha_\lambda(d^*) \right\|_2^2 \right) \\
 &= \sum_{\lambda \in \Lambda} \left(\|e_\lambda\|_2^2 + \left\| \sum_{F,G} \tau(v_{\lambda_1 \lambda_2^{-1}} b^* u_\gamma c v_{\lambda_3 \lambda_4^{-1} \lambda^{-1}}) a \alpha_\lambda(d^*) \right\|_2^2 \right. \\
 &\quad \left. - 2 \operatorname{Re} \tau \left(e_\lambda \left| \sum_{F,G} \tau(v_{\lambda_1 \lambda_2^{-1}} b^* u_\gamma c v_{\lambda_3 \lambda_4^{-1} \lambda^{-1}}) a \alpha_\lambda(d^*) \right| \right) \right). \quad (4.1.5)
 \end{aligned}$$

From Lemma 4.3 we also have that $\sum_{\lambda \in \Lambda} e_\lambda = 1$ and hence $\sum_{\lambda \in \Lambda} \|e_\lambda\|_2^2 = 1$. Combining this with (4.1.4) and (4.1.5) we get

$$\begin{aligned}
 \sum_{\lambda \in \Lambda} \varepsilon \|e_\lambda\|_2^2 &\geq \sum_{\lambda} \|e_\lambda\|_2^2 + \left\| \sum_{F,G} \tau(v_{\lambda_1 \lambda_2^{-1}} b^* u_\gamma c v_{\lambda_3 \lambda_4^{-1} \lambda^{-1}}) a \alpha_\lambda(d^*) \right\|_2^2 \\
 &\quad - 2 \operatorname{Re} \tau \left(e_\lambda \left| \sum_{F,G} \tau(v_{\lambda_1 \lambda_2^{-1}} b^* u_\gamma c v_{\lambda_3 \lambda_4^{-1} \lambda^{-1}}) a \alpha_\lambda(d^*) \right| \right).
 \end{aligned}$$

Hence, for every $\gamma \in \Gamma$ there exists $\lambda \in \Lambda$ such that $e_\lambda \neq 0$ satisfies

$$\begin{aligned}
 \varepsilon \|e_\lambda\|_2^2 &\geq \|e_\lambda\|_2^2 + \left\| \sum_{F,G} \tau(v_{\lambda_1 \lambda_2^{-1}} b^* u_\gamma c v_{\lambda_3 \lambda_4^{-1} \lambda^{-1}}) a \alpha_\lambda(d^*) \right\|_2^2 \\
 &\quad - 2 \operatorname{Re} \tau \left(e_\lambda \left| \sum_{F,G} \tau(v_{\lambda_1 \lambda_2^{-1}} b^* u_\gamma c v_{\lambda_3 \lambda_4^{-1} \lambda^{-1}}) a \alpha_\lambda(d^*) \right| \right). \quad (4.1.6)
 \end{aligned}$$

Using (4.1.6) and the operatorial inequality $(\sum_{i=1}^n x_i)^* (\sum_{i=1}^n x_i) \leq 2^{n-1} \sum_{i=1}^n x_i^* x_i$ we get

$$\begin{aligned}
 &(2 - 2\sqrt{\varepsilon}) \|e_\lambda\|_2^2 \\
 &\leq (1 - \varepsilon) \|e_\lambda\|_2^2 + \left\| \sum_{F,G} \tau(v_{\lambda_1 \lambda_2^{-1}} b^* u_\gamma c v_{\lambda_3 \lambda_4^{-1} \lambda^{-1}}) a \alpha_\lambda(d^*) \right\|_2^2 \\
 &\leq 2 \operatorname{Re} \tau \left(e_\lambda \left| \sum_{F,G} \tau(v_{\lambda_1 \lambda_2^{-1}} b^* u_\gamma c v_{\lambda_3 \lambda_4^{-1} \lambda^{-1}}) a \alpha_\lambda(d^*) \right| \right)
 \end{aligned}$$

$$\begin{aligned}
&\leq 2^{|F|^4|G|^4+1} \operatorname{Re} \tau \left(e_\lambda \left(\sum_{F,G} |\tau(v_{\lambda_1\lambda_2^{-1}} b^* u_\gamma c v_{\lambda_3\lambda_4^{-1}\lambda^{-1}})|^2 |a\alpha_\lambda(d^*)|^2 \right)^{1/2} \right) \\
&\leq 2^{|F|^4|G|^4+1} \operatorname{Re} \tau \left(\left(\sum_{F,G} |\tau(v_{\lambda_1\lambda_2^{-1}} b^* u_\gamma c v_{\lambda_3\lambda_4^{-1}\lambda^{-1}})|^2 \|a\|_\infty^2 \|d\|_\infty^2 \right)^{1/2} e_\lambda \right) \\
&\leq 2^{|F|^4|G|^4+1} \left(\max_{a \in F} \|a\|_\infty \right)^2 |G|^4 |F|^4 \max_{\mu \in GG^{-1}\lambda^{-1}GG^{-1}, b, c \in F} |\tau(b^* u_\gamma c v_\mu)| \|e_\lambda\|_2^2.
\end{aligned} \tag{4.1.7}$$

Letting $0 < D_0 := 2^{|F|^4|G|^4+1} (\max_{a \in F} \|a\|_\infty)^2 |G|^4 |F|^4$ and using that $\|e_\lambda\|_2 \neq 0$, the previous equation gives that $\max_{\mu \in GG^{-1}\lambda^{-1}GG^{-1}, b, c \in F} |\tau(b^* u_\gamma c v_\mu)| \geq \frac{1 - \sqrt{\varepsilon}}{D_0} > 0$, which finishes the proof. \square

The previous technical result on height can be successfully exploited in combination with some soft analysis arising from icc property for groups in order to derive the conjugacy criterion for actions.

Theorem 4.5 *Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be free ergodic actions where Γ is icc. Assume that $\Theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$ is a $*$ -isomorphism such that $\Theta(L^\infty(X)) = L^\infty(Y)$ and there exists a unitary $u \in L^\infty(Y) \rtimes \Lambda$ such that $\Theta(L(\Gamma)) = uL(\Lambda)u^*$. Then one can find $x \in \mathcal{N}_{L^\infty(Y) \rtimes \Lambda}(L^\infty(Y))$, a character $\eta : \Gamma \rightarrow \mathbb{T}$, and a group isomorphism $\delta : \Gamma \rightarrow \Lambda$ such that $xu \in \mathcal{U}(L(\Lambda))$ and for all $a \in L^\infty(X)$, $\gamma \in \Gamma$ we have*

$$\Theta(au_\gamma) = \eta(\gamma)\Theta(a)x^*v_{\delta(\gamma)}x. \tag{4.1.8}$$

Here $\{u_\gamma\}_{\gamma \in \Gamma}$ and $\{v_\lambda\}_{\lambda \in \Lambda}$ are the canonical group unitaries implementing the actions in $L^\infty(X) \rtimes \Gamma$ and $L^\infty(Y) \rtimes \Lambda$, respectively.

In particular, it follows that $\Gamma \curvearrowright X$ is conjugate to $\Lambda \curvearrowright Y$.

Proof For the ease of presentation we first introduce some notations. After suppressing Θ from the notation we assume that $A = L^\infty(X) = L^\infty(Y)$ and hence $M = A \rtimes_\sigma \Gamma = A \rtimes_\alpha \Lambda$. Also letting $C = uAu^*$ and $\Lambda_1 = u\Lambda u^*$ we also have $M = C \rtimes_{\alpha'} \Lambda_1$ and $L(\Gamma) = L(\Lambda_1)$. Throughout the proof we denote by $t_\lambda = uv_\lambda u^*$ and $\alpha'_\lambda(c) = t_\lambda c t_{\lambda^{-1}}$ for all $c \in C$. Note that the condition ii) in Theorem 4.4 is automatically satisfied and hence by the conclusion of Theorem 4.4 there exists a $D > 0$ and a finite subset $F \subset A$ so that for every $\gamma \in \Gamma$, there is $\lambda \in \Lambda$ such that

$$D \leq \max_{b, c \in F} |\tau(b^* u_\gamma c v_\lambda)| = \max_{b, c \in F} |\tau((ub^*)u_\gamma(cu^*)uv_\lambda u^*)| = \max_{b, c \in F} |\tau((ub^*)u_\gamma(cu^*)t_\lambda)|. \tag{4.1.9}$$

Approximating b^*u, u^*c in (4.1.9) via Kaplansky's theorem with elements in $C \rtimes_{\text{alg}} \Lambda_1$ and then diminishing D if necessary, we can in fact assume the following: there exists

$D > 0$ and $K \subset C$ finite, such that for every $\gamma \in \Gamma$, there exists $\lambda \in \Lambda_1$ satisfying

$$\max_{d,e \in K} |\tau(du_\gamma et_\lambda)| \geq D. \quad (4.1.10)$$

On the other hand since $E_C(x) = \tau(x)1$ for all $x \in L(\Lambda_1)$ we can see that

$$\begin{aligned} \max_{d,e \in K} |\tau(du_\gamma et_\lambda)| &= \max_{d,e \in K} |\tau(du_\gamma t_\lambda \alpha'_{\lambda-1}(e))| = \max_{d,e \in K} |\tau(E_C(du_\gamma t_\lambda \alpha'_{\lambda-1}(e)))| \\ &= \max_{d,e \in K} |\tau(dE_C(u_\gamma t_\lambda) \alpha'_{\lambda-1}(e))| = \max_{d,e \in K} |\tau(u_\gamma v_\lambda)| |\tau(d\alpha'_{\lambda-1}(e))| \\ &\leq \max_{d \in K} \|d\|_\infty^2 |\tau(u_\gamma t_\lambda)|. \end{aligned}$$

Combining this with (4.1.10), for every $\gamma \in \Gamma$ there exists $\lambda_1 \in \Lambda_1$ such that $|\tau(u_\gamma t_{\lambda_1})| \geq \frac{D}{\max_{d \in K} \|d\|_\infty^2} > 0$. Since $L(\Gamma) = L(\Lambda_1)$, in the notation of [42] this implies $h_{\Lambda_1}(\Gamma) > 0$. Then by [42, Theorem 3.1] there is $w \in \mathcal{U}(L(\Lambda_1))$, a character $\eta : \Gamma \rightarrow \mathbb{T}$, and a group isomorphism $\delta_1 : \Gamma \rightarrow \Lambda_1$ satisfying $wu_\gamma w^* = \eta(\gamma)t_{\delta_1(\gamma)}$. Since $t_\lambda = uv_\lambda u^*$, letting $x = u^*w$, we further get that there is a group isomorphism $\delta : \Gamma \rightarrow \Lambda$ satisfying

$$xu_\gamma x^* = \eta(\gamma)v_{\delta(\gamma)}, \text{ for all } \gamma \in \Gamma. \quad (4.1.11)$$

As $v_\lambda A v_{\lambda-1} = A$, using (4.1.11) we get $xu_h x^* A x u_{h-1} x^* = A$ for all $h \in \Gamma$. Fix arbitrary $a \in A$ with $\|a\| \leq 1$, and note $u_h x^* a x u_{h-1} = x^* E_A(xu_h x^* a x u_{h-1} x^*)x$. Applying the expectation we also have $u_h E_A(x^* a x) u_{h-1} = E_A(x^* E_A(xu_h x^* a x u_{h-1} x^*)x)$. Subtracting these relations, for every $h \in \Gamma$ we have

$$u_h(x^* a x - E_A(x^* a x))u_{h-1} = x^* E_A(xu_h x^* a x u_{h-1} x^*)x - E_A(x^* E_A(xu_h x^* a x u_{h-1} x^*)x). \quad (4.1.12)$$

Fix $\varepsilon > 0$. By Kaplansky Density Theorem there exist finite subsets $K \subset \Gamma \setminus \{1\}$, $L \subset \Gamma$ and elements $y_K \in \text{span}AK$ and $x_L \in \text{span}AL$ such that

$$\begin{aligned} \|y_K\|_\infty &\leq 2, \quad \|x_L\|_\infty \leq 1 \\ \|x^* a x - E_A(x^* a x) - y_K\|_2 &\leq \varepsilon, \quad \|x - x_L\|_2 \leq \varepsilon \end{aligned} \quad (4.1.13)$$

Using (4.1.12) and (4.1.13) together with basic calculations we see that for every $h \in \Gamma$ we have

$$\begin{aligned} \|x^* a x - E_A(x^* a x)\|_2^2 &= \|u_h(x^* a x - E_A(x^* a x))u_{h-1}\|_2^2 \\ &= |\langle u_h(x^* a x - E_A(x^* a x))u_{h-1}, x^* E_A(xu_h x^* a x u_{h-1} x^*)x \\ &\quad - E_A(x^* E_A(xu_h x^* a x u_{h-1} x^*)x) \rangle| \\ &\leq 2\varepsilon + |\langle u_h y_K u_{h-1}, x^* E_A(xu_h x^* a x u_{h-1} x^*)x - E_A(x^* E_A(xu_h x^* a x u_{h-1} x^*)x) \rangle| \\ &\leq 2\varepsilon + |\langle u_h y_K u_{h-1}, x^* E_A(xu_h x^* a x u_{h-1} x^*)x \rangle| \\ &\leq 6\varepsilon + |\langle u_h y_K u_{h-1}, x_L^* E_A(xu_h x^* a x u_{h-1} x^*)x_L \rangle| \end{aligned} \quad (4.1.14)$$

Since Γ is icc and $K \subset \Gamma \setminus \{1\}$, $L \subset \Gamma$ are finite then by [19, Proposition 2.4] there is $h \in \Gamma$ so that $hKh^{-1} \cap L^{-1}L = \emptyset$. Hence $\langle u_h y_K u_{h^{-1}}, x_L^* E_A(xu_h x^* a x u_{h^{-1}} x^*) x_L \rangle = 0$ and using (4.1.14) we conclude that $\|x^* a x - E_A(x^* a x)\|^2 \leq 4\varepsilon$. Since this holds for all $\varepsilon > 0$ then $x^* a x = E_A(x^* a x)$ for all $a \in A$. Therefore $x^* A x \subseteq A$ and since A is a MASA we obtain $x^* A x = A$; thus $x \in \mathcal{N}_M(A)$. This together with (4.1.11) give (4.1.8). In addition, for every $a \in A$ and $\gamma \in \Gamma$ we have $x\sigma_\gamma(a)x^* = xu_\gamma a u_{\gamma^{-1}} x^* = v_{\delta(\gamma)} x a x^* v_{\delta(\gamma)^{-1}} = \alpha_{\delta(\gamma)}(x a x^*)$; in particular $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are conjugate. \square

Remarks The Theorem 4.5 actually holds in a greater generality, namely, for all actions $\Gamma \curvearrowright A$, $\Lambda \curvearrowright B$ that are properly outer over the center. The proof is essentially the same with the one presented above. We highlighted only the more particular case of free ergodic actions solely because this is what we will mainly use to derive the main results of this section.

4.2 Applications to the generalized Neshveyev–Størmer rigidity question

In this subsection we show that large families of group actions verify the conjugacy criterion presented in Theorem 4.5 and therefore will satisfy the generalized Neshveyev–Størmer rigidity question. Our examples appear as mixing extensions of free distal actions. Our method of proof rely on combining the persistence of mixing through von Neumann equivalence from Sect. 2.4 and the von Neumann algebraic description of compactness using quasinormalizers from [15,36,52,53,57].

Theorem 4.6 *Let $\Gamma \curvearrowright X$ be a ergodic pmp action whose distal quotient $\Gamma \curvearrowright X_d$ is free and the extension $\pi : X \rightarrow X_d$ is nontrivial and mixing. Let $\Lambda \curvearrowright Y$ be an ergodic pmp action whose distal quotient $\Lambda \curvearrowright Y_d$ is also free. Assume that $\Theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$ is a $*$ -isomorphism such that $\Theta(L(\Gamma)) = L(\Lambda)$. Then there exists a unitary $u \in L^\infty(Y_d) \rtimes \Lambda$ such that $\Theta(L^\infty(X_d)) = uL^\infty(Y_d)u^*$ and $\Theta(L^\infty(X)) = uL^\infty(Y)u^*$.*

Proof To ease our presentation we assume that $M := L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda$ with $P = L(\Gamma) = L(\Lambda)$. Using Theorem 2.5 it follows that $N := L^\infty(X_d) \rtimes \Gamma = L^\infty(Y_d) \rtimes \Lambda$. Next we argue that $L^\infty(Y_d) \prec_N L^\infty(X_d)$. Indeed, if we assume $L^\infty(Y_d) \not\prec_N L^\infty(X_d)$, since the extension $\pi : X \rightarrow X_d$ is assumed to be mixing, by Theorem 2.11 we have that $\mathcal{QN}_M(L^\infty(Y_d))'' \subseteq N$. However since $\mathcal{QN}_M(L^\infty(Y_d))'' = M$ it would imply that $M \subseteq N$ which is a contradiction.

Since $\Gamma \curvearrowright X_d$ and $\Lambda \curvearrowright Y_d$ are free and $L^\infty(Y_d) \prec_N L^\infty(X_d)$ then by [64, Appendix A] one can find a unitary $u \in N$ so that $L^\infty(X_d) = uL^\infty(Y_d)u^*$. Passing to relative commutants and using freeness of $\Gamma \curvearrowright X_d$, $\Lambda \curvearrowright Y_d$ again we also get $L^\infty(X) = uL^\infty(Y)u^*$, as desired. \square

Theorem 4.7 *Let Γ be an icc group and let $\Gamma \curvearrowright X$ be an ergodic pmp action whose distal quotient $\Gamma \curvearrowright X_d$ is free and the extension $\pi : X \rightarrow X_d$ is nontrivial and mixing. Let $\Lambda \curvearrowright Y$ be any free ergodic pmp action. Assume that $\Theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$ is a $*$ -isomorphism such that $\Theta(L(\Gamma)) = L(\Lambda)$. Then there exists a unitary $u \in L^\infty(Y_d) \rtimes \Lambda$ such that $\Theta(L^\infty(X_d)) = uL^\infty(Y_d)u^*$ and $\Theta(L^\infty(X)) = uL^\infty(Y)u^*$.*

Proof As before we assume that $M := L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda$ with $L(\Gamma) = L(\Lambda)$. Using Theorem 2.5 it follows that $N := L^\infty(X_d) \rtimes \Gamma = L^\infty(Y_d) \rtimes \Lambda$. Next we argue that $L^\infty(X_d) \prec_N L^\infty(Y_d)$. First notice since $\pi : X \rightarrow X_d$ is mixing it follows from Theorem 2.10 that $\pi : Y \rightarrow Y_d$ is also mixing. If we would have $L^\infty(Y_d) \not\prec_N L^\infty(X_d)$, since the extension $\pi : Y \rightarrow Y_d$ is mixing, then Theorem 2.11 would imply that $\mathcal{QN}_M(L^\infty(X_d))'' \subseteq N$. However since $\mathcal{QN}_M(L^\infty(X_d))'' = M$ it would imply that $M \subseteq N$ which is a contradiction.

Notice that since Γ is icc and $L(\Gamma) = L(\Lambda)$ it follows that Λ is icc as well. Also since $L^\infty(X_d) \prec_N L^\infty(Y_d)$ and $L^\infty(X_d)$ is a Cartan subalgebra in N it follows from [55, Lemma 4.1] that $\Lambda \curvearrowright Y_d$ is free and then the desired conclusion follows from Theorem 4.6. \square

Remarks 1) If in the statements of Theorems 4.6 and 4.7 one only requires that the distal factor $\Gamma \curvearrowright X_d$ is actually compact, then in the proof of Theorem 4.6 we don't need to use Theorem 2.5. Instead one can just directly apply [36, Proposition 6.10].

2) If in the statement of Theorem 4.7 one requires that the first element $\Gamma \curvearrowright X_0$ of the distal tower $\Gamma \curvearrowright X_d$ is free profinite then one can show the action $\Lambda \curvearrowright Y_d$ is free without appealing to [55, Lemma 4.1]. Briefly, using the mixing we have $L^\infty(X_0) \prec_M L^\infty(Y)$ and employing some basic intertwining properties one can further show that $L^\infty(X_0) \prec_{L^\infty(X_0) \rtimes \Gamma} L^\infty(Y_0)$ and hence $L^\infty(Y_0)' \cap (L^\infty(Y_0) \rtimes \Lambda) \prec_{L^\infty(Y_0) \rtimes \Lambda} L^\infty(X_0) (*)$. However using the same calculations from the proof of part 2. in Theorem 4.12 we have $L^\infty(Y_0)' \cap (L^\infty(Y_0) \rtimes \Lambda) = L^\infty(Y_0) \rtimes \Sigma$ for some normal subgroup $\Sigma \triangleleft \Lambda$. However since $L(\Sigma) \subseteq L(\Gamma)$ the intertwining $(*)$ implies that Σ is finite and since Λ is icc we further have $\Sigma = 1$; hence $\Lambda \curvearrowright Y_0$ must be free.

Combining the previous theorems with Theorem 4.5 we obtain the following

Theorem 4.8 *Let Γ be an icc group and let $\Gamma \curvearrowright X$ be a free, ergodic pmp action whose distal quotient $\Gamma \curvearrowright X_d$ is free and the extension $\pi : X \rightarrow X_d$ is nontrivial and mixing. Let $\Lambda \curvearrowright Y$ be any free ergodic pmp action. Assume that $\Theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$ is a $*$ -isomorphism such that $\Theta(L(\Gamma)) = L(\Lambda)$. Then there exists $y \in \mathcal{U}(L(\Lambda))$, $\omega : \Gamma \rightarrow \mathbb{T}$ a character, and $\delta : \Gamma \rightarrow \Lambda$ a group isomorphism such that $y\Theta(L^\infty(X))y^* = L^\infty(Y)$, and for all $a \in L^\infty(X)$, $\gamma \in \Gamma$, we have*

$$\Theta(au_\gamma) = \omega(\gamma)\Theta(a)y^*v_{\delta(\gamma)}y.$$

In particular, we have $y\Theta(\sigma_\gamma(a))y^ = \alpha_{\delta(\gamma)}(y\Theta(a)y^*)$ and hence $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are conjugate.*

Here $\{u_\gamma\}_{\gamma \in \Gamma}$ and $\{v_\lambda\}_{\lambda \in \Lambda}$ are the canonical group unitaries implementing the actions in $L^\infty(X) \rtimes \Gamma$ and $L^\infty(Y) \rtimes \Lambda$, respectively.

Proof To ease our presentation, we assume, as before, that $L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda$, and $L(\Gamma) = L(\Lambda)$. Theorem 4.7 yields that there is a unitary $u \in L^\infty(Y_d) \rtimes \Lambda$ such that $L^\infty(X) = uL^\infty(Y)u^*$. This is equivalent to assuming that $L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda$, $L(\Gamma) = uL(\Lambda)u^*$ and $L^\infty(X) = L^\infty(Y)$. We are now exactly in the set up of Theorem 4.5, which yields the desired conclusions. \square

Examples Theorem 4.8 implies that if Γ be an icc group and $\Gamma \curvearrowright X$ is any ergodic pmp action that admits a free profinite quotient $\Gamma \curvearrowright X_d$ and the extension $\pi : X \rightarrow X_d$ is nontrivial and mixing then $\Gamma \curvearrowright X$ satisfies Neshveyev–Størmer rigidity question. For instance if Γ is icc residually finite then this is the case for any diagonal action $\Gamma \curvearrowright Z \times T$ where $\Gamma \curvearrowright Z$ is a Gaussian action associated to a mixing orthogonal representation of Γ and $\Gamma \curvearrowright T$ is any free ergodic profinite action.

Corollary 4.9 *Let Γ be an icc group, let $\Gamma \curvearrowright X$ be a free, mixing pmp action and let $\Lambda \curvearrowright Y$ be any free ergodic pmp action. Also let $\Gamma \curvearrowright X_0$ be a free factor of $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y_0$ be a factor of $\Lambda \curvearrowright Y$. Assume that $\Theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$ is a $*$ -isomorphism such that $\Theta(L(\Gamma)) = L(\Lambda)$ and $\Theta(L^\infty(X_0) \rtimes \Gamma) = L^\infty(Y_0) \rtimes \Lambda$. Then there exists $y \in \mathcal{U}(L(\Lambda))$, $\omega : \Gamma \rightarrow \mathbb{T}$ a character, and $\delta : \Gamma \rightarrow \Lambda$ a group isomorphism such that $y\Theta(L^\infty(X))y^* = L^\infty(Y)$, and for all $a \in L^\infty(X)$, $\gamma \in \Gamma$, we have*

$$\Theta(au_\gamma) = \omega(\gamma)\Theta(a)y^*v_{\delta(\gamma)}y.$$

In particular, we have $y\Theta(\sigma_\gamma(a))y^ = \alpha_{\delta(\gamma)}(y\Theta(a)y^*)$ and hence $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are conjugate.*

Here $\{u_\gamma\}_{\gamma \in \Gamma}$ and $\{v_\lambda\}_{\lambda \in \Lambda}$ are the canonical group unitaries implementing the actions in $L^\infty(X) \rtimes \Gamma$ and $L^\infty(Y) \rtimes \Lambda$, respectively.

Proof Since $\Gamma \curvearrowright X$ is mixing then by Theorem 2.10 so is $\Lambda \curvearrowright Y$. In particular the extensions $\Gamma \curvearrowright (L^\infty(X_0) \subset L^\infty(X))$ and $\Gamma \curvearrowright (L^\infty(X_0) \subset L^\infty(X))$ are mixing. Since $\Theta(L^\infty(X_0) \rtimes \Gamma) = L^\infty(Y_0) \rtimes \Lambda$ the conclusion follows using the same arguments as in the proof of Theorem 4.8. \square

Following the terminology from [71] a free ergodic action $\Gamma \curvearrowright X$ is called \mathcal{C}_{gms} -superrigid if up to unitary conjugacy $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma = M$ is the only group measure space Cartan subalgebra of M . Over the last decade many classes of examples of such actions have been discovered via deformation/rigidity theory. For some concrete examples the reader is referred to [12, 17, 18, 39, 55, 72, 73] and the survey [41]. An immediate consequence of [67, Theorem 5.1] is that all weakly mixing \mathcal{C}_{gms} -superrigid actions satisfy the statement of Theorem 4.8. Using our Theorem 4.5 we obtain the following generalization

Corollary 4.10 *Any \mathcal{C}_{gms} -superrigid action $\Gamma \curvearrowright X$ of any icc group Γ satisfies the statement of Theorem 4.8.*

In particular the generalized Neshveyev–Størmer rigidity holds for all action $\Gamma \curvearrowright X$ of icc groups Γ that are: hyperbolic groups, [73], free products [39] or finite step extensions of such groups [12].

At this point it is increasingly evident that all the above Neshveyev–Størmer type rigidity results were achieved by heavily exploiting, at the von Neumann algebra level, the natural tension between mixing and compactness properties for action. It would be interesting to understand whether such results could still be obtained only in the compact regime. Specifically, we would like to propose for study the following

Problem 4.11 *If Γ is icc does every free ergodic profinite action $\Gamma \curvearrowright X$ satisfy the statement of Theorem 4.8?*

While providing a complete answer to this question seems hard at the moment, one can show there are many aspects of $\Gamma \curvearrowright X$ that are shared by $\Lambda \curvearrowright Y$ through this equivalence (e.g. compactness, profiniteness, etc). In fact we have the following result.

Theorem 4.12 *Let $\Gamma \curvearrowright X$ be a free ergodic action and let $\Lambda \curvearrowright Y$ be any action. Let $\Theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$ be a $*$ -isomorphism such that $\Theta(L(\Gamma)) = L(\Lambda)$. Then the following hold*

1. *If $\Gamma \curvearrowright X$ is (weakly) compact then $\Lambda \curvearrowright Y$ is also (weakly) compact.*
2. *If Γ is icc and $\Gamma \curvearrowright X$ is profinite then $\Lambda \curvearrowright Y$ is also ergodic and profinite. Specifically, if $\Gamma \curvearrowright X$ is the inverse limit of $\Gamma \curvearrowright X_n$ with X_n finite then $\Lambda \curvearrowright Y$ is the inverse limit of $\Lambda \curvearrowright Y_n$ with Y_n finite so that for every n we have $\Theta(L^\infty(X_n) \rtimes \Gamma) = L^\infty(Y_n) \rtimes \Lambda$. In addition, the stabilizer $\text{Stab}_\Lambda(Y_n) \triangleleft \Lambda$ is normal and we have that $\Gamma/\text{Stab}_\Gamma(X_n) \cong \Lambda/\text{Stab}_\Lambda(Y_n)$ for all n . Finally, there exists a normal subgroup $\Sigma \triangleleft \Lambda$ so that $L^\infty(Y)' \cap L^\infty(Y) \rtimes \Lambda = L^\infty(Y) \rtimes \Sigma$.*

Proof 1. As before we assume that $L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda = M$ and $L(\Gamma) = L(\Lambda)$. Since $\Gamma \curvearrowright X$ is compact, [36, Theorem 6.10] implies that the quasinormalizer algebra satisfies $\mathcal{QN}_M(L(\Gamma))'' = M$. Since canonically $\mathcal{QN}_M(L(\Gamma))'' = \mathcal{QN}_M(L(\Lambda))''$ then $\mathcal{QN}_M(L(\Lambda))'' = M$ which by [36, Theorem 6.10] again implies that $\Lambda \curvearrowright Y$ is also compact. The statement on weak compactness follows from [55, Proposition 3.2].

2. Since Γ is icc then Λ is also icc. Hence $\Lambda \curvearrowright Y$ is ergodic (otherwise M will not be a factor). Next we show that $\Lambda \curvearrowright Y$ is profinite. As $\Gamma \curvearrowright X$ is profinite, it is the inverse limit of ergodic actions $\Gamma \curvearrowright X_n$ on finite spaces. Thus $A_n = L^\infty(X_n)$ form a tower of finite dimensional abelian Γ -invariant subalgebras $A_0 \subset \dots \subset A_n \subset A_{n+1} \subset \dots \subset L^\infty(X)$ such that $\overline{\bigcup_n A_n}^{SOT} = L^\infty(X)$. Moreover $\Gamma \curvearrowright A_n$ is transitive for every n . Since $L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda$ and $L(\Gamma) = L(\Lambda)$ using Theorem 3.10 for every n one can find a Λ -invariant subalgebra $B_n \subset L^\infty(Y)$ such that $A_n \rtimes \Gamma = B_n \rtimes \Lambda$. Factoriality of $A_n \rtimes \Gamma$ and Λ being icc imply that the action $\Lambda \curvearrowright B_n$ is ergodic. Since $L(\Gamma) \subseteq A_n \rtimes \Gamma$ is a finite index inclusion of II_1 factors so is $L(\Lambda) \subseteq B_n \rtimes \Lambda$. Using Lemma 2.6 we get that B_n is finite dimensional and the action $\Lambda \curvearrowright B_n$ is transitive. One can easily check that $B_0 \subset \dots \subset B_n \subset B_{n+1} \subset \dots \subset L^\infty(Y)$ and also $\overline{\bigcup_n B_n}^{SOT} = L^\infty(Y)$. Thus there exist factors $\Lambda \curvearrowright Y_n$ of $\Lambda \curvearrowright Y$ with Y_n finite such that $\Lambda \curvearrowright Y$ is the inverse limit of $\Lambda \curvearrowright Y_n$.

Denote by $\{p_i^n \mid 1 \leq i \leq k_n\} = \text{At}(B_n)$. Since $\text{Stab}_\Gamma(q)$ is assumed normal in Γ for every $q \in \text{At}(A_n)$ it follows from Proposition 2.7 that $\text{Stab}_\Lambda(p_i^n)$ is normal in Λ for every i . Moreover, since the action $\Lambda \curvearrowright B_n$ is transitive one can easily see that we actually have $\text{Stab}_\Lambda(p_i^n) = \text{Stab}_\Lambda(B_n)$ for all $1 \leq i \leq k_n$. Finally, by Proposition 2.7 we also have that $\Gamma/\text{Stab}_\Gamma(X_n) \cong \Lambda/\text{Stab}_\Lambda(Y_n)$ for all n .

In the remaining part we describe the relative commutant $L^\infty(Y)' \cap M$. So fix $b \in L^\infty(Y)' \cap M$ and consider its Fourier decomposition $b = \sum_{\lambda \in \Lambda} b_\lambda v_\lambda$. Since b commutes with $L^\infty(Y)$ we get that $y b_\lambda = \alpha_\lambda(y) b_\lambda$ for all $\lambda \in \Lambda$ and $y \in L^\infty(Y)$. Letting e_λ be the support projection of $b_\lambda b_\lambda^*$ this further implies that for all $y \in L^\infty(Y)$ and $\lambda \in \Lambda$ we have

$$y e_\lambda = \alpha_\lambda(y) e_\lambda. \quad (4.2.1)$$

Fix λ such that $b_\lambda \neq 0$ (and hence $e_\lambda \neq 0$). Denote by $e_\lambda^n := E_{B_n}(e_\lambda)$ and applying the conditional expectation E_{B_n} in (4.2.1), for all $y \in B_n$ we have

$$ye_\lambda^n = \alpha_\lambda(y)e_\lambda^n. \quad (4.2.2)$$

Since $e_\lambda \neq 0$ then $e_\lambda^n \neq 0$ and hence there is $p_i^n \in At(B_n)$ satisfying $e_\lambda^n p_i^n = c p_i^n$ for some scalar $c > 0$. Multiplying (4.2.2) by $c^{-1} p_i^n$ we get $yp_i^n = \alpha_\lambda(y)p_i^n$ for all $y \in B_n$. This entails that $\alpha_\lambda(p_i^n) = p_i^n$ and hence $\lambda \in Stab_\Lambda(p_i^n) = Stab_\Lambda(B_n)$. Altogether, we have shown that for every λ with $b_\lambda \neq 0$ we have $\lambda \in Stab_\Lambda(B_n)$. Applying this for every n we conclude that $\lambda \in \cap_n Stab_\Lambda(B_n) =: \Sigma$. In particular $b \in L^\infty(Y) \rtimes \Sigma$ and hence $L^\infty(Y)' \cap M \subseteq L^\infty(Y) \rtimes \Sigma$. Since the reverse containment canonically holds we get $L^\infty(Y)' \cap M = L^\infty(Y) \rtimes \Sigma$. As $Stab_\Lambda(B_n)$'s are normal in Λ then Σ is also normal in Λ . \square

These results can be used to produce additional examples of actions satisfying the statement of Theorem 4.8. For example part 1. of the previous theorem in combination with Theorem 4.5 and [5, Theorem 4.16], [16, Theorem 5.1] shows that any free ergodic weakly compact action $\Gamma \curvearrowright X$ satisfies the generalized Neshveyev-Størmer rigidity whenever Γ is an icc group in one of the following classes

1. Γ is any properly proximal group [5, Definition 4.1], in particular when $\Gamma = PSL_n(\mathbb{Z})$, $n \geq 2$ or any Γ that admits a proper array into a nonamenable representation (see [17, Definition 2.1]). In fact the latter also follows by using the results in [17, 18];
2. $\Gamma = H \wr G$ is a wreath product where H is nontrivial abelian and G nonamenable [16].

5 Some applications to strong rigidity results in von Neumann algebras and orbit equivalence

Theorem 5.1 *Let Γ and Λ be icc property (T) groups. Let $\Gamma \curvearrowright X = \lim X_n$ be a free ergodic profinite action and let $\Lambda \curvearrowright Y$ be a free ergodic compact action. Assume that $\Theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$ is a $*$ -isomorphism. Then $\Lambda \curvearrowright Y = \lim Y_n$ is a profinite action. Moreover there exists $l \in \mathbb{N}$ and a unitary $w \in L^\infty(Y) \rtimes \Lambda$ such that $\Theta(L^\infty(X_{k+l}) \rtimes \Gamma) = w(L^\infty(Y_{k+l}) \rtimes \Lambda)w^*$ for every positive integer k .*

Proof To simplify the notations let $A = L^\infty(X)$, $B = L^\infty(Y)$ and notice that $M = A \rtimes \Gamma = B \rtimes \Lambda$. Moreover if $L^\infty(X_n) = A_n$, then $A_n \subseteq A_{n+1}$ and $A = \overline{\bigcup_n A_n}^{SOT}$. Also if $M_n = A_n \rtimes \Gamma$ then $M_n \subseteq M_{n+1}$ and $M = \overline{\bigcup_n M_n}^{SOT}$. Note that by [36, Theorem 6.9(b)], $L^\infty(X) \rtimes \Gamma$ has Haagerup's property relative to $L(\Gamma)$. Since $L(\Lambda)$ has property (T), the same arguments as in the proof of [64, 5.4(2°)] (see also [64, Remarks 6.3.1°]) and [66, Theorem 2.1] implies that $L(\Lambda) \prec_M L(\Gamma)$. Hence one can find nonzero projections $p \in L(\Lambda)$, $q \in L(\Gamma)$ a nonzero partial isometry $v \in M$ and an injective $*$ -homomorphism $\phi : pL(\Lambda)p \rightarrow qL(\Gamma)q$ satisfying $\phi(x)v = vx$ for all $x \in pL(\Lambda)p$. Since $L(\Lambda) \subset M$ is a irreducible subfactor by Theorem 2.2, we may assume that $\phi(pL(\Lambda)p)' \cap qL(\Gamma)q = \mathbb{C}q$. Also, since $L(\Lambda)' \cap M = \mathbb{C}$ we have that

$v^*v = p$. Denoting by $Q = \phi(pL(\Lambda)p)$ we also have that $r = vv^* \in Q' \cap qMq$. Letting $u \in M$ a unitary such that $uv^*v = v$ we have that

$$upL(\Lambda)pu^* = Qr. \quad (5.0.3)$$

Next we prove the following

Claim 5.2 $Q \subseteq qL(\Gamma)q$ is a finite index subfactor.

Proof of Claim 5.2. Since $L(\Gamma) \subset M$ is a rigid subalgebra and M has Haagerup's property relative to $L(\Lambda)$, by the same arguments as in the second paragraph of this proof, we also have that $L(\Gamma) \prec_M L(\Lambda)$. Since $L(\Lambda)$ is a factor this further entails that $L(\Gamma) \prec_M upL(\Lambda)pu^* = Qr$. Hence by Popa's intertwining techniques there exist finitely many $x_j, y_j \in M$ and $c > 0$ such that $\sum_j \|E_{Qr}(x_j u y_j)\|_2^2 \geq c$ for all $u \in \mathcal{U}(L(\Gamma))$. Since $E_Q(r) = \tau_q(r)q$ then we have $E_{Qr}(x) = E_Q(r)^{-1}E_Q(qxq)r = \tau_q(r)^{-1}E_Q(qxq)r$ for all $x \in L(\Gamma)$. Using this formula in the previous inequality we further get that $\sum_j \|E_Q(qx_j u y_j q)\|_2^2 \geq c\tau_q(r) > 0$. Approximating x_j and y_j with their Fourier decompositions one can find finitely many $a_i, b_i \in A$ and $\gamma_i, \delta_i \in \Gamma$ such that for all $u \in \mathcal{U}(L(\Gamma))$ we have $\sum_i \|E_Q(qu_{\gamma_i} a_i u b_i u_{\delta_i} q)\|_2^2 \geq \frac{c\tau_q(r)}{2}$. Using this together with $E_Q = E_Q \circ E_{qL(\Gamma)q}$ we see that for all $\gamma \in \Gamma$ we have

$$\begin{aligned} \frac{c\tau_q(r)}{2} &\leq \sum_{i=1}^s \|E_Q(qu_{\gamma_i} a_i u_{\gamma} b_i u_{\delta_i} q)\|_2^2 = \sum_{i=1}^s \|E_Q(qu_{\gamma_i} a_i \sigma_{\gamma}(b_i) u_{\gamma \delta_i} q)\|_2^2 \\ &= \sum_{i=1}^s \|E_Q(qE_{L(\Gamma)}(u_{\gamma_i} a_i \sigma_{\gamma}(b_i) u_{\gamma \delta_i})q)\|_2^2 \\ &= \sum_{i=1}^s |\tau(a_i \sigma_{\gamma}(b_i))|^2 \|E_Q(qu_{\gamma_i \gamma \delta_i} q)\|_2^2 \\ &\leq \left(\max_{i=1,s} \|a_i\|_{\infty}^2 \|b_i\|_{\infty}^2 \right) \left(\sum_i \|E_Q(qu_{\gamma_i \gamma \delta_i} q)\|_2^2 \right). \end{aligned}$$

Thus letting $d = \frac{c\tau_q(r)}{2 \max_{i=1,s} \|a_i\|_{\infty}^2 \|b_i\|_{\infty}^2}$ we have that $\sum_i \|E_Q(qu_{\gamma_i \gamma \delta_i} q)\|_2^2 \geq d > 0$ for all $\gamma \in G$. Hence by Theorem 2.1 we get $L(\Gamma) \prec_{L(\Gamma)} Q$ and since $L(\Gamma)$ is a II_1 factor we actually have $qL(\Gamma)q \prec_{L(\Gamma)} Q$ and hence $qL(\Gamma)q \prec_{qL(\Gamma)q} Q$. Since $Q' \cap qL(\Gamma)q = \mathbb{C}q$, this entails $[qL(\Gamma)q : Q] < \infty$ by [9, Proposition 2.3], and the claim follows. \square

Combining the Claim 5.2 with [65, Lemma 3.1] it follows the inclusion $qL(\Gamma)q' \cap qMq \subseteq Q' \cap qMq$ has finite Pimsner-Popa probabilistic index. Since Γ is icc and $\Gamma \curvearrowright X$ is free it follows that $L(\Gamma)' \cap M = \mathbb{C}$ and thus $qL(\Gamma)q' \cap qMq = \mathbb{C}q$. Combining with the above we conclude that $Q' \cap qMq$ is a finite dimensional von Neumann algebra. Since $Q \subseteq qL(\Gamma)q \subset qM_1q \subset \dots \subset qM_nq \subset qM_{n+1}q \subset \dots \subset qMq$ and $qMq = \overline{\cup_n qM_nq}^{SOT}$ one can check that $Q' \cap qM_1q \subset \dots \subset Q' \cap qM_nq \subset Q' \cap$

$qM_{n+1}q \subset \dots \subset Q' \cap qMq$ and also $Q' \cap qMq = \overline{\cup_n Q' \cap qM_n q}^{SOT}$. Since $Q' \cap qMq$ is finite dimensional there must be a minimal integer l so that $Q' \cap qM_l q = Q' \cap qMq$. In particular, we have $r \in qM_n q$ and by (5.0.3) we obtain $upL(\Lambda)pu^* \subseteq M_l$. As M_l is a factor one can find $w \in \mathcal{U}(M)$ so that $wL(\Lambda)w^* \subseteq M_l$.

Since the action $\Lambda \curvearrowright B$ is compact the using Theorem 3.10 there is a Λ -invariant von Neumann subalgebra $B_1 \subset B$ satisfying $w(B_1 \rtimes \Lambda)w^* = A_l \rtimes \Gamma = M_l$. Since $L(\Gamma)$ has property (T) and $[M_l : L(\Gamma)] < \infty$ it follows that M_l has property (T). Thus $B_1 \rtimes \Lambda$ is a factor with property (T) and as $B_1 \rtimes \Lambda$ has Haagerup property relative to $L(\Lambda)$ we conclude that $B_1 \rtimes \Lambda \prec L(\Lambda)$ and hence by [9, Proposition 2.3] we have $[B_1 \rtimes \Lambda : L(\Lambda)] < \infty$. Hence by Lemma 2.6 B_1 is finite dimensional and the action $\Lambda \curvearrowright B_1$ is transitive. Finally, using Theorem 3.10 successively there exist a tower of Λ -invariant finite dimensional abelian von Neumann subalgebras $B_1 \subset \dots \subset B_n \subset B_{n+1} \subset \dots \subset B$ such that $\overline{\cup_{n \geq 1} B_n}^{SOT} = B$ and also $w(B_{k+1} \rtimes \Lambda)w^* = A_{k+1} \rtimes \Gamma$ for all $k \geq 0$. Thus there exists a sequence of factors $\Lambda \curvearrowright Y_n$ of $\Lambda \curvearrowright Y$ into finite probability spaces Y_n such that $L^\infty(Y_n) = B_n$ for all $n \geq 1$. From the previous relations one can check $\Lambda \curvearrowright Y$ is the inverse limit of $\Lambda \curvearrowright Y_n$ which gives the desired statement. \square

The von Neumann algebraic methods developed in the previous sections can be used effectively to derive the following version of Ioana's *OE*-superrigidity theorem [37, Theorem A] for profinite actions of icc groups.

Theorem 5.3 *Let $\Gamma \curvearrowright X$ be a profinite free ergodic action of an icc property (T) group Γ and let $\Lambda \curvearrowright Y$ be an arbitrary free ergodic action of an icc group Λ . Assume that $\Theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$ is a $*$ -isomorphism such that $\Theta(L^\infty(X)) = L^\infty(Y)$. Then there exist projections $p \in L^\infty(X)$ and $q \in L^\infty(Y)$, a unitary $u \in \mathcal{N}_{L^\infty(Y) \rtimes \Lambda}(L^\infty(Y))$ with $u\Theta(p)u^* = q$, normal subgroups $\Gamma' \triangleleft \Gamma$, $\Lambda' \triangleleft \Lambda$ with $[\Gamma : \Gamma'] = [\Lambda : \Lambda'] < \infty$, a character $\eta : \Gamma' \rightarrow \mathbb{T}$ and a group isomorphism $\delta : \Gamma' \rightarrow \Lambda'$ such that for all $\gamma \in \Gamma'$ and $a \in A$ we have*

$$\Theta(apu_\gamma) = \eta(\gamma)\Theta(ap)u^*v_{\delta(\gamma)}u.$$

In particular the actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ are virtually conjugate.

Proof Suppressing Θ we can assume $L^\infty(X) = L^\infty(Y) = A$ and $A \rtimes \Gamma = A \rtimes \Lambda = M$. Since property (T) is an *OE*-invariant [27, Corollary 1.4] it follows that Λ is also a property (T) icc group. Since $\Gamma \curvearrowright A$ is profinite then it is weakly compact in the sense of Ozawa-Popa and by [55, Proposition 3.4] it follows that $\Lambda \curvearrowright B$ is also weakly compact. Since Λ has property (T) then [37, Remark 6.4] implies that $\Lambda \curvearrowright B$ is compact. Thus using Theorem 5.1 there exist increasing towers of Γ -invariant, finite dimensional algebras $(A_n)_n \subseteq A$ and $(B_n)_n \subseteq A$ such that $\overline{\cup_n A_n}^{SOT} = A = \overline{\cup_n B_n}^{SOT}$. Also there is a unitary $w \in M$ and an integer s such that for all k we have

$$w(A_{s+k} \rtimes \Gamma)w^* = B_k \rtimes \Lambda. \quad (5.0.4)$$

Since $A_s \rtimes \Gamma \subseteq A_{s+k} \rtimes \Gamma$ is a finite index inclusion of II_1 factors then so is $B_0 \rtimes \Lambda \subseteq B_k \rtimes \Lambda$. Thus by Proposition 2.7 it follows that B_k is finite dimensional. Since

$\Gamma_k := \text{Stab}_\Gamma(A_{s+k}) \triangleleft \Gamma$ is a finite index normal subgroup so is $\Lambda_k := \text{Stab}_\Lambda(B_k) \triangleleft \Lambda$. Since $w \in M = \bigcup_k A_k \rtimes \Gamma^{\text{SOT}}$ is a unitary there exists a sequence $w_k \in \mathcal{U}(A_k \rtimes \Gamma)$ such that $\|w - w_k\|_2 \rightarrow 0$ as $k \rightarrow \infty$.

For the remaining part of the proof for every $m \geq k$ we will keep in mind the following diagram of inclusions

$$\begin{array}{ccc} ww_k^*(A_{s+m} \rtimes \Gamma) w_k w^* & = & B_m \rtimes \Lambda \\ \cup & & \cup \\ ww_k^*(A_{s+k} \rtimes \Gamma) w_k w^* & = & B_k \rtimes \Lambda \\ \cup & & \cup \\ w(A_s \rtimes \Gamma) w^* & = & B_0 \rtimes \Lambda \end{array} \quad (5.0.5)$$

Pick k large enough such that $\|1 - ww_k^*\|_2 \leq 10^{-9}$. Denote by $At(A_{s+l}) = \{a_l^i : 1 \leq i \leq r_l\}$ and $At(B_l) = \{b_l^j : 1 \leq j \leq t_l\}$. Also we can assume without any loss of generality that $\dim B_0 \leq \dim A_s$ (hence $\dim B_k \leq \dim A_{s+k}$ for all k); in particular we have $\tau(b_l^j) \geq \tau(a_l^i)$. Fix $1 \leq i \leq r_k$ such that $\|a_k^i(1 - ww_k^*)\|_2 + \|(1 - ww_k^*)a_k^i\|_2 \leq 2\|a_k^i\|_2\|1 - ww_k^*\|_2$. Hence if we denote by $\delta_k^i = \|a_k^i(1 - ww_k^*)\|_2 + \|(1 - ww_k^*)a_k^i\|_2\|a_k^i\|_2^{-1}$ then we have that

$$\delta_k^i \leq 2\|1 - ww_k^*\|_2 < 10^{-8}. \quad (5.0.6)$$

With this notations at hand we show that

Claim 5.4 *There is a unique $1 \leq j \leq t_k$ such that for every $\gamma \in \Gamma_k$ one can find $\lambda, \lambda' \in \Lambda$ such that*

$$(1 - 2\delta_k^i)\|b_k^j\|_2^2 \leq |\tau(a_k^i u_\gamma v_\lambda b_k^j)|, \text{ and} \quad (5.0.7)$$

$$(1 - 3\delta_k^i)\|b_k^j\|_2^2 \leq |\tau(ww_k^* a_k^i u_\gamma w_k w^* v_{\lambda'} b_k^j)|. \quad (5.0.8)$$

Proof of Claim 5.4. Fix $\gamma \in \Gamma_k$. By triangle inequality we have $\|a_k^i u_\gamma - ww_k^* a_k^i u_\gamma w_k w^*\|_2 \leq \delta_k^i \|a_k^i\|_2$. Applying the conditional expectation and using (5.0.5) we also have $\|E_{B_k \rtimes \Lambda}(a_k^i u_\gamma) - ww_k^* a_k^i u_\gamma w_k w^*\|_2 \leq \delta_k^i \|a_k^i\|_2$. Then the triangle inequality further gives

$$\|a_k^i u_\gamma - E_{B_k \rtimes \Lambda}(a_k^i u_\gamma)\|_2 \leq 2\delta_k^i \|a_k^i\|_2 \quad (5.0.9)$$

$$\|a_k^i u_\gamma - E_{B_k \rtimes \Lambda}(ww_k^* a_k^i u_\gamma w_k w^*)\|_2 \leq 3\delta_k^i \|a_k^i\|_2. \quad (5.0.10)$$

By Lemma 4.3 there exist orthogonal projections $e_\lambda \in A$ so that $\sum_\lambda e_\lambda = 1$ and unitaries $x_\lambda \in A$ so that $u_\gamma = \sum_{\lambda \in \Lambda} e_\lambda x_\lambda v_\lambda$. This combined with (5.0.9) yield

$$\begin{aligned} \sum_{\lambda \in \Lambda} \|a_k^i e_\lambda x_\lambda - E_{B_k}(a_k^i u_\gamma v_{\lambda^{-1}})\|_2^2 &= \|a_k^i u_\gamma - E_{B_k \rtimes \Lambda}(a_k^i u_\gamma)\|_2^2 \leq 4(\delta_k^i)^2 \|a_k^i\|_2^2 \\ &= \sum_{\lambda \in \Lambda} 4(\delta_k^i)^2 \|a_k^i e_\lambda x_\lambda\|_2^2. \end{aligned}$$

Thus one can find $\lambda \in \Lambda$ so that $a_k^i e_\lambda x_\lambda \neq 0$ and $\|a_k^i e_\lambda x_\lambda - E_{B_k}(a_k^i u_\gamma v_{\lambda^{-1}})\|_2 \leq 2\delta_k^i \|a_k^i e_\lambda x_\lambda\|_2$. This inequality and basic calculations show that

$$(2 - 4\delta_k^i) \|a_k^i e_\lambda x_\lambda\|_2^2 \leq (1 - 4(\delta_k^i)^2) \|a_k^i e_\lambda x_\lambda\|_2^2 + \|E_{B_k}(a_k^i u_\gamma v_{\lambda^{-1}})\|_2^2 \leq 2 \operatorname{Re} \tau(a_k^i e_\lambda x_\lambda^* E_{B_k}(a_k^i u_\gamma v_{\lambda^{-1}})). \quad (5.0.11)$$

Using the formulas $\sum_j b_k^j = 1$ and $E_{B_k}(x) = \sum_j \tau(x b_k^j) \tau(b_k^j)^{-1} b_k^j$, relation (5.0.11) implies that $(1 - 2\delta_k^i) \sum_j \|a_k^i e_\lambda x_\lambda b_k^j\|_2^2 \leq \sum_j |\tau(a_k^i e_\lambda x_\lambda b_k^j)| |\tau(a_k^i u_\gamma v_{\lambda^{-1}} b_k^j)| |\tau(b_k^j)^{-1}|$. Hence there is j (which at this point may depend on γ !) so that $a_k^i e_\lambda x_\lambda b_k^j \neq 0$ and $(1 - 2\delta_k^i) \|a_k^i e_\lambda x_\lambda b_k^j\|_2^2 \leq |\tau(a_k^i e_\lambda x_\lambda b_k^j)| |\tau(a_k^i u_\gamma v_{\lambda^{-1}} b_k^j)| |\tau(b_k^j)^{-1}|$. Using $\|a_k^i e_\lambda b_k^j\|_2^2 \geq |\tau(a_k^i e_\lambda x_\lambda b_k^j)| \neq 0$ this further gives

$$(1 - 2\delta_k^i) \|b_k^j\|_2^2 \leq |\tau(a_k^i u_\gamma v_{\lambda^{-1}} b_k^j)|. \quad (5.0.12)$$

Proceeding in a similar manner inequality (5.0.10) implies there exist $\lambda' \in \Lambda$ and $1 \leq j' \leq r_k$ such that

$$(1 - 3\delta_k^i) \|b_k^{j'}\|_2^2 \leq |\tau(w w_k^* a_k^i u_\gamma w_k w^* v_{\lambda'} b_k^{j'})|. \quad (5.0.13)$$

To finish the proof it suffices to argue that $j = j'$ and j is unique (hence does not depend on γ). Since $\tau(a_k^i u_\gamma v_{\lambda^{-1}} b_k^j) = \tau(a_k^i E_A(u_\gamma v_{\lambda^{-1}}) b_k^j) = \tau(a_k^i e_\lambda x_\lambda b_k^j)$ then (5.0.12) implies $\|b_k^j - a_k^i\|_2^2 = \tau(b_k^j + a_k^i - 2b_k^j a_k^i) \leq \tau(b_k^j) + \tau(a_k^i) - 2|\tau(b_k^j e_\lambda x_\lambda a_k^i)| \leq 4\delta_k^i \tau(b_k^j)$ and hence

$$\|b_k^j - a_k^i\|_2 \leq 2(\delta_k^i)^{\frac{1}{2}} \|b_k^j\|_2. \quad (5.0.14)$$

By triangle inequality this also yields

$$\|b_k^j - w w_k^* a_k^i w_k w^*\|_2 \leq (2(\delta_k^i)^{\frac{1}{2}} + \delta_k^i) \|b_k^j\|_2. \quad (5.0.15)$$

Then Cauchy–Schwarz inequality in combination with (5.0.13) and (5.0.15) show that

$$\begin{aligned} \|b_k^{j'} b_k^j\|_2 &\geq |\tau(b_k^j w w_k^* u_\gamma w_k w^* v_{\lambda'} b_k^{j'})| \\ &\geq |\tau(w w_k^* a_k^i u_\gamma w_k w^* v_{\lambda'} b_k^{j'})| - \|b_k^j - w w_k^* a_k^i w_k w^*\|_2 \|b_k^{j'}\|_2 \\ &\geq (1 - 4\delta_k^i - 2(\delta_k^i)^{\frac{1}{2}}) \|b_k^j\|_2^2. \end{aligned}$$

As b_k^j 's are orthogonal this forces that $j = j'$. Uniqueness of j (hence independence of γ) follows from (5.0.14). \square

Next we show the following

Claim 5.5 *There exist $1 \leq j \leq t_k$ and a unitary $s_k^j \in B_k \rtimes \Lambda$ such that $s_k^j w w_k^* a_k^i w_k w^* (s_k^j)^* = c_k^j \leq p_k^j$ and*

$$s_k^j w w_k^* a_k^i (A_{s+k} \rtimes \Gamma) a_k^i w_k w^* (s_k^j)^* = c_k^j (B_k \rtimes \Lambda) c_k^j \quad (5.0.16)$$

and for every $\gamma \in \Gamma_k$ there is $\lambda' \in \Lambda_k$ satisfying

$$(1 - 3\delta_k^i - 36(\delta_k^i)^{\frac{1}{4}}) \|b_k^j\|_2^2 \leq |\tau(s_k^j w w_k^* a_k^i u_\gamma w_k w^* (s_k^j)^* v_{\lambda'} b_k^j)|. \quad (5.0.17)$$

Proof of Claim 5.5. As $\tau(a_k^i) = \tau(w w_k^* a_k^i w_k w^*) \leq \tau(p_k^j)$ there is a subprojection $c_k^j \in B_k \rtimes \Lambda$ of b_k^j that is equivalent (in $B_k \rtimes \Lambda$) to $w w_k^* a_k^i w_k w^*$. By [22, Lemma 4.1] one can find a unitary $s_k^j \in B_k \rtimes \Lambda$ satisfying $s_k^j w w_k^* a_k^i w_k w^* (s_k^j)^* = c_k^j$, $[s_k^j, |w w_k^* a_k^i w_k w^* - c_k^j|] = 0$ and $|s_k^j - 1| \leq 3|w w_k^* a_k^i w_k w^* - c_k^j|$.

Using (5.0.15) we see that

$$\begin{aligned} \|w w_k^* a_k^i w_k w^* - c_k^j\|_2^2 &\leq 2(2(\delta_k^i)^{\frac{1}{2}} + \delta_k^i)^2 \|b_k^j\|_2^2 + 2\|b_k^j - c_k^j\|_2^2 \\ &= 2(2(\delta_k^i)^{\frac{1}{2}} + \delta_k^i)^2 \|b_k^j\|_2^2 + 2\tau(b_k^j - c_k^j) \\ &= (2(2(\delta_k^i)^{\frac{1}{2}} + \delta_k^i)^2 + 2) \|b_k^j\|_2^2 - 2\|a_k^i\|_2^2 \\ &\leq ((2(2(\delta_k^i)^{\frac{1}{2}} + \delta_k^i)^2 + 2) - 2(1 - 2(\delta_k^i)^{\frac{1}{2}})^2) \|b_k^j\|_2^2 \\ &\leq 18(\delta_k^i)^{\frac{1}{2}} \|b_k^j\|_2^2. \end{aligned}$$

Combining with the previous inequality we get $\|s_k^j - 1\|_2 \leq 18(\delta_k^i)^{\frac{1}{4}} \|b_k^j\|_2$. In turn this together with Cauchy-Schwarz inequality and (5.0.8) show that for every $\gamma \in \Gamma_k$ there is $\lambda' \in \Lambda$ such that

$$\begin{aligned} |\tau(s_k^j w w_k^* a_k^i u_\gamma w_k w^* (s_k^j)^* v_{\lambda'} b_k^j)| &\geq |\tau(w w_k^* u_\gamma a_k^i w_k w^* v_{\lambda'} b_k^j)| - 2\|1 - s_k^j\|_2 \|b_k^j\|_2 \\ &\geq (1 - 3\delta_k^i - 36(\delta_k^i)^{\frac{1}{4}}) \|b_k^j\|_2^2. \end{aligned}$$

This shows (5.0.17). Also since $\|c_k^j v_{\lambda'} b_k^j\|_2 \geq |\tau(s_k^j w w_k^* u_\gamma a_k^i w_k w^* (s_k^j)^* v_{\lambda'} b_k^j)|$ the above inequality also shows that $\lambda' \in \Lambda_k$. The rest of the statement follows from the previous relations. \square

Since $a_k^i (A_{s+k} \rtimes \Gamma) a_k^i = L(\Gamma_k) a_k^i$ and $b_k^j (B_k \rtimes \Lambda) b_k^j = L(\Lambda_k) b_k^j$ then (5.0.16) of Claim 5.5 implies that $s_k^j w w_k^* L(\Gamma_k) a_k^i w_k w^* (s_k^j)^* = c_k^j L(\Lambda_k) b_k^j c_k^j \subseteq L(\Lambda_k) b_k^j$. Since Λ_k and Γ_k are icc groups we see that the conditions (1) and (2) in [50, Theorem 4.1] are satisfied, where $\mathcal{G} = s_k^j w w_k^* \Gamma_k a_k^i w_k w^* (s_k^j)^*$. Also (5.0.17) shows that (3) in [50, Theorem 4.1] is also satisfied. Therefore using the conclusion of that theorem we get that $c_k^j = b_k^j$.

In conclusion we have that $s_k^j w w_k^* L(\Gamma_k) a_k^i w_k w^* (s_k^j)^* = L(\Lambda_k) b_k^j$. Notice we also have $s_k^j w w_k^* (A \rtimes \Gamma_k) a_k^i w_k w^* (s_k^j)^* = s_k^j w w_k^* a_k^i (A \rtimes \Gamma) a_k^i w_k w^* (s_k^j)^* = b_k^j (A \rtimes$

$\Gamma)b_k^j = (A \rtimes \Gamma_k)b_k^j$. Let $z \in \mathcal{N}_M(A)$ such that $z^*z = a_k^i$ and $z^*z = b_k^j$ and denote by $y = b_k^j z w_k w^*(s_k^j)^*$ one can check that y is a unitary in $(A \rtimes \Lambda_k)b_k^j$ satisfying $y(s_k^j w_k w^* A a_k^i w_k w^*(s_k^j)^*)y^* = A b_k^j$. Thus applying Theorem 4.5 (working with the algebra $(A \rtimes \Lambda_k)b_k^j$) we get the desired conclusion by letting $p = a_k^i$ $q = b_k^j$ etc. \square

Final remarks. We notice that (5.0.17) can be used directly to show that Γ_k is isomorphic to finite index subgroup of Λ_k . It is plausible that one can exploit this further and show the conclusion directly, without appealing to the results in [42,50].

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