ISOTOPY OF SURFACES IN 4-MANIFOLDS AFTER A SINGLE STABILIZATION

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Abstract. Any two homologous surfaces of the same genus embedded in a smooth 4-manifold X with simply-connected complements are shown to be smoothly isotopic in $X \# S^2 \times S^2$ if the surfaces are ordinary, and in $X \# S^2 \tilde{\times} S^2$ if they are characteristic.

1. Introduction

By the 4-dimensional light bulb theorem of Gabai [11, Theorem 1.2], any two homologous 2-spheres embedded with a common geometric dual in a smooth simply-connected 4-manifold X are isotopic; here a *geometric dual* for a surface $F \subset X$ is an embedded 2-sphere Σ of *square* zero (meaning self-intersection zero) intersecting F transversely in a single point. The same result holds for homologous closed surfaces F_0 and F_1 of the same genus embedded in X under a mild fundamental group condition (that the F_i should be "Σ–inessential"; see [11, Theorem 9.7]).

It has been known for some time that this result fails without a common geometric dual for the surfaces F_i, or even without the weaker condition that each surface should have an *immersed* geometric dual, i.e. a simply connected complement. Examples arise easily from the existence of exotic smooth structures on closed simply-connected 4-manifolds (Donaldson [7]) and the fact that such manifolds become diffeomorphic after sufficiently many stabilizations (Wall [22]); here a *stabilization* means a connected sum with a single $S^2 \times S^2$. Work of Quinn [18] and Perron [16, 17] shows that the surfaces F_i always become isotopic after sufficiently many *external* stabilizations, where the connected sums are taken away from $F_0 \cup F_1$. This raises the question of how many stabilizations are needed. In particular, *is one enough*? If n is the minimal number of stabilizations needed, the surfaces are said to be *strictly n-stably isotopic*; the first explicit examples of families of strictly 1-stably isotopic surfaces were given by the authors in [4] (see also Akbulut [1]).

Analogous questions have been asked about many exotic phenomena in 4-dimensional topology which are known to dissipate after sufficiently many stabilizations. To the authors' knowledge, there are no instances known where it can be shown that one is *not* enough.

In this note it is shown using Gabai's results that, indeed, one is always enough if the surfaces have simply-connected complements and are *ordinary*. Recall that a surface is ordinary if it is not *characteristic*, that is, not dual to the second Stiefel-Whitney class $w_2(X)$; geometrically, a surface is characteristic if it intersects even classes evenly and odd classes oddly. Furthermore, this one-is-enough result still holds for *characteristic* surfaces under 'twisted' stabilization, meaning connected sum with the twisted bundle $S^2 \tilde{\times} S^2$.

2. One is enough

Theorem. *If X is a smooth simply-connected 4-manifold and $\alpha \in H_2(X)$ is an ordinary class, then any two closed oriented surfaces F_0 and F_1 in X of the same genus representing α, both with simply-connected complement, are smoothly isotopic in $X \# S^2 \times S^2$ (summing away from $F_0 \cup F_1$). When α is characteristic, the same result holds if one stabilizes by summing with $S^2 \tilde{\times} S^2$.*

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Proof. It can be assumed that F_0 and F_1 intersect transversely. When α is ordinary, the strategy is to find an embedded sphere in $X \# (S^2 \times S^2)$ of square zero geometrically dual to both F_0 and F_1. The result will then follow from Gabai's theorem. When α is characteristic, no such sphere exists since the image of α under the natural map $H_2(X) \to H_2(X \# S^2 \times S^2)$ is still characteristic. But it will be seen to exist in $X \# S^2 \widetilde{\times} S^2$, and the proof will follow as before. Here is how one carries out this strategy, assuming first that α is ordinary.

Start with an *immersed* sphere $\Sigma \subset X$ meeting F_0 transversally in one point, i.e. an immersed geometric dual for F_0, that is transverse to F_1. Such a sphere exists since $\pi_1(X - F_0) = 1$. If X is even, then Σ has even square. If X is odd, then Σ may have odd square, but it can be modified to have even square by connected summing with an immersed sphere S in $X - F_0$ of odd square. To see that such an S exists, note that since F_0 is ordinary, there exists an immersed surface $E \subset X$ whose square is of the opposite parity from the algebraic intersection number $n = E \cdot F_0$. Now by the immersed Norman trick [14], E can be tubed to parallel copies of Σ along arcs in F_0 to remove its intersections with F_0, giving a surface $E + n\Sigma$ of *odd* square in $X - F_0$, since $E + n\Sigma$ is homologically the sum of an even and odd class. But since $X - F_0$ is simply-connected, this last surface is homologous to an immersed sphere S in $X - F_0$. Thus it can always be arranged for the self-intersection of Σ to be even.

Orient Σ so that $\Sigma \cdot F_0 = 1$. Then $\Sigma \cdot F_1 = 1$ as well, since F_1 is homologous to F_0. If the geometric intersection number $|\Sigma \cap F_1|$ is greater than 1, then finger and Whitney moves can be used to reduce this number, thus inductively moving Σ to meet F_1 in only one point. This is accomplished as follows:

For any oppositely oriented pair p, q of intersection points in $\Sigma \cap F_1$, consider a Whitney circle on $\Sigma \cup F_1$ disjoint from F_0, made up of two arcs $\gamma_0 \subset \Sigma$ and $\gamma_1 \subset F_1$ meeting at their endpoints p and q. Since $X - F_1$ is simply-connected, this circle bounds an immersed disk D with interior disjoint from F_1, but not necessarily disjoint from F_0 and Σ. See Figure 1 for a schematic of the intersections and self-intersections between F_0, F_1, Σ and D.

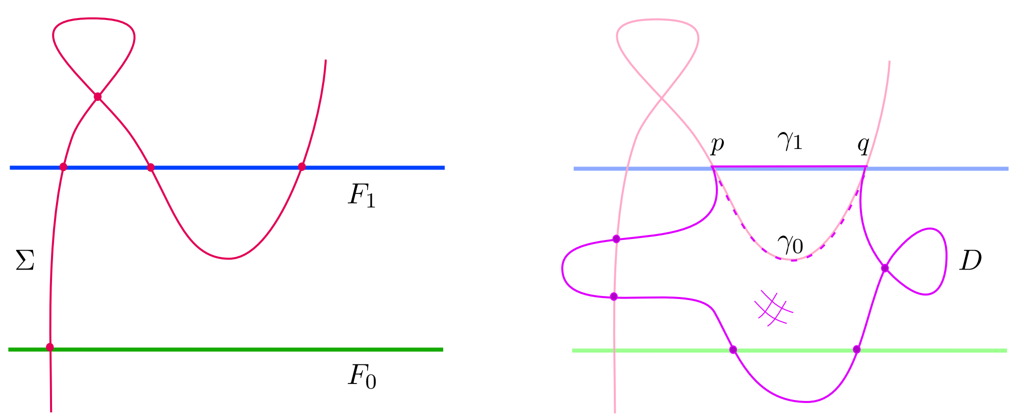

FIGURE 1. The surfaces F_0 and F_1, the immersed dual sphere Σ, and a Whitney disk D

Now perform finger moves of F_0 across F_1 to remove the intersections of F_0 with D, guided by disjoint arcs in $D - \Sigma$ from the points of $D \cap F_0$ to γ_1. To perform a Whitney move of Σ across D one must first fix the framing, that is, ensure that the restriction to ∂D of the framing of the normal bundle of D matches the framing induced by $\Sigma \cup F_1$. This is achieved by "boundary twisting" D around Σ along γ_0; see Freedman and Quinn [9, §1.3–1.4] for more details. This introduces additional intersection points between Σ and D, but keeps the interior of D and $F_0 \cup F_1$ disjoint. Once the framing has been fixed, one may push Σ over the immersed Whitney disk D, thus removing its intersection points p and q with F_1. Repeating this process, one obtains an immersed sphere $\Sigma \subset X$ with even self-intersection that is geometrically dual to both F_0 and F_1.

To arrange for Σ to be embedded, the ambient manifold X must be stabilized. In particular, take the connected sum with $S^2 \times S^2$ at a point in the complement of $F_0 \cup F_1 \cup \Sigma$. Choose coordinate 2-spheres $S = S^2 \times \{\text{pt}\}$ and $T = \{\text{pt}\} \times S^2$ in $S^2 \times S^2$. Now replace Σ by its connected sum with S along a tube disjoint from $F_0 \cup F_1$, so that Σ now has an embedded geometric dual sphere T in $X \# S^2 \times S^2$. Then eliminate the double points in Σ by the Norman trick, tubing to parallel copies of T. The result is an *embedded* sphere, still denoted Σ, that intersects both F_0 and F_1 geometrically in exactly one point. This sphere still has even square and an embedded geometric dual sphere T of square zero. Tubing with additional copies of T, the dual sphere Σ can be made to have self-intersection zero.

This entire argument can be repeated when α is characteristic, except that the self-intersection of the immersed dual Σ for F_0 and F_1 will now necessarily be odd. In this case, stabilize by summing with $S^2 \widetilde{\times} S^2$, viewed as a Hirzebruch surface with a section S of odd square, and fiber T. Now tube Σ to S to obtain an immersed sphere of even square and a geometric dual T, and proceed as in the ordinary case by tubing with T to make Σ embedded.

The proof is finished by applying Gabai's results [11]. The technical assumption needed for surfaces of higher genus (that $\pi_1(F_i - \Sigma) \to \pi_1(Y - \Sigma)$ for $F_i \subset Y$ should be trivial; see Theorem 9.7 in [11]) holds since Σ has a geometric dual T, so the complement of Σ in $Y = X \# S^2 \times S^2$ (or $Y = X \# S^2 \widetilde{\times} S^2$) is simply-connected. ☐

As a consequence of this theorem, one can find infinite families of strictly 1-stably isotopic 2-spheres in many once-stabilized 4-manifolds.

Corollary. *Let $X_1, X_2 \ldots$ be any (possibly infinite) list of pairwise non-diffeomorphic, smooth, closed, simply-connected 4-manifolds, all homeomorphic to one such X, such that $X_i \# S^2 \times S^2$ is diffeomorphic to $X \# S^2 \times S^2$ for each i. Also assume that the X_i remain distinct after connected summing with any number of copies of $\overline{\mathbb{C}P}^2$. If either*

1) *X is even and n is any even nonnegative integer, or*

2) *X is odd and indefinite, and n is any nonnegative integer,*

then there is a corresponding family of strictly 1-stably isotopic 2-spheres $S_1, S_2, \ldots$ of square n smoothly embedded in $X \# S^2 \times S^2$.

Proof. Observe that if X is even, then it is indefinite by Donaldson's Theorem A [6]. Thus in either case **1)** or **2)**, every automorphism of the quadratic form of $X \# S^2 \times S^2$ is induced by a diffeomorphism of $X \# S^2 \times S^2$, by Wall [21].

First assume that n is even. Then $S^2 \times S^2$ is diffeomorphic to the S^2-bundle over S^2 of Euler class n. Let S denote the zero section of this bundle, which is an embedded 2-sphere of square n, and T denote the fiber. By abuse of notation, S and T will also denote the corresponding 2-spheres in the $S^2 \times S^2$ factor in $X \# S^2 \times S^2$, and also in $X_i \# S^2 \times S^2$ for each i. By Wall's result, there exist diffeomorphisms $h_i \colon X_i \# S^2 \times S^2 \to X \# S^2 \times S^2$ such that the 2-spheres $S_i = h_i(S)$ are all homologous to S in $X \# S^2 \times S^2$. However, these spheres are not smoothly isotopic in $X \# S^2 \times S^2$, since blowing up n points on S_i and then surgering the resulting sphere yields $X_i \# n\overline{\mathbb{C}P}^2$ (seen for example by a Kirby calculus exercise), and these manifolds are distinct by hypothesis.

If n is odd and X is odd, use the facts that $X \# S^2 \times S^2$ is diffeomorphic to $X \# S^2 \widetilde{\times} S^2$ and that $S^2 \widetilde{\times} S^2$ is diffeomorphic to the S^2-bundle over S^2 of Euler class n. As above, one then constructs a family of homologous but non-isotopic spheres $S_i \subset X \# S^2 \times S^2$ of square n.

Now in either case, the spheres S_i have geometric dual spheres $T_i = h_i(T)$ of self-intersection zero, so the S_i are ordinary with simply-connected complements. It follows from the theorem that

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the S_i become isotopic in $X \# S^2 \times S^2 \# S^2 \times S^2$. Since the S_i are not isotopic in $X \# S^2 \times S^2$, they are strictly 1-stably isotopic. $\square$

Examples. 1) Let K_i be a sequence of knots with distinct Alexander polynomials, and X_i be the 4-manifolds obtained from the elliptic surface $E(2)$ by K_i-knot surgery along a regular fiber. Then the X_i are pairwise non-diffeomorphic [8, 19] and satisfy the stability hypothesis of the corollary [2, 3]. Thus the spin manifold $E(2) \# S^2 \times S^2$ contains strictly 1-stably isotopic families of spheres of any even, nonnegative self-intersection.

2) Let X_i be the family of Dolgacev surfaces obtained from the rational elliptic surface $E(1)$ by a pair of logarithmic transforms of orders 2 and $2i+1$. Then the X_i are pairwise non-diffeomorphic 4-manifolds [7, 10, 15] that satisfy the stability hypothesis of the corollary [12]. Thus $2\mathbb{C}P^2 \# 10\overline{\mathbb{C}P}^2$ contains strictly 1-stably isotopic families of spheres of any nonnegative self-intersection.

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