

D -MODULE AND F -MODULE LENGTH OF LOCAL COHOMOLOGY MODULES

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ABSTRACT. Let R be a polynomial or power series ring over a field k . We study the length of local cohomology modules $H_I^j(R)$ in the category of D -modules and F -modules. We show that the D -module length of $H_I^j(R)$ is bounded by a polynomial in the degree of the generators of I . In characteristic $p > 0$ we obtain upper and lower bounds on the F -module length in terms of the dimensions of Frobenius stable parts and the number of special primes of local cohomology modules of R/I . The obtained upper bound is sharp if R/I is an isolated singularity, and the lower bound is sharp when R/I is Gorenstein and F -pure. We also give an example of a local cohomology module that has different D -module and F -module lengths.

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1. INTRODUCTION

Since its introduction by Grothendieck, local cohomology has become a major part of commutative algebra that has been studied from different points of view. When R is a polynomial or power series ring over a field k , each local cohomology

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module $H_I^j(R)$ admits a natural module structure over $D(R, k)$, the ring of k -linear differential operators ($D(R, k)$ -modules will be reviewed in §2). In characteristic 0, [Lyu93] shows that $H_I^j(R)$ has finite length as a $D(R, k)$ -module, though $H_I^j(R)$ is rarely finitely generated as an R -module. To this day, using the finite length property of $H_I^j(R)$ in the category of $D(R, k)$ -modules is still the only way to prove that $H_I^j(R)$ has finitely many associated primes in characteristic 0. In characteristic p , Frobenius action on local cohomology modules was used with great success by a number of authors, e.g., [PS73], [HS77], and [HS93]. Lyubeznik ([Lyu97]) conceptualizes the previous work to develop a theory of F -modules in characteristic p (the reader may find an overview in §2).

As we have seen, in characteristic p , local cohomology modules $H_I^j(R)$ can be viewed as both $D(R, k)$ -modules and F -modules; [Lyu97] compares these two points of view. It's shown that each F -module M admits a natural $D(R, k)$ -module structure and its length as an F -module, $l_{F_R}(M)$, is no more than its length as a $D(R, k)$ -module, $l_{D(R, k)}(M)$. The comparison of these two points of view was continued in [Bli03], where Blickle shows that over an algebraically closed field the $D(R, k)$ -module length is equal to the F^∞ -module length (F^∞ -modules will be reviewed in §2). The fact that local cohomology modules have finite length as $D(R, k)$ -modules has found many applications; for instance [NBW14] introduces numerical invariants of local rings using the length of local cohomology modules as $D(R, k)$ -modules and shows that there are close connections between these invariants and F -singularities.

Despite the importance of the finiteness of the length of local cohomology modules as $D(R, k)$ -modules and F -modules, finding the actual length of local cohomology modules as such modules remains an intriguing and difficult open question. In this paper we provide partial answers to this question in characteristic p .

Theorem 1.1 (Theorems 4.3 and 5.1). *Let $R = k[[x_1, \dots, x_n]]$ (or $k[x_1, \dots, x_n]$) with $\mathfrak{m} = (x_1, \dots, x_n)$, where k is a field of characteristic $p > 0$. Let $A = R/I$ be reduced and equidimensional (respectively, graded reduced and equidimensional) of dimension $d \geq 1$.*

- (1) *If A has an isolated non- F -rational point at $\mathfrak{m}$ (e.g., A has an isolated singularity at $\mathfrak{m}$), then*

$$l_{D(R, k)}(H_I^{n-d}(R)) = \dim_k(0_{H_{\mathfrak{m}}^d(A)}^*)_s + c = \dim_k((H_{\mathfrak{m}}^d(A))_s) + c,$$

where c is the number of minimal primes of A . Moreover, if k is separably closed, then

$$\begin{aligned} l_{F_R}(H_I^{n-d}(R)) &= l_{F_R^e}(H_I^{n-d}(R)) = l_{F_R^\infty}(H_I^{n-d}(R)) = l_{D_R}(H_I^{n-d}(R)) \\ &= \dim_k(0_{H_{\mathfrak{m}}^d(A)}^*)_s + c. \end{aligned}$$

- (2) *If A is F -pure and quasi-Gorenstein, then $l_{F_R}(H_I^{n-d}(R))$ is exactly the number of special primes of $H_{\mathfrak{m}}^d(A)$.*

One ought to remark that there is an effective algorithm to compute special primes of $H_{\mathfrak{m}}^d(A)$ ([KZ14]); hence the result above provides a practical tool to compute $l_{F_R}(H_I^{n-d}(R))$ when $A = R/I$ is F -pure and quasi-Gorenstein.

We also construct the first example of a local cohomology module over an algebraically closed field whose $D(R, k)$ -module length disagrees with its F -module length.

Theorem 1.2 (Proposition 7.5). *Let $R = \overline{\mathbb{F}}_p[x, y, z, t]$ with $p \equiv 4 \pmod{7}$ and let $f = tx^7 + ty^7 + z^7$. Then*

$$l_{F_R}(\mathrm{H}_f^1(R)) = 3 < 7 = l_{F_R^\infty}(\mathrm{H}_f^1(R)) = l_{D(R, \overline{\mathbb{F}}_p)}(\mathrm{H}_f^1(R)).$$

While finding the actual length remains elusive in its full generality, we provide both upper and lower bounds.

Theorem 1.3 (Theorem 3.5). *Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field k and let $f \in R$ be a polynomial of degree d . Then*

$$l_{D(R, k)}(\mathrm{H}_f^1(R)) \leq (d+1)^n - 1.$$

Theorem 1.4 (Theorem 4.8). *Let $R = k[x_1, \dots, x_n]$ and $\mathfrak{m} = (x_1, \dots, x_n)$. Let I be a homogeneous reduced and equidimensional ideal of R . Set $A = R/I$ with $\dim A = d \geq 2$. Suppose the non- F -rational locus of A has dimension ≤ 1 (e.g., the nonsingular locus has dimension ≤ 1). Then we have*

$$\begin{aligned} l_{D(R, k)}(\mathrm{H}_I^{n-d}(R)) &\leq c + \sum_{\dim R/P=1} \dim_{\kappa(P)}(\mathrm{H}_{P\widehat{A}_P}^{d-1}(\widehat{A}_P))_s + \dim_k(\mathrm{H}_{\mathfrak{m}}^d(A))_s \\ &= c + \sum_{\dim R/P=1} \dim_{\kappa(P)}(0_{\mathrm{H}_{P\widehat{A}_P}^{d-1}(\widehat{A}_P)}^*)_s + \dim_k(0_{\mathrm{H}_{\mathfrak{m}}^d(A)}^*)_s, \end{aligned}$$

where c is the number of minimal primes of I .

Theorem 1.5 (Theorem 5.1). *Let k be a field of positive characteristic, let R denote the local ring $k[[x_1, \dots, x_n]]$ (or the graded ring $k[x_1, \dots, x_n]$), and let $\mathfrak{m} = (x_1, \dots, x_n)$. Let $A = R/I$ be reduced, equidimensional (or, respectively, graded reduced and equidimensional), and F -pure of dimension $d \geq 1$. Then $l_{F_R}(\mathrm{H}_I^{n-j}(R))$ is at least the number of special primes of $\mathrm{H}_{\mathfrak{m}}^j(A)$.*

The upper and lower bounds on the $D(R, k)$ -module and F_R -module length in the above results are sharp in many cases (see §§4, 5, 6), and we can explicitly describe an F_R -submodule filtration of $\mathrm{H}_I^{n-d}(R)$ in terms of the generating morphisms when R/I is Cohen–Macaulay, which is maximal when R/I is Gorenstein and F -pure (see Theorems 5.8 and 5.5).

We also construct an example (Example 4.11) of a simple $D(R, k)$ -module whose completion at a prime ideal P is not a simple $D(\widehat{R}_P, \kappa(P))$ -module.

Our paper is organized as follows. In §2, we recall some basic notions and results regarding $D(R, k)$ -modules, F -modules, and tight closure theory. Section 3 is concerned with Theorem 1.3. Section 4 is devoted to proving Theorem 1.4. In §5 we prove Theorem 1.5, and we also describe explicitly the maximal F_R -module filtration of $\mathrm{H}_I^{n-d}(R)$ in terms of their generating morphisms when R/I is Gorenstein and F -injective (the Cohen–Macaulay F -injective case will also be discussed). In §6 we compute the dimension of the stable part (under the natural Frobenius action) of the top local cohomology of Fermat hypersurfaces. Section 7 proves Theorem 1.2 and related results. Examples and remarks showing the sharpness of our bounds will be given throughout.

2. PRELIMINARIES

Throughout this paper, we always assume that $R = k[[x_1, \dots, x_n]]$ or $R = k[x_1, \dots, x_n]$, where k is a field (not necessarily algebraically closed or perfect)

and A is a reduced and equidimensional quotient or graded reduced and equidimensional quotient of R of dimension $d \geq 1$. We set $\mathfrak{m} = (x_1, \dots, x_n)$. In this section we collect some notation and preliminary results from [Lyu97] and [Bli04].

D -modules. The differential operators $\delta: R \rightarrow R$ of order $\leq n$ can be defined inductively as follows. A differential operator of order 0 is just multiplication by an element of R . A differential operator of order $\leq n$ is an additive map $\delta: R \rightarrow R$ such that for every $r \in R$, the commutator $[\delta, r] = \delta \circ r - r \circ \delta$ is a differential operator of order $\leq n - 1$. The differential operators form a ring with the multiplication defined via the composition. We denote this ring by D_R .

We denote by $D(R, k) \subseteq D_R$ the subring of D_R consisting of all k -linear differential operators. Since $R = k[[x_1, \dots, x_n]]$ or $R = k[x_1, \dots, x_n]$, it can be verified that $D(R, k)$ is generated by all operators of the form $\frac{1}{j!} \frac{\partial^j}{\partial x_i^j}$. By a D_R -module or a $D(R, k)$ -module we mean a left module over D_R or $D(R, k)$.

When k is a field of characteristic p , it is not hard to show that every differential operator of order $\leq p^e - 1$ is R^{p^e} -linear, where $R^{p^e} \subseteq R$ is the subring of all the p^e th powers of all the elements of R . In other words, we always have D_R is a subring of $\bigcup_e \text{Hom}_{R^{p^e}}(R, R)$.¹ In particular, all differential operators are automatically k -linear if k is perfect; thus $D_R = D(R, k)$ if k is perfect.

F -modules. Assume that k is a field of characteristic p . The notion of F -modules was introduced in [Lyu97] and further investigated and generalized in [Bli01, Bli03]. We use $R^{(e)}$ to denote the target ring of the e th Frobenius map $F^e: R \rightarrow R$. We shall let $F^e(-)$ denote the Peskine–Szpiro Frobenius functor from R -modules to R -modules. In detail, $F^e(M)$ is given by base change to $R^{(e)}$ and then identifying $R^{(e)}$ with R , i.e., $F^e(M) = R^{(e)} \otimes_R M$.

An F_R^e -module is an R -module M equipped with an R -linear isomorphism $\theta: M \rightarrow F^e(M)$, which we call the structure morphism of M . A homomorphism of F_R^e -modules is an R -module homomorphism $f: M \rightarrow M'$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ \theta \downarrow & & \downarrow \theta' \\ F^e(M) & \xrightarrow{F^e(f)} & F^e(M') \end{array}$$

When $e = 1$ we simply say M is an F_R -module (or F -module if R is clear from the context). It is easy to see that every F_R^e -module is also an F_R^{er} -module for every $r \geq 1$ by iterating the structure isomorphism r times. The union of the categories of F_R^e -modules over all e forms is what we call the category of F_R^∞ -modules.² With these definitions, the categories of F_R^e -modules and F_R^∞ -modules are abelian categories.

When $R = k[x_1, \dots, x_n]$ and M is a graded R -module, there is a natural grading on $F(M) = R^{(e)} \otimes_R M$ given by $\deg(r \otimes m) = \deg r + p^e \cdot \deg m$ for homogeneous

¹In fact $D_R = \bigcup_e \text{Hom}_{R^{p^e}}(R, R)$ when R is F -finite; i.e., R is finitely generated as an R^{p^e} -module [Yek92].

²In [Bli01, Bli03, Bli04], F_R^e -modules and F_R^∞ -modules are called unit $R[F^e]$ -modules and unit $R[F]$ -modules respectively. In this paper we will use Lyubeznik's notation [Lyu97, Remark 5.6] since we think this is more natural in comparison with the usual F_R -modules.

elements $r \in R$ and $m \in M$. With this grading, a *graded F_R^e -module* is an F_R^e -module M such that the structure isomorphism θ is degree-preserving. A morphism of graded F_R^e -modules is a degree-preserving morphism of F_R^e -modules. It is not hard to see that graded F_R^e -modules form an abelian subcategory of the category of F_R^e -modules. Graded F_R^e -modules, at least for $e = 1$, were introduced and studied in detail in [Zha12] and [MZ14].

A generating morphism of an F_R^e -module M is an R -module homomorphism $\beta: M_0 \rightarrow F^e(M_0)$, where M_0 is some R -module such that M is the limit of the inductive system in the top row of the commutative diagram

$$\begin{array}{ccccccc} M_0 & \xrightarrow{\beta} & F^e(M_0) & \xrightarrow{F^e(\beta)} & F^{2e}(M_0) & \xrightarrow{F^{2e}(\beta)} & \cdots \\ \downarrow \beta & & \downarrow F^e(\beta) & & \downarrow F^{2e}(\beta) & & \\ F^e(M_0) & \xrightarrow{F^e(\beta)} & F^{2e}(M_0) & \xrightarrow{F^{2e}(\beta)} & F^{3e}(M_0) & \xrightarrow{F^{3e}(\beta)} & \cdots \end{array}$$

and $\theta: M \rightarrow F^e(M)$, the structure isomorphism of M , is induced by the vertical arrows in this diagram. M_0 is called a *root* of M if $\beta: M_0 \rightarrow F^e(M_0)$ is injective. An F_R^e -module M is called *F -finite* if M has a generating morphism $\beta: M_0 \rightarrow F^e(M_0)$ with M_0 a finitely generated R -module. When M_0 is graded and β is degree-preserving, we say that M is a *graded F -finite F_R^e -module*.

It is a fundamental result of Lyubeznik ([Lyu97]) that local cohomology modules $H_I^i(R)$ have a natural structure of F -finite F_R^e -modules for every $e \geq 1$. Moreover, when $R = k[x_1, \dots, x_n]$ and I is a homogeneous ideal of R , $H_I^i(R)$ are graded F -finite F_R^e -modules [Zha12].

Following [Lyu97], for any F -finite F_R -module M , there exists a smallest F_R -submodule $N \subseteq M$ with the property that M/N is supported only at $\mathfrak{m}$. Hence M/N is isomorphic (as an R -module) to $E^{\oplus r}$ where $E = E_R(k)$ denotes the injective hull of k . We define $\text{crk}(M)$, the *corank* of M , to be r .

One important feature of F_R -modules is that they have a natural structure of D_R -modules and thus $D(R, k)$ -modules. We briefly recall this here, and we refer to [Lyu97, §5] for details. Let M be an F_R -module with structure isomorphism θ . We set θ_e to be the e th iterate of θ , i.e.,

$$\theta_e = F^{e-1}(\theta) \circ \cdots \circ F(\theta) \circ \theta : M \rightarrow F^e(M).$$

Now every element $\delta \in \text{Hom}_{R^p^e}(R, R)$ acts on $F^e(M) = R^{(e)} \otimes_R M$ via $\delta \otimes \text{id}_M$. We let δ act on M via $\theta_e^{-1} \circ (\delta \otimes \text{id}_M) \circ \theta_e$. It is not very difficult to check that this action is well-defined. Moreover, an entirely similar construction shows that every F_R^e -module also has a canonical structure of a D_R -module. It follows that F_R^∞ -modules are naturally D_R -modules and thus $D(R, k)$ -modules. In sum, we have the following inclusion of abelian categories:

$$\begin{aligned} \{F_R - \text{modules}\} &\subseteq \{F_R^e - \text{modules}\} \subseteq \{F_R^\infty - \text{modules}\} \subseteq \{D_R - \text{modules}\} \\ &\subseteq \{D(R, k) - \text{modules}\}. \end{aligned}$$

Therefore for any F_R -module M we have the following inequalities on its length considered in the corresponding categories:

$$(2.0.1) \quad l_{F_R}(M) \leq l_{F_R^e}(M) \leq l_{F_R^\infty}(M) \leq l_{D_R}(M) \leq l_{D(R, k)}(M).$$

$A\{f\}$ -modules. Recall that we always assume A is a reduced and equidimensional or graded reduced and equidimensional quotient of R of dimension $d \geq 1$. Assume also that k is a field of characteristic p . Let M be a (in most cases Artinian) module over A . We say that M is an $A\{f\}$ -module if there is an additive map $f: M \rightarrow M$ such that $f(am) = a^p f(m)$. We will use $M_s = \bigcap k\langle f^i(M) \rangle$ to denote the Frobenius stable part of M , note that this is a k -vector space, and that it depends not only on M but also on the action f on M . It is well-known that when M is either Noetherian or Artinian over A , M_s is a finite dimensional k -vector space [HS77], [Lyu97].

Let $\mathcal{H}_{R,A}$ denote the Lyubeznik functor introduced in [Lyu97]: for every Artinian $A\{f\}$ -module M , we have a natural induced map $\alpha: F_R(M) \rightarrow M$. Since $F_R(M)^\vee \cong F_R(M^\vee)$ by [Lyu97, Lemma 4.1], we define

$$\mathcal{H}_{R,A}(M) = \varinjlim (M^\vee \xrightarrow{\alpha^\vee} F_R(M^\vee) \xrightarrow{F(\alpha^\vee)} F_R^2(M^\vee) \rightarrow \cdots),$$

which is an F -finite F_R -module. In the graded case we have a similar functor ${}^*\mathcal{H}_{R,A}$ that takes a graded Artinian $A\{f\}$ -module to a graded F -finite F_R -module: one needs to replace Matlis dual by graded Matlis dual in the construction of ${}^*\mathcal{H}_{R,A}$ (see [LSW16] for details). One important example that we will use repeatedly is that $\mathcal{H}_{R,A}(H_{\mathfrak{m}}^d(A)) \cong H_I^{n-d}(R)$, and in the graded case we also have ${}^*\mathcal{H}_{R,A}(H_{\mathfrak{m}}^d(A)) \cong H_I^{n-d}(R)$ [Lyu97, Example 4.8], [LSW16, Proposition 2.8].

Tight closure and F -singularities. Tight closure theory was introduced by Hochster-Huneke in [HH90]. In this article, we need only some basic properties of tight closure of zero in the top local cohomology module, $0_{H_{\mathfrak{m}}^d(A)}^*$. Under mild conditions, for example, when $(A, \mathfrak{m})$ is an excellent local domain of dimension d , $0_{H_{\mathfrak{m}}^d(A)}^*$ is the largest proper A -submodule of $H_{\mathfrak{m}}^d(A)$ that is stable under the natural Frobenius action on $H_{\mathfrak{m}}^d(A)$ (cf. [Smi97]). A local ring $(A, \mathfrak{m})$ is called F -rational if A is Cohen–Macaulay and $0_{H_{\mathfrak{m}}^d(A)}^* = 0$.³ Under mild conditions on the ring, for example, when $(A, \mathfrak{m})$ is an excellent, reduced, and equidimensional local ring, A is F -rational on the punctured spectrum if and only if $0_{H_{\mathfrak{m}}^d(A)}^*$ has finite length.

A local ring $(A, \mathfrak{m})$ of characteristic $p > 0$ is called F -pure if the Frobenius endomorphism $F: A \rightarrow A$ is pure.⁴ Under mild conditions, for example, when the Frobenius map $A \xrightarrow{F} A$ is a finite map or when A is complete, F -purity of A is equivalent to the condition that the Frobenius endomorphism $A \xrightarrow{F} A$ is split [HR76, Corollary 5.3]. The Frobenius endomorphism on A induces a natural Frobenius action on each local cohomology module $H_{\mathfrak{m}}^i(A)$, and we say $(A, \mathfrak{m})$ is F -injective if this natural Frobenius action on $H_{\mathfrak{m}}^i(A)$ is injective for every i ([Fed83]). This holds if A is F -pure [HR76, Lemma 2.2]. For some other basic properties of F -pure and F -injective singularities, see [HR76], [Fed83], [EH08].

3. A GENERAL BOUND OF LOCAL COHOMOLOGY MODULES AS D -MODULES

In this section, we will establish a bound of $H_I^j(R)$ as a $D(R, k)$ -module when $R = k[x_1, \dots, x_n]$ is a polynomial ring over a field k (of any characteristic) in terms

³This is not the original definition of F -rationality but is shown to be equivalent ([Smi97]).

⁴A map of A -modules $N \rightarrow N'$ is pure if for every A -module M the map $N \otimes_A M \rightarrow N' \otimes_A M$ is injective. This implies that $N \rightarrow N'$ is injective and is weaker than the condition that $0 \rightarrow N \rightarrow N'$ be split.

of the degrees of generators of I . To this end, we begin with recalling the notion of the Bernstein filtration and k -filtration.

Denote $\frac{1}{t!} \frac{\partial^t}{\partial x_i^t}$ by $D_{t,i}$ for $t \in \mathbb{N}, 1 \leq i \leq n$. Then $D(R, k) = R\langle D_{t,i} \mid t \in \mathbb{N}, 1 \leq i \leq n \rangle$. We set $\mathcal{F}_j$ to be the k -linear span of the set of products

$$\{x_1^{i_1} \cdots x_n^{i_n} \cdot D_{t_1,1} \cdots D_{t_n,n} \mid i_1 + \cdots + i_n + t_1 + \cdots + t_n \leq j\}.$$

Then the *Bernstein filtration* on $D(R, k)$ ([Lyu11, Definition 2.6]) is defined to be $k = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \cdots$.

Definition 3.1 (Definition 3.2 in [Lyu11]). A *k-filtration* on a $D(R, k)$ -module M is an ascending chain of finite-dimensional k -vector spaces $M_0 \subset M_1 \subset \cdots$ such that $\bigcup_i M_i = M$ and $\mathcal{F}_i M_j \subset M_{i+j}$ for all i and j .

We need the following result of Lyubeznik. We note that in characteristic 0, the D -module length of holonomic D -modules has been studied before; see [Ber71], [Bjo79].

Theorem 3.2 (Theorem 3.5 in [Lyu11]). *Let M be a $D(R, k)$ -module with a k -filtration $M_0 \subset M_1 \subset \cdots$. Assume there is a constant C such that $\dim_k(M_i) \leq Ci^n$ for sufficiently large i . Then the length of M as a $D(R, k)$ -module is at most $n!C$.*

The statement of [Lyu11, Theorem 3.5] assumes that there is a constant C such that $\dim_k(M_i) \leq Ci^n$ for all $i \geq 0$; however, the proof of [Lyu11, Theorem 3.5] only uses the fact that there is a constant C such that $\dim_k(M_i) \leq Ci^n$ for sufficiently large i . Hence the proof of Theorem 3.2 is identical to the one of [Lyu11, Theorem 3.5] and is omitted.

To illustrate the advantage of requiring only that $\dim_k(M_i) \leq Ci^n$ for sufficiently large i , we consider a simple example.

Example 3.3. Set R_i to be the k -span of monomials in $x_1, \dots, x_n$ of degree at most i . It is clear that $R_0 \subset R_1 \subset \cdots$ is a k -filtration of R . It is well-known that $\dim_k(R_i) = \binom{n+i}{i}$, which is a polynomial in i of degree n with leading coefficient $\frac{1}{n!}$. Hence, given any $\varepsilon > 0$, we have $\dim_k(R_i) \leq \frac{1+\varepsilon}{n!} i^n$ for sufficiently large⁵ i . Since the length of a module is an integer, it follows from Theorem 3.2 that the length of R in the category of $D(R, k)$ -modules is 1. On the other hand, if one requires $\dim_k(R_i) \leq Ci^n$ for all i , then one will need $C \geq \frac{n+1}{n!}$ (consider the case when $i = 1$) and consequently cannot deduce the correct length of R from [Lyu11, Theorem 3.5].

Remark 3.4. In the proof of [Lyu11, Corollary 3.6], the following statement is proved: *Let M be a $D(R, k)$ -module with a k -filtration $M_0 \subseteq M_1 \subseteq \cdots$ and let $f \in R$ be a polynomial of degree d . If there is a constant C such that $\dim_k(M_i) \leq Ci^n$ for sufficiently large i , then $M'_i = \{\frac{m}{f^i} \mid m \in M_{i(d+1)}\}$ induces a k -filtration of M_f such that $\dim_k(M'_i) \leq C(d+1)^n i^n$ for sufficiently large i .*

Theorem 3.5. *Let $f \in R$ be a polynomial of degree d . Then the length of R_f in the category of $D(R, k)$ -modules is at most $(d+1)^n$. Also, the length of $H_f^1(R)$ is at most $(d+1)^n - 1$.*

Proof. Combining Example 3.3 and Remark 3.4, we see that R_f admits a k -filtration $R'_0 \subset R'_1 \subset \cdots$ such that, for any any $\varepsilon > 0$, one has $\dim_k(R'_i) \leq \frac{1+\varepsilon}{n!} (d+1)^n i^n$.

⁵We should remark that how large i needs to be certainly depends on ε .

According to Theorem 3.2, we have that the length of R_f in the category of $D(R, k)$ -modules is at most $(1 + \varepsilon)(d + 1)^n$ for any $\varepsilon > 0$; it follows that the length of R_f in the category of $D(R, k)$ -modules is at most $(d + 1)^n$.

The second conclusion follows from the exact sequence $0 \rightarrow R \rightarrow R_f \rightarrow H_f^1(R)$ and the fact that the length of R is 1. $\square$

Corollary 3.6. *Let I be an ideal of R . If I is generated by $f_1, \dots, f_t$ with $\deg(f_i) = d_i$, then the length of $H_I^j(R)$ in the category of $D(R, k)$ -modules is at most*

$$\sum_{1 \leq i_1 \leq \dots \leq i_j \leq t} (d_{i_1} + \dots + d_{i_j} + 1)^n - 1.$$

Proof. It is clear from Theorem 3.5 that the length of $\bigoplus R_{f_{i_1} \dots f_{i_j}}$ is at most

$$\sum_{1 \leq i_1 \leq \dots \leq i_j \leq t} (d_{i_1} + \dots + d_{i_j} + 1)^n.$$

Our corollary follows from the fact that $H_I^j(R)$ is a proper subquotient of $\bigoplus R_{f_{i_1} \dots f_{i_j}}$ in the category of $D(R, k)$ -modules. $\square$

The bounds in Theorem 3.5 and Corollary 3.6, though general, are very coarse. In the rest of the paper, we will focus on the length of $H_I^c(R)$ where c is the height of I and k is of prime characteristic p , and where F -module theory and tight closure theory can be used to produce sharper bounds.

4. FORMULAS AND UPPER BOUNDS ON THE D -MODULE AND F -MODULE LENGTH

Notation. Henceforth R denotes $k[[x_1, \dots, x_n]]$ or $k[x_1, \dots, x_n]$, where k is a field of characteristic $p > 0$, $\mathfrak{m} = (x_1, \dots, x_n)$, and we let $A = R/I$ be reduced and equidimensional or graded reduced and equidimensional of dimension $d \geq 1$.

We first analyze the case when A has an isolated non- F -rational point at $\{\mathfrak{m}\}$. We start with a few lemmas.

Lemma 4.1. *Let $0 \rightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0$ be an exact sequence of Artinian $A\{f\}$ -modules. Let f_L, f_M, f_N denote the Frobenius actions on L, M, N respectively. Then the stable parts form a left exact sequence of finite dimensional vector spaces: $0 \rightarrow L_s \xrightarrow{\alpha} M_s \xrightarrow{\beta} N_s$.*

Proof. Let $\mathbb{K}$ be the perfect closure of k . Define $A^{\mathbb{K}} = A \otimes_k \mathbb{K}$ and for any $A\{f\}$ -module X , let $X^{\mathbb{K}}$ be the $A^{\mathbb{K}}\{f\}$ -module $A^{\mathbb{K}} \otimes_k X$ where the action of f is given by $f(\sum_{i=1}^{\mu} x_i \otimes \lambda_i) = \sum_{i=1}^{\mu} f(x_i) \otimes \lambda_i^p$ for $x_1, \dots, x_{\mu} \in X$ and $\lambda_1, \dots, \lambda_{\mu} \in \mathbb{K}$. [Lyu97, Proposition 4.9] implies that $X_s^{\mathbb{K}} = X_s \otimes_k \mathbb{K}$, so if we could show that $0 \rightarrow L_s^{\mathbb{K}} \xrightarrow{\alpha^{\mathbb{K}}} M_s^{\mathbb{K}} \xrightarrow{\beta^{\mathbb{K}}} N_s^{\mathbb{K}}$, the fact that $\mathbb{K}$ is faithfully flat over k would imply the exactness of $0 \rightarrow L_s \xrightarrow{\alpha} M_s \xrightarrow{\beta} N_s$. Therefore we may assume that k is a perfect field.

The exactness at L_s and $\beta\alpha = 0$ are obvious. Hence to prove exactness it suffices to show that $\ker \beta \subseteq \operatorname{im} \alpha$. Since N is Artinian, by [Lyu97, Proposition 4.4] $\bigcup_{r \geq 1} \ker f_N^r = \ker f_N^{r_0}$ for some r_0 sufficiently large and $\ker f_N \subseteq \ker f_N^2 \subseteq \dots$ stabilizes.

Pick $x \in M_s$ such that $\beta(x) = 0$. Since we are assuming that k is perfect, the k -vector space generated by $f^r(M)$ is just $f^r(M)$ for all $r \geq 0$; therefore, for each

$r \geq r_0$, there exists $y \in M$ such that $x = f_M^r(y)$. We have $f_N^r(\beta(y)) = \beta(f_M^r(y)) = 0$. So $f_N^{r_0}(\beta(y)) = 0$ and hence $z = f_M^{r_0}(y) \in L$. Therefore for each $r \geq r_0$, there exists $z \in L$ such that $x = f_M^{r-r_0}(z)$, hence $x \in L_s$. $\square$

Lemma 4.2. *Suppose $\dim A = d \geq 1$. Then $(0_{H_m^d(A)}^*)_s \cong (H_m^d(A))_s$.*

Proof. We have $0 \rightarrow 0_{H_m^d(A)}^* \rightarrow H_m^d(A) \rightarrow H_m^d(A)/0_{H_m^d(A)}^* \rightarrow 0$. By Lemma 4.1, it suffices to show that $(H_m^d(A)/0_{H_m^d(A)}^*)_s = 0$. But by [Lyu97, Proposition 4.10], $\dim(H_m^d(A)/0_{H_m^d(A)}^*)_s = \text{crk } \mathcal{H}_{R,A}(H_m^d(A)/0_{H_m^d(A)}^*)$. Let $P_1, \dots, P_c$ be all the minimal primes of A . By [Bli04, Theorems 4.3 and 4.4], $\mathcal{H}_{R,A}(H_m^d(A)/0_{H_m^d(A)}^*)$ is a direct sum of simple F_R -modules, each of which has P_i as its unique associated prime. This implies that $\text{crk } \mathcal{H}_{R,A}(H_m^d(A)/0_{H_m^d(A)}^*) = 0$ because $\dim A \geq 1$, and hence $(H_m^d(A)/0_{H_m^d(A)}^*)_s = 0$ as desired. $\square$

Theorem 4.3. *Let $R = k[[x_1, \dots, x_n]]$ (or $k[x_1, \dots, x_n]$) with $\mathfrak{m} = (x_1, \dots, x_n)$, where k is a field of characteristic $p > 0$. Let $A = R/I$ be reduced and equidimensional (respectively, graded reduced and equidimensional) of dimension $d \geq 1$. Assume that A has an isolated non- F -rational point at $\mathfrak{m}$ (e.g., A has an isolated singularity at $\mathfrak{m}$). Then*

$$l_{D(R,k)}(H_I^{n-d}(R)) = \dim_k(0_{H_m^d(A)}^*)_s + c = \dim_k((H_m^d(A))_s) + c,$$

where c is the number of minimal primes of A . Moreover, if k is separably closed, then we also have

$$\begin{aligned} l_{F_R}(H_I^{n-d}(R)) &= l_{F_R^e}(H_I^{n-d}(R)) = l_{F_R^\infty}(H_I^{n-d}(R)) = l_{D_R}(H_I^{n-d}(R)) \\ &= \dim_k(0_{H_m^d(A)}^*)_s + c. \end{aligned}$$

Proof. The second equality follows immediately from Lemma 4.2. Thus it suffices to show that

$$l_{D(R,k)}(H_I^{n-d}(R)) = \dim_k(0_{H_m^d(A)}^*)_s + c$$

and

$$l_{F_R}(H_I^{n-d}(R)) = \dim_k(0_{H_m^d(A)}^*)_s + c$$

when k is separably closed, since the other equalities would follow from (2.0.1).

The short exact sequence $0 \rightarrow 0_{H_m^d(A)}^* \rightarrow H_m^d(A) \rightarrow H_m^d(A)/0_{H_m^d(A)}^* \rightarrow 0$ induces

$$0 \rightarrow \mathcal{H}_{R,A}(H_m^d(A)/0_{H_m^d(A)}^*) \rightarrow \mathcal{H}_{R,A}(H_m^d(A)) \cong H_I^{n-d}(R) \rightarrow \mathcal{H}_{R,A}(0_{H_m^d(A)}^*) \rightarrow 0.$$

Now by [Bli04, Corollary 4.2 and Theorem 4.4], $\mathcal{H}_{R,A}(H_m^d(A)/0_{H_m^d(A)}^*)$ is a direct sum of simple $D(R, k)$ -modules, each supported at a different minimal prime of A , so its $D(R, k)$ -module length is c , and in fact each of these simple $D(R, k)$ -modules are (simple) F_R -modules [Bli04, Theorem 4.3]. Thus $l_{D(R,k)}(\mathcal{H}_{R,A}(H_m^d(A)/0_{H_m^d(A)}^*)) = l_{F_R}(\mathcal{H}_{R,A}(H_m^d(A)/0_{H_m^d(A)}^*)) = c$.

Since A has an isolated non- F -rational point at $\mathfrak{m}$, $0_{H_m^d(A)}^*$ has finite length as an A -module. This implies that $\mathcal{H}_{R,A}(0_{H_m^d(A)}^*)$ is supported only at $\mathfrak{m}$. Hence $\mathcal{H}_{R,A}(0_{H_m^d(A)}^*)$, as a $D(R, k)$ -module, is a direct sum of finitely many copies of $H_m^n(R)$ by [Lyu00, Lemma (c)]. Moreover, when k is separably closed, $\mathcal{H}_{R,A}(0_{H_m^d(A)}^*)$ is a direct sum of copies of $E = H_m^n(R)$ even as an F_R -module [Ma14a, Lemma 4.3].

The number of copies of $H_{\mathfrak{m}}^n(R)$ is exactly its $D(R, k)$ -module length (F_R -module length when k is separably closed). However, since $\mathcal{H}_{R,A}(0_{H_{\mathfrak{m}}^d(A)}^*)$ is an F_R -module supported only at $\mathfrak{m}$, the number of copies of $H_{\mathfrak{m}}^n(R)$ is by definition $\text{crk } \mathcal{H}_{R,A}(0_{H_{\mathfrak{m}}^d(A)}^*)$, which is $\dim_k(0_{H_{\mathfrak{m}}^d(A)}^*)_s$ by [Lyu97, Proposition 4.10]. Hence $l_{D(R,k)}(\mathcal{H}_{R,A}(0_{H_{\mathfrak{m}}^d(A)}^*)) = \dim_k(0_{H_{\mathfrak{m}}^d(A)}^*)_s$, and $l_{F_R}(\mathcal{H}_{R,A}(0_{H_{\mathfrak{m}}^d(A)}^*)) = \dim_k(0_{H_{\mathfrak{m}}^d(A)}^*)_s$ when k is separably closed. $\square$

Remark 4.4. When k is not separably closed, $l_{F_R}(H_I^{n-d}(R)) = \dim_k(0_{H_{\mathfrak{m}}^d(A)}^*)_s + c$ fails to hold in general; see Corollary 7.4. However, if k is a *finite field*, then we always have $l_{F_R^\infty}(H_I^{n-d}(R)) = \dim_k(0_{H_{\mathfrak{m}}^d(A)}^*)_s + c$; see Proposition 7.7.

Remark 4.5. When I is a homogeneous reduced and equidimensional ideal in $R = k[x_1, \dots, x_n]$ (i.e., A is a graded domain), it is easy to check that $(0_{H_{\mathfrak{m}}^d(A)}^*)_s = (H_{\mathfrak{m}}^d(A))_s = (H_{\mathfrak{m}}^d(A))_0$ since the stable part will be concentrated in degree 0. Therefore, we have an upper bound of the $D(R, k)$ -module length of $H_I^{n-d}(R)$ in the graded isolated singularity case: it is at most $\dim_k(H_{\mathfrak{m}}^d(A)_0) + c$. Geometrically, it is at most $\dim_k(H^{d-1}(X, \mathcal{O}_X)) + c$ where $X = \text{Proj}(A)$.

We have the following application.

Example 4.6. Let $A = k[x_1, \dots, x_n]/(f)$ where k is a field of prime characteristic p and $\deg(f) = n$. Denote $k[x_1, \dots, x_n]$ by R . Then there is a commutative diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{\mathfrak{m}}^{n-1}(A)_0 & \longrightarrow & H_{\mathfrak{m}}^n(R)_{-n} & \xrightarrow{f} & H_{\mathfrak{m}}^n(R)_0 = 0 \longrightarrow 0 \\ & & \downarrow F & & \downarrow f^{p-1}F & & \downarrow F \\ 0 & \longrightarrow & H_{\mathfrak{m}}^{n-1}(A)_0 & \longrightarrow & H_{\mathfrak{m}}^n(R)_{-n} & \xrightarrow{f} & H_{\mathfrak{m}}^n(R)_0 = 0 \longrightarrow 0 \end{array}$$

where F denotes the natural Frobenius maps. It follows that $\dim_k(H_{\mathfrak{m}}^{n-1}(A)_0) = \dim_k(H_{\mathfrak{m}}^n(R)_{-n}) = 1$. It also follows from the diagram that $F : H_{\mathfrak{m}}^{n-1}(A)_0 \rightarrow H_{\mathfrak{m}}^{n-1}(A)_0$ is injective if and only if so is $f^{p-1}F : H_{\mathfrak{m}}^n(R)_{-n} \rightarrow H_{\mathfrak{m}}^n(R)_{-n}$, which holds if and only if $f^{p-1} \notin \mathfrak{m}^{[p]}$.

Assume further that f is irreducible with an isolated singularity at $\mathfrak{m}$ over a perfect field k ; i.e., A is a Calabi–Yau hypersurface over k . Then by Fedder’s Criterion, $F : H_{\mathfrak{m}}^{n-1}(A)_0 \rightarrow H_{\mathfrak{m}}^{n-1}(A)_0$ is injective if and only if A is F -pure. Consequently, $\dim_k(H_{\mathfrak{m}}^{n-1}(A)_0)_s = 1$ if and only if A is F -pure. Thus it follows from Theorem 4.3 that

$$l_{D(R,k)}(H_f^1(R)) = \begin{cases} 1 & A \text{ is not } F\text{-pure,} \\ 2 & \text{otherwise.} \end{cases}$$

In particular, when $\text{Proj}(A)$ happens to be an elliptic curve, there are infinitely many primes p such that $H_f^1(R)$ is a simple $D(R, k)$ -module and infinitely many primes p such that $H_f^1(R)$ has length 2 as a $D(R, k)$ -module.

Next we partially generalize Theorem 4.3 to the case that the dimension of the singular locus of R/I is 1. We first prove a lemma, which should be well known to experts.

Lemma 4.7. *Let M be an (F -finite) F_R -module (resp., F_R^∞ -module). Then M_P and $M \otimes \widehat{R_P}$ are (F -finite) F_{R_P} and $F_{\widehat{R_P}}$ -modules (resp., $F_{R_P}^\infty$ and $F_{\widehat{R_P}}^\infty$ -modules). Moreover, if M is a simple F_R -module (resp., simple F_R^∞ -module), then M_P and $M \otimes \widehat{R_P}$, if not zero, are simple as F_{R_P} and $F_{\widehat{R_P}}$ -modules (resp., simple as $F_{R_P}^\infty$ and $F_{\widehat{R_P}}^\infty$ -modules).*

Proof. The conclusion for F -modules follows from Proposition 2.7 and Corollary 2.9 of [Bli04]. The argument for F_R^∞ -modules is very similar, so we omit the details. $\square$

Theorem 4.8. *Let $R = k[x_1, \dots, x_n]$ and $\mathfrak{m} = (x_1, \dots, x_n)$. Let I be a homogeneous reduced and equidimensional ideal of R . Set $A = R/I$ with $\dim A = d \geq 2$. Suppose the non- F -rational locus of A has dimension ≤ 1 (e.g., the nonsingular locus has dimension ≤ 1). Then we have*

$$\begin{aligned} l_{D(R,k)}(\mathrm{H}_I^{n-d}(R)) &\leq c + \sum_{\dim R/P=1} \dim_{\kappa(P)}(\mathrm{H}_{P\widehat{A_P}}^{d-1}(\widehat{A_P}))_s + \dim_k(\mathrm{H}_{\mathfrak{m}}^d(A))_s \\ &= c + \sum_{\dim R/P=1} \dim_{\kappa(P)}(0_{\mathrm{H}_{P\widehat{A_P}}^{d-1}(\widehat{A_P})}^*)_s + \dim_k(0_{\mathrm{H}_{\mathfrak{m}}^d(A)}^*)_s, \end{aligned}$$

where c is the number of minimal primes of I .

Proof. Clearly the second equality follows from Lemma 4.2. Therefore it suffices to prove the first inequality. We may assume that the dimension of the non- F -rational locus is 1, since otherwise the result follows from Theorem 4.3 (the second term is 0). We begin with the following claim.

Claim 4.8.1. There exist graded F_R -submodules

$$0 \subseteq L \subseteq M \subseteq \mathrm{H}_I^{n-d}(R)$$

such that every $D(R, k)$ -module composition factor of L is supported at a minimal prime of A , every $D(R, k)$ -module composition factor of M/L is supported at a dimension 1 prime, and $\mathrm{H}_I^{n-d}(R)/M$ is supported only at $\mathfrak{m}$.

Proof of claim. We have a short exact sequence

$$0 \rightarrow 0_{\mathrm{H}_{\mathfrak{m}}^d(A)}^* \rightarrow \mathrm{H}_{\mathfrak{m}}^d(A) \rightarrow \mathrm{H}_{\mathfrak{m}}^d(A)/0_{\mathrm{H}_{\mathfrak{m}}^d(A)}^* \rightarrow 0$$

that induces

$$0 \rightarrow {}^*\mathcal{H}_{R,A}(\mathrm{H}_{\mathfrak{m}}^d(A)/0_{\mathrm{H}_{\mathfrak{m}}^d(A)}^*) \rightarrow {}^*\mathcal{H}_{R,A}(\mathrm{H}_{\mathfrak{m}}^d(A)) \cong \mathrm{H}_I^{n-d}(R) \rightarrow {}^*\mathcal{H}_{R,A}(0_{\mathrm{H}_{\mathfrak{m}}^d(A)}^*) \rightarrow 0.$$

We have, by the graded version of [Bli04, Corollary 4.2 and Theorem 4.4], that ${}^*\mathcal{H}_{R,A}(\mathrm{H}_{\mathfrak{m}}^d(A)/0_{\mathrm{H}_{\mathfrak{m}}^d(A)}^*)$ is a direct sum of simple $D(R, k)$ -modules, each supported at a different minimal prime of A . So we set $L = {}^*\mathcal{H}_{R,A}(\mathrm{H}_{\mathfrak{m}}^d(A)/0_{\mathrm{H}_{\mathfrak{m}}^d(A)}^*)$. The existence of M follows by applying [LSW16, Theorem 2.9(3)] to ${}^*\mathcal{H}_{R,A}(0_{\mathrm{H}_{\mathfrak{m}}^d(A)}^*)$, which is a graded F -finite F_R -module. Note that the support of each $D(R, k)$ -module composition factor of M/L has dimension 1. This is because the support of ${}^*\mathcal{H}_{R,A}(0_{\mathrm{H}_{\mathfrak{m}}^d(A)}^*)$ has dimension 1 since the dimension of the non- F -rational locus is 1. $\square$

We know that $l_{D(R,k)}(L) = c$. Moreover, it is clear from the above claim that M is the smallest F_R -submodule of $\mathrm{H}_I^{n-d}(R)$ such that $\mathrm{H}_I^{n-d}(R)/M$ is only supported at $\mathfrak{m}$. So $\mathrm{H}_I^{n-d}(R)/M$ is isomorphic, as a $D(R, k)$ -module, to $E^{\oplus r} \cong \mathrm{H}_{\mathfrak{m}}^d(R)^{\oplus r}$,

where $r = \text{crk } H_I^{n-d}(R)$ by [Lyu00, Lemma (c)] and the definition of corank. Thus we have $l_{D(R,k)}(H_I^{n-d}(R)/M) = \text{crk } H_I^{n-d}(R) = \dim_k(H_m^d(A))_s$ by [Lyu97, Proposition 4.10].

It remains to estimate the $D(R, k)$ -module length of M/L . Suppose we have

$$(4.8.2) \quad 0 \subseteq L = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_t = M \subseteq H_I^{n-d}(R)$$

such that each $N_i = M_i/M_{i-1}$ is a simple $D(R, k)$ -module. We know that each N_i has a unique associated prime P with $\dim R/P = 1$ and A_P not F -rational.

Claim 4.8.3. The number of N_i such that $\text{Ass}(N_i) = P$ is at most $\text{crk } H_{I\widehat{R}_P}^{n-d}(\widehat{R}_P)$.

Proof of claim. We localize (4.8.2) at P and complete. We have

$$0 \subseteq L \otimes \widehat{R}_P = M_0 \otimes \widehat{R}_P \subseteq M_1 \otimes \widehat{R}_P \subseteq \cdots \subseteq M_t \otimes \widehat{R}_P = H_{I\widehat{R}_P}^{n-d}(\widehat{R}_P)$$

with successive quotients $N_i \otimes \widehat{R}_P$ (the last equality follows because $H_I^{n-d}(R)/M$ is supported only at $\mathfrak{m}$). Each $N_i \otimes \widehat{R}_P$ is either 0 or a $D(\widehat{R}_P, k)$ -module supported only at $P\widehat{R}_P$ (and thus a direct sum of $E(\widehat{R}_P/P\widehat{R}_P)$), depending on whether $\text{Ass}(N_i) \neq P$ or $\text{Ass}(N_i) = P$. Therefore $H_{I\widehat{R}_P}^{n-d}(\widehat{R}_P)/(L \otimes \widehat{R}_P)$, at least as an $\widehat{R}_P$ -module, is isomorphic to $E(\widehat{R}_P/P\widehat{R}_P)^r$. The number of N_i such that $\text{Ass}(N_i) = P$ is thus $\leq r$.

But $L \otimes \widehat{R}_P$ is a direct sum of simple $F_{\widehat{R}_P}$ -submodules of $H_{I\widehat{R}_P}^{n-d}(\widehat{R}_P)$ supported at minimal primes of $\widehat{A}_P$ by Lemma 4.7, so we have $r = \text{crk } H_{I\widehat{R}_P}^{n-d}(\widehat{R}_P)$ by the definition of corank. This finishes the proof of the claim. $\square$

Applying the above claim to (4.8.2) we get

$$l_{D(R,k)}(M/L) \leq \sum_{\dim R/P=1} \text{crk } H_{I\widehat{R}_P}^{n-d}(\widehat{R}_P).$$

Because $\mathcal{H}_{\widehat{R}_P, \widehat{A}_P}(H_{P\widehat{A}_P}^{d-1}(\widehat{A}_P)) \cong H_{I\widehat{R}_P}^{n-d}(\widehat{R}_P)$ (the indices match because I is equidimensional), by [Lyu97, Proposition 4.10] we have

$$\text{crk } H_{I\widehat{R}_P}^{n-d}(\widehat{R}_P) = \dim_{\kappa(P)}(H_{P\widehat{A}_P}^{d-1}(\widehat{A}_P))_s.$$

Finally, summing up the $D(R, k)$ -module length of L , M/L , and $H_I^{n-d}(R)/M$, we have:

$$l_{D(R,k)}(H_I^{n-d}(R)) \leq c + \sum_{\dim R/P=1} \dim_{\kappa(P)}(H_{P\widehat{A}_P}^{d-1}(\widehat{A}_P))_s + \dim_k(H_m^d(A))_s. \quad \square$$

We end this section with some remarks and questions regarding Theorem 4.8:

Remark 4.9. It is clear that the sum $\sum_{\dim R/P=1} \dim_{\kappa(P)}(H_{P\widehat{A}_P}^{d-1}(\widehat{A}_P))_s$ in Theorem 4.8 is a finite sum: in fact we only need to consider those primes P such that A_P is not F -rational (which form a finite set by our assumption). We ought to point out that, more generally, without any assumption on the F -rational locus, [Lyu97, Proposition 4.14] shows that there are only finitely many prime ideals P such that $(H_{P\widehat{A}_P}^j(\widehat{A}_P))_s \neq 0$.

Remark 4.10. We do not know whether the inequality in Theorem 4.8 is an equality. This is due to the fact that, in the proof of Claim 4.8.3, we do not know whether the $D(\widehat{R}_P, k)$ -module $N_i \otimes \widehat{R}_P$ is isomorphic to a *single copy* of $E(\widehat{R}_P/P\widehat{R}_P)$.

In general, a simple $D(R, k)$ -module may not stay simple as a $D(\widehat{R}_P, \kappa(P))$ -module after taking localization and completion. We point out the following example, which is derived from [Bli03, Example 5.1].

Example 4.11. Let $R = k[x]$ where k is an algebraically closed field of positive characteristic. Let $M = R \oplus R$ be a free R -module of rank 2. We give M an F_R -module structure by setting the composition map

$$R \oplus R \xrightarrow{\theta_M} F(R) \oplus F(R) \xrightarrow{\theta_R^{-1} \oplus \theta_R^{-1}} R \oplus R$$

to be the map represented by the matrix

$$\begin{pmatrix} -x & 1 \\ 1 & 0 \end{pmatrix}$$

where θ_R denotes the standard isomorphism $R \cong F(R)$.

Next we pick a nonzero simple F_R^∞ -module $N \subseteq M$ (note that the only associated prime of N is 0). By [Bli03, Corollary 4.7], N must be a simple $D(R, k)$ -module since k is algebraically closed. However, after we localize at 0, that is, tensor with the fraction field $k(x)$ of R , $M \otimes_R k(x)$ becomes a simple $F_{k(x)}^\infty$ -module because $M \otimes_R k(x)^{1/p^\infty}$ is a simple $F_{k(x)^{1/p^\infty}}^\infty$ -module by [Bli03, Example 5.1].⁶ Thus we must have $N \otimes_R k(x) \cong M \otimes_R k(x)$, but $M \otimes_R k(x)$ is not a simple $D(k(x), k(x))$ -module because obviously every 1-dimensional $k(x)$ -subspace is a nontrivial $D(k(x), k(x))$ -submodule.

Remark 4.12. One approach to generalizing Theorem 4.8 is to find an F -submodule M of $H_I^{n-d}(R)$ such that none of the composition factors of M has 0-dimensional support and the support of $H_I^{n-d}(R)/M$ is contained in $\{\mathfrak{m}\}$. In the graded case, the existence of such an M follows from the proof of [LSW16, Theorem 2.9]. We don't know whether [LSW16, Theorem 2.9] can be extended to the nongraded case.

Hence it is natural to ask the following.

Question 4.13. Does there always exist an F -submodule M of $H_I^{n-d}(R)$ such that none of the composition factors of M has 0-dimensional support while the support of $H_I^{n-d}(R)/M$ is contained in $\{\mathfrak{m}\}$?

Despite the above remarks and questions, we still expect that there should be an analogue of Theorems 4.3 and 4.8 or similar estimates in the local case and without the restriction on the non- F -rational locus.

5. A LOWER BOUND ON F -MODULE LENGTH OF LOCAL COHOMOLOGY MODULES

In this section we will give lower bounds on $l_{F_R}(H_I^c(R))$. Throughout this section we will assume $R = k[[x_1, \dots, x_n]]$ with $\mathfrak{m} = (x_1, \dots, x_n)$, where k is a field of characteristic $p > 0$, and that $A = R/I$ is reduced and equidimensional of dimension $d > 1$. Henceforth in this section $E = E_R(k)$ will denote the injective hull of the residue field of R , and $E_A = E_A(k) = \text{Ann}_E I$ will denote the injective hull of the residue field of A .

⁶Note that the matrix we used here is the inverse of the matrix in [Bli03, Example 5.1]. This is because we are describing the matrix representing the F -module structure on $M \otimes_R k(x)^{1/p^\infty}$, while Blickle was working with the matrix representing the Frobenius action on $M \otimes_R k(x)^{1/p^\infty}$. We leave the reader to check that they define the same F -module structure on $M \otimes_R k(x)^{1/p^\infty}$.

We first collect definitions and facts from [Sha07] and [Kat08]. Given an Artinian $A\{f\}$ -module W , a *special ideal* of W is an ideal of A that is also the annihilator of some $A\{f\}$ -submodule $V \subseteq W$, and a *special prime* is a special ideal that is also a prime ideal (note that the special ideals depend on the $A\{f\}$ -module structure on W , i.e., the Frobenius action f on W). An important result of Sharp [Sha07, Corollary 3.7] and Enescu-Hochster [EH08, Theorem 3.6] shows that when f acts injectively on W , the number of special primes of W is finite.

The module of inverse polynomials E comes equipped with a natural Frobenius map T given by $T(\lambda x_1^{-\alpha_1} \dots x_n^{-\alpha_n}) = \lambda^p x_1^{-p\alpha_1} \dots x_n^{-p\alpha_n}$ for all $\lambda \in k$ and $\alpha_1, \dots, \alpha_n \geq 0$. Any Frobenius map on E has the form uT where $u \in R$ and any Frobenius map on E_A has that form with $u \in (I^{[p]} : I)$ (cf. [Bli01, Proposition 3.36]). Such an action uT on E_A is injective if and only if $u \notin \mathfrak{m}^{[p]}$ and is nonzero if and only if $u \notin I^{[p]}$. If we now specialize the notion of special ideals to the $A\{f\}$ -module E_A where $f = uT$, we see that these are ideals J such that $u \in J^{[p]} : J$ and we refer to these as *u -special ideals* ([Kat08, Theorem 4.3]). A *u -special prime* is a u -special ideal that is also a prime ideal.

5.1. F -pure case. Our main result in this subsection is the following.

Theorem 5.1. *Assume $A = R/I$ is reduced and equidimensional of dimension $d \geq 1$. Suppose A is F -pure. Then $l_{F_R}(H_I^{n-j}(R))$ is at least the number of special primes of $H_{\mathfrak{m}}^j(A)$. Moreover, when A is quasi-Gorenstein, $l_{F_R}(H_I^{n-d}(R))$ is exactly the number of special primes of $H_{\mathfrak{m}}^d(A)$.*

Proof. Let P be a special prime of $H_{\mathfrak{m}}^j(A)$. Take an $A\{f\}$ -submodule $N \subseteq H_{\mathfrak{m}}^j(A)$ such that $\text{Ann } N = P$. Recall that the Frobenius action on N induces a map $F(N) \rightarrow N$. We claim that this map is surjective: letting $N' \subseteq N$ be the image, we have $N' \subseteq N \subseteq H_{\mathfrak{m}}^j(A)$ are $A\{f\}$ -submodules such that the Frobenius action on N/N' is nilpotent. But $H_{\mathfrak{m}}^j(A)$ is anti-nilpotent (i.e., the Frobenius action on $H_{\mathfrak{m}}^j(R)/N'$ is injective) by [Ma14b, Theorem 3.7]. So we must have $N' = N$, and thus $F(N) \rightarrow N$ is surjective.

Taking the Matlis dual, we get $N^\vee \hookrightarrow F(N)^\vee \cong F(N^\vee)$. This shows that $N^\vee$ is a root of $\mathcal{H}_{R,A}(N)$ (recall that by definition, $\mathcal{H}_{R,A}(N) = \varinjlim (N^\vee \rightarrow F(N^\vee) \rightarrow F^2(N^\vee) \rightarrow \dots)$). In particular, we know that the set of associated primes of $N^\vee$ is the same as the set of associated primes of $\mathcal{H}_{R,A}(N)$ (this follows easily from the argument in [Lyu97, Remark 2.13]). But $\text{Ann } N = \text{Ann } N^\vee = P$ and $N^\vee$ is a finitely generated R -module; thus P is a minimal associated prime of $N^\vee$ and hence a minimal associated prime of $\mathcal{H}_{R,A}(N)$. This implies that $\mathcal{H}_{R,A}(N)$ must have a simple F_R -module composition factor with P its unique associated prime. But we have $H_I^{n-j}(R) \cong \mathcal{H}_{R,A}(H_{\mathfrak{m}}^j(A)) \twoheadrightarrow \mathcal{H}_{R,A}(N)$. Hence for every special prime P of $H_{\mathfrak{m}}^j(A)$, $H_I^{n-j}(R)$ has a simple F_R -module composition factor with P its unique associated prime. This proves that $l_{F_R}(H_I^{n-j}(R))$ is at least the number of special primes of $H_{\mathfrak{m}}^j(A)$.

Finally, when A is quasi-Gorenstein, $H_{\mathfrak{m}}^d(A) \cong E_A$, the injective hull of the residue field. So there is a one-one correspondence between $A\{f\}$ -submodules of $H_{\mathfrak{m}}^d(A)$ and their annihilator ideals. Let $P_1, \dots, P_m$ be all the special primes with $\text{ht } P_1 \geq \text{ht } P_2 \geq \dots \geq \text{ht } P_m$. Let $Q_j = P_1 \cap P_2 \cap \dots \cap P_j$. We have an ascending chain of $A\{f\}$ -submodules of $H_{\mathfrak{m}}^d(A) = E_A$:

$$0 \subsetneq \text{Ann}_E Q_1 \subsetneq \text{Ann}_E Q_2 \subsetneq \dots \subsetneq \text{Ann}_E Q_m = E_A.$$

It suffices to show that $\mathcal{H}_{R,A}(\text{Ann}_E Q_j / \text{Ann}_E Q_{j-1})$ is a nonzero simple F -module. It is nonzero because the Frobenius action on $\text{Ann}_E Q_j / \text{Ann}_E Q_{j-1}$ is not nilpotent (in fact it is injective because $H_{\mathfrak{m}}^d(A)$ is anti-nilpotent). If it is not simple, then there exists another $A\{f\}$ -submodule M such that $\text{Ann}_E Q_{j-1} \subsetneq M \subsetneq \text{Ann}_E Q_j$, which implies that $Q_j \subsetneq \text{Ann } M \subsetneq Q_{j-1}$. But since the Frobenius action on $H_{\mathfrak{m}}^d(A) = E_A$ is injective and M is an $A\{f\}$ -submodule, $\text{Ann } M$ is an intersection of the special primes by [Sha07, Corollary 3.7] or [EH08, Theorem 3.6]. This is then impossible by our height hypothesis on P_i and the definition of Q_j . $\square$

Remark 5.2. When A is not F -pure, the number of special primes of $H_{\mathfrak{m}}^j(A)$ is not necessarily a lower bound of $l_{F_R}(H_I^{n-j}(R))$. In fact, in Example 4.6, when $A = R/f$ is a Calabi–Yau hypersurface that is not F -pure, then $l_{F_R}(H_f^1(R)) = 1$, while the number of special primes of $H_{\mathfrak{m}}^1(A)$ is 2: (f) and $\mathfrak{m}$ are both special primes of $H_{\mathfrak{m}}^1(A)$. So the first conclusion of Theorem 5.1 need not hold when A is not F -pure.

Example 5.3. Let $A = k[x_1, \dots, x_n]/(x_i x_j \mid 1 \leq i < j \leq n) = R/I$. Then A is a 1-dimensional F -pure ring, and A is not Gorenstein when $n \geq 3$. A straightforward computation using [EH08, Theorem 5.1] shows that the special primes of $H_{\mathfrak{m}}^1(A)$ are $P_i = (x_1, \dots, \hat{x}_i, \dots, x_n)$ and $\mathfrak{m}$ (thus there are $n+1$ special primes) but $l_{F_R}(H_I^{n-1}(R)) = 2n-1$. Hence $l_{F_R}(H_I^{n-1}(R))$ is strictly bigger than the number of special primes when $n \geq 3$ (and the difference can be arbitrarily large when $n \gg 0$). This shows that the second conclusion of Theorem 5.1 need not hold when A is not quasi-Gorenstein.

5.2. Gorenstein case: A second approach. In this subsection we assume that $A = R/I$ is Gorenstein and F -injective (equivalently, Gorenstein and F -pure). In this case $H_I^i(R)$ vanishes unless $i = n-d$, and we already know from Theorem 5.1 that $l_{F_R}(H_I^{n-j}(R))$ is equal to the number of special primes of $H_{\mathfrak{m}}^d(A)$. Our goal here is to give a more detailed analysis on the F_R -submodules of $H_I^{n-j}(R)$ in terms of their generating morphisms, and in particular we recover the second conclusion of Theorem 5.1.

Since A is Gorenstein, $E_A \cong H_{\mathfrak{m}}^d(A)$, and thus there is a natural Frobenius action on E_A . In this case the module $\frac{I^{[p]}:I}{I^{[p]}}$ is a cyclic A -module, and the natural Frobenius action on E_A is given up to sign by uT , where we fix $u \in I^{[p]} : I$ whose image in $\frac{I^{[p]}:I}{I^{[p]}}$ generates it as an A -module (and T denotes the natural Frobenius on E). The u -special ideals (resp., u -special primes) are thus the special ideals (resp., special primes), and they are finite by [Sha07, Corollary 3.7] or [EH08, Theorem 3.6].

Following the construction in [Lyu97, §4], we obtain a generating morphism for $H_I^{n-d}(R)$ of the form $R/I \xrightarrow{u} R/I^{[p]}$ with u as above. To obtain a root, we let $K = \bigcup_{e \geq 1} (I^{[p^e]} : u^{1+\dots+p^{e-1}})$, and now $R/K \xrightarrow{u} R/K^{[p]}$ is a root of $H_I^{n-d}(R)$.

Lemma 5.4. *The proper F -finite F -submodules of $H_I^{n-d}(R)$ have roots $J/K \xrightarrow{u} J^{[p]}/K^{[p]}$ as J ranges over all proper u -special ideals, and, furthermore, distinct special ideals J define distinct F -finite F -submodules of $H_I^{n-d}(R)$.*

Proof. [Lyu97, Corollary 2.6] establishes a bijection between F -finite F -submodules $\mathcal{N}$ of $H_I^{n-d}(R)$ and R -submodules of the root R/K which is given by $\mathcal{N} \mapsto \mathcal{N} \cap R/K$.

Fix such $\mathcal{N}$ and write $J/K = \mathcal{N} \cap R/K$. The fact that $\mathcal{N}$ is an F -finite F -submodule of $H_I^{n-d}(R)$ implies that the image of the restriction of the map

$R/K \xrightarrow{u} R/K^{[p]}$ to J/K is in $F(J/K) = J^{[p]}/K^{[p]}$, and hence J is a u -special ideal. Clearly, any such u -special ideal defines an F -finite F -submodule of $H_I^{n-d}(R)$, i.e.,

$$\lim_{\rightarrow} \left(J/K \xrightarrow{u} J^{[p]}/K^{[p]} \xrightarrow{u^p} J^{[p^2]}/K^{[p^2]} \xrightarrow{u^{p^2}} \dots \right).$$

To finish the proof we need to show that any two distinct u -special ideals J_1 and J_2 define different F -finite F -submodules of $H_I^{n-d}(R)$. If this is not the case, then for some $e \geq 0$, $u^{\nu_e} J_1 / K^{[p^e]} = u^{\nu_e} J_2 / K^{[p^e]}$, where $\nu_e = 1 + p + \dots + p^{e-1}$ and $u^{\nu_e} J_1 + K^{[p^e]} = u^{\nu_e} J_2 + K^{[p^e]}$.

The fact that f is injective on E is equivalent to $u^{\nu_e} \notin L^{[p^e]}$ for all $e \geq 1$ and all proper ideals $L \subsetneq R$ ([Kat08, Theorem 4.6]), and in particular $u^{\nu_e} \notin J_1^{[p^e]}$ and $u^{\nu_e} \notin J_2^{[p^e]}$. But now $u^{\nu_e} J_1 \subseteq u^{\nu_e} J_2 + K^{[p^e]} \subseteq J_2^{[p^e]}$, and $J_2^{[p^e]}$ is a primary ideal (because R is regular); hence $J_1 \subseteq \sqrt{J_2^{[p^e]}} = J_2$. Similarly, also $J_2 \subseteq J_1$, contradicting the fact that $J_1 \neq J_2$. $\square$

Theorem 5.5. *Let $A=R/I$ be Gorenstein and F -injective where $R=k[[x_1, \dots, x_n]]$ or $k[x_1, \dots, x_n]$ with $\mathfrak{m} = (x_1, \dots, x_n)$. Let $\{P_1, \dots, P_m\}$ be the set of all the special prime ideals of $H_{\mathfrak{m}}^d(A)$ which contain K , and assume that these were ordered so that $\text{ht } P_1 \geq \text{ht } P_2 \geq \dots \geq \text{ht } P_m$. Write $Q_j = P_1 \cap \dots \cap P_j$ for all $1 \leq j \leq m$. The chain of roots*

$$\begin{array}{ccccccccc} 0 & \subset & \frac{Q_m}{K} & \subset & \frac{Q_{m-1}}{K} & \subset & \dots & \subset & \frac{Q_1}{K} & \subset & \frac{R}{K} \\ & & \downarrow u & & \downarrow u & & & & \downarrow u & & \downarrow u \\ 0 & \subset & \frac{Q_m^{[p]}}{K^{[p]}} & \subset & \frac{Q_{m-1}^{[p]}}{K^{[p]}} & \subset & \dots & \subset & \frac{Q_1^{[p]}}{K^{[p]}} & \subset & \frac{R}{K^{[p]}} \end{array}$$

corresponds to a maximal filtration of $H_I^{n-d}(R)$ in the category of F_R -modules.

Proof. Since $uQ_j \subseteq Q_j^{[p]}$ and $uK \subseteq K^{[p]}$, the vertical maps are well defined, and the diagram is clearly commutative.

To show that the factors are nonzero, note that if $Q_{j+1} = P_1 \cap \dots \cap P_{j+1} = P_1 \cap \dots \cap P_j = Q_j$, then $P_{j+1} \supseteq P_1 \cap \dots \cap P_j$ and $P_{j+1} \supseteq P_i$ for some $1 \leq i \leq j$. But the ordering of $P_1, \dots, P_m$ implies that $\text{ht } P_{j+1} \leq \text{ht } P_i$, giving $P_i = P_{j+1}$, a contradiction.

If the factors are not simple, then for some $1 \leq j \leq m$ there exists a special ideal J such that $P_1 \cap \dots \cap P_j \cap P_{j+1} \subsetneq J \subsetneq P_1 \cap \dots \cap P_j$. Being special, J is radical and has the form $P_1 \cap \dots \cap P_j \cap P_{k_1} \cap P_{k_2} \cap \dots \cap P_{k_s}$ for $j < k_1, \dots, k_s \leq m$. Now for every $1 \leq \ell \leq s$, $P_{k_\ell} \supseteq P_1 \cap \dots \cap P_{j+1}$ so $P_{k_\ell} \supseteq P_w$ for some $1 \leq w \leq j+1$, and the height condition implies $P_{k_\ell} = P_w$, and we conclude that $J \subseteq P_1 \cap \dots \cap P_{j+1}$, a contradiction.

It remains to show that the F_R -submodules defined by the roots $Q_j/K \xrightarrow{u} Q_j^{[p]}/K^{[p]}$ are distinct: this follows from Lemma 5.4. $\square$

Corollary 5.6. *Suppose $A = R/I$ is Gorenstein and F -injective. The length of $H_I^{n-d}(R)$ in the category of F_R -modules equals the number of u -special primes of $H_{\mathfrak{m}}^d(A)$ that contain K .*

Remark 5.7. Suppose $A = R/I$ is Gorenstein and F -injective. We can actually prove that the length of $H_I^{n-d}(R)$ in the category of F_R^e -modules for every e (and hence in the category of F_R^∞ -modules) equals the number of u -special primes of

$H_{\mathfrak{m}}^d(A)$ that contain K . It suffices to prove that every F_R^e -submodule of $H_I^{n-d}(R)$ is already an F_R -submodule of $H_I^{n-d}(R)$. By the same argument as in Lemma 5.4, all F_R^e -submodules of $H_I^{n-d}(R)$ have roots of the form J/K where J is an ideal containing K with the property $u^{1+p+\cdots+p^{e-1}}J \subseteq J^{[p^e]}$. Since $A = R/I$ is Gorenstein and F -injective (and hence F -pure), u is a generator of $(I^{[p]} : I)/I^{[p]}$ as an R/I -module and $u \notin \mathfrak{m}^{[p]}$ by Fedder's criterion [Fed83]. We will show that these imply $uJ \subseteq J^{[p]}$, and thus J/K already generates an F_R -submodule of $H_I^{n-d}(R)$. Since $u \notin \mathfrak{m}^{[p]}$ and R is regular, there is a p^{-1} -linear map (i.e., a Frobenius splitting) $\phi: R \rightarrow R$ such that $\phi(u) = 1$. Therefore $u^{1+p+\cdots+p^{e-1}} \in J^{[p^e]} : J$ implies that

$$\begin{aligned} u^{1+p+\cdots+p^{e-2}} &= \phi(u^{p(1+p+\cdots+p^{e-2})} \cdot u) = \phi(u^{1+p+\cdots+p^{e-1}}) \in \phi(J^{[p^e]} : J) \\ &\subseteq \phi(J^{[p^e]} : J^{[p]}) = J^{[p^{e-1}]} : J, \end{aligned}$$

and thus by an easy induction we have $u \in J^{[p]} : J$ (note that we have used $(J^{[p^{e-1}]} : J)^{[p]} = J^{[p^e]} : J^{[p]}$ because R is regular so the Frobenius endomorphism is flat).

5.3. Cohen–Macaulay case. In this subsection we assume that $A = R/I$ is reduced and Cohen–Macaulay. In this case the canonical module of A can be identified with an ideal $\omega \subseteq A$. Let Ω be the pre-image of ω in R , that is, $\Omega/I = \omega \subseteq A$. The inclusion $\omega \subseteq A$ is compatible with the Frobenius endomorphism, and the short exact sequence $0 \rightarrow \omega \subseteq A \rightarrow A/\omega \rightarrow 0$ induces an A -linear map

$$0 \rightarrow H_{\mathfrak{m}}^{d-1}(A/\omega) \rightarrow H_{\mathfrak{m}}^d(\omega) \rightarrow H_{\mathfrak{m}}^d(A) \rightarrow H_{\mathfrak{m}}^d(A/\omega) \rightarrow 0.$$

Now each of the Artinian A -modules is equipped with a Frobenius map induced by the Frobenius endomorphism acting on the short exact sequence, and $H_{\mathfrak{m}}^d(A/\omega)$ vanishes since $\dim A/\omega < \dim A = d$. So we obtain a short exact sequence of $A\{f\}$ -modules

$$0 \rightarrow H_{\mathfrak{m}}^{d-1}(A/\omega) \rightarrow H_{\mathfrak{m}}^d(\omega) \rightarrow H_{\mathfrak{m}}^d(A) \rightarrow 0.$$

We can now identify $H_{\mathfrak{m}}^d(\omega)$ with $E_A = \text{Ann}_E I$. Note that the annihilator of $H_{\mathfrak{m}}^{d-1}(A/\omega)$ in A is ω ; hence the annihilator of $H_{\mathfrak{m}}^{d-1}(A/\omega)$ in R is Ω (since $R/I = A$ and $\Omega/I = \omega$). Thus we may identify $H_{\mathfrak{m}}^{d-1}(A/\omega)$ with $\text{Ann}_E \Omega$.

We now have a short exact sequence

$$0 \rightarrow \text{Ann}_E \Omega \rightarrow \text{Ann}_E I \rightarrow H_{\mathfrak{m}}^d(A) \rightarrow 0$$

of $A\{f\}$ -modules, and we can also write $H_{\mathfrak{m}}^d(A) = \text{Ann}_E I / \text{Ann}_E \Omega$. Recall that any Frobenius action on $\text{Ann}_E I$ has the form uT where T is the natural Frobenius on E and $u \in (I^{[p]} : I)$. Fix $u \in R$ to be such that uT is the Frobenius action on $\text{Ann}_E I$ in the exact sequence above. We now can obtain an F_R -module filtration of $H_I^{n-d}(R) = \mathcal{H}_{R,A}(\text{Ann}_E I / \text{Ann}_E \Omega)$ by applying the Lyubeznik functor $\mathcal{H}_{R,A}$ to a chain of surjections

$$\text{Ann}_E I / \text{Ann}_E \Omega \rightarrow \text{Ann}_E I / \text{Ann}_E J_1 \rightarrow \cdots \rightarrow \text{Ann}_E I / \text{Ann}_E J_m$$

where $J_1, \dots, J_m$ are u -special ideals such that $\Omega \supseteq J_1 \supseteq \cdots \supseteq J_m \supseteq I$. We let $K = \bigcup_{e \geq 1} (I^{[p^e]} : u^{1+p+\cdots+p^{e-1}})$ so that $\Omega/K \xrightarrow{u} \Omega^{[p]}/K^{[p]}$ is a root for $H_I^{n-d}(R) = \mathcal{H}_{R,A}(\text{Ann}_E I / \text{Ann}_E \Omega)$.

Theorem 5.8. *Assume that $A = R/I$ is Cohen–Macaulay. Let $\{P_1, \dots, P_m\}$ be the set of all the u -special prime ideals $P \supseteq K$ such that $P \not\supseteq \Omega$ and $u \notin P^{[p]}$,*

and assume that these were ordered so that $\text{ht } P_1 \geq \text{ht } P_2 \geq \cdots \geq \text{ht } P_m$. Write $Q_j = \Omega \cap P_1 \cap \cdots \cap P_j$ for all $1 \leq j \leq m$. The chain of roots

$$\begin{array}{ccccccccc} 0 & \subset & \frac{Q_m}{K} & \subset & \frac{Q_{m-1}}{K} & \subset & \cdots & \subset & \frac{Q_1}{K} & \subset & \frac{\Omega}{K} \\ & & \downarrow u & & \downarrow u & & & & \downarrow u & & \downarrow u \\ 0 & \subset & \frac{Q_m^{[p]}}{K^{[p]}} & \subset & \frac{Q_{m-1}^{[p]}}{K^{[p]}} & \subset & \cdots & \subset & \frac{Q_1^{[p]}}{K^{[p]}} & \subset & \frac{\Omega}{K^{[p]}} \end{array}$$

corresponds to a filtration of $H_I^{n-d}(R)$ in the category of F -modules with nonzero factors.

Proof. The homomorphic images $\text{Ann}_E I / \text{Ann}_E Q_1, \dots, \text{Ann}_E I / \text{Ann}_E Q_1$ of $\text{Ann}_E I / \text{Ann}_E \Omega$ are preserved by the natural Frobenius action on $H_m^d(A) \cong \text{Ann}_E I / \text{Ann}_E \Omega$ because each Q_i , being the intersection of u -special ideals, is itself special.

An application of the Lyubeznik functor $\mathcal{H}_{R,A}$ to the chain of surjections

$$\text{Ann}_E I / \text{Ann}_E \Omega \rightarrow \text{Ann}_E I / \text{Ann}_E J_1 \rightarrow \cdots \rightarrow \text{Ann}_E I / \text{Ann}_E J_m$$

yields a filtration of $H_I^{n-d}(R)$ whose generating morphisms are the vertical maps in the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \subset & \frac{Q_m}{I} & \subset & \frac{Q_{m-1}}{I} & \subset & \cdots & \subset & \frac{Q_1}{I} & \subset & \frac{\Omega}{I} \\ & & \downarrow u & & \downarrow u & & & & \downarrow u & & \downarrow u \\ 0 & \subset & \frac{Q_m^{[p]}}{I^{[p]}} & \subset & \frac{Q_{m-1}^{[p]}}{I^{[p]}} & \subset & \cdots & \subset & \frac{Q_1^{[p]}}{I^{[p]}} & \subset & \frac{\Omega}{I^{[p]}} \end{array}$$

We can replace these generating morphisms by their corresponding roots and obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \subset & \frac{Q_m}{K} & \subset & \frac{Q_{m-1}}{K} & \subset & \cdots & \subset & \frac{Q_1}{K} & \subset & \frac{\Omega}{K} \\ & & \downarrow u & & \downarrow u & & & & \downarrow u & & \downarrow u \\ 0 & \subset & \frac{Q_m^{[p]}}{K^{[p]}} & \subset & \frac{Q_{m-1}^{[p]}}{K^{[p]}} & \subset & \cdots & \subset & \frac{Q_1^{[p]}}{K^{[p]}} & \subset & \frac{\Omega}{K^{[p]}} \end{array}$$

where now all vertical maps are roots, and once we show that the inclusions in this diagram are strict, this gives a filtration of $H_I^{n-d}(R)$ with nonzero factors.

We need to show that for all $e \geq 1$ and all $1 \leq i < m$,

$$(Q_{i+1}^{[p^e]} : u^{\nu_e}) \subsetneq (Q_i^{[p^e]} : u^{\nu_e}),$$

where $\nu_e = 1 + p + \cdots + p^{e-1}$. If we have equality, we may take radicals of both sides to obtain

$$\sqrt{(\Omega^{[p^e]} : u^{\nu_e})} \cap \bigcap_{j=1}^{i+1} \sqrt{(P_j^{[p^e]} : u^{\nu_e})} = \sqrt{(\Omega^{[p^e]} : u^{\nu_e})} \cap \bigcap_{j=1}^i \sqrt{(P_j^{[p^e]} : u^{\nu_e})},$$

and so

$$\sqrt{(P_{i+1}^{[p^e]} : u^{\nu_e})} \supseteq \sqrt{(\Omega^{[p^e]} : u^{\nu_e})} \cap \bigcap_{j=1}^i \sqrt{(P_j^{[p^e]} : u^{\nu_e})}.$$

We claim that $u^{\nu_e} \notin P_j^{[p^e]}$ for each j and we will use induction on e to prove this. When $e = 1$, this is precisely our assumption that $u \notin P_j^{[p]}$. We assume that $u^{\nu_e} \notin P_j^{[p^e]}$ and we wish to prove that $u^{\nu_{e+1}} \notin P_j^{[p^{e+1}]}$. If $u^{\nu_{e+1}} \in P_j^{[p^{e+1}]}$, then we would have

$$u^{\nu_e} \in (P_j^{[p^{e+1}]} : u^{p^e}) = (P_j^{[p]} : u)^{[p^e]} \subseteq P_j^{[p^e]},$$

a contradiction, where the last inclusion follows from the fact that $P_j^{[p]}$ is P -primary and our assumption that $u \notin P_j^{[p]}$.

Consequently $(P_{i+1}^{[p^e]} : u^{\nu_e}) \neq R$, and we must have $(P_{i+1}^{[p^e]} : u^{\nu_e}) \subseteq P_{i+1}$. So $P_{i+1} \supseteq \sqrt{(\Omega^{[p^e]} : u^{\nu_e})} \cap \bigcap_{j=1}^i \sqrt{(P_j^{[p^e]} : u^{\nu_e})}$, and hence P_{i+1} must contain one of the ideals in the intersection. Since these ideals are among the unit ideal, $P_1, \dots, P_i$ and $\sqrt{(\Omega^{[p^e]} : u^{\nu_e})} \supseteq \Omega$, this is impossible. $\square$

We have the following immediate corollary of Theorem 5.8.

Corollary 5.9. *Let A, R, I, u, K, Ω be as in Theorem 5.8. The length of $H_I^{n-d}(R)$ in the category of F -modules is at least the number of u -special prime ideals $P \supseteq K$ of $H_m^d(A)$ such that $P \not\supseteq \Omega$ and $u \notin P^{[p]}$.*

Remark 5.10. If A is quasi-Gorenstein and we take $\Omega = R$, the prime special ideals in the statement of Theorem 5.8 are the prime special ideals $P \supseteq I$ of $H_m^d(A)$ such that $P \supseteq K$, and $u \notin P^{[p]}$. The set of all such primes has been known to be finite ([KS12, Remark 5.3]), and if A is also F -injective, we obtain the same set of primes as in Theorem 5.5; thus Theorem 5.8 generalizes Theorem 5.5.

6. A COMPUTATION OF FERMAT HYPERSURFACES

We have seen from Theorem 4.3 and Remark 4.5 that the problem of computing the $D(R, k)$ -module length of $H_I^{n-d}(R)$ when $A = R/I$ is a graded isolated singularity comes down to computing the dimension of the Frobenius stable part of $H_m^d(A)_0$. In this section we study this problem for Fermat hypersurfaces $A = k[x_0, \dots, x_d]/(x_0^n + x_1^n + \dots + x_d^n)$ with $d \geq 2$. We express the dimension of the Frobenius stable part of $H_m^d(A)$ explicitly in terms of the number of solutions to a system of equations on remainders. These results generalize earlier computations of Blickle in [Bli01, Examples 5.26–5.29].

Remark 6.1. Let $A = k[x_0, x_1, \dots, x_d]/(x_0^n + x_1^n + \dots + x_d^n)$. Then the degree 0 part of the top local cohomology $H_m^d(A)$ has a k -basis consisting of the elements of the form $\frac{x_0^c}{x_1^{a_1} \dots x_d^{a_d}}$, where $a_1, \dots, a_d$ are positive integers and $a_1 + \dots + a_d = c \leq n - 1$. Therefore, its dimension is $\binom{n-1}{d}$.

In the following we will use $s \% t$ to denote the remainder of s mod t .

Remark 6.2. We want to record an elementary observation. Let $n \geq 2$ be an integer and let p be a prime. Let $p = nk + r$ where r is the remainder. If for some positive integer $a < n$, $n|ar$, then we claim that $p|n$.

This is because $n|ar$ and $a < n$ implies that n and r must have a nontrivial common divisor. But $p = nk + r$ is prime, so the only nontrivial common divisor that n and r could have is p , in which case we must have p divides n .

Theorem 6.3. Let $A = k[x_0, x_1, \dots, x_d]/(x_0^n + x_1^n + \dots + x_d^n)$ with $d \geq 2$. Suppose $p \equiv r \pmod n$. If p does not divide n , then the dimension of the stable part of $H_m^d(A)$ can be computed as the number of solutions of the following system of inequalities on $1 \leq a_i \leq n-1$:

$$\begin{cases} a_1 + \dots + a_d < n, \\ (ra_1)\%n + \dots + (ra_d)\%n < n, \\ (r^2a_1)\%n + \dots + (r^2a_d)\%n < n, \\ \vdots \\ (r^{\varphi(n)-1}a_1)\%n + \dots + (r^{\varphi(n)-1}a_d)\%n < n. \end{cases}$$

Proof. A basis of the degree 0 part of $H_m^d(A)$ is formed by the elements

$$\frac{x_0^c}{x_1^{a_1} \dots x_d^{a_d}},$$

where $a_1 + \dots + a_d = c < n$ and $a_i \geq 1$. On such elements, Frobenius acts as

$$\frac{x_0^c}{x_1^{a_1} \dots x_d^{a_d}} \mapsto \frac{x_0^{cp}}{x_1^{a_1 p} \dots x_d^{a_d p}} = \frac{x_0^{(cr)\%n} \cdot (-x_1^n - \dots - x_d^n)^{\lfloor \frac{cp}{n} \rfloor}}{x_1^{a_1 p} \dots x_d^{a_d p}}.$$

After expanding the expression we obtain the sum of monomials of the form

$$(-1)^{\lfloor \frac{cp}{n} \rfloor} \binom{\lfloor \frac{cp}{n} \rfloor}{\alpha_1, \dots, \alpha_d} \frac{x_0^{(cr)\%n} \cdot (x_1^{n\alpha_1} \dots x_d^{n\alpha_d})}{x_1^{a_1 p} \dots x_d^{a_d p}}$$

for $\alpha_1 + \dots + \alpha_d = \lfloor \frac{cp}{n} \rfloor$. This element will be zero unless $\alpha_i n < a_i p$ for all i . Hence it is zero if $\alpha_i > \lfloor \frac{a_i p}{n} \rfloor$ for some i . In particular, the element $\frac{x_0^c}{x_1^{a_1} \dots x_d^{a_d}}$ is in the kernel of the Frobenius map if

$$\alpha_1 + \dots + \alpha_d = \left\lfloor \frac{cp}{n} \right\rfloor > \left\lfloor \frac{a_1 p}{n} \right\rfloor + \dots + \left\lfloor \frac{a_d p}{n} \right\rfloor;$$

i.e., $a_1 p \% n + \dots + a_d p \% n = a_1 r \% n + \dots + a_d r \% n \geq n$. Similarly, if

$$\left\lfloor \frac{cp}{n} \right\rfloor = \left\lfloor \frac{a_1 p}{n} \right\rfloor + \dots + \left\lfloor \frac{a_d p}{n} \right\rfloor,$$

the only term that can possibly survive is

$$\begin{aligned} & \binom{\lfloor \frac{cp}{n} \rfloor}{\lfloor \frac{a_1 p}{n} \rfloor, \dots, \lfloor \frac{a_d p}{n} \rfloor} \frac{(-1)^{\lfloor \frac{cp}{n} \rfloor} x_0^{(cr)\%n} \cdot (x_1^{\lfloor \frac{a_1 p}{n} \rfloor} \dots x_d^{\lfloor \frac{a_d p}{n} \rfloor})^n}{x_1^{a_1 p} \dots x_d^{a_d p}} \\ &= \binom{\lfloor \frac{cp}{n} \rfloor}{\lfloor \frac{a_1 p}{n} \rfloor, \dots, \lfloor \frac{a_d p}{n} \rfloor} \frac{(-1)^{\lfloor \frac{cp}{n} \rfloor} x_0^{(cr)\%n}}{x_1^{(a_1 r)\%n} \dots x_d^{(a_d r)\%n}}. \end{aligned}$$

Since $\lfloor \frac{cp}{n} \rfloor < p$, the binomial coefficient $\binom{\lfloor \frac{cp}{n} \rfloor}{\lfloor \frac{a_1 p}{n} \rfloor, \dots, \lfloor \frac{a_d p}{n} \rfloor}$ is nonzero. Thus the last possibility that the above term is zero is that $a_i r$ is divisible by n for some i . But this cannot happen as explained in Remark 6.2 (since this implies $p|n$).

In sum, an element of the basis

$$\frac{x_0^c}{x_1^{a_1} \dots x_d^{a_d}}$$

is not in the kernel of Frobenius if and only if

$$(ra_1)\%n + \dots + (ra_k)\%n < n.$$

Thus the claim follows after considering further iterates of Frobenius (we only need to consider the first $\varphi(n) - 1$ iterates because $r^{\varphi(n)} \equiv 1 \pmod n$ by Fermat's little theorem, so further iterates would repeat this pattern), because if a basis element is not in the kernel of all the iterates of the Frobenius map, then it contributes to an element in $H_m^d(A)_s$. $\square$

Corollary 6.4. *Suppose $p \nmid n$. Then Frobenius acts injectively on $H_m^d(A)_0$ if and only if $p \equiv 1 \pmod n$. As a consequence, $(H_m^d(A)_0)_s = H_m^d(A)_0$ if and only if $p \equiv 1 \pmod n$.*

Proof. If $p \equiv 1 \pmod n$ the claim immediately follows from Theorem 6.3. For the other direction, we need to show that there are integers $a_i \geq 1$ such that $a_1 + \cdots + a_d < n$ but $(ra_1) \% n + \cdots + (ra_d) \% n \geq n$.

Now r is invertible modulo n , take $1 \leq a_1 < n - 1$ such that $a_1 r \equiv -1 \pmod n$. Then since $ra_i \geq 1$ for all $i > 1$, we always have $(ra_1) \% n + \cdots + (ra_d) \% n \geq n$ as $d \geq 2$. $\square$

Corollary 6.5. *If $p^h \equiv -1 \pmod n$ for some h , then Frobenius acts nilpotently on $H_m^d(A)_0$. As a consequence, $(H_m^d(A)_0)_s = 0$ in this case.*

Proof. Consider the equation

$$(-a_1) \% n + \cdots + (-a_d) \% n < n$$

corresponding to r^h (which is $\equiv p^h \equiv -1 \pmod n$). Since $1 \leq a_i \leq n - 1$, $(-a_1) \% n = n - a_1$, so the equation becomes

$$dn - a_1 - \cdots - a_d < n.$$

But this equation has no solution since $a_1 + \cdots + a_d < n$ and $d \geq 2$. $\square$

Remark 6.6. The converse to the last corollary does not hold. For example, if $n = 11$ and $d = 2$, then a direct computation shows that Frobenius acts nilpotently on $H_m^2(A)_0$ unless $p \equiv 1 \pmod n$.

7. D -MODULE LENGTH VS. F -MODULE LENGTH

We continue to use the notation as in the beginning of §§4 and 5. In [Bli03], Blickle made a deep study on the comparison of D -module and F -module length. For example, in [Bli03, Theorem 1.1 or Corollary 4.7] it was proved that if k is algebraically closed, then for every F -finite F_R^∞ -module M , we have $l_{F_R^\infty}(M) = l_{D_R}(M)$, which is also $l_{D(R,k)}(M)$ since $D_R = D(R, k)$ when k is perfect. Moreover, when k is perfect but not algebraically closed, Blickle constructed an example [Bli03, 5.1] of a simple F_R^∞ -module that is not D_R -simple (equivalently, not $D(R, k)$ -simple since k is perfect). In particular, even the F_R^∞ -module length may differ from the $D(R, k)$ -module length in general.

However it is not clear whether these pathologies are artificial; i.e., can they occur for local cohomology modules with their natural F_R -module structure? In this section we will construct an example of a local cohomology module of R , with k algebraically closed, such that its F_R -module length is strictly less than its D_R -module length.

To begin, let V be a vector space over a field k of positive characteristic p . Then we can describe an eth Frobenius action f on V in the following way. Choose

a basis $e_1, \dots, e_n$ of V . Let $f(e_i) = a_{1i}e_1 + \dots + a_{ni}e_n$. Then for any element $b = (b_1, \dots, b_n)^T$ written in the basis e_i , we can write

$$f(b) = Ab^{[p^e]},$$

where $A = (a_{ij})$ and $[b^{p^e}]$ raises all entries to the p^e th power. Or explicitly,

$$f(b) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b_1^{p^e} \\ \vdots \\ b_n^{p^e} \end{pmatrix}.$$

Via this description it is very easy to see the following result.

Lemma 7.1. *Let k be a field of positive characteristic p and let V be a finite dimensional vector space with an e th Frobenius action f . Let A be a matrix describing the action of f in some basis e_i . Then in a new basis obtained by an orthogonal matrix O , f is represented by the matrix $OA(O^\tau)^{[p^e]}$, where all entries of the transpose O^τ are raised to the p^e th power.*

Proposition 7.2. *Let $R = k[x_1, \dots, x_n]$ or $k[[x_1, \dots, x_n]]$ and let V be a k -vector space with an e th Frobenius action f . Then $l_{F_R^e}(\mathcal{H}_{R,R}(V))$ is the length of any longest flag of f -subspaces*

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_m = V$$

such that f is not nilpotent on V_i/V_{i-1} for every i .

In particular, $l_{F_R^e}(\mathcal{H}_{R,R}(V)) = \dim V$ if and only if there is a basis of V such that, with the notation as in Lemma 7.1, f can be represented by an upper-triangular matrix A with nonzero entries on the main diagonal.

Proof. The first claim is [Lyu97, Theorem 4.7].

If $l_{F_R^e}(\mathcal{H}_{R,R}(V)) = \dim V$, then we must have $\dim V_i = i$. Then we will choose a compatible basis for the flag, i.e., $V_i = k\langle e_1, \dots, e_i \rangle$. Since $f(V_i) \subseteq V_i$, now $f(e_i) = a_{i1}e_1 + \dots + a_{ii}e_i$. Thus the matrix representing f is upper-triangular. Moreover, since f acts nontrivially on V_i/V_{i-1} , we must have $a_{ii} \neq 0$. So the matrix has nonzero entries on the main diagonal.

Conversely, if the matrix is upper-triangular with nonzero entries on the main diagonal in some basis, it is easy to see that $V_i = k\langle e_1, \dots, e_i \rangle$ form a flag of f -subspaces where f acts nontrivially on V_i/V_{i-1} . $\square$

Remark 7.3. Before proceeding further, we need a simple result in linear algebra. Over a finite field $\mathbb{F}_p$ where $p \neq 3$, consider the matrix

$$M = \begin{pmatrix} 0 & 0 & a \\ a & 0 & 0 \\ 0 & a & 0 \end{pmatrix}$$

where $a \neq 0$ in $\mathbb{F}_p$. The characteristic polynomial of this matrix is

$$\lambda^3 - a^3 = (\lambda - a)(\lambda^2 + \lambda a + a^2).$$

The discriminant of the quadratic polynomial is $D = -3a^2$. Thus if -3 is a quadratic residue in $\mathbb{F}_p$, then the characteristic polynomial has three distinct eigenvalues. On the other hand, if -3 is not a quadratic residue, then $(1, 1, 1)$ is the only

eigenvector and the restriction of the matrix M on the subspace of $\mathbb{F}_p^3$ orthogonal to $(1, 1, 1)$ can be expressed as

$$\begin{pmatrix} 0 & -a \\ a & -a \end{pmatrix},$$

which has no eigenvalues over $\mathbb{F}_p$.

By quadratic reciprocity, -3 is a quadratic residue if and only if $p \equiv 1 \pmod{3}$. Thus if $p \not\equiv 1 \pmod{3}$, this matrix has no eigenvalues over $\mathbb{F}_p$ and thus cannot be transformed in an upper-triangular form by a change of basis. Otherwise, it has three eigenvectors and can be transformed in an upper-triangular form.

Our first example shows that $l_{F_R}(\mathrm{H}_I^{n-c}(R))$ can be strictly less than $l_{F_R^\infty}(\mathrm{H}_I^{n-c}(R))$ and thus strictly less than $l_{D(R,k)}(\mathrm{H}_I^{n-c}(R))$ by (2.0.1) if k is not separably closed, even when $A = R/I$ has isolated singularities.

Corollary 7.4. *Let p be a prime number, let $R = \mathbb{F}_p[x, y, z]$, and let $f = x^7 + y^7 + z^7$. Then*

$$l_{F_R^\infty}(\mathrm{H}_f^1(R)) = l_{D_R}(\mathrm{H}_f^1(R)) = \begin{cases} 16 & \text{if } p \equiv 1 \pmod{7}, \\ 7 & \text{if } p \equiv 2 \text{ or } 4 \pmod{7}, \\ 1 & \text{otherwise.} \end{cases}$$

On the other hand,

$$l_{F_R}(\mathrm{H}_f^1(R)) = \begin{cases} 16 & \text{if } p \equiv 1 \pmod{7}, \\ 7 & \text{if } p \equiv 2 \text{ or } 4 \pmod{7} \text{ and } p \equiv 1 \pmod{3}, \\ 5 & \text{if } p \equiv 2 \text{ or } 4 \pmod{7} \text{ and } p \not\equiv 1 \pmod{3}, \\ 1 & \text{otherwise.} \end{cases}$$

In particular, $l_{D_R}(\mathrm{H}_f^1(R)) \neq l_{F_R}(\mathrm{H}_f^1(R))$ for any $p \equiv 11 \pmod{21}$.

Proof. Let V denote the k -vector space $(0_{\mathrm{H}_m^2(R/f)}^*)_s = (\mathrm{H}_m^2(R/f)_0)_s$. By Corollaries 6.4 and 6.5, Frobenius acts injectively on $\mathrm{H}_m^2(R/f)_0$ if $p \equiv 1 \pmod{7}$ and nilpotently if $p \equiv 3, 5, 6 \pmod{7}$. When $p \equiv 2$ or $4 \pmod{7}$, using the algorithm described in Theorem 6.3 it can be checked that the Frobenius map (i.e., $e = 1$) on V is spanned by two 3-cycles. If $p \equiv 4 \pmod{7}$, the cycles are

$$\frac{z^3}{x^2y} \rightarrow \frac{z^5}{xy^4} \rightarrow \frac{z^6}{x^4y^2} \quad \text{and} \quad \frac{z^3}{xy^2} \rightarrow \frac{z^5}{x^4y} \rightarrow \frac{z^6}{x^2y^4}.$$

While if $p \equiv 2 \pmod{7}$, the cycles become

$$\frac{z^3}{x^2y} \rightarrow \frac{z^6}{x^4y^2} \rightarrow \frac{z^5}{xy^4} \quad \text{and} \quad \frac{z^3}{xy^2} \rightarrow \frac{z^6}{x^2y^4} \rightarrow \frac{z^5}{x^4y}.$$

In particular, one obtains that

$$\dim V = \begin{cases} 15 & \text{if } p \equiv 1 \pmod{7}, \\ 6 & \text{if } p \equiv 2 \text{ or } 4 \pmod{7}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $R/(f)$ is an isolated singularity, $l_{D_R}(\mathrm{H}_f^1(R)) = \dim V + 1$ by Theorem 4.3. In the cases when Frobenius acts injectively or nilpotently on $\mathrm{H}_m^2(R/f)_0$, we also deduce from Proposition 7.2 that $l_{F_R}(\mathrm{H}_f^1(R)) = \dim V + 1$ (note that when Frobenius acts injectively on $\mathrm{H}_m^2(R/f)_0$, the proof of Theorem 6.3 shows that

Frobenius sends each canonical basis element of $H_m^2(R/f)_0$ to a multiple of itself, so the representing matrix is diagonal).

In the remaining cases, in order to compute its F_R^e -module length we will study a matrix which represents the Frobenius map on V by Proposition 7.2. We can use the proof of Theorem 6.3 to describe the Frobenius action on the cycles. If $p = 7k + 4$, one obtains that the Frobenius action on both cycles is described by the matrix

$$A = \begin{pmatrix} 0 & 0 & (-1)^{6k+3} \binom{6k+3}{2k+1} \\ (-1)^{3k+1} \binom{3k+1}{k} & 0 & 0 \\ 0 & (-1)^{5k+2} \binom{5k+2}{k} & 0 \end{pmatrix},$$

where we have chosen the natural bases, e.g., $e_1 = \frac{z^3}{x^2y}$, $e_2 = \frac{z^5}{xy^4}$, $e_3 = \frac{z^6}{x^4y^2}$ for the first cycle. Similarly, if $p = 7k + 2$, the matrix is

$$\begin{pmatrix} 0 & 0 & (-1)^{5k+1} \binom{5k+1}{4k+1} \\ (-1)^{3k} \binom{3k}{k} & 0 & 0 \\ 0 & (-1)^{6k+1} \binom{6k+1}{4k+1} & 0 \end{pmatrix}.$$

We claim that the nonzero entries of A are equal. Observe that by Wilson's theorem

$$\begin{aligned} (n-1)!(p-n)! &= (n-1)!(p-n)(p-n-1)\cdots 1 \\ &= (n-1)!(-n)(-n-1)\cdots(-p+1) = (-1)^n \end{aligned}$$

mod p . Furthermore, because $p = 7k + 4$ is odd, k is odd. Thus we can rewrite

$$\begin{aligned} \binom{5k+2}{k} &= \frac{(5k+2)!}{(k)!(4k+2)!} = \frac{(p-2k-2)!}{(p-6k-4)!(4k+2)!} \\ &= (-1)^{4k+2} \binom{6k+3}{2k+1} = \binom{6k+3}{2k+1} \end{aligned}$$

and

$$\binom{5k+2}{k} = \frac{(5k+2)!}{(k)!(4k+2)!} = \frac{(p-2k-2)!}{k!(p-3k-2)!} = (-1)^{5k} \binom{3k+1}{k} = -\binom{3k+1}{k}.$$

The case of $p = 7k + 2$ is identical.

Since $a^p = a$ for any element $a \in \mathbb{F}_p$, the Frobenius action is linear. Thus by Lemma 7.1 and Remark 7.3, the matrix associated to the Frobenius map on the chosen basis can be transformed into upper-triangular form if and only if $p \equiv 1 \pmod{3}$. Thus by Proposition 7.2, in the case $p = 7k + 4$ or $p = 7k + 2$, we obtain that $l_{F_R}(H_f^1(R)) = l_{F_R}(\mathcal{H}_{R,R}(V)) + 1 = 7$ when $p \equiv 1 \pmod{3}$, and otherwise $l_{F_R}(H_f^1(R)) = 5$.

Lastly, it is easy to see that the third iterate of the Frobenius map on V can be represented by a diagonal matrix; hence

$$l_{F_R^3}(\mathcal{H}_{R,R}(V)) = l_{F_R^3}(\mathcal{H}_{R,R}(0_{H_m^2(R/f)}^*)) = l_{F_R^3}(\mathcal{H}_{R,R}(0_{H_m^2(R/f)}^*)) = 6$$

and $l_{F_R^\infty}(H_f^1(R)) = 7$. □

Finally, we exhibit an example of a local cohomology module of R , with k algebraically closed, such that its $D(R, k)$ -module length (equivalently, its F_R^∞ -module length) is strictly bigger than its F_R -module length. Recall that by Theorem 4.3, this cannot happen if $A = R/I$ has isolated singularities.

Proposition 7.5. *Let $p = 7k + 4$ be a prime number. Let $R = \overline{\mathbb{F}}_p[x, y, z, t]$ and let $f = tx^7 + ty^7 + z^7$. Then*

$$l_{F_R}(\mathrm{H}_f^1(R)) = 3 < 7 = l_{F_R^\infty}(\mathrm{H}_f^1(R)) = l_{D(R, \overline{\mathbb{F}}_p)}(\mathrm{H}_f^1(R)).$$

Proof. Denote $R/(f)$ by A . First we claim that $(x, y, z)A$ is the only height-2 prime ideal of A that contains the test ideal $\tau(A)$ of A . By [ST12, Theorem 6.4], we have

$$\tau(A) = \sum_e \sum_{\phi \in \mathrm{Hom}_A(A^{1/p^e}, A)} \phi(c^{1/p^e}),$$

where c is a test element for A . According to [Hoc07, Theorem on page 184], z^6 is a test element for A and we may set $c = z^6$. Since $A = R/(f)$, we have

$$\tau(A) = \left(\sum_e \sum_{\varphi \in \mathrm{Hom}_R(R^{1/p^e}, R)} \varphi((f^{p^e-1}z^6)^{1/p^e}) \right) A.$$

It is straightforward to check that $(x^6, y^6, z^6) \subset \tau(A)$ and $\tau(A)$ has height 2. Hence $(x, y, z)A$ is the only height-2 prime ideal that contains the test ideal $\tau(A)$ of A . Consequently, $\widehat{A}_P$ is F -rational for each height-2 prime $P \neq (x, y, z)A$ or, equivalently, $0_{\mathrm{H}_{P\widehat{A}_P}^2(\widehat{A}_P)}^* = 0$.

Next we calculate the stable part of $\mathrm{H}_{\mathfrak{m}}^3(A)$ where $\mathfrak{m} = (x, y, z, t)$. To this end, we assign the grading $\deg(x) = \deg(y) = 1$, $\deg(z) = 2$, and $\deg(t) = 7$ degrees 1, 1, 2, 7 to R , and, consequently, f is homogeneous. It is straightforward to check that $\mathrm{H}_{\mathfrak{m}}^3(A)_0$ has an $\overline{\mathbb{F}}_p$ -basis:

$$\left[\frac{z^5}{tx^2y} \right], \left[\frac{z^5}{txy^2} \right], \left[\frac{z^6}{tx^4y} \right], \left[\frac{z^6}{tx^3y^2} \right], \left[\frac{z^6}{tx^2y^3} \right], \left[\frac{z^6}{txy^4} \right].$$

Because the degree of z is larger than the degrees of x and y , each of these elements is nilpotent under the natural Frobenius action. For example, raising the last generator to the power $p = 7k + 4$, we get

$$\left[\frac{z^{42k+24}}{t^{7k+4}x^{7k+4}y^{28k+16}} \right] = \left[\frac{z^3(z^7)^{6k+3}}{t^{7k+4}x^{7k+4}y^{28k+16}} \right] = \left[\frac{z^3(x^7 + y^7)^{6k+3}}{t^{k+1}x^{7k+4}y^{28k+16}} \right],$$

which necessarily equals 0 since the degree of any monomial in x, y in the numerator is $7(6k + 3)$, which is greater than $(7k + 4) + (28k + 16)$. Hence $\mathrm{H}_{\mathfrak{m}}^3(A)_s = 0$.

Given the grading on R , we are in the situation of Theorem 4.8. By Claim 4.8.1, there exists a graded F_R -module filtration $0 \subseteq L \subseteq M \subseteq \mathrm{H}_f^1(R)$ where L is supported at (f) , each $D(R, \overline{\mathbb{F}}_p)$ -module (equivalently, D_R -module or F_R^∞ -module) composition factor of M/L is supported at (x, y, z) , and $\mathrm{H}_f^1(R)/M$ is supported only at $\mathfrak{m} = (x, y, z, t)$.

By [Bli04, Corollary 4.2 and Theorem 4.4],

$$l_{D(R, \overline{\mathbb{F}}_p)}(L) = l_{F_R^\infty}(L) = l_{F_R}(L) = 1,$$

because there is only one minimal prime (f) of A . Moreover, we have

$$l_{D(R, \overline{\mathbb{F}}_p)}(\mathrm{H}_f^1(R)/M) = l_{F_R^\infty}(\mathrm{H}_f^1(R)/M) = l_{F_R}(\mathrm{H}_f^1(R)/M) = 0$$

since $\dim_{\overline{\mathbb{F}}_p}(\mathrm{H}_{\mathfrak{m}}^3(A))_s = 0$. Thus we actually have $M = \mathrm{H}_f^1(R)$ in this example.

It remains to compute $l_{D(R, \overline{\mathbb{F}}_p)}(M/L)$, $l_{F_R^\infty}(M/L)$, and $l_{F_R}(M/L)$. Note that the first two are equal because we are working over an algebraically closed field

[Bli03, Theorem 1.1]. Moreover, if we take an F_R^∞ -module (resp., F_R -module) filtration of M/L , say

$$L = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_l = M = H_f^1(R),$$

such that each $N_i = M_i/M_{i-1}$ is a simple F_R^∞ -module (resp., simple F_R -module) supported at $P = (x, y, z)$ and then if we localize at P and complete, we have

$$L \otimes \widehat{R_P} = M_0 \otimes \widehat{R_P} \subseteq M_1 \otimes \widehat{R_P} \subseteq \cdots \subseteq M_l \otimes \widehat{R_P} = H_f^1(\widehat{R_P})$$

such that each successive quotient $N_i \otimes \widehat{R_P}$ is still simple as an $F_{\widehat{R_P}}^\infty$ -module (resp., as an $F_{\widehat{R_P}}$ -module) by Lemma 4.7.

Observe that

$$\widehat{R_P}/(f) \cong \overline{\mathbb{F}}_p(t)[[x, y, z]]/(tx^7 + ty^7 + z^7),$$

which is an isolated singularity. Hence by Proposition 7.2 (and the proof of Theorem 4.3), the $F_{\widehat{R_P}}^\infty$ -module (resp., $F_{\widehat{R_P}}$ -module) length of $H_f^1(\widehat{R_P})/(L \otimes \widehat{R_P})$ is the longest flag of Frobenius-stable subspaces of

$$V = \left(0_{H_m^2(\overline{\mathbb{F}}_p(t)[[x, y, z]]/(tx^7 + ty^7 + z^7))}^* \right)_s.$$

Via a direct computation similar to the proof of Theorem 6.3 one can show that $\dim V = 6$ and V is a direct sum of two 3-dimensional Frobenius-stable subspaces. In the natural bases as in the proof of Corollary 7.4, the Frobenius action on each cycle is represented by the matrix:

$$A = \begin{pmatrix} 0 & 0 & \binom{3k+1}{k} t^{6k+3} \\ \binom{3k+1}{k} t^{3k+1} & 0 & 0 \\ 0 & \binom{3k+1}{k} t^{5k+2} & 0 \end{pmatrix} = \begin{pmatrix} 3k+1 \\ k \end{pmatrix} \begin{pmatrix} 0 & 0 & t^{6k+3} \\ t^{3k+1} & 0 & 0 \\ 0 & t^{5k+2} & 0 \end{pmatrix}.$$

We can easily see that the third iterate of the Frobenius map on V can be represented by a diagonal matrix. Hence by Proposition 7.2,

$$l_{F_{\widehat{R_P}}^3}(H_f^1(\widehat{R_P})/(L \otimes \widehat{R_P})) = l_{F_{\widehat{R_P}}^\infty}(H_f^1(\widehat{R_P})/(L \otimes \widehat{R_P})) = 6,$$

and thus by the above discussion,

$$l_{D(R, \overline{\mathbb{F}}_p)}(H_f^1(R)) = l_{F_R^\infty}(H_f^1(R)) = 1 + 6 = 7.$$

Finally let us show that V has no proper subspace stable under the Frobenius action. If U is a proper Frobenius-stable subspace and $v \in U$, then we must have that $\langle v, F(v), F^2(v) \rangle \subseteq U \subsetneq V$. Thus if $v = (a, b, c)$ in the standard basis, then

$$\det \begin{pmatrix} a & t^{6k+3} c^p & t^{6k+3+(5k+2)p} b^{p^2} \\ b & t^{3k+1} a^p & t^{3k+1+(6k+3)p} c^{p^2} \\ c & t^{5k+2} b^p & t^{5k+2+(3k+1)p} a^{p^2} \end{pmatrix} = 0,$$

Observe that if $w = \lambda v$, then

$$\det(w, Fw, F^2w) = \lambda^{p^2+p+1} \det(v, Fv, F^2v),$$

so we can multiply v by the common denominator of a, b, c and assume that $a, b, c \in \overline{\mathbb{F}}_p[t]$.

By a direct computation, we have

$$\begin{aligned}\Delta &= \det \begin{pmatrix} a & t^{6k+3}c^p & t^{6k+3+(5k+2)p}b^{p^2} \\ b & t^{3k+1}a^p & t^{3k+1+(6k+3)p}c^{p^2} \\ c & t^{5k+2}b^p & t^{5k+2+(3k+1)p}a^{p^2} \end{pmatrix} \\ &= t^{(3k+1)p+8k+3}a^{p^2+p+1} - t^{(6k+3)p+8k+3}ab^pc^{p^2} - t^{(3k+1)p+11k+5}a^{p^2}bc^p \\ &\quad + t^{(6k+3)p+9k+4}c^{p^2+p+1} + t^{(5k+2)p+11k+5}b^{p^2+p+1} - t^{(5k+2)p+9k+4}a^pb^{p^2}c.\end{aligned}$$

Note that we may factor out $t^{(3k+1)p+8k+3}$ and obtain $\Delta = t^{(3k+1)p+8k+3}\Delta'$, where

$$\begin{aligned}\Delta' &= a^{p^2+p+1} - t^{(3k+2)p}ab^pc^{p^2} - t^{3k+2}a^{p^2}bc^p \\ &\quad + t^{(3k+2)p+k+1}c^{p^2+p+1} + t^{(2k+1)p+3k+2}b^{p^2+p+1} - t^{(2k+1)p+k+1}a^pb^{p^2}c.\end{aligned}$$

Let $\deg a = \alpha$, $\deg b = \beta$, $\deg c = \gamma$. Then we have

$$\begin{aligned}\deg a^{p^2+p+1} &= p^2\alpha + p\alpha + \alpha, \\ \deg t^{(3k+2)p}ab^pc^{p^2} &= p^2\gamma + p(\beta + 3k + 2) + \alpha, \\ \deg t^{3k+2}a^{p^2}bc^p &= p^2\alpha + p\gamma + \beta + 3k + 2, \\ \deg t^{(3k+2)p+k+1}c^{p^2+p+1} &= p^2\gamma + p(\gamma + 3k + 2) + \gamma + k + 1, \\ \deg t^{(2k+1)p+3k+2}b^{p^2+p+1} &= p^2\beta + p(\beta + 2k + 1) + \beta + 3k + 2, \\ \deg t^{(2k+1)p+k+1}a^pb^{p^2}c &= p^2\beta + p(\alpha + 2k + 1) + \gamma + k + 1.\end{aligned}$$

Now we prove that there is always a nonzero term with highest degree in Δ' (hence Δ' cannot be zero). We consider the following three cases:

- (1) $\gamma \geq \max\{\alpha, \beta\}$.
- (2) $\beta > \gamma$ and $\beta \geq \alpha$.
- (3) $\alpha > \max\{\beta, \gamma\}$.

The point is that in case (1) (similarly, in cases (2) and (3)), $\deg t^{(3k+2)p+k+1}c^{p^2+p+1}$ (resp., $\deg t^{(2k+1)p+3k+2}b^{p^2+p+1}$, $\deg a^{p^2+p+1}$) is strictly bigger than the degree of the other five terms. We give a detailed explanation in case (2) and leave the other cases to the reader to check (they are all very similar). In case (2), since $p = 7k + 4$, we have

$$\begin{aligned}\deg t^{(2k+1)p+3k+2}b^{p^2+p+1} - \deg a^{p^2+p+1} &\geq 3k + 2 > 0, \\ \deg t^{(2k+1)p+3k+2}b^{p^2+p+1} - \deg t^{(3k+2)p}ab^pc^{p^2} \\ &> p^2(\beta - \gamma) - p(k + 1) \geq p^2 - p(k + 1) > 0, \\ \deg t^{(2k+1)p+3k+2}b^{p^2+p+1} - \deg t^{3k+2}a^{p^2}bc^p &> p^2(\beta - \alpha) + p(\beta - \gamma) > 0, \\ \deg t^{(2k+1)p+3k+2}b^{p^2+p+1} - \deg t^{(3k+2)p+k+1}c^{p^2+p+1} \\ &> (p^2 + p)(\beta - \gamma) - p(k + 1) \geq p^2 + p - p(k + 1) > 0, \\ \deg t^{(2k+1)p+3k+2}b^{p^2+p+1} - \deg t^{(2k+1)p+k+1}a^pb^{p^2}c &= p(\beta - \alpha) + 2k + 1 > 0.\end{aligned}$$

This finishes the proof that V does not have Frobenius stable subspaces and thus

$$l_{F_{\widehat{R_P}}}(\mathrm{H}_f^1(\widehat{R_P})/(L \otimes \widehat{R_P})) = 2,$$

and hence by the above discussion

$$l_{D(R, \mathbb{F}_p)}(H_f^1(R)) = l_{F_R^\infty}(H_f^1(R)) = 1 + 2 = 3. \quad \square$$

Remark 7.6. Blickle [Bli03, Section 5] showed that for a general F -finite F_R^∞ -module M the two lengths $l_{F_R^\infty}(M)$ and $l_{D(R, k)}(M)$ might be different if the residue field k is perfect but not algebraically closed. However, we do not know whether this can happen when $M = H_f^c(R)$. The Fermat hypersurfaces cannot provide such an example: the natural Frobenius action decomposes into cycles, and thus the matrix representing some large iterate of the Frobenius action will be a diagonal matrix. Hence Theorem 4.3 and Proposition 7.2 show that the two lengths always coincide in this case.

We end with a slight generalization of Blickle's result ([Bli03, Theorem 1.1]) to local cohomology modules of rings with an isolated non- F -rational point over finite fields.

Proposition 7.7. *Let $(R, \mathfrak{m}, k)$ and $A = R/I$ be as in the Notation at the beginning of §4. Assume k is a finite field and A has an isolated non- F -rational point at $\mathfrak{m}$. Then $l_{F_R^\infty}(H_I^c(R)) = l_{D(R, k)}(H_I^c(R))$.*

Proof. Following the proof of Theorem 4.3 and Proposition 7.2, it is enough to show that there exists $e > 0$ such that the matrix B_e representing the e th Frobenius action on $(0_{H_{\mathfrak{m}}^d(A)}^*)_s$ is an upper triangular matrix. We will show that in fact B_e is the identity matrix for $e \gg 0$ sufficiently divisible. It is easy to observe that $B_e = B_1 \cdot B_1^{[p]} \cdots B_1^{[p^{e-1}]}$ where $B^{[p^i]}$ denotes the matrix obtained by raising each entry of B to its p^i th power. Suppose $k = \mathbb{F}_{p^n}$ and thus $B^{[p^n]} = B$. Hence we know that $B_{ne} = B_1^e \cdot (B_1^{[p]})^e \cdots (B_1^{[p^{n-1}]})^e$. Now the result follows from the elementary fact that over $\mathbb{F}_{p^n}$, every invertible matrix B has a power that is the identity matrix: look at the Jordan normal form of B in $\mathbb{F}_p$; taking a large p^m th power will make each Jordan block into a diagonal matrix. But an invertible diagonal matrix over $\mathbb{F}_{p^n}$ can be raised to a large power to make each diagonal entry be 1. $\square$

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