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CONTENT OF LOCAL COHOMOLOGY, PARAMETER IDEALS, AND ROBUST ALGEBRAS

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Abstract. This paper continues the investigation of quasilength, of content of local cohomology with respect to generators of the support ideal, and of robust algebras begun in joint work of Hochster and Huneke. We settle several questions raised by Hochster and Huneke. In particular, we give a family of examples of top local cohomology modules both in equal characteristic 0 and in positive prime characteristic that are nonzero but have content 0. We use the notion of a robust forcing algebra (the condition turns out to be strictly stronger than the notion of a solid forcing algebra in, for example, equal characteristic 0) to define a new closure operation on ideals. We prove that this new notion of closure coincides with tight closure for ideals in complete local domains of positive characteristic, which requires proving that forcing algebras for instances of tight closure are robust, and study several related problems. This gives, in effect, a new characterization of tight closure in complete local domains of positive characteristic. As a byproduct, we also answer a question of Lyubeznik in the negative.

1. Introduction

This paper continues the study of the notions of quasilength, content, and robustness initiated in [HH09] and resolves questions raised in that paper. We use these ideas to give a new characterizaton of tight closure in equal characteristic $p > 0$. The condition we use is similar to the characterization of tight closure in [Hoc94] using solid closure, but the condition we impose on the forcing algebras that arise is *a priori* stronger than being solid. The stronger condition, which is that the forcing algebra be *robust* (defined briefly two paragraphs below, and with more detail in Definition 5.9), nonetheless gives the usual notion of tight closure for ideals in complete local domains.

There is great interest in extending the notion of tight closure to rings of mixed characteristic. An example of Paul Roberts [Rob94] shows that solid closure is not the right notion, in that even in a regular local ring of dimension 3 in equal characteristic 0, it is not true that every ideal is solidly closed. Roberts proves this by showing that a certain forcing algebra is solid when one hopes it should not be. Specifically, let k be a field of characteristic 0, let A be the formal power series ring $K[[x_1, x_2, x_3]]$, and let $R = A[z_1, z_2, z_3]/(g)$ where $g = x_1^2x_2^2x_3^2 - \sum_{i=1}^3 z_i x_i^3$. R is a generic forcing algebra over the element $x_1^2x_2^2x_3^2$ and the ideal $(x_1^3, x_2^3, x_3^3)A$.

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Roberts shows that $H^3_{(x_1,x_2,x_3)}(R) \neq 0$, so that R is a solid A-algebra; see §5 and Definition 5.6 in particular. However, this algebra is *not* robust (*cf.* [HH09, Example 3.11] and Definition 5.9). It remains an intriguing open question to determine the content c of $H^3_{(x_1,x_2,x_3)}(R)$ with respect to x_1, x_2, x_3. The fact that g vanishes in R implies at once that $c \leq 26/27$, but so far as the authors know, c might be 0.

In §2 we review the needed background concerning quasilength, content, and robust algebras. In §3 we study quasilength and content for modules over a local ring with respect to a system of parameters. We give some surprising examples of the failure of additivity on direct sums. However, over an equicharacteristic local domain of Krull dimension d, it is true that the content of the d-th local cohomology module of a module of torsion-free rank r with respect to a system of parameters is r. Theorem 3.7 gives a more general statement.

A sequence of elements $x_1, \ldots, x_d$ generating a proper ideal of a ring S is called a Q-*sequence* if for every positive integer t, any finite filtration of $S/(x_1^t, \ldots, x_d^t)$ with factors that are quotients of $S/(x_1, \ldots, x_d)S$ requires at least d^t factors. An algebra S over a local ring (R, m, K) of Krull dimension d is *robust* if every system of parameters $x_1, \ldots, x_d$ of R is a Q-sequence in S. That S is robust implies that $H^d_m(S) \neq 0$. The condition that $H^d_m(S) \neq 0$ is equivalent to the condition that S be solid if R is complete (*cf.* [Hoc94, Corollary 2.4]).

In §4 we show that there are nonzero local cohomology modules whose content is 0 in all characteristics, at the same time answering a question raised in [HH09, Example 3.14]. Specifically, let $S = A[x, y, u, v]$ be a polynomial ring in 4 variables over a Noetherian commutative ring A, and let $R = A[xu, yv, xv + yu]$. Then $H^3_{(xu,yv,xv+yu)}(S) \neq 0$, and hence S is a solid R-algebra. In [HH09], it was asked whether xu, yv, $xv + yu$ is a Q-sequence. It is shown here that the content with respect to xu, yv, $xv + yu$ is 0 in all characteristics. If xu, yv, $xv + yu$ were a Q-sequence, the content would be 1.

Section 5 gives the definition of robust closure, and in §6 we prove that it agrees with tight closure in complete local domains of characteristic $p > 0$. This is equivalent to the following statement, which is one of our main results.

Theorem 1.1 (Theorem 6.1)**.** *Let R be a local ring of prime characteristic $p > 0$. Suppose that $I = (f_1, \ldots, f_h)$ is an ideal of R, that $g \in R$, that $g \in I^*$, the tight closure of I, and that*

$$S = R[Z_1, \ldots, Z_h]/(g - \sum_{i=1}^{h} f_i Z_i)$$

(which is a generic forcing algebra for $(R, (f_1, \ldots, f_h), g)$; see Definition 5.1*). Then S is a robust R-algebra.*

Section 7 discusses the relationship between being a Q-sequence and superheight. In a Noetherian ring R, whether $x_1, \ldots, x_d$ is a Q-sequence is related to whether the height of $(x_1, \ldots, x_d)S$ becomes d in some Noetherian R-algebra S. In equal characteristic, the latter condition is sufficient for $x_1, \ldots, x_d$ to be a Q-sequence. We show, however, that it is not necessary. The examples are subtle, and the proofs depend on difficult theorems. Our examples also answer negatively a question of Gennady Lyubeznik (page 144 in [Lyu02]) on the vanishing of local cohomology modules; *cf.* Proposition 7.7.

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Section 8 describes some conjectures and questions that are related to the results of this paper. In particular, it discusses several conjectures that, so far as we know, are strictly stronger than the direct summand conjecture. While the direct summand conjecture has now been proved by Y. André [And16b, And16a] and B. Bhatt [Bha16] (Bhatt uses part of André's results but gives a simpler proof as well as a proof of a derived version suggest by de Jong), it would be desirable to have other proofs that do not use perfectoid geometry. Moreover, the results of André and Bhatt do not solve the problem of giving a closure operation in mixed characteristic that has properties analogous to tight closure.

2. Preliminaries

In this section we collect some basic facts about content of local cohomology from [HH09]. We begin with the definition of *quasilength*.

Definition 2.1 (Quasilength)**.** Let R be a commutative ring (not necessarily Noetherian), I a finitely generated ideal of R, and M an R-module.

M is defined to have *finite I-quasilength* if there is a finite filtration of M in which the factors are cyclic modules killed by I.

The *I-quasilength* of M, denoted by $\mathcal{L}_I(M)$, is defined to be the minimum number of factors in such a filtration. If M does not have finite I-quasilength, then its I-quasilength is defined to be ∞.

One can check that M has finite I-quasilength if and only if M is finitely generated and is killed by a power of I and $\mathcal{L}_I(M)$ is bounded below by the least number of generators of M (*cf.* Proposition 1.1(a) of [HH09]).

Assume R is a Noetherian commutative ring and M is a finitely generated R-module. Let $x_1, \ldots, x_d$ be elements of R and let $I = (x_1, \ldots, x_d)$. We will use $\underline{t}$ to denote the d-tuple of positive integers $(t_1, \ldots, t_d)$, $\underline{t} + k$ to denote the d-tuple $(t_1 + k, \ldots, t_d + k)$, and $I_{\underline{t}}$ to denote $(x_1^{t_1}, \ldots, x_d^{t_d})$. One can define

$$(I_{\underline{t}}M)^{\lim} = \bigcup_{k=0}^{\infty} ((I_{\underline{t}+k}M) :_M (x_1 \cdots x_d)^k).$$

If we set $\underline{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$, then one also has that

$$(I_{\underline{t}}M)^{\lim} = \bigcup_{\underline{k} \in \mathbb{N}^d} ((I_{\underline{t}+k}M) :_M (x_1^{k_1} \cdots x_d^{k_d})).$$

As we will see $(I_{\underline{t}}M)^{\lim}$ is closely related to the local cohomology module $H^d_I(M)$. It is well-known that $H^d_I(M) = \varinjlim_{\underline{t}} M/I_{\underline{t}}M$ where in the direct system the map $M/I_{\underline{t}}M \to M/I_{\underline{t}+\underline{k}}M$ is induced by multiplication by $x_1^{k_1} \cdots x_d^{k_d}$ on the numerators. It follows that $(I_{\underline{t}}M)^{\lim}$ is the kernel of the canonical map $M \to M/I_{\underline{t}}M \to H^d_I(M)$. Therefore

$$H^d_I(M) = \varinjlim_{\underline{t}} M/(I_{\underline{t}}M)^{\lim}$$

in which all maps in the direct system are injective and each $M/(I_{\underline{t}}M)^{\lim}$ in this direct system can be viewed as a submodule of $H^d_I(M)$.

Now we are in position to introduce the $\underline{\mathfrak{h}}$-*content* of $H^d_I(M)$ *with respect to* $\underline{x}$ ([HH09, page 3177]).

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Definition 2.2. With $\underline{t} \geq s$ for $s \in \mathbb{N}$ meaning that every $t_j \geq s$, we define

$$\mathfrak{h}^d_{\underline{x}}(M) = \lim_{s\to\infty} \inf\{\frac{\mathcal{L}_I(M/(I_{\underline{t}}M)^{\lim})}{t_1 \cdots t_d} | \underline{t} \geq s\}.$$

The existence of $\underline{\mathfrak{h}}^d_{\underline{x}}(M)$ is guaranteed by [HH09, Proposition 2.1]:

$$\underline{\mathfrak{h}}^d_{\underline{x}}(M) = \lim_{s\to\infty} \inf\{\frac{\mathcal{L}_I(M/(I_{\underline{t}}M))}{t_1 \cdots t_d} | \underline{t} \geq s\}.$$

If $\mathfrak{h}^d_{\underline{x}}(R) = 1$, then according to [HH09, Theorem 3.8] one also has $\underline{\mathfrak{h}}^d_{\underline{x}}(R) = 1$; in this case $\underline{x}$ is called a Q-*sequence* and $H^d_I(R)$ is called *robust* for $\underline{x}$.

By [HH09, Theorem 5.1], if R is a local ring and $x_1, \ldots, x_d$ is a system of parameters, then one has $\underline{\mathfrak{h}}^d_{\underline{x}}(M) = \mathfrak{h}^d_{\underline{x}}(M)$. This is the reason we will mainly work with $\mathfrak{h}^d_{\underline{x}}(M)$ in the next section, where we are only concerned with parameters.

3. Content of local cohomology with respect to parameters

Proposition 3.1. *Let $x_1, \ldots, x_d$ be elements of a commutative ring R and let $0 \to L \to M \to N \to 0$ be a short exact sequence of finite R-modules. Then*

$$\mathfrak{h}^d_{\underline{x}}(M) \leq \mathfrak{h}^d_{\underline{x}}(L) + \mathfrak{h}^d_{\underline{x}}(N).$$

Proof. Let $I = (x_1, \ldots, x_d)$ and denote $(x_1^t, \ldots, x_d^t)$ by I_t for all integers $t \geq 1$. It is clear that one has the following short exact sequence:

$$0 \to L/L \cap I_tM \to M/I_tM \to N/I_tN \to 0.$$

Hence $\mathcal{L}_I(M/I_tM) \leq \mathcal{L}_I(L/L \cap I_tM) + \mathcal{L}_I(N/I_tN)$ by [HH09, Proposition 1.1(d)]. Since there is a natural surjection $L/I_tL \twoheadrightarrow L/L \cap I_tM$, one has $\mathcal{L}_I(L/L \cap I_tM) \leq \mathcal{L}_I(L/I_tL)$ by [HH09, Proposition 1.1(d)]. Therefore,

$$\mathcal{L}_I(M/I_tM) \leq \mathcal{L}_I(L/I_tL) + \mathcal{L}_I(N/I_tN).$$

Dividing both sides by t^d and taking limit over $t \to \infty$ give us the desired inequality. □

Proposition 3.2. *Let R be a d-dimensional equicharacteristic local ring and let $x_1, \ldots, x_d \in R$ be a system of parameters. Then*

$$\mathfrak{h}^d_{\underline{x}}(R^n) = n$$

for all integers $n \geq 1$.

Proof. It follows directly from [HH09, Proposition 3.4] that the condition that $\mathfrak{h}^d_{\underline{x}}(R^n) = n$ is equational, and hence reduction to characteristic $p > 0$ is applicable. Therefore, it suffices to prove $\mathfrak{h}^d_{\underline{x}}(R^n) = n$ in characteristic $p > 0$.

Use induction on n. When $n = 1$, this is precisely [HH09, Theorem 4.7].

Assume that $n \geq 2$. The short exact sequence $0 \to R \to R^n \to R^{n-1} \to 0$ and Proposition 3.1 imply that $\mathfrak{h}^d_{\underline{x}}(R^n) \leq \mathfrak{h}^d_{\underline{x}}(R) + \mathfrak{h}^d_{\underline{x}}(R^{n-1})$. By induction, $\mathfrak{h}^d_{\underline{x}}(R^{n-1}) = n - 1$, and hence $\mathfrak{h}^d_{\underline{x}}(R^n) \leq n$.

Assume that $\mathfrak{h}^d_{\underline{x}}(R^n) < n$. Let $I = (x_1, \ldots, x_d)$ and denote $(x_1^t, \ldots, x_d^t)$ by I_t for all integers $t \geq 1$. By [HH09, Theorem 2.4] and [HH09, Remark 3.7], there exist an integer $q = p^e$ and a filtration

$$0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_h = R^n/I_qR^n$$

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of R^n/I_qR^n with M_{j+1}/M_j a homomorphic image of R/I and $h < nq^d$. As a consequence, one has

$$\lambda(R^n/I_qR^n) \leq \sum_{j=0}^{h-1} \lambda(M_{j+1}/M_j) \leq h\lambda(R/I).$$

Applying the e-th Frobenius functor to the above filtration, one has a filtration (with a slight abuse of notation)

$$0 \subsetneq M_1^{[q]} \subsetneq \cdots \subsetneq M_h^{[q]} = R^n/I_{q^2}R^n$$

of $R^n/I_{q^2}R^n$ with $M_{j+1}^{[q]}/M_j^{[q]}$ a homomorphic image of R/I_q. Consequently, one has the following filtration of $R^{n^2}/I_{q^2}R^{n^2}$:

$$0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_h = R^{n^2}/I_{q^2}R^{n^2},$$

where F_j is the direct sum of n copies of $M_j^{[q]}$. It is clear that F_{j+1}/F_j is a homomorphic image of R^n/I_qR^n and hence

$$\lambda(R^{n^2}/I_{q^2}R^{n^2}) \leq \sum_{j=0}^{h-1} \lambda(F_{j+1}/F_j) \leq h\lambda(R^n/I_qR^n) \leq h^2\lambda(R/I).$$

Similarly, one can prove that

$$n^l\lambda(R/I_{q^l}) = \lambda(R^{n^l}/I_{q^l}R^{n^l}) \leq h^l\lambda(R/I)$$

for all integers $l \geq 1$. Dividing both sides by $(nq^d)^l$, one has

$$\frac{\lambda(R/I_{q^l})}{q^{ld}} \leq (\frac{h}{nq^d})^l\lambda(R/I)$$

for all integers $l \geq 1$. Since $h < nq^d$, it follows that

$$\lim_{l\to\infty} \frac{\lambda(R/I_{q^l})}{q^{ld}} = 0,$$

which is absurd because $\lim_{l\to\infty} \frac{\lambda(R/I_{q^l})}{q^{ld}}$ is exactly $e(I_q, R)$ (the multiplicity of R with respect to I_q), which is positive.

Therefore, one has $\mathfrak{h}^d_{\underline{x}}(R^n) = n$. □

Corollary 3.3. *Let R be an equicharacteristic local ring and let $x_1, \ldots, x_s \in R$ be part of a system of parameters of R. Then*

$$\mathfrak{h}^s_{\underline{x}}(R^n) = n$$

for all integers $n \geq 1$.

Proof. Use induction on n.

When $n = 1$, it follows from [HH09, Proposition 1.2(a)] that $\mathfrak{h}^s_{\underline{x}}(R) \leq 1$. Let P be a prime ideal minimal over $I = (x_1, \ldots, x_s)$ with $\mathrm{ht}(P) = s$. According to [HH09, Proposition 2.5], we have $\mathfrak{h}^s_{\underline{x}}(R) \geq \mathfrak{h}^s_{\underline{x}}(R_P)$. But $\mathfrak{h}^s_{\underline{x}}(R_P) = 1$ by Proposition 3.2 since $x_1, \ldots, x_s$ is a system of parameters of R_P. Hence $\mathfrak{h}^s_{\underline{x}}(R) \geq 1$. Therefore,

$$\mathfrak{h}^s_{\underline{x}}(R) = 1.$$

Assume that $n \geq 2$ and $\mathfrak{h}^s_{\underline{x}}(R^{n-1}) = n - 1$. It follows from Proposition 3.1 that $\mathfrak{h}^s_{\underline{x}}(R^n) \leq n$. According to [HH09, Proposition 2.5], we have $\mathfrak{h}^s_{\underline{x}}(R^n) \geq \mathfrak{h}^s_{\underline{x}}(R^n_P)$.

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But $\mathfrak{h}^s_{\underline{x}}(R^n_P) = n$ by Proposition 3.2 since $x_1, \ldots, x_s$ is a system of parameters of R_P. Hence $\mathfrak{h}^s_{\underline{x}}(R^n) \geq n$. Therefore,

$$\mathfrak{h}^s_{\underline{x}}(R^n) = n.$$

□

Corollary 3.4. *Let R be a d-dimensional equicharacteristic local domain and let $x_1, \ldots, x_d \in R$ be a system of parameters. Then*

$$\mathfrak{h}^d_{\underline{x}}(M) = \operatorname{rank}(M)$$

for all finitely generated R-modules M.

In particular, under the same hypotheses, $\mathfrak{h}^d_{\underline{x}}$ is additive; i.e., given a short exact sequence $0 \to L \to M \to N \to 0$ of finitely generated R-modules, we have

$$\mathfrak{h}^d_{\underline{x}}(M) = \mathfrak{h}^d_{\underline{x}}(L) + \mathfrak{h}^d_{\underline{x}}(N).$$

Proof. Let $I = (x_1, \ldots, x_d)$ and let T denote the torsion submodule of M. Then $\dim(T) < d$ and hence $H^d_I(T) = 0$. By [HH09, Proposition 2.2], one has $\mathfrak{h}^d_{\underline{x}}(T) = 0$. The short exact sequence $0 \to T \to M \to M/T \to 0$ and Proposition 3.1 imply that

$$\mathfrak{h}^d_{\underline{x}}(M/T) \leq \mathfrak{h}^d_{\underline{x}}(M) \leq \mathfrak{h}^d_{\underline{x}}(T) + \mathfrak{h}^d_{\underline{x}}(M/T) = \mathfrak{h}^d_{\underline{x}}(M/T).$$

Hence, we may assume that M is torsion-free. Consequently, there is a short exact sequence $0 \to M \to R^r \to R^r/M \to 0$ where $r = \operatorname{rank}(M)$ and $\dim(R^r/M) < d$ (hence $\mathfrak{h}^d_{\underline{x}}(R^r/M) = 0$). Thus, one has

$$\operatorname{rank}(M) = r = \mathfrak{h}^d_{\underline{x}}(R^r) \leq \mathfrak{h}^d_{\underline{x}}(M) + \mathfrak{h}^d_{\underline{x}}(R^r/M) = \mathfrak{h}^d_{\underline{x}}(M).$$

Let $g_1, \ldots, g_r$ be elements in M whose images in $(R\backslash 0)^{-1}M$ form a basis, and let $M' = \sum_{j=1}^r Rg_j$. Then one has a short exact sequence $0 \to M' \to M \to M/M' \to 0$. Since $(R\backslash 0)^{-1}(M/M') = 0$, it follows that $\dim(M/M') < d$ and hence $\mathfrak{h}^d_{\underline{x}}(M/M') = 0$. By Proposition 3.1, one has

$$\mathfrak{h}^d_{\underline{x}}(M) \leq \mathfrak{h}^d_{\underline{x}}(M') + \mathfrak{h}^d_{\underline{x}}(M/M') = \mathfrak{h}^d_{\underline{x}}(M').$$

It is clear that there is a surjection $R^r \to M'$ and hence $\mathfrak{h}^d_{\underline{x}}(M') \leq \mathfrak{h}^d_{\underline{x}}(R^r) = r$. Therefore, $\mathfrak{h}^d_{\underline{x}}(M) \leq r$. This finishes the proof. □

From Corollary 3.4, one may expect quasilength to be additive under the same hypotheses as well. However, quasilength does not respect direct sum even when R is a DVR, as shown in the following example.

Example 3.5. Let $R = k[[x]]$, let $I = (x^2)$, let $M = R/(x^3)$, and let $N = R/(x) = k$. Then it is easy to see that $\mathcal{L}_I(N) = 1$. Since M is not killed by I, one has $\mathcal{L}_I(M) \geq 2$; on the other hand, one has an I-filtration

$$0 \subsetneq M_1 = R \cdot x^2 \subsetneq M_2 = M,$$

hence $\mathcal{L}_I(M) = 2$.

We will prove that $\mathcal{L}_I(M \oplus N) = 2 \neq 3$ by constructing a filtration of $M \oplus N$ with only 2 factors that are homomorphic images of R/I (*a priori*, $\mathcal{L}_I(M \oplus N) \geq \max\{\mathcal{L}_I(M), \mathcal{L}_I(N)\} = 2$).

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Let $L_1 = R \cdot (x, 1)$ and $L_2 = R \cdot (x, 1) + R \cdot (1, 1)$. It is easy to check that

(1) $IL_1 = 0$;
(2) $IL_2 \subset L_1$ and L_2/L_1 is cyclic;
(3) $L_2 = M \oplus N$.

Therefore

$$0 \subsetneq L_1 \subsetneq L_2 = M \oplus N$$

is the desired filtration.

When R is not a domain, we can't expect $\mathfrak{h}^d_{\underline{x}}$ to be additive.

Example 3.6. Let $R = k[[u, v]]/(uv)$ where u and v are indeterminates over a field k. Set $x = u + v$. Then x is a system of parameters of R ($\dim(R) = 1$). Since R is reduced and has only two minimal prime ideals (u) and (v), we have a short exact sequence

$$0 \to R \to R/(u) \oplus R/(v) \to R/(u, v) \to 0.$$

It is clear that $\mathfrak{h}^1_{\underline{x}}(R/(u, v)) = 0$. As a consequence of Proposition 3.1, we have $\mathfrak{h}^1_{\underline{x}}(R/(u) \oplus R/(v)) \leq \mathfrak{h}^1_{\underline{x}}(R) = 1$. But, it is easy to check that

$$\mathfrak{h}^1_{\underline{x}}(R/(u)) = \mathfrak{h}^1_{\underline{x}}(R/(v)) = 1,$$

and hence

$$1 = \mathfrak{h}^1_{\underline{x}}(R/(u) \oplus R/(v)) \neq \mathfrak{h}^1_{\underline{x}}(R/(u)) + \mathfrak{h}^1_{\underline{x}}(R/(v)) = 2.$$

This example suggests that $\mathfrak{h}^d_{\underline{x}}(R)$ should not exceed $\mathfrak{h}^d_{\underline{x}}(R/P)$ for all minimal prime ideals P. The precise formula is given in the following theorem.

Theorem 3.7. *Let R be a d-dimensional equicharacteristic local ring and suppose that $x_1, \ldots, x_d$ is a system of parameters of R. Let $\mathrm{Min}_d(R)$ denote the set of minimal prime ideals P of R with $\dim(R/P) = d$. Then for all finitely generated R-modules M,*

$$\mathfrak{h}^d_{\underline{x}}(M) = \max\{\mathrm{rank}_{R/P}(M/PM) | P \in \mathrm{Min}_d(R)\}.$$

Proof. If $\dim(M) < d$, then $H^d_{(\underline{x})}(M) = 0$, and hence $\mathfrak{h}^d_{\underline{x}}(M) = 0$. It is clear that $\mathrm{rank}_{R/P}(M/PM) = 0$ for all $P \in \mathrm{Min}(R)$. Thus, the theorem is true when $\dim(M) < d$.

Assume that $\dim(M) = d$. Let N be the largest submodule of M with dimension smaller than d. The short exact sequence $0 \to N \to M \to M/N \to 0$ and Proposition 3.1 imply that $\mathfrak{h}^d_{\underline{x}}(M) = \mathfrak{h}^d_{\underline{x}}(M/N)$. Hence, we may assume that M is of pure dimension d. Since M is naturally an $R/\operatorname{Ann}_R(M)$-module, replacing R by $R/\operatorname{Ann}_R(M)$, we may assume that M is faithful (consequently R is also of pure dimension d). Let P be a minimal prime ideal of R and let y_i denote the image of x_i in R/P. Then by [HH09, Proposition 2.5], $\mathfrak{h}^d_{\underline{x}}(M) \geq \mathfrak{h}^d_{\underline{y}}(M/PM)$; the latter is $\mathrm{rank}_{R/P}(M/PM)$ by Corollary 3.4. Therefore,

$$\mathfrak{h}^d_{\underline{x}}(M) \geq \max\{\mathrm{rank}_{R/P}(M/PM) | P \in \mathrm{Min}_d(R)\}.$$

Without loss of generality, we may assume that $\mathrm{Min}_d(R) \subset \mathrm{Supp}(M)$. For each $P \in \mathrm{Min}_d(R)$, we have a filtration

$$0 \subsetneq L(P)_1 \subsetneq \cdots \subsetneq L(P)_{\mathrm{rank}_{R/P}(M/PM)} = M_P$$

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with $L(P)_{j+1}/L(P)_j \cong R_P/PR_P (= ((R/P)\backslash\{0\})^{-1}R/P)$. Set

$$h = \max\{\operatorname{rank}_{R/P}(M/PM) | P \in \operatorname{Min}_d(R)\}$$

and extend the above filtration trivially (if necessary) as follows:

$$0 \subsetneq L(P)_1 \subsetneq \cdots \subsetneq L(P)_{\operatorname{rank}_{R/P}(M/PM)} = \cdots = L(P)_h = M_P.$$

Let $L_j = \bigoplus_{P\in\operatorname{Min}_d(R)} L(P)_j$. Then we have a filtration

$$0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_h = \bigoplus_{P\in\operatorname{Min}_d(R)} M_P \tag{3.7.1}$$

with L_{j+1}/L_j a homomorphic image of $\bigoplus_{P\in\operatorname{Min}_d(R)} R_P$.

Let $W = R\backslash\bigcup_{P\in\operatorname{Min}_d(R)} P$. It is clear that $W^{-1}R = \bigoplus_{P\in\operatorname{Min}_d(R)} R_P$ and $W^{-1}M = \bigoplus_{P\in\operatorname{Min}_d(R)} M_P$ (since $W^{-1}R$ is artinian). Thus, the filtration (3.7.1) gives us a filtration of $W^{-1}M$ with each factor a homomorphic image of $W^{-1}R$, which in turn gives us a filtration of M,

$$0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_h = M,$$

such that there is a surjection $W^{-1}R \to W^{-1}(M_{j+1}/M_j)$ for each j.

Let g be an element of M_{j+1}/M_j whose image in $W^{-1}(M_{j+1}/M_j)$ generates $W^{-1}(M_{j+1}/M_j)$. Then we have a short exact sequence $0 \to Rg \to M_{j+1}/M_j \to (M_{j+1}/M_j)/Rg \to 0$. Since $W^{-1}Rg = W^{-1}(M_{j+1}/M_j)$, we have

$$W^{-1}((M_{j+1}/M_j)/Rg) = 0.$$

Thus, $\dim((M_{j+1}/M_j)/Rg) < d$ and hence $\mathfrak{h}^d_{\underline{x}}((M_{j+1}/M_j)/Rg) = 0$. It follows from Proposition 3.1 that

$$\mathfrak{h}^d_{\underline{x}}(M_{j+1}/M_j) \le \mathfrak{h}^d_{\underline{x}}(Rg) + \mathfrak{h}^d_{\underline{x}}((M_{j+1}/M_j)/Rg) = \mathfrak{h}^d_{\underline{x}}(Rg) \le 1.$$

Furthermore, as a consequence of Proposition 3.1, we have

$$\mathfrak{h}^d_{\underline{x}}(M) \le \sum_{j=1}^{h-1} \mathfrak{h}^d_{\underline{x}}(M_{j+1}/M_j) \le h = \max\{\operatorname{rank}_{R/P}(M/PM) | P \in \operatorname{Min}_d(R)\},$$

which finishes the proof. □

Recall that if M is a finitely generated module over a local ring R, then a *system of parameters* of M is a sequence of $d = \dim(M)$ elements $x_1, \ldots, x_d$ in R such that $\dim(M/(x_1, \ldots, x_d)M) = 0$.

Corollary 3.8. *Let R be an equicharacteristic local ring and let M be a finite R-module of dimension d. Let $\operatorname{Min}_d(M)$ denote the set of prime ideals P minimal over $\operatorname{Ann}_R(M)$ with $\dim(R/P) = \dim(M)$. Then for any system of parameters $x_1, \ldots, x_d$ of M,*

$$\mathfrak{h}^d_{\underline{x}}(M) = \max\{\operatorname{rank}_{R/P}(M/PM) | P \in \operatorname{Min}(M)\}.$$

Proof. It is clear that M is naturally an $R/\operatorname{Ann}_R(M)$-module. By the remark on page 3179 in [HH09], we may replace R by $R/\operatorname{Ann}_R(M)$ and assume that $x_1, \ldots, x_d$ is also a system of parameters of R. Then Theorem 3.7 finishes the proof. □

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4. A nonzero local cohomology module whose content is 0

Let $R = A[x, y, u, v]$, where A is any reduced Noetherian commutative ring. Let $P = (x, y)$, $Q = (u, v)$, and $I = (x_1, x_2, x_3)$, where $x_1 = xu$, $x_2 = yv$, and $x_3 = xv + yu$. Then one can check that $J = P \cap Q$ is the radical of I. It is left open in [HH09, Example 3.14] whether x_1, x_2, x_3 forms a Q-sequence when $A = k$ is a field. Our goal in this section is to prove that

$$\mathfrak{h}^3_{\underline{x}}(R) = \mathfrak{h}^3_{\underline{x}}(R) = 0$$

no matter what A is, and consequently that $\underline{x}$ is not a Q-sequence for any choice of A. To the best of our knowledge, this is the first example of a nonzero local cohomology module whose content is 0.

We begin with an easy comparison between the ideals $I_t = (x_1^t, x_2^t, x_3^t)$ and $I'_t = (x_1^t, x_2^t, x^t v^t + y^t u^t)$.

Proposition 4.1. *We have*

(1) $(xv)^2, (yu)^2 \in I$,
(2) $I_{4t} \subseteq I'_t$,
(3) $I'_{12t} \subseteq I_t$.

Proof. (1) It is clear that we have

$$(xv)^2 = x_3^2 - yux_3 - x_1x_2 \in I \quad \text{and} \quad (yu)^2 = x_3^2 - xvx_3 - x_1x_2 \in I.$$

(2) It suffices to show that $x_3^{4t} \in I'_t$. It is easy to see that $(xv)^{2t}, (yu)^{2t} \in I'_t$. Since x_3^{4t} is a linear combination of $(xv)^i(yu)^{4t-i}$ in which at least either $i \geq 2t$ or $4t - i \geq 2t$, we have that $x_3^{4t} \in I'_t$.

(3) It suffices to show that $(xv)^{12t} + (yu)^{12t} \in I_t$. First we note that $I^{3t} \subseteq I_t$ since I is generated by 3 elements. Consequently,

$$(xv)^{6t} = ((xv)^2)^{3t} \in I^{3t} \subseteq I_t.$$

And similarly, $(yu)^{6t} \in I_t$. Now we have

$$(xv)^{12t} + (yu)^{12t} = (xv + yu)^{12t} - (xv)^{6t}\alpha - (yu)^{6t}\beta,$$

for some $\alpha, \beta \in R$. Hence, $(xv)^{12t} + (yu)^{12t} \in I_t$. □

We ought to remark that the subscripts $4t$ and $12t$ in Proposition 4.1(2) and (3), respectively, are not optimal.

Let T be a commutative Noetherian ring and let $\mathfrak{a} = (a_1, \ldots, a_t)$ and $\mathfrak{b} = (b_1, \ldots, b_t)$ be ideals of R. Let $H^i(\underline{a})$ (respectively, $H^i(\underline{b})$) denote the $(t - i)$-th homology of the Koszul complex associated to $a_1, \ldots, a_t$ (respectively, to $b_1, \ldots, b_t$). We recall the following result from [SV86].

Lemma 4.2 (Lemma 1.5 in [SV86])**.** *Let $T, \mathfrak{a}, \mathfrak{b}, H^i(\underline{a}), H^i(\underline{b})$ be as above. If $\mathfrak{a} \subseteq \mathfrak{b}$, then we have $t + 1$ commutative diagrams:*

$$\begin{array}{ccc} H^i(\underline{b}) & \longrightarrow & H^i(\underline{a}) \\ \downarrow & & \downarrow \\ H^i_{\mathfrak{b}}(T) & \longrightarrow & H^i_{\mathfrak{a}}(T) \end{array}$$

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As a consequence of this lemma, one can see that the map $R/\mathfrak{a} \to H^t_{\mathfrak{a}}(R)$ is canonical in the sense that if $(a_1, \ldots, a_t) = (\alpha_{ij})(b_1, \ldots, b_t)$ for a matrix (α_{ij}) with $\alpha_{ij} \in T$ (i.e., $\mathfrak{a} \subseteq \mathfrak{b}$) and $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$, then we have a commutative diagram

$$\begin{array}{ccc} R/\mathfrak{b} & \xrightarrow{\det(\alpha_{ij})} & R/\mathfrak{a} \\ & \searrow \quad \swarrow & \\ & H^t_{\mathfrak{a}}(R) & \end{array} \tag{4.2.1}$$

Theorem 4.3. *Let $R, I, J, P, Q, \underline{x}$ be as above. Then $\mathfrak{h}^3_{\underline{x}}(R) = \underline{\mathfrak{h}}^3_{\underline{x}}(R) = 0$.*

Proof. It is clear that $\varinjlim_t R/I_t = H^3_I(R) = H^3_J(R)$, where the last equality holds since J is the radical of I. We also note that, by the Mayer-Vietoris sequence

$$\cdots \to 0 = H^3_P(R) \oplus H^3_Q(R) \to H^3_J(R) \to H^4_{P+Q}(R) \to H^4_P(R) \oplus H^4_Q(R) = 0 \to \cdots,$$

we have $H^3_J(R) = H^4_{(x,y,u,v)}(R)$. Let $\mathfrak{m} = (x, y, u, v)$ and $\mathfrak{m}_t = (x^t, y^t, u^t, v^t)$ for $t \in \mathbb{N}$.

Since every element of $H^4_{\mathfrak{m}}(R)$ is killed by $\mathfrak{m}_t$ for some suitable t, we know that the image of $R/(xu, yv, xv + yu)$ in $H^3_I(R) = H^3_j(R) = H^4_{\mathfrak{m}}(R)$ is killed by $\mathfrak{m}_c$ for some $c \in \mathbb{N}$.

Let $F_t : R \to R$ be the endomorphism that sends x (y, u, v, respectively) to x^t (y^t, u^t, v^t, respectively) and is the identity on A. Applying F_t to the map

$$R/(xu, yv, xv + yu) \to H^4_{\mathfrak{m}}(R),$$

we see that the image of R/I'_t in $H^4_{\mathfrak{m}_t}(R) = H^4_{\mathfrak{m}}(R)$ is killed by $\mathfrak{m}_{ct}$.

Since we always have a surjection $R/P_{ct} \to R/(P_{ct} + Q_{ct}) = R/\mathfrak{m}_{ct}$, we have that

$$\mathcal{L}_J(R/\mathfrak{m}_{ct}) \leq \mathcal{L}_J(R/P_{ct}).$$

Moreover we have that

$$\mathcal{L}_J(R/P_{ct}) \leq \mathcal{L}_P(R/P_{ct}) = c^2t^2$$

since $J \subset P$. Therefore, we have that

$$\mathcal{L}_J(R/\mathfrak{m}_{ct}) \leq c^2t^2,$$

and hence

$$\mathcal{L}_J(\mathrm{Im}(R/I'_t \to H^4_{\mathfrak{m}}(R))) \leq c^2t^2.$$

Since $I'_{12t} \subseteq I_t$, there is a surjection $\mathrm{Im}(R/I'_{12t} \to H^4_{\mathfrak{m}}(R)) \to \mathrm{Im}(R/I_t \to H^4_{\mathfrak{m}}(R))$. Hence

$$\begin{aligned} \mathcal{L}_I(\mathrm{Im}(R/I_t \to H^4_{\mathfrak{m}}(R))) &\leq \mathcal{L}_J(\mathrm{Im}(R/I_t \to H^4_{\mathfrak{m}}(R))) \\ &\leq \mathcal{L}_J(\mathrm{Im}(R/I'_{12t} \to H^4_{\mathfrak{m}}(R))) \\ &\leq c^2(12t)^2 = 144c^2t^2, \end{aligned}$$

where the first inequality holds since $I \subset J$.

As discussed in §2, $\mathrm{Im}(R/I_t \to H^3_I(R) = H^4_{\mathfrak{m}}(R))$ is isomorphic to $R/(I_t)^{\mathrm{lim}}$. Therefore,

$$\underline{\mathfrak{h}}^3_{\underline{x}}(R) \leq \lim_{t\to\infty} \frac{144c^2t^2}{t^3} = 0.$$

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To finish the proof of our theorem, we will show that $\mathfrak{h}^3_{\underline{x}}(R)$ is also 0;[1] i.e., we will prove that

$$\lim_{t\to\infty}\inf\{\frac{\mathcal{L}_I(R/I_t)}{t^3}\}=0.$$

For any $s\geq t$, we have that

$$\begin{aligned}&((xu)^{t+s},(yv)^{t+s},x^{t+s}v^{t+s}+y^{t+s}u^{t+s})\\&=((xu)^t,(yv)^t,x^tv^t+y^tu^t)\begin{pmatrix}(xu)^s & 0 & -y^su^{s-t}v^t\\ 0 & (yv)^s & -x^su^tv^{s-t}\\ 0 & 0 & x^sv^s+y^su^s\end{pmatrix}.\end{aligned}$$

Hence by the commutative diagram (4.2.1), we have that $\mathrm{Im}(R/I'_t\to H^3_I(R))$ is H_t, the cyclic module generated by the image of $(xu)^s(yv)^s(x^sv^s+y^su^s)$ in $H^3_I(R)$ through $R/I'_{s+t}\to H^3_I(R)$. For $s\gg 0$, one has that H_t is isomorphic to H'_{s+t}, the submodule of R/I'_{s+t} generated by $(xu)^s(yv)^s(x^sv^s+y^su^s)$. Therefore, we have

$$\mathcal{L}_J(H'_{s+t})=\mathcal{L}_J(H_t)=\mathcal{L}_J(\mathrm{Im}(R/I'_t\to H^3_I(R)))\leq c^2t^2.$$

Next we want to get an upper bound for the quasilength of

$$R/(I'_{s+t},(xu)^s(yv)^s(x^sv^s+y^su^s))\cong\frac{R/I'_{s+t}}{H'_{s+t}}.$$

More precisely, we wish to show that

$$\mathcal{L}_I(R/(I'_{s+t},(xu)^s(yv)^s(x^sv^s+y^su^s)))\leq(t+s)^3-t^3.$$

To this end we will construct an I-filtration of $R/(I'_{s+t},(xu)^s(yv)^s(x^sv^s+y^su^s))$ with $(s+t)^3-t^3$ factors. First we linearly order the elements in

$$\mathcal{S}:=\{(xu)^{e_1}(yv)^{e_2}(x^{e_3}v^{e_3}+y^{e_3}u^{e_3})|0\leq e_1,e_2,e_3\leq t+s-1\}$$

as follows: we set

$$(xu)^{e_1}(yv)^{e_2}(x^{e_3}v^{e_3}+y^{e_3}u^{e_3})\succeq(xu)^{e'_1}(yv)^{e'_2}(x^{e'_3}v^{e'_3}+y^{e'_3}u^{e'_3})$$

if $e_1+e_2>e'_1+e'_2$, or $e_1+e_2=e'_1+e'_2$ and $e_1>e'_1$, or $e_1=e'_1$ and $e_2=e'_2$ and $e_3\geq e'_3$. Now we remove all elements $(xu)^{e_1}(yv)^{e_2}(x^{e_3}v^{e_3}+y^{e_3}u^{e_3})$ with $e_1,e_2,e_3\geq s$ from $\mathcal{S}$. Hence according to our order, the remaining $(t+s)^3-t^3$ elements can be listed as

$$(xu)^{t+s-1}(yv)^{t+s-1}(xv+yu)^{s-1},(xu)^{t+s-1}(yv)^{t+s-1}(xv+yu)^{s-2},\cdots. \tag{4.3.1}$$

Let M_h be the submodule of $R/(I'_{s+t},(xu)^s(yv)^s(x^sv^s+y^su^s))$ generated by the first h elements in (4.3.1). To see that the M_h give an I-filtration of

$$R/(I'_{s+t},(xu)^s(yv)^s(x^sv^s+y^su^s)),$$

it suffices to see that $IM_h\subseteq M_{h-1}$. It is clear that $(xu,yv)M_h\subseteq M_{h-1}$. It remains to prove that $(xv+yu)M_h\subseteq M_{h-1}$. Let

$$g(e_1,e_2,e_3)=(xu)^{e_1}(yv)^{e_2}(x^{e_3}v^{e_3}+y^{e_3}u^{e_3}).$$

Assume that $g(e_1,e_2,e_3)\in M_h\backslash M_{h-1}$. If $e_3=0$, then it is clear that

$$(xv+yu)g(e_1,e_2,0)=g(e_1,e_2,1)\in M_{h-1}.$$

[1]Note that $\underline{\mathfrak{h}}^3_{\underline{x}}(R)=0$ already implies that x_1,x_2,x_3 do not form a Q-sequence in R by [HH09, Theorem 3.8].

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If $0 < e_3 < s-1$, then

$$\begin{aligned}&(xv+yu)g(e_1,e_2,e_3)\\&=(xu)^{e_1}(yv)^{e_2}(x^{e_3+1}v^{e_3+1}+y^{e_3+1}u^{e_3+1})+(xu)^{e_1}(yv)^{e_2}(xvy^{e_3}u^{e_3}+yux^{e_3}v^{e_3})\\&=g(e_1,e_2,e_3+1)+(xu)^{e_1+1}(yv)^{e_2+1}(y^{e_3-1}u^{e_3-1}+x^{e_3-1}v^{e_3-1})\in M_{h-1}.\end{aligned}$$

When $e_3 = s-1$, a similar calculation shows that

$$\begin{aligned}(xv+yu)g(e_1,e_2,s-1)&=g(e_1,e_2,s)+(xu)^{e_1+1}(yv)^{e_2+1}(y^{s-2}u^{s-2}+x^{s-2}v^{s-2})\\&=g(e_1,e_2,s)+g(e_1+1,e_2+1,s-2).\end{aligned}$$

When $e_1, e_2 \geq s$, we know that $g(e_1,e_2,s)=0$ in $R/(I'_{s+t},(xu)^s(yv)^s(x^sv^s+y^su^s))$. When one of e_1 and e_2 is less than s, we have that $g(e_1,e_2,s)\in M_{h-1}$. And it is clear that $g(e_1+1,e_2+1,s-2)\in M_{h-1}$. Hence $(xv+yu)g(e_1,e_2,s-1)\in M_{h-1}$. This finishes the proof that $\mathcal{L}_I(R/(I'_{s+t},(xu)^s(yv)^s(x^sv^s+y^su^s)))\leq (t+s)^3-t^3$.

By the short exact sequence

$$0\to H'_{s+t}\to R/I'_{s+t}\to R/(I'_{s+t},(xu)^s(yv)^s(x^sv^s+y^su^s))\to 0,$$

we have that

$$\mathcal{L}_I(R/I'_{s+t})\leq \mathcal{L}_I(H'_{s+t})+\mathcal{L}_I(R/(I'_{s+t},(xu)^s(yv)^s(x^sv^s+y^su^s)))\leq c^2t^2+(s+t)^3-t^3.$$

Therefore, for $n\gg 0$, we have $\mathcal{L}_I(R/I'_n)<n^3$. We may choose such an integer n that is divisible by 12, i.e., $n=12n_0$. We can write $\mathcal{L}_I(R/I'_n)=\epsilon n^3$ with $0\leq\epsilon<1$. After applying F_n repeatedly, we have that $\mathcal{L}_I(R/I'_{n^l})\leq \epsilon^l n^{3l}$. Now according to Proposition 4.1(3), we have

$$\mathcal{L}_I(R/I_{12^{l-1}n_0^l})\leq \mathcal{L}_I(R/I'_{12^l n_0^l})=\mathcal{L}_I(R/I'_{n^l})\leq \epsilon^l n^{3l}$$

and hence

$$\mathfrak{h}^3_{\underline{x}}(R)\leq \frac{\mathcal{L}_I(R/I_{12^{l-1}n_0^l})}{(12^{l-1}n_0^l)^3}\leq \frac{\epsilon^l n^{3l}}{(12^{l-1}n_0^l)^3}=12^3\epsilon^l$$

for all $l\geq 1$. Therefore,

$$\mathfrak{h}^3_{\underline{x}}(R)=0.$$

□

5. Robust closure

Definition 5.1. We beginning by recalling the definitions of forcing algebras and generic forcing algebras for a triple (M,N,u) ([Hoc94, 4.3]) where $N\subseteq M$ are finitely generated modules over a Noetherian ring R and u is an element of M. An R-algebra S is called a *forcing algebra* for the triple (M,N,u) if the image $1\otimes u$ of u in $S\otimes_R M$ is in $\mathrm{Im}(S\otimes_R N\to S\otimes_R M)$. Let $R^\beta\xrightarrow{A}R^\alpha\to M/N\to 0$ be a finite presentation of M/N, where $A=(a_{ij})$ is an $\alpha\times\beta$ matrix over R. Let $r=(r_1,\ldots,r_\alpha)\in R^\alpha$ represent $\bar{u}\in R^\alpha/\mathrm{Im}(A)\cong M/N$. Let $Z_1,\ldots,Z_\beta$ be indeterminates over R. Then

$$\frac{R[Z_1,\ldots,Z_\beta]}{(r_i-\sum_{j=1}^\beta a_{ij}Z_j : 1\leq i\leq\alpha)}$$

is called a *generic forcing algebra* for the triple (M,N,u) for the data A,r. It is clear that the image $1\otimes u$ of u in $S\otimes_R M$ is contained in $\mathrm{Im}(S\otimes_R N\to S\otimes_R M)$. Given any other forcing algebra S' for the triple (M,N,u), there is a ring homomorphism $S\to S'$; this justifies the word 'generic'.

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Remark 5.2. It is important to note that a forcing algebra (respectively, a generic forcing algebra) for (M, N, u) is the same as a forcing algebra (respectively, a generic forcing algebra) for $(M/N, 0, \overline{u})$, where $\overline{u}$ is the image of u in M/N.

Note also that if S is a forcing algebra for (M, N, u) and $S \to T$ is an R-algebra homomorphism, then T is also a forcing algebra for (M, N, u).

Furthermore, note that if $N \subseteq N' \subseteq M \subseteq M'$ are finitely generated R-modules and $u \in M$, then a forcing algebra for (M, N, u) is a forcing algebra for (M', N', u), since we have an R-linear map $M/N \to M'/N'$ that sends the image of u in the first module to the image of u in the second module. The case where $M = M'$ and the case where $N = N'$ are of particular interest.

The following proposition ([Hoc94, Proposition 4.6]) will be useful to us.

Proposition 5.3. *If R is a Noetherian ring, $N \subseteq M$ are finitely generated R-modules, $u \in M$, and S, T are two generic forcing algebras for (M, N, u) for possibly different data, then there are finite sets of indeterminates $\boldsymbol{Y}$, $\boldsymbol{Z}$ such that $S[\boldsymbol{Y}] = T[\boldsymbol{Z}]$ as R-algebras.*

For any Noetherian ring R, following [Hoc94, 3.1], we refer to the R-algebra obtained by completing R localized at a maximal ideal and then killing a minimal prime as a *complete local domain* of R. Note that if R is local and $\mathfrak{p}$ is a minimal prime of R, then every minimal prime of $\mathfrak{p}\,\widehat{R}$ is a minimal prime of (0) in $\widehat{R}$ lying over $\mathfrak{p}$. (If $a \notin \mathfrak{p}$ it is a nonzerodivisor on $R/\mathfrak{p}$ and, hence, of the flat $(R/\mathfrak{p})$-algebra $\widehat{R}/\mathfrak{p}\,\widehat{R}$. Thus $\mathfrak{q}$ lies over $\mathfrak{p}$. Moreover, we can choose $a \in R - \mathfrak{p}$ such that $a\,\mathfrak{p}^n = 0$ for some integer $n \geq 1$. When we localize at $\mathfrak{q}$, $\mathfrak{p}$ therefore becomes nilpotent, and $\mathfrak{q}$ becomes nilpotent modulo $\mathfrak{p}\,\widehat{R}$.) It follows that the set of complete local domains D of the local ring R is the same as the set of complete local domains of the $R/\mathfrak{p}$ for $\mathfrak{p}$ a minimal prime of R: the map $R \to \widehat{R} \to D$ factors uniquely $R \to \widehat{R} \to \widehat{R}/\mathfrak{p}\,R \to \widehat{R}/\mathfrak{q} = D$, where $\mathfrak{q}$ is a minimal prime both of $\mathfrak{p}\,\widehat{R}$ and of (0) in $\widehat{R}$.

Next we collect the definition and some basic facts of solid closure.

Definition 5.4 (Definition 1.1 in [Hoc94])**.** If R is a domain, we shall say that an R-module M is *solid* if $\operatorname{Hom}_R(M, R) \neq 0$. We shall say that an R-algebra S is *solid* if it is solid as an R-module.

Remark 5.5. S is solid as an R-algebra if and only if there is an R-module map $\theta : S \to R$ such that $\theta(1) \neq 0$. (If $\theta(s_0) \neq 0$, one may replace θ by θ' where $\theta'(s) := \theta(s_0 s)$.) A module-finite extension S of a Noetherian domain R is always a solid R-algebra ([Hoc94, Proposition 2.1(i)]). If $(R, \mathfrak{m})$ is a complete local domain of Krull dimension d, then an R-module is solid if and only if $H^d_{\mathfrak{m}}(M) \neq 0$ ([Hoc94, Corollary 2.4]).

Definition 5.6 (Definition 5.1 in [Hoc94])**.** Let R be a Noetherian ring, let $N \subseteq M$ be finitely generated R-modules, and let $u \in M$. If R is a complete local domain we say that u is in the *solid closure* $N^{\bigstar}$ of N in M over R if there is a solid R-algebra S such that the image $1 \otimes u$ of u in $S \otimes_R M$ is in $\operatorname{Im}(S \otimes_R N \to S \otimes_R M)$. In other words, $u \in N^{\bigstar}$ if and only if (M, N, u) has a forcing algebra S that is solid as an R-algebra. If there is a solid forcing algebra, then the generic forcing algebra is solid.

In the general case, we say that x is in the solid closure $N^{\bigstar}$ of N in M over R if for every complete local domain D of R, the image of $1 \otimes x$ in $D \otimes_R M$ is in the

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solid closure of $\mathrm{Im}(D \otimes_R N \to B \otimes_R M)$ in $D \otimes_R M$ over B. In other words, every complete local domain D of R has a solid D-algebra S such that the image of x in $S \otimes_R M$ is in $\mathrm{Im}(S \otimes_R N \to S \otimes_R M)$.

The following facts about solid closure will be used in what follows:

(1) Let R be a Noetherian ring and let I be an ideal of R. Then $I^\bigstar \subseteq \bar{I}$. Moreover if I is principal, then $I^\bigstar = \bar{I}$ ([Hoc94, Theorem 5.10]).
(2) If R is a Noetherian ring of positive characteristic and admits a completely stable test element and $N \subseteq M$ are finitely generated R-modules, then $N^*_M = N^\bigstar_M$. In particular, this is true in positive characteristic if R is reduced and essentially of finite type over an excellent semilocal ring, and, in particular, it is true in positive characteristic for every complete local domain ([Hoc94, Theorem 8.6]).

Recall the following definition from [Bre03].

Definition 5.7 ([Bre03])**.** Let $(A, \mathfrak{m})$ be a d-dimensional Noetherian commutative local ring. A *paraclass* in $H^d_\mathfrak{m}(A)$ is the cohomology class $[\frac{1}{x_1 \dots x_d}] \in H^d_\mathfrak{m}(A)$ where $x_1, \dots, x_d$ is a system of parameters of A and we think of $H^d_\mathfrak{m}(A)$ as the quotient of $A_{x_1 \cdots x_d}$ by $\sum_{i=1}^d A_{f_i}$ where $f_i = \prod_{j \neq i} x_j$.

Let R be an A-algebra. R is called a *parasolid* A-algebra if the image of each nonzero paraclass $c \in H^d_\mathfrak{m}(A)$ in $H^d_{\mathfrak{m}R}(R)$ does not vanish.

More generally, let A be a Noetherian commutative ring and let R be an A-algebra. R is called a *parasolid* A-algebra if $R_\mathfrak{m}$ is parasolid over $A_\mathfrak{m}$ for each maximal ideal $\mathfrak{m}$ of A.

Remark 5.8. It is clear that a parasolid algebra over a complete local domain is always solid from the local cohomology characterization of being solid given in Remark 5.5. Note also that if $A \to R$ is split or even pure as a map of A-modules, then R is parasolid over A: the condition of being split or pure is preserved by base change to a complete local domain of A. But when $(A, \mathfrak{m})$ is a complete local domain the map of local cohomology $H^d_\mathfrak{m}(A) \to H^d_\mathfrak{m}(R)$ is injective.

Definition 5.9. Let R be a complete local domain. An R-algebra S is called *robust* if the image of every system of parameters for R in S forms a Q-sequence in S.

In general, for any Noetherian ring R, an R-algebra S is called *robust* if $D \otimes_R A$ is a robust D-algebra for every complete local domain D of R.

Note that a robust algebra S over a complete local domain $(R, \mathfrak{m})$ of dimension d is solid, since even the fact that one system of parameters $\underline{x} = x_1, \dots, x_d$ is a Q-sequence on S implies that $H^d_{(\underline{x})}(S) = H^d_\mathfrak{m}(S) \neq 0$. We make the following conjecture:

Conjecture 5.10. *Every system of parameters for every local ring is a Q-sequence. Hence, every Noetherian ring is robust as an algebra over itself.*

The issue for Noetherian rings of Krull dimension at most d reduces to the case of complete local domains of Krull dimension at most d, where it is equivalent to asking whether every system of parameters is a Q-sequence. One may pass to the normalization of the complete local domain. The results of Hochster and Huneke [HH09] show that this conjecture is true in equal characteristic ([HH09, Theorem

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4.1]) and in dimension at most 2 (the normalized complete local domain is Cohen-Macaulay and one may apply [HH09, Proposition 1.2(c)]). Hence, by the results of Hochster and Huneke we have:

Theorem 5.11. *Let R be a Noetherian ring such that R_{red} contains a field or such that R is of Krull dimension at most 2. Then R is robust.*

The difficult result on the vanishing of content in §4 enables us to show that parasolid algebras are not, in general, robust. In the example given in the result below, $A \to R$ is actually split as a map of A-modules.

Theorem 5.12. *Let $R = \mathbb{C}[[x, y, u, v]]$ and $A = \mathbb{C}[[xu, yv, xv + yu]]$. Then R is a parasolid A-algebra; in fact $A \to R$ splits as a map of A-modules, but R is not a robust A-algebra.*

Proof. Let $B = \mathbb{C}[[xu, xv, yu, yv]]$. Then it is easy to see that B is Cohen-Macaulay and $xu, yv, xv + yu$ is a system of parameters of B. Therefore B is a free A-module and hence the natural inclusion $A \hookrightarrow B$ splits (over A). On the other hand, B is the invariant subring of R under the $\mathbb{C}^*$-action given by

$$c \circ x = cx, \qquad c \circ y = cy, \qquad c \circ u = \frac{1}{c}u, \qquad c \circ v = \frac{1}{c}v$$

for each $c \in \mathbb{C}^*$. Hence $B \hookrightarrow R$ splits (over B). Therefore $A \hookrightarrow R$ splits (over A), and consequently R is a parasolid A-algebra by Remark 5.8.

However, from §4 we know that R is not a robust A-algebra. □

Definition 5.13. Let $(R, \mathfrak{m})$ be a complete local domain and let $N \subseteq M$ be finitely generated R-modules. Let $N^{\square}$ (or $N^{\square}{}_M$ if M is not clear from context) denote the submodule of M generated by N and all elements $u \in M$ such that a generic forcing S for the triple (M, N, u) is a robust R-algebra. (It is equivalent to saying that there exists a forcing algebra for (M, N, u) that is robust.) We refer to $N^{\square}$ as the *immediate robust closure* of N in M. Let $N^{\square^j}$ be defined recursively by $N^{\square^0} = N$ and $N^{\square^{j+1}} = (N^{\square^j})^{\square}$. The *robust closure* of N in M is defined to be $\bigcup_j N^{\square^j}$, denoted by $N^{\blacksquare}$ or $N^{\blacksquare}{}_M$. Note that since the sequence $N^{\square^j}$ is clearly nondecreasing as j increases and M is Noetherian, we have that $N^{\blacksquare} = N^{\square^j}$ for all $j \gg 0$.

In general, for any Noetherian ring R and finitely generated R-modules $N \subseteq M$, we say an element $u \in M$ is in the *robust closure* of N in M if $1 \otimes u$ is contained in $\mathrm{Im}(R' \otimes N \to D \otimes_R M)^{\blacksquare}{}_{D \otimes_R M}$ for every complete local domain D of R.

Remark 5.14. In Definition 5.13, if we use a generic forcing algebra T for (M, N, u) for possibly different data, then whether the generic forcing algebra obtained is robust is not affected. To see this, note that by Proposition 5.3, it suffices to prove that a system of parameters for R forms a Q-sequence in a generic forcing algebra S for (M, N, u) if and only if it forms a Q-sequence in the polynomial ring $S[\mathbf{Y}]$ over S. This is clear because we have S-algebra homomorphisms $S \to S[\mathbf{Y}] \to S$ whose composition is the identity, and these are also R-algebra homomorphisms.

Moreover, in defining $N^{\square}$ (and, hence, in defining all the $N^{\square^j}$), we may take S to be any forcing algebra rather than a generic forcing algebra, for then we may choose a generic forcing algebra S_0 such that $R \to S$ factors $R \to S_0 \to S$, and the fact that S is robust implies that S_0 is robust.

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There are technical difficulties in working with robust closure. One is that we do not know that Noetherian rings are robust as algebras over themselves, which was discussed above. Here are two others. We do not know whether the tensor product of two robust algebras over a complete local domain R is robust nor whether, if S is robust over R and $R \to R'$ is a local homomorphism of complete local domains, $R' \otimes_R S$ is robust over R'. But the notion is well-behaved in many other ways.

Let R be a complete local domain. We will see later that in positive characteristic both robust closure and solid closure agree with tight closure.

For every $n \geq 2$ it is an open question in the non-Noetherian case whether regular sequences of length n are Q-sequences, and if this is true then the same is true for latent regular sequences. Assuming this, then according to [Hoc94, Theorem 12.5], in characteristic 0, when $\dim(R) = 2$, robust closure coincides with solid closure. We do not know whether this is true.

In dimension 3 robust closure can be strictly smaller than solid closure in equal characteristic 0. For example, in $R = \mathbb{C}[[x_1, x_2, x_3]]$, one has that $x_1^2x_2^2x_3^2 \in (x_1^3, x_2^3, x_3^3)^{\bigstar}$, using the result of [Rob94]. See also the second paragraph of §1. However, every ideal of R is its own robust closure by Theorem 5.24 below.

Proposition 5.15. *If R is Noetherian and S is a robust R-algebra, then the contraction of IS to R is contained in $I^{\blacksquare}$ for all ideals I of R.*

Proof. Let $u \in R$ have image in IS. Let D be a complete local domain of R. Then $D \otimes_R S$ is robust, and the image of u is in $I(D \otimes_R S)$, which is the image of IS. Hence, u is in $I^{\blacksquare}$. □

Proposition 5.16. *If R is Noetherian and R_{red} contains a field, then every module-finite extension S of R is robust. Hence, $IS \cap R \subseteq I^{\blacksquare}$ for every ideal of R. Therefore, if R is reduced and contains a field, $I^+ \subseteq I^{\blacksquare}$.*

Proof. If $\mathfrak{m}$ is maximal in R and R_1 denotes the completion of $R_{\mathfrak{m}}$, then $S_1 = R_1 \otimes S$ is a module-finite extension of R_1. Let P be a minimal prime of R_1 and let Q be a prime of S_1 lying over P_1. Let $D = R/P_1$. We need to show that $D \otimes S_1 = S_1/PS_1$ is robust over D. But S_1/PS_1 maps to S/Q, and $R/P = D \to S/Q$ is injective and module-finite. Hence, any system of parameters for R/P is a system of parameters for the equicharacteristic complete local domain S/Q, and the result follows from Theorem 5.11. The final statement is immediate from the fact that every integral extension of R is a directed union of module-finite extensions. □

Remark 5.17. There are results for finitely generated modules corresponding to both Proposition 5.15 and Proposition 5.16. If $N \subseteq M$ and we have a homomorphism $R \to S$, we work with the inverse image in M of $\mathrm{Im}(S \otimes_R N \to S \otimes_R M)$ under the map $M \to S \otimes_R M$ (sending $u \mapsto 1 \otimes u$). That is, we think of the contracted expansion of N as the set of $u \in M$ such that $1 \otimes u$ is in the image of $S \otimes_R N$ in $S \otimes_R M$.

Remark 5.18. Proposition 5.16 holds for module-finite extensions of Noetherian rings that are robust as algebras over themselves. Thus, if Conjecture 5.10 is true, Proposition 5.16 will hold for all Noetherian rings. Coupled with Theorem 5.24 below, this gives a new proof of the direct summand conjecture.

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Proposition 5.19. *Let R be a Noetherian ring, and let $N \subseteq M$ and $N' \subseteq M'$ be Noetherian R-modules. Then:*

(a) $N \subseteq N^{\blacksquare}{}_M$.
(b) *If $u \in M$, then $u \in N^{\blacksquare}{}_M$ iff $\overline{u} \in 0^{\blacksquare}{}_{M/N}$, where $\overline{v}$ denotes the image of $v \in M$ in M/N.*
(c) *If $u \in M$, $u' \in M'$, and there is an R-linear map $M/N \to M'/N'$ such that $u \mapsto u'$, then if $u \in N^{\blacksquare}{}_M$ we have that $u' \in N'^{\blacksquare}{}_{M'}$. In particular, if $N \subseteq N' \subseteq M$, then $N^{\blacksquare}{}_M \subseteq N'^{\blacksquare}{}_M$, and if $N \subseteq M \subseteq M'$, then $N^{\blacksquare}{}_M \subseteq N^{\blacksquare}{}_{M'}$.*
(d) $(N^{\blacksquare}{}_M)^{\blacksquare}{}_M = N^{\blacksquare}{}_M$.
(e) *If $N' \subseteq M$, then $(N \cap N')^{\blacksquare}{}_M \subseteq N^{\blacksquare}{}_M \cap N'^{\blacksquare}{}_M$.*
(f) $(N + N')^{\blacksquare}{}_M \subseteq (N^{\blacksquare}{}_M + N'^{\blacksquare}{}_M)^{\blacksquare}{}_M$.
(g) *If $N_i \subseteq M_i$ are Noetherian, $1 \leq i \leq n$, $N = \bigoplus_i N_i$, and $M = \bigoplus_i M_i$, then $N^{\blacksquare}{}_M$ may be identified in the obvious way with $\bigoplus_i N_i{}^{\blacksquare}{}_{M_i}$.*

Proof. (a). This is clear from the definition.

(b) The submodules N'/N of M/N are in a bijective correspondence with the submodules N' of M with $N \subseteq N'$, and by Remark 5.2 for each element $v \in M$, a forcing algebra for (M, N', v) is the same as a forcing algebra for $(M/N, N'/N, \overline{v})$. Again by Remark 5.2, this remains true when we make a base change to a complete local domain R' of R, replacing M by $R' \otimes_R M$ and N by the image of $R' \otimes_R N$ in $R' \otimes_R M$. We use the notation R, M, N for what we obtain after the base change. It follows that $0^{\square^j}{}_{M/N} = (N^{\square^j}{}_M)/N$, and the result follows.

(c) It suffices to prove the result when $N = 0$ and $N' = 0$ (we may replace M, M' by M/N, M'/N'). The result then says that a linear map $M \to M'$ takes $0^{\blacksquare}{}_M$ into $0^{\blacksquare}{}_{M'}$. It suffices to prove the result after base change to a complete local domain of R, and so we may assume that R is a complete local domain. We then need only show that the image $0^{\square^j}_M$ maps into $0^{\square^j}_{M'}$ for all j. We use induction on j. Again, we may replace M and M' by quotients, $M/0^{\square^j}{}_M$ and $M'/0^{\square^j}{}_{M'}$, respectively. Thus, we need only consider the case $j = 1$ to do the inductive step. But this follows from the fact that by the third paragraph of Remark 5.2, a forcing algebra for $(M, 0, u)$ is a forcing algebra for $(M', 0, u')$. The corollary statements in the second sentence are obvious.

(d) Let $u \in (N^{\blacksquare}{}_M)^{\blacksquare}{}_M$. We must show that $u \in N^{\blacksquare}{}_M$. It suffices to consider the issue after base change to a complete local domain of R. After the base change, the image of $N^{\blacksquare}$ is in the robust closure of the image of N. Therefore, using part (c), it suffices to show that $(N^{\blacksquare}{}_M)^{\blacksquare}{}_M = N^{\blacksquare}{}_M$ in the case of a complete local domain R. But then $N^{\blacksquare}{}_M = N^{\square^i}{}_M$ for i sufficiently large, and so $(N^{\blacksquare}{}_M)^{\blacksquare}{}_M = (N^{\square^i}{}_M)^{\square^j}{}_M = N^{\square^{i+j}}{}_M = N^{\blacksquare}{}_M$, as required.

Parts (e) and (f) are immediate from the second statement in part (c) (note that $N + N' \subseteq N^{\blacksquare}{}_M + N'^{\blacksquare}{}_M$).

(g) Since $N_i{}^{\blacksquare}{}_{M_i} \subseteq N^{\blacksquare}{}_M$ is clear from part (c), it suffices to show that if $u = u_1 \oplus \cdots \oplus u_n \in N^{\blacksquare}{}_M$, then $u_i \in N_i{}^{\blacksquare}{}_{M_i}$ for each i. Identify M with $\prod_i M_i$. The needed fact also follows from part (c), along with the fact that the product projection $M \to M_i$ takes u to u_i and N onto N_i. □

Proposition 5.20. *Let R be a Noetherian ring and let $N \subseteq M$ be finitely generated R-modules. Let S be a Noetherian R-algebra such that every maximal ideal $\mathfrak{M}$ of S lies over a maximal ideal $\mathfrak{m}$ of R in such a way that the contraction of every minimal prime $\mathfrak{q}$ of $\widehat{S}_{\mathfrak{M}}$ under $\theta : \widehat{R}_{\mathfrak{m}} \to \widehat{S}_{\mathfrak{M}}$ is a minimal prime $\mathfrak{p}$ of $\widehat{R}_{\mathfrak{m}}$ in such*

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a way that $\widehat{R}_{\mathfrak{m}}/\mathfrak{p} \to \widehat{S}_{\mathfrak{M}}/\mathfrak{q}$ *is an isomorphsim. If* $u \in N^{\blacksquare}{}_M$, *then* $1 \otimes u \in S \otimes M$ *is in* $\big(Im(S \otimes_R N \to S \otimes_R M)\big)^{\blacksquare}{}_{S \otimes_R M}$.

In particular, the conclusion holds when S *is the localization of* R *at one or at finitely many maximal ideals, the completion of* $R_{\mathfrak{m}}$ *for a maximal ideal* $\mathfrak{m}$ *of* R, *or when* S *is the quotient of* R *by a minimal prime.*

Proof. The hypothesis at once implies that every complete local domain of S is a complete local domain of R, which can be checked instead for the map $R_{\mathfrak{m}} \to S_{\mathfrak{M}}$ or for the induced map of completions. This immediately yields the first conclusion.

We next consider the statement of the second paragraph. In the case of localization at a finite set of maximal ideals, each localization $S_{\mathfrak{M}}$ is isomorphic to the corresponding $R_{\mathfrak{m}}$, from which the result is obvious. If we consider $R \to \widehat{R}_{\mathfrak{m}}$, the induced map of completions is the identity on $\widehat{R}_{\mathfrak{m}}$. Therefore we need only consider the case where $S = R/P$ for a minimal prime P of R. Then $\mathfrak{M} = \mathfrak{m}/P$ for a maximal ideal of $\mathfrak{m}$ of R, and we can reduce to studying the local case; i.e., we may assume that $(R, \mathfrak{m})$ is local and $S = R/P$ for P minimal. Then $\widehat{S} \cong \widehat{R}/P\widehat{R}$, and it suffices to observe that every minimal prime $\mathfrak{q}$ of $P\widehat{R}$ is a minimal prime of $\widehat{R}$, by the discussion in the paragraph following Proposition 5.3. □

Proposition 5.21. *Let* R *be a Noetherian ring, let* $\mathfrak{N}$ *denote the radical of the ideal* (0) *of* R, *i.e., the nilradical of* R, *let* $R_{red} = R/\mathfrak{N}$, *and let* $N \subseteq M$ *be finitely generated* R*-modules.*

(a) *The robust closure of the ideal* (0) *is the nilradical* $\sqrt{(0)}$ *of* R. *The robust closure* $0^{\blacksquare}{}_M$ *of* 0 *in* M *is contained in* $\sqrt{(0)}M$.

(b) *Let* $R_{red} = R/\sqrt{(0)}$. *Then* $N^{\blacksquare}{}_M$ *is the inverse image of the robust closure of* $\mathrm{Im}(R_{red} \otimes_R N \to R_{red} \otimes_R M)$ *in* $R_{red} \otimes_R M = M/\sqrt{(0)}M$.

(c) *An element* $u \in M$ *is in* $N^{\blacksquare}{}_M$ *if and only if for every minimal prime* $\mathfrak{p}$ *of* R, *the image of* u *in* $M/\mathfrak{p}M$ *is in the robust closure of* $\mathrm{Im}(N/\mathfrak{p}N \to M/\mathfrak{p}M)$ *in* $M/\mathfrak{p}M$, *computed over* $R/\mathfrak{p}$.

Proof. (a) First note that $\mathfrak{N}$ maps to 0 in every complete local domain of R, which shows that $\mathfrak{N} \subseteq (0)^{\blacksquare}$. If $r \in R$ is not nilpotent choose a minimal prime $\mathfrak{p}$ of R that does not contain r and a maximal ideal $\mathfrak{m}$ of R that contains $\mathfrak{p}$. Then $r \notin \mathfrak{p}R_{\mathfrak{m}}$, and so r is not nilpotent in $R_{\mathfrak{m}}$. It follows that the image of r is not nilpotent in the completion of $R_{\mathfrak{m}}$, and so it is not in a minimal prime of the completion. It therefore suffices to show that (0) is solidly closed in a complete local domain R. This is clear because every nonzero element x in a complete local domain is a parameter or a unit: the generic forcing algebra for $(R, (0), x)$ is a polynomial ring over R/xR, and R/xR is obviously not a robust R-algebra. Parts (b) and (c) are clear from Proposition 5.20 and the fact that every map $R \to D$, where D is a complete local domain of R, factors $R_{\mathrm{red}} \to D$ (respectively, $R/\mathfrak{p} \to D$) for some minimal prime $\mathfrak{p}$ of R in such a way that D is a complete local domain of R_{red} (respectively, of $R/\mathfrak{p}$). □

Proposition 5.22. *For all ideals* I *of a Noetherian ring* R, $I^{\blacksquare} \subseteq I^{\bigstar} \subseteq \overline{I}$.

Proof. The fact that $I^{\bigstar} \subseteq \overline{I}$ is [Hoc94, Theorem 5.10]. Thus, we only need to show that if $u \in I^{\blacksquare}$, then $u \in I^{\bigstar}$, and it suffices to show this after base change to a complete local domain of R, where the issue becomes showing that $I^{\square^j} \subseteq I^{\bigstar}$. By induction on j this reduces to the case where $j = 1$, where it follows from the

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fact that a robust algebra over a complete local domain is solid; see the comment following Definition 5.9. □

Proposition 5.23. *Let R be a Noetherian ring. If I is principal and every module-finite extension of a complete local domain D of R is a robust D-algebra, then $I^{\blacksquare} = \overline{I}$. In particular, if R_{red} contains a field or R has dimension at most 2 and I is principal, then $I^{\blacksquare} = \overline{I}$.*

Theorem 5.24. *Let R be a regular Noetherian ring. Then for any two finitely generated R-modules $N \subseteq M$, ${N^{\blacksquare}}_M = N$.*

Proof. Suppose $u \in M$ is such that $u \in {N^{\blacksquare}}_M \backslash N$. Localize at a maximal ideal $\mathfrak{m}$ of R such that $u \notin N_{\mathfrak{m}}$ and complete. Thus, we may assume without loss of generality that R is a complete regular local ring.

Choose $N' \supseteq N$ maximal in M with respect to not containing u. Then by Proposition 5.19(d) we still have $u \in {N'^{\blacksquare}}_M$. But then the image of u is in every submodule of M/N'. We may replace M, N, and u by M/N', 0, and the image of u in M/N'. Hence, we may assume that M has finite length, $N = 0$, and that u generates the socle. Let $x_1, \ldots, x_d$ be a regular system of parameters for R. Then the injective hull of $R/\mathfrak{m}$ over R is the directed union of modules of the form R/I_t where $I = (x_1^t, \ldots, x_d^t)R$, and M is contained in R/I_t for $t \gg 0$. Moreover, up to a unit factor, u must be the socle generator $(x_1 \cdots x_d)^{t-1}$ for R/I_t. It will therefore suffice to show that $0 = {0^{\blacksquare}}_{R/I_t}$, and for this it will suffice to show that $0^{\square} = 0$. This will follow if we can show that there is no forcing algebra that is robust for the triple $(R/I_t, 0, v)$ where v is a nonzero element of R/I_t. Note that any such forcing algebra must also be a forcing algebra for $(R/I_t, 0, u)$, since u is a multiple of v in R/I_t. It therefore suffices to show that the generic forcing algebra $S = R[Z_1, \ldots, Z_d]/(x_1^{t-1} \cdots x_d^{t-1} - \sum_{i=1}^d Z_i x_i^t)$ is not robust over R. But this is clear: $S/(x_1^t, \cdots, x_d^t)$ will have a filtration using ideals generated by monomials in the x_i of length $t^d - 1$, and so $x_1, \ldots, x_d$ is not a Q-sequence in S. □

Proposition 5.25. *Let $R \to S$ be a homomorphism of Noetherian rings. Suppose that*

(†) $\operatorname{MaxSpec}(S) \to \operatorname{MaxSpec}(R)$ *is surjective and that for every maximal ideal $\mathfrak{m}$ of R and every minimal prime ideal $\mathfrak{p}$ of $\widehat{R_m}$, we have $ht(\mathfrak{m}\,\widehat{S}^m/\,\mathfrak{p}\,\widehat{S}^m) \geq ht(\mathfrak{m}\,/\,\mathfrak{p})$.*

Let $N \subseteq M$ be finitely generated R-modules. If $u \in M$ is such that $1 \otimes u$ is in the robust closure of the image of $S \otimes_R N$ in $S \otimes_R M$, then $u \in {N^{\blacksquare}}_M$. In particular, for an ideal $I \subseteq R$, the contraction to R of $(IS)^{\blacksquare}$, calculated over S, is contained in $I^{\blacksquare}$.

The hypothesis (†) holds if S is faithfully flat over R, or if $\widehat{R_m}$ has a unique minimal prime for every maximal ideal $\mathfrak{m}$ of R (i.e., if R is excellent normal) and $ht(\mathfrak{m}\,S) \geq ht(\mathfrak{m})$.

Proof. We claim that for every complete local domain $D = \widehat{R_m}/\,\mathfrak{p}$ there is a complete local domain D' of S such that $R \to S \to D'$ factors $R \to D \to D'$ and that $ht(\mathfrak{m}\,D') = ht(\mathfrak{m}\,D)$, and we reason as follows. Note that $S \to \widehat{S}^m$ is flat, and that the maximal ideals of $\widehat{S}^m$ are in bijective correspondence with the maximal ideals of S containing $\mathfrak{m}$. Moreover, the completion of S at a maximal ideal containing $\mathfrak{m}$ is isomorphic with the completion of $\widehat{S}^m$ at its corresponding maximal ideal.

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Since the expansion of the maximal ideal of D to $\widehat{S^m}/\mathfrak{p}\,\widehat{S^m}$ has height equal to the dimension of D, this remains true when we localize at maximal ideal of $\widehat{S^m}$ that contains $\mathfrak{m}\,\widehat{S^m} = \mathfrak{m}_D\,\widehat{S^m}$ and complete: call the resulting complete local ring T. It suffices to observe that the height of a proper ideal J of T cannot drop modulo every minimal prime of T, so that for some minimal prime $\mathfrak{q}$ of T, $D' = T/\mathfrak{q}$ is a complete local domain of S with the required property.

The height condition implies that every system of parameters for D is part of a system of parameters for D' and, hence, that every robust D'-algebra is robust as a D-algebra. It is then immediate that if the image of $1 \otimes u$ is in the robust closure of the image of $D' \otimes_R N$ in $D' \otimes_R M$, then this is also true for the image when D replaces D': the sequence of robust D'-algebras needed to establish the required instances of immediate robust closure over D' consists of robust D-algebras, and so proves the same result over D. □

6. Comparison with tight closure in positive characteristic

In this section, we compare robust closure with tight closure in positive characteristic. In particular, we prove the following theorem.

Theorem 6.1. *Let $(R, \mathfrak{m})$ be a d-dimensional complete local domain of characteristic $p > 0$. Let I be an ideal of R, u an element of R, and S a generic forcing algebra for (R, I, u). Then $H^d_{\mathfrak{m}}(S) \neq 0$ if and only if the image of each system of parameters for R in S forms a* Q*-sequence in S.*

Proof. The sufficiency is clear.

Assume that $H^d_{\mathfrak{m}}(S) \neq 0$. From the local cohomology criterion for solid closure ([Hoc94]), we have that $u \in I^{\bigstar}$. And by [Hoc94, Theorem 8.6(b)] $I^{\bigstar} = I^*$ under our hypotheses, hence u is an element of I^*; i.e., for $q \gg 0$, we have $cu^q \in I^{[q]} = (g_1^q, \ldots, g_n^q)$ where $\{g_1, \ldots, g_n\}$ is a set of generators of I. We need to prove that the image of a system of parameters $x_1, \ldots, x_d$ in $S = \frac{R[Z_1,\ldots,Z_n]}{(u-\sum_{l=1}^n Z_l g_l)}$ forms a Q-sequence in S. Assume otherwise; then there would exist an integer $q_0 = p^{e_0}$ and a short filtration of $\frac{S}{(x_1^{q_0},\ldots,x_d^{q_0})}$ with $h < q_0^d$ factors. This short filtration would be given by $\tilde{s}_1, \ldots, \tilde{s}_h \in S$ ($\tilde{s}_h = 1$) satisfying

$$(x_1, \ldots, x_d)\tilde{s}_1 \subseteq (x_1^{q_0}, \ldots, x_d^{q_0}),$$
$$(x_1, \ldots, x_d)\tilde{s}_{i+1} \subseteq (\tilde{s}_1, \ldots, \tilde{s}_i) + (x_1^{q_0}, \ldots, x_d^{q_0}), 1 \leq i \leq h-1.$$

Pick and fix s_i a lifting of $\tilde{s}_i$ in $R[Z_1, \ldots, Z_n]$. Then there would be the following equations:

(6.1.1)
$$x_j s_i = \sum_{k=1}^{i-1} \alpha_{ijk} s_k + \sum_{k=1}^{d} \beta_{ijk} x_k^{q_0} + \gamma_{ij}(u - \sum_{l=1}^{n} Z_l g_l),\ \alpha_{ijk}, \beta_{ijk}, \gamma_{ij} \in R[Z_1, \ldots, Z_n]$$

where $\alpha_{1jk} = 0$. Since $cu^q \in I^{[q]} = (g_1^q, \ldots, g_n^q)$, we have $c^{\frac{1}{q}} u = \sum_{l=1}^n r_l^{\frac{1}{q}} g_l$ in $R^{\frac{1}{q}}$ for some $r_i \in R$. After the substitution $Z_l = c^{-\frac{1}{q}} r_l^{\frac{1}{q}}$ in each equation appearing in (6.1.1), we set N to be the least integer such that $c^{\frac{N}{q}}$ can clear all denominators appearing in all equations in (6.1.1). Next we define $\varphi : R[Z_1, \ldots, Z_n] \to R^{\frac{1}{q}}$ by $Z_l \mapsto c^{-\frac{1}{q}} r_l^{\frac{1}{q}}$ and $r \mapsto r$ for all $r \in R$. Set $\sigma_i = \varphi(s_i) c^{\frac{iN}{q}} \in R^{\frac{1}{q}}$ for $1 \leq i \leq h$. Then

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we have the following equations:

$$x_j\sigma_i = c^{\frac{(i-1)N}{q}}\varphi(\alpha_{ij1})\sigma_1 + \cdots + c^{\frac{N}{q}}\varphi(\alpha_{ij(i-1)})\sigma_{i-1} + c^{\frac{iN}{q}}\sum_{k=1}^{d}\varphi(\beta_{ijk})x_k^{q_0}l. \tag{6.1.2}$$

Set

$$\tau_i = \begin{cases} c^{\frac{hN}{q}}\sigma_{\frac{i+1}{2}} \text{ when } i \text{ odd and } i < 2h+1, \\ \sigma_{\frac{i}{2}} \text{ when } i \text{ even}, \\ 1 \text{ when } i = 2h+1, \end{cases}$$

and $M_j = R^{\frac{1}{q}}\tau_1 + \cdots + R^{\frac{1}{q}}\tau_j + (x_1^{q_0}, \ldots, x_d^{q_0})R^{\frac{1}{q}}$ for $j = 1, \ldots, 2h+1$. Then by equations in (6.1.2) we have a filtration of $\frac{R^{\frac{1}{q}}}{(x_1^{q_0},\ldots,x_d^{q_0})R^{\frac{1}{q}}}$ as

$$0 = M_0 \subseteq \frac{M_1}{(x_1^{q_0}, \ldots, x_d^{q_0})R^{\frac{1}{q}}} \subseteq \cdots \subseteq \frac{M_{2h+1}}{(x_1^{q_0}, \ldots, x_d^{q_0})R^{\frac{1}{q}}} = \frac{R^{\frac{1}{q}}}{(x_1^{q_0}, \ldots, x_d^{q_0})R^{\frac{1}{q}}},$$

in which $\frac{M_{j+1}}{M_j}$ is killed by $(c^{\frac{hN}{q}}, x_1^{q_0}, \ldots, x_d^{q_0})$ when j is even and it is killed by $(x_1, \ldots, x_d)$ when j is odd. Consequently, we have

$$\lambda(\frac{R^{\frac{1}{q}}}{(x_1^{q_0}, \ldots, x_d^{q_0})R^{\frac{1}{q}}}) \leq h\lambda(\frac{R^{\frac{1}{q}}}{(x_1, \ldots, x_d)R^{\frac{1}{a}}}) + \lambda(\frac{R^{\frac{1}{q}}}{(c^{\frac{hn}{q}}x_1^{q_0}, \ldots, x_d^{q_0})R^{\frac{1}{q}}}),$$

which is equivalent to

$$\lambda(\frac{R}{(x_1^{qq_0}, \ldots, x_d^{qq_0})}) \leq h\lambda(\frac{R}{(x_1^{q}, \ldots, x_d^{q})}) + (h+1)\lambda(\frac{\frac{R}{(c^{\frac{hN}{q}})}}{(x_1^{qq_0}, \ldots, x_d^{qq_0})\frac{R}{(c^{\frac{hN}{q}})}})$$

and hence

$$\frac{\lambda(\frac{R}{(x_1^{qq_0},\ldots,x_d^{qq_0})})}{(qq_0)^d} \leq h\frac{\lambda(\frac{R}{(x_1^{q},\ldots,x_d^{q})})}{(qq_0)^d} + (h+1)\frac{\lambda(\frac{\frac{R}{(c^{\frac{hN}{q}})}}{(x_1^{qq_0},\ldots,x_d^{qq_0})\frac{R}{(c^{\frac{hN}{q}})}})}{(qq_0)^d}. \tag{6.1.3}$$

Letting $q \to \infty$, we have

$$e_{\underline{x}}(R) \leq \frac{h}{q_0^d}e_{\underline{x}}(R) + 0,$$

a contradiction since $h < q_0^d$, where $e_{\underline{x}}(R)$ denotes the multiplicity of R with respect to the ideal $(x_1, \ldots, x_d)$ and the limit of the second summand in the right hand side in (6.1.3) is 0 because $\dim(\frac{R}{(c^{\frac{hN}{q}})}) < \dim(R)$. This completes the proof. □

The following corollary follows immediately from Theorem 6.1.

Corollary 6.2. *Let R be a Noetherian ring of characteristic $p > 0$ and assume that R has a completely stable test element c. Then for all ideals I of R one has $I^{\square^j} = I^*$ for all $j \geq 1$, where I_j is as in Definition* 5.13. *In particular,*

$$I^{\blacksquare} = I^*$$

for all ideals I.

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Let u be an element of R and I be an ideal of R with a set of generators $\{g_1, \ldots, g_n\}$. Let $S = \frac{R[Z_1,\ldots,Z_n]}{(u-\sum_{l=1}^{n} Z_l g_l)}$ be the generic forcing algebra. If one particular system of parameters for R forms a Q*-sequence S, so does every system of parameters for R.*

7. Q-sequences, superheight, and Lyubeznik's question

In this section, we investigate the connections between Q-sequences and superheight. First we recall the definition of superheight.

Definition 7.1 (Superheight)**.** Let R be a Noetherian commutative ring and I an ideal of R. The *superheight* of I, denoted by $\operatorname{supht}(I)$, is defined as

$$\sup\{\operatorname{ht}_S(IS) | \text{for all Noetherian } R\text{-algebras } S\}.$$

Remark 7.2. It follows immediately from [HH09, Theorem 4.7] that if R contains a field and $\operatorname{supht}((x_1, \ldots, x_n)) = n$, then $x_1, \ldots, x_n$ form a Q-sequence.

Given Remark 7.2, it is natural to ask:

Question 7.3. Assume that R contains a field and $x_1, \ldots, x_n$ is a Q-sequence. Is it true that $\operatorname{supht}((x_1, \ldots, x_n)) = n$?

As we will see from the examples in this section, the answer to Question 7.3 is no. We will give two examples, one in positive characteristic and the other in characteristic 0.

Example 7.4 (Brenner-Monsky example)**.** Let $A = \frac{\bar{\mathbb{F}}_2(t)[x,y,z]}{(z^4+xyz^2+x^3z+y^3z+tx^2y^2)}$. As shown in [BM10, Remark 4.1], the element x^3y^3 is contained in the tight closure of (x^4, y^4, z^4), but not in the plus closure of (x^4, y^4, z^4). Since the sequence x, y is a system of parameters in A, it follows immediately from Theorem 6.1 that the sequence x, y forms a Q-sequence in the forcing algebra

$$R = \frac{\bar{\mathbb{F}}_2(t)[x, y, z, u, v, w]}{(z^4 + xyz^2 + x^3z + y^3z + tx^2y^2, x^3y^3 - ux^4 - vy^4 - wz^4)}.$$

However, since x^3y^3 is not in the plus closure of (x^4, y^4, z^4), we claim that the superheight of (x, y) as an ideal in R is 1 and we reason as follows. Assume that $\operatorname{supht}((x, y)) = 2$. According to a result by Koh ([Koh87, Fact (1.4)]), since R is a finite algebra over a field, the finite superheight of (x, y) is also 2; i.e., there exists a module-finite R-algebra R' such that $\operatorname{ht}((x, y)R') = 2$. After localizing R' at a height-2 minimal prime over $(x, y)R'$, we may assume that the dimension of R' is 2 and x, y is a system of parameters in R'. Since x, y is also a system of parameters in A, we see that R' is a module-finite A-algebra. Since $x^3y^3 \in (x^4, y^4, z^4)$ in R and R' is an R-algebra, we have $x^3y^3 \in (x^4, y^4, z^4)$ in R'. This implies that x^3y^3 is in the plus closure of (x^4, y^4, z^4) in A, a contradiction.

Proposition 7.5. *Let $R = \mathbb{C}[[x, y, z, u, v]]/(x^3 + y^3 + z^3, z^2 - ux - vy)$. Then the sequence x, y is a Q-sequence, but* $\operatorname{supht}((x, y)) = 1$.

Proof. It is clear that R is the forcing algebra of the ring $A = \frac{\mathbb{C}[[x,y,z]]}{(x^3+y^3+z^3)}$, the ideal $I = (x, y)$ in A, and the element z^2 in A. Now x, y is a Q-sequence in R follows from the fact that in A the element z^2 belongs to the big equational tight closure of I (*cf.* [Hoc94, Example 12.7]).

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We assume that $\operatorname{supht}((x,y))$ is 2 and we seek a contradiction. Let S be a Noetherian R-algebra in which the height of $(x,y)S$ is 2. After localizing and completing S at a height-2 prime ideal containing $(x,y)S$, we may assume that S is a 2-dimensional complete local ring. Let $T = \mathbb{C}[[x,y]]$ be the subring of S generated by x, y over $\mathbb{C}$. Since x, y is a system of parameters of S, we know that T is a regular local ring. Still denote the image of z in S by z. Let $T' = T[z] \subseteq S$. Since z satisfies $x^3 + y^3 + z^3 = 0$ in R, it still satisfies the equation in S. Hence there is a surjection $A \twoheadrightarrow T'$. Since A is a 2-dimensional domain and the dimension of T' is also 2, we must have $A \cong T'$. Since A is normal, so is T'. Since in R we have $z^2 = ux + vy$, the same equation holds in S. This implies that the element z^2 in T' is in $(x,y)S \cap T'$. Since T' is normal, z^2 must be contained in (x,y) in $T' \cong A$. However, this is impossible since $z^2 \notin (x,y)$ in A, a contradiction. □

It turns out that Proposition 7.5 can be used to give a negative answer to part (2) of the following question raised by Gennady Lyubeznik in [Lyu02].

Question 7.6 (Page 144 in [Lyu02])**.** Let $(R, \mathfrak{m})$ be a complete local domain with a separably closed residue field.

(1) Find necessary and sufficient conditions on I such that $H^j_I(M) = 0$ for all integers $j \geq \dim(R) - 1$ and all R-modules M.
(2) Let I be a prime ideal. Is it true that $H^j_I(M) = 0$ for all integers $j \geq \dim(R) - 1$ and all R-modules M if and only if $(P + I)$ is not primary to the maximal ideal for any prime ideal P of height 1?

Proposition 7.7. *Let* $R = \frac{\mathbb{C}[[x,y,z,u,v]]}{(x^3+y^3+z^3, z^2-ux-vy)}$ *and* $I = (x,y,z)$. *Then*

(1) $(P + I)$ *is not primary to the maximal ideal of* R *for any prime ideal* P *of height* 1 *and*
(2) $H^2_I(R) \neq 0$.

In particular, the answer to Question 7.6(2) *is negative.*

Proof. Since we have proved that the superheight of (x,y) in the ring

$$R = \frac{\mathbb{C}[[x,y,z,u,v]]}{(x^3 + y^3 + z^3, z^2 - ux - vy)}$$

is 1, it is clear that $(x,y,z) + P$ is not $\mathfrak{m}$-primary for any height-1 prime ideal P of R (otherwise, in the ring R/P, the height of (x,y) would be 2). However, we claim that $H^2_{(x,y,z)}(R) \neq 0$. One can either see this from the fact that in A the element z^2 is contained in the solid closure of (x,y) or verify directly that the class $[\frac{1}{xy}]$ is not 0 in $H^2_{(x,y,z)}(R)$. □

Remark 7.8. Note that in the ring R_I, the ideal IR_I is 2-generated; hence R is not normal by Serre's criterion (I is a height-1 prime). Or, by the Jacobian criterion, one can see that the singular locus is actually defined by I, and hence R is not normal. To obtain a counterexample to Question 7.6(2) that is a normal domain, we consider the normalization R' of R.

It is straightforward to check that the fraction $w = \frac{y^2+zv}{x} = \frac{-x^2-zu}{y}$ satisfies the equation

$$w^3 - xzu + 2yzv + y^3 - u^3 + v^3 = 0$$

and hence w is integral over R. Also one can check that $R' = R[w]$; i.e., $R' = \mathbb{C}[[x,y,z,u,v,w]]/\mathfrak{P}$, where $\mathfrak{P}$ is the ideal of $\mathbb{C}[[x,y,z,u,v,w]]$ generated by the

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elements $z^2 - xu - yv$, $yw + x^2 + zu$, $xw - y^2 - zv, zw^2 + xyz + yu^2 + xv^2$, and $w^3 - xzu + 2yzv + y^3 - u^3 + v^3$. By an argument similar to the one above, we see that in R' we have that $I+Q$ is not primary to the maximal ideal for each height-1 prime Q. We claim that $H^2_I(R') \neq 0$ and we reason as follows. Since z is contained in the radical of $(x,y)R'$, we know that $H^t_I(M) = H^t_{(x,y)}(M)$ for all integers t and all R'-modules M. It is clear that xw and yw are in R. Thus, $(x,y)\frac{R'}{R} = 0$ since $R' = R[w]$. This implies that R'/R is (x,y)-torsion; therefore $H^t_{(x,y)}(R'/R) = 0$ for all $t \geq 1$. In particular, $H^1_{(x,y)}(R'/R) = H^2_{(x,y)}(R'/R) = 0$. Now the short exact sequence $0 \to R \to R' \to R'/R \to 0$ induces an exact sequence of local cohomology modules

$$\cdots \to H^1_{(x,y)}(R'/R) \to H^2_{(x,y)}(R) \to H^2_{(x,y)}(R') \to H^2_{(x,y)}(R'/R) \to \cdots .$$

Therefore, $H^2_I(R') = H^2_{(x,y)}(R') \cong H^2_{(x,y)}(R) = H^2_I(R) \neq 0$.

In Proposition 7.7, R is a 3-dimensional complete intersection where $\operatorname{embdim}(R) - \operatorname{mdim}(R/I) = 2 = d - 1$ and $H^2_I(R) \neq 0$. This indicates that, without further assumptions on R and I, the bound $\operatorname{embdim}(R) - \operatorname{mdim}(R/I) < d - 1$ is sharp.

8. Questions, conjectures, and related problems

In this section we have collected some questions and conjectures related to the behavior of quasilength and content.

Conjecture 8.1. *In a complete local domain R of mixed characteristic, every system of parameters is a Q-sequence (equivalently, R is robust as an R-algebra).*

The conjecture above would give a new proof of the direct summand conjecture at once (*cf.* [HH09, Remark 4.8]) but does not follow from it. So far as we know, it does not follow even from the existence of big Cohen-Macaulay algebras.

Question 8.2. Let R be a complete local domain of mixed characteristic, and let $z_1, \ldots, x_d$ be a system of parameters. Is it true that $(x_1, \ldots, x_k)R :_R x_{k+1} \subseteq \big((x_1, \ldots, x_k)R\big)^{\blacksquare}$? That is, does robust closure have the colon-capturing property?

Question 8.3. If $R \to S$ is a local homomorphism of complete local domains and $r \in I^{\blacksquare}$, where $I \subseteq R$, is the image of r in S in $(IS)^{\blacksquare}$? An affirmative answer gives a form of persistence for robust closure.

Affirmative answers to these questions would imply the following conjecture, whose statement does not refer to robust closure.

Conjecture 8.4. *Let $(R, \mathfrak{m})$ be a d-dimensional complete local domain of mixed characteristic p. Assume that $p, x_2, \ldots, x_d$ is a system of parameters. Let $\overline{R} := R/pR$. Let $J = (x_2, \ldots, x_d)R$, and let $J_n = (x_2, \ldots, x_d) :_R p^n \subseteq R$. Then for all $n > 0$, $J_n\overline{R} \subseteq (J\overline{R})^*$.*

The point is that J_n would be in $J^{\blacksquare}$, and this would persist when we map to R/pR. But in R/pR, robust closure coincides with tight closure. We prove below that Conjecture 8.4 suffices to give a new proof of the direct summand conjecture. We also prove that it holds in dimension at most 3. This argument in dimension 3 needs very difficult results of Heitmann ([Hei05, Theorem 0.2]).

So far as we know, Conjecture 8.4 does not follow from the direct summand conjecture.

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Theorem 8.5. *Conjecture* 8.4 *implies the Monomial Conjecture* (*or equivalently the Direct Summand Conjecture*).

Proof. Without loss of generality, we may assume that $(R, \mathfrak{m})$ is a complete local domain of mixed characteristic. Assume the Monomial Conjecture does not hold in R; i.e., there is a system of parameters $p = x_1, \dots, x_d$ of R such that $(x_1 \cdots x_d)^{t-1} \in (x_1^t, \dots, x_d^t)$ for some integer $t \geq 2$ (we may choose $x_1 = p$ by [Hoc83, page 539]). This will imply that there are elements $r_1, \dots, r_d$ such that

$$x_1^{t-1}((x_2 \cdots x_d)^{t-1} - r_1 x_1) = \sum_{i=2}^{d} r_i x_i^t.$$

Hence $(\bar{x}_2 \cdots \bar{x}_d)^{t-1} \in (\bar{x}_2^t, \dots, \bar{x}_d^t)^*$ in $\bar{R} := R/x_1R = R/pR$, where $\bar{x}_i$ denotes the image of x_i in $\bar{R}$, and hence, $(\bar{x}_2 \cdots \bar{x}_d)^{t-1} \in (\bar{x}_2^t, \dots, \bar{x}_d^t)^*$ in $R/\sqrt{pR}$. Then by the colon-capture property of tight closure, we have $1 \in (\bar{x}_2^t, \dots, \bar{x}_d^t)^* : (\bar{x}_2 \cdots \bar{x}_d)^{t-1} \subseteq (\bar{x}_2, \dots, \bar{x}_d)^* \subseteq \mathfrak{m}$, a contradiction. □

Theorem 8.6. *Conjecture* 8.4 *holds in dimension* 3.

Proof. Let $(R, \mathfrak{m})$ be a 3-dimensional complete local domain of mixed characteristic. Let p, x, y be a system of parameters of R and $J_n = (x, y) :_R p^n$. It suffices to show that $J_n\overline{R} \subseteq (J\overline{R})^*$, where $\overline{R} = R/\mathfrak{p}$, for each minimal prime ideal $\mathfrak{p}$ of p. Assume that $u \in J_n$ for some n, i.e., $up^n \in (x, y)$ for some n. Then by [Hei05, Theorem 0.2], for any nonzero element $c \in \mathfrak{m}$ and every $q = p^e$ one has $c^{1/q}u \in (x, y)R^+$. This holds true after modding out $\mathfrak{p}$, and therefore we have in $\overline{R} := R/\mathfrak{p}$ that $\bar{c}^{1/q}\bar{u} \in (\bar{x}, \bar{y})\overline{R}^+$. Then it follows from [HH91, Theorem 3.1] that $\bar{u} \in (\bar{x}, \bar{y})^* = (J\overline{R})^*$. The finishes the proof. □

Question 8.7. Is it always true that $\underline{\mathfrak{h}}_{\underline{x}}^d(R) = \mathfrak{h}_{\underline{x}}^d(R)$? This is true in positive characteristic as shown in [HH09, Theorem 3.9]. We do not even know whether the weaker statement that if $\underline{\mathfrak{h}}_{\underline{x}}^d(R) = 0$, then $\mathfrak{h}_{\underline{x}}^d(R) = 0$ is true.

Question 8.8. Let R be regular of characteristic $p > 0$, and let I be a prime ideal with $x_1, \dots, x_n$ a minimal set of generators. If $h < n$, is it true that $\mathfrak{h}_{\underline{x}}^n(R) = 0$?

Question 8.9. Assume that

$$S_1 = R[Y_1, \dots, Y_n]/(u - \sum_{l=1}^{n} Y_l g_l) \quad \text{and} \quad S_2 = R[Z_1, \dots, Z_n]/(v - \sum_{l=1}^{n} Z_l g_l)$$

satisfy $\mathfrak{h}_{\underline{x}}^d(S_1) = \mathfrak{h}_{\underline{x}}^d(S_2) = 1$. Then do we have

$$\mathfrak{h}_{\underline{x}}^d(S_1 \otimes_R S_2) = 1?$$

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