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A splitting theorem for the Seiberg-Witten invariant of a homology $S^1 \times S^3$

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We study the Seiberg–Witten invariant $\lambda_{SW}(X)$ of smooth spin 4–manifolds X with the rational homology of $S^1 \times S^3$ defined by Mrowka, Ruberman and Saveliev as a signed count of irreducible monopoles amended by an index-theoretic correction term. We prove a splitting formula for this invariant in terms of the Frøyshov invariant $h(X)$ and a certain Lefschetz number in the reduced monopole Floer homology of Kronheimer and Mrowka. We apply this formula to obstruct the existence of metrics of positive scalar curvature on certain 4–manifolds, and to exhibit new classes of homology 3–spheres of infinite order in the homology cobordism group.

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1 Introduction

Let X be a smooth oriented spin 4–manifold with the rational homology of $S^1 \times S^3$. Such manifolds play an important role in the study of homology cobordisms of homology 3–spheres and in addressing certain classification problems in 4–dimensional topology; see discussion in Ruberman and Saveliev [55]. Their study, however, represents a challenge because the usual count of the Seiberg–Witten monopoles on X generally depends on the auxiliary choices of metric and perturbation and hence does not result in a smooth invariant of X. This problem has been remedied by Mrowka, Ruberman and Saveliev [41], who defined an invariant $\lambda_{\mathrm{SW}}(X)$ for integral homology $S^1 \times S^3$ by adding an index-theoretic correction term to the count of the Seiberg–Witten monopoles on X; we will extend their definition to the case of rational homology $S^1 \times S^3$ in this paper. Because of its gauge-theoretic nature, the invariant $\lambda_{\mathrm{SW}}(X)$ is difficult to compute directly from its definition. We address this problem in the current paper by expressing $\lambda_{\mathrm{SW}}(X)$ in terms of Floer-theoretic invariants via a gluing theory, the way it is done for the classical Seiberg–Witten invariants.

An invariant relevant to this gluing theory was defined by Frøyshov [12]: under the additional hypothesis that a generator of $H_3(X;\mathbb{Z})$ is carried by an embedded rational homology sphere Y, he showed [12, Theorem 8] that the invariant $h(Y,\mathfrak{s})$ arising from the monopole Floer homology of Y with the induced spin structure $\mathfrak{s}$ is an invariant of the spin manifold X alone; we will denote this invariant by $h(X)$. The invariants $\lambda_{\mathrm{SW}}(X)$ and $h(X)$ are certainly different: for instance, the mod 2 reduction of $\lambda_{\mathrm{SW}}(X)$ equals the Rokhlin invariant of X, while this is not the case for $h(X)$.

The following theorem establishes a precise relation between $\lambda_{\mathrm{SW}}(X)$ and $h(X)$. It is followed by some strong applications (Theorems B, C and D below) to the study of metrics of positive scalar curvature and of the homology cobordism group of homology 3–spheres.

Theorem A *Let X be a smooth oriented spin rational homology $S^1 \times S^3$ which is homology oriented by a choice of generator $\alpha \in H^1(X;\mathbb{Z})$. Assume that the Poincaré dual of α is realized by a rational homology sphere $Y \subset X$. Let $\mathfrak{s}$ be the induced spin structure on Y, and let W be the spin cobordism from Y to itself obtained by cutting X open along Y. Then*

$$\lambda_{\mathrm{SW}}(X) + h(X) = -\mathrm{Lef}(W_*\colon \mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s}) \to \mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s})). \tag{1}$$

The reduced monopole Floer homology $\mathrm{HM}^{\mathrm{red}}(Y, \mathfrak{s})$ that appears in the statement of this theorem is the Floer theory defined by Kronheimer and Mrowka [23], with rational coefficients. We could use instead the monopole Floer homology defined by Frøyshov [12], which would change the sign of the Lefschetz number because of different grading conventions in the two theories. The rational homology sphere Y in Theorem A is oriented by the rule that the orientation on a curve Hom-dual to α followed by the orientation on Y gives the orientation on X. The precise sign convention for $\lambda_{\mathrm{SW}}(X)$ is described in Section 2.1.

In the special case of $X = S^1 \times Y$, where Y is an integral homology sphere, Theorem A reduces to Theorem 5 of Frøyshov [12], which relates the Casson invariant $\lambda(Y)$, the Frøyshov invariant $h(Y)$ and the Euler characteristic of $\mathrm{HM}^{\mathrm{red}}(Y)$. An analogous theorem [12, Theorem 7] holds for manifolds X with $b_+^2(X) > 1$ once the term $\lambda_{\mathrm{SW}}(X) + h(X)$ in formula (1) is replaced by the usual Seiberg–Witten invariant.

1.1 An outline of the proof

The proof of Theorem A relies on the calculation of the two terms in the definition of $\lambda_{\mathrm{SW}}(X)$, the count of the Seiberg–Witten monopoles on X and the index-theoretic correction term, using metrics on X with long necks $(0, R) \times Y$. This neck-stretching technique for counting monopoles is well known in gauge theory, although mainly in the separating case. The nonseparating case at hand was studied in Frøyshov [11, Section 11.1] under the technical assumption of the absence of reducible monopoles on the noncompact manifold W_∞ obtained by attaching infinite product ends to W. In our case, this assumption fails because $b_2^+(W) = 0$. We could overcome this problem by adapting the proof of the gluing theorem for framed moduli spaces from [11] to our situation. Instead, for the sake of completeness, we decided to take a different approach and show that such a reducible monopole does not cause any trouble because it is isolated in the sense that a sequence of irreducible monopoles on X cannot converge to it as $R \to \infty$. This is proved using an a priori estimate on the smallest eigenvalue of the Dirac Laplacian on a manifold with long neck. We provide a detailed argument in Sections 7, 8 and 9, which all use the setup of Kronheimer and Mrowka [23]. As a result, we are able to match the count of monopoles on X with a certain Lefschetz number in the monopole Floer chain complex. The truly novel part of the proof of Theorem A, however, is the calculation of the correction term in $\lambda_{\mathrm{SW}}(X)$, which boils down to calculating the index of the spin Dirac operator on a manifold with periodic end modeled on the infinite cyclic cover of X as $R \to \infty$. This is done in two substantially

different ways, one direct, in Section 6, and the other using the end-periodic index theorem of Mrowka, Ruberman and Saveliev [42] in Section 11.

A very useful technical result underpinning the proof of Theorem A is the existence of Riemannian metrics on the manifold W_∞ with infinite product ends which make the L^2 Sobolev completion of the spin Dirac operator invertible. This existence result, which is proved in Section 10 in all dimensions divisible by four, is an extension of the generic metric theorem of Ammann, Dahl and Humbert [1] to certain noncompact manifolds. One advantage of working with such generic metrics is that they greatly simplify the treatment of perturbations needed to ensure the regularity of the Seiberg–Witten moduli spaces, and allow us to avoid perturbations on manifolds with periodic ends altogether. See the discussion at the end of Section 2.1.

1.2 Calculations and applications

The splitting formula of Theorem A makes the invariant $\lambda_{\mathrm{SW}}(X)$ computable in a number of cases. This is due, on one hand, to the availability of advanced computational tools in monopole Floer homology (such as Floer exact triangles and the Pin(2) symmetry), and on the other, to the identification between monopole Floer homology and Heegaard Floer homology, by the work of Kutluhan, Lee and Taubes [25; 26; 27; 28; 29] or, alternatively, the work of Colin, Ghiggini and Honda [6; 7; 8] and Taubes [63]. This newly found computability of $\lambda_{\mathrm{SW}}(X)$ leads to a number of applications, of which we present two in Sections 4 and 5. In both applications, the Lefschetz number in formula (1) vanishes, albeit for different reasons.

The first application gives an obstruction to a 4–manifold having a Riemannian metric of positive scalar curvature. Historically (see Witten [64]), the Seiberg–Witten invariants have been used to produce many obstructions of this nature that go well beyond the classical index-theoretic obstruction of Lichnerowicz [31]. We add to this body of knowledge the following theorem, which was originally proved by the first author [33, Theorem 1.2] using different techniques. It was conjectured in [33, Remark 1] that there should exist a proof along the lines of this paper.

Theorem B *Let X be a smooth oriented spin rational homology $S^1 \times S^3$ which is homology oriented by a choice of generator in $H^1(X;\mathbb{Z})$. Assume that the Poincaré dual of this generator is realized by a rational homology sphere $Y \subset X$ with the induced spin structure $\mathfrak{s}$. Then X admits no Riemannian metric of positive scalar curvature unless $\lambda_{\mathrm{SW}}(X) + h(Y,\mathfrak{s}) = 0$.*

It was shown in Mrowka, Ruberman and Saveliev [41] that $\lambda_{SW}(X)$ reduces modulo 2 to the Rokhlin invariant $\rho(X)$. As in [33], this fact leads to the corollary that, if X admits a metric of positive scalar curvature, any rational homology sphere Y carrying the generator of $H_3(X;\mathbb{Z})$ must satisfy the relation $h(Y,\mathfrak{s}) = \rho(X) \pmod 2$ with respect to the induces spin structure $\mathfrak{s}$. For example, if a generator of $H_3(X;\mathbb{Z})$ is carried by the Brieskorn homology sphere $\Sigma(2,3,7)$, then X cannot admit a positive scalar curvature metric.

The second, and more elaborate, application is to the study of homology cobordisms. Recall that oriented 3–manifolds Y_0 and Y_1 are called homology cobordant (resp. $\mathbb{Z}/2$ homology cobordant) if there exists a smooth, compact, oriented cobordism W from Y_0 to Y_1 such that $H_*(W, Y_i;\mathbb{Z}) = 0$ (resp. $H_*(W, Y_i;\mathbb{Z}/2) = 0$) for $i = 0, 1$. The homology cobordism group $\Theta^3_{\mathbb{Z}}$ is generated by oriented integral homology 3–spheres, modulo the equivalence relation of being homology cobordant. Similarly, the $\mathbb{Z}/2$ homology cobordism group $\Theta^3_{\mathbb{Z}/2}$ is generated by oriented $\mathbb{Z}/2$ homology 3–spheres, modulo the equivalence relation of being $\mathbb{Z}/2$ homology cobordant.

Let us first consider the group $\Theta^3_{\mathbb{Z}}$. Recall that the Rokhlin invariant provides a surjective homomorphism $\rho\colon \Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2$. Manolescu [38] used Pin(2)–equivariant Seiberg–Witten theory to show that this homomorphism does not split, that is, no integral homology sphere Y with $\rho(Y) = 1$ has order two in $\Theta^3_{\mathbb{Z}}$. It seems reasonable to conjecture that $\rho(Y) = 1$ in fact implies that Y has infinite order in $\Theta^3_{\mathbb{Z}}$; indeed, this was shown to be true for all Seifert fibered homology spheres by the third author [58] following Fukumoto, Furuta and Ue [13]. Generalizing this result, we show that the conjecture holds under an additional assumption that Y is h–positive or h–negative: a $\mathbb{Z}/2$ homology sphere Y is said to be h–positive (resp. h–negative) if the reduced monopole homology $\mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s})$ corresponding to the unique spin structure $\mathfrak{s}$ on Y is supported in degrees $\geq -2h(Y,\mathfrak{s})$ (resp. $\leq -2h(Y,\mathfrak{s}) - 1$); see Definition 5.3 for more details. All Seifert fibered homology spheres satisfy this assumption, and many more examples can be found in Section 5.

Theorem C *Any h–positive or h–negative integral homology sphere Y with $\rho(Y) = 1$ has infinite order in $\Theta^3_{\mathbb{Z}}$.*

It is worth mentioning that, instead of using Theorem A, one could follow similar arguments to deduce Theorem C from Stoffregen's connected Seiberg–Witten Floer homology [61].

Let us now turn our attention to the group $\Theta^3_{\mathbb{Z}/2}$, where we can prove a stronger result. Let Θ^3_L be the subgroup of $\Theta^3_{\mathbb{Z}/2}$ generated by all the L–spaces (over $\mathbb{Q}$) that are $\mathbb{Z}/2$ homology spheres. It was shown by Stoffregen [62] (using the invariants α, β and γ) and Hendricks, Manolescu and Zemke [18, Proposition 1.4] (using the invariants $\overline{d}$ and $\underline{d}$) that the Brieskorn homology sphere $\Sigma(2,3,7)$ has infinite order in the quotient group $\Theta^3_{\mathbb{Z}/2}/\Theta^3_L$. In particular, this implies that $\Theta^3_{\mathbb{Z}/2}/\Theta^3_L$ is an infinite group. The following theorem generalizes this result by exhibiting a large family of such manifolds.

Theorem D *Any h–positive or h–negative $\mathbb{Z}/2$ homology sphere Y with $\rho(Y) \neq h(Y) \pmod 2$ has infinite order in the group $\Theta^3_{\mathbb{Z}/2}/\Theta^3_L$.*

A further study of the h–positive (h–negative) condition using the rational surgery formula of Ozsváth and Szabó [51] leads to the following corollary.

Corollary E *Let Y be an integral homology sphere obtained by $1/n$ surgery on a knot K in S^3 with n odd, and suppose that $\operatorname{Arf}(K) = 1$. Then Y has infinite order in $\Theta^3_{\mathbb{Z}}$ in the following cases:*

(1) *K is the figure-eight knot.*

(2) *K is a quasialternating knot with nonzero signature $\sigma(K)$.*

We remark that the Frøyshov invariant $h(Y)$ in the above corollary vanishes both in case (1) and in case (2) whenever $n\cdot\sigma(K) > 0$ (see Lemma 5.13), hence the result of the corollary cannot be proved using $h(Y)$ alone. It may be possible, however, to give an alternative proof using other Frøyshov-type invariants involving the Pin(2)–symmetry, such as the invariants α, β and γ of Manolescu [38] — see also Lin [32] — and the invariants $\overline{d}$ and $\underline{d}$ of Hendricks and Manolescu [17], once their behavior under the connected sum operation is better understood.

Example Let Y be the integral homology sphere obtained by $1/n$ surgery on a two-bridge knot $K(24m \pm 5, 3)$ or $K(24m \pm 11, 3)$ with $m > 0$ and odd $n > 0$. Then $h(Y) = 0$ and Y has infinite order in the groups $\Theta^3_{\mathbb{Z}}$ and $\Theta^3_{\mathbb{Z}/2}/\Theta^3_L$.

Another major reason to study the group $\Theta^3_{\mathbb{Z}/2}$ is to gain information about the smooth knot concordance group $\mathcal{C}_s$ via passing to the double branched cover of the knot. Recall that a knot K is called Khovanov-homology thin (over $\mathbb{Z}/2$) if its reduced Khovanov homology $\widetilde{\mathrm{Kh}}(K;\mathbb{Z}/2)$ is supported in a single δ–grading (see Khovanov [21; 22]). Such knots are very common: all quasialternating knots [39] and 238 of the 250 prime

knots with up to 10 crossings are Khovanov-homology thin. Let $\mathcal{C}_{\text{thin}}$ be the subgroup of $\mathcal{C}_s$ generated by the Khovanov-homology thin knots. Theorem D has the following curious corollary:

Corollary F *Any knot K whose double branched cover satisfies the conditions of Theorem D generates an infinite cyclic subgroup in $\mathcal{C}_s/\mathcal{C}_{\text{thin}}$.*

Example Let K be the $(3, 12n-5)$ or $(3, 12n-1)$ torus knot, with $n > 0$. Then the connected sum $\#_m K$ (with $m > 0$) is never smoothly concordant to any Khovanov-homology thin knot.

In our subsequent paper [34], we use similar techniques to compute the invariant $\lambda_{\text{SW}}(X)$ for the mapping tori X of all smooth orientation-preserving involutions $\tau\colon Y \to Y$ on integral homology spheres Y with the quotient S^3. That calculation will confirm the conjecture of Mrowka, Ruberman and Saveliev [41, Conjecture B] for all such mapping tori X by identifying $\lambda_{\text{SW}}(X)$ with its Yang–Mills counterpart, the invariant $\lambda_{\text{FO}}(X)$ of Furuta and Ohta [14].

It is worth mentioning that the invariant $\lambda_{\text{SW}}(X)$ was extended in [41, Section 10] to the wider class of smooth closed oriented 4–manifolds X satisfying the hypotheses $H_1(X;\mathbb{Z}) = \mathbb{Z}$, $b_2^+(X) = 0$ and that $H_3(X)$ is generated by an integral homology 3–sphere $Y \subset X$. According to Frøyshov [12, Theorem 8], the same set of hypotheses on X makes for the well-defined Lefschetz number and h–invariant of X. It is then natural to wonder if Theorem A still holds in this more general situation.

1.3 Organization of the paper

We begin by reviewing the definitions of the monopole Floer homology and of the invariants $\lambda_{\text{SW}}(X)$ and $h(Y, \mathfrak{s})$ that go into the statement of Theorem A in Section 2. We also use this section to settle various technical matters and introduce some notations. The proofs of Theorems A, B, C and D, as well as Corollaries E and F, are given in the three sections that follow. They rely on certain technical results whose proofs are postponed until later in the paper for the sake of exposition. The first of these results, discussed in Section 6, is a calculation of the index-theoretic correction term $w(X, g)$ on manifolds with long necks. An alternative calculation using the end-periodic index theorem of Mrowka, Ruberman and Saveliev [42] is given in Section 11. Both calculations boil down to computing the L^2 index of a spin Dirac operator on

certain manifolds with periodic ends, which we do in all dimensions $n \equiv 0 \pmod 4$. Similar techniques lead to the first eigenvalue estimates in Section 7. These are later used for the compactness and gluing results in Sections 8 and 9. Several of these results rely on the generic metric theorem, whose proof uses a rather different set of techniques and for this reason is postponed until Section 10.

Although many results in this paper can be generalized to spin^c structures, we will be mainly concerned with spin structures. We will omit spin structures on 4–manifolds from our notations. We will usually include spin structures on 3–manifolds, to be consistent with Kronheimer and Mrowka [23]. One exception is $\mathbb{Z}/2$ homology spheres: each of these has a unique spin structure, which will be omitted.

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2 Preliminaries

We begin by recalling the definitions of all the invariants involved and settling some technical matters.

2.1 The invariant $\lambda_{SW}(X)$

Let X be an oriented smooth 4–manifold with the rational homology of $S^1 \times S^3$, equipped with a fixed spin structure. We will review the definition of $\lambda_{SW}(X)$ following [41]. Note that the invariant $\lambda_{SW}(X)$ was defined in [41] only for an integral homology $S^1 \times S^3$ but a careful check of the details shows that the construction of [41] extends to a rational $S^1 \times S^3$ essentially word for word. There are only two exceptions, which will be discussed in Remarks 2.1 and 2.2. With a slight abuse of language, we will cite [41] directly.

Fix a homology orientation on X by choosing a generator $1 \in H^1(X;\mathbb{Z}) = \mathbb{Z}$. Given a metric g on X and a coclosed 1–form $\beta \in \Omega^1(X; i\mathbb{R})$ orthogonal to $\mathcal{H}^1(X; i\mathbb{R})$ (the space of harmonic 1–forms on X), consider the triples (A, s, φ) consisting of a $U(1)$ connection A on the determinant bundle of the spin bundle, a real number

$s \geq 0$ and a positive spinor φ such that $\|\varphi\|_{L^2(X)} = 1$. The gauge group $\mathrm{Map}(X, S^1)$ acts freely on such triples by the rule $u(A, s, \varphi) = (A - u^{-1}du, s, u\varphi)$. The blown-up Seiberg–Witten moduli space $\mathcal{M}(X, g, \beta)$ consists of the gauge equivalence classes of triples (A, s, φ) that solve the perturbed Seiberg–Witten equations

$$\begin{cases} F_A^+ - s^2\tau(\varphi) = d^+\beta, \\ \mathcal{D}_A^+(X, g)(\varphi) = 0. \end{cases}$$

The solutions are referred to as monopoles. Monopoles with $s = 0$ are called reducible, and all other monopoles are called irreducible. The latter are identified with the irreducible monopoles in the usual moduli Seiberg–Witten moduli space via the map $(A, s, \varphi) \to (A, s\varphi)$.

According to [41, Proposition 2.2], the moduli space $\mathcal{M}(X, g, \beta)$ is regular for a generic choice of (g, β). In particular, $\mathcal{M}(X, g, \beta)$ is a compact oriented manifold of dimension zero that contains only irreducible solutions. The count of points in $\mathcal{M}(X, g, \beta)$ with signs determined by the orientation and homology orientation is denoted by $\#\mathcal{M}(X, g, \beta)$. The invariant $\lambda_{\mathrm{SW}}(X)$ is defined in [41] by the formula

$$\lambda_{\mathrm{SW}}(X) = \#\mathcal{M}(X, g, \beta) - w(X, g, \beta),$$

where $w(X, g, \beta)$ is the correction term which cancels the dependence of $\#\mathcal{M}(X, g, \beta)$ on the parameters (g, β). The definition of $w(X, g, \beta)$ is as follows.

Let $Y \subset X$ be a connected manifold which is Poincaré dual to the chosen generator in $H^1(X; \mathbb{Z})$. Note that Y is canonically oriented, and inherits a spin structure $\mathfrak{s}$ from X. Denote by W the cobordism from Y to itself obtained by cutting X open along Y. For any smooth compact spin manifold Z with spin boundary Y, consider the manifold

$$Z_\infty(X) = Z \cup W_0 \cup W_1 \cup W_2 \cup \cdots \tag{2}$$

with periodic end modeled on the infinite cyclic cover of X; each of the manifolds W_i in this formula is just a copy of W. Choose a metric and a perturbation on $Z_\infty(X)$ which match the metric and the perturbation over the end lifted from those on X. Then the operator $\mathcal{D}^+(Z_\infty(X), g, \beta) = \mathcal{D}^+(Z_\infty(X), g) + \beta$ is Fredholm with respect to the usual Sobolev L^2 completion, and the correction term

$$w(X, g, \beta) = \operatorname{ind} \mathcal{D}^+(Z_\infty(X), g, \beta) + \tfrac{1}{8} \operatorname{sign} Z \tag{3}$$

is independent of the choice of Y and Z.

Remark 2.1 The proof of well-definedness of $w(X,g,\beta)$ in [41, Proposition 3.2] makes use of the following fact: if X is an integral homology $S^1\times S^3$, its p–fold cyclic cover X_p is a rational homology $S^1\times S^3$ for any prime number p. This need not be true if X is a rational homology $S^1\times S^3$. However, one can use the argument of [41] to show that X_p is a rational homology $S^1\times S^3$ as long as the prime number p is large enough that $H_*(X;\mathbb{Z}/p)=H_*(S^1\times S^3;\mathbb{Z}/p)$. This is sufficient to complete the proof of well-definedness of $w(X,g,\beta)$.

Theorem A of [41] asserts that $\lambda_{\mathrm{SW}}(X)$ is independent of the choice of metric g and generic perturbation β, and that the reduction of $\lambda_{\mathrm{SW}}(X)$ modulo 2 is the Rokhlin invariant of X.

Remark 2.2 Unlike in the case of an integral homology $S^1\times S^3$ treated in [41], different spin structures on X may lead to different invariants $\lambda_{\mathrm{SW}}(X)$. To keep our notations clean, we will not include the spin structure in the notation. Note that when X is a $\mathbb{Z}/2$ homology $S^1\times S^3$, different spin structures are all equivalent as spin^c structures and hence give the same invariant $\lambda_{\mathrm{SW}}(X)$.

There are several implicit orientation conventions that go into the definition of $\lambda_{\mathrm{SW}}(X)$. We will not discuss them here but notice that altering these conventions consistently only changes $\lambda_{\mathrm{SW}}(X)$ by an overall sign. The sign of $\lambda_{\mathrm{SW}}(X)$ was fixed in [41, Section 11.2] by the condition

$$\lambda_{\mathrm{SW}}(S^1\times Y)=-\lambda(Y),$$

where $\lambda(Y)$ is the Casson invariant of an integral homology sphere Y normalized so that $\lambda(\Sigma(2,3,5))=-1$ for the Brieskorn homology sphere $\Sigma(2,3,5)$ oriented as a link of complex surface singularity.

We conclude this section by addressing an important technical point about choices of metrics and perturbations. According to [56], the operator $\mathcal{D}^+(Z_\infty(X),g,\beta)$ can be made Fredholm by choosing a generic metric g and letting $\beta=0$. This choice of metric also guarantees [41, Proposition 7.2] that the moduli space $\mathcal{M}(X,g,0)$ has no reducibles but not that it is regular. One way to ensure regularity is to choose a generic perturbation β small enough that $\operatorname{ind}\mathcal{D}^+(Z_\infty(X),g)=\operatorname{ind}\mathcal{D}^+(Z_\infty(X),g,\beta)$. Then

$$\lambda_{\mathrm{SW}}(X)=\#\mathcal{M}(X,g,\beta)-w(X,g), \tag{4}$$

where $w(X,g)$ stands for $w(X,g,0)$. The perturbation β in this formula can be replaced by a more general perturbation γ as in [41, Section 9.2] or in Section 8.2

below. As long as γ ensures regularity and is small enough, one can connect γ and β by a generic path $\{\beta_t \mid 0 \le t \le 1\}$ of small perturbations. Since the perturbations β_t are small, the union

$$\bigcup_{0 \le t \le 1} \mathcal{M}(X, g, \beta_t)$$

contains no reducibles (just like $\mathcal{M}(X, g, 0)$) and provides an oriented cobordism between $\mathcal{M}(X, g, \beta)$ and $\mathcal{M}(X, g, \gamma)$, as explained in [41, Proposition 9.4]. This gives us the formula

$$\lambda_{\mathrm{SW}}(X) = \#\mathcal{M}(X, g, \gamma) - w(X, g). \tag{5}$$

2.2 Monopole Floer homology

In this subsection, we will briefly recall the definition of the monopole Floer homology $\mathrm{HM}(Y, \mathfrak{s})$. We will focus on the special case when Y is a rational homology sphere and $\mathfrak{s}$ is a spin structure, which will suffice for the purpose of this paper. The general definition can be found in Kronheimer and Mrowka [23, Chapter 1]. We will work with rational coefficients and omit the coefficient ring from our notations.

Let Y be an oriented rational homology sphere with a Riemannian metric h and a spin structure $\mathfrak{s}$. An important example to have in mind is the rational homology sphere $Y \subset X$ of Theorem A with the induced spin structure $\mathfrak{s}$. Trivialize the spinor bundle $\mathcal{S}$ and choose the product connection B_0 to be our reference connection. We will make the following assumption, which according to [1, Theorem 1.1] holds for a generic metric h.

Assumption 2.3 *The spin Dirac operator $\mathcal{D}(Y, h)$ has zero kernel.*

Let $\mathcal{C}(Y)$ be the Sobolev $L^2_{k-1/2}$ completion of the affine space of configurations (B, ψ), where B is a connection in $\mathcal{S}$, ψ is a spinor and $k \ge 3$ is an integer which will be fixed throughout the paper. We refer to $\mathcal{C}(Y)$ as the configuration space. We also introduce the blown-up configuration space $\mathcal{C}^\sigma(Y)$, which consists of the triples (B, s, ψ), where B is a connection in $\mathcal{S}$, $s \ge 0$ is a real number and ψ is a spinor with $\|\psi\|_{L^2(Y)} = 1$.

Let $p\colon \mathcal{C}^\sigma(Y) \to \mathcal{C}(Y)$ be the natural projection sending (B, s, ψ) to $(B, s\psi)$. Using the terminology of [23], configurations in $\mathcal{C}(Y)$ are said to be downstairs and those in $\mathcal{C}^\sigma(Y)$ are said to be upstairs. A downstairs configuration (B, ψ) is called irreducible

if $\psi \neq 0$, while an upstairs configuration (B, s, ψ) is called irreducible if $s \neq 0$. All other configurations are called reducible. The projection p provides a diffeomorphism between the spaces of irreducible configurations upstairs and downstairs; we denote both of these spaces by $\mathcal{C}^*(Y)$.

The group of $L^2_{k+1/2}$ gauge transformations $u\colon Y \to S^1$ acts on $\mathcal{C}(Y)$ by the formula $u(B, \psi) = (B - u^{-1}du, u\psi)$ and on $\mathcal{C}^\sigma(Y)$ by the formula $u(B, s, \psi) = (B - u^{-1}du, s, u\psi)$. Both actions restrict to an action on $\mathcal{C}^*(Y)$. The corresponding quotient spaces will be denoted by $\mathcal{B}(Y)$, $\mathcal{B}^\sigma(Y)$ and $\mathcal{B}^*(Y)$. Note that $\mathcal{B}^*(Y)$ is a Hilbert manifold, while $\mathcal{B}^\sigma(Y)$ is a Hilbert manifold with boundary. The boundary of $\mathcal{B}^\sigma(Y)$, given by the equation $s = 0$, contains the gauge equivalence classes of all reducible configurations $[(B, 0, \psi)]$.

The monopole Floer homology was defined in [23] as a variant of Morse homology of the Chern–Simons–Dirac functional $\mathcal{L}\colon \mathcal{C}(Y) \to \mathbb{R}$. The definition of $\mathcal{L}$ can be found in [23, Definition 4.1.1]. To ensure that transversality holds, $\mathcal{L}$ is perturbed using a perturbation $\mathfrak{q}$ which is the formal gradient of a gauge-invariant functional $f\colon \mathcal{C}(Y) \to \mathbb{R}$. Note that $\mathfrak{q}$ has two well-defined components, the connection component $\mathfrak{q}^0$ and the spinor component $\mathfrak{q}^1$. The perturbed Chern–Simons–Dirac functional is denoted by $\mathcal{L}_\mathfrak{q} = \mathcal{L} + f$. Its gradient $\operatorname{grad}\mathcal{L}_\mathfrak{q} = \operatorname{grad}\mathcal{L} + \mathfrak{q}$ gives rise to a vector field $v^\sigma_\mathfrak{q}$ on $\mathcal{B}^\sigma(Y)$.

Let $\epsilon_0 > 0$ be any small number such that $\mathcal{D}(Y, h)$ has no eigenvalues in the interval $[-\epsilon_0, \epsilon_0]$ (the existence of ϵ_0 follows from Assumption 2.3). Then one can prove, as in [33, Proposition 2.8], that there exist perturbations $\mathfrak{q}$ satisfying the following assumption:

Assumption 2.4 *The perturbation $\mathfrak{q}$ satisfies the following three conditions:*

(a) *$\mathfrak{q}$ is nice, that is, $\mathfrak{q}(B, 0) = 0$ for all B; in other words, $\mathfrak{q}$ equals zero when restricted to reducible configurations.*

(b) *$\mathfrak{q}$ is admissible, that is, the critical points of $v^\sigma_\mathfrak{q}$ are nondegenerate and the moduli spaces of trajectories connecting them are regular; see* [23, Definition 22.1.1].

(c) *The derivative of the spinor component $\mathfrak{q}^1$ of $\mathfrak{q}$ satisfies the inequality*

$$\|D_{(B_0,0)}\mathfrak{q}^1(0, \psi)\|_{L^2(Y)} \leq \tfrac{1}{4}\epsilon_0 \cdot \|\psi\|_{L^2(Y)} \tag{6}$$

for any $\psi \in L^2_{k-1/2}(Y; \mathcal{S})$.

Under Assumption 2.4, the set $\mathfrak{C}$ of critical points of $v^\sigma_\mathfrak{q}$ is discrete and can be decomposed into the disjoint union of three subsets:

- $\mathfrak{C}^o$, the set of irreducible critical points.
- $\mathfrak{C}^s$, the set of reducible, boundary stable critical points (ie reducible critical points near which $v_{\mathfrak{q}}^{\sigma}$ points inside the boundary).
- $\mathfrak{C}^u$, the set of reducible, boundary unstable critical points (ie reducible critical points near which $v_{\mathfrak{q}}^{\sigma}$ points outside the boundary).

We will next take a closer look at the reducible critical points. According to [23, Corollary 4.2.2], the vector field $\operatorname{grad}\mathcal{L} + \mathfrak{q}$ has a unique reducible critical point $[(B_0, 0)]$ downstairs, which we call $[\theta]$. The situation with the reducible critical points upstairs is quite different. To describe it, consider the perturbed Dirac operator

$$\mathcal{D}_{B_0,\mathfrak{q}}(Y,h) = \mathcal{D}_{B_0}(Y,h) + D_{(B_0,0)}\mathfrak{q}^1(0,-)\colon L_1^2(Y;\mathcal{S}) \to L^2(Y;\mathcal{S}).$$

This is a self-adjoint elliptic operator. Since $\mathfrak{q}$ is admissible, its eigenvalues are all nonzero and have multiplicity one. We enumerate them so that

$$\cdots < \lambda_{-2} < \lambda_{-1} < 0 < \lambda_0 < \lambda_1 < \cdots.$$

For each λ_i pick an eigenvector ψ_i of unit L^2–norm and let $[\theta_i] = [(B_0, 0, \psi_i)] \in \mathcal{B}^{\sigma}(Y)$. Then

$$\mathfrak{C}^s = \{[\theta_i] \mid i \geq 0\} \quad \text{and} \quad \mathfrak{C}^u = \{[\theta_i] \mid i < 0\}.$$

Let C^o (resp. C^s and C^u) be a vector space over $\mathbb{Q}$ with the basis $\{e_{[\alpha]}\}$ indexed by the critical points $[\alpha]$ in $\mathfrak{C}^o$ (resp. $\mathfrak{C}^s$ and $\mathfrak{C}^u$). Define a linear map $\partial_o^o\colon C^o \to C^o$ by the formula

$$\partial_o^o e_{[\alpha]} = \sum_{[\beta]\in\mathfrak{C}^o} \#\check{\mathcal{M}}([\alpha],[\beta]) \cdot e_{[\beta]},$$

where $\check{\mathcal{M}}([\alpha],[\beta])$ is the moduli space of unparametrized flow lines going from $[\alpha]$ to $[\beta]$ and $\#\check{\mathcal{M}}([\alpha],[\beta])$ is the signed count of points in this moduli space. (This number is set to be zero if the dimension of the moduli space is positive.) One defines maps $\partial_s^o\colon C^o \to C^s$ and $\partial_o^u\colon C^u \to C^o$ similarly. Consider the vector spaces

$$\overline{C} = C^s \oplus C^u, \quad \check{C} = C^o \oplus C^s \quad \text{and} \quad \widehat{C} = C^o \oplus C^u.$$

The monopole Floer homologies $\overline{\mathrm{HM}}(Y,\mathfrak{s})$, $\widecheck{\mathrm{HM}}(Y,\mathfrak{s})$ and $\widehat{\mathrm{HM}}(Y,\mathfrak{s})$ are defined as the homology of the chain complexes $(\overline{C},0)$, $(\check{C},\check{\partial})$ and $(\widehat{C},\widehat{\partial})$, respectively, with the differentials

$$\check{\partial} = \begin{pmatrix} \partial_o^o & 0 \\ \partial_s^o & 0 \end{pmatrix} \quad \text{and} \quad \widehat{\partial} = \begin{pmatrix} \partial_o^o & \partial_o^u \\ 0 & 0 \end{pmatrix}.$$

(Note that our formulas are simpler than those in [23] because we are working with a rational homology sphere Y and a nice perturbation $\mathfrak{q}$.) The chain map $i\colon \overline{C} \to \check{C}$ with the matrix

$$\begin{pmatrix} 0 & \partial_o^u \\ 1 & 0 \end{pmatrix}$$

induces a natural map

$$i_*\colon \overline{\mathrm{HM}}(Y,\mathfrak{s}) \to \widetilde{\mathrm{HM}}(Y,\mathfrak{s}). \tag{7}$$

We define the reduced monopole Floer homology as $\mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s}) = \operatorname{coker} i_*$. This is a finite-dimensional vector space. Note that this definition matches the definition of the reduced monopole Floer homology in [23, Definition 3.6.3] because of the long exact sequence [23, (3.4)]. A rational homology sphere Y is called an L–space (over $\mathbb{Q}$) if the reduced monopole Floer homology $\mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s})$ vanishes for all spin^c structures $\mathfrak{s}$.

All these different versions of monopole Floer homology are modules over the polynomial ring $\mathbb{Q}[U]$, where U is a formal variable of degree -2. In fact, we have canonical isomorphisms $\overline{\mathrm{HM}}(Y,\mathfrak{s}) \cong \mathbb{Q}[U,U^{-1}]$ and $\operatorname{im} i_* \cong \mathbb{Q}[U,U^{-1}]/U\cdot\mathbb{Q}[U]$ as well as a noncanonical splitting $\widetilde{\mathrm{HM}}(Y,\mathfrak{s}) \cong \operatorname{im} i_* \oplus \mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s})$. In addition, the monopole Floer homology has the so-called "TQFT property". More precisely, any spin^c cobordism $(W,\mathfrak{s}_W)$ from $(Y_0,\mathfrak{s}_0)$ to $(Y_1,\mathfrak{s}_1)$ induces a morphism

$$(W,\mathfrak{s}_W)_*\colon \mathrm{HM}(Y_0,\mathfrak{s}_0) \to \mathrm{HM}(Y_1,\mathfrak{s}_1) \tag{8}$$

of $\mathbb{Q}[U]$–modules, where HM stands for any one of the monopole Floer homologies $\widetilde{\mathrm{HM}}$, $\widehat{\mathrm{HM}}$, $\overline{\mathrm{HM}}$ or $\mathrm{HM}^{\mathrm{red}}$. Each of these morphisms is induced by a respective chain map, whose definition requires further perturbations as described in Section 8.2. Note that in the current paper, we only consider spin structures $\mathfrak{s}_W$ and omit them from our notations.

Next, we need to discuss the canonical gradings in monopole Floer homology. With each critical point $[\alpha] \in \mathfrak{C}$ one associates two gradings,

$$\mathrm{gr}^{\mathbb{Q}}([\alpha]) \in \mathbb{Q} \quad \text{and} \quad \mathrm{gr}^{(2)}([\alpha]) \in \mathbb{Z}/2;$$

see [23, page 587] for the former and [23, page 427] for the latter. These naturally induce (absolute) $\mathbb{Q}$– and $\mathbb{Z}/2$–gradings on $\widetilde{\mathrm{HM}}(Y,\mathfrak{s})$, $\widehat{\mathrm{HM}}(Y,\mathfrak{s})$ and $\mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s})$. The generators $[\theta_i] \in \overline{\mathrm{HM}}(Y,\mathfrak{s})$ are graded by $\mathrm{gr}^{\mathbb{Q}}([\theta_i])$ and $\mathrm{gr}^{(2)}([\theta_i])$ if $i \geq 0$, and by $\mathrm{gr}^{\mathbb{Q}}([\theta_i]) - 1$ and $\mathrm{gr}^{(2)}([\theta_i]) - 1$ if $i < 0$. In all cases, the U–action decreases the $\mathbb{Q}$–grading by two and preserves the $\mathbb{Z}/2$–grading. We will use the $\mathbb{Z}/2$–grading to define various Lefschetz numbers, and use the $\mathbb{Q}$–grading to define the Frøyshov invariant.

Definition 2.5 The Frøyshov invariant $h(Y, \mathfrak{s})$ is defined as negative one-half of the lowest $\mathbb{Q}$–grading of elements in $\mathrm{im}(i_*)$, where i_* is the map (7). If X is a spin homology $S^1 \times S^3$ as in Theorem A, we define its Frøyshov invariant by the formula $h(X) = h(Y, \mathfrak{s})$, where $Y \subset X$ is an oriented rational homology sphere Poincaré dual to $1 \in H^1(X; \mathbb{Z})$, with the induced spin structure $\mathfrak{s}$. It follows from [12, Theorem 8] that $h(X)$ is well-defined.

The Frøyshov invariant $h(Y, \mathfrak{s})$ is an invariant of spin^c rational homology cobordism. It is also known [12, Theorem 3] that it changes sign with the change of orientation and that it is additive with respect to connected sums. A Heegaard Floer version of $h(X)$ was defined in [30] without the assumption that Y be a rational homology sphere.

We will conclude this section by computing the gradings of the generators $[\theta_i] \in \overline{\mathrm{HM}}(Y, \mathfrak{s})$. To this end, consider a smooth compact spin manifold Z with spin boundary Y and define

$$n(Y, h, \mathfrak{s}) = \mathrm{ind}\, \mathcal{D}^+(Z_\infty) + \tfrac{1}{8} \operatorname{sign} Z, \tag{9}$$

where $\mathcal{D}^+(Z_\infty)$ is the spin Dirac operator on the manifold Z_∞ with cylindrical end obtained by setting $X = S^1 \times Y$ in formula (2). In essence, $n(Y, h, \mathfrak{s})$ is a special case of the correction term (3) and, like the latter, it is independent of the arbitrary choices in its definition.

Lemma 2.6 *For any $i \geq 0$, we have $\mathrm{gr}^{(2)}([\theta_i]) = 0$ and $\mathrm{gr}^{\mathbb{Q}}([\theta_i]) = -2n(Y, h, \mathfrak{s}) + 2i$.*

Proof This follows directly from the definition of $\mathrm{gr}^{(2)}$ and $\mathrm{gr}^{\mathbb{Q}}$ (see the discussion at the end of [23, page 421]). The only nontrivial point is the use of Assumption 2.4(c) to ensure that $\mathcal{D}_{B_0,\mathfrak{q}}(Y, h)$ can be deformed into $\mathcal{D}_{B_0}(Y, h)$ without acquiring nonzero spectral flow, which allows one to compute the grading with zero perturbation. □

3 Proof of Theorem A

Our proof will use the neck-stretching operation, which is well-known in gauge theory; we will use its nonseparating version.

3.1 Manifolds with long necks

Let X be a spin rational homology $S^1 \times S^3$ and $Y \subset X$ a rational homology sphere Poincaré dual to the choice of homology orientation $1 \in H^1(X; \mathbb{Z})$. The spin structure

on X induces a spin structure $\mathfrak{s}$ on Y. Let h be a metric on Y satisfying Assumption 2.3, and extend it to a metric g on X which takes the form $g = dt^2 + h$ in a product region $[-\varepsilon, \varepsilon] \times Y$, with $\varepsilon > 0$. Given a real number $R > 0$, consider the spin manifold "with long neck"

$$X_R = W \cup ([0, R] \times Y) \tag{10}$$

obtained by cutting X open along $\{0\} \times Y$ and gluing in the cylinder $[0, R] \times Y$ along the two copies of Y. We also consider the noncompact manifold

$$W_\infty = ((-\infty, 0] \times Y) \cup W \cup ([0, +\infty) \times Y) \tag{11}$$

with two product ends. The metric g induces metrics on X_R and W_∞, which will be denoted by g_R and g_∞, respectively.

Assumption 3.1 *The metric g on X has the form $g = dt^2 + h$ in a product region $[-\varepsilon, \varepsilon] \times Y$, with $\varepsilon > 0$, and makes the spin Dirac operator*

$$\mathcal{D}^+(W_\infty, g_\infty)\colon L^2_1(W_\infty; \mathcal{S}^+) \to L^2(W_\infty; \mathcal{S}^-) \tag{12}$$

invertible.

The existence of metrics satisfying Assumption 3.1 will be proved in Theorem 10.3; see also Remark 10.4. Note that, once Assumption 3.1 is satisfied, the metric h that shows up in its statement will automatically satisfy Assumption 2.3.

The proof of Theorem A will rely on the following two theorems about manifolds with long necks, whose proofs occupy Sections 6–11.

Theorem 3.2 *Let g be a metric on X satisfying Assumption 3.1. Then, for all sufficiently large R, the correction term $w(X_R, g_R)$ in formula (4) is well-defined, and we have the equality* (see (9))

$$w(X_R, g_R) = n(Y, h, \mathfrak{s}).$$

Theorem 3.3 *Let g be a metric on X satisfying Assumption 3.1. Then, for all sufficiently large R and all sufficiently small perturbations $\widehat{\mathfrak{p}}_R$* (defined in Section 8.2) *which make the moduli space $\mathcal{M}(X_R, g_R, \widehat{\mathfrak{p}}_R)$ regular, we have the equality*

$$\#\mathcal{M}(X_R, g_R, \widehat{\mathfrak{p}}_R) = -\mathrm{Lef}(W_*\colon C^o \to C^o).$$

3.2 The proof

Let g be a metric on X satisfying Assumption 3.1. Fix a large number

$$N \in -2n(Y,h,\mathfrak{s}) + 2\mathbb{Z}$$

and consider the truncated monopole chain complex $(\check{C}_{\leq N}, \check{\partial}_{\leq N})$ for Y. It is generated by the irreducible critical points and the boundary stable reducible critical points of grading $\leq N$. Then we have a short exact sequence

$$0 \to C^s_{\leq N} \to \check{C}_{\leq N} \to C^o \to 0,$$

where the chain complex $C^s_{\leq N}$ is generated by the boundary stable reducible critical points of grading $\leq N$ and has trivial differential. By Lemma 2.6, each $m \in -2n(Y,h,\mathfrak{s}) + 2\mathbb{Z}$ with $-2n(Y,h,\mathfrak{s}) \leq m \leq N$ contributes a generator to $C^s_{\leq N}$. Therefore, we have

$$\dim(C^s_{\leq N}) = \tfrac{1}{2}(N + 2n(Y,h,\mathfrak{s})) + 1.$$

The cobordism W induces chain maps W_* on the three chain complexes in the above exact sequence making the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^s_{\leq N} & \longrightarrow & \check{C}_{\leq N} & \longrightarrow & C^o & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle W_*} & & \downarrow{\scriptstyle W_*} & & \downarrow{\scriptstyle W_*} & & \\
0 & \longrightarrow & C^s_{\leq N} & \longrightarrow & \check{C}_{\leq N} & \longrightarrow & C^o & \longrightarrow & 0
\end{array}$$

With the obvious abuse of notation, the Lefschetz numbers of the three maps W_* in this diagram are therefore related by the equation

$$\mathrm{Lef}(\check{C}_{\leq N}) = \mathrm{Lef}(C^s_{\leq N}) + \mathrm{Lef}(C^o).$$

Lemma 3.4 *The restriction of W_* to the chain complex $C^s_{\leq N}$ is the identity map.*

Proof This is essentially proved in [23, Proposition 39.1.2]. The result is stated there for homology but it holds as well for the chain complex because the boundary map is trivial for grading reasons. □

By Lemma 2.6, the $\mathbb{Z}/2$–gradings of the generators of $C^s_{\leq N}$ are all zero. Then Lemma 3.4 implies that $\mathrm{Lef}(C^s_{\leq N}) = \dim(C^s_{\leq N})$ and therefore

$$\mathrm{Lef}(\check{C}_{\leq N}) = \tfrac{1}{2}(N + 2n(Y,h,\mathfrak{s})) + 1 + \mathrm{Lef}(C^o). \tag{13}$$

On the other hand, for all sufficiently large N, the group $\mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s})$ can be identified with the cokernel of the map

$$i_{\le N}\colon \overline{\mathrm{HM}}_{\le N}(Y,\mathfrak{s}) \to \widetilde{\mathrm{HM}}_{\le N}(Y,\mathfrak{s}).$$

Therefore, we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{im}(i_{\le N}) & \longrightarrow & \widetilde{\mathrm{HM}}_{\le N}(Y,\mathfrak{s}) & \longrightarrow & \mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle W_*} & & \downarrow{\scriptstyle W_*} & & \downarrow{\scriptstyle W_*} & & \\
0 & \longrightarrow & \mathrm{im}(i_{\le N}) & \longrightarrow & \widetilde{\mathrm{HM}}_{\le N}(Y,\mathfrak{s}) & \longrightarrow & \mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s}) & \longrightarrow & 0
\end{array}$$

with exact rows, which gives us the identity

$$\mathrm{Lef}(\widetilde{\mathrm{HM}}_{\le N}(Y,\mathfrak{s})) = \mathrm{Lef}(\mathrm{im}(i_{\le N})) + \mathrm{Lef}(\mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s})).$$

Since $\mathrm{im}(i_{\le N})$ is a finite-length U–tail whose top grading is N and lowest grading is $-2h(Y,\mathfrak{s})$, we have

$$\dim(\mathrm{im}(i_{\le N})) = \tfrac{1}{2}(N + 2h(Y,\mathfrak{s})) + 1.$$

The restriction of W_* on $\mathrm{im}(i_{\le N})$ is the identity map by Lemma 3.4; therefore, $\mathrm{Lef}(\mathrm{im}(i_{\le N})) = \dim(\mathrm{im}(i_{\le N}))$ and

$$\mathrm{Lef}(\widetilde{\mathrm{HM}}_{\le N}(Y,\mathfrak{s})) = \tfrac{1}{2}(N + 2h(Y,\mathfrak{s})) + 1 + \mathrm{Lef}(\mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s})). \tag{14}$$

Combining (13) and (14) with the fact that the Lefschetz number of a chain map equals the Lefschetz number of the induced map on homology, we obtain the identity

$$\tfrac{1}{2}(N + 2n(Y,h,\mathfrak{s})) + 1 + \mathrm{Lef}(C^o) = \tfrac{1}{2}(N + 2h(Y,\mathfrak{s})) + 1 + \mathrm{Lef}(\mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s}))$$

and, after simplification,

$$-\mathrm{Lef}(C^o) - n(Y,h,\mathfrak{s}) + h(Y,\mathfrak{s}) = -\mathrm{Lef}(\mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s})).$$

The proof is now complete because it follows from Theorems 3.2 and 3.3 that, for all sufficiently large R and the small perturbation $\widehat{\mathfrak{p}}_R$,

$$\lambda_{\mathrm{SW}}(X) = \#\mathcal{M}(X_R, g_R, \widehat{\mathfrak{p}}_R) - w(X_R, g_R) = -\mathrm{Lef}(C^o) - n(Y,h,\mathfrak{s}).$$

4 Proof of Theorem B

In this section we prove Theorem B, which is an application of Theorem A to the question of existence of metrics of positive scalar curvature.

Let X be as in the statement of Theorem B and suppose that it admits a metric of positive scalar curvature. According to a theorem of Schoen and Yau [60], the Poincaré dual to the generator of $H^1(X;\mathbb{Z})$ can be realized by an embedded manifold $M \subset X$ which admits a metric of positive scalar curvature. Since the first Chern class of the spin structure $\mathfrak{s}_M$ induced on M vanishes, it follows from [23, Proposition 36.1.3] that $\mathrm{HM}^{\mathrm{red}}(M, \mathfrak{s}_M) = 0$. Note that the manifold M need not be a rational homology sphere; however, we will prove that its existence implies that

$$\mathrm{Lef}(W_*\colon \mathrm{HM}^{\mathrm{red}}(Y,s) \to \mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s})) = 0.$$

Our proof will adapt the argument of Frøyshov [12, Section 13] that shows the well-definedness of the Lefschetz number. Since M generates $H_3(X;\mathbb{Z})$, the standard covering space theory implies that the manifold M lifts to the infinite cyclic cover $\widetilde{X}$, and that this lift can be arranged to be disjoint from a copy of Y. It follows that, for some $k \geq 0$, the manifold

$$W^{(k)} = W \cup_Y W \cup_Y \cdots \cup_Y W \quad (k \text{ times})$$

contains a copy of M separating its two boundary components. Therefore, the map

$$(W_*)^k = W_*^{(k)}\colon \mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s}) \to \mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s})$$

factors through $\mathrm{HM}^{\mathrm{red}}(M, \mathfrak{s}_M) = 0$ making W_* nilpotent. It then follows that the trace of W_* vanishes in each $\mathbb{Z}/2$–grading, and that the Lefschetz number of W_* must therefore be zero.

5 Proof of Theorems C and D

We now prove Theorems C and D from the introduction, which assert that, in a number of circumstances, a homology sphere must have infinite order in the homology cobordism groups $\Theta^3_{\mathbb{Z}}$ or $\Theta^3_{\mathbb{Z}/2}/\Theta^3_L$. The proofs can be found at the end of Section 5.1. The part of Section 5 after that is dedicated to examples and the proofs of Corollaries E and F.

5.1 A homology cobordism obstruction from $\mathrm{HM}^{\mathrm{red}}$

Let $(Y,\mathfrak{s})$ be a rational homology sphere with a spin^c structure $\mathfrak{s}$. Then $\mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s})$ is graded by the rational numbers, and we define the support of $\mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s})$ by the formula

$$S(Y,\mathfrak{s}) = \{a \in \mathbb{Q} \mid \mathrm{HM}^{\mathrm{red}}_a(Y,\mathfrak{s}) \neq 0\}.$$

For the rest of this section, whenever we are considering the unique spin structure $\mathfrak{s}$ on a $\mathbb{Z}/2$ homology sphere Y, we will usually drop $\mathfrak{s}$ from our notations. In particular, the notations $\rho(Y)$, $S(Y)$ and $h(Y)$ will be used, respectively, to denote the Rokhlin invariant, the support of reduced monopole Floer homology and the Frøyshov invariant for the unique spin structure.

Proposition 5.1 *Let Y_1 and Y_2 be $\mathbb{Z}/2$ homology spheres such that $S(Y_1)\cap S(Y_2)=\varnothing$ and $h(Y_1)\neq\rho(Y_1) \pmod 2$. Then Y_1 is not $\mathbb{Z}/2$ homology cobordant to Y_2.*

Proof Suppose to the contrary that we have an $\mathbb{Z}/2$ homology cobordism W_1 from Y_1 to Y_2. It carries a unique spin structure, which restricts to $\mathfrak{s}_i$ on Y_i. Reversing the orientation, we obtain a spin cobordism $-W_1$ from Y_2 to Y_1. Now, consider the composite cobordism

$$W = W_1 \cup_{Y_2} (-W_1)$$

from Y_1 to itself. The morphism

$$(W_1)_*\colon \mathrm{HM}^{\mathrm{red}}(Y_1,\mathfrak{s}_1)\to \mathrm{HM}^{\mathrm{red}}(Y_2,\mathfrak{s}_2)$$

induced by the cobordism W_1 as in (8) preserves the absolute grading. Since the intersection $S(Y_1)\cap S(Y_2)$ is empty, we conclude that the map $(W_1)_*$ must be zero. By functoriality, the map

$$W_*\colon \mathrm{HM}^{\mathrm{red}}(Y_1,\mathfrak{s}_1)\to \mathrm{HM}^{\mathrm{red}}(Y_1,\mathfrak{s}_1)$$

is also zero; in particular, its Lefschetz number vanishes. Let X be the homology $S^1\times S^3$ obtained by identifying the two boundary components of W via the identity map. Then, using Theorem A, we obtain

$$\lambda_{\mathrm{SW}}(X) = -h(Y_1) - \mathrm{Lef}(W_*) = -h(Y_1)\neq \rho(Y_1) \pmod 2.$$

This contradicts [41, Theorem A], which asserts that $\lambda_{\mathrm{SW}}(X)$ equals the Rokhlin invariant of X modulo 2. □

Remark 5.2 For a $\mathbb{Z}/2$ homology sphere Y with spin structure $\mathfrak{s}$, the condition $h(Y)\neq\rho(Y) \pmod 2$ is equivalent to the condition that $\dim_{\mathbb{Q}} \mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s})$ is odd.

The following definition is a slight generalization of the one given in the introduction.

Definition 5.3 Let Y be a rational homology sphere with spin^c–structure $\mathfrak{s}$.

- $(Y,\mathfrak{s})$ is *h–positive* if $\mathrm{HM}^{\mathrm{red}}(Y,\mathfrak{s})$ is supported in degrees $\geq -2h(Y,\mathfrak{s})$, that is, $S(Y,\mathfrak{s})\subset[-2h(Y,\mathfrak{s}),+\infty)$.

- $(Y, \mathfrak{s})$ is *h–negative* if $\mathrm{HM}^{\mathrm{red}}(Y, \mathfrak{s})$ is supported in degrees $\leq -2h(Y, \mathfrak{s}) - 1$, that is, $S(Y, s) \subset (-\infty, -2h(Y, \mathfrak{s}) - 1]$.

Because Heegaard Floer homology is (at present) easier to compute, we would prefer to use it in place of the monopole Floer homology in our calculations whenever possible. In fact, these two theories are known to be isomorphic. Furthermore, by combining the main results of [52; 20; 9], the absolute $\mathbb{Q}$–gradings in the two theories coincide. Therefore, the relation $d(Y, \mathfrak{s}) = -2h(Y, \mathfrak{s})$ between the Frøyshov invariant and the Heegaard Floer correction term holds for any rational homology sphere. It then follows that the property of $(Y, \mathfrak{s})$ being h–positive (resp. h–negative) can be characterized by saying that the reduced Heegaard Floer homology $\mathrm{HF}^{\mathrm{red}}(Y, \mathfrak{s})$ is supported in degrees $\geq d(Y, \mathfrak{s})$ (resp. in degrees $\leq d(Y, \mathfrak{s}) - 1$).

Example 5.4 By a positive orientation on a Seifert fibered homology sphere we will mean the canonical orientation of it as a link of singularity. Using the graded roots model for computing Heegaard Floer homology [43, Section 11.13], one can show that all Seifert fibered homology spheres with positive orientation are h–negative, while the ones with negative orientation are h–positive. According to [45, Proposition 8.3], the homology sphere obtained by $1/n$ surgery on the figure-eight knot is h–negative if $n \geq 0$ and h–positive if $n \leq 0$. (Note that L–spaces are both h–positive and h–negative.)

Lemma 5.5 *A $\mathbb{Z}/2$ homology sphere Y is h–positive if and only if the $\mathbb{Z}/2$ homology sphere $-Y$ obtained from Y by orientation reversal is h–negative.*

Proof We use $\mathfrak{s}$ and $-\mathfrak{s}$ to denote the spin structure on Y and $-Y$, respectively. Recall from [23, (3.4)] that there is a long exact sequence[1]

$$\cdots \to \widehat{\mathrm{HM}}_{a+1}(Y, \mathfrak{s}) \xrightarrow{p_{Y,a}} \overline{\mathrm{HM}}_a(Y, \mathfrak{s}) \xrightarrow{i_{Y,a}} \widetilde{\mathrm{HM}}_a(Y, \mathfrak{s}) \xrightarrow{j_{Y,a}} \widehat{\mathrm{HM}}_a(Y, \mathfrak{s}) \to \cdots;$$

therefore, the set $S(Y)$ can be equivalently defined as $S(Y) = \{a \in \mathbb{Q} \mid j_{Y,a} \neq 0\}$. Under the natural duality isomorphisms

$$\widetilde{\mathrm{HM}}_a(Y, \mathfrak{s}) \cong (\widehat{\mathrm{HM}}_{-1-a}(-Y, -\mathfrak{s}))^*,$$
$$\widehat{\mathrm{HM}}_a(Y, \mathfrak{s}) \cong (\widetilde{\mathrm{HM}}_{-1-a}(-Y, -\mathfrak{s}))^*,$$

[1] While the grading conventions for $\widetilde{\mathrm{HM}}$ and $\overline{\mathrm{HM}}$ are consistent with those for HF^+ and HF^∞, respectively, the grading convention for $\widehat{\mathrm{HM}}$ differs from that for HF^- by 1. For example, the generator of $\widehat{\mathrm{HM}}(S^3)$ as a $\mathbb{Q}[U]$–module has grading -1, while the generator of $\mathrm{HF}^-(S^3)$ has grading -2. We follow here the conventions of [24].

the dual map

$$(j_{Y,a})^*\colon \widehat{\mathrm{HM}}_{-1-a}(-Y,-\mathfrak{s}) \to \widecheck{\mathrm{HM}}_{-1-a}(Y,-\mathfrak{s})$$

is exactly the map $j_{-Y,-1-a}$; therefore, $S(-Y) = \{-1-a \mid a \in S(Y)\}$. The result now follows because $h(Y) = -h(-Y)$. □

Lemma 5.6 *For $\mathbb{Z}/2$ homology spheres Y_1 and Y_2, the connected sum $Y_1 \# Y_2$ is h–positive (resp. h–negative) if Y_1 and Y_2 are both h–positive (resp. h–negative).*

Proof Because of Lemma 5.5, we only need to treat the h–negative case. Let us introduce the notations

$$\mathcal{T}_a(b) = (\mathbb{Q}[U]/U^b)[-a] \quad \text{and} \quad \mathcal{T}_a^- = (\mathbb{Q}[U])[-a], \tag{15}$$

where $1 \in \mathbb{Q}[U]$ has degree 0 and, for a graded module M, we follow the convention $(M[c])_k = M_{k+c}$ for the grading shift. Let $\mathfrak{s}_j$ for $j = 1, 2$ be the spin structure on Y_j. For both Y_1 and Y_2, we have a (noncanonical) splitting of the $\mathbb{Q}[U]$–modules,

$$\begin{aligned}\widehat{\mathrm{HM}}(Y_1,\mathfrak{s}_1) &= \mathcal{T}^-_{-2h(Y_1)-1} \oplus \mathcal{T}_{a_1}(b_1) \oplus \cdots \oplus \mathcal{T}_{a_k}(b_k),\\ \widehat{\mathrm{HM}}(Y_2,\mathfrak{s}_2) &= \mathcal{T}^-_{-2h(Y_2)-1} \oplus \mathcal{T}_{c_1}(d_1) \oplus \cdots \oplus \mathcal{T}_{c_\ell}(b_\ell).\end{aligned}$$

By combining the connected sum formula for Heegaard Floer homology [48, Proposition 6.2] with the identification between monopole Floer homology and Heegaard Floer homology (or alternatively, by directly using the connected sum formula in [5; 3]) we obtain[2]

$$\begin{aligned}&\widehat{\mathrm{HM}}(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2)\\ &= (\widehat{\mathrm{HM}}(Y_1,\mathfrak{s}_1) \otimes_{\mathbb{Q}[U]} \widehat{\mathrm{HM}}(Y_2,\mathfrak{s}_2))[-1] \oplus \big(\mathrm{Tor}_{\mathbb{Q}[U]}(\widehat{\mathrm{HM}}(Y_1,\mathfrak{s}_1), \widehat{\mathrm{HM}}(Y_2,\mathfrak{s}_2))\big)[-2].\end{aligned}$$

We will now trace the contributions of each of the summands of $\widehat{\mathrm{HM}}(Y_1)$ and $\widehat{\mathrm{HM}}(Y_2)$ to $\widehat{\mathrm{HM}}(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2)$:

- The tensor product

$$(\mathcal{T}^-_{-2h(Y_1)-1} \otimes_{\mathbb{Q}[U]} \mathcal{T}^-_{-2h(Y_2)-1})[-1] = \mathcal{T}^-_{-2h(Y_1)-2h(Y_2)-1}$$

  contributes the infinite U–tail to $\widehat{\mathrm{HM}}(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2)$.

[2] Only the relatively graded version of this formula appears in [48]. One obtains the absolutely graded version with the help of the Frøyshov invariant, which is additive under connected sum.

- Each of the tensor products

$$\begin{aligned}(\mathcal{T}_{a_i}(b_i)\otimes_{\mathbb{Q}[U]}\mathcal{T}^-_{-2h(Y_2)-1})[-1]&=\mathcal{T}_{-2h(Y_2)+a_i}(b_i),\\ (\mathcal{T}^-_{-2h(Y_1)-1}\otimes_{\mathbb{Q}[U]}\mathcal{T}_{c_j}(d_j))[-1]&=\mathcal{T}_{-2h(Y_1)+c_j}(d_j),\\ (\mathcal{T}_{a_i}(b_i)\otimes_{\mathbb{Q}[U]}\mathcal{T}_{c_j}(d_j))[-1]&=\mathcal{T}_{a_i+c_j+1}(\min(b_i,d_j))\end{aligned}$$

  contributes a summand to the kernel of the map $p_{Y_1\#Y_2,*}$.

- To compute $\operatorname{Tor}(\mathcal{T}_{a_i}(b_i)\otimes_{\mathbb{Q}[U]}\mathcal{T}_{c_j}(d_j))[-2]$, consider the *grading-preserving* free resolution

$$0\to\mathcal{T}^-_{a_i-2b_i}\xrightarrow{U^{b_i}}\mathcal{T}^-_{a_i}\to\mathcal{T}_{a_i}(b_i)\to 0.$$

  By taking tensor product with $\mathcal{T}_{c_j}(d_j)[-2]$ and computing homology of the resulting chain complex, we obtain

$$\operatorname{Tor}(\mathcal{T}_{a_i}(b_i)\otimes_{\mathbb{Q}[U]}\mathcal{T}_{c_j}(d_j))[-2]=\mathcal{T}_{a_i+c_j+2-2\max(b_i,d_j)}(\min(b_i,d_j)),$$

  which contributes another summand to the kernel of $p_{Y_1\#Y_2,*}$.

Since Y_1 and Y_2 are both h–negative, we have $a_i\le -2h(Y_1)-1$ and $c_j\le -2h(Y_2)-1$. Also note that $b_i,d_j\ge 1$. It is now easy to check that all the summands in $\ker p_{Y_1\#Y_2,*}$ are supported in degrees at most $-2h(Y_1)-2h(Y_2)-1=-2h(Y_1\#Y_2)-1$. Therefore, $Y_1\#Y_2$ is h–negative, and the lemma is proved. □

Corollary 5.7 *Let Y_1 be an h–positive (resp. h–negative) $\mathbb{Z}/2$ homology sphere, and suppose that $h(Y_1)\neq\rho(Y_1)\pmod 2$. Then Y_1 is not $\mathbb{Z}/2$ homology cobordant to any h–negative (resp. h–positive) $\mathbb{Z}/2$ homology sphere Y_2.*

Proof Suppose to the contrary that Y_1 is $\mathbb{Z}/2$ homology cobordant to an h–negative $\mathbb{Z}/2$ homology sphere Y_2. Since both h and ρ are invariants of $\mathbb{Z}/2$ homology cobordism,

$$h(Y_1)=h(Y_2)\quad\text{and}\quad\rho(Y_1)=\rho(Y_2)\pmod 2$$

and, in particular, $h(Y_2)\neq\rho(Y_2)\pmod 2$. Let $h=h(Y_1)=h(Y_2)$; then $S(Y_1)\subset[-2h,+\infty)$ and $S(Y_2)\subset(-\infty,-2h-1]$, so that $S(Y_1)\cap S(Y_2)=\varnothing$. This contradicts Proposition 5.1. □

Corollary 5.8 *Let $Y_1,\dots,Y_n$ be h–positive $\mathbb{Z}/2$ homology spheres, and $a_1,\dots,a_n$ positive integers. Suppose that at least one of the Y_j satisfies the condition $h(Y_j)\neq$*

$\rho(Y_j) \pmod 2$. *Then the connected sum*

$$(\#_{a_1} Y_1) \# \cdots \# (\#_{a_n} Y_n) \tag{16}$$

cannot be $\mathbb{Z}/2$ homology cobordant to S^3. A similar result holds for h–negative homology spheres.

Proof Suppose to the contrary that the connected sum (16) is $\mathbb{Z}/2$ homology cobordant to S^3. Without loss of generality, we may assume that $h(Y_1) \neq \rho(Y_1) \pmod 2$. Then the manifold $-Y_1$, which is h–negative by Lemma 5.5, is $\mathbb{Z}/2$ homology cobordant to the manifold

$$(\#_{a_1-1} Y_1) \# \cdots \# (\#_{a_n} Y_n),$$

which is h–positive by Lemma 5.6. This contradicts Corollary 5.7. □

Proof of Theorem C Suppose to the contrary that Y is of finite order in $\Theta^3_{\mathbb{Z}}$. Then $h(Y) = 0 \neq \rho(Y) \pmod 2$. This contradicts Corollary 5.8, since a $\mathbb{Z}$ homology cobordism is also a $\mathbb{Z}/2$ homology cobordism. □

Proof of Theorem D Suppose to the contrary that Y is of finite order in $\Theta^3_{\mathbb{Z}/2}/\Theta^3_L$. Then there exists an integer $n > 0$ and an L–space Y_1 which is a $\mathbb{Z}/2$ homology sphere, such that $(\#_n Y) \# Y_1$ is $\mathbb{Z}/2$ homology cobordant to S^3. This contradicts Corollary 5.8, since Y_1 is both h–positive and h–negative. □

Proof of Corollary F Using the spectral sequences of Ozsváth and Szabó [49] and Bloom [4] one can easily see that, for any Khovanov-homology thin knot K_1, the double branched cover $\Sigma(K_1)$ is an L–space over $\mathbb{Z}/2$. The universal coefficient theorem then implies that $\Sigma(K_1)$ is also an L–space over $\mathbb{Q}$. Note that the double branched cover of $I \times S^3$ with branch set a smooth concordance between two knots is a $\mathbb{Z}/2$ homology cobordism between the double branched covers of the knots. The result now follows from Theorem D. □

5.2 Surgery on knots

In this section, we will use the rational surgery formula of Ozsváth and Szabó [51] to obtain a sufficient condition for a surgered manifold to be h–positive. Corollary E will be proved in the next section by checking this condition and applying Theorem C.

Let K be a knot in S^3. Given coprime integers p and q, denote by $S^3_{p/q}(K)$ the manifold obtained by the p/q surgery on K. For any $p > 0$, the manifold $S^3_p(K)$ is

a rational homology sphere. It admits exactly p distinct spin^c structures, which can be naturally identified [50] with the elements of $\mathbb{Z}/p$. The spin^c structure on $S^3_p(K)$ corresponding to an integer s will be denoted by $[s]$.

Theorem 5.9 (Ozsváth and Szabó [47]; Rasmussen [53]) *For all sufficiently large $p>0$ and all $s\in\mathbb{Z}$ with $|s|\le \frac{1}{2}p$ the Heegaard Floer homology group $\mathrm{HF}^+(S^3_p(K),[s])$, viewed as a relatively graded $\mathbb{Q}[U]$–module, is independent of p.*

Proposition 5.10 *Suppose that, for all sufficiently large $p>0$ and all $s\in\mathbb{Z}$ with $|s|\le \frac{1}{2}p$, the rational homology sphere $(S^3_p(K),[s])$ is h–positive. Then, for any positive integer n, the integral homology sphere $S^3_{1/n}(K)$ is h–positive.*

Proof This is a straightforward corollary of Ozsváth and Szabó's rational surgery formula [51]. For the sake of completeness, we will sketch the argument here and refer the reader to [44] for a concise summary. (See also [19], which treats a similar situation as here).

For a sufficiently large $p>0$ and any $s\in\mathbb{Z}$, consider n copies $A^+_{s,1},\dots,A^+_{s,n}$ of the Heegaard Floer homology $\mathrm{HF}^+(S^3_p(K),[s])$ (this notation is justified by Theorem 5.9), and n copies $B^+_{s,1},\dots,B^+_{s,n}$ of the module $\mathcal{T}^+=\mathbb{Q}[U,U^{-1}]/(U\cdot\mathbb{Q}[U])$. By [51, Theorem 1.1] and [19, Remark 2.3], one can recover the Heegaard Floer homology $\mathrm{HF}^+(S^3_{1/n}(K))$ as the homology of the mapping cone of a certain map

$$\Phi_{1/n}\colon \Bigg(\bigoplus_{s\in\mathbb{Z},\,1\le i\le n} A^+_{s,i}\Bigg)\to\Bigg(\bigoplus_{s\in\mathbb{Z},\,1\le i\le n} B^+_{s,i}\Bigg).$$

In practice, one can take a large enough integer N and instead consider the mapping cone of the truncated map

$$\Phi^N_{1/n}\colon A^+_{-N-1,n}\oplus\Bigg(\bigoplus_{-N\le s\le N,\,1\le i\le n} A^+_{s,i}\Bigg)\to\Bigg(\bigoplus_{-N\le s\le N,\ 1\le i\le n} B^+_{s,i}\Bigg).$$

This map is surjective, so one has an isomorphism

$$\ker\Phi^N_{1/n}\cong\mathrm{HF}^+(S^3_{1/n}(K)).$$

Furthermore, one can impose suitable absolute gradings on $A^+_{s,i}$ such that the above isomorphism preserves the absolute grading. Recall that $A^+_{s,i}$ admits a splitting $A^+_{s,i}=\mathcal{T}^+\oplus A^{\mathrm{red}}_{s,i}$. Let $a_{s,i}$ be the absolute grading of the bottom term in $\mathcal{T}^+\subset A^+_{s,i}$.

Then

$$a_{0,i} = d(S^3_{1/n}(K)) \quad \text{for all } 1 \le i \le n,$$
$$a_{s,i} \ge d(S^3_{1/n}(K)) \quad \text{for all } s \in \mathbb{Z} \text{ and all } 1 \le i \le n,$$

by [19, (2.1)–(2.4)]. Using the assumption that the spin^c structure $[s]$ is h–positive for all sufficiently large $p > 0$, we conclude that $A^+_{s,i}$ is supported in degrees $\ge a_{s,i}$. This implies that $\ker \Phi^N_{1/n}$ is supported in degrees greater than or equal to

$$\inf_{s,i}(a_{s,i}) = d(S^3_{1/n}(K)).$$

Therefore, the integral homology sphere $S^3_{1/n}(K)$ is h–positive. □

5.3 L–space knots and thin knots

In this section, we will apply Proposition 5.10 to the classes of L–space knots and Floer homology thin knots, and then prove Corollary E.

Recall that a knot K is called an L–space knot (over the rationals) if there is a rational number $p/q > 0$ such that the manifold $S^3_{p/q}(K)$ is an L–space, that is,

$$\widehat{\mathrm{HF}}_{p/q}(S^3_{p/q}(K)) = \mathbb{Q}^p.$$

This condition actually implies that $S^3_{p'/q'}(K)$ is an L–space for all $p'/q' \ge p/q$. Proposition 5.10 has the following corollary:

Corollary 5.11 *Let K be an L–space knot. Then $S^3_{1/n}(K)$ is h–positive for all $n > 0$.*

Remark 5.12 Using Corollary 5.11, one can derive a result similar to Corollary E for all L–space knots. However, this can be proved directly using the Heegaard Floer correction term.

Now we turn to the case of surgeries on Floer homology thin knots. We will need a number of constructions involving Heegaard Floer homology of knots, for which we refer to the original paper [47], as well as to the survey [37].

Recall that, for an even integer τ, a knot K is called Floer homology τ–thin (over the rationals) if the bigraded knot Floer homology group $\widehat{\mathrm{HFK}}_*(K, *)$ satisfies the condition

$$\widehat{\mathrm{HFK}}_i(K, j) = 0 \quad \text{unless } i = j + \tfrac{1}{2}\tau. \tag{17}$$

The following lemma summarizes properties of thin knots that are useful for the application we have in mind.

Lemma 5.13 *Let τ be an even integer, K a Floer homology τ–thin knot and n a positive integer. Then:*

(1) $d(S^3_{1/n}(K)) = 2\min\big(0, -\lceil -\frac{1}{4}\tau \rceil\big)$. *In particular, $d(S^3_{1/n}(K)) \neq 0$ if and only if $\tau < 0$.*

(2) *If $\tau > 0$ then for all sufficiently large $p > 0$ and all integers s such that $|s| \leq \frac{1}{2}p$, the rational homology sphere $(S^3_p(K), [s])$ is h–positive.*

Proof Recall that the knot Floer complexes are generated by triples $[x, i, j]$ satisfying various conditions, where i and j are integers and x is an intersection point between Lagrangian tori in the symmetric product of the Heegaard surface. For any $a, b \in \mathbb{Z}$, we will denote by $C\{i = a, j = b\}$ the complex generated by triples $[x, i, j]$ with $i = a$ and $j = b$. We will use similar notations for the other complexes. It follows from (17) that $H(C\{i = a, j = b\})$ is supported in degree (absolute Maslov grading) $a + b + \frac{1}{2}\tau$. Via a basic spectral sequence argument, this implies that

$$\begin{aligned} H(C\{\min(i, j - s)\} \geq 0) &\quad \text{is supported in degrees} \geq s + \tfrac{1}{2}\tau, \\ H(C\{\max(i, j - s)\} \leq -1) &\quad \text{is supported in degrees} \leq s + \tfrac{1}{2}\tau - 2. \end{aligned}$$

With these two facts established, we can prove that

$$d(S^3_1(K)) = 2\min\big(0, -\lceil -\tfrac{1}{4}\tau \rceil\big)$$

by repeating word for word the proof of [46, Corollary 1.5] (which deals with the special case of an alternating knot K). Since $d(S^3_{1/n}(K)) = d(S^3_1(K))$ for any $n > 0$ (see [44, Proposition 1.6]), claim (1) is proved.

We now turn to claim (2). Since the spin^c structures $[s]$ and $[-s]$ are conjugate to each other, one has an isomorphism $\mathrm{HF}^+(S^3_p(K), [s]) \cong \mathrm{HF}^+(S^3_p(K), [-s])$. Therefore, it is sufficient to consider the case of $s \geq 0$. Recall from [47; 53] that there is an isomorphism

$$\mathrm{HF}^+(S^3_p(K), [s]) \cong H(C\{\max(i, j - s) \geq 0\})$$

of relatively graded $\mathbb{Q}[U]$–modules. For any integer a, denote by $\mathcal{T}^+_a$ the graded module $(\mathbb{Q}[U, U^{-1}]/U \cdot \mathbb{Q}[U])[-a]$ (cf (15)). Then we have a decomposition of absolutely graded $\mathbb{Q}[U]$–modules,

$$H(C\{\max(i, j - s) \geq 0\}) \cong \mathcal{T}^+_e \oplus V$$

for some integer e and a finite-dimensional $\mathbb{Q}$–vector space V. Consider the short exact sequence

$$0 \to C\{\max(i, j-s) \geq 0\} \to C\{i \geq 0\} \oplus C\{j \geq s\} \to C\{\min(i, j-s) \geq 0\} \to 0.$$

Since $H(C\{\min(i, j-s) \geq 0)$ is supported in degrees $\geq s + \frac{1}{2}\tau \geq 1$, we obtain

$$H_{\leq -1}\big(C\{\max(i, j-s) \geq 0\}\big) \cong H_{\leq -1}(C\{i \geq 0\}) \oplus H_{\leq -1}(C\{j \geq s\}) = 0,$$

with the last equality following from the isomorphisms

$$H(C\{i \geq 0\}) \cong \mathrm{HF}^+(S^3) \cong \mathcal{T}_0^+ \quad \text{and} \quad H(C\{j \geq s\}) \cong \mathrm{HF}^+(S^3)[-2s] \cong \mathcal{T}_{2s}^+.$$

Therefore, $H\big(C\{\max(i, j-s) \geq 0\}\big)$ is supported in degrees ≥ 0. The proof will be complete once we show that $e = 0$. To this end, consider another short exact sequence

$$0 \to C\{\min(-1-i, j-s) \geq 0\} \to C\{\max(i, j-s) \geq 0\} \xrightarrow{p} C\{i \geq 0\} \to 0.$$

Since $C\{\max(-1-i, j-s) \geq 0\}$ is finite-dimensional, for any sufficiently large integer m we have isomorphisms

$$p_*\colon H_{2m}\big(C\{\min(i, j-s) \geq 0\}\big) \cong H_{2m}(C\{i \geq 0\}) \cong \mathbb{Q}.$$

Let $\xi \in H_{2m}\big(C\{\min(i, j-s) \geq 0\}\big)$ be any nonzero element. Since $H(C\{i \geq 0\}) \cong \mathcal{T}_0^+$, we have $p_*(U^m\xi) = U^m p_*(\xi) \neq 0$. This implies that $U^m\xi \neq 0$ and

$$e = 2m - 2 \cdot \max\{n \in \mathbb{Z} \mid U^n\xi \neq 0\} \leq 2m - 2m = 0.$$

Because $H\big(C\{\max(i, j-s) \geq 0\}\big)$ is supported in degrees ≥ 0, we conclude that $e = 0$, which completes the proof. □

Corollary 5.14 *Let K be a Floer homology τ–thin knot with $\tau > 0$. Then $S^3_{1/n}(K)$ is h–positive for all $n > 0$.*

Proof This is immediate from Lemma 5.13(2) and Proposition 5.10. □

Proof of Corollary E Since $\mathrm{Arf}(K) = 1$ and n is odd, it follows from the surgery formula for the Rokhlin invariant [16; 59] that $\rho(S^3_{1/n}(K)) = 1 \pmod 2$. Claim (1) now follows from Example 5.4. To prove claim (2), consider the mirror image $\overline{K}$ of the knot K. Since $\overline{K}$ is quasialternating and $S^3_{-1/n}(K) = -S^3_{1/n}(\overline{K})$, it is sufficient to consider the case of $n > 0$. According to [39], any quasialternating knot K is Floer homology $\sigma(K)$–thin over $\mathbb{Z}/2$, where $\sigma(K)$ stands for the knot signature. By the universal coefficient theorem, this implies that K is also Floer homology $\sigma(K)$–thin

over $\mathbb{Q}$. If $\sigma(K) < 0$, it follows from Lemma 5.13(1) that $d(S^3_{1/n}(K)) \neq 0$ and hence $S^3_{1/n}(K)$ has infinite order in $\Theta^3_{\mathbb{Z}}$. If $\sigma(K) > 0$, it follows from Corollary 5.14 that $S^3_{1/n}(K)$ is h–positive and hence it has infinite order in $\Theta^3_{\mathbb{Z}}$. □

6 The correction term

In this section, we will prove Theorem 3.2. The index theory that will go into our proof is not specific to dimension four; therefore, we will work in more generality than strictly necessary.

Let X be a connected smooth spin compact manifold of dimension $n \equiv 0 \pmod 4$ with a primitive cohomology class $\gamma \in H^1(X;\mathbb{Z})$. Let $Y \subset X$ be a connected manifold Poincaré dual to γ with the induced spin structure $\mathfrak{s}$. Choose a metric g on X which takes the form $g = dt^2 + h$ in a product region $[-\varepsilon, \varepsilon] \times Y$, with $\varepsilon > 0$. We will assume that $(Y, \mathfrak{s})$ is a spin boundary and that the $\widehat{A}$–genus of X vanishes; both of these conditions are automatic when X is a homology $S^1 \times S^3$. Given a real number $R > 0$, construct the spin manifold

$$X_R = W \cup ([0, R] \times Y) \tag{18}$$

as in (10) by cutting X open along $\{0\} \times Y$ and gluing in the cylinder $[0, R] \times Y$ along the two copies of Y. The metric g defines a metric g_R on X_R, which lifts to a metric g_R on the infinite cyclic cover of X_R determined by γ. Following (2), denote by $Z_\infty(X_R)$ the manifold with *periodic* end modeled on this infinite cyclic cover, and by Z_∞ and W_∞ the manifolds with *product* ends modeled on the product $\mathbb{R} \times Y$ with metric $dt^2 + h$. Note that W_∞ has two ends, corresponding to the two boundary components of W. The metrics will often be suppressed in our notations.

Theorem 6.1 *Assume that the spin Dirac operator*

$$\mathcal{D}^+(W_\infty)\colon L^2_1(W_\infty; \mathcal{S}^+) \to L^2(W_\infty; \mathcal{S}^-)$$

is an isomorphism. Then, for all sufficiently large R, the end-periodic operator $\mathcal{D}^+(Z_\infty(X_R))$ is Fredholm of index

$$\operatorname{ind} \mathcal{D}^+(Z_\infty(X_R)) = \operatorname{ind} \mathcal{D}^+(Z_\infty).$$

The existence of metrics on W_∞ making the operator $\mathcal{D}^+(W_\infty)$ invertible is addressed in Theorem 10.3. When applied to a spin 4–manifold X with the rational homology of $S^1 \times S^3$, Theorem 3.2 is a straightforward corollary of Theorem 6.1.

6.1 Preliminaries

We begin by proving two technical lemmas which will be used later in the argument.

Lemma 6.2 *Suppose $A\colon H \to H'$ is a surjective bounded operator between Hilbert spaces. Then there exists a constant $C > 0$ such that for any vector $v \in H'$ one can find a vector $u \in A^{-1}(v)$ with $\|u\|_H \le C \cdot \|v\|_{H'}$.*

Proof By the open mapping theorem, the map $H/\ker A \to H_1$ is an isomorphism. □

Lemma 6.3 *Let $A_1\colon H \to H_1$ and $A_2\colon H \to H_2$ be bounded linear operators between Hilbert spaces, and assume that A_2 is surjective. Then the operator $A = (A_1, A_2)\colon H \to H_1 \oplus H_2$ is Fredholm if and only if the operator $A_1|_{\ker A_2}$ is Fredholm and*

$$\operatorname{ind} A = \operatorname{ind}(A_1|_{\ker A_2}).$$

Proof The projection map $\operatorname{im} A \to H_2$ can be included in the short exact sequence $0 \to \operatorname{im}(A_1|_{\ker A_2}) \to \operatorname{im} A \to H_2 \to 0$, which is naturally a subsequence of the short exact sequence $0 \to H_1 \to H_1 \oplus H_2 \to H_2 \to 0$. The quotient sequence $0 \to H_1/\operatorname{im}(A_1|_{\ker A_2}) \to (H_1 \oplus H_2)/\operatorname{im} A \to 0 \to 0$ is exact by the snake lemma, which proves the equality of the cokernels of the two operators in question. The equality of their kernels is clear. □

We will find it convenient to introduce the notation $M = [0, R] \times Y$ and write $X_R = W \cup M$ and

$$Z_\infty(X_R) = Z \cup_{Y_1^-} M_1 \cup_{Y_1^+} W_1 \cup_{Y_2^-} M_2 \cup_{Y_2^+} W_2 \cup \cdots$$

with $W_n = W$ and $M_n = M$ for all $n \ge 1$. Each of the manifolds $Y_n^\pm$ is just a copy of Y but the notations are chosen so that M_n is a cobordism from Y_n^- to Y_n^+ while W_n is a cobordism from Y_n^+ to Y_{n+1}^-.

The spin Dirac operator $\mathcal{D}(Y)$ is a self-adjoint elliptic operator on a compact manifold, hence it has a discrete spectrum with real eigenvalues of finite multiplicity. Denote by $V_\pm(Y) \subset L^2_{1/2}(Y; \mathcal{S})$ the subspaces spanned by the eigenspinors of $\mathcal{D}(Y)$ with the positive and the negative eigenvalues, respectively. The L^2 orthogonal projections onto these subspaces will be denoted by $\pi_\pm$.

Lemma 6.4 *The operator $\mathcal{D}^+(W_\infty)$ of Theorem 6.1 is invertible if and only if the following two conditions are satisfied:*

(1) *The Dirac operator $\mathcal{D}(Y)$ has zero kernel.*

(2) *The Dirac operator*

$$(\mathcal{D}^+(W), \pi_+ \circ r^-, \pi_- \circ r^+)\colon L^2_1(W;\mathcal{S}^+) \to L^2(W;\mathcal{S}^-) \oplus V_+(Y) \oplus V_-(Y)$$

with the Atiyah–Patodi–Singer boundary conditions is an isomorphism. Here, $r^{\pm}\colon L^2_1(W;\mathcal{S}^+) \to L^2_{1/2}(Y;\mathcal{S})$ denote the restriction maps to the boundary components $Y = \partial^- W$ and $Y = \partial^+ W$ of W.

Proof The condition on $\mathcal{D}(Y)$ to have zero kernel is equivalent to the condition on $\mathcal{D}^+(W_\infty)$ to be Fredholm. The relation between $\mathcal{D}^+(W_\infty)$ and the operator $\mathcal{D}^+(W)$ with the Atiyah–Patodi–Singer boundary conditions is well known; see Proposition 3.11 of [2]. □

From now on, we will assume that the operator $\mathcal{D}^+(W_\infty)$ is invertible or, equivalently, that the conditions (1) and (2) of Lemma 6.4 are satisfied.

Given a family of Hilbert spaces H_i for $i \geq 1$, their direct sum $\bigoplus H_i$ is the Hilbert space which consists of all the sequences $(u_1, u_2, \dots)$ of vectors $u_i \in H_i$ such that $\sum \|u_i\|^2_{H_i} < \infty$, the inner product of sequences $(u_1, u_2, \dots)$ and $(v_1, v_2, \dots)$ being $\sum (u_i, v_i)_{H_i}$. Any uniformly bounded family of bounded operators $T_i\colon H_i \to H_i'$ gives rise to a well-defined bounded operator

$$\bigoplus A_i\colon \bigoplus H_i \to \bigoplus H_i'$$

of norm $\sup \|A_i\|$. An application of this abstract construction to the above splitting of $Z_\infty(X_R)$ yields the following result (we suppress spinor bundles in our notations):

Lemma 6.5 *The natural restriction maps provide Hilbert space isomorphisms*

$$L^2(Z_\infty(X_R)) = L^2(Z) \oplus \Big(\bigoplus L^2(W_i)\Big) \oplus \Big(\bigoplus L^2(M_i)\Big), \quad L^2_1(Z_\infty(X_R)) = \ker r,$$

where r is the restriction map

$$L^2_1(Z) \oplus \Big(\bigoplus L^2_1(W_i)\Big) \oplus \Big(\bigoplus L^2_1(M_i)\Big) \to \Big(\bigoplus L^2_{1/2}(Y_i^-)\Big) \oplus \Big(\bigoplus L^2_{1/2}(Y_i^+)\Big)$$

which sends $\varphi_0 \oplus (\varphi_1, \varphi_2, \dots) \oplus (\psi_1, \psi_2, \dots)$ to

$$(\varphi_0|_{Y_1^-} - \psi_1|_{Y_1^-}, \varphi_1|_{Y_2^-} - \psi_2|_{Y_2^-}, \dots) \oplus (\varphi_1|_{Y_1^+} - \psi_1|_{Y_1^+}, \varphi_2|_{Y_2^+} - \psi_2|_{Y_2^+}, \dots).$$

Proof Claim (1) is straightforward. To prove (2), observe that there is an obvious norm-preserving inclusion of $L^2_1(Z_\infty(X_R))$ into $\ker r$. The result now follows from the fact that all spinors in $\ker r$ belong to $L^2_{1,\mathrm{loc}}(Z_\infty(X_R))$; see for instance Manolescu [36, Lemma 3]. □

Lemma 6.6 (1) *The Dirac operator* $\mathcal{D}^+(M)\colon L^2_1(M)\to L^2(M)$ *is surjective.*

(2) *The operator*

$$\ker \mathcal{D}^+(M)\to V_+(Y^-)\oplus V_-(Y^+)$$

taking ψ *to* $(\pi_+(\psi|_{Y^-}),\pi_-(\psi|_{Y^+}))$ *is an isomorphism.*

(3) *The restriction maps* $r^{\pm}\colon L^2_1(W;\mathcal{S}^+)\to L^2_{1/2}(Y;\mathcal{S})$ *of Lemma 6.4 are surjective.*

(4) *The restriction map* r *of Lemma 6.5 is surjective.*

(5) *The operator* $\bigoplus \mathcal{D}^+(M_i)\colon \bigoplus L^2_1(M_i)\to \bigoplus L^2(M_i)$ *is surjective.*

Proof Claim (1) is proved in [23, Corollary 17.1.5]; (2) can be easily verified using the spectral decomposition of $\mathcal{D}(Y)$ and the fact that $\ker \mathcal{D}(Y)=0$; (3) is a standard fact about Sobolev spaces; (4) follows from (3) and Lemma 6.2; (5) follows from (1) and Lemma 6.2. □

6.2 Proof of Theorem 6.1

The proof will essentially be a repeated application of Lemma 6.3 to the Dirac operator

$$\mathcal{D}_0=\mathcal{D}^+(Z_\infty(X_R))\colon L^2_1(Z_\infty(X_R);\mathcal{S}^+)\to L^2(Z_\infty(X_R);\mathcal{S}^-).$$

Step 1 Consider the operator

$$\begin{aligned}\mathcal{D}_1\colon\ & L^2_1(Z)\oplus\Big(\bigoplus L^2_1(W_i)\Big)\oplus\Big(\bigoplus L^2_1(M_i)\Big)\\ &\to L^2(Z)\oplus\Big(\bigoplus L^2(W_i)\Big)\oplus\Big(\bigoplus L^2(M_i)\Big)\oplus\Big(\bigoplus L^2_{1/2}(Y_i^-)\Big)\oplus\Big(\bigoplus L^2_{1/2}(Y_i^+)\Big)\end{aligned}$$

sending $\varphi_0\oplus(\varphi_1,\varphi_2,\dots)\oplus(\psi_1,\psi_2,\dots)$ to

$$\begin{aligned}&\mathcal{D}^+\varphi_0\oplus(\mathcal{D}^+\varphi_1,\mathcal{D}^+\varphi_2,\dots)\oplus(\mathcal{D}^+\psi_1,\mathcal{D}^+\psi_2,\dots)\\ &\oplus(\varphi_0|_{Y_1^-}-\psi_1|_{Y_1^-},\varphi_1|_{Y_2^-}-\psi_2|_{Y_2^-},\dots)\oplus(\varphi_1|_{Y_1^+}-\psi_1|_{Y_1^+},\varphi_2|_{Y_2^+}-\psi_2|_{Y_2^+},\dots).\end{aligned}$$

It follows from Lemmas 6.5, 6.6(4) and 6.3 that $\mathcal{D}_0$ is Fredholm if and only if $\mathcal{D}_1$ is Fredholm, and

$$\operatorname{ind}\mathcal{D}_0=\operatorname{ind}\mathcal{D}_1.$$

Step 2 Observe that the kernel of $\bigoplus \mathcal{D}^+(M_i)$ equals $\bigoplus \ker \mathcal{D}^+(M_i)$ and consider the operator

$$\begin{aligned}\mathcal{D}_2\colon L^2_1(Z) \oplus \Big(\bigoplus L^2_1(W_i)\Big) \oplus \Big(\bigoplus \ker \mathcal{D}^+(M_i)\Big)\\ \to L^2(Z) \oplus \Big(\bigoplus L^2(W_i)\Big) \oplus \Big(\bigoplus L^2_{1/2}(Y_i^-)\Big) \oplus \Big(\bigoplus L^2_{1/2}(Y_i^+)\Big)\end{aligned}$$

sending $\varphi_0 \oplus (\varphi_1, \varphi_2, \dots) \oplus (\psi_1, \psi_2, \dots)$ to

$$\begin{aligned}\mathcal{D}^+\varphi_0 \oplus (\mathcal{D}^+\varphi_1, \mathcal{D}^+\varphi_2, \dots) \oplus (\varphi_0|_{Y_1^-} - \psi_1|_{Y_1^-}, \varphi_1|_{Y_2^-} - \psi_2|_{Y_2^-}, \dots)\\ \oplus (\varphi_1|_{Y_1^+} - \psi_1|_{Y_1^+}, \varphi_2|_{Y_2^+} - \psi_2|_{Y_2^+}, \dots).\end{aligned}$$

It follows from Lemmas 6.3 and 6.6(5) that the operator $\mathcal{D}_1$ is Fredholm if and only if $\mathcal{D}_2$ is Fredholm, and

$$\operatorname{ind} \mathcal{D}_1 = \operatorname{ind} \mathcal{D}_2.$$

Step 3 Using the subspaces $V_+(Y)$ and $V_-(Y)$ spanned by the positive and negative eigenspinors of the operator $\mathcal{D}(Y)$, and the respective L^2 orthogonal projections π_+ and π_-, the operator $\mathcal{D}_2$ can be written as the operator

$$\begin{aligned}\mathcal{D}_3\colon L^2_1(Z) \oplus \Big(\bigoplus L^2_1(W_i)\Big) \oplus \Big(\bigoplus \ker \mathcal{D}^+(M_i)\Big)\\ \to L^2(Z) \oplus \Big(\bigoplus L^2(W_i)\Big) \oplus \Big(\bigoplus V_-(Y_i^-)\Big) \oplus \Big(\bigoplus V_+(Y_i^+)\Big)\\ \oplus \Big(\bigoplus V_+(Y_i^-)\Big) \oplus \Big(\bigoplus V_-(Y_i^+)\Big)\end{aligned}$$

sending $\varphi_0 \oplus (\varphi_1, \varphi_2, \dots) \oplus (\psi_1, \psi_2, \dots)$ to

$$\begin{aligned}\mathcal{D}^+\varphi_0 \oplus (\mathcal{D}^+\varphi_1, \mathcal{D}^+\varphi_2, \dots) &\oplus (\pi_-\varphi_0|_{Y_1^-} - \pi_-\psi_1|_{Y_1^-}, \pi_-\varphi_1|_{Y_2^-} - \pi_-\psi_2|_{Y_2^-}, \dots)\\ &\oplus (\pi_+\varphi_1|_{Y_1^+} - \pi_+\psi_1|_{Y_1^+}, \pi_+\varphi_2|_{Y_2^+} - \pi_+\psi_2|_{Y_2^+}, \dots)\\ &\oplus (\pi_+\varphi_0|_{Y_1^-} - \pi_+\psi_1|_{Y_1^-}, \pi_+\varphi_1|_{Y_2^-} - \pi_+\psi_2|_{Y_2^-}, \dots)\\ &\oplus (\pi_-\varphi_1|_{Y_1^+} - \pi_-\psi_1|_{Y_1^+}, \pi_-\varphi_2|_{Y_2^+} - \pi_-\psi_2|_{Y_2^+}, \dots).\end{aligned}$$

Since the operators $\mathcal{D}_2$ and $\mathcal{D}_3$ are isomorphic, we have

$$\operatorname{ind} \mathcal{D}_2 = \operatorname{ind} \mathcal{D}_3.$$

Step 4 By Lemma 6.6(2), for each $i \geq 1$ we have an isomorphism

$$\Big(\bigoplus V_+(Y_i^-)\Big) \oplus \Big(\bigoplus V_-(Y_i^+)\Big) = \bigoplus \ker \mathcal{D}^+(M_i).$$

Compose this isomorphism with the restrictions to respective boundary components and spectral projections to obtain the operator

$$\Big(\bigoplus V_+(Y_i^-)\Big)\oplus\Big(\bigoplus V_-(Y_i^+)\Big)\to\bigoplus\ker\mathcal{D}^+(M_i)\\ \to\Big(\bigoplus V_-(Y_i^-)\Big)\oplus\Big(\bigoplus V_+(Y_i^+)\Big)$$

sending $(s_{1,+},s_{2,+},\dots)\oplus(s_{1,-},s_{2,-},\dots)$ to

$$(e^{R\mathcal{D}}s_{1,-},e^{R\mathcal{D}}s_{2,-},\dots)\oplus(e^{-R\mathcal{D}}s_{1,+},e^{-R\mathcal{D}}s_{2,+},\dots).$$

Here, we used the notation $\mathcal{D}=\mathcal{D}(Y)$. Note that $e^{-R\mathcal{D}}$ is a smoothing operator on V_+ while $e^{R\mathcal{D}}$ is a smoothing operator on V_-. The operator $\mathcal{D}_3$ can now be written as

$$\mathcal{D}_4\colon L^2_1(Z)\oplus\Big(\bigoplus L^2_1(W_i)\Big)\oplus\Big(\bigoplus V_+(Y_i^-)\Big)\oplus\Big(\bigoplus V_-(Y_i^+)\Big)\\ \to L^2(Z)\oplus\Big(\bigoplus L^2(W_i)\Big)\oplus\Big(\bigoplus V_-(Y_i^-)\Big)\oplus\Big(\bigoplus V_+(Y_i^+)\Big)\\ \oplus\Big(\bigoplus V_+(Y_i^-)\Big)\oplus\Big(\bigoplus V_-(Y_i^+)\Big)$$

sending $\varphi_0\oplus(\varphi_1,\varphi_2,\dots)\oplus(s_{1,+},s_{2,+},\dots)\oplus(s_{1,-},s_{2,-},\dots)$ to

$$\mathcal{D}^+\varphi_0\oplus(\mathcal{D}^+\varphi_1,\mathcal{D}^+\varphi_2,\dots)\oplus(\pi_-\varphi_0|_{Y_1^-}-e^{R\mathcal{D}}s_{1,-},\pi_-\varphi_1|_{Y_2^-}-e^{R\mathcal{D}}s_{2,-},\dots)\\ \oplus(\pi_+\varphi_1|_{Y_1^+}-e^{-R\mathcal{D}}s_{1,+},\pi_+\varphi_2|_{Y_2^+}-e^{-R\mathcal{D}}s_{2,+},\dots)\\ \oplus(\pi_+\varphi_0|_{Y_1^-}-s_{1,+},\pi_+\varphi_1|_{Y_2^-}-s_{2,+},\dots)\\ \oplus(\pi_-\varphi_1|_{Y_1^+}-s_{1,-},\pi_-\varphi_2|_{Y_2^+}-s_{2,-},\dots).$$

Since the operators $\mathcal{D}_3$ and $\mathcal{D}_4$ are isomorphic, we again conclude that

$$\operatorname{ind}\mathcal{D}_3=\operatorname{ind}\mathcal{D}_4.$$

Step 5 Consider the last two components of $\mathcal{D}_4$, that is, the operator

$$L^2_1(Z)\oplus\Big(\bigoplus L^2_1(W_i)\Big)\oplus\Big(\bigoplus V_+(Y_i^-)\Big)\oplus\Big(\bigoplus V_-(Y_i^+)\Big)\\ \to\Big(\bigoplus V_+(Y_i^-)\Big)\oplus\Big(\bigoplus V_-(Y_i^+)\Big)$$

sending $\varphi_0\oplus(\varphi_1,\varphi_2,\dots)\oplus(s_{1,+},s_{2,+},\dots)\oplus(s_{1,-},s_{2,-},\dots)$ to

$$(\pi_+\varphi_0|_{Y_1^-}-s_{1,+},\pi_+\varphi_1|_{Y_2^-}-s_{2,+},\dots)\oplus(\pi_-\varphi_1|_{Y_1^+}-s_{1,-},\pi_-\varphi_2|_{Y_2^+}-s_{2,-},\dots).$$

This operator is obviously surjective. Therefore, we can apply Lemma 6.3 to the first four components of $\mathcal{D}_4$ restricted to the kernel of the last two components. The

resulting operator

$$\mathcal{D}_5\colon L^2_1(Z) \oplus \Big(\bigoplus L^2_1(W_i)\Big) \to L^2(Z) \oplus \Big(\bigoplus L^2(W_i)\Big) \oplus \Big(\bigoplus V_-(Y_i^-)\Big) \oplus \Big(\bigoplus V_+(Y_i^+)\Big)$$

sends $\varphi_0 \oplus (\varphi_1, \varphi_2, \dots)$ to

$$\begin{aligned}&\mathcal{D}^+\varphi_0 \oplus (\mathcal{D}^+\varphi_1, \mathcal{D}^+\varphi_2, \dots)\\ &\qquad \oplus (\pi_-\varphi_0|_{Y_1^-} - e^{R\mathcal{D}}\pi_-\varphi_1|_{Y_1^+}, \pi_-\varphi_1|_{Y_2^-} - e^{R\mathcal{D}}\pi_-\varphi_2|_{Y_2^+}, \dots)\\ &\qquad \oplus (\pi_+\varphi_1|_{Y_1^+} - e^{-R\mathcal{D}}\pi_+\varphi_0|_{Y_1^-}, \pi_+\varphi_2|_{Y_2^+} - e^{-R\mathcal{D}}\pi_+\varphi_1|_{Y_2^-}, \dots).\end{aligned}$$

It follows from Lemma 6.3 that the operator $\mathcal{D}_4$ is Fredholm if and only if $\mathcal{D}_5$ is Fredholm, and

$$\operatorname{ind}\mathcal{D}_4 = \operatorname{ind}\mathcal{D}_5.$$

Step 6 The operator $\mathcal{D}_5$ splits as $\mathcal{D}_5 = \mathcal{D}_6 + K$, where the operator $\mathcal{D}_6$ sends $\varphi_0 \oplus (\varphi_1, \varphi_2, \dots)$ to

$$\mathcal{D}^+\varphi_0 \oplus (\mathcal{D}^+\varphi_1, \mathcal{D}^+\varphi_2, \dots) \oplus (\pi_-\varphi_0|_{Y_1^-}, \pi_-\varphi_1|_{Y_2^-}, \dots) \oplus (\pi_+\varphi_1|_{Y_1^+}, \pi_+\varphi_2|_{Y_2^+}, \dots)$$

and the operator K sends $\varphi_0 \oplus (\varphi_1, \varphi_2, \dots)$ to

$$\begin{aligned}&0 \oplus 0 \oplus (-e^{R\mathcal{D}}\pi_-\varphi_1|_{Y_1^+}, -e^{R\mathcal{D}}\pi_-\varphi_2|_{Y_2^+}, \dots)\\ &\qquad \oplus (-e^{-R\mathcal{D}}\pi_+\varphi_0|_{Y_1^-}, -e^{-R\mathcal{D}}\pi_+\varphi_1|_{Y_2^-}, \dots).\end{aligned}$$

According to Lemma 6.4, the operator $\mathcal{D} = \mathcal{D}(Y)$ has zero kernel. Denote by $\mu > 0$ the smallest absolute value of the eigenvalues of $\mathcal{D}(Y)$ then the operator norm of K does not exceed $C \cdot e^{-\mu R}$, where C is a constant independent of R. Therefore, if R is sufficiently large, the operator $\mathcal{D}_5$ is Fredholm if $\mathcal{D}_6$ is Fredholm, and in this case

$$\operatorname{ind}\mathcal{D}_5 = \operatorname{ind}\mathcal{D}_6.$$

The operator $\mathcal{D}_6$ further splits as a direct sum of the Dirac operator with the Atiyah–Patodi–Singer boundary conditions,

$$\mathcal{D}_6^0\colon L^2_1(Z) \to L^2(Z) \oplus V_-(Y_1^-), \quad \varphi_0 \mapsto (\mathcal{D}^+\varphi_0, \pi_-\varphi_0|_{Y_1^-}),$$

and an infinite family of operators

$$\begin{aligned}\mathcal{D}_6^i\colon L^2_1(W_i) &\to L^2(W_i) \oplus V_+(Y_i^+) \oplus V_-(Y_{i+1}^-),\\ \varphi_i &\mapsto (\mathcal{D}^+\varphi_i, \pi_+\varphi_i|_{Y_i^+}, \pi_-\varphi_i|_{Y_{i+1}^-}),\end{aligned}$$

for $i = 1, 2, \dots$. By Lemma 6.4, each of the operators $\mathcal{D}_6^i$ with $i \geq 1$ is an isomorphism. Therefore, the operator $\mathcal{D}_6$ is Fredholm if and only if $\mathcal{D}_6^0$ is Fredholm, and

$$\operatorname{ind} \mathcal{D}_6 = \operatorname{ind} \mathcal{D}_6^0.$$

The operator $\mathcal{D}_6^0$ is precisely the Dirac operator $\mathcal{D}^+(Z)$ with the Atiyah–Patodi–Singer boundary conditions. Since $\ker \mathcal{D}(Y) = 0$, the operator $\mathcal{D}^+(Z_\infty)$ is a Fredholm operator of index

$$\operatorname{ind} \mathcal{D}_6^0 = \operatorname{ind} \mathcal{D}^+(Z_\infty);$$

see Atiyah, Patodi and Singer [2, Proposition 3.11]. This completes the proof of Theorem 6.1.

7 First eigenvalue estimate

In this section, we continue the study of manifolds X_R defined in (18) by stretching the neck of a spin manifold X of dimension $n \equiv 0 \pmod 4$. We will be interested in estimating the first eigenvalue of $\mathcal{D}^-(X_R)\mathcal{D}^+(X_R)$ as $R \to \infty$. This estimate will be used in the compactness argument in Section 8.

Proposition 7.1 *Let us assume that the spin Dirac operator*

$$\mathcal{D}^+(W_\infty)\colon L_2^2(W_\infty; \mathcal{S}^+) \to L_1^2(W_\infty; \mathcal{S}^-)$$

is an isomorphism. Then there exist constants $R_0 > 0$ and $\epsilon_1 > 0$ such that for any $R \geq R_0$, the operator

$$\Delta_R = \mathcal{D}^-(X_R)\mathcal{D}^+(X_R)\colon L_2^2(X_R; \mathcal{S}^+) \to L^2(X_R; \mathcal{S}^+)$$

has no eigenvalues in the interval $[0, \epsilon_1^2)$.

Proof For the purpose of this proof, we will view W as a cobordism from Y_1 to Y_2 with $Y_1 = Y_2 = Y$. The manifold $X_R = W \cup ([0, R] \times Y)$ is then obtained from W by gluing $\{0\} \times Y$ to Y_2 and $\{R\} \times Y$ to Y_1. Denote by $V_\pm(Y_1) \subset L^2_{3/2}(Y_1; \mathcal{S})$ and $V_\pm(Y_2) \subset L^2_{3/2}(Y_2; \mathcal{S})$ the subspaces spanned by the eigenspinors of $\mathcal{D}(Y_1)$ and $\mathcal{D}(Y_2)$ with positive and negative eigenvalues, respectively. The L^2 orthogonal projections onto these subspaces will be denoted by $\pi_\pm$, and the restriction maps $L_2^2(W; \mathcal{S}^+) \to L^2_{3/2}(Y_i; \mathcal{S})$ will be denoted by r_i for $i = 1, 2$. As in Lemma 6.4, the operator $\mathcal{D}^+(W_\infty)$ is an isomorphism if and only if the following two conditions are satisfied:

(1) The Dirac operator $\mathcal{D}(Y)$ has zero kernel.

(2) The Dirac operator

$$(\mathcal{D}^+(W), \pi_+ \circ r_1, \pi_- \circ r_2)\colon L^2_2(W;\mathcal{S}^+) \to L^2_1(W;\mathcal{S}^-) \oplus V_+(Y_1) \oplus V_-(Y_2)$$

with the Atiyah–Patodi–Singer boundary conditions is an isomorphism.

The operator Δ_R is a nonnegative self-adjoint elliptic differential operator, hence all of its eigenvalues have the form λ^2 with a real $\lambda \geq 0$. Using the fact that $\mathcal{D}^+(X_R)$ has zero index, one can easily check that λ^2 is an eigenvalue of Δ_R if and only if the operator

$$\begin{aligned}\mathcal{D}_1\colon L^2_2(X_R;\mathcal{S}^+ \oplus \mathcal{S}^-) &\to L^2_1(X_R;\mathcal{S}^+ \oplus \mathcal{S}^-),\\ \mathcal{D}_1(\psi^+,\psi^-) &= (\mathcal{D}^-\psi^- - \lambda\psi^+, \mathcal{D}^+\psi^+ - \lambda\psi^-),\end{aligned}$$

has nonzero kernel. We denote the restriction of (ψ^+,ψ^-) to W and to $[0,R]\times Y$ by (φ^+,φ^-) and $(\widetilde{\varphi}^+,\widetilde{\varphi}^-)$, respectively. Supposing that (ψ^+,ψ^-) belongs to the kernel of $\mathcal{D}_1$, the following conditions are satisfied:

(i) $\mathcal{D}^+\varphi^+ = \lambda\varphi^-$ and $\mathcal{D}^-\varphi^- = \lambda\varphi^+$ on W.

(ii) $\mathcal{D}^+\widetilde{\varphi}^+ = \lambda\widetilde{\varphi}^-$ and $\mathcal{D}^-\widetilde{\varphi}^- = \lambda\widetilde{\varphi}^+$ on $[0,R]\times Y$.

(iii) $$\begin{aligned}&(\pi_+\varphi^+|_{Y_1}, \pi_-\varphi^+|_{Y_2}, \pi_-\varphi^-|_{Y_1}, \pi_+\varphi^-|_{Y_2})\\ &\qquad = (\pi_+\widetilde{\varphi}^+|_{Y_1}, \pi_-\widetilde{\varphi}^+|_{Y_2}, \pi_-\widetilde{\varphi}^-|_{Y_1}, \pi_+\widetilde{\varphi}^-|_{Y_2}).\end{aligned}$$

(iv) $$\begin{aligned}&(\pi_-\varphi^+|_{Y_1}, \pi_+\varphi^+|_{Y_2}, \pi_+\varphi^-|_{Y_1}, \pi_-\varphi^-|_{Y_2})\\ &\qquad = (\pi_-\widetilde{\varphi}^+|_{Y_1}, \pi_+\widetilde{\varphi}^+|_{Y_2}, \pi_+\widetilde{\varphi}^-|_{Y_1}, \pi_-\widetilde{\varphi}^-|_{Y_2}).\end{aligned}$$

Lemma 7.2 (1) *There exists a linear operator $T_+(\lambda,R)\colon V_+(Y_2)\oplus V_+(Y_1) \to V_+(Y_2)\oplus V_+(Y_1)$ such that, for any $(\widetilde{\varphi}^+,\widetilde{\varphi}^-)$ satisfying* (ii),

$$T_+(\lambda,R)(\pi_+\widetilde{\varphi}^+|_{Y_2}, \pi_+\widetilde{\varphi}^-|_{Y_1}) = (\pi_+\widetilde{\varphi}^-|_{Y_2}, \pi_+\widetilde{\varphi}^+|_{Y_1}).$$

(2) *There exists a linear operator $T_-(\lambda,R)\colon V_-(Y_1)\oplus V_-(Y_2) \to V_-(Y_1)\oplus V_-(Y_2)$ such that, for any $(\widetilde{\varphi}^+,\widetilde{\varphi}^-)$ satisfying* (ii),

$$T_-(\lambda,R)(\pi_-\widetilde{\varphi}^+|_{Y_1}, \pi_-\widetilde{\varphi}^-|_{Y_2}) = (\pi_-\widetilde{\varphi}^-|_{Y_1}, \pi_-\widetilde{\varphi}^+|_{Y_2}).$$

(3) *For any $\epsilon > 0$, there exist constants $R_0 > 0$ and $\epsilon_2 > 0$ such that, for any $R \geq R_0$ and $0 \leq \lambda < \epsilon_2$,*

$$|T_\pm(\lambda,R)| < \epsilon.$$

Proof We focus on the case of $T_+(\lambda, R)$ since the other case is similar. Over $[0, R]\times Y$, use Clifford multiplication with $\partial/\partial t$ to identify the bundles $\mathcal{S}^+$ and $\mathcal{S}^-$ with each other and with the pullback of the bundle $\mathcal{S}$. This identifies the operators $\mathcal{D}^+([0, R]\times Y)$ and $\mathcal{D}^-([0, R]\times Y)$ with the operators $\partial/\partial t + \mathcal{D}(Y)$ and $\partial/\partial t - \mathcal{D}(Y)$, respectively.

Choose a complete system of orthonormal eigenspinors φ_i for $\mathcal{D}(Y)$, with corresponding eigenvalues λ_i for $i \geq 1$. Let $V_i(Y_1)$ (resp. $V_i(Y_2)$) denote the vector space spanned by φ_i, treated as a section over Y_1 (resp. Y_2). It is sufficient to define $T_+(\lambda, R)$ on each of the spaces $V_i(Y_2) \oplus V_i(V_1)$ with $\lambda_i > 0$, which we will do next.

Let us write $\widetilde{\varphi}^+(t, y) = \sum a_i(t)\varphi_i(y)$ and $\widetilde{\varphi}^-(t, y) = \sum b_i(t)\varphi_i(y)$; then condition (ii) takes the form

$$\frac{d}{dt}\begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} -\lambda_i & \lambda \\ \lambda & \lambda_i \end{pmatrix}\begin{pmatrix} a_i \\ b_i \end{pmatrix} \tag{19}$$

for all i and $t \in [0, R]$. The matrix of this system will be denoted by A_i. Recall that Y_2 is identified with $\{0\}\times Y$ and Y_1 is identified with $\{R\}\times Y$, and express $(b_i(0), a_i(R))$ in terms of $(a_i(0), b_i(R))$. The computation that follows is elementary if a bit tedious.

The eigenvalues of A_i are $\pm\omega_i$, where $\omega_i = \sqrt{\lambda_i^2 + \lambda^2}$, corresponding to the eigenvectors $(\lambda, \lambda_i \pm \omega_i)$. The solutions of (19) are explicitly given by the formula

$$x \cdot e^{\omega_i t} \cdot \begin{pmatrix} \lambda \\ \lambda_i + \omega_i \end{pmatrix} + y \cdot e^{\omega_i (R-t)} \cdot \begin{pmatrix} \lambda \\ \lambda_i - \omega_i \end{pmatrix} \quad \text{with } x, y \in \mathbb{C},$$

from which we obtain

$$\begin{pmatrix} b_i(0) \\ a_i(R) \end{pmatrix} = \begin{pmatrix} \lambda_i + \omega_i & e^{\omega_i R}(\lambda_i - \omega_i) \\ e^{\omega_i R}\lambda & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$\begin{pmatrix} a_i(0) \\ b_i(R) \end{pmatrix} = \begin{pmatrix} \lambda & e^{\omega_i R}\lambda \\ e^{\omega_i R}(\lambda_i + \omega_i) & \lambda_i - \omega_i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} b_i(0) \\ a_i(R) \end{pmatrix} = B_i \begin{pmatrix} a_i(0) \\ b_i(R) \end{pmatrix},$$

where

$$B_i = \frac{1}{(\lambda_i - \omega_i) - (\lambda_i + \omega_i)e^{2\omega_i R}} \cdot \begin{pmatrix} \lambda(e^{2\omega_i R} - 1) & -2\omega_i e^{\omega_i R} \\ -2\omega_i e^{\omega_i R} & \lambda(1 - e^{2\omega_i R}) \end{pmatrix}.$$

We define $T_+(\lambda, R)$ on each of the spaces $V_i(Y_2)\oplus V_i(Y_1)$ by the respective matrix B_i. To derive estimate (3), we let λ_0 be the smallest absolute value of the eigenvalues of

the operator $\mathcal{D}(Y)$. Note that λ_0 is positive by our assumption on the kernel of $\mathcal{D}(Y)$ and that $\omega_i \geq \lambda_0$ for all i. For any $\lambda_i > 0$ we obviously have

$$|(\lambda_i - \omega_i) - (\lambda_i + \omega_i)e^{2\omega_i R}| > (\lambda_i + \omega_i)(e^{2\omega_i R} - 1) > \omega_i(e^{2\omega_i R} - 1),$$

hence

$$\left| \frac{2\omega_i e^{\omega_i R}}{(\lambda_i - \omega_i) - (\lambda_i + \omega_i)e^{2\omega_i R}} \right| \leq 2 \cdot \frac{e^{\omega_i R}}{e^{2\omega_i R} - 1} \leq 2 \cdot \frac{e^{\lambda_0 R}}{e^{2\lambda_0 R} - 1}$$

and

$$\left| \frac{\lambda(e^{2\omega_i R} - 1)}{(\lambda_i - \omega_i) - (\lambda_i + \omega_i)e^{2\omega_i R}} \right| \leq \frac{\lambda}{\omega_i} \leq \frac{\lambda}{\lambda_0}.$$

Therefore, the norms of B_i approach zero uniformly over i as $R \to \infty$ and $\lambda \to 0$. This proves claim (3) for the operator $T_+(\lambda, R)$. □

We now return to the proof of Proposition 7.1. It follows from Lemma 7.2 together with conditions (iii) and (iv) that

$$\begin{aligned}(\pi_+\varphi^-|_{Y_2}, \pi_+\varphi^+|_{Y_1}, \pi_-\varphi^-|_{Y_1}, \pi_-\varphi^+|_{Y_2}) \\ = \big(T_+(\lambda, R)(\pi_+\varphi^+|_{Y_2}, \pi_+\varphi^-|_{Y_1}), T_-(\lambda, R)(\pi_-\varphi^+|_{Y_1}, \pi_-\varphi^-|_{Y_2})\big).\end{aligned}$$

Therefore, the pair (φ^+, φ^-) belongs to the kernel of the operator $\mathcal{D}_2 = \mathcal{D}_3 - K$, where the operators

$$\mathcal{D}_3, K\colon L^2_2(W; \mathcal{S}^+ \oplus \mathcal{S}^-) \to L^2_1(W; \mathcal{S}^+ \oplus \mathcal{S}^-) \oplus V_+(Y_2) \oplus V_+(Y_1) \oplus V_-(Y_1) \oplus V_-(Y_2)$$

are given by the formulas

$$\mathcal{D}_3(\varphi^+, \varphi^-) = (\mathcal{D}^-\varphi^-, \mathcal{D}^+\varphi^+, \pi_+\varphi^-|_{Y_2}, \pi_+\varphi^+|_{Y_1}, \pi_-\varphi^-|_{Y_1}, \pi_-\varphi^+|_{Y_2})$$

and

$$\begin{aligned}K(\varphi^+, \varphi^-) \\ = \big(\lambda\varphi^+, \lambda\varphi^-, T_+(\lambda, R)(\pi_+\varphi^+|_{Y_2}, \pi_+\varphi^-|_{Y_1}), T_-(\lambda, R)(\pi_-\varphi^+|_{Y_1}, \pi_-\varphi^-|_{Y_2})\big).\end{aligned}$$

One can easily see that $\mathcal{D}_3$ is isomorphic to the operator $\mathcal{D}_0 \oplus \mathcal{D}_0^*$, hence its kernel is zero by our assumption on the kernel of $\mathcal{D}_0$ (note that the operator $\mathcal{D}_0$ has zero index). Therefore, there exists a constant C_0 such that the operator $\mathcal{D}_2 = \mathcal{D}_3 - K$ has zero kernel as long as $\|K\| \leq C_0$, and so does the operator $\mathcal{D}_1$. The proposition now follows from Lemma 7.2(3). □

The following result, which will be used in Section 11, is a straightforward extension of Proposition 7.1 to the holomorphic family of operators

$$\mathcal{D}_z^{\pm}(X_R) = \mathcal{D}^{\pm}(X_R) - \ln z \cdot df \quad \text{for } z \in \mathbb{C}^*,$$

where $f\colon X \to S^1$ is an arbitrary smooth function such that $[df] = \gamma \in H^1(X;\mathbb{Z})$. Note that the operators $\mathcal{D}_z^+(X_R)$ and $\mathcal{D}_z^-(X_R)$ are adjoint to each other whenever $|z| = 1$.

Proposition 7.3 *Let us assume that the spin Dirac operator*

$$\mathcal{D}^+(W_\infty)\colon L_2^2(W_\infty;\mathcal{S}^+) \to L_1^2(W_\infty;\mathcal{S}^-)$$

is an isomorphism. Then there exist constants $R_0 > 0$ and $\epsilon_1 > 0$ such that for any $R \geq R_0$, the operators

$$\mathcal{D}_z^-(X_R)\mathcal{D}_z^+(X_R)\colon L_2^2(X_R;\mathcal{S}^+) \to L^2(X_R;\mathcal{S}^-) \quad \text{for } |z| = 1,$$

have no eigenvalues in the interval $[0, \epsilon_1^2)$.

Proof The above proof can easily be adapted by introducing an extra parameter z into the matching of the spinor bundles over Y_1. This preserves the conditions (i) and (ii) but replaces the conditions (iii) and (iv) with:

(iii) $$(\pi_+\varphi^+|_{Y_1}, \pi_-\varphi^+|_{Y_2}, \pi_-\varphi^-|_{Y_1}, \pi_+\varphi^-|_{Y_2}) = (z\pi_+\widetilde{\varphi}^+|_{Y_1}, \pi_-\widetilde{\varphi}^+|_{Y_2}, z\pi_-\widetilde{\varphi}^-|_{Y_1}, \pi_+\widetilde{\varphi}^-|_{Y_2}).$$

(iv) $$(\pi_-\varphi^+|_{Y_1}, \pi_+\varphi^+|_{Y_2}, \pi_+\varphi^-|_{Y_1}, \pi_-\varphi^-|_{Y_2}) = (z\pi_-\widetilde{\varphi}^+|_{Y_1}, \pi_+\widetilde{\varphi}^+|_{Y_2}, z\pi_+\widetilde{\varphi}^-|_{Y_1}, \pi_-\widetilde{\varphi}^-|_{Y_2}).$$

The new operators $T_+(\lambda, R)$ and $T_-(\lambda, R)$ that show up in the formula for K are obtained from the old ones by multiplying them on the left by

$$\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix},$$

respectively. Since $|z| = 1$, this does not change the operator norm of K, and the rest of the proof goes through with no change. □

8 Compactness

The proof of Theorem 3.3 naturally divides into two steps: compactness and gluing. In this section, we provide the necessary compactness results; the proof of Theorem 3.3 will be completed in Section 9.

8.1 Notations

Let X be a connected smooth spin compact 4–manifold with a primitive cohomology class $\gamma \in H^1(X;\mathbb{Z})$. We will assume that the Poincaré dual of γ is realized by a rational homology 3–sphere $Y \subset X$ and choose a metric g on X which takes the form $g = dt^2 + h$ in a product region $[-\varepsilon, \varepsilon] \times Y$. Given a real number $T > 0$, consider the manifold with long neck

$$X_T = W \cup ([-T, T] \times Y)$$

obtained by cutting X open along $\{0\} \times Y$ and gluing in the cylinder $[-T, T] \times Y$ along the two copies of Y. This differs from the notation X_R used in Section 6 by a simple reparametrization. In addition, for any $0 < T' < T < \infty$ we will write $X_T = W_{T'} \cup I_{T',T}$, where

$$W_{T'} = ([-T', 0] \times Y) \cup W \cup ([0, T'] \times Y) \quad \text{and} \quad I_{T',T} = [-T + T', T - T'] \times Y.$$

We will find it convenient to extend these notations to the case of $T = \infty$ by letting I_∞ be the disjoint union $(-\infty, 0] \times Y \cup [0, +\infty) \times Y$ and using X_∞ to denote the manifold W_∞ with infinite product ends, as in Section 6. When $T = \infty$ and T' is finite, the notation $I_{T',\infty}$ will mean $W_\infty - \mathrm{int}(W_{T'})$.

8.2 Perturbations and regularity of moduli spaces

Recall that, in order to define the monopole Floer homology of a rational homology sphere Y in Section 2.2, we introduced perturbations $\mathfrak{q}$. We will assume that our perturbation $\mathfrak{q}$ satisfies Assumption 2.4 with respect to a constant ϵ_0 satisfying $0 < \epsilon_0 < \epsilon_1$, where $\epsilon_1 > 0$ is the constant from Proposition 7.1.

To define the morphisms (8) induced on the Floer homology of Y by the spin cobordism W, we will need to introduce further perturbations. To this end, consider a collar neighborhood $U = [0, 1] \times \partial W$, with $\{1\} \times \partial W$ identified with the actual boundary $\partial W = -Y \cup Y$. Let ζ be a cut-off function which equals 1 near $t = 1$ and equals 0 near $t = 0$, and let ζ_0 be a bump function with compact support in $(-1, 0)$. Pick another perturbation $\mathfrak{p}_0$ as in Section 2.2, and let

$$\widehat{\mathfrak{p}} = \zeta \cdot \widehat{\mathfrak{q}} + \zeta_0 \cdot \widehat{\mathfrak{p}}_0, \tag{20}$$

where $\widehat{\mathfrak{q}}$ and $\widehat{\mathfrak{p}}_0$ are the 4–dimensional perturbations corresponding to $\mathfrak{q}$ and $\mathfrak{p}_0$, respectively; see [23, Definition 10.1.1]. This is a perturbation on W, supported in U. By gluing the perturbations $\widehat{\mathfrak{p}}$ on W and $\widehat{\mathfrak{q}}$ on $I_{0,T}$ together, we obtain a perturbation

$\widehat{\mathfrak{p}}_T$ on X_T. Similarly, we define a perturbation $\widehat{\mathfrak{p}}_\infty$ on W_∞ by gluing together the perturbations $\widehat{\mathfrak{p}}$ on W and $\widehat{\mathfrak{q}}$ on $I_{0,\infty}$. These perturbations give rise to the perturbed Seiberg–Witten equations, whose solutions will be referred to as monopoles.

Let $\mathcal{L}_q$ be the perturbed Chern–Simons–Dirac functional as in Section 2.2. Downstairs, the gauge equivalence classes of its critical points form the finite set

$$\widetilde{\mathfrak{C}} = \mathfrak{C}^* \cup [\theta],$$

where $[\theta]$ is the unique reducible class and $\mathfrak{C}^*$ consists of the irreducible classes. Given $[\alpha], [\beta] \in \widetilde{\mathfrak{C}}$, consider the following moduli spaces:

(1) The moduli space $\check{\mathcal{M}}([\alpha], [\beta])$ of unparametrized (downstairs) trajectories of the perturbed Chern–Simons–Dirac gradient flow (that is, monopoles on $\mathbb{R} \times Y$) limiting to $[\alpha]$ and $[\beta]$ at minus and plus infinity, in other words, the quotient of $\mathcal{M}([\alpha], [\beta])$ by translations, excluding the constant trajectory if $[\alpha] = [\beta]$.

(2) The moduli space $\mathcal{M}(W_\infty, [\alpha], [\beta])$ of (downstairs) monopoles on W_∞ limiting to $[\alpha]$ and $[\beta]$ at minus and plus infinity. We will write $\mathcal{M}(W_\infty, [\alpha])$ for $\mathcal{M}(W_\infty, [\alpha], [\alpha])$.

Since $\mathfrak{q}$ is admissible by Assumption 2.4, the moduli space $\check{\mathcal{M}}([\alpha], [\beta])$ is always regular. The regularity of the moduli space $\mathcal{M}(W_\infty, [\alpha], [\beta])$ is proved in the following lemma.

Lemma 8.1 *For any nice admissible perturbation $\mathfrak{q}$, there exists a nice perturbation $\mathfrak{p}_0$ such that, for the perturbation (20), the following conditions are satisfied:*

- *The various moduli spaces of upstairs monopoles on W_∞ are all regular. As a result, the cobordism induced map $W_*\colon \mathrm{HM}^{\mathrm{red}}(Y, \mathfrak{s}) \to \mathrm{HM}^{\mathrm{red}}(Y, \mathfrak{s})$ can be defined using this perturbation.*
- *The moduli space $\mathcal{M}(W_\infty, [\alpha], [\beta])$ is regular for all $[\alpha], [\beta] \in \widetilde{\mathfrak{C}}$.*

Furthermore, we may assume that $\mathfrak{p}_0$ is nice and that it satisfies the estimate

$$\|D_{(B_0,0)}\mathfrak{p}_0^1(0,\psi)\|_{L^2(Y)} \le \tfrac{1}{4}\epsilon_0 \cdot \|\psi\|_{L^2(Y)} \tag{21}$$

for any $\psi \in L^2_{k-1/2}(Y; \mathcal{S})$, where B_0 is the product connection and $\epsilon_0 > 0$ is the constant fixed in the beginning of this section.

Proof The proof is a careful check that the arguments of [23, Proposition 24.4.7] hold in our situation. We first introduce a large Banach space $\mathcal{P}$ of *nice* perturbations and

form the parametrized moduli space

$$\mathcal{M}_{\mathcal{P}}(*,*) = \bigcup_{\mathfrak{p}_0 \in \mathcal{P}} \mathcal{M}_{\mathfrak{p}_0}(*,*).$$

After proving the regularity of $\mathcal{M}_{\mathcal{P}}(*,*)$, we can apply the Sard–Smale lemma to find a residual subset $U \subset \mathcal{P}$ with the property that $\mathcal{M}_{\mathfrak{p}_0}(*,*)$ is regular for any $\mathfrak{p}_0 \in U$. In particular, we can choose a $\mathfrak{p}_0 \in U$ satisfying the estimate (21). There is one new feature in this argument: since $\mathcal{P}$ only consists of nice perturbations, at a reducible monopole, we can only obtain the transversality in the spinor direction by repeating the arguments in [23]. This does not cause a problem for the following reason: At a reducible monopole, the linearization of the curvature equation $F^+_{A^t} = 0$ is the operator

$$d^+ \colon L^2_k(W_\infty; iT^*W_\infty) \to L^2_{k-1}(W_\infty; i\Lambda^2_+ T^*W_\infty).$$

Since $b_2^+(W) = 0$, this operator is surjective (without any perturbation). As a result, the transversality in directions tangent to the space of connections is automatically satisfied. □

8.3 Statement of the theorem

From now on, we will fix a perturbation $\mathfrak{q}$ satisfying Assumption 2.4 and a perturbation $\mathfrak{p}_0$ as in Lemma 8.1. The following compactness theorem is the main result of this section.

Theorem 8.2 *Let T_n be a sequence of positive real numbers such that $T_n \to \infty$. Then, for any sequence $[(A_n, \varphi_n)] \in \mathcal{M}(X_{T_n})$, there exist*

$$[\alpha] \in \mathfrak{C}^* \quad \text{and} \quad [(A_\infty, \varphi_\infty)] \in \mathcal{M}(W_\infty, [\alpha])$$

such that, after passing to a subsequence, $[(A_n, \varphi_n)]$ converges to $[(A_\infty, \varphi_\infty)]$ in the sense of Definition 8.3 below.

Definition 8.3 Let T_n be a sequence of positive real numbers such that $T_n \to \infty$. We will say that $[(A_n, \varphi_n)] \in \mathcal{M}(X_{T_n})$ converges to $[(A_\infty, \varphi_\infty)] \in \mathcal{M}(W_\infty, [\alpha])$ if the following two conditions hold:

(1) There exists a sequence of L^2_{k+1} gauge transformation $u_n \colon X_{T_n} \to S^1$ such that

$$u_n \cdot (A_n, \varphi_n) \to (A_\infty, \varphi_\infty) \quad \text{in } L^2_{k,\mathrm{loc}}(W_\infty),$$

where by $L^2_{k,\mathrm{loc}}(W_\infty)$ convergence we mean L^2_k convergence on compact subsets of W_∞.

(2) For any $\epsilon > 0$, there exist a real number $T > 0$ and an integer $N > 0$ such that, for all $n > N$, we have $T_n > T$ and there is a sequence of L^2_{k+1} gauge transformations $v_n\colon I_{T,T_n} \to S^1$ such that

$$\|v_n \cdot (A_n, \varphi_n)|_{I_{T,T_n}} - \gamma_\alpha\|_{L^2_k(I_{T,T_n})} < \epsilon.$$

Here, γ_α stands for the constant trajectory of the Chern–Simons–Dirac gradient flow (that is, a translation-invariant monopole on the cylinder I_{T,T_n}) connecting the critical point α to itself.

Condition (2) roughly says that, up to a gauge transformation, (A_n, φ_n) is a "near-constant trajectory" when restricted to the middle of the long neck. Note that the definition of convergence given on page 486 of [23] only includes condition (1). Actually, condition (2) follows from condition (1) in our case but proving this would require some additional work. Instead of doing this, we simply include condition (2) in our definition of convergence.

The rest of this section will be dedicated to the proof of Theorem 8.2. We begin with some preparations.

8.4 Topological energy

The perturbed topological energy of a configuration (A, φ) on a 4–manifold was defined by Kronheimer and Mrowka [23, Definition 24.6.3]. On a cylinder $[a, b] \times Y$, the perturbed topological energy of (A, φ) is given by

$$\mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}(A, \varphi) = 2\big(\mathcal{L}_{\mathfrak{q}}((A, \varphi)|_{\{a\}\times Y}) - \mathcal{L}_{\mathfrak{q}}((A, \varphi)|_{\{b\}\times Y})\big).$$

More generally, the topological energy of a configuration (A, φ) on the cobordism W_T is given by

$$\mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}(A, \varphi) = 2\big(\mathcal{L}_{\mathfrak{q}}((A, \varphi)|_{\{T\}\times Y}) - \mathcal{L}_{\mathfrak{q}}((A, \varphi)|_{\{-T\}\times Y})\big).$$

Lemma 8.4 (1) *If (A, φ) is a monopole on a cylinder $[a, b]\times Y$, then $\mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}(A, \varphi) \geq 0$, with equality if and only if (A, φ) is gauge equivalent to a constant trajectory.*

(2) *There exists a constant C such that $\mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}(A, \varphi) \geq C$ for any monopole (A, φ) on W_T.*

Proof Claim (1) is clear because any monopole on a cylinder is gauge-equivalent to a downward gradient flow line of $\mathcal{L}_{\mathfrak{q}}$. Claim (2) is a direct consequence of [23, Lemma 24.5.1] and the equality of the topological and analytic energies for monopoles; see [23, Definition 4.5.4]. □

Lemma 8.5 (1) *Let (A_n, φ_n) be a sequence of monopoles on $[a,b]\times Y$ satisfying a uniform bound $\mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}(A_n,\varphi_n) \le M$. After passing to a subsequence, there exist gauge transformations $u_n\colon [a,b]\times Y \to S^1$ and a monopole $(A_\infty,\varphi_\infty)$ on $[a,b]\times Y$ such that the sequence $u_n\cdot(A_n,\varphi_n)$ converges to $(A_\infty,\varphi_\infty)$ in the L^2_k–norm on every interior domain in $[a,b]\times Y$.*

(2) *Let $I_n=[a_n,b_n]$ be a sequence of intervals with $\lim b_n=-\infty$ and $\lim a_n=+\infty$, and (A_n,φ_n) a sequence of monopoles on $I_n\times Y$ satisfying a uniform bound $\mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}(A_n,\varphi_n)\le M$. After passing to a subsequence, there exist gauge transformations $u_n\colon [a_n,b_n]\times Y\to S^1$ and a monopole $[(A_\infty,\varphi_\infty)]\in\mathcal{M}([\alpha],[\beta])$ with $[\alpha],[\beta]\in\mathfrak{C}$ such that the sequence $u_n\cdot(A_n,\varphi_n)$ converges to $(A_\infty,\varphi_\infty)$ in $L^2_{k,\mathrm{loc}}(\mathbb{R}\times Y)$.*

(3) *Let T_n be a sequence of positive real numbers with $\lim T_n=+\infty$, and (A_n,φ_n) a sequence of monopoles on W_{T_n} satisfying a uniform bound $\mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}(A_n,\varphi_n)\le M$. After passing to a subsequence, there exist gauge transformations $u_n\colon W_{T_n}\to S^1$ and a monopole $[(A_\infty,\varphi_\infty)]\in\mathcal{M}(W_\infty,[\alpha],[\beta])$ with $[\alpha],[\beta]\in\mathfrak{C}$ such that the sequence $u_n\cdot(A_n,\varphi_n)$ converges to $(A_\infty,\varphi_\infty)$ in $L^2_{k,\mathrm{loc}}(W_\infty)$.*

Proof This follows from [23, Theorems 10.7.1 and 24.5.2]. □

8.5 Near-constant trajectories

For every $[\alpha]\in\widetilde{\mathfrak{C}}$, choose an open neighborhood $U_{[\alpha]}\subset\mathcal{B}(Y)$ such that $U_{[\alpha]}\cap U_{[\beta]}=\varnothing$ when $[\alpha]\neq[\beta]$. In addition, for every $[\alpha]\in\widetilde{\mathfrak{C}}$, choose an open neighborhood $U_{[\gamma_\alpha]}$ of the constant trajectory $[\gamma_\alpha]\in\mathcal{B}([0,1]\times Y)$ such that $[(A,\varphi)|_{\{t\}\times Y}]\in U_{[\alpha]}$ for all $t\in[0,1]$ and all $[(A,\varphi)]\in U_{[\gamma_\alpha]}$. Here, $\mathcal{B}([0,1]\times Y)$ denotes the space of gauge equivalent classes of L^2_k configurations over $[0,1]\times Y$.

Lemma 8.6 *There exists a constant $\epsilon_0>0$ such that for any monopole (A,φ) on $[-1,2]\times Y$ of energy $\mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}(A,\varphi)\le 3\epsilon_0$ we have $[(A,\varphi)|_{[0,1]\times Y}]\subset U_{[\gamma_\alpha]}$ for some $[\alpha]\in\widetilde{\mathfrak{C}}$.*

Proof Suppose that this is not true. Then we can find a sequence of monopoles (A_n,φ_n) on $[-1,2]\times Y$ such that $\lim\mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}(A_n,\varphi_n)=0$ as $n\to\infty$ but $[(A_n,\varphi_n)|_{[0,1]\times Y}]$ does not belong to any $U_{[\gamma_\alpha]}$. By Lemma 8.5(1), after passing to a subsequence,

$[(A_n,\varphi_n)|_{[0,1]\times Y}]$ will converge to a monopole of zero energy, which by Lemma 8.4(1) must be of the form $[\gamma_\alpha]$ for some $[\alpha]\in\mathfrak{C}$. This leads to a contradiction. □

Lemma 8.7 *For any real numbers $M>0$ and $\epsilon>0$ there exists a constant $T'>0$ with the following significance: Let $T>T'$ and let (A,φ) be a monopole on $[-T,T]\times Y$ such that*

$$\mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}(A,\varphi)\le M \qquad \text{and} \qquad [(A,\varphi)|_{\{t\}\times Y}]\in U_{[\alpha]} \quad \text{for all } t\in[-T,T].$$

Then, for any interval $[t,t+4]\subset[-T+T',T-T']$, there exists a gauge transformation $u\colon [t,t+4]\times Y\to S^1$ such that

$$\|u\cdot(A,\varphi)|_{[t,t+4]\times Y}-\gamma_\alpha\|_{L^2_k([t,t+4]\times Y)}<\epsilon.$$

Proof Suppose that this is not true. Then there exist a sequence $T'_n\to\infty$ and a sequence (A_n,φ_n) of monopoles on $[-T_n,T_n]\times Y$ with $T_n>T'_n$ such that $\mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}(A_n,\varphi_n)\le M$ and $[(A_n,\varphi_n)|_{\{t\}\times Y}]\in U_{[\alpha]}$ for all $t\in[-T_n,T_n]$ but, for some interval $[t_n,t_n+4]\subset[-T_n+T'_n,T_n-T'_n]$, we have

$$\|u\cdot(A_n,\varphi_n)|_{[t_n,t_n+4]\times Y}-\gamma_\alpha\|_{L^2_k([t_n,t_n+4]\times Y)}\ge\epsilon>0$$

for all n and all gauge transformations $u\colon[t_n,t_n+4]\times Y\to S^1$. Using the translation-invariance of the Seiberg–Witten equations on the cylinder, reparametrize (A_n,φ_n) to obtain a monopole on $[-T_n-t_n,T_n-t_n]$, again called (A_n,φ_n), such that

$$\|u\cdot(A_n,\varphi_n)|_{[0,4]\times Y}-\gamma_\alpha\|_{L^2_k([0,4]\times Y)}\ge\epsilon>0$$

for all n and all gauge transformations $u\colon[0,4]\times Y\to S^1$. Note that $\lim(-T_n-t_n)=-\infty$ and $\lim(T_n-t_n)=\infty$. Therefore, by Lemma 8.5(2), after passing to a subsequence, we can find gauge transformations $u_n\colon[-T_n,T_n]\times Y\to S^1$ such that $u_n\cdot(A_n,\varphi_n)\to(A_\infty,\varphi_\infty)$ in $L^2_{k,\mathrm{loc}}(\mathbb{R}\times Y)$, where $(A_\infty,\varphi_\infty)$ is a monopole on $\mathbb{R}\times Y$ limiting to critical points in $\widetilde{\mathfrak{C}}$ at plus and minus infinity. Since the gauge equivalence class of the restriction of $(A_\infty,\varphi_\infty)$ to each slice $\{t\}\times Y$ is contained in $U_{[\alpha]}$, and since $[\alpha]$ is the only critical point in $U_{[\alpha]}$, the monopole $(A_\infty,\varphi_\infty)$ must be gauge-equivalent to γ_α (note that if Y were not a rational homology sphere, we would need to impose the extra condition that $U_{[\alpha]}$ is contractible). Without loss of generality, we may assume that $(A_\infty,\varphi_\infty)=\gamma_\alpha$. But then the $L^2_{k,\mathrm{loc}}(\mathbb{R}\times Y)$ convergence implies that

$$\|u_n\cdot(A_n,\varphi_n)|_{[0,4]\times Y}-\gamma_\alpha\|_{L^2_k([0,4]\times Y)}\to 0,$$

which leads to a contradiction. □

Lemma 8.8 *For any $\epsilon > 0$ there exists $\delta > 0$ with the following significance: For any $T > 5$ and any irreducible critical point $[\alpha] \in \mathfrak{C}^*$, let (A, φ) be a monopole on $[-T, T] \times Y$ with the property that, for any $[t, t+4] \subset [-T, T]$, there exist a gauge transformation $u_t \colon [t, t+4] \times Y \to S^1$ such that*

$$\|u_t \cdot (A, \varphi)|_{[t,t+4]\times Y} - \gamma_\alpha\|_{L^2_k([t,t+4]\times Y)} \leq \delta.$$

Then there exists a gauge transformation $u \colon [-T+5, T-5] \times Y \to S^1$ such that

$$\|u \cdot (A, \varphi)|_{[-T+5,T-5]\times Y} - \gamma_\alpha\|_{L^2_k([-T+5,T-5]\times Y)} \leq \epsilon.$$

Proof This is essentially Lemma 19.3.2 of [23]. In fact, the patching argument in the proof of Lemma 8.10 will be extracted, as in the proof of this lemma, from [23, Lemma 13.6.]. □

8.6 Broken trajectories on W_∞

Let $\mathrm{gr}^{\mathbb{Q}} \colon \mathfrak{C}^* \to \mathbb{Q}$ be the absolute grading function [23, page 587] on the irreducible critical points, and extend it to the unique reducible $[\theta]$ by the formula

$$\mathrm{gr}^{\mathbb{Q}}([\theta]) = -2n(Y, h, \mathfrak{s})$$

(compare with the grading in Lemma 2.6). Then, for any $[\alpha], [\beta] \in \mathfrak{C}^*$, the expected dimensions (denoted by e-dim) of the moduli spaces are as follows:

(1) $\text{e-dim}(\breve{\mathcal{M}}([\alpha], [\beta])) = \mathrm{gr}^{\mathbb{Q}}([\alpha]) - \mathrm{gr}^{\mathbb{Q}}([\beta]) - 1$.

(2) $\text{e-dim}(\breve{\mathcal{M}}([\alpha], [\theta])) = \mathrm{gr}^{\mathbb{Q}}([\alpha]) - \mathrm{gr}^{\mathbb{Q}}([\theta]) - 1$.

(3) $\text{e-dim}(\breve{\mathcal{M}}([\theta], [\alpha])) = \mathrm{gr}^{\mathbb{Q}}([\theta]) - \mathrm{gr}^{\mathbb{Q}}([\alpha]) - 2$.

(4) $\text{e-dim}(\mathcal{M}(W_\infty, [\alpha], [\beta])) = \mathrm{gr}^{\mathbb{Q}}([\alpha]) - \mathrm{gr}^{\mathbb{Q}}([\beta])$.

(5) $\text{e-dim}(\mathcal{M}(W_\infty, [\alpha], [\theta])) = \mathrm{gr}^{\mathbb{Q}}([\alpha]) - \mathrm{gr}^{\mathbb{Q}}([\theta])$.

(6) $\text{e-dim}(\mathcal{M}(W_\infty, [\theta], [\alpha])) = \mathrm{gr}^{\mathbb{Q}}([\theta]) - \mathrm{gr}^{\mathbb{Q}}([\alpha]) - 1$.

(7) $\text{e-dim}(\mathcal{M}(W_\infty, [\theta], [\theta])) = -1$.

By our regularity assumption, the actual dimensions of the moduli spaces are equal to their expected dimensions except in case (7): in this case, there is always a one-dimensional cokernel of the corresponding Fredholm operator. As a result, the manifold $\mathcal{M}(W_\infty, [\theta], [\theta])$ is zero-dimensional and only contains reducible monopoles. Note that the moduli space $\breve{\mathcal{M}}([\theta], [\theta])$ is empty because we only allow nonconstant trajectories in our definition of the moduli spaces $\breve{\mathcal{M}}([\alpha], [\alpha])$.

Lemma 8.9 *Let $d \geq 2$ and suppose that $[\alpha_1], \dots, [\alpha_{d-1}]$ are critical points such that the moduli spaces $\breve{\mathcal{M}}([\alpha_1],[\alpha_2]), \dots, \breve{\mathcal{M}}([\alpha_{d-2}],[\alpha_{d-1}])$ and $\mathcal{M}(W_\infty,[\alpha_{d-1}],[\alpha_1])$ are all nonempty. Then $d = 2$.*

Proof This follows from our regularity assumption by a simple dimension count. □

8.7 Proof of Theorem 8.2

We will follow closely the argument of [23, Section 16.2]. Let (A_n, φ_n) be a sequence of monopoles as in the statement of Theorem 8.2. Since

$$\begin{aligned}\mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}((A_n,\varphi_n)|_{[-T_n,T_n]\times Y}) &= 2\big(\mathcal{L}_{\mathfrak{q}}((A_n,\varphi_n)|_{\{-T_n\}\times Y}) - \mathcal{L}_{\mathfrak{q}}((A_n,\varphi_n)|_{\{T_n\}\times Y})\big)\\ &= -\mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}((A_n,\varphi_n)|_W),\end{aligned}$$

we conclude from Lemma 8.4 that

$$\mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}((A_n,\varphi_n)|_W) \leq 0 \quad \text{and} \quad \mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}((A_n,\varphi_n)|_{[-T_n,T_n]\times Y}) \leq M$$

for some constant $M \geq 0$ independent of n. As a result, we obtain the uniform bounds

$$\mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}((A_n,\varphi_n)|_{W_T}) \leq M \quad \text{and} \quad \mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}((A_n,\varphi_n)|_{I\times Y}) \leq M \tag{22}$$

for any $0 < T < T_n$ and any interval $I \subset [-T_n, T_n]$, which will allow us to apply Lemma 8.5.

Choose neighborhoods $U_{[\alpha]}$ and $U_{[\gamma_\alpha]}$ as in Section 8.5, and let $\epsilon_0 > 0$ be the constant provided by Lemma 8.6. Restricting (A_n, φ_n) to the slices $\{t\} \times Y$ gives rise to a path $\gamma_n\colon [-T_n, T_n] \to \mathcal{B}(Y)$. For each n, consider the set

$$S_n = \{p \in \mathbb{Z} \mid [p, p+1] \subset [-T_n, T_n] \text{ and } \mathcal{E}^{\mathrm{top}}_{\mathfrak{q}}((A_n,\varphi_n)|_{[p,p+1]\times Y}) \geq \epsilon_0 > 0\}.$$

This set contains at most M/ϵ_0 elements. By passing to a subsequence, we may find an integer j such that every set S_n contains exactly j elements, $p_1^n < p_2^n < \cdots < p_j^n$. Also introduce the integers $p_0^n = \lceil -T_n \rceil$ and $p_{j+1}^n = \lfloor T_n \rfloor - 1$. After passing to a subsequence one more time, we may assume that, for each integer m between 0 and j, either $\lim_{n\to\infty}(p_{m+1}^n - p_m^n) = +\infty$ or $p_{m+1}^n - p_m^n$ is independent of n. On the set $\{0, 1, \dots, j+1\}$, define an equivalence relation

$$m_1 \sim m_2 \quad \text{if and only if} \quad \lim_{n\to\infty} |p_{m_1}^n - p_{m_2}^n| < \infty.$$

Denote by d the number of the equivalence classes; since 0 is not equivalent to j, we must have $d \geq 2$. Pick representatives $m_0 < m_1 < m_2 < \cdots < m_{d-1}$, one for each equivalence class, and let

$$a_i^n = \min_m\{p_m^n \mid m \sim m_i\} \quad \text{and} \quad b_i^n = \max_m\{p_m^n \mid m \sim m_i\} \quad \text{for } 0 \leq i \leq d-1.$$

Then

$$\lceil -T_n \rceil = a_0^n \leq b_0^n < a_1^n \leq b_1^n < \cdots < a_{d-1}^n \leq b_{d-1}^n = \lfloor T_n \rfloor - 1,$$

the difference $b_m^n - a_m^n$ is independent of n for $0 \leq m \leq d-1$, and

$$\lim_{n\to\infty}(a_{m+1}^n - b_m^n) = \infty \quad \text{for } 0 \leq m \leq d-2.$$

Using Lemma 8.6 together with the translation-invariance of the Seiberg–Witten equations, we obtain

$$\gamma_n([b_m^n + 2, a_{m+1}^n - 1]) \subset U_{[\alpha_{m+1}]} \tag{23}$$

for some critical point $[\alpha_{m+1}]$. Here, we passed to a subsequence again to ensure that $[\alpha_{m+1}]$ is independent of n. Using Lemma 8.5 and passing to a subsequence if necessary, we conclude the following:

- There exist gauge transformations $u_n\colon X_{T_n} \to S^1$ such that $u_n \cdot (A_n, \varphi_n)$ converges in $L^2_{k,\mathrm{loc}}(W_\infty)$ to a monopole $(A_\infty, \varphi_\infty)$ on W_∞. Using (23), it is not difficult to see that $[(A_\infty, \varphi_\infty)] \in \mathcal{M}(W_\infty, [\alpha_{d-1}], [\alpha_1])$.

- Let τ_t denote the translation on trajectories defined by $(\tau_t \cdot \gamma)(x) = \gamma(x+t)$. Then, for every $n \geq 0$ and $1 \leq m \leq d-2$, we have

$$\tau_{(a_m^n + b_m^n)/2} \cdot \gamma_n \to \gamma_{\infty,m} \quad \text{in } L^2_{k,\mathrm{loc}}(\mathbb{R} \times Y),$$

where $\gamma_{\infty,m}$ is a trajectory on $\mathbb{R} \times Y$. Since the topological energy of $\gamma_n([a_m^n, a_m^n + 1])$ is greater than or equal to $\epsilon_0 > 0$, we conclude that $\gamma_{\infty,m}$ is not a constant trajectory. Using (23), it is not difficult to see that $\gamma_{\infty,m}$ represents a monopole in $\mathcal{M}([\alpha_m], [\alpha_{m+1}])$.

Since the moduli spaces

$$\mathcal{M}([\alpha_1], [\alpha_2]), \ldots, \mathcal{M}([\alpha_{d-2}], [\alpha_{d-1}]) \quad \text{and} \quad \mathcal{M}(W_\infty, [\alpha_{d-1}], [\alpha_1])$$

are all nonempty, it follows from Lemma 8.9 that $d = 2$. Keeping this in mind, our earlier discussion implies the following two results:

(A) There exist gauge transformations $u_n\colon X_{T_n} \to S^1$ such that $u_n \cdot (A_n, \varphi_n) \to (A_\infty, \varphi_\infty)$ in $L^2_{k,\mathrm{loc}}(W_\infty)$ for some $[(A_\infty, \varphi_\infty)] \in \mathcal{M}(W_\infty, [\alpha])$ with $[\alpha] = [\alpha_1]$.

(B) There exists a constant $T' > 0$ such that $\gamma_n([-T_n + T', T_n - T']) \subset U_{[\alpha]}$ for all n (this is implied by (23)).

Using Lemma 8.7, we can replace (B) by the following:

(C) For any $\epsilon > 0$, there exists a constant $C > 0$ with the following significance: for any n large enough that $T_n > C+4$ and any interval $[t, t+4] \subset [-T_n+C, T_n-C]$, there exists a gauge transformation $u_{n,t}\colon [t, t+4] \times Y \to S^1$ such that

$$\|u_{n,t} \cdot (A_n, \varphi_n)|_{[t,t+4]\times Y} - \gamma_\alpha\|_{L^2_k([t,t+4]\times Y)} \leq \epsilon.$$

Lemma 8.10 *The critical point $[\alpha]$ is irreducible.*

This lemma will be proved in Section 8.8. For now, let us assume it and finish the proof of Theorem 8.2. It is clear that condition (1) of Definition 8.3 follows from (A) and Lemma 8.10. Therefore, we only need to verify condition (2) of Definition 8.3. For any $\epsilon > 0$, let C be the constant from (C). Choose $T = C + 5$ and let N be an integer large enough that $T_n > T$ for all $n \geq N$. A straightforward application of Lemma 8.8 finishes the proof.

8.8 Convergence to reducible

In this subsection we prove Lemma 8.10. The proof will be based on the following three lemmas.

Lemma 8.11 *The moduli space $\mathcal{M}(W_\infty, [\theta])$ contains a single point $[(A_0, 0)]$, where A_0 is a trivial connection on W_∞.*

Proof Since both $\mathfrak{q}$ and $\mathfrak{p}_0$ are nice, we may disregard the perturbations when studying reducible monopoles downstairs. We saw in Section 8.6 that the moduli space $\mathcal{M}(W_\infty, [\theta])$ contains only reducible monopoles $[(A, 0)]$ with $F_A^+ = 0$. Write $A = A_0 + a$, where a is an L^2_k differential 1–form on W_∞ with coefficients in $i\mathbb{R}$ satisfying $d^+a = 0$. Integration by parts shows that

$$\int_{W_\infty} \langle da, da\rangle = 0.$$

Therefore, the 1–form a is closed. Since $H^1(W_\infty, \mathbb{R}) = 0$, there exists $\xi\colon W_\infty \to \mathbb{R}$ such that $a = i\,d\xi$, and $(A, 0)$ is gauge-equivalent to $(A_0, 0)$ via the gauge transformation $u = e^{i\xi} \in L^2_{k+1,\mathrm{loc}}(W_\infty)$. Now we use [23, Definition 24.2.1] to conclude that $[(A, 0)] = [(A_0, 0)]$. □

Lemma 8.12 *For any $\epsilon > 0$ there exists a positive integer N such that, for any $n \geq N$:*

(1) $T_n \geq 4$.

(2) *There exists a gauge transformation $u_n\colon W_4 \to S^1$ such that*

$$\|u_n \cdot (A_n, \varphi_n)|_{W_4} - (A_0, 0)\|_{L^2_k(W_4)} < \epsilon.$$

(3) *For any interval $[t, t+4] \subset [-T_n, T_n]$, there exists a gauge transformation $u_{n,t}\colon [t, t+4] \times Y \to S^1$ such that*

$$\|u_{n,t} \cdot (A_n, \varphi_n)|_{[t,t+4]\times Y} - (A_0, 0)\|_{L^2_k([t,t+4]\times Y)} < \epsilon.$$

Proof Claim (1) follows trivially from $T_n \to \infty$. Since W_4 is a compact subset of W_∞, claim (2) follows from (A). To prove claim (3), let $C > 0$ be the constant from (C). If $[t, t+4]$ belongs to $[-T_n + C, T_n - C]$, claim (3) follows from (C). Otherwise, $[t, t+4]$ belongs to either $[-T_n, -T_n + C + 4]$ or $[T_n - C - 4, T_n]$. For every n, these are identified with the fixed compact subsets $[0, C+4]$ and $[-C-4, 0]$ of W_∞, hence the result follows from (A). □

Lemma 8.13 *Let $\epsilon_0 > 0$ be the constant fixed in the beginning of Section 8.2. Then there exists an integer $N > 0$ such that, for any $n \geq N$, we have*

$$\|\mathcal{D}^+_{A_n}(\varphi_n)\|_{L^2(X_{T_n})} \leq \tfrac{1}{2}\epsilon_0 \cdot \|\varphi_n\|_{L^2(X_{T_n})}.$$

Proof Since (A_n, φ_n) solves the perturbed Seiberg–Witten equations, we have the equality

$$\mathcal{D}^+_{A_n}(\varphi_n) = \widehat{\mathfrak{p}}^1_{T_n}(A_n, \varphi_n),$$

where $\widehat{\mathfrak{p}}^1_{T_n}(A_n, \varphi_n)$ denotes the spinor component of the perturbation term $\widehat{\mathfrak{p}}_{T_n}(A_n, \varphi_n)$; see Section 8.2. It is supported in $I_{0,T_n} \cup U$, where U is a collar neighborhood of ∂W. By our definition of $\widehat{\mathfrak{p}}_{T_n}$,

$$\widehat{\mathfrak{p}}^1_{T_n}(A_n, \varphi_n)|_{\{t\}\times Y} = \mathfrak{q}^1((A_n, \varphi_n)|_{\{t\}\times Y}) \quad \text{for all } t \in [-T_n, T_n].$$

Since $\mathfrak{q}$ satisfies Assumption 2.4, it follows from (6) that there exists a neighborhood V of $[\theta] \in \mathcal{B}(Y)$ such that

$$\|\mathfrak{q}^1(B, \psi)\|_{L^2(Y)} \leq \tfrac{1}{2}\epsilon_0 \cdot \|\psi\|_{L^2(Y)}$$

for any configuration (B, ψ) with $[(B, \psi)] \in V$. But then, by Lemma 8.12, there exists a positive integer N such that for any $n \geq N$, we have $[(A_n, \varphi_n)|_{\{t\}\times Y}] \in V$ for all

$t \in [-T_n, T_n]$. Therefore, we have the estimate

$$\|\mathcal{D}^+_{A_n}(\varphi_n)|_{\{t\}\times Y}\|_{L^2(Y)} \le \tfrac{1}{2}\epsilon_0 \cdot \|\varphi_n|_{\{t\}\times Y}\|_{L^2(Y)} \quad \text{for all } t \in [-T_n, T_n],$$

which implies that

$$\|\mathcal{D}^+_{A_n}(\varphi_n)\|_{L^2(I_{0,T_n})} \le \tfrac{1}{2}\epsilon_0 \cdot \|\varphi_n\|_{L^2(I_{0,T_n})}.$$

A similar argument involving estimate (21) shows that

$$\|\mathcal{D}^+_{A_n}(\varphi_n)\|_{L^2(U)} \le \tfrac{1}{2}\epsilon_0 \cdot \|\varphi_n\|_{L^2(U)}.$$

This completes the proof of the lemma because $\mathcal{D}^+_{A_n}(\varphi_n)$ is supported on $I_{0,T_n} \cup U$. □

Lemma 8.14 *For any $\epsilon > 0$ there exists a positive integer N such that, for any $n \ge N$, there is a global gauge transformation $\widetilde{u}_n\colon X_{T_n} \to S^1$ such that*

$$\|A_n - \widetilde{u}_n^{-1} d\widetilde{u}_n\|_{C^0(X_{T_n})} < \epsilon.$$

Before we go on to prove this lemma, we will show how it implies Lemma 8.10. We know from Proposition 7.1 that, as $n \to \infty$, the smallest eigenvalue of the operator $\mathcal{D}^-\mathcal{D}^+$ on X_{T_n} is bounded from below by $\epsilon_1^2 > 0$. Therefore, for all sufficiently large n and all positive spinors ψ,

$$\|\mathcal{D}^+\psi\|_{L^2(X_{T_n})} \ge \epsilon_0 \cdot \|\psi\|_{L^2(X_{T_n})}.$$

On the other hand, consider the sequence of gauge transformations $\widetilde{u}_n$ from Lemma 8.14 and the sequence of spinors $\psi_n = \widetilde{u}_n\varphi_n$ with $\epsilon = \frac{1}{4}\epsilon_0$. Lemma 8.13 implies that

$$\|\mathcal{D}^+_{A_n - \widetilde{u}_n^{-1}d\widetilde{u}_n}(\psi_n)\|_{L^2(X_{T_n})} \le \tfrac{1}{2}\epsilon_0 \cdot \|\psi_n\|_{L^2(X_{T_n})}$$

We then conclude using Lemma 8.14 that

$$\begin{aligned}\|\mathcal{D}^+\psi_n\|_{L^2(X_{T_n})} &= \|\mathcal{D}^+_{A_n - \widetilde{u}_n^{-1}d\widetilde{u}_n}(\psi_n) - (A_n - \widetilde{u}_n^{-1}d\widetilde{u}_n)\cdot\psi_n\|_{L^2(X_{T_n})}\\ &< \epsilon_0 \cdot \|\psi_n\|_{L^2(X_{T_n})}\end{aligned}$$

for all sufficiently large n, which gives a contradiction.

Proof of Lemma 8.14 Let N be as in Lemma 8.12; then, for any $n \ge N$, there exists an integer $m_n \ge 1$ such that $-T_n + 2m_n \in [T_n - 4, T_n - 2)$. For $1 \le j \le m_n - 1$, we denote the gauge transformation $u_{n,-T_n+2j}$ of Lemma 8.12 by $u(n, j)$. We wish to glue the gauge transformations u_n, $u(n,1)$, $u(n,2)$, …, $u(n, m_n - 1)$ together with the help of cutoff functions.

First, we pick a basepoint $o \in Y$. After multiplying by suitable constant gauge transformations, we may assume that

$$u(n, j)(\{-T_n + 2j + 2\} \times o) = u(n, j+1)(\{-T_n + 2j + 2\} \times o).$$

Since Y is a rational homology sphere, we have

$$u(n, j)|_{[-T_n+2j+2,-T_n+2j+4]\times Y} = e^{i\xi(n,j)} \cdot u(n, j+1)|_{[-T_n+2j+2,-T_n+2j+4]\times Y},$$

where

$$\xi(n, j)\colon [-T_n + 2j + 2, -T_n + 2j + 4] \times Y \to \mathbb{R}$$

satisfies

$$\xi(n, j)(\{-T_n + 2j + 2\} \times o) = 0. \tag{24}$$

Now, since

$$\|A_n - u(n, j)^{-1} du(n, j)\|_{L^2_k([-T_n+2j+2,-T_n+2j+4]\times Y)} < \epsilon$$

and

$$\|A_n - u(n, j+1)^{-1} du(n, j+1)\|_{L^2_k([-T_n+2j+2,-T_n+2j+4]\times Y)} < \epsilon,$$

we have

$$\begin{aligned}\|d\xi(n, j)\|&_{L^2_k([-T_n+2j+2,-T_n+2j+4]\times Y)} \\ &= \|u(n, j)^{-1} du(n, j) - u(n, j+1)^{-1} du(n, j+1)\|_{L^2_k([-T_n+2j+2,-T_n+2j+4]\times Y)} \\ &< 2\epsilon.\end{aligned}$$

Together with (24), this implies that there exists a constant $C_2 > 0$ such that

$$\|\xi(n, j)\|_{L^2_{k+1}([-T_n+2j+2,-T_n+2j+4]\times Y)} < C_2 \cdot \epsilon. \tag{25}$$

Next, choose a bump function $\tau\colon [0, 2] \to [0, 1]$ such that $\tau([0, 1]) = 1$ and $\tau\left(\left[\frac{3}{2}, 2\right]\right) = 0$. We let $\tau_t\colon [t, t+2] \times Y \to [0, 1]$ be the function defined by the formula $\tau_t(s) = \tau(s-t)$. Extend the function $\tau_{-T_n+2j+2} \cdot \xi(n, j)$ defined on

$$[-T_n + 2j + 2, -T_n + 2j + 4] \times Y$$

by zero to obtain a function

$$\widetilde{\xi}(n, j)\colon [-T_n + 2j + 2, -T_n + 2j + 5] \times Y \to \mathbb{R}.$$

We have $\|\widetilde{\xi}(n,j)\|_{L^2_{k+1}([-T_n+2j+2,-T_n+2j+5])} < C_3\epsilon$ for a constant $C_3>0$, which implies that

$$\|d\widetilde{\xi}(n,j)\|_{C^0([-T_n+2j+2,-T_n+2j+5]\times Y)} < C_4\cdot\epsilon \tag{26}$$

for another constant $C_4>0$. Now, for $2\le j\le m_n-1$, consider the gauge transformations

$$\overline{u}(n,j) = e^{i\widetilde{\xi}(n,j-1)}\cdot u(n,j)\colon [-T_n+2j,-T_n+2j+3]\times Y\to S^1.$$

One can easily see that $\overline{u}(n,j)$ equals $u(n,j-1)$ over $[-T_n+2j,-T_n+2j+1]\times Y$ and equals $u(n,j)$ over $[-T_n+2j+2,-T_n+2j+3]\times Y$. Therefore, the functions $\overline{u}(n,2)$, $\overline{u}(n,3),\dots,\overline{u}(n,m_n-1)$, together with the functions $u(n,1)|_{[-T_n+2,-T_n+5]\times Y}$ and $u(n,m_n-1)|_{[-T_n+2m_n,-T_n+2m_n+2]\times Y}$, agree with each other on the overlaps. As a result, we can glue them together to obtain a gauge transformation

$$\overline{u}_n\colon [-T_n+2,-T_n+2m_n+2]\times Y\to S^1.$$

Since $\overline{u}_n$ is obtained from $u(n,j)$ by multiplying by $e^{i\widetilde{\xi}(n,j-1)}$, there is a constant $C_5>0$ such that

$$\|A_n-\overline{u}_n^{-1}d\overline{u}_n^{-1}\|_{C^0([-T_n+2,-T_n+2m_n+2]\times Y)} < C_5\cdot\epsilon.$$

In our next step, we introduce $\widehat{u}_n = e^{i(a+bt)}\overline{u}_n$, where t is the coordinate in the cylindrical direction, and $a\in[0,2\pi]$ and $b\in[0,\pi/(T_n-2)]$ are chosen so that

$$\widehat{u}_n(\{-T_n+2\}\times o) = u_n(\{-T_n+2\}\times o)$$

and

$$\widehat{u}_n(\{-T_n+2m_n+2\}\times o) = u_n(\{-T_n+2m_n+2\}\times o).$$

There is a constant $C_6>0$ such that

$$\|A_n-\widehat{u}_n^{-1}d\widehat{u}_n^{-1}\|_{C^0([-T_n+2,-T_n+2m_n+2]\times Y)} \le C_6\cdot\epsilon+\frac{\pi}{T_n-2},$$

where, by choosing $N>0$ large enough, we may assume $\pi/(T_n-2)<\epsilon$ for all $n\ge N$. Arguing as before, we can find

$$\xi_n\colon ([-T_n,-T_n+2]\cup[-T_n+2m_n+2,-T_n+2m_n+4])\times Y\to\mathbb{R}$$

such that $e^{i\xi_n}\widehat{u}_n$ equals u_n over the domain of ξ_n and

$$\xi_n(\{-T_n\}\times o) = \xi_n(\{-T_n+2m_n+4\}\times o) = 0.$$

As before, we have the estimate

$$\|d\xi_n\|_{L^2_k} < 4\epsilon + b < 5\epsilon$$

on the domain of ξ_n. Let τ_n be a cut-off function on the domain of ξ_n which equals 1 when restricted to $([-T_n+1,-T_n+2]\cup[-T_n+2m_n+2,-T_n+2m_n+3])\times Y$ and equals 0 when restricted to $\big(\big[-T_n,-T_n+\frac{1}{2}\big]\cup\big[-T_n+2m_n+\frac{7}{2},-T_n+2m_n+4\big]\big)\times Y$. Assume that the C^∞–norm of τ_n is uniformly bounded for all $n \geq N$ and extend $\tau_n \cdot \xi_n$ by zero to a function $\widetilde{\xi}_n\colon W_4 \to \mathbb{R}$. The gauge transformations $e^{i\widetilde{\xi}(n)}u_n$ and $\widehat{u}_n|_{[-T_n+1,-T_n+2m_n+3]\times Y}$ match on the overlap of their domains; therefore, we can glue them together to the desired gauge transformation $\widetilde{u}_n\colon X_{T_n} \to S^1$. Since $\widetilde{u}_n$ is obtained by modifying u_n and $\overline{u}_{n,j}$ using the cutoff functions $\widetilde{\xi}_n$, $\widetilde{\xi}_{n,j}$ and the function $e^{i(a+bt)}$, the estimate of the lemma can be easily verified. □

9 Gluing results

In this section, we will finish the proof of Theorem 3.3 by first establishing a bijective correspondence between monopoles on X_R and monopoles on W_∞ for all sufficiently large R, and then matching the signs to identify $\#\mathcal{M}(X_R, g_R, \widehat{\mathfrak{p}}_R)$ with the Lefschetz number in the monopole chain complex C^o. To simplify notations, we will continue writing $\mathcal{M}(X_R)$ for $\mathcal{M}(X_R, g_R, \widehat{\mathfrak{p}}_R)$.

Theorem 9.1 *Assume that the spin Dirac operator*

$$\mathcal{D}^+(W_\infty)\colon L^2_1(W_\infty;\mathcal{S}^+) \to L^2(W_\infty;\mathcal{S}^-)$$

is an isomorphism. Then, for all sufficiently large $R>0$, the moduli space $\mathcal{M}(X_R)$ is regular, and there exists a homeomorphism

$$\rho\colon \mathcal{M}(X_R) \to \bigcup_{[\alpha]\in\mathfrak{C}^*} \mathcal{M}(W_\infty,[\alpha]). \tag{27}$$

We will first prove Theorem 9.1 by adopting the gluing techniques from [23] to the nonseparating case at hand. Theorem 3.3 will be proved at the end of this section.

9.1 Fiber products

We will be using notations from Section 8.1. Denote by $\mathcal{M}^*(X_T)$ the moduli space of irreducible monopoles on X_T. It follows from Proposition 7.3 that $\mathcal{M}^*(X_T) = \mathcal{M}(X_T)$

for all sufficiently large $T > 0$. The similarly defined moduli spaces $\mathcal{M}^*(W_{T'})$ and $\mathcal{M}^*(I_{T',T})$ will be infinite-dimensional Hilbert manifolds because both $W_{T'}$ and $I_{T',T}$ have nonempty boundary, but we are not imposing any boundary conditions. By the unique continuation theorem [23, Section 10.8], the restriction of an irreducible monopole to the boundary is irreducible; therefore, for all $0 < T' < T < \infty$ we have well-defined restriction maps

$$R^-_{T'}\colon \mathcal{M}^*(W_{T'}) \to \mathcal{B}^*(Y) \times \mathcal{B}^*(Y) \quad \text{and} \quad R^+_{T',T}\colon \mathcal{M}^*(I_{T',T}) \to \mathcal{B}^*(Y) \times \mathcal{B}^*(Y),$$

where $\mathcal{B}^*(Y)$ consists of irreducible configurations in $\mathcal{B}(Y)$. One can show that these maps are embeddings of Hilbert manifolds. It will be convenient to extend these notations to the case of $T = \infty$. Recall that in Section 8.1 we defined $I_{T',\infty} = W_\infty - \operatorname{int}(W_{T'})$. Let $\mathcal{M}^*(I_{T',\infty})$ be the moduli space of monopoles (A, φ) on $I_{T',\infty}$ satisfying

$$\lim_{t\to+\infty}[(A,\varphi)|_{\{t\}\times Y}] = \lim_{t\to-\infty}[(A,\varphi)|_{\{t\}\times Y}] \in \mathfrak{C}^*.$$

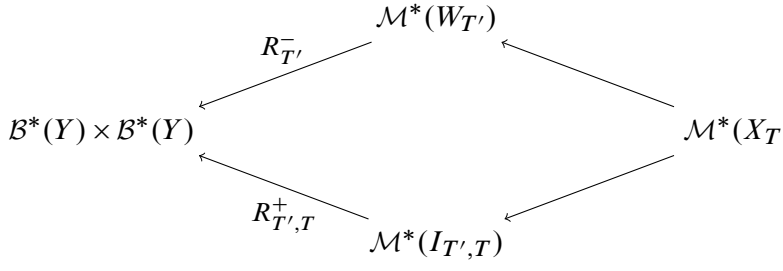
Then the restriction map

$$R^+_{T',\infty}\colon \mathcal{M}^*(I_{T',\infty}) \to \mathcal{B}^*(Y) \times \mathcal{B}^*(Y)$$

is still well defined and is an embedding of Hilbert manifolds. To unify the notations, we will write $X_\infty = W_\infty$ and

$$\mathcal{M}^*(X_\infty) = \bigcup_{[\alpha]\in\mathfrak{C}^*} \mathcal{M}^*(W_\infty, [\alpha]).$$

Then, for all $0 < T' < T \le \infty$, we have the following commutative diagram, whose unmarked arrows are given by restriction to submanifolds:



Lemma 9.2 *For all $0 < T' < T \le \infty$, the above diagram is a Cartesian square, that is, $\mathcal{M}^*(X_T)$ is homeomorphic to the fiber product*

$$\operatorname{Fib}(R^-_{T'}, R^+_{T',T}) = \{(x, y) \in \mathcal{M}^*(W_{T'}) \times \mathcal{M}^*(I_{T',T}) \mid R^-_{T'}(x) = R^+_{T',T}(y)\}.$$

Proof The proof is identical to that of [23, Lemma 19.1.1], which deals with the separating case, and will be omitted. Note that some new issues would appear if we were to glue reducible monopoles in the blown-up moduli space but we do not deal with them here. □

In less formal terms, Lemma 9.2 asserts that the moduli space $\mathcal{M}^*(X_T)$ is the intersection of the moduli spaces $\mathcal{M}^*(W_{T'})$ and $\mathcal{M}^*(I_{T',T})$ viewed as submanifolds of $\mathcal{B}^*(Y) \times \mathcal{B}^*(Y)$. We will prove Theorem 9.1 by showing that all of these intersections occur in small neighborhoods of constant trajectories γ_α, where the intersection points for sufficiently large T can be matched with those for $T = \infty$ using the implicit function theorem.

9.2 Technical lemmas

The central role in our proof will be played by the following theorem, which is a special case of [23, Theorem 18.2.1]. When T is finite, I_T will stand for $[-T, T] \times Y$ and R_T^+ for the corresponding restriction map.

Theorem 9.3 *There exists a constant $T_1 > 0$ such that for all $T \geq T_1$ and $[\alpha] \in \mathfrak{C}^*$, there exist smooth maps*

$$u_{[\alpha]}(T, -)\colon B([\alpha]) \to \mathcal{M}^*(I_T), \quad u_{[\alpha]}(\infty, -)\colon B([\alpha]) \to \mathcal{M}^*(I_\infty)$$

which are diffeomorphisms from an open neighborhood $B([\alpha]) \subset \mathcal{B}^*(Y)$, which is independent of T, onto neighborhoods of the constant solutions $[\gamma_\alpha]$. Moreover, the maps

$$\mu_{[\alpha],T} = R_T^+ \circ u_{[\alpha]}(T, -)\colon B([\alpha]) \to \mathcal{B}^*(Y) \times \mathcal{B}^*(Y)$$

are smooth embeddings for all $T \in [T_1, \infty]$, and we have a C^∞_{loc} convergence

$$\mu_{[\alpha],T} \to \mu_{[\alpha],\infty} \quad \text{as } T \to \infty.$$

Finally, there exists a constant $\eta > 0$, independent of T, such that the image of the map $u_{[\alpha]}(T, -)$ contains all the trajectories $[\gamma] \in \mathcal{M}^*(I_T)$ with $\|\gamma - \gamma_\alpha\|_{L^2_k(I_T)} \leq \eta$.

Remark 9.4 In addition, we will assume that, for all $S, T \in [T_1, \infty]$ and all $[\alpha] \neq [\beta] \in \mathfrak{C}^*$,

$$\operatorname{im}(u_{[\alpha]}(S, -)) \cap \operatorname{im}(u_{[\beta]}(T, -)) = \varnothing, \quad \operatorname{im}(\mu_{[\alpha]}(S, -)) \cap \operatorname{im}(\mu_{[\beta]}(T, -)) = \varnothing.$$

Lemma 9.5 *Let $\eta > 0$ be as in Theorem 9.3. Then one can find constants $0 < T_2 < T_3 < \infty$ with the following significance: for any $T \in [T_3, \infty]$, any element of $\mathcal{M}^*(X_T)$ can be represented by a monopole (A, φ) such that $\|(A,\varphi)|_{I_{T_2,T}} - \gamma_\alpha\|_{L^2_k(I_{T_2,T})} \le \frac{1}{2}\eta$ for some $[\alpha] \in \mathfrak{C}^*$.*

Proof The case $T = \infty$ should be clear once we remember that, for $T = \infty$, the notation $I_{T_2,\infty}$ means $W_\infty - \mathrm{int}(W_{T_2})$. Let us now assume that $T < \infty$ and suppose to the contrary that the constants T_2 and T_3 do not exist. Then we can find two sequences of real numbers $T_n' < T_n$, both going to infinity as $n \to \infty$, and a sequence of monopoles (A_n, φ_n) on X_{T_n} such that, for any gauge transformation $u_n\colon I_{T_n',T_n} \times Y \to S^1$ and any $\alpha \in \mathfrak{C}^*$, we have

$$\|u_n \cdot (A_n,\varphi_n)|_{I_{T_n',T_n}} - \gamma_\alpha\|_{L^2_k(I_{T_n',T_n})} > \tfrac{1}{2}\eta$$

for all n. According to Theorem 8.2, after passing to a subsequence, we may assume that $[(A_n,\varphi_n)]$ converges to an element in $\mathcal{M}(W_\infty, [\alpha'])$ for some $[\alpha'] \in \mathfrak{C}^*$. This leads to a contradiction with part (2) of Definition 8.3, the definition of convergence. □

Lemma 9.6 *Let T_2 and T_3 be the constants from Lemma 9.5. Then for any $T \in [T_3, +\infty]$, there exists a homeomorphism*

$$\mathcal{M}^*(X_T) = \bigcup_{[\alpha]\in\mathfrak{C}^*} \mathrm{Fib}(R^-_{T_2}, \mu_{[\alpha],T-T_2})$$

between $\mathcal{M}^*(X_T)$ and a disjoint union of the fiber products. Moreover, the moduli space $\mathcal{M}^*(X_T)$ is regular if and only if, for any $[\alpha] \in \mathfrak{C}^*$, the images of the maps $R^-_{T_2}$ and $\mu_{[\alpha],T-T_2}$ intersect transversely in $\mathcal{B}^*(Y) \times \mathcal{B}^*(Y)$.

Proof The first assertion is a straightforward corollary of Theorem 9.3 and Lemmas 9.2 and 9.5. The second assertion is essentially [23, Theorem 19.1.4]. □

Lemma 9.7 *Let $B([\alpha]) \subset \mathcal{B}^*(Y)$ be an open neighborhood as in Theorem 9.3, and let $x_n \in B([\alpha])$ be a sequence such that $\mu_{[\alpha],T_n}(x_n) \to \mu_{[\alpha],\infty}(x)$ for some $T_n \to \infty$ and some $x \in B([\alpha])$. Then $x_n \to x$ in the topology of $\mathcal{B}^*(Y)$.*

Proof This will be clear once we recall the construction of the map $u_{[\alpha]}(T, -)$ from Section 18.4 of [23]. The tangent space $T_{[\alpha]}\mathcal{B}^*(Y)$, denoted by $\mathcal{K}$, has a decomposition $\mathcal{K} = \mathcal{K}^+_{\mathfrak{q}} \oplus \mathcal{K}^-_{\mathfrak{q}}$ given by the spectral decomposition of the Hessian of $\mathcal{L}_{\mathfrak{q}}$. We will denote by $\Pi^\pm\colon \mathcal{K} \to \mathcal{K}^\pm_{\mathfrak{q}}$ the corresponding projections. We will also identify small open balls $B_\epsilon([\alpha]) \subset \mathcal{K}$ of radius $\epsilon > 0$ with open neighborhoods $B([\alpha]) \subset \mathcal{B}^*(Y)$.

According to [23, Section 18.4], for any sufficiently small $\epsilon > 0$, there exist constants $\epsilon' > 0$ and $T_1 > 0$ with the following significance: for any $T \in [T_1, \infty]$ and any $(a, b) \in B_\epsilon([\alpha]) \subset \mathcal{K}_{\mathfrak{q}}^+ \oplus \mathcal{K}_{\mathfrak{q}}^-$, there exists a unique

$$[\gamma_{(a,b)}^T] \in (R_T^+)^{-1}(B([\alpha]) \times B([\alpha])) \subset \mathcal{M}^*(I_T)$$

such that $\|\gamma_{(a,b)}^T - \gamma_\alpha\|_{L_k^2(I_T)} < \epsilon'$ and $(\Pi_+, \Pi_-)(R_T^+([\gamma_{(a,b)}^T])) = (a, b)$. The map $u_{[\alpha]}(T, -)$ is then defined by the formula

$$u_{[\alpha]}(T, (a, b)) = [\gamma_{(a,b)}^T].$$

With this definition in place, write $x_n = (a_n, b_n) \in \mathcal{K}_{\mathfrak{q}}^+ \oplus \mathcal{K}_{\mathfrak{q}}^-$ and similarly $x_\infty = (a_\infty, b_\infty) \in \mathcal{K}_{\mathfrak{q}}^+ \oplus \mathcal{K}_{\mathfrak{q}}^-$. Then

$$\mu_{[\alpha],T_n}(x_n) = (a_n, *, *, b_n) \quad \text{and} \quad \mu_{[\alpha],\infty}(x_\infty) = (a_\infty, *, *, b_\infty),$$

where we embedded $B([\alpha]) \times B([\alpha])$ into $\mathcal{K}_{\mathfrak{q}}^+ \oplus \mathcal{K}_{\mathfrak{q}}^- \oplus \mathcal{K}_{\mathfrak{q}}^+ \oplus \mathcal{K}_{\mathfrak{q}}^-$. From this we clearly see that $\mu_{[\alpha],T_n}(x_n) \to \mu_{[\alpha],\infty}(x)$ implies $x_n \to x_\infty$. □

9.3 Proof of Theorem 9.1

Let T_2 and T_3 be the constants from Lemma 9.5. Using Lemma 9.6 and our regularity assumption on $\mathcal{M}(W_\infty)$, we can claim that, for any $[\alpha] \in \mathfrak{C}^*$, the images of the maps $R_{T_2}^-$ and $\mu_{[\alpha],\infty}$ intersect each other transversely and we have a homeomorphism (bijection)

$$\mathcal{M}^*(X_\infty) = \bigcup_{[\alpha] \in \mathfrak{C}^*} \mathrm{Fib}(R_{T_2}^-, \mu_{[\alpha],\infty}).$$

By Theorem 9.3, the maps $\mu_{[\alpha],T}$ converge to $\mu_{[\alpha],\infty}$ in the C_{loc}^∞ topology as $T \to \infty$. The implicit function theorem now implies that there exists a constant $T_4 \geq T_3$ such that any $(y, x) \in \mathrm{Fib}(R_{T_2}^-, \mu_{[\alpha],\infty})$ has an open neighborhood $U(y, x) \subset \mathcal{M}^*(W_{T_2}) \times B_{[\alpha]}$ with the following significance: for any $T \geq T_4$, the images of embeddings $R_{T_2}^-$ and $\mu_{[\alpha],T-T_2}$ intersect each other in exactly one point in $U(y, x)$, and this intersection is transverse. Therefore, our proof will be finished once we prove the following lemma:

Lemma 9.8 *There exists a constant $T_5 \geq T_4$ such that, for any $T \geq T_5$ and any element of $\mathcal{M}^*(X_T)$ represented by*

$$(y', x') \in \bigcup_{[\alpha] \in \mathfrak{C}^*} \mathrm{Fib}(R_{T_2}^-, \mu_{[\alpha],T-T_2}),$$

there exists a point $(y, x) \in \mathrm{Fib}(R_{T_2}^-, \mu_{[\alpha],\infty})$ such that $(y', x') \in U(y, x)$.

Proof Suppose to the contrary that this is not the case. Then there is a sequence

$$(y_n, x_n) \in \mathrm{Fib}(R^-_{T_2}, \mu_{[\alpha_n],T_n}), \quad T_n \to \infty,$$

representing monopoles (A_n, φ_n) on X_{T_n} such that $(y_n, x_n) \notin U(y, x)$ for any (y, x) in the fiber product $\mathrm{Fib}(R^-_{T_2}, \mu_{[\alpha],\infty})$. By Theorem 8.2, after passing to a subsequence, we may assume that $[(A_n, \varphi_n)]$ converges to a monopole $[(A_\infty, \varphi_\infty)]$ on X_∞. Represent the latter by $(y_\infty, x_\infty) \in \mathrm{Fib}(R^-_{T_2}, \mu_{[\alpha],\infty})$. Then:

(a) $y_n \to y_\infty$. This follows from part (1) of Definition 8.3, the definition of convergence, because W_{T_2} is a compact subset of W_∞ and

$$y_n = [(A_n, \varphi_n)|_{W_{T_2}}] \quad \text{and} \quad y_\infty = [(A_\infty, \varphi_\infty)|_{W_{T_2}}].$$

(b) $x_n \to x_\infty$. This follows from Remark 9.4, which implies that $[\alpha_n] = [\alpha]$ for all large enough n, and from Lemma 9.7 applied to the convergent sequence

$$\mu_{[\alpha_n],T_n}(x_n) = R^-_{T_2}(y_n) \to R^-_{T_2}(y_\infty) = \mu_{[\alpha],\infty}(x_\infty).$$

Therefore, $(y_n, x_n) \in U(y_\infty, x_\infty)$ for all sufficiently large n. This gives a contradiction. □

9.4 Proof of Theorem 3.3

All we need to do is compare the signs with which the monopoles corresponding to each other under the map (27) are counted in $\#\mathcal{M}(X_R)$ and $\mathrm{Lef}(W_*\colon C^o \to C^o)$. This is done in [12, Proposition 3] under a different $\mathbb{Z}/2$–grading convention. Since our setting here is slightly different, we give an alternative argument using the excision principle.

In the product case, $X = S^1 \times Y$, the orientation transport argument of [41] (see also [54] in the instanton setting) can be used to show that

$$\#\mathcal{M}(X) = \pm\chi(C^o) \tag{28}$$

up to an overall sign which is independent of the choice of Y. The sign is determined by the sign fixing condition

$$\lambda_{\mathrm{SW}}(X) = -\lambda(Y) \tag{29}$$

of [41, Section 11.2], where $\lambda(Y)$ is the Casson invariant normalized by

$$\lambda(\Sigma(2,3,5)) = -1.$$

To calculate the sign in (28), we will let $X = S^1 \times Y$, where $Y = \Sigma(2,3,7)$ is the Brieskorn homology sphere endowed with a natural metric h realizing its Thurston geometry, and compute

$$\lambda_{\mathrm{SW}}(X) = \#\mathcal{M}(X,g) - w(X,g) \tag{30}$$

with respect to the product metric $g = dt^2 + h$. The correction term in this formula equals

$$w(X,g) = \operatorname{ind} \mathcal{D}^+(Z_\infty) + \tfrac{1}{8} \operatorname{sign} Z,$$

where Z can be any smooth compact spin manifold with boundary Y. Let us choose Z to be the plumbed manifold with the intersection form isomorphic to $E_8 \oplus H$, where E_8 is positive definite. According to [57, Section 6], the index of $\mathcal{D}^+(Z_\infty)$ vanishes; therefore, $w(X,g) = 1$. Since $\lambda(Y) = -1$, we conclude from formulas (29) and (30) that $\#\mathcal{M}(X,g) = 2$. This needs to be compared to the Euler characteristic of the chain complex C^o. The latter complex is known [40] to have exactly two generators of the same grading. Note that

$$-2h(Y) = 0 > \mathrm{gr}^{\mathbb{Q}}([\theta_0]) = -2w(X,g) = -2.$$

By [33, Lemma 2.9], the boundary map $\partial^o_s \colon C^o \to C^s$ must be nonzero. Therefore, both generators of C^o must be of odd grading, and the overall sign in formula (28) is a minus.

The general case now follows by using the excision principle for determinant bundles as in [23, Section 25].

10 Generic metrics

This section contains two results about metrics with no harmonic spinors on two types of spin manifolds: compact manifolds with product regions and noncompact manifolds with cylindrical ends. These results are similar to those of Amman, Dahl and Humbert [1] and are proved by a modification of their argument.

10.1 Manifolds with product regions

Let X be a connected smooth spin compact manifold of dimension $n \equiv 0 \pmod 4$ and $f \colon X \to S^1$ a smooth map with the primitive cohomology class $[df] \in H^1(X;\mathbb{Z})$. Let $Y \subset X$ be a connected manifold Poincaré dual to $[df]$, and introduce Riemannian metrics h on Y and g on X such that $g = dt^2 + h$ in a product region $[-\varepsilon, \varepsilon] \times Y \subset X$.

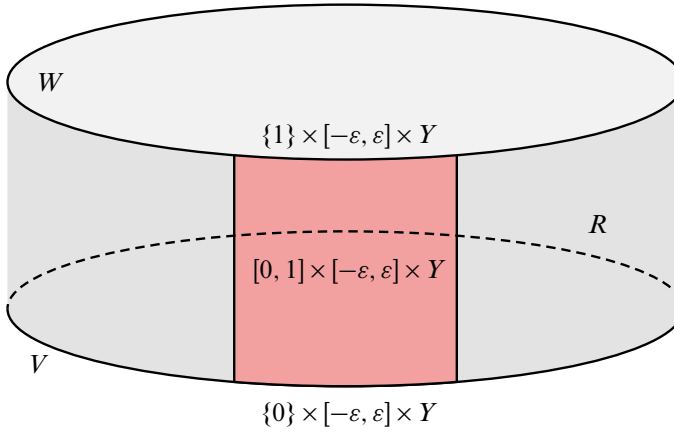


Figure 1: The spin cobordism

According to [1, Theorem 1.1], the spin Dirac operator $\mathcal{D}(Y,h)$ is invertible in the Sobolev L^2 completion for a generic choice of h.

Theorem 10.1 *Let X be a manifold as above with a product region $[-\varepsilon,\varepsilon]\times Y$ and suppose that the metric h is such that $\ker \mathcal{D}(Y,h)=0$. If X is spin cobordant to zero then there exists a metric g on X such that $g=dt^2+h$ in the product region and $\mathcal{D}^+(X,g)$ is invertible in the Sobolev L^2 completion.*

Proof We will use the method of transporting invertibility of the operator $\mathcal{D}^+(X,g)$ across a spin cobordism as in [1] but we will be more specific in choosing the cobordism.

Start with the manifold $X'=S^1\times Y$ with the product metric $g'=dt^2+h$ and the spin Dirac operator $\mathcal{D}^+(X',g')=dt\cdot(\partial/\partial t+\mathcal{D}(Y,h))$. Since the operator $\mathcal{D}(Y,h)$ is invertible, so is the operator $\mathcal{D}^+(X',g')$ by a direct calculation.

We claim that there is a spin cobordism from X' to X that contains a product region $I\times[-\varepsilon,\varepsilon]\times Y$, and furthermore has handles of index at most $n-1$. To prove this, write

$$X = W\cup_{\{-\varepsilon,\varepsilon\}\times Y}([-\varepsilon,\varepsilon]\times Y) \quad \text{and} \quad X' = V\cup_{\{-\varepsilon,\varepsilon\}\times Y}[-\varepsilon,\varepsilon]\times Y;$$

see Figure 1. We assume of course that V has a product structure as well. Let us consider

$$U = W\cup([0,1]\times\{-\varepsilon,\varepsilon\}\times Y)\cup V,$$

where $\{0\}\times\{-\varepsilon,\varepsilon\}\times Y$ is identified with ∂V and $\{1\}\times\{-\varepsilon,\varepsilon\}\times Y$ is identified with ∂W. After rounding corners, U is a spin manifold diffeomorphic to X, hence U

is the boundary of a connected spin manifold R of dimension $n+1$. View R as a cobordism from V to W, relative to the product region $[0,1]\times\partial V$, and give it a handle decomposition.

By surgeries on generators of $\pi_1(R)$, preserving the spin condition, we can make R simply connected, and canceling the 0–handles and $(n+1)$–handles, we may assume that all the handles have index between 1 and n. Then a standard handle-trading argument can be used to eliminate the handles of index 1 and n. Gluing this to the product region $I\times[-\varepsilon,\varepsilon]\times Y$ gives the desired cobordism.

According to [1, Theorem 1.1], the manifold X admits a metric g with $\ker \mathcal{D}(X,g)=0$. This explicitly constructed metric matches the original product metric on X' away from arbitrarily thin tubular neighborhoods of the surgery spheres. By construction, these spheres lie outside of the product region, so that the metric remains a product there. Since X is spin cobordant to zero, we conclude that $\widehat{A}(X)=0$. Therefore, the operator $\mathcal{D}^+(X,g)$ has zero index and must be invertible. □

Remark 10.2 The existence of just a single metric as in Theorem 10.1 implies that the set of such metrics is in fact generic (that is, C^0 open and C^∞ dense) in the space of all metrics on X with a fixed product metric on $[-\varepsilon,\varepsilon]\times Y$. The proof of [35, Proposition 3.1] goes through after one notes that fixing a product metric on $[-\varepsilon,\varepsilon]\times Y$ defines a convex subset in the space of all metrics.

10.2 Manifolds with product ends

Let X be a manifold from the previous section with a product region $[-\varepsilon,\varepsilon]\times Y$ and a metric g which in the product region takes the form $g=dt^2+h$. Cut X open along $Y=\{0\}\times Y$ into a cobordism W, and attach infinite product ends to W. This results in the noncompact manifold

$$W_\infty=((-\infty,0]\times Y)\cup W\cup([0,\infty)\times Y).$$

Theorem 10.3 *Let us suppose that* $\ker\mathcal{D}(Y,h)=0$ *and that* X *is spin cobordant to zero. Then one can find a metric* g *on* W_∞ *such that* $g=dt^2+h$ *on the product ends and* $\ker\mathcal{D}^+(W_\infty,g)=0$ *in the Sobolev* L^2 *completion.*

Remark 10.4 The conditions of this theorem are obviously satisfied for spin 4–manifolds X with integral homology of $S^1\times S^3$ because the spin cobordism class of such a manifold is determined by its $\widehat{A}$–genus,

$$\widehat{A}(X)=-\tfrac{1}{8}\operatorname{sign}(X)=0.$$

The rest of this section will be dedicated to the proof of Theorem 10.3. We showed in the proof of Theorem 10.1 that X can be obtained from $X' = S^1 \times Y$ by performing surgery along spheres disjoint from the product region $[-\varepsilon, \varepsilon] \times Y$. In the language of manifolds with product ends, this implies that W_∞ can be obtained from the product manifold $\mathbb{R} \times Y$ by performing surgeries inside a compact region in $\mathbb{R} \times Y$. Since $\ker \mathcal{D}(Y, h) = 0$, the Sobolev L^2 completion of the operator $\mathcal{D}^+(X')$ is obviously invertible. Therefore, we can proceed with the construction of a desired metric on W_∞ exactly as in [1] making some changes along the way to account for the noncompactness of W_∞.

The first change comes up in the proof of [1, Lemma 3.4] which uses the Rellich lemma to conclude that an L^2_1 bounded sequence of spinors contains a strongly convergent subsequence in L^2. While the Rellich lemma fails on noncompact manifolds, the sequence of *harmonic* spinors in Lemma 3.4 still admits a strongly convergent subsequence: we first apply the Rellich lemma on the compact manifold $([-1, 0] \times Y) \cup W \cup ([0, 1] \times Y)$, and then use the following estimate:

Lemma 10.5 *Let $X = [0, \infty) \times Y$ be a manifold with product metric $g = dt^2 + h$ such that $\ker \mathcal{D}(Y, h) = 0$. Then there exists a constant $C > 0$ such that, for any harmonic L^2–spinor φ on X,*

$$\|\varphi\|_{L^2([0,\infty)\times Y)} \le C \cdot \|\varphi\|_{L^2([0,1]\times Y)}.$$

Proof Let $\{\psi_\lambda\}$ be an orthonormal basis of eigenspinors of $\mathcal{D}(Y, h)$ and let $\lambda_0 > 0$ stand for the smallest positive λ. Since φ is an L^2–spinor in the kernel of the operator $\mathcal{D}^+(X, g) = dt \cdot (\partial/\partial t + \mathcal{D}(Y, h))$, it takes the form

$$\varphi(t, y) = \sum_{\lambda > 0} a_\lambda \cdot e^{-\lambda t} \cdot \psi_\lambda(y).$$

A direct calculation with this formula gives

$$\|\varphi\|^2_{L^2([0,\infty)\times Y)} = \int_0^\infty \|\varphi\|^2_{L^2(Y)}\, dt = \sum_{\lambda > 0} \frac{|a_\lambda|^2}{2\lambda}$$

and

$$\|\varphi\|^2_{L^2([0,1]\times Y)} = \int_0^1 \|\varphi\|^2_{L^2(Y)}\, dt = \sum_{\lambda > 0} \frac{|a_\lambda|^2}{2\lambda} \cdot (1 - e^{-2\lambda}).$$

This leads to the desired estimate with the constant $C = 1/\sqrt{1 - e^{-2\lambda_0}}$. □

The rest of the proof of [1, Lemma 3.4] goes through using exhaustion of the complement of the surgery sphere S in W_∞ by the compact sets

$$([-1/\varepsilon, 0] \times Y) \cup W \cup ([0, 1/\varepsilon] \times Y) - U_S(\varepsilon),$$

where $U_S(\varepsilon)$ is the open tubular neighborhood of S of radius $\varepsilon > 0$. All the lemmas used in that proof are already proved in [1, Section 2] without the compactness assumption.

The second change comes up in the proof of Step 2 on page 537 of [56]. That step goes through using exhaustion by the compact sets

$$([-Z, 0] \times Y) \cup W \cup ([0, Z] \times Y) - U_S(1/Z)$$

for positive integers Z. In Step 3, convergence in $C^1_{\mathrm{loc}}(W_\infty - S)$ of a sequence of harmonic spinors implies only L^2 convergence on compact subsets of $W_\infty - U_S(s)$. To obtain the desired L^2 convergence on the entire $W_\infty - U_S(s)$, we use Lemma 10.5 one more time.

11 Periodic η–invariants

Let X be a connected smooth spin compact manifold of dimension $n \equiv 0 \pmod 4$ and $f\colon X \to S^1$ a smooth map such that the cohomology class $[df] \in H^1(X; \mathbb{Z})$ is primitive. Choose a connected manifold $Y \subset X$ Poincaré dual to $[df]$. We will assume that the manifold Y with the induced spin structure is a spin boundary and that the $\widehat{A}$–genus of X vanishes; both of these conditions are automatic when X is a homology $S^1 \times S^3$. Define the Riemannian manifold X_R with long neck as in (18). Recall that the metric g_R on X_R takes the form $g_R = dt^2 + h$ along the neck. Consider the holomorphic family

$$\mathcal{D}^{\pm}_z = \mathcal{D}^{\pm}(X_R, g_R) - \ln z \cdot df \quad \text{for } z \in \mathbb{C}^*.$$

Under the assumption that $\ker \mathcal{D}^+_z = 0$ for all z on the unit circle $|z| = 1$, the periodic η–invariant $\eta(X_R)$ was defined in [42] by the formula

$$\eta(X_R) = \frac{1}{\pi i} \int_0^\infty \oint_{|z|=1} \operatorname{Tr}(df \cdot \mathcal{D}^+_z \exp(-t\mathcal{D}^-_z \mathcal{D}^+_z)) \frac{dz}{z}\, dt. \tag{31}$$

It follows from [42, Theorem A and Remark 5.4] that the so-defined periodic η–invariant is independent of the choice of f as long as df is supported in the product region in X_R, which we will assume from now on.

Theorem 11.1 *Let $\eta(Y)$ be the Atiyah–Patodi–Singer η–invariant [2] of the Dirac operator $\mathcal{D} = -\mathcal{D}(Y,h)$.*[3] *Assume that the operator $\mathcal{D}$ has zero kernel, as does the L^2–completion of the operator $\mathcal{D}^+(W_\infty)$ on the manifold W_∞ obtained from W by attaching infinite product ends. Then the invariants $\eta(X_R)$ are well defined and, for all sufficiently large R,*

$$\eta(X_R) = \eta(Y).$$

Proof The well-definedness of $\eta(X_R)$ for all sufficiently large R follows from Proposition 7.3. According to [42, Theorem A and Remark 5.4], for any spin manifold $Z_\infty(X_R)$ whose periodic end is modeled on (X_R, g_R) we have

$$\operatorname{ind}\mathcal{D}^+(Z_\infty(X_R)) = \int_Z \widehat{A}(Z) - \tfrac{1}{2}\eta(X_R).$$

A similar formula applied to an end-periodic manifold Z_∞ whose end is modeled on $(S^1\times Y, dt^2+h)$ yields

$$\operatorname{ind}\mathcal{D}^+(Z_\infty) = \int_Z \widehat{A}(Z) - \tfrac{1}{2}\eta(S^1\times Y).$$

We know from [42, Section 6.3] that $\eta(S^1\times Y) = \eta(Y)$. By subtracting the above formulas from each other, we conclude that $\eta(X_R)$ must differ from $\eta(Y)$ by an even integer. The statement of the theorem will follow as soon as we prove that $\eta(X_R)$ and $\eta(S^1\times Y)$ can be made arbitrarily close by choosing sufficiently large R. The proof of this will occupy the rest of this section. □

Remark 11.2 For any X which is spin cobordant to zero, all of the conditions of Theorem 11.1 are satisfied for the right choice of metric; see Theorem 10.3.

Remark 11.3 Because of the periodic index theorem [42] the statements of Theorems 6.1 and 11.1 are essentially equivalent to each other. What follows is an independent proof of Theorem 11.1 using heat kernel techniques. We chose to include this proof, inspired by [10], because it may be of interest in its own right.

11.1 Heat kernel estimates on $S^1_r \times Y$

Let S^1_r a circle of length r and consider a smooth function f on its universal cover, the real line, such that $f(t+r) = f(t)+1$. The cohomology class of df generates

[3] The change in sign is dictated by the different conventions for the spin Dirac operator with respect to the product metric $g = dt^2+h$ used in this paper and in [42].

$H^1(S^1_r;\mathbb{Z})$, and we will write $df = f'(u)\,du$ with respect to the natural parameter u on the circle S^1_r. Consider the family of elliptic operators $\partial/\partial u - \ln z \cdot f'(u)$ with $z \in \mathbb{C}^*$. If $K^r_z(t;u,v)$ is the kernel of the operator $\exp(t\cdot(\partial/\partial u - \ln z\cdot f')^2)$ then $(\partial/\partial u - \ln z\cdot f')K^r_z(t;u,v)$ is the kernel of the operator

$$\Big(\frac{\partial}{\partial u} - \ln z\cdot f'\Big)\exp\Big(t\cdot\Big(\frac{\partial}{\partial u} - \ln z\cdot f'\Big)^2\Big).$$

Lemma 11.4 *There are positive constants γ and $C>0$, independent of z and r, such that the following estimates hold for all unitary z, $r\geq 1$ and $t>0$:*

$$|K^r_z(t;u,v)| \leq C\cdot t^{-1/2}\cdot e^{-\gamma(u-v)^2/t}$$
$$\Big|\Big(\frac{\partial}{\partial u} - \ln z\cdot f'\Big)K^r_z(t;u,v)\Big| \leq C\cdot t^{-1}\cdot e^{-\gamma(u-v)^2/t}.$$

Proof We begin by observing that, for the purpose of making kernel estimates on the circle $|z|=1$, one may assume that $f'(u)$ is constant and is therefore equal to $1/r$. This can be seen from the formula

$$z^h\cdot\Big(\frac{\partial}{\partial u} - \ln z\cdot f'(u)\Big)\cdot z^{-h} = \frac{\partial}{\partial u} - \ln z\cdot(f'(u)+h'(u)),$$

which holds for any function h defined on S^1_r, and the fact that adding dh does not change the cohomology class of df. Choosing $h(u) = u/r - f(u)$ will then do the job. Conjugating the operator by the unitary complex number $z^{h(x)}$ multiplies the kernel by $z^{h(x)-h(y)}$, hence preserves its norm.

Let $z = e^{is}$; then the kernel $K^r_z(t;u,v)$ of $\exp(t\cdot(\partial/\partial u - is/r))$ is given by the formula

$$K^r_z(t;u,v) = \frac{1}{r}\cdot\sum_{k\in\mathbb{Z}} e^{-t(2\pi k - s)^2/r^2}\cdot e^{2\pi iku/r}\cdot e^{-2\pi ikv/r}$$

with respect to the orthonormal basis $r^{-1/2}e^{2\pi iku/r}$ on the circle S^1_r. One can easily verify that

$$K^r_z(t;u,v) = \frac{1}{r}\cdot K^1_z\Big(\frac{t}{r^2};\frac{u}{r},\frac{v}{r}\Big),$$

where $K^1_z(t;u,v)$ stands for the heat kernel of $\exp(t\cdot(\partial/\partial u - is)^2)$ on the circle of length one. It is well known that there exist positive constants γ and C independent of z such that

$$|K^1_z(t;u,v)| \leq C\cdot t^{-1/2}\cdot e^{-\gamma(u-v)^2/t} \quad \text{for all } t>0.$$

But then

$$|K_z^r(t;u,v)| \le C\cdot\frac{1}{r}\cdot\left(\frac{t}{r^2}\right)^{-1/2}\cdot e^{-\gamma(u/r-v/r)^2/(t/r^2)}$$
$$\le C\cdot t^{-1/2}\cdot e^{-\gamma(u-v)^2/t} \quad \text{for all } t>0$$

with the same constants γ and C independent of z and r. A similar calculation proves the estimate on $(\partial/\partial u - is/r)K_z^r(t;u,v)$ as well; compare with [10, Example 2.5]. □

Given a closed spin Riemannian manifold Y of dimension $n-1$, consider the chiral spin Dirac operators $\mathcal{D}^{\pm}$ on $S^1_r\times Y$ and their twisted versions $\mathcal{D}_z^{\pm}=\mathcal{D}^{\pm}-\ln z\cdot df$. We wish to derive estimates on the kernels of the operators $\exp(-t\mathcal{D}_z^-\mathcal{D}_z^+)$ and $\mathcal{D}_z^+\exp(-t\mathcal{D}_z^-\mathcal{D}_z^+)$ which are uniform in z on the unit circle $|z|=1$ and in r. Denote by $K_z(t;x,y)$ the kernel of the operator $\exp(-t\mathcal{D}_z^-\mathcal{D}_z^+)$; then $\mathcal{D}_z^+K_z(t;x,y)$ is the kernel of $\mathcal{D}_z^+\exp(-t\mathcal{D}_z^-\mathcal{D}_z^+)$.

Lemma 11.5 *Suppose that the Dirac operator $\mathcal{D}$ on Y has zero kernel. Then there exist positive constants γ and C independent of z and r such that the following estimates hold for all unitary z, $r\ge 1$ and $t>0$:*

$$|K_z(t;x,y)| \le C\cdot t^{-n/2}\cdot e^{-\gamma\, d^2(x,y)/t}, \tag{32}$$

$$|\mathcal{D}_z K_z(t;x,y)| \le C\cdot t^{-(n+1)/2}\cdot e^{-\gamma\, d^2(x,y)/t}. \tag{33}$$

Proof On the circle $|z|=1$, we have $\mathcal{D}_z^+ = du\,(\partial/\partial u - \mathcal{D} - \ln z\cdot f')$ and $\mathcal{D}_z^- = (\partial/\partial u + \mathcal{D} - \ln z\cdot f')\,du$, hence $\mathcal{D}_z^-\mathcal{D}_z^+ = -(\partial/\partial u - \ln z\cdot f')^2 + \mathcal{D}^2$ and the kernel of $\exp(-t\mathcal{D}_z^-\mathcal{D}_z^+)$ is the product of the kernels of $\exp(t\cdot(\partial/\partial u - \ln z\cdot f')^2)$ and $\exp(-t\mathcal{D}^2)$. To obtain estimate (32), simply combine the estimate $C\cdot t^{-1/2}\cdot e^{-(u-v)^2/4t}$ on the former kernel with the estimate $C\cdot t^{-(n-1)/2}\cdot e^{-\gamma\, d^2(x',y')/t}$ for $x', y'\in Y$ on the latter; see Lemma 11.4 and [10, Proposition 1.1], respectively. To obtain estimate (33), write the operator $\mathcal{D}_z^+\exp(-t\mathcal{D}_z^-\mathcal{D}_z^+)$ in the form

$$du\left[\left(\frac{\partial}{\partial u}-\ln z\cdot f'\right)\exp\left(t\cdot\left(\frac{\partial}{\partial u}-\ln z\cdot f'\right)^2\right)\right]\cdot\exp(-t\mathcal{D}^2) \tag{34}$$
$$-\,du\cdot\exp\left(t\cdot\left(\frac{\partial}{\partial u}-\ln z\cdot f'\right)^2\right)\cdot[\mathcal{D}\exp(-t\mathcal{D}^2)]$$

and apply the estimates of Lemma 11.4 and [10, Proposition 1.1] twice. □

Lemma 11.6 *Suppose that the Dirac operator $\mathcal{D}$ on Y has zero kernel. Then there are positive constants μ and C independent of z and r such that the following estimates*

hold for all unitary z, $r \geq 1$ and $t \geq 8$:

$$|K_z(t; x, y)| \leq C \cdot e^{-\mu t} \quad \text{and} \quad |\mathcal{D}_z K_z(t; x, y)| \leq C \cdot e^{-\mu t}.$$

Proof We use again the fact that the kernel of $\exp(-t\mathcal{D}_z^- \mathcal{D}_z^+)$ is the product of the kernels of $\exp(t \cdot (\partial/\partial u - \ln z \cdot f')^2)$ and $\exp(-t\mathcal{D}^2)$. The former is uniformly bounded for all $t \geq 8$ by Lemma 11.4. As for the latter, note that the smallest eigenvalue of $\mathcal{D}^2$ is positive. Denote this eigenvalue by λ^2; then the kernel of $\exp(-t\mathcal{D}^2)$ can be estimated from above by $C \cdot e^{-\lambda^2 t/2}$ by [10, Proposition 1.1]. The argument for the kernel of $\mathcal{D}_z K_z(t; x, y)$ is similar using (34) together with [10, Proposition 1.1]. □

11.2 Heat kernel estimates on X_R

In this section, we will prove certain estimates on the heat kernels on manifolds X_R. To begin with, we will give a description of X_R which differs notationally from that in (18).

Let Y be a connected submanifold of X which is Poincaré dual to df, and let W be the cobordism obtained by cutting X open along Y. Assume that the Riemannian metric on X is a product metric in a normal neighborhood $[-1, 1] \times Y$. The induced metric on W will have product regions $[-1, 0] \times Y$ and $[0, 1] \times Y$ near its boundary components. We will use these product regions to define, for every real number $R \geq 1$, the manifold

$$X_R = W \cup ([-R - 1, R + 1] \times Y)$$

by gluing the product region $[-1, 0] \times Y$ of the first summand to $[-R - 1, -R] \times Y$ of the second, and the product region $[0, 1] \times Y$ of the first summand to $[R, R + 1] \times Y$ of the second. The gluing functions we use are linear on the first factor and are the identity on the second. We will view the manifold $[-R - 1, R + 1] \times Y$ with the identified boundary components as the product $S^1_R \times Y$ with the circle S^1_R of circumference $2R + 2$, cut open along a copy of Y.

Throughout this section we assume that the Dirac operator $\mathcal{D}$ on Y has zero kernel, df is supported in the product region $[-1, 1] \times Y$, and f is a function of the normal coordinate u in that region.

11.2.1 Gaussian estimates Denote by $K_z(t; x, y)$, $K^1_z(t; x, y)$ and $K^2_z(t; x, y)$ the kernels of the operator $\exp(-t\mathcal{D}_z^- \mathcal{D}_z^+)$ on X_R, X and $S^1_R \times Y$, respectively. We wish to compare the functions $K_z(t; x, x)$ and $K^2_z(t; x, x)$ over the product region $[-1, 1] \times Y$

shared by the manifolds X_R and $S^1_R \times Y$. To this end, define an approximate kernel of the operator $\exp(-t\mathcal{D}_z^-\mathcal{D}_z^+)$ on X_R by the formula

$$K_z^a(t;x,y) = \varphi_1(x)\cdot K^1(t;x,y)\cdot\psi_1(y) + \varphi_2(x)\cdot K_z^2(t;x,y)\cdot\psi_2(y). \tag{35}$$

The functions ψ_1 and ψ_2 here form a smooth partition of unity on X_R such that $\operatorname{supp}(\psi_2) = \left[-R-\frac{4}{7}, R+\frac{4}{7}\right]\times Y$ and $\psi_2 = 1$ on $\left[-R-\frac{3}{7}, R+\frac{3}{7}\right]\times Y$. The function φ_1 equals zero on $\left[-R-\frac{1}{7}, R+\frac{1}{7}\right]\times Y$ and one outside of $\left[-R-\frac{2}{7}, R+\frac{2}{7}\right]\times Y$, and the function φ_2 equals one on $\left[-R-\frac{5}{7}, R+\frac{5}{7}\right]\times Y$ and zero outside of $\left[-R-\frac{6}{7}, R+\frac{6}{7}\right]\times Y$. Note that $\varphi_j = 1$ on $\operatorname{supp}(\psi_j)$ and that the distance between $\operatorname{supp}(\partial\varphi_j/\partial u)$ and $\operatorname{supp}(\psi_j)$ is greater than or equal to $\frac{1}{7}$ for $j = 1, 2$.

Remark 11.7 It is important to note that in (35) we did not twist the Dirac operators on X because df is supported away from W in the manifold X_R.

The advantage of having the approximate smoothing kernel $K_z^a(t;x,x)$ is that it is defined on the same manifold X_R as $K_z(t;x,x)$, while

$$K_z(t;x,x) - K_z^2(t;x,x) = K_z(t;x,x) - K_z^a(t;x,x)$$

in the region $[-1,1]\times Y$ of our interest. To calculate the latter difference, consider the error term

$$-E_z(t;x,y) = \Big(\frac{\partial}{\partial t} + \mathcal{D}_z^-\mathcal{D}_z^+\Big)K_z^a(t;x,y),$$

where the operator $\mathcal{D}_z^-\mathcal{D}_z^+$ acts on the variable x for any fixed t and y. Since

$$(\mathcal{D}_z^-\mathcal{D}_z^+)(\varphi_j K_z^j) = (\Delta\varphi_j)K_z^j - 2\nabla_{\nabla\varphi_j}K_z^j + \varphi_j(\mathcal{D}_z^-\mathcal{D}_z^+)K_z^j$$

and both K_z^j solve the heat equation, we obtain

$$\begin{aligned} -E_z(t;x,y) = {}& \Delta\varphi_1(x)\cdot K^1(t;x,y)\cdot\psi_1(y) - 2\nabla_{\nabla\varphi_1(x)}K^1(t;x,y)\cdot\psi_1(y) \\ & + \Delta\varphi_2(x)\cdot K_z^2(t;x,y)\cdot\psi_2(y) - 2\nabla_{\nabla\varphi_2(x)}K_z^2(t;x,y)\cdot\psi_2(y). \end{aligned} \tag{36}$$

In particular, $E_z(t;x,y) = 0$ whenever $d(x,y) < \frac{1}{7}$. Following the standard argument — see for instance [42, Section 10.4] — we obtain

$$K_z(t;x,y) - K_z^a(t;x,y) = \int_0^t\int_{X_R} K_z(s;x,w)\cdot E_z(t-s;w,y)\,dw\,ds.$$

The w–integration in this formula extends only to $\operatorname{supp}_w E_z(t-s;w,y)$, which is contained in $N = \big(\big[-R-\frac{6}{7}, -R-\frac{1}{7}\big]\times Y\big)\cup\big(\big[R+\frac{1}{7}, R+\frac{6}{7}\big]\times Y\big)$. Therefore, the

above integral can be written in the form

$$K_z(t;x,y) - K_z^a(t;x,y) = \int_0^t \int_N K_z(s;x,w) \cdot E_z(t-s;w,y)\, dw\, ds. \tag{37}$$

To obtain an equation on the kernel of $\mathcal{D}_z^+ e^{-t\mathcal{D}_z^-\mathcal{D}_z^+}$ on X_R similar to (37), apply $\mathcal{D}_z^+$ to both sides of that equation:

$$\begin{aligned}\mathcal{D}_z^+ K_z(t;x,y) - \mathcal{D}_z^+ K_z^a(t;x,y) \\ = \int_0^t \int_N \mathcal{D}_z^+ K_z(s;x,w) \cdot E_z(t-s;w,y)\, dw\, ds.\end{aligned} \tag{38}$$

We will use this formula to obtain our first on-diagonal estimate on the difference between the kernels $\mathcal{D}_z^+ K_z$ and $\mathcal{D}_z^+ K_z^a$. The second such estimate will be coming up in Proposition 11.11.

Proposition 11.8 *There are positive constants α, γ and C independent of unitary z and R such that the following estimate holds for $x \in [-1,1] \times Y \subset X_R$ and $t > 0$:*

$$|\mathcal{D}_z^+ K_z(t;x,x) - \mathcal{D}_z^+ K_z^a(t;x,x)| \le C \cdot e^{\alpha t} \cdot e^{-\gamma R^2/t}.$$

The proof of this proposition will use the following two lemmas, which provide us with estimates on $\mathcal{D}_z^+ K_z$ and E_z which are uniform in z and R.

Lemma 11.9 *There are positive constants α, γ and C independent of unitary z and R such that the following estimates hold for $x, y \in X_R$ and $t > 0$:*

$$|K_z(t;x,y)| \le C \cdot e^{\alpha t} \cdot t^{-n/2} \cdot e^{-\gamma\, d^2(x,y)/t}, \tag{39}$$

$$|\mathcal{D}_z^+ K_z(t;x,y)| \le C \cdot e^{\alpha t} \cdot t^{-(n+1)/2} \cdot e^{-\gamma\, d^2(x,y)/t}. \tag{40}$$

Proof According to Lemma 11.5, such estimates hold on $S^1_R \times Y$ with $\alpha = 0$ and the constants γ and C independent of z and R. According to [10, Proposition 1.1], the same estimates with $\alpha = 0$ hold on the manifold X but only for nontwisted Dirac operators, which are exactly the operators that X contributes to the definition (35) of the approximate kernel. Now, the kernels of $\exp(-t\mathcal{D}_z^-\mathcal{D}_z^+)$ and $\mathcal{D}_z^+ \exp(-t\mathcal{D}_z^-\mathcal{D}_z^+)$ on X_R can be constructed from this approximate kernel by an iterative procedure using the Duhamel principle as in the proof of [10, Theorem 2.4]. In the process, one obtains the estimates (39) and (40) on X_R from the respective estimates on $S^1_R \times Y$ and X. The constants in these estimates will be independent of z and R because they were already independent of z and R on $S^1_R \times Y$ and X. One also acquires in the process a

possibly nonzero constant α which has to do with the volume of Y and is therefore independent of z and R. □

Lemma 11.10 *There exist positive constants γ and C independent of unitary z and R such that the following estimate holds for $x, y \in X_R$ and $t > 0$:*

$$|E_z(t; x, y)| \le C \cdot e^{-\gamma\, d^2(x,y)/t}. \tag{41}$$

Proof This follows from formula (36) for the error term and the usual estimates on the kernels $K^1(t; x, y)$ and $K^2_z(t; x, y)$ and their space derivatives. That the estimates for $K^2_z(t; x, y)$ are independent of z and R follows as in the proof of Lemma 11.4 from an explicit formula for the heat kernel on $S^1_R \times Y$. The negative powers of t that show up in the estimates are absorbed into the factor $e^{-\gamma\, d^2(x,y)/t}$ using the observation that $E_z(t; x, y) = 0$ whenever $d(x, y) < \frac{1}{7}$. □

Proof of Proposition 11.8 We proceed by estimating the difference

$$\mathcal{D}^+_z K_z(t; x, x) - \mathcal{D}^+_z K^a_z(t; x, x)$$

for $x \in [-1, 1] \times Y$ using formula (38). For all $z \in N$ we have $d(x, z) \ge (R-1) + \frac{1}{7} \ge \frac{1}{2}R + \frac{1}{7}$, hence $d^2(x, z) \ge \frac{1}{4}R^2 + \frac{1}{49}$ and (40) gives

$$\begin{aligned}|\mathcal{D}^+_z K_z(s; x, z)| &\le C_1 \cdot e^{\alpha_1 s} \cdot s^{-(n+1)/2} \cdot e^{-\gamma_1\, d^2(x,z)/s} \\ &\le C_1 \cdot e^{\alpha_1 s} \cdot s^{-(n+1)/2} \cdot e^{-1/49s} \cdot e^{-\gamma_1 R^2/4s} \le C_2 \cdot e^{\alpha_1 s} \cdot e^{-\gamma_1 R^2/4s}.\end{aligned}$$

Similarly, using (41), we obtain $|E_z(t-s; z, x)| \le C_3 \cdot e^{-\gamma_2 R^2/4(t-s)}$, hence

$$\begin{aligned}|\mathcal{D}^+_z K_z(t; x, x) - \mathcal{D}^+_z K^a_z(t; x, x)| &\le \int_0^t \int_N C_4 \cdot e^{\alpha_1 s} \cdot e^{-\gamma R^2/s} \cdot e^{-\gamma R^2/(t-s)}\, dz\, ds \\ &\le \int_0^t \int_N C_4 \cdot e^{\alpha_1 t} \cdot e^{-\gamma R^2/t}\, dz\, ds \\ &\le C \cdot e^{\alpha t} \cdot e^{-\gamma R^2/t},\end{aligned}$$

where we used the obvious fact that $1/t \le 1/s + 1/(t-s)$ for $s \in (0, t)$. □

11.2.2 Large-time estimates The second estimate that goes into the proof of our theorem has to do with the smallest eigenvalue of the operator $\mathcal{D}$ on Y. Such estimates are well known for the L^2–norms of heat kernels; the following proposition claims pointwise estimates.

Proposition 11.11 *There exist positive constants μ and C independent of R such that*

$$|\mathcal{D}_z^+ K_z(t;x,y)| \le C \cdot e^{-\mu t} \quad \text{and} \quad |\mathcal{D}_z^+ K_z^a(t;x,y)| \le C \cdot e^{-\mu t}$$

for all $x, y \in [-1,1] \times Y \subset X_R$ and $t \ge 8$.

Proof The estimate for $\mathcal{D}_z^+ K_z^a(t;x,y)$ is precisely the second estimate of Lemma 11.6. The following proof is modeled after the proof of [10, Proposition 1.1].

We begin with an observation about the Sobolev spaces on X_R with a fixed R. For a unitary z and a nonnegative integer s, define the Sobolev norm

$$\|\varphi\|_{L_s^2(z)} = \|\varphi\|_{L^2} + \|\mathcal{D}_z^s \varphi\|_{L^2}.$$

This norm is equivalent to the Sobolev norm

$$\|\varphi\|_{L_s^2} = \|\varphi\|_{L^2} + \|\mathcal{D}^s \varphi\|_{L^2},$$

meaning that the identity operators Id: $L_s^2(z) \to L_s^2$ and Id: $L_s^2 \to L_s^2(z)$ are bounded. The norms of these operators are continuous functions of z which achieve their absolute minimum and maximum on the circle $|z| = 1$. Therefore, there exist positive constants m and M independent of $z \in S^1$ such that for all φ we have

$$m \cdot \|\varphi\|_{L_s^2} \le \|\varphi\|_{L_s^2(z)} \le M \cdot \|\varphi\|_{L_s^2}. \tag{42}$$

With this understood, let $\mathcal{D}_z \varphi_k = \lambda_k \varphi_k$ be the spectral decomposition of the full Dirac operator $\mathcal{D}_z$ with $\|\varphi_k\|_{L^2} = 1$ (both φ_k and λ_k depend on z but we omit this dependence from our notation). We first estimate

$$K_z(t;x,y) = \sum_k e^{-t\lambda_k^2} \cdot \varphi_k(x) \cdot \overline{\varphi}_k(y).$$

Use the Sobolev embedding theorem with $s = 2n$ — see for instance Lemma 1.1.4 of [15] — and inequality (42) to obtain the pointwise estimates

$$|\varphi_k(x)| \le a \cdot \|\varphi_k\|_{L_{2n}^2} \le \frac{a}{m} \cdot \|\varphi_k\|_{L_{2n}^2(z)} = \frac{a}{m} \cdot (1 + \lambda_k^{2n})$$

with the constants a and m independent of z and k. For any $t \ge 8$, we have

$$\begin{aligned} |\mathcal{D}_z K_z(t;x,y)| &\le \frac{a}{m} \cdot \sum |\lambda_k| e^{-t\lambda_k^2} \cdot (1 + \lambda_k^{2n})^2 \\ &\le \frac{a}{m} \cdot ((2n)!)^2 \cdot \sum e^{-(t-3)\lambda_k^2} \\ &\le \frac{a}{m} \cdot ((2n)!)^2 \cdot \sum e^{-(t-4)\lambda^2} \cdot e^{-\lambda_k^2} \le C_1 \cdot e^{-\lambda^2 t/2} \end{aligned}$$

with some positive constants μ and C_1 independent of z. In the last line, we used the condition that $\ker \mathcal{D}_z = 0$ on the circle $|z| = 1$ to guarantee that the smallest eigenvalue of the family $\mathcal{D}_z^2$ over $z \in S^1$ is positive; we call this eigenvalue λ^2. We also used the fact that

$$\sum |\lambda_k| e^{-\lambda_k^2} \tag{43}$$

is a continuous function of $z \in S^1$ to estimate it from above by a constant independent of z. The constants in the estimates depend on R in two different ways. One is via the smallest eigenvalue λ which by Proposition 7.3 is bounded away from zero by a positive constant for all sufficiently large R. The other is via the function (43), which can be shown to be bounded for $t \geq 8$ using [42, Lemma 10.13] and Proposition 7.3. □

11.3 Proof of Theorem 11.1

For our choice of df, the difference between $\eta(X_R)$ and $\eta(S^1_R \times Y)$ is given by integrating the quantity

$$\int_{[-1,1]\times Y} \operatorname{Tr}\big(df \cdot (\mathcal{D}_z^+ K_z(t;x,x) - \mathcal{D}_z^+ K_z^a(t;x,x))\big)\, dx,$$

with respect to t and z as in formula (31) (in the above formula, Tr stands for the matrix trace). The quantity $\mathcal{D}_z^+ K_z(t;x,x) - \mathcal{D}_z^+ K_z^a(t;x,x)$ has been estimated twice, first by $C \cdot e^{\alpha t} \cdot e^{-\gamma R^2/t}$ for $t > 0$ in Proposition 11.8 and then by $C \cdot e^{-\mu t}$ for $t \geq 8$ in Proposition 11.11. The graphs of $e^{\alpha t - \gamma R^2/t}$ and $e^{-\mu t}$ intersect at the point $t = \beta R$, where

$$\beta = \sqrt{\frac{\gamma}{\alpha + \mu}}.$$

Therefore, for sufficiently large R and after adjusting the constants C, we can use the first estimate on the interval $0 < t \leq \beta R$ and the second on the interval $\beta R \leq t < \infty$. Since both estimates are independent of x and z, we can integrate them with respect to these two variables. The integration results in estimating the distance between $\eta(X_R)$ and $\eta(S^1_R \times Y)$ by a uniform constant times the integral

$$\int_0^{\beta R} e^{\alpha t - \gamma R^2/t}\, dt + \int_{\beta R}^{\infty} e^{-\mu t}\, dt.$$

The latter integral can easily be estimated from above by

$$\beta \cdot R \cdot e^{(\alpha\beta - \gamma/\beta)R} + \Big(\frac{1}{\mu}\Big) \cdot e^{-\mu\beta R}.$$

Since $\alpha\beta - \gamma/\beta$ is negative, the difference between $\eta(X_R)$ and $\eta(S^1_R \times Y)$ must approach zero as $R \to \infty$. This completes the proof of Theorem 11.1.

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