ARITHMETIC OF BORCHERDS PRODUCTS

BENJAMIN HOWARD AND KEERTHI MADAPUSI PERA

Abstract. We compute the divisors of Borcherds products on integral models of orthogonal Shimura varieties. As an application, we obtain an integral version of a theorem of Borcherds on the modularity of a generating series of special divisors.

Contents

1. Introduction

In the series of papers [Bor95, Bor98, Bor99], Borcherds introduced a family of meromorphic modular forms on orthogonal Shimura varieties, whose zeroes and poles are prescribed linear combinations of special divisors arising from embeddings of smaller orthogonal Shimura varieties. These meromorphic modular forms are the Borcherds products of the title.

After work of Kisin [Kis10] on integral models of general Hodge and abelian type Shimura varieties, the theory of integral models of orthogonal Shimura varieties and their special divisors was developed further in [Hör10, Hör14] and [Mad16, AGHMP17, AGHMP18].

The goal of this paper is to combine the above theories to compute the divisor of a Borcherds product on the integral model of an orthogonal Shimura

1991 *Mathematics Subject Classification.* 14G35, 14G40, 11F55, 11F27, 11G18.

Key words and phrases. Shimura varieties, Borcherds products.

B.H. was supported in part by NSF grants DMS-1501583 and DMS-1801905.

K.M.P. was supported in part by NSF grants DMS-1502142 and DMS-1802169.

variety. We show that such a divisor is given as a prescribed linear combination of special divisors, exactly as in the generic fiber.

The first such results were obtained by Bruinier, Burgos Gil, and Kühn [BBK], who worked on Hilbert modular surfaces (a special type of signature $(2,2)$ orthogonal Shimura variety). Those results were later extended to more general orthogonal Shimura varieties by Hörmann [Hör10, Hör14], but with some restrictions.

Our results extend Hörmann's, but with substantially weaker hypotheses. For example, our results include cases where the integral model is not smooth, cases where the divisors in question may have irreducible components supported in nonzero characteristics, and even cases where the Shimura variety is compact (so that one has no theory of q-expansions with which to analyze the arithmetic properties of Borcherds products).

1.1. **Orthogonal Shimura varieties.** Given an integer $n \geqslant 1$ and a quadratic space (V, Q) over $\mathbb{Q}$ of signature $(n, 2)$, one can construct a Shimura datum $(G, \mathcal{D})$ with reflex field $\mathbb{Q}$.

The group $G = \mathrm{GSpin}(V)$ is a subgroup of the group of units in the Clifford algebra $C(V)$, and sits in a short exact sequence

$$1 \to \mathbb{G}_m \to G \to \mathrm{SO}(V) \to 1.$$

The hermitian symmetric domain is

$$\mathcal{D} = \{z \in V_{\mathbb{C}} : [z, z] = 0,\ [z, \overline{z}] < 0\}/\mathbb{C}^\times \subset \mathbb{P}(V_{\mathbb{C}}),$$

where the bilinear form

$$[x, y] = Q(x + y) - Q(x) - Q(y) \tag{1.1.1}$$

on V has been extended $\mathbb{C}$-bilinearly to $V_{\mathbb{C}}$.

To define a Shimura variety, fix a $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V$ on which the quadratic form is $\mathbb{Z}$-valued, and a compact open subgroup $K \subset G(\mathbb{A}_f)$ such that

$$K \subset G(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}})^\times. \tag{1.1.2}$$

Here $C(V_{\widehat{\mathbb{Z}}})$ is the Clifford algebra of the $\widehat{\mathbb{Z}}$-quadratic space $V_{\widehat{\mathbb{Z}}} = V_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}}$. The canonical model of the complex orbifold

$$\mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C}) = G(\mathbb{Q}) \backslash \big(\mathcal{D} \times G(\mathbb{A}_f)/K\big)$$

is a smooth n-dimensional Deligne-Mumford stack

$$\mathrm{Sh}_K(G, \mathcal{D}) \to \mathrm{Spec}(\mathbb{Q}).$$

As in work of Kudla [Kud97, Kud04], our Shimura variety carries a family of effective Cartier divisors

$$Z(m, \mu) \to \mathrm{Sh}_K(G, \mathcal{D})$$

indexed by positive $m \in \mathbb{Q}$ and $\mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}$, and a metrized line bundle

$$\widehat{\boldsymbol{\omega}} \in \widehat{\mathrm{Pic}}(\mathrm{Sh}_K(G, \mathcal{D}))$$

of weight one modular forms. Under the complex uniformization of the Shimura variety, this line bundle pulls back to the tautological bundle on $\mathcal{D}$, with the metric defined by (4.2.3).

We say that $V_{\mathbb{Z}}$ is *maximal* if there is no proper superlattice in V on which Q takes integer values, and is *maximal at* p if the $\mathbb{Z}_p$-quadratic space $V_{\mathbb{Z}_p} = V_{\mathbb{Z}} \otimes \mathbb{Z}_p$ has the analogous property. It is clear that $V_{\mathbb{Z}}$ is maximal at every prime not dividing the discriminant $[V_{\mathbb{Z}}^\vee : V_{\mathbb{Z}}]$.

Let Ω be a finite set of rational primes containing all primes at which $V_{\mathbb{Z}}$ is not maximal, and abbreviate

$$\mathbb{Z}_\Omega = \mathbb{Z}[1/p : p \in \Omega].$$

Assume that (1.1.2) factors as $K = \prod_p K_p$, in such a way that

$$K_p = G(\mathbb{Q}_p) \cap C(V_{\mathbb{Z}_p})^\times$$

for every prime $p \notin \Omega$. For such K there is a flat and normal integral model

$$\mathcal{S}_K(G, \mathcal{D}) \to \mathrm{Spec}(\mathbb{Z}_\Omega)$$

of $\mathrm{Sh}_K(G, \mathcal{D})$. It is a Deligne-Mumford stack of finite type over $\mathbb{Z}_\Omega$, and is a scheme if K is sufficiently small. At any prime $p \notin \Omega$, it satisfies the following properties:

(1) If the lattice $V_{\mathbb{Z}}$ is self-dual at a prime p (or even *almost self-dual* in the sense of Definition 6.1.1) then the restriction of the integral model to $\mathrm{Spec}(\mathbb{Z}_{(p)})$ is smooth.
(2) If p is odd and p^2 does not divide the discriminant $[V_{\mathbb{Z}}^\vee : V_{\mathbb{Z}}]$, then the restriction of the integral model to $\mathrm{Spec}(\mathbb{Z}_{(p)})$ is regular.
(3) If $n \geqslant 6$ then the reduction mod p is geometrically normal.

The integral model carries over it a metrized line bundle

$$\widehat{\boldsymbol{\omega}} \in \widehat{\mathrm{Pic}}(\mathcal{S}_K(G, \mathcal{D}))$$

of weight one modular forms, extending the one already available in the generic fiber, and a family of effective Cartier divisors

$$\mathcal{Z}(m, \mu) \to \mathcal{S}_K(G, \mathcal{D})$$

indexed by positive $m \in \mathbb{Q}$ and $\mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}$.

Remark 1.1.1. If $V_{\mathbb{Z}}$ is itself maximal, one can take $\Omega = \varnothing$, choose

$$K = G(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}})^\times$$

for the level subgroup, and obtain an integral model of $\mathrm{Sh}_K(G, \mathcal{D})$ over $\mathbb{Z}$.

1.2. **Borcherds products.** In §5.1, we recall the Weil representation

$$\rho_{V_{\mathbb{Z}}} : \widetilde{\mathrm{SL}}_2(\mathbb{Z}) \to \mathrm{Aut}_{\mathbb{C}}(S_{V_{\mathbb{Z}}})$$

of the metaplectic double cover of $\mathrm{SL}_2(\mathbb{Z})$ on the $\mathbb{C}$-vector space

$$S_{V_{\mathbb{Z}}} = \mathbb{C}[V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}].$$

Any weakly holomorphic form

$$f(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} c(m) \cdot q^m \in M^!_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}})$$

valued in the complex-conjugate representation has Fourier coefficients

$$c(m) \in S_{V_{\mathbb{Z}}},$$

and we denote by $c(m,\mu)$ the value of $c(m)$ at the coset $\mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}$. Fix such an f, assume that f is *integral* in the sense that $c(m,\mu) \in \mathbb{Z}$ for all m and μ.

Using the theory of regularized theta lifts, Borcherds [Bor98] constructs a Green function $\Theta^{\mathrm{reg}}(f)$ for the analytic divisor

$$\sum_{\substack{m>0 \\ \mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}}} c(-m,\mu) \cdot Z(m,\mu)(\mathbb{C}) \tag{1.2.1}$$

on $\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})$, and shows (after possibly replacing f by a suitable multiple) that some power of $\boldsymbol{\omega}^{an}$ admits a meromorphic section $\psi(f)$ satisfying

$$-2\log\|\psi(f)\| = \Theta^{\mathrm{reg}}(f). \tag{1.2.2}$$

This implies that the divisor of $\psi(f)$ is (1.2.1). These meromorphic sections are the *Borcherds products* of the title.

Our main result, stated in the text as Theorem 9.1.1, asserts that the Borcherds product $\psi(f)$ is algebraic, defined over $\mathbb{Q}$, and has the expected divisor when viewed as a rational section over the integral model.

Theorem A. *After possibly replacing f by a positive integer multiple, there is a rational section $\psi(f)$ of the line bundle $\boldsymbol{\omega}^{c(0,0)}$ on $\mathcal{S}_K(G,\mathcal{D})$ whose norm under the metric (4.2.3) satisfies (1.2.2), and whose divisor is*

$$\mathrm{div}(\psi(f)) = \sum_{\substack{m>0 \\ \mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}}} c(-m,\mu) \cdot \mathcal{Z}(m,\mu).$$

The unspecified positive integer by which one must multiply f can be made at least somewhat more explicit. For example, it depends only on the lattice $V_{\mathbb{Z}}$, and not on the form f. See the discussion of §9.3.

As noted earlier, similar results can be found in the work of Hörmann [Hör10, Hör14]. Hörmann only considers self-dual lattices, so that the corresponding integral model $\mathcal{S}_K(G,\mathcal{D})$ is smooth, and always assumes that the quadratic space V admits an isotropic line. This allows him to prove the flatness of $\mathrm{div}(\psi(f))$ by examining the q-expansion of $\psi(f)$ at a cusp. As Hörmann's special divisors $\mathcal{Z}(m,\mu)$, unlike ours, are defined as the Zariski closures of their generic fibers, the equality of divisors stated in Theorem A is then a formal consequence of the same equality in the generic fiber.

In contrast, we can prove Theorem A even in cases where the divisors in question may not be flat, and in cases where V is anisotropic, so no theory of q-expansions is available.

The reader may be surprised to learn that even the descent of $\psi(f)$ to $\mathbb{Q}$ was not previously known in full generality. Indeed, there is a product formula for the Borcherds product giving its q-expansions at every cusp, and so one should be able to detect the field of definition of $\psi(f)$ from a suitable q-expansion principle.

If V is anisotropic then $\mathrm{Sh}_K(G,\mathcal{D})$ is proper over $\mathbb{Q}$, no theory of q-expansions exists, and the above strategy fails completely. But even when V is isotropic there is a serious technical obstruction to this argument. The product formula of Borcherds is not completely precise, in that the q-expansion of $\psi(f)$ at a given cusp is only specified up to multiplication by an unknown constant of absolute value 1, and there is no a priori relation between the different constants at different cusps. These constants are the $\kappa^{(a)}$ appearing in Proposition 5.4.2.

If $\mathrm{Sh}_K(G,\mathcal{D})$ admits (a toroidal compactification with) a cusp defined over $\mathbb{Q}$ there is no problem: simply rescale the Borcherds product by a constant of absolute value 1 to remove the mysterious constant at that cusp, and now $\psi(f)$ is defined over $\mathbb{Q}$. But if $\mathrm{Sh}_K(G,\mathcal{D})$ has no rational cusps, then to prove that $\psi(f)$ descends to $\mathbb{Q}$ one must compare the q-expansions of $\psi(f)$ at all points in a Galois orbit of cusps. One can rescale the Borcherds product to trivialize the constant at one cusp, but then one has no control over the constants at other cusps in the Galois orbit.

Using the q-expansion principle alone, is seems that the best one can prove is that $\psi(f)$ descends to the minimal field of definition of a cusp. Our strategy to improve on this is sketched in §1.4 below.

Remark 1.2.1. As in the statement and proof of [Hör10, Theorem 10.4.12], there is an elementary argument using Hilbert's Theorem 90 that allows one to rescale the Borcherds product so that it descends to $\mathbb{Q}$, but in this argument one has no control over the scaling factor, and it need not have absolute value 1. In particular this rescaling may destroy the norm relation (1.2.2). Even worse, rescaling by such factors may introduce unwanted and unknown vertical components into the divisor of the Borcherds product on the integral model of the Shimura variety, and understanding what's happening on the integral model is the central concern of this work.

1.3. **Modularity of generating series.** The family of special divisors determines a family of line bundles

$$\mathcal{Z}(m,\mu) \in \mathrm{Pic}(\mathcal{S}_K(G,\mathcal{D}))$$

indexed by positive $m \in \mathbb{Q}$ and $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. We extend the definition to $m=0$ by setting

$$\mathcal{Z}(0,\mu) = \begin{cases} \boldsymbol{\omega}^{-1} & \text{if } \mu = 0 \\ \mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})} & \text{if } \mu \neq 0. \end{cases}$$

Exactly as in the work of Borcherds [Bor99], Theorem A produces enough relations in the Picard group to prove the modularity of the generating series of these line bundles. Let

$$\phi_\mu \in S_{V_{\mathbb{Z}}} = \mathbb{C}[V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}]$$

denote the characteristic function of the coset $\mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}$.

Theorem B. *The formal q-expansion*

$$\sum_{\substack{m \geqslant 0 \\ \mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}}} \mathcal{Z}(m,\mu) \otimes \phi_\mu \cdot q^m$$

is a modular form valued in $\mathrm{Pic}(\mathcal{S}_K(G,\mathcal{D})) \otimes S_{V_{\mathbb{Z}}}$. *More precisely, we have*

$$\sum_{\substack{m \geqslant 0 \\ \mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}}} \alpha(\mathcal{Z}(m,\mu)) \cdot \phi_\mu \cdot q^m \in M_{1+\frac{n}{2}}(\rho_{V_{\mathbb{Z}}})$$

for any $\mathbb{Z}$-linear map $\alpha : \mathrm{Pic}(\mathcal{S}_K(G,\mathcal{D})) \to \mathbb{C}$.

Theorem B is stated in the text as Theorem 9.4.1. After endowing the special divisors with Green functions as in [Bru02], we also prove a modularity result in the group of metrized line bundles. See Theorem 9.5.1.

1.4. **Idea of the proof.** We first prove Theorem A assuming that $n \geqslant 6$, and that $V_{\mathbb{Z}}$ splits an integral hyperbolic plane. This assumption has three crucial consequences. First, it guarantees the existence of cusps of $\mathrm{Sh}_K(G,\mathcal{D})$ defined over $\mathbb{Q}$. Second, it guarantees that our integral model has geometrically normal fibers, so that we may use the results of [Mad19] to fix a toroidal compactification in such a way that every irreducible component of every mod p fiber of $\mathcal{S}_K(G,\mathcal{D})$ meets a cusp. Finally, it guarantees the flatness of all special divisors $\mathcal{Z}(m,\mu)$.

As noted above, the existence of cusps over $\mathbb{Q}$ allows us to deduce the descent of $\psi(f)$ to $\mathbb{Q}$ using the q-expansion principle. Moreover, by examining the q-expansions of $\psi(f)$ at the cusps, one can show that its divisor is flat over $\mathbb{Z}_\Omega$, and the equality of divisors in Theorem A then follows from the known equality in the generic fiber.

Remark 1.4.1. In fact, we prove that our divisors are flat over $\mathbb{Z}$ as soon as $n \geqslant 4$. When $n \in \{1,2,3\}$ the orthogonal Shimura varieties and their special divisors can be interpreted as a moduli space of abelian varieties with additional structure, as in the work of Kudla-Rapoport [KR99, KR00a, KR00b]. Already in the case of $n = 1$, Kudla and Rapoport [KR00b] provide examples in which the special divisors are not flat.

To understand how to deduce the general case from the special case above, we first recall how Borcherds constructs $\psi(f)$ in the complex fiber. If V contains an isotropic line, the construction boils down to explicitly writing down its q-expansion as an infinite product. This gives the desired $\psi(f)$, along with the norm relation (1.2.2), on the region of convergence. The right hand side of (1.2.2) is a pluriharmonic function defined on the complement of the support of (1.2.1), and the meromorphic continuation of $\psi(f)$ follows more-or-less formally from this.

Suppose now that V is anisotropic. The idea of Borcherds is to fix isometric embeddings of V into two (very particular) quadratic spaces $V^{[1]}$ and $V^{[2]}$ of signature $(n+24,2)$. From this one can construct morphisms of orthogonal Shimura varieties

$$\begin{tikzcd} & \mathrm{Sh}_K(G,\mathcal{D}) \arrow[dl, "j^{[1]}"'] \arrow[dr, "j^{[2]}"] & \\ \mathrm{Sh}_{K^{[1]}}(G^{[1]},\mathcal{D}^{[1]}) & & \mathrm{Sh}_{K^{[2]}}(G^{[2]},\mathcal{D}^{[2]}). \end{tikzcd}$$

As both $V^{[1]}$ and $V^{[2]}$ contain isotropic lines, one already has Borcherds products on their associated Shimura varieties.

The next step should be to define

$$\psi(f) = \frac{(j^{[2]})^*\psi(f^{[2]})}{(j^{[1]})^*\psi(f^{[1]})} \tag{1.4.1}$$

for (very particular) weakly holomorphic forms $f^{[1]}$ and $f^{[2]}$. The problem is that the quotient on the right hand side is nearly always either $0/0$ or ∞/∞, and so doesn't really make sense.

Borcherds gets around this via an analytic construction on the level of hermitian domains. On the hermitian domain

$$\mathcal{D}^{[i]} = \{z \in V^{[i]}_{\mathbb{C}} : [z,z]=0,\ [z,\overline{z}]<0\}/\mathbb{C}^\times \subset \mathbb{P}(V^{[i]}_{\mathbb{C}}),$$

every irreducible component of every special divisor has the form

$$\mathcal{D}^{[i]}(x) = \{z \in \mathcal{D}^{[i]} : z \perp x\}$$

for some $x \in V^{[i]}$, and the dual of the tautological line bundle $\boldsymbol{\omega}_{\mathcal{D}^{[i]}}$ on $\mathcal{D}^{[i]}$ admits a canonical section

$$\mathrm{obst}^{an}_x \in H^0(\mathcal{D}^{[i]}, \boldsymbol{\omega}^{-1}_{\mathcal{D}^{[i]}})$$

with zero locus $\mathcal{D}^{[i]}(x)$. See the discussion at the beginning of §6.5.

Whenever there is an $x \in V^{[i]}$ such that $\mathcal{D} \subset \mathcal{D}^{[i]}(x)$, Borcherds multiplies $\psi(f^{[i]})$ by a suitable power of obst^{an}_x in order to remove the component $\mathcal{D}^{[i]}(x)$ from $\mathrm{div}(\psi(f^{[i]})$. After modifying both $\psi(f^{[1]})$ and $\psi(f^{[2]})$ in this way, the quotient (1.4.1) is defined. This process is what Borcherds calls the *embedding trick* in [Bor98]. As understood by Borcherds, the embedding trick is a purely analytic construction. The sections obst^{an}_x over $\mathcal{D}^{[i]}$ do not descend to the Shimura varieties, and have no obvious algebraic properties. In particular, even if one knows that the $\psi(f^{[i]})$ are defined over $\mathbb{Q}$, it is not obvious that the renormalized quotient (1.4.1) is defined over $\mathbb{Q}$.

One of the main contributions of this paper is an algebraic analogue of the embedding trick, which works even on the level of integral models. This is based on the methods used to compute improper intersections in [BHY15, AGHMP17, How19]. The idea is to use deformation theory to construct an analogue of the section obst^{an}_x, not over all of $\mathrm{Sh}_{K^{[i]}}(G^{[i]}, \mathcal{D}^{[i]})$, but only over the first order infinitesimal neighborhood of $\mathrm{Sh}_K(G, \mathcal{D})$ in $\mathrm{Sh}_{K^{[i]}}(G^{[i]}, \mathcal{D}^{[i]})$. This section is the *obstruction to deforming x* appearing in §6.5.

With this algebraic analogue of the embedding trick in hand, we can make sense of the quotient (1.4.1), and compute the divisor of the left hand side in terms of the divisors of the numerator and denominator on the right. This allows us to deduce the general case of Theorem A from the special case explained above.

1.5. **Organization of the paper.** Ultimately, all arithmetic information about Borcherds products comes from their q-expansions, and so we must make heavy use of the arithmetic theory of toroidal compactifications of Shimura varieties of [Pin89, Mad19]. This theory requires introducing a substantial amount of notation just to state the main results. Also, because Borcherds products are rational sections of powers of the line bundle $\boldsymbol{\omega}$, we need the theory of automorphic vector bundles on toroidal compactifications. This theory is distributed across a series of papers of Harris [Har84, Har85, Har86, Har89] and Harris-Zucker [HZ94a, HZ94b, HZ01].

Accordingly, before we even begin to talk about orthogonal Shimura varieties, we first recall in §2 the main results on toroidal compactification from Pink's thesis [Pin89], and in §3 the results of Harris and Harris-Zucker on automorphic vector bundles. All of this is in the generic fiber of fairly general Shimura varieties.

Beginning in §4 we specialize to case of orthogonal Shimura varieties. We consider their toroidal compactifications, and give a purely algebraic definition of q-expansions of modular forms on them. In particular, we prove the q-expansion principle Proposition 4.6.3, which can be used to detect their fields of definition.

In §5 we introduce Borcherds products and, when V admits an isotropic line, describe their q-expansions.

In §6 we introduce integral models of orthogonal Shimura varieties over $\mathbb{Z}_{(p)}$, along with their line bundles of modular forms and special divisors. This material is drawn from [Mad16, AGHMP17, AGHMP18], although we work here in slightly more generality. The main new result in §6 is the pullback formula of Proposition 6.6.3, which explains how the special divisors behave under pullback via embeddings of orthogonal Shimura varieties. This formula, whose proof is similar to calculations of improper intersections found in [BHY15, AGHMP17, How19], is essential to our algebraic version of the embedding trick.

In §7 we prove some technical properties of the integral models over $\mathbb{Z}_{(p)}$. We show that the special divisors are flat when $n \geqslant 4$, and the integral model has geometrically normal fibers when $n \geqslant 6$. When $p \neq 2$ these results already appear in [AGHMP18]. The methods here are similar, except that we appeal to the work of Ogus [Ogu79] instead of [HP17] (which excludes $p = 2$) to control the dimension of the supersingular locus.

In §8 we extend the theory of toroidal compactifications and q-expansions to our integral models, making use of the general theory of toroidal compactifications of Hodge type Shimura varieties found in [Mad19]. The culmination of the discussion is Corollary 8.2.4, which allows one to use q-expansions to detect the flatness of divisors of rational sections of $\boldsymbol{\omega}$ and its powers.

Finally, in §9 we put everything together to prove Theorem A. The modularity result of Theorem B (and its extension to the group of metrized line bundles) follows immediately from Theorem A and the modularity criterion of Borcherds.

1.6. **Notation and conventions.** For every $a \in \mathbb{A}_f^\times$ there is a unique factorization

$$a = \mathrm{rat}(a) \cdot \mathrm{unit}(a)$$

in which $\mathrm{rat}(a)$ is a positive rational number and $\mathrm{unit}(a) \in \widehat{\mathbb{Z}}^\times$.

Class field theory provides us with a reciprocity map

$$\mathrm{rec} : \mathbb{Q}_{>0}^\times \backslash \mathbb{A}_f^\times \cong \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}),$$

which we normalize as follows. Let μ_∞ be the set of all roots of unity in $\mathbb{C}$, so that $\mathbb{Q}^{ab} = \mathbb{Q}(\mu_\infty)$ is the maximal abelian extension of $\mathbb{Q}$. The group $(\mathbb{Z}/M\mathbb{Z})^\times$ acts on the set of M^{th} roots of unity in the usual way, by letting $u \in (\mathbb{Z}/M\mathbb{Z})^\times$ act by $\zeta \mapsto \zeta^u$. Passing to the limit yields an action of $\widehat{\mathbb{Z}}^\times$ on μ_∞, and the reciprocity map is characterized by

$$\zeta^{\mathrm{rec}(a)} = \zeta^{\mathrm{unit}(a)}$$

for all $a \in \mathbb{A}_f^\times$ and $\zeta \in \mu_\infty$.

We follow the conventions of [Del79] and [Pin89, Chapter 1] for Hodge structures and mixed Hodge structures. As usual, $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m\mathbb{C}}$ is Deligne's torus, so that $\mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$, with complex conjugation acting by $(t_1, t_2) \mapsto (\bar{t}_2, \bar{t}_1)$. In particular, $\mathbb{S}(\mathbb{R}) \cong \mathbb{C}^\times$ by $(t, \bar{t}) \mapsto t$. If V is a rational vector space endowed with a Hodge structure $\mathbb{S} \to \mathrm{GL}(V_\mathbb{R})$, then $V^{(p,q)} \subset V_\mathbb{C}$ is the subspace on which $(t_1, t_2) \in \mathbb{C}^\times \times \mathbb{C}^\times = \mathbb{S}(\mathbb{C})$ acts via $t_1^{-p} t_2^{-q}$. There is a distinguished cocharacter

$$\mathrm{wt} : \mathbb{G}_{m\mathbb{R}} \to \mathbb{S}$$

defined on complex points by $t \mapsto (t^{-1}, t^{-1})$. The composition

$$\mathbb{G}_{m\mathbb{R}} \xrightarrow{\mathrm{wt}} \mathbb{S} \to \mathrm{GL}(V_\mathbb{R})$$

encodes the weight grading on $V_\mathbb{R}$, in the sense that

$$\bigoplus_{p+q=k} V^{(p,q)} = \{ v \in V_\mathbb{C} : \mathrm{wt}(z) \cdot v = z^k \cdot v,\ \forall z \in \mathbb{C}^\times \}.$$

Now suppose that V is endowed with a mixed Hodge structure. This consists of an increasing weight filtration $\mathrm{wt}_\bullet V$ on V, and a decreasing Hodge filtration $F^\bullet V_\mathbb{C}$ on $V_\mathbb{C}$, whose induced filtration on every graded piece

$$\mathrm{gr}_k(V) = \mathrm{wt}_k V / \mathrm{wt}_{k-1} V \tag{1.6.1}$$

is a pure Hodge structure of weight k. By [PS08, Lemma-Definition 3.4] there is a canonical bigrading $V_{\mathbb{C}} = \bigoplus V^{(p,q)}$ with the property that

$$\mathrm{wt}_k V_{\mathbb{C}} = \bigoplus_{p+q\leqslant k} V^{(p,q)}, \quad F^i V_{\mathbb{C}} = \bigoplus_{p\geqslant i} V^{(p,q)}.$$

This bigrading is induced by a morphism $\mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(V_{\mathbb{C}})$.

2. Toroidal compactification

This section is a (relatively) short summary of Pink's thesis [Pin89] on toroidal compactifications of canonical models of Shimura varieties. See also [Hör10] and [HZ94a, HZ01]. We limit ourselves to what is needed in the sequel, and simplify the discussion somewhat by only dealing with those mixed Shimura varieties that appear at the boundaries of pure Shimura varieties.

2.1. **Shimura varieties.** Throughout §2 and §3 we let $(G,\mathcal{D})$ be a (pure) Shimura datum in the sense of [Pin89, §2.1]. Thus G is a reductive group over $\mathbb{Q}$, and $\mathcal{D}$ is a $G(\mathbb{R})$-homogeneous space equipped with a finite-to-one $G(\mathbb{R})$-equivariant map

$$\mathtt{h} : \mathcal{D} \to \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$$

such that the pair $(G, \mathtt{h}(\mathcal{D}))$ satisfies Deligne's axioms [Del79, (2.1.1.1)-(2.1.1.3)]. We often abuse notation and confuse $z \in \mathcal{D}$ with its image $\mathtt{h}(z)$.

The *weight cocharacter*

$$w \stackrel{\mathrm{def}}{=} \mathtt{h}(z) \circ \mathrm{wt} : \mathbb{G}_{m\mathbb{R}} \to G_{\mathbb{R}} \tag{2.1.1}$$

of $(G,\mathcal{D})$ is independent of $z \in \mathcal{D}$, and takes values in the center of $G_{\mathbb{R}}$.

Hypothesis 2.1.1. Because it will simplify much of what follows, and because it is assumed throughout [HZ01], we always assume that our Shimura datum $(G,\mathcal{D})$ satisfies:

(1) The weight cocharacter (2.1.1) is defined over $\mathbb{Q}$.
(2) The connected center of G is isogenous to the product of a $\mathbb{Q}$-split torus with a torus whose group of real points is compact.

Suppose $K \subset G(\mathbb{A}_f)$ is any compact open subgroup. The associated Shimura variety

$$\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C}) = G(\mathbb{Q})\backslash\big(\mathcal{D} \times G(\mathbb{A}_f)/K\big)$$

is a complex orbifold. Its canonical model $\mathrm{Sh}_K(G,\mathcal{D})$ is a Deligne-Mumford stack over the reflex field $E(G,\mathcal{D}) \subset \mathbb{C}$. If K is neat in the sense of [Pin89, §0.6], then $\mathrm{Sh}_K(G,\mathcal{D})$ is a quasi-projective scheme. By slight abose of notation, the image of a point $(z,g) \in \mathcal{D} \times G(\mathbb{A}_f)$ is again denoted

$$(z,g) \in \mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C}).$$

Remark 2.1.2. Let $\mathbb{G}_m(\mathbb{R}) = \mathbb{R}^\times$ act on the two point set

$$\mathcal{H}_0 \stackrel{\text{def}}{=} \{2\pi\epsilon \in \mathbb{C} : \epsilon^2 = -1\}$$

via the unique continuous transitive action: positive real numbers act trivially, and negative real numbers swap the two points. If we define

$$\mathcal{H}_0 \to \mathrm{Hom}(\mathbb{S}, \mathbb{G}_{m\mathbb{R}})$$

by sending both points to the norm map $\mathbb{C}^\times \to \mathbb{R}^\times$, then $(\mathbb{G}_m, \mathcal{H}_0)$ is a Shimura datum in the sense of [Pin89].

2.2. **Mixed Shimura varieties.** Toroidal compactifications of Shimura varieties are obtained by gluing together certain mixed Shimura varieties, which we now define.

Recall from [Pin89, Definition 4.5] the notion of an *admissible* parabolic subgroup $P \subset G$. If G^{ad} is simple, this just means that P is either a maximal proper parabolic subgroup, or is all of G. In general, it means that if we write $G^{ad} = G_1 \times \cdots \times G_s$ as a product of simple groups, then P is the preimage of a subgroup $P_1 \times \cdots \times P_s$, where each $P_i \subset G_i$ is an admissible parabolic.

Definition 2.2.1. A *cusp label representative* $\Phi = (P, \mathcal{D}^\circ, h)$ for $(G, \mathcal{D})$ is a triple consisting of an admissible parabolic subgroup P, a connected component $\mathcal{D}^\circ \subset \mathcal{D}$, and an $h \in G(\mathbb{A}_f)$.

As in [Pin89, §4.11 and §4.12], any cusp label representative $\Phi = (P, \mathcal{D}^\circ, h)$ determines a mixed Shimura datum $(Q_\Phi, \mathcal{D}_\Phi)$, whose construction we now recall.

Let $W_\Phi \subset P$ be the unipotent radical, and let U_Φ be the center of W_Φ. According to [Pin89, §4.1] there is a distinguished central cocharacter $\lambda : \mathbb{G}_m \to P/W_\Phi$. The weight cocharacter $w : \mathbb{G}_m \to G$ is central, so takes values in P, and therefore determines a new central cocharacter

$$w \cdot \lambda^{-1} : \mathbb{G}_m \to P/W_\Phi. \tag{2.2.1}$$

Suppose $G \to \mathrm{GL}(N)$ is a faithful representation on a finite dimensional $\mathbb{Q}$-vector space. Each point $z \in \mathcal{D}$ determines a Hodge filtration $F^\bullet N_\mathbb{C}$ on N. Any lift of (2.2.1) to a cocharacter $\mathbb{G}_m \to P$ determines a grading $N = \bigoplus N^k$, and the associated *weight filtration*

$$\mathrm{wt}_\ell N = \bigoplus_{k \leqslant \ell} N^k$$

is independent of the lift. The triple $(N, F^\bullet N_\mathbb{C}, \mathrm{wt}_\bullet N)$ is a mixed Hodge structure [Pin89, §4.12, Remark (i)], and the associated bigrading of $N_\mathbb{C}$ determines a morphism $\mathtt{h}_\Phi(z) \in \mathrm{Hom}(\mathbb{S}_\mathbb{C}, P_\mathbb{C})$ independent of the choice of faithful representation N.

As in [Pin89, §4.7], define $Q_\Phi \subset P$ to be the smallest closed normal subgroup through which every such $\mathtt{h}_\Phi(z)$ factors. Thus we have normal subgroups

$$U_\Phi \lhd W_\Phi \lhd Q_\Phi \lhd P,$$

and a map

$$\mathtt{h}_\Phi : \mathcal{D} \to \mathrm{Hom}(\mathbb{S}_\mathbb{C}, Q_{\Phi\mathbb{C}}).$$

The cocharacter (2.2.1) takes values in Q_Φ/W_Φ, defining the *weight cocharacter*

$$w_\Phi : \mathbb{G}_m \to Q_\Phi/W_\Phi. \tag{2.2.2}$$

Remark 2.2.2. Being an abelian unipotent group, $\mathrm{Lie}(U_\Phi) \cong U_\Phi$ has the structure of a $\mathbb{Q}$-vector space. By [Pin89, Proposition 2.14], the conjugation action of Q_Φ on U_Φ is through a character

$$\nu_\Phi : Q_\Phi \to \mathbb{G}_m. \tag{2.2.3}$$

By [Pin89, Proposition 4.15(a)], the map $\mathtt{h}_\Phi$ restricts to an open immersion on every connected component of $\mathcal{D}$, and so the diagonal map

$$\mathcal{D} \to \pi_0(\mathcal{D}) \times \mathrm{Hom}(\mathbb{S}_\mathbb{C}, Q_{\Phi\mathbb{C}})$$

is a $P(\mathbb{R})$-equivariant open immersion. The action of the subgroup $U_\Phi(\mathbb{R})$ on $\pi_0(\mathcal{D})$ is trivial, and we extend it to the trivial action of $U_\Phi(\mathbb{C})$ on $\pi_0(\mathcal{D})$. Now define

$$\mathcal{D}_\Phi = Q_\Phi(\mathbb{R})U_\Phi(\mathbb{C})\mathcal{D}^\circ \subset \pi_0(\mathcal{D}) \times \mathrm{Hom}(\mathbb{S}_\mathbb{C}, Q_{\Phi\mathbb{C}}).$$

Projection to the second factor defines a finite-to-one map

$$\mathtt{h}_\Phi : \mathcal{D}_\Phi \to \mathrm{Hom}(\mathbb{S}_\mathbb{C}, Q_{\Phi\mathbb{C}}),$$

and we usually abuse notation and confuse $z \in \mathcal{D}_\Phi$ with its image $\mathtt{h}_\Phi(z)$.

Having now defined the mixed Shimura datum $(Q_\Phi, \mathcal{D}_\Phi)$, the compact open subgroup

$$K_\Phi \stackrel{\mathrm{def}}{=} hKh^{-1} \cap Q_\Phi(\mathbb{A}_f)$$

determines a mixed Shimura variety

$$\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) = Q_\Phi(\mathbb{Q})\backslash\big(\mathcal{D}_\Phi \times Q_\Phi(\mathbb{A}_f)/K_\Phi\big), \tag{2.2.4}$$

which has a canonical model $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)$ over its reflex field. Note that the reflex field is again $E(G, \mathcal{D})$, by [Pin89, Proposition 12.1]. The canonical model is a quasi-projective scheme if K (hence K_Φ) is neat.

Remark 2.2.3. If we choose our cusp label representative to have the form $\Phi = (G, \mathcal{D}^\circ, h)$, then $(Q_\Phi, \mathcal{D}_\Phi) = (G, \mathcal{D})$ and

$$\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) = \mathrm{Sh}_{hKh^{-1}}(G, \mathcal{D}) \cong \mathrm{Sh}_K(G, \mathcal{D}).$$

As a consequence, all of our statements about the mixed Shimura varieties $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)$ include the Shimura variety $\mathrm{Sh}_K(G, \mathcal{D})$ as a special case.

2.3. **The torsor structure.** Define $\bar{Q}_\Phi = Q_\Phi/U_\Phi$ and $\bar{\mathcal{D}}_\Phi = U_\Phi(\mathbb{C})\backslash\mathcal{D}_\Phi$. The pair

$$(\bar{Q}_\Phi, \bar{\mathcal{D}}_\Phi) = (Q_\Phi, \mathcal{D}_\Phi)/U_\Phi$$

is the quotient mixed Shimura datum in the sense of [Pin89, §2.9]. Let $\bar{K}_\Phi$ be the image of K_Φ under the quotient map $Q_\Phi(\mathbb{A}_f) \to \bar{Q}_\Phi(\mathbb{A}_f)$, so that we have a canonical morphism

$$\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) \to \mathrm{Sh}_{\bar{K}_\Phi}(\bar{Q}_\Phi, \bar{\mathcal{D}}_\Phi), \tag{2.3.1}$$

where the target mixed Shimura variety is defined in the same way as (2.2.4).

Proposition 2.3.1. *Define a $\mathbb{Z}$-lattice in $U_\Phi(\mathbb{Q})$ by $\Gamma_\Phi = K_\Phi \cap U_\Phi(\mathbb{Q})$. The morphism (2.3.1) is canonically a torsor for the relative torus*

$$T_\Phi \stackrel{\mathrm{def}}{=} \Gamma_\Phi(-1) \otimes \mathbb{G}_m$$

with cocharacter group $\Gamma_\Phi(-1) = (2\pi i)^{-1}\Gamma_\Phi$.

Proof. This is proved in [Pin89, §6.6]. In what follows we only want to make the torsor structure explicit on the level of complex points.

The character (2.2.3) factors through a character $\bar{\nu}_\Phi : \bar{Q}_\Phi \to \mathbb{G}_m$. A pair $(z, g) \in \mathcal{D}_\Phi \times Q_\Phi(\mathbb{A}_f)$ determines points

$$(z, g) \in \mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}), \quad (\bar{z}, \bar{g}) \in \mathrm{Sh}_{\bar{K}_\Phi}(\bar{Q}_\Phi, \bar{\mathcal{D}}_\Phi)(\mathbb{C}),$$

and we define $\mathbf{T}_\Phi(\mathbb{C}) \to \mathrm{Sh}_{\bar{K}_\Phi}(\bar{Q}_\Phi, \bar{\mathcal{D}}_\Phi)(\mathbb{C})$ as the relative torus with fiber

$$U_\Phi(\mathbb{C})/(gK_\Phi g^{-1} \cap U_\Phi(\mathbb{Q})) = U_\Phi(\mathbb{C})/\mathrm{rat}(\bar{\nu}_\Phi(\bar{g})) \cdot \Gamma_\Phi \tag{2.3.2}$$

at $(\bar{z}, \bar{g})$. There is a natural action of $\mathbf{T}_\Phi(\mathbb{C})$ on (2.2.4) defined as follows: using the natural action of $U_\Phi(\mathbb{C})$ on $\mathcal{D}_\Phi$, a point u in the fiber (2.3.2) acts as $(z, g) \mapsto (uz, g)$.

It now suffices to construct an isomorphism

$$\mathbf{T}_\Phi(\mathbb{C}) \cong T_\Phi(\mathbb{C}) \times \mathrm{Sh}_{\bar{K}_\Phi}(\bar{Q}_\Phi, \bar{\mathcal{D}}_\Phi)(\mathbb{C}), \tag{2.3.3}$$

and this is essentially [Pin89, §3.16]. First choose a morphism

$$\bar{\mathcal{D}}_\Phi \xrightarrow{\bar{z} \mapsto 2\pi\epsilon(\bar{z})} \mathcal{H}_0 \tag{2.3.4}$$

in such a way that it, along with the character $\bar{\nu}_\Phi$, induces a morphism of mixed Shimura data $(\bar{Q}_\Phi, \bar{\mathcal{D}}_\Phi) \to (\mathbb{G}_m, \mathcal{H}_0)$. Such a morphism always exists, by the Remark of [Pin89, §6.8]. The fiber (2.3.2) is

$$U_\Phi(\mathbb{C})/\mathrm{rat}(\bar{\nu}_\Phi(\bar{g})) \cdot \Gamma_\Phi \xrightarrow{2\pi\epsilon(\bar{z})/\mathrm{rat}(\bar{\nu}_\Phi(\bar{g}))} U_\Phi(\mathbb{C})/\Gamma_\Phi(1),$$

and this identifies $\mathbf{T}_\Phi(\mathbb{C})$ fiber-by-fiber with the constant torus

$$U_\Phi(\mathbb{C})/\Gamma_\Phi(1) \cong \Gamma_\Phi \otimes \mathbb{C}/\mathbb{Z}(1) \cong \Gamma_\Phi \otimes \mathbb{C}^\times \xrightarrow{(-2\pi\epsilon^\circ)^{-1}} \Gamma_\Phi(-1) \otimes \mathbb{C}^\times. \tag{2.3.5}$$

Here $2\pi\epsilon^\circ$ is the image of $\mathcal{D}^\circ$ under $\mathcal{D}_\Phi \to \bar{\mathcal{D}}_\Phi \to \mathcal{H}_0$, and the minus sign is included so that (2.6.5) holds below; compare with the definition of the function "ord" in [Pin89, §5.8].

One can easily check that the trivialization (2.3.3) does not depend on the choice of (2.3.4). □

Remark 2.3.2. Our $\mathbb{Z}$-lattice $\Gamma_\Phi \subset U_\Phi(\mathbb{Q})$ agrees with the seemingly more complicated lattice of [Pin89, §3.13], defined as the image of

$$\{(c,\gamma) \in Z_\Phi(\mathbb{Q})_0 \times U_\Phi(\mathbb{Q}) : c\gamma \in K_\Phi\} \xrightarrow{(c,\gamma)\mapsto\gamma} U_\Phi(\mathbb{Q}).$$

Here Z_Φ is the center of Q_Φ, and $Z_\Phi(\mathbb{Q})_0 \subset Z_\Phi(\mathbb{Q})$ is the largest subgroup acting trivially on $\mathcal{D}_\Phi$ (equivalently, acting trivially on $\pi_0(\mathcal{D}_\Phi)$). This follows from the final comments of [*loc. cit.*] and the simplifying Hypothesis 2.1.1, which implies that the connected center of Q_Φ/U_Φ is isogenous to the product of a $\mathbb{Q}$-split torus and a torus whose group of real points is compact (see the proof of [Pin89, Corollary 4.10]).

Denoting by $\langle -,-\rangle : \Gamma_\Phi^\vee(1) \times \Gamma_\Phi(-1) \to \mathbb{Z}$ the tautological pairing, define an isomorphism

$$\Gamma_\Phi^\vee(1) \xrightarrow{\alpha\mapsto q_\alpha} \mathrm{Hom}(\Gamma_\Phi(-1)\otimes\mathbb{G}_m, \mathbb{G}_m) = \mathrm{Hom}(T_\Phi, \mathbb{G}_m)$$

by $q_\alpha(\beta\otimes z) = z^{\langle\alpha,\beta\rangle}$. This determines an isomorphism

$$T_\Phi \cong \mathrm{Spec}\Big(\mathbb{Q}[q_\alpha]_{\alpha\in\Gamma_\Phi^\vee(1)}\Big),$$

and hence, for any rational polyhedral cone[1] $\sigma \subset U_\Phi(\mathbb{R})(-1)$, a partial compactification

$$T_\Phi(\sigma) \stackrel{\mathrm{def}}{=} \mathrm{Spec}\Big(\mathbb{Q}[q_\alpha]_{\substack{\alpha\in\Gamma_\Phi^\vee(1)\\ \langle\alpha,\sigma\rangle\geqslant 0}}\Big). \tag{2.3.6}$$

More generally, the T_Φ-torsor structure on (2.3.1) determines, by the general theory of torus embeddings [Pin89, §5], a partial compactification

$$\begin{tikzcd} \mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi) \arrow[r] \arrow[d] & \mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma) \arrow[dl] \\ \mathrm{Sh}_{\bar{K}_\Phi}(\bar{Q}_\Phi,\bar{\mathcal{D}}_\Phi) & \end{tikzcd} \tag{2.3.7}$$

with a stratification by locally closed substacks

$$\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma) = \bigsqcup_\tau Z^\tau_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma) \tag{2.3.8}$$

indexed by the faces $\tau \subset \sigma$. The unique open stratum

$$Z^{\{0\}}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma) = \mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi)$$

corresponds to $\tau = \{0\}$. The unique closed stratum corresponds to $\tau = \sigma$.

[1] By which we mean a *convex rational polyhedral cone* in the sense of [Pin89, §5.1]. In particular, each σ is a closed subset of the real vector space $U_\Phi(\mathbb{R})(-1)$.

2.4. **Rational polyhedral cone decompositions.** Let $\Phi = (P, \mathcal{D}^\circ, h)$ be a cusp label representative for $(G, \mathcal{D})$, with associated mixed Shimura datum $(Q_\Phi, \mathcal{D}_\Phi)$. We denote by $\mathcal{D}_\Phi^\circ = U_\Phi(\mathbb{C})\mathcal{D}^\circ$ the connected component of $\mathcal{D}_\Phi$ containing $\mathcal{D}^\circ$.

Define the *projection to the imaginary part* $c_\Phi : \mathcal{D}_\Phi \to U_\Phi(\mathbb{R})(-1)$ by

$$c_\Phi(z)^{-1} \cdot z \in \pi_0(\mathcal{D}) \times \mathrm{Hom}(\mathbb{S}, Q_{\Phi\mathbb{R}})$$

for every $z \in \mathcal{D}_\Phi$. By [Pin89, Proposition 4.15] there is an open convex cone

$$C_\Phi \subset U_\Phi(\mathbb{R})(-1) \tag{2.4.1}$$

characterized by $\mathcal{D}^\circ = \{z \in \mathcal{D}_\Phi^\circ : c_\Phi(z) \in C_\Phi\}$.

Definition 2.4.1. Suppose $\Phi = (P, \mathcal{D}^\circ, h)$ and $\Phi_1 = (P_1, \mathcal{D}_1^\circ, h_1)$ are cusp label representatives. A K-*morphism*

$$\Phi \xrightarrow{(\gamma,q)} \Phi_1 \tag{2.4.2}$$

is a pair $(\gamma, q) \in G(\mathbb{Q}) \times Q_{\Phi_1}(\mathbb{A}_f)$, such that

$$\gamma Q_\Phi \gamma^{-1} \subset Q_{\Phi_1}, \quad \gamma\mathcal{D}^\circ = \mathcal{D}_1^\circ, \quad \gamma h \in q h_1 K.$$

A K-morphism is a K-*isomorphism* if $\gamma Q_\Phi \gamma^{-1} = Q_{\Phi_1}$.

Remark 2.4.2. The Baily-Borel compactification of $\mathrm{Sh}_K(G, \mathcal{D})$ admits a stratification by locally closed substacks, defined over the reflex field, whose strata are indexed by the K-isomorphism classes of cusp label representatives. Whenever there is a K-morphism $\Phi \to \Phi_1$, the stratum indexed by Φ is "deeper into the boundary" than the stratum indexed by Φ_1, in the sense that the Φ-stratum is contained in the closure of the Φ_1-stratum. The unique open stratum, which is just the Shimura variety $\mathrm{Sh}_K(G, \mathcal{D})$, is indexed by the K-isomorphism class consisting of all cusp label representatives of the form $(G, \mathcal{D}^\circ, h)$ as $\mathcal{D}^\circ$ and h vary.

Suppose we have a K-morphism (2.4.2) of cusp label representatives. It follows from [Pin89, Proposition 4.21] that $U_{\Phi_1} \subset \gamma U_\Phi \gamma^{-1}$, and the image of the open convex cone C_{Φ_1} under

$$U_{\Phi_1}(\mathbb{R})(-1) \xrightarrow{u \mapsto \gamma^{-1} u \gamma} U_\Phi(\mathbb{R})(-1) \tag{2.4.3}$$

lies in the closure of the open convex cone C_Φ. Define, as in [Pin89, Definition-Proposition 4.22],

$$C_\Phi^* = \bigcup_{\Phi \to \Phi_1} \gamma^{-1} C_{\Phi_1} \gamma \subset U_\Phi(\mathbb{R})(-1),$$

where the union is over all K-morphisms with source Φ. This is a convex cone lying between C_Φ and its closure, but in general C_Φ^* is neither open nor closed. For every K-morphism $\Phi \to \Phi_1$ as above, the injection (2.4.3) identifies $C_{\Phi_1}^* \subset C_\Phi^*$.

Definition 2.4.3. A *(rational polyhedral) partial cone decomposition* of C^*_Φ is a collection $\Sigma_\Phi = \{\sigma\}$ of rational polyhedral cones $\sigma \subset U_\Phi(\mathbb{R})(-1)$ such that

- each $\sigma \in \Sigma_\Phi$ satisfies $\sigma \subset C^*_\Phi$,
- every face of every $\sigma \in \Sigma_\Phi$ is again an element of Σ_Φ,
- the intersection of any $\sigma, \tau \in \Sigma_\Phi$ is a face of both σ and τ,
- $\{0\} \in \Sigma_\Phi$.

We say that Σ_Φ is *smooth* if it is smooth, in the sense of [Pin89, §5.2], with respect to the lattice $\Gamma_\Phi(-1) \subset U_\Phi(\mathbb{R})(-1)$. It is *complete* if

$$C^*_\Phi = \bigcup_{\sigma \in \Sigma_\Phi} \sigma.$$

Definition 2.4.4. A *K-admissible (rational polyhedral) partial cone decomposition* $\Sigma = \{\Sigma_\Phi\}_\Phi$ for $(G, \mathcal{D})$ is a collection of partial cone decompositions Σ_Φ for C^*_Φ, one for every cusp label representative Φ, such that for any K-morphism $\Phi \to \Phi_1$, the induced inclusion $C^*_{\Phi_1} \subset C^*_\Phi$ identifies

$$\Sigma_{\Phi_1} = \{\sigma \in \Sigma_\Phi : \sigma \subset C^*_{\Phi_1}\}.$$

We say that Σ is *smooth* if every Σ_Φ is smooth, and *complete* if every Σ_Φ is complete.

Fix a K-admissible complete cone decomposition Σ of $(G, \mathcal{D})$.

Definition 2.4.5. A *toroidal stratum representative* for $(G, \mathcal{D}, \Sigma)$ is a pair (Φ, σ) in which Φ is a cusp label representative and $\sigma \in \Sigma_\Phi$ is a rational polyhedral cone whose interior is contained in C_Φ. In other words, σ is not contained in any proper subset $C^*_{\Phi_1} \subsetneq C^*_\Phi$ determined by a K-morphism $\Phi \to \Phi_1$.

We now extend Definition 2.4.1 from cusp label representatives to toroidal stratum representatives.

Definition 2.4.6. A *K-morphism* of toroidal stratum representatives

$$(\Phi, \sigma) \xrightarrow{(\gamma, q)} (\Phi_1, \sigma_1)$$

consists of a pair $(\gamma, q) \in G(\mathbb{Q}) \times Q_{\Phi_1}(\mathbb{A}_f)$ such that

$$\gamma Q_\Phi \gamma^{-1} \subset Q_{\Phi_1}, \quad \gamma \mathcal{D}^\circ = \mathcal{D}^\circ_1, \quad \gamma h \in q h_1 K,$$

and such that the injection (2.4.3) identifies σ_1 with a face of σ. Such a K-morphism is a *K-isomorphism* if $\gamma Q_\Phi \gamma^{-1} = Q_{\Phi_1}$ and $\gamma^{-1} \sigma_1 \gamma = \sigma$.

The set of K-isomorphism classes of toroidal stratum representatives will be denoted $\mathrm{Strat}_K(G, \mathcal{D}, \Sigma)$.

Definition 2.4.7. We say that Σ is *finite* if $\#\mathrm{Strat}_K(G, \mathcal{D}, \Sigma) < \infty$.

Definition 2.4.8. We say that Σ has the *no self-intersection property* if the following holds: whenever we are given toroidal stratum representatives (Φ,σ) and (Φ_1,σ_1), and two K-morphisms

$$(\Phi,\sigma) \rightrightarrows (\Phi_1,\sigma_1),$$

the two injections

$$U_{\Phi_1}(\mathbb{R})(-1) \rightrightarrows U_{\Phi}(\mathbb{R})(-1)$$

of (2.4.3) send σ_1 to the same face of σ.

The no self-intersection property is just a rewording of the condition of [Pin89, §7.12]. If Σ has the no self-intersection property then so does any refinement (in the sense of [Pin89, §5.1]).

Remark 2.4.9. Any finite and K-admissible cone decomposition Σ for $(G,\mathcal{D})$ acquires the no self-intersection property after possibly replacing K by a smaller compact open subgroup [Pin89, §7.13]. Moreover, by examining the proof one can see that if K factors as $K = K_\ell K^\ell$ for some prime ℓ with $K_\ell \subset G(\mathbb{Q}_\ell)$ and $K^\ell \subset G(\mathbb{A}_f^\ell)$, then it suffices to shrink K_ℓ while holding K^ℓ fixed.

2.5. **Functoriality of cone decompositions.** Suppose that we have an embedding $(G,\mathcal{D}) \to (G',\mathcal{D}')$ of Shimura data.

As explained in [Mad19, (2.1.28)], every cusp label representative

$$\Phi = (P,\mathcal{D}^\circ, g)$$

for $(G,\mathcal{D})$ determines a cusp label representative

$$\Phi' = (P',\mathcal{D}'^{,\circ}, g')$$

for $(G',\mathcal{D}')$. More precisely, we define $g' = g$, let $\mathcal{D}'^{,\circ} \subset \mathcal{D}'$ be the connected component containing $\mathcal{D}^\circ$, and let $P' \subset G'$ be the smallest admissible parabolic subgroup containing P. In particular,

$$Q_\Phi \subset Q_{\Phi'}, \quad U_\Phi \subset U_{\Phi'}, \quad C_\Phi \subset C_{\Phi'}.$$

If $K \subset G(\mathbb{A}_f)$ is a compact open subgroup contained in a compact open subgroup $K' \subset G'(\mathbb{A}_f)$, then every K-morphism

$$\Phi \xrightarrow{(\gamma,q)} \Phi_1$$

determines a K'-morphism

$$\Phi' \xrightarrow{(\gamma,q)} \Phi_1'.$$

Any K'-admissible rational cone decomposition Σ' for $(G',\mathcal{D}')$ pulls back to a K-admissible rational cone decomposition Σ for $(G,\mathcal{D})$, defined by

$$\Sigma_\Phi = \{\sigma' \cap C_\Phi^* : \sigma' \in \Sigma'_{\Phi'}\}$$

for every cusp label representative Φ of $(G,\mathcal{D})$. It is shown in [Har89, §3.3] that Σ is finite whenever Σ' is so. It is also not hard to check that Σ has the no self-intersection property whenever Σ' does, and that it is complete when Σ' is so.

Given a cusp label representative Φ for $(G,\mathcal{D})$ and a $\sigma \in \Sigma_\Phi$, there is a unique rational polyhedral cone $\sigma' \in \Sigma'_{\Phi'}$ such that $\sigma \subset \sigma'$, but σ is not contained in any proper face of σ'. The assignment $(\Phi,\sigma) \mapsto (\Phi',\sigma')$ induces a function

$$\mathrm{Strat}_K(G,\mathcal{D},\Sigma) \to \mathrm{Strat}_{K'}(G',\mathcal{D}',\Sigma')$$

on K-isomorphism classes of toroidal stratum representatives.

2.6. **Compactification of canonical models.** In this subsection we assume that $K \subset G(\mathbb{A}_f)$ is neat. Suppose Σ is a finite and K-admissible complete cone decomposition for $(G,\mathcal{D})$.

Remark 2.6.1. A Σ with the above properties always exists, and may be refined, in the sense of [Pin89, §5.1], to make it smooth. This is the content of [Pin89, Theorem 9.21].

The main result of [Pin89, §12] is the existence of a proper toroidal compactification

$$\mathrm{Sh}_K(G,\mathcal{D}) \hookrightarrow \mathrm{Sh}_K(G,\mathcal{D},\Sigma),$$

in the category of algebraic spaces over $E(G,\mathcal{D})$, along with a stratification

$$\mathrm{Sh}_K(G,\mathcal{D},\Sigma) = \bigsqcup_{(\Phi,\sigma)\in \mathrm{Strat}_K(G,\mathcal{D},\Sigma)} Z_K^{(\Phi,\sigma)}(G,\mathcal{D},\Sigma) \tag{2.6.1}$$

by locally closed subspaces indexed by the finite set $\mathrm{Strat}_K(G,\mathcal{D},\Sigma)$ appearing in Definition 2.4.7. The stratum indexed by (Φ,σ) lies in the closure of the stratum indexed by (Φ_1,σ_1) if and only if there is a K-morphism of toroidal stratum representatives $(\Phi,\sigma) \to (\Phi_1,\sigma_1)$.

If Σ is smooth then so is the toroidal compacification.

After possibly shrinking K, we may assume that Σ has the no self-intersection property (see Remark 2.4.9). The no self-intersection property guarantees that the strata appearing in (2.6.1) have an especially simple shape. Fix one $(\Phi,\sigma) \in \mathrm{Strat}_K(G,\mathcal{D},\Sigma)$ and write $\Phi = (P,\mathcal{D}^\circ,h)$. Pink shows that there is a canonical isomorphism

$$\begin{tikzcd} Z^\sigma_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma) \arrow[r, "\cong"] \arrow[d] & Z_K^{(\Phi,\sigma)}(G,\mathcal{D},\Sigma) \arrow[d] \\ \mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma) & \mathrm{Sh}_K(G,\mathcal{D},\Sigma) \end{tikzcd} \tag{2.6.2}$$

such that the two algebraic spaces in the bottom row become isomorphic after formal completion along their common locally closed subspace in the top row. See [Pin89, Corollary 7.17] and [Pin89, Theorem 12.4].

In other words, if we abbreviate

$$\widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma) = \mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma)^\wedge_{Z^\sigma_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma)}$$

for the formal completion of $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)$ along its closed stratum, and abbreviate[2]

$$\widehat{\mathrm{Sh}}_K(G, \mathcal{D}, \Sigma) = \mathrm{Sh}_K(G, \mathcal{D}, \Sigma)^\wedge_{Z_K^{(\Phi,\sigma)}(G,\mathcal{D},\Sigma)}$$

for the formal completion of $\mathrm{Sh}_K(G, \mathcal{D}, \Sigma)$ along the locally closed stratum $Z_K^{(\Phi,\sigma)}(G, \mathcal{D}, \Sigma)$, there is an isomorphism of formal algebraic spaces

$$\widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) \cong \widehat{\mathrm{Sh}}_K(G, \mathcal{D}, \Sigma). \tag{2.6.3}$$

Remark 2.6.2. In [Pin89] the isomorphism (2.6.3) is constructed after the left hand side is replace by its quotient by a finite group action. Thanks to Hypothesis 2.1.1 and the assumption that K is neat, the finite group in question is trivial. See [Wil00, Lemma 1.7 and Remark 1.8].

We can make the above more explicit on the level of complex points. Suppose (Φ, σ) is a toroidal stratum representative with underlying cusp label representative $\Phi = (P, \mathcal{D}^\circ, h)$, and denote by $Q_\Phi(\mathbb{R})^\circ \subset Q_\Phi(\mathbb{R})$ the stabilizer of the connected component $\mathcal{D}^\circ \subset \mathcal{D}$. The complex manifold

$$\mathscr{U}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) = Q_\Phi(\mathbb{Q})^\circ \backslash (\mathcal{D}^\circ \times Q_\Phi(\mathbb{A}_f)/K_\Phi)$$

sits in a diagram

$$\begin{tikzcd} \mathscr{U}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) \arrow[r] \arrow[d, "(z,g)\mapsto(z,gh)"'] & \mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) \\ \mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C}) & \end{tikzcd} \tag{2.6.4}$$

in which the horizontal arrow is an open immersion, and the vertical arrow is a local isomorphism. This allows us to define a partial compactification

$$\mathscr{U}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) \hookrightarrow \mathscr{U}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)$$

as the interior of the closure of $\mathscr{U}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)$ in $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)(\mathbb{C})$.

Any K-morphism as in Definition 2.4.6 induces a morphism of complex manifolds

$$\mathscr{U}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) \xrightarrow{(z,g)\mapsto(\gamma z, \gamma g \gamma^{-1} q)} \mathscr{U}_{K_{\Phi_1}}(Q_{\Phi_1}, \mathcal{D}_{\Phi_1}),$$

which extends uniquely to

$$\mathscr{U}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) \to \mathscr{U}_{K_{\Phi_1}}(Q_{\Phi_1}, \mathcal{D}_{\Phi_1}, \sigma_1).$$

Complex analytically, the toroidal compactification is defined as the quotient

$$\mathrm{Sh}_K(G, \mathcal{D}, \Sigma)(\mathbb{C}) = \Big(\bigsqcup_{(\Phi,\sigma)\in \mathrm{Strat}_K(G,\mathcal{D},\Sigma)} \mathscr{U}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) \Big) \Big/ \sim,$$

where $\sim$ is the equivalence relation generated by the graphs of all such morphisms.

[2] In order to limit the already burdensome notation, we choose to suppress the dependence on (Φ, σ) of the left hand side. The meaning will always be clear from context.

By [Pin89, §6.13] the closed stratum appearing in (2.3.8) satisfies

$$Z^{\sigma}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)(\mathbb{C}) \subset \mathscr{U}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma). \tag{2.6.5}$$

The morphisms in (2.6.4) extend continuously to morphisms

$$\begin{tikzcd} \mathscr{U}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) \arrow[r] \arrow[d] & \mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)(\mathbb{C}) \\ \mathrm{Sh}_K(G, \mathcal{D}, \Sigma)(\mathbb{C}) & \end{tikzcd} \tag{2.6.6}$$

in such a way that the vertical map identifies

$$Z^{\sigma}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)(\mathbb{C}) \cong Z^{(\Phi,\sigma)}_{K}(G, \mathcal{D}, \Sigma)(\mathbb{C}).$$

This agrees with the analytification of the isomorphism (2.6.2).

Now pick any point $z \in Z^{\sigma}_{\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)(\mathbb{C})$. Let R be the completed local ring of $\mathrm{Sh}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}}$ at z, and let R_Φ be the completed local ring of $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)_{/\mathbb{C}}$ at z. Each completed local ring can be computed with respect to the étale or analytic topologies, and the results are canonically identified. Working in the analytic topology, the morphisms in (2.6.6) induce an isomorphism $R \cong R_\Phi$, as they identify both rings with the completed local ring of $\mathscr{U}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)$ at z. This analytic isomorphism agrees with the one induced by the algebraic isomorphism (2.6.3).

3. Automorphic vector bundles

Throughout §3 we fix a Shimura datum $(G, \mathcal{D})$ satisfying Hypothesis 2.1.1, and a compact open subgroup $K \subset G(\mathbb{A}_f)$.

We recall the theory of automorphic vector bundles on the Shimura variety $\mathrm{Sh}_K(G, \mathcal{D})$, on its toroidal compactification, and on the mixed Shimura varieties appearing along the boundary. The main reference is [HZ01].

3.1. Holomorphic vector bundles.

Let $\Phi = (P, \mathcal{D}^\circ, h)$ be a cusp label representative for $(G, \mathcal{D})$. As in §2, this determines a mixed Shimura datum $(Q_\Phi, \mathcal{D}_\Phi)$ and a compact open subgroup $K_\Phi \subset Q_\Phi(\mathbb{A}_f)$.

Suppose we have a representation $Q_\Phi \to \mathrm{GL}(N)$ on a finite dimensional $\mathbb{Q}$-vector space. Given a point $z \in \mathcal{D}_\Phi$, its image under

$$\mathcal{D}_\Phi \to \mathrm{Hom}(\mathbb{S}_\mathbb{C}, Q_{\Phi\mathbb{C}})$$

determines a mixed Hodge structure $(N, F^\bullet N_\mathbb{C}, \mathrm{wt}_\bullet N)$. The weight filtration is independent of z, and is split by any lift $\mathbb{G}_m \to Q_\Phi$ of the weight cocharacter (2.2.2).

Denote by $(\boldsymbol{N}^{an}_{dR}, F^\bullet \boldsymbol{N}^{an}_{dR}, \mathrm{wt}_\bullet \boldsymbol{N}^{an}_{dR})$ the doubly filtered holomorphic vector bundle on $\mathcal{D}_\Phi \times Q_\Phi(\mathbb{A}_f)/K_\Phi$ whose fiber at (z, g) is the vector space $N_\mathbb{C}$ endowed with the Hodge and weight filtrations determined by z. There is a natural action of $Q_\Phi(\mathbb{Q})$ on this doubly filtered vector bundle, covering the action on the base. By taking the quotient, we obtain a functor

$$N \mapsto (\boldsymbol{N}^{an}_{dR}, F^\bullet \boldsymbol{N}^{an}_{dR}, \mathrm{wt}_\bullet \boldsymbol{N}^{an}_{dR}) \tag{3.1.1}$$

from finite dimensional representations of Q_Φ to doubly filtered holomorphic vector bundles on $\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi)(\mathbb{C})$. Ignoring the double filtration, this functor is simply

$$N \mapsto \boldsymbol{N}^{an}_{dR} = Q_\Phi(\mathbb{Q})\backslash\big(\mathcal{D}_\Phi \times N_\mathbb{C} \times Q_\Phi(\mathbb{A}_f)/K_\Phi\big). \tag{3.1.2}$$

Given a K_Φ-stable $\widehat{\mathbb{Z}}$-lattice $N_{\widehat{\mathbb{Z}}} \subset N \otimes \mathbb{A}_f$, we may define a $\mathbb{Z}$-lattice

$$gN_\mathbb{Z} = gN_{\widehat{\mathbb{Z}}} \cap N$$

for every $g \in Q_\Phi(\mathbb{A}_f)$, along with a weight filtration

$$\mathrm{wt}_\bullet(gN_\mathbb{Z}) = gN_{\widehat{\mathbb{Z}}} \cap \mathrm{wt}_\bullet N.$$

Denote by $(\boldsymbol{N}_{Be}, \mathrm{wt}_\bullet \boldsymbol{N}_{Be})$ the filtered $\mathbb{Z}$-local system on $\mathcal{D}_\Phi \times Q_\Phi(\mathbb{A}_f)/K_\Phi$ whose fiber at (z,g) is $(gN_\mathbb{Z}, \mathrm{wt}_\bullet(gN_\mathbb{Z}))$. This local system has an obvious action of $Q_\Phi(\mathbb{Q})$, covering the action on the base. Passing to the quotient, we obtain a functor

$$N_{\widehat{\mathbb{Z}}} \mapsto (\boldsymbol{N}_{Be}, \mathrm{wt}_\bullet \boldsymbol{N}_{Be})$$

from K_Φ-stable $\widehat{\mathbb{Z}}$-lattices in $N \otimes \mathbb{A}_f$ to filtered $\mathbb{Z}$-local systems on (2.2.4).

By construction there is a canonical isomorphism

$$(\boldsymbol{N}^{an}_{dR}, \mathrm{wt}_\bullet \boldsymbol{N}^{an}_{dR}) \cong (\boldsymbol{N}_{Be} \otimes \mathcal{O}^{an}, \mathrm{wt}_\bullet \boldsymbol{N}_{Be} \otimes \mathcal{O}^{an}), \tag{3.1.3}$$

where $\mathcal{O}^{an}$ denotes the structure sheaf on $\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi)(\mathbb{C})$.

3.2. **The Borel morphism.** Suppose $G \to \mathrm{GL}(N)$ is any faithful representation of G on a finite dimensional $\mathbb{Q}$-vector space. A point $z \in \mathcal{D}$ determines a Hodge structure $\mathbb{S} \to \mathrm{GL}(N_\mathbb{R})$ on N, and we denote by $F^\bullet N_\mathbb{C}$ the induced Hodge filtration. As in [Mil90, §III.1] and [HZ01, §1], define the *compact dual*

$$\check{M}(G,\mathcal{D})(\mathbb{C}) = \left\{ \begin{array}{c} \text{descending filtrations on } N_\mathbb{C} \\ \text{that are } G(\mathbb{C})\text{-conjugate to } F^\bullet N_\mathbb{C} \end{array} \right\}. \tag{3.2.1}$$

By construction, there is a canonical $G(\mathbb{R})$-equivariant finite-to-one *Borel morphism*

$$\mathcal{D} \to \check{M}(G,\mathcal{D})(\mathbb{C})$$

sending a point of $\mathcal{D}$ to the induced Hodge filtration on $N_\mathbb{C}$. The compact dual is the space of complex points of a smooth projective variety $\check{M}(G,\mathcal{D})$ defined over the reflex field $E(G,\mathcal{D})$, and admitting an action of $G_{E(G,\mathcal{D})}$ inducing the natural action of $G(\mathbb{C})$ on complex points. It is independent of the choice of z, and of the choice of faithful representation N.

More generally, there is an analogue of (3.2.1) for the mixed Shimura datum $(Q_\Phi,\mathcal{D}_\Phi)$, as in [Hör10, Main Theorem 3.4.1] and [Hör14, Main Theorem 2.5.12]. Let $Q_\Phi \to \mathrm{GL}(N)$ be a faithful representation on a finite dimensional $\mathbb{Q}$-vector space. Any point $z \in \mathcal{D}_\Phi$ then determines a mixed Hodge structure $(N, F^\bullet N_\mathbb{C}, \mathrm{wt}_\bullet N)$, and we define the *dual* of $(Q_\Phi,\mathcal{D}_\Phi)$ by

$$\check{M}(Q_\Phi,\mathcal{D}_\Phi)(\mathbb{C}) = \left\{ \begin{array}{c} \text{descending filtrations on } N_\mathbb{C} \\ \text{that are } Q_\Phi(\mathbb{C})\text{-conjugate to } F^\bullet N_\mathbb{C} \end{array} \right\}.$$

It is the space of complex points of an open $Q_{\Phi,E(G,\mathcal{D})}$-orbit

$$\check{M}(Q_\Phi,\mathcal{D}_\Phi) \subset \check{M}(G,\mathcal{D}),$$

independent of the choice of $z \in \mathcal{D}_\Phi$ and N. By construction, there is a $Q_\Phi(\mathbb{C})$-equivariant *Borel morphism*

$$\mathcal{D}_\Phi \to \check{M}(Q_\Phi,\mathcal{D}_\Phi)(\mathbb{C}).$$

3.3. **The standard torsor.** We want to give a more algebraic interpretation of the functor (3.1.1).

Harris and Zucker [HZ01, §1] prove that the mixed Shimura variety (2.2.4) carries a *standard torsor*[3]. This consists of a diagram of $E(G,\mathcal{D})$-stacks

$$\begin{tikzcd} J_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi) \arrow[r, "b"] \arrow[d, "a"'] & \check{M}(Q_\Phi,\mathcal{D}_\Phi) \\ \mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi), & \end{tikzcd} \tag{3.3.1}$$

in which a is a relative Q_Φ-torsor, and b is Q_Φ-equivariant. See also the papers of Harris [Har84, Har85, Har86], Harris-Zucker [HZ94a, HZ94b], and Milne [Mil88, Mil90]. Complex analytically, the standard torsor is the complex orbifold

$$J_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi)(\mathbb{C}) = Q_\Phi(\mathbb{Q})\backslash\big(\mathcal{D}_\Phi \times Q_\Phi(\mathbb{C}) \times Q_\Phi(\mathbb{A}_f)/K_\Phi\big),$$

with $Q_\Phi(\mathbb{C})$ acting by $s\cdot(z,t,g) = (z,ts^{-1},g)$. The morphisms a and b are, respectively,

$$(z,t,g) \mapsto (z,g) \quad \text{and} \quad (z,t,g) \mapsto t^{-1}z.$$

Exactly as in [HZ01], we can use the standard torsor to define models of the vector bundles (3.1.1) over the reflex field. First, we require a lemma.

Lemma 3.3.1. *Suppose $\check{N} \to \check{M}(Q_\Phi,\mathcal{D}_\Phi)$ is a Q_Φ-equivariant vector bundle; that is, a finite rank vector bundle endowed with an action of $Q_{\Phi,E(G,\mathcal{D})}$ covering the action on the base. There are canonical Q_Φ-equivariant filtrations $\mathrm{wt}_\bullet\check{N}$ and $F^\bullet\check{N}$ on $\check{N}$, and the construction*

$$\check{N} \mapsto (\check{N}, F^\bullet\check{N}, \mathrm{wt}_\bullet\check{N})$$

is functorial in $\check{N}$.

Proof. Fix a faithful representation $Q_\Phi \to \mathrm{GL}(H)$. Suppose we are given an étale neighborhood $U \to \check{M}(Q_\Phi,\mathcal{D}_\Phi)$ of some geometric point x of $\check{M}(Q_\Phi,\mathcal{D}_\Phi)$. By the very definition of $\check{M}(Q_\Phi,\mathcal{D}_\Phi)$, U determines a $Q_{\Phi U}$-stable filtration $F^\bullet H_U$ on $H_U = H \otimes \mathcal{O}_U$. After possibly shrinking U we may choose a cocharacter $\mu_x : \mathbb{G}_m \to Q_{\Phi U}$ splitting this filtration.

As $Q_{\Phi U}$ acts on $\check{N}_U$, the cocharacter μ_x determines a filtration $F^\bullet\check{N}_U$, which does not depend on the choice of splitting. Glueing over an étale cover of $\check{M}(Q_\Phi,\mathcal{D}_\Phi)$ defines the desired filtration $F^\bullet\check{N}$. The definition of

[3] a.k.a. *standard principal bundle*

$\mathrm{wt}_\bullet \check{N}$ is similar, but easier: it is the filtration split by any lift $\mathbb{G}_m \to Q_\Phi$ of the weight cocharacter (2.2.2). $\square$

Now suppose we have a representation $Q_\Phi \to \mathrm{GL}(N)$ on a finite dimensional $\mathbb{Q}$-vector space. Applying Lemma 3.3.1 to the constant Q_Φ-equivariant vector bundle

$$\check{N} = \check{M}(Q_\Phi, \mathcal{D}_\Phi) \times_{\mathrm{Spec}(E(G,\mathcal{D}))} N_{E(G,\mathcal{D})}$$

yields a Q_Φ-equivariant doubly filtered vector bundle $(\check{N}, F^\bullet\check{N}, \mathrm{wt}_\bullet\check{N})$ on $\check{M}(Q_\Phi, \mathcal{D}_\Phi)$. The construction

$$N \mapsto (\boldsymbol{N}_{dR}, F^\bullet \boldsymbol{N}_{dR}, \mathrm{wt}_\bullet \boldsymbol{N}_{dR}) = Q_\Phi \backslash b^*(\check{N}, F^\bullet\check{N}, \mathrm{wt}_\bullet\check{N}) \tag{3.3.2}$$

defines a functor from representations of Q_Φ to doubly filtered vector bundles on $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)$. Passing to the complex fiber recovers the functor (3.1.1).

The following proposition extends the above functor to partial compactifications.

Proposition 3.3.2. *For any rational polyhedral cone $\sigma \subset U_\Phi(\mathbb{R})(-1)$ there is a functor*

$$N \mapsto (\boldsymbol{N}_{dR}, F^\bullet \boldsymbol{N}_{dR}, \mathrm{wt}_\bullet \boldsymbol{N}_{dR}),$$

extending (3.3.2), from representations of Q_Φ on finite dimensional $\mathbb{Q}$-vector spaces to doubly filtered vector bundles on $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)$.

Proof. This is part of [HZ01, Definition-Proposition 1.3.5]. Here we sketch a different argument.

Recall the T_Φ-torsor structure on (2.3.1). On complex points, this action was deduced from the natural left action of $U_\Phi(\mathbb{C})$ on $\mathcal{D}_\Phi$. Of course the group $U_\Phi(\mathbb{C})$ also acts on both factors of $\mathcal{D}_\Phi \times Q_\Phi(\mathbb{C})$ on the left, and imitating the proof of Proposition 2.3.1 yields action of the relative torus $T_\Phi(\mathbb{C})$ on the standard torsor $J_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C})$, covering the action on $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C})$.

To see that the action is algebraic and defined over the reflex field, one can reduce, exactly as in the proof of [HZ01, Proposition 1.2.4], to the case in which $(Q_\Phi, \mathcal{D}_\Phi)$ is either a pure Shimura datum, or is a mixed Shimura datum associated with a Siegel Shimura datum. The pure case is vacuous (the relative torus is trivial). The Siegel mixed Shimura varieties are moduli spaces of polarized 1-motives, and it is not difficult to give a moduli-theoretic interpretation of the torus action, along the lines of [Mad19, §2.2.8]. From this interpretation the descent to the reflex field is obvious.

In the diagram (3.3.1), the arrow a is T_Φ-equivariant, and the arrow b is constant on T_Φ-orbits. This is clear from the complex analytic description.

Taking the quotient of the standard torsor by this action, we obtain a diagram

$$\begin{tikzcd} T_\Phi \backslash J_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) \arrow[r, "b"] \arrow[d, "a"'] & \check{M}(Q_\Phi, \mathcal{D}_\Phi) \\ \mathrm{Sh}_{\bar{K}_\Phi}(\bar{Q}_\Phi, \bar{\mathcal{D}}_\Phi), & \end{tikzcd}$$

in which a is a relative Q_Φ-torsor and b is Q_Φ-equivariant. Pulling back the quotient $T_\Phi \backslash J_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)$ along the diagonal arrow in (2.3.7) defines the upper left entry in the diagram

$$\begin{tikzcd} J_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) \arrow[r, "b"] \arrow[d, "a"'] & \check{M}(Q_\Phi, \mathcal{D}_\Phi) \\ \mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) & \end{tikzcd}$$

extending (3.3.1), in which a is a Q_Φ-torsor, and b is Q_Φ-equivariant. Now simply repeat the construction (3.3.2) to obtain the desired functor. □

Remark 3.3.3. The proof actually shows more: because the standard torsor admits a canonical descent to $\mathrm{Sh}_{\bar{K}_\Phi}(\bar{Q}_\Phi, \bar{\mathcal{D}}_\Phi)$, the same is true of all doubly filtered vector bundles (3.3.2). Compare with [HZ01, (1.2.11)].

3.4. **Automorphic vector bundles on toroidal compactifications.** Assume that K is neat, and that Σ is a finite K-admissible complete cone decomposition for $(G, \mathcal{D})$ having the no self-intersection property.

By results of Harris and Harris-Zucker, see especially [HZ01], one can glue together the diagrams in the proof of Proposition 3.3.2 as (Φ, σ) varies in order to obtain a diagram

$$\begin{tikzcd} J_K(G, \mathcal{D}, \Sigma) \arrow[r, "b"] \arrow[d, "a"'] & \check{M}(G, \mathcal{D}) \\ \mathrm{Sh}_K(G, \mathcal{D}, \Sigma) & \end{tikzcd} \tag{3.4.1}$$

in which a is a G-torsor and b is G-equivariant. This implies the following:

Theorem 3.4.1. *There is a functor $N \mapsto (\boldsymbol{N}_{dR}, F^\bullet \boldsymbol{N}_{dR})$ from representations of G on finite dimensional $\mathbb{Q}$-vector spaces to filtered vector bundles on $\mathrm{Sh}_K(G, \mathcal{D}, \Sigma)$, compatible, in the obvious sense, with the isomorphism*

$$\widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) \cong \widehat{\mathrm{Sh}}_K(G, \mathcal{D}, \Sigma)$$

of (2.6.3) and the functor of Proposition 3.3.2, for every toroidal stratum representative

$$(\Phi, \sigma) \in \mathrm{Strat}_K(G, \mathcal{D}, \Sigma).$$

In other words, there is an arithmetic theory of automorphic vector bundles on toroidal compactifications.

Remark 3.4.2. Over the open Shimura variety $\mathrm{Sh}_K(G,\mathcal{D})$ there is also a weight filtration $\mathrm{wt}_\bullet \boldsymbol{N}_{dR}$ on $\boldsymbol{N}_{dR}$, but it is not compatible with the weight filtrations along the boundary. It is also not very interesting. On an irreducible representation N the (central) weight cocharacter $w : \mathbb{G}_m \to G$ acts through $z \mapsto z^k$ for some k, and the weight filtration has a unique nonzero graded piece $\mathrm{gr}_k \boldsymbol{N}_{dR}$.

3.5. **A simple Shimura variety.** Let $(\mathbb{G}_m, \mathcal{H}_0)$ be the Shimura datum of Remark 2.1.2. For any compact open subgroup $K \subset \mathbb{A}_f^\times$, we obtain a 0-dimensional Shimura variety

$$\mathrm{Sh}_K(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C}) = \mathbb{Q}^\times \backslash (\mathcal{H}_0 \times \mathbb{A}_f^\times / K), \tag{3.5.1}$$

with a canonical model $\mathrm{Sh}_K(\mathbb{G}_m, \mathcal{H}_0)$ over $\mathbb{Q}$.

The action of $\mathrm{Aut}(\mathbb{C})$ on its complex points satisfies

$$\tau \cdot (2\pi\epsilon, a) = (2\pi\epsilon, a a_\tau) \tag{3.5.2}$$

whenever $\tau \in \mathrm{Aut}(\mathbb{C})$ and $a_\tau \in \mathbb{A}_f^\times$ are related by $\tau|_{\mathbb{Q}^{ab}} = \mathrm{rec}(a_\tau)$. This implies that

$$\mathrm{Sh}_K(\mathbb{G}_m, \mathcal{H}_0) \cong \mathrm{Spec}(F),$$

where $F/\mathbb{Q}$ is the abelian extension characterized by

$$\mathrm{rec} : \mathbb{Q}_{>0}^\times \backslash \mathbb{A}_f^\times / K \cong \mathrm{Gal}(F/\mathbb{Q}).$$

The following proposition shows that all automorphic vector bundles on (3.5.1) are canonically trivial. The particular trivializations will be essential in our later discussion of q-expansions. See especially Proposition 4.6.1.

Proposition 3.5.1. *For any representation $\mathbb{G}_m \to \mathrm{GL}(N)$ there is a canonical isomorphism*

$$N \otimes \mathcal{O}_{\mathrm{Sh}_K(\mathbb{G}_m,\mathcal{H}_0)} \xrightarrow{n\otimes 1 \mapsto \boldsymbol{n}} \boldsymbol{N}_{dR}$$

of vector bundles. If $\mathbb{G}_m$ acts on N through the character $z \mapsto z^k$, the global section $\boldsymbol{n} = n \otimes 1$ is given, in terms of the complex parametrization

$$\boldsymbol{N}_{dR}^{an} = \mathbb{Q}^\times \backslash (\mathcal{H}_0 \times N_{\mathbb{C}} \times \mathbb{A}_f^\times / K)$$

of (3.1.2), by

$$(2\pi\epsilon, a) \mapsto \left(2\pi\epsilon, \frac{\mathrm{rat}(a)^k}{(2\pi\epsilon)^k} \cdot n, a \right).$$

Proof. First set $N = \mathbb{Q}$ with $\mathbb{G}_m$ acting via the identity character $z \mapsto z$, and set $N_{\widehat{\mathbb{Z}}} = \widehat{\mathbb{Z}}$. Recalling (3.1.3), the quotient $\boldsymbol{N}_{Be} \backslash \boldsymbol{N}_{dR}^{an}$ defines an analytic family of rank one tori over $\mathrm{Sh}_K(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C})$, whose relative Lie algebra is the line bundle

$$\mathrm{Lie}(\boldsymbol{N}_{Be} \backslash \boldsymbol{N}_{dR}^{an}) = \boldsymbol{N}_{dR}^{an} = \mathbb{Q}^\times \backslash (\mathcal{H}_0 \times \mathbb{C} \times \mathbb{A}_f^\times / K).$$

Using this identification, we may identify the standard $\mathbb{C}^\times$-torsor

$$J_K(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C}) = \mathbb{Q}^\times \backslash \big(\mathcal{H}_0 \times \mathbb{C}^\times \times \mathbb{A}_f^\times / K \big) \tag{3.5.3}$$

with the $\mathbb{C}^\times$-torsor of trivializations of $\mathrm{Lie}(\boldsymbol{N}_{Be} \backslash \boldsymbol{N}_{dR}^{an})$.

On the other hand, the isomorphisms

$$(N \cap aN_{\widehat{\mathbb{Z}}})\backslash N_{\mathbb{C}} = (\mathbb{Q} \cap a\widehat{\mathbb{Z}})\backslash \mathbb{C} \xrightarrow{2\pi\epsilon/\mathrm{rat}(a)} \mathbb{Z}(1)\backslash \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times$$

identify $\boldsymbol{N}_{Be}\backslash \boldsymbol{N}_{dR}^{an}$, fiber-by-fiber, with the constant torus $\mathbb{C}^\times$, and so identify (3.5.3) with the $\mathbb{C}^\times$-torsor of trivializations of $\mathrm{Lie}(\mathbb{C}^\times)$. The canonical model of (3.5.3) is now concretely realized as the $\mathbb{G}_m$-torsor

$$J_K(\mathbb{G}_m, \mathcal{H}_0) = \underline{\mathrm{Iso}}\big(\mathrm{Lie}(\mathbb{G}_m), \mathcal{O}_{\mathrm{Sh}_K(\mathbb{G}_m,\mathcal{H}_0)}\big).$$

For any ring R, the Lie algebra of $\mathbb{G}_m = \mathrm{Spec}(R[q,q^{-1}])$ is canonically trivialized by the invariant derivation $q \cdot d/dq$. Thus the standard torsor admits a canonical section which, in terms of the uniformization (3.5.3), is

$$(2\pi\epsilon, a) \mapsto \left(2\pi\epsilon, \frac{\mathrm{rat}(a)}{2\pi\epsilon}, a\right).$$

This section trivializes the standard torsor, and induces the desired trivialization of any automorphic vector bundle. □

Remark 3.5.2. Let $\mathbb{G}_m$ act on N via $z \mapsto z^k$. What the above proof actually shows is that there are canonical isomorphisms

$$N \otimes \mathcal{O}_{\mathrm{Sh}_K(\mathbb{G}_m,\mathcal{H}_0)} \cong N \otimes \mathrm{Lie}(\mathbb{G}_m)^{\otimes k} \cong \boldsymbol{N}_{dR}.$$

4. Orthogonal Shimura varieties

In §4 we specialize the preceding theory to the case of Shimura varieties associated to the group of spinor similitudes of a quadratic space (V,Q) over $\mathbb{Q}$ of signature $(n,2)$ with $n \geqslant 1$. This will allow us to define q-expansions of modular forms on such Shimura varieties, and prove the q-expansion principle of Proposition 4.6.3.

4.1. **The GSpin Shimura variety.** Let $G = \mathrm{GSpin}(V)$ as in [Mad16]. This is a reductive group over $\mathbb{Q}$ sitting in an exact sequence

$$1 \to \mathbb{G}_m \to G \to \mathrm{SO}(V) \to 1.$$

There is a distinguished character $\nu : G \to \mathbb{G}_m$, called the *spinor similitude*. Its kernel is the usual spin double cover of $\mathrm{SO}(V)$, and its restriction to $\mathbb{G}_m$ is $z \mapsto z^2$.

The group $G(\mathbb{R})$ acts on the hermitian domain

$$\mathcal{D} = \big\{z \in V_{\mathbb{C}} : [z,z] = 0 \text{ and } [z,\overline{z}] < 0\big\}/\mathbb{C}^\times \subset \mathbb{P}(V_{\mathbb{C}}) \tag{4.1.1}$$

in the obvious way. This hermitian domain has two connected components, interchanged by the action of any $\gamma \in G(\mathbb{R})$ with $\nu(\gamma) < 0$. The pair $(G,\mathcal{D})$ is the *GSpin Shimura datum*. Its reflex field is $\mathbb{Q}$.

By construction, G is a subgroup of the multiplicative group of the Clifford algebra $C(V)$. As such, G has two distinguished representations. One is the standard representation $G \to \mathrm{SO}(V)$, and the other is the faithful action on

$H = C(V)$ defined by left multiplication in the Clifford algebra. These two representations are related by a G-equivariant injection

$$V \to \mathrm{End}_{\mathbb{Q}}(H) \tag{4.1.2}$$

defined by the left multiplication action of $V \subset C(V)$ on H. A point $z \in \mathcal{D}$ determines a Hodge structure on any representation of G. For the representations V and H, the induced Hodge filtrations are

$$F^2 V_{\mathbb{C}} = 0, \quad F^1 V_{\mathbb{C}} = \mathbb{C}z, \quad F^0 V_{\mathbb{C}} = (\mathbb{C}z)^{\perp}, \quad F^{-1} V_{\mathbb{C}} = V_{\mathbb{C}}, \tag{4.1.3}$$

and

$$F^1 H_{\mathbb{C}} = 0, \quad F^0 H_{\mathbb{C}} = zH_{\mathbb{C}}, \quad F^{-1} H_{\mathbb{C}} = H_{\mathbb{C}}. \tag{4.1.4}$$

Here we are using (4.1.2) to view $\mathbb{C}z \subset \mathrm{End}_{\mathbb{C}}(H_{\mathbb{C}})$.

In order to obtain a Shimura variety $\mathrm{Sh}_K(G, \mathcal{D})$ as in 1.1, we fix a $\mathbb{Z}$-lattice $V_{\mathbb{Z}} \subset V$ on which Q is $\mathbb{Z}$-valued and assume that the compact open subgroup $K \subset G(\mathbb{A}_f)$ is chosen as in (1.1.2). According to [Mad16, Lemma 2.6], any such K stabilizes both $V_{\widehat{\mathbb{Z}}}$ and its dual, and acts trivially on the discriminant group

$$V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}} \cong V_{\widehat{\mathbb{Z}}}^{\vee}/V_{\widehat{\mathbb{Z}}}. \tag{4.1.5}$$

4.2. **The line bundle of modular forms.** Applying the functor of Proposition 3.3.2 to the standard representation $G \to \mathrm{SO}(V)$ yields a filtered vector bundle $(\boldsymbol{V}_{dR}, F^{\bullet}\boldsymbol{V}_{dR})$ on $\mathrm{Sh}_K(G, \mathcal{D})$. The filtration has the form

$$0 = F^2\boldsymbol{V}_{dR} \subset F^1\boldsymbol{V}_{dR} \subset F^0\boldsymbol{V}_{dR} \subset F^{-1}\boldsymbol{V}_{dR} = \boldsymbol{V}_{dR},$$

in which $F^1\boldsymbol{V}_{dR}$ is a line, isotropic with respect to the bilinear form

$$[-,-] : \boldsymbol{V}_{dR} \otimes \boldsymbol{V}_{dR} \to \mathcal{O}_{\mathrm{Sh}_K(G,\mathcal{D})} \tag{4.2.1}$$

induced by (1.1.1), and $F^0\boldsymbol{V}_{dR} = (F^1\boldsymbol{V}_{dR})^{\perp}$. These properties are clear from the complex analytic definition (3.1.1) of $\boldsymbol{V}_{dR}^{an}$, and the explicit description of the Hodge filtration (4.1.3). In particular, the filtration on $\boldsymbol{V}_{dR}$ is completely determined by the isotropic line $F^1\boldsymbol{V}_{dR}$.

Definition 4.2.1. The *line bundle of weight one modular forms* on $\mathrm{Sh}_K(G, \mathcal{D})$ is defined by

$$\boldsymbol{\omega} = F^1\boldsymbol{V}_{dR}.$$

For any $g \in G(\mathbb{A}_f)$, the pullback of $\boldsymbol{\omega}$ via the complex uniformization

$$\mathcal{D} \xrightarrow{z \mapsto (z,g)} \mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C})$$

is just the tautological bundle on the hermitian domain (4.1.1). In particular, the line bundle $\boldsymbol{\omega}$ carries a metric, inherited from the metric

$$\|z\|^2_{\mathrm{naive}} = -[z, \overline{z}] \tag{4.2.2}$$

on the tautological bundle. We will more often use the rescaled metric

$$\|z\|^2 = -\frac{[z, \overline{z}]}{4\pi e^{\gamma}} \tag{4.2.3}$$

where $\gamma = -\Gamma'(1)$ is the Euler-Mascheroni constant.

4.3. **The Hodge embedding.** As above, let $H = C(V)$ viewed as a faithful 2^{n+2}-dimensional representation of $G \subset C(V)^\times$ via left multiplication. If we define a $\mathbb{Z}$-lattice in H by

$$H_{\mathbb{Z}} = C(V_{\mathbb{Z}}),$$

the inclusion (1.1.2) implies that $H_{\widehat{\mathbb{Z}}} = H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ is K-stable.

The discussion of §3 provides a filtered vector bundle $(\boldsymbol{H}_{dR}, F^\bullet \boldsymbol{H}_{dR})$ on $\mathrm{Sh}_K(G, \mathcal{D})$, and a $\mathbb{Z}$-local system $\boldsymbol{H}_{Be}$ over the complex fiber endowed with an isomorphism

$$\boldsymbol{H}_{Be} \otimes \mathcal{O}_{\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})} \cong \boldsymbol{H}_{dR}^{an}.$$

The double quotient

$$A(\mathbb{C}) = \boldsymbol{H}_{Be} \backslash \boldsymbol{H}_{dR}^{an} / F^0 \boldsymbol{H}_{dR}^{an} \tag{4.3.1}$$

defines an analytic family of complex tori over $\mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C})$. In fact, this arises from an abelian scheme over $\mathrm{Sh}_K(G, \mathcal{D})$, as we now explain.

As in [AGHMP17, §2.2], one may choose a symplectic form ψ on H such that the representation of G factors through $G^{\mathrm{Sg}} = \mathrm{GSp}(H)$, and induces a Hodge embedding

$$(G, \mathcal{D}) \to (G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}})$$

into the Siegel Shimura datum determined by (H, ψ). Explicitly, choose any vectors $v, w \in V$ of negative length with $[v, w] = 0$ and set

$$\delta = vw \in C(V).$$

If we denote by $c \mapsto c^*$ the $\mathbb{Q}$-algebra involution on $C(V)$ fixing pointwise the subset $V \subset C(V)$, then $\delta^* = -\delta$. Denoting by $\mathrm{Trd} : C(V) \to \mathbb{Q}$ the reduced trace, the symplectic form

$$\psi(x, y) = \mathrm{Trd}(x\delta y^*)$$

has the desired properties.

As in (3.2.1), we may describe the compact dual $\check{M}(G, \mathcal{D})$ as a G-orbit of descending filtrations on the faithful representation H. It is more convenient to characterize the compact dual as the $\mathbb{Q}$-scheme with functor of points

$$\check{M}(G, \mathcal{D})(S) = \{\text{isotropic lines } z \subset V \otimes \mathcal{O}_S\},$$

where *line* means a locally free $\mathcal{O}_S$-module direct summand of rank one. In order to realize $\check{M}(G, \mathcal{D})$ as a space of filtrations on H, first define

$$\check{M}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}})(S) = \{\text{Lagrangian subsheaves } F^0 \subset H \otimes \mathcal{O}_S\}$$

and then use (4.1.2) to define a closed immersion

$$\check{M}(G, \mathcal{D}) \to \check{M}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}})$$

sending the isotropic line $z \subset V$ to the Lagrangian $zH \subset H$.

By rescaling, we may assume that ψ is $\mathbb{Z}$-valued on $H_{\mathbb{Z}}$, and so the Hodge embedding defines a morphism from $\mathrm{Sh}_K(G,\mathcal{D})$ to a moduli stack of polarized abelian varieties of dimension 2^{n+1}. Pulling back the universal object defines the *Kuga-Satake abelian scheme*

$$\pi : A \to \mathrm{Sh}_K(G,\mathcal{D}).$$

The Kuga-Satake abelian scheme does not depend on the choice of ψ, but the polarization on it does. Passing to the complex analytic fiber recovers the family of complex tori defined by (4.3.1).

The first relative de Rham homology of A, with its Hodge filtration, is related to the vector bundle $\boldsymbol{H}_{dR}$ by a canonical isomorphism of filtered vector bundles

$$\boldsymbol{H}_{dR} \cong \underline{\mathrm{Hom}}\big(R^1\pi_*\Omega^\bullet_{A/\mathrm{Sh}_K(G,\mathcal{D})}, \mathcal{O}_{\mathrm{Sh}_K(G,\mathcal{D})}\big).$$

4.4. **Cusp label representatives: isotropic lines.** We wish to make more explicit the structure of the mixed Shimura datum $(Q_\Phi, \mathcal{D}_\Phi)$ associated to a cusp label representative

$$\Phi = (P, \mathcal{D}^\circ, h)$$

for $(G,\mathcal{D})$. See §2.2 for the definitions.

The admissible parabolic $P \subset G$ is the stabilizer of a totally isotropic subspace $I \subset V$ with $\dim(I) \in \{0,1,2\}$. In this subsection we assume that $P \subset G$ is the stabilizer of an isotropic line $I \subset V$. The case of isotropic planes will be considered in §4.5.

The P-stable weight filtration on V defined by

$$\mathrm{wt}_{-3}V = 0, \quad \mathrm{wt}_{-2}V = \mathrm{wt}_{-1}V = I, \quad \mathrm{wt}_0V = \mathrm{wt}_1V = I^\perp, \quad \mathrm{wt}_2V = V,$$

and the Hodge filtration (4.1.3) determined by a point $z \in \mathcal{D}$, together determine a mixed Hodge structure

$$\mathbb{S}_{\mathbb{C}} \xrightarrow{\mathbf{h}_\Phi(z)} P_{\mathbb{C}} \to \mathrm{SO}(V_{\mathbb{C}})$$

on V of type $\{(-1,-1),(0,0),(1,1)\}$.

Similarly,the P-stable weight filtration on H defined by

$$\mathrm{wt}_{-3}H = 0, \quad \mathrm{wt}_{-2}H = \mathrm{wt}_{-1}H = IH, \quad \mathrm{wt}_0H = H,$$

and the Hodge filtration (4.1.4) determined by a point $z \in \mathcal{D}$, together determine a mixed Hodge structure

$$\mathbb{S}_{\mathbb{C}} \xrightarrow{\mathbf{h}_\Phi(z)} P_{\mathbb{C}} \to \mathrm{GSp}(H_{\mathbb{C}})$$

on H of type $\{(-1,-1),(0,0)\}$. In the definition of the weight filtration we are using the inclusion $I \subset \mathrm{End}_{\mathbb{Q}}(H)$ determined by (4.1.2), and setting

$$IH = \mathrm{Span}_{\mathbb{Q}}\{\ell x : \ell \in I,\, x \in H\}.$$

The proof of the following lemma is left as an exercise to the reader.

Lemma 4.4.1. *Recalling the notation (1.6.1), the largest closed normal subgroup $Q_\Phi \subset P$ through which every such $\mathtt{h}_\Phi(z)$ factors is*

$$Q_\Phi = \ker\big(P \to \mathrm{GL}(\mathrm{gr}_0(H))\big).$$

The action $Q_\Phi \to \mathrm{SO}(V)$ is faithful, and is given on the graded pieces of $\mathrm{wt}_\bullet V$ by the commutative diagram

$$\begin{tikzcd} Q_\Phi \arrow[rr, "\nu_\Phi"] \arrow[d] & & \mathbb{G}_m \arrow[d, "t \mapsto (t,1,t^{-1})"] \\ P \arrow[r] & \mathrm{GL}(I) \times \mathrm{SO}(I^\perp/I) \times \mathrm{GL}(V/I^\perp), & \end{tikzcd} \tag{4.4.1}$$

in which ν_Φ is the restriction to Q_Φ of the spinor similitude on G. This agrees with the character (2.2.3). The groups U_Φ and W_Φ are

$$U_\Phi = W_\Phi = \ker(\nu_\Phi : Q_\Phi \to \mathbb{G}_m),$$

and there is an isomorphism of $\mathbb{Q}$-vector spaces

$$(I^\perp/I) \otimes I \cong U_\Phi(\mathbb{Q}) \tag{4.4.2}$$

sending $v \otimes \ell \in (I^\perp/I) \otimes I$ to the unipotent transformation of V defined by

$$x \mapsto x + [x,\ell]v - [x,v]\ell - Q(v)[x,\ell]\ell.$$

The dual of $(Q_\Phi, \mathcal{D}_\Phi)$ is the $\mathbb{Q}$-scheme with functor of points

$$\check{M}(Q_\Phi, \mathcal{D}_\Phi)(S) = \left\{ \begin{array}{c} \text{isotropic lines } z \subset V \otimes \mathcal{O}_S \text{ such that} \\ V \to V/I^\perp \\ \text{identifies } z \cong (V/I^\perp) \otimes \mathcal{O}_S \end{array} \right\}. \tag{4.4.3}$$

Every point

$$z \in \mathcal{D}_\Phi \subset \pi_0(\mathcal{D}) \times \mathrm{Hom}(\mathbb{S}_\mathbb{C}, Q_{\Phi\mathbb{C}})$$

determines a mixed Hodge structure on V of type $\{(-1,-1),(0,0),(1,1)\}$, and the Borel morphism

$$\mathcal{D}_\Phi \to \check{M}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C})$$

sends z to the isotropic line $F^1 V_\mathbb{C} \subset V_\mathbb{C}$. This induces an isomorphism

$$\mathcal{D}_\Phi \cong \pi_0(\mathcal{D}) \times \check{M}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}). \tag{4.4.4}$$

4.5. **Cusp label representatives: isotropic planes.** In this subsection we fix a cusp label representative $\Phi = (P, \mathcal{D}^\circ, h)$ with $P \subset G$ the stabilizer of an isotropic plane $I \subset V$.

The P-stable weight filtrations on V defined by

$$\mathrm{wt}_{-2}V = 0, \quad \mathrm{wt}_{-1}V = I, \quad \mathrm{wt}_0 V = I^\perp, \quad \mathrm{wt}_1 V = V,$$

and the Hodge filtration (4.1.3) determined by a point $z \in \mathcal{D}$, together determine a mixed Hodge structure

$$\mathbb{S}_\mathbb{C} \xrightarrow{\mathtt{h}_\Phi(z)} P_\mathbb{C} \to \mathrm{SO}(V_\mathbb{C})$$

on V of type $\{(-1,0),(0,-1),(0,0),(1,0),(0,1)\}$.

Similarly, the P-stable weight filtration on H defined by

$$\mathrm{wt}_{-3}H = 0, \quad \mathrm{wt}_{-2}H = I^2H, \quad \mathrm{wt}_{-1}H = IH, \quad \mathrm{wt}_0 H = H,$$

and the Hodge filtration (4.1.4) determined by a point $z \in \mathcal{D}$, together determine a mixed Hodge structure

$$\mathbb{S}_{\mathbb{C}} \xrightarrow{\mathtt{h}_\Phi(z)} P_{\mathbb{C}} \to \mathrm{GSp}(H_{\mathbb{C}})$$

on H of type $\{(-1,-1),(-1,0),(0,-1),(0,0)\}$. In the definition of the weight filtration we are using the inclusion $I \subset \mathrm{End}_{\mathbb{Q}}(H)$ determined by (4.1.2), and setting

$$\begin{aligned} IH &= \mathrm{Span}_{\mathbb{Q}}\{\ell x : \ell \in I,\, x \in H\} \\ I^2H &= \mathrm{Span}_{\mathbb{Q}}\{\ell\ell' x : \ell, \ell' \in I,\, x \in H\}. \end{aligned}$$

The proof of the following lemma is left as an exercise to the reader.

Lemma 4.5.1. *Recalling the notation (1.6.1), the largest closed normal subgroup $Q_\Phi \subset P$ through which every such $\mathtt{h}_\Phi(z)$ factors is*

$$Q_\Phi = \ker\big(P \to \mathrm{GL}(\mathrm{gr}_0(H))\big).$$

The natural action $Q_\Phi \to \mathrm{SO}(V)$ is faithful, and is trivial on the quotient $I^\perp/I$. The groups $U_\Phi \lhd W_\Phi \lhd Q_\Phi$ are

$$W_\Phi = \ker(Q_\Phi \to \mathrm{GL}(I)),$$

and

$$U_\Phi \cong \bigwedge^2 I,$$

where we identify $a \wedge b \in \bigwedge^2 I$ with the unipotent transformation of V defined by

$$x \mapsto x + [x,a]b - [x,b]a.$$

The dual of $(Q_\Phi, \mathcal{D}_\Phi)$ is the $\mathbb{Q}$-scheme with functor of points

$$\check{M}(Q_\Phi,\mathcal{D}_\Phi)(S) = \left\{ \begin{array}{c} \text{isotropic lines } z \subset V \otimes \mathcal{O}_S \text{ such that} \\ V \to V/I^\perp \text{ identifies } z \text{ with a rank one} \\ \text{local direct summand of } (V/I^\perp) \otimes \mathcal{O}_S \end{array} \right\}.$$

Every point $z \in \mathcal{D}_\Phi$ determines a mixed Hodge structure on V of type $\{(-1,0),(0,-1),(0,0),(0,1),(1,0)\}$, and again the Borel morphism

$$\mathcal{D}_\Phi \to \check{M}(Q_\Phi,\mathcal{D}_\Phi)(\mathbb{C})$$

sends $z \mapsto F^1V_{\mathbb{C}}$. It identifies $\mathcal{D}_\Phi$ with the open subset

$$\mathcal{D}_\Phi = U_\Phi(\mathbb{C})\mathcal{D} \subset \pi_0(\mathcal{D}) \times \check{M}(Q_\Phi,\mathcal{D}_\Phi)(\mathbb{C}).$$

4.6. **The q-expansion principle.** Now suppose the compact open subgroup K of (1.1.2) is neat, and small enough that there exists a finite K-admissible complete cone decomposition Σ for $(G, \mathcal{D})$ having the no self-intersection property. See §2.4 for the definitions.

The results of Pink recalled in §2.6 provide us with a toroidal compactification

$$\mathrm{Sh}_K(G, \mathcal{D}, \Sigma) = \bigsqcup_{(\Phi,\sigma)\in \mathrm{Strat}_K(G,\mathcal{D},\Sigma)} Z_K^{(\Phi,,\sigma)}(G, \mathcal{D}, \Sigma), \tag{4.6.1}$$

and the result of Harris-Zucker recalled as Theorem 3.4.1 gives a filtered vector bundle

$$0 = F^2\boldsymbol{V}_{dR} \subset F^1\boldsymbol{V}_{dR} \subset F^0\boldsymbol{V}_{dR} \subset F^{-1}\boldsymbol{V}_{dR} = \boldsymbol{V}_{dR}$$

on the compactification, endowed with a quadratic form

$$[-,-] : \boldsymbol{V}_{dR} \to \mathcal{O}_{\mathrm{Sh}_K(G,\mathcal{D},\Sigma)}$$

induced by the bilinear form on V. Exactly as in 4.2, the *line bundle of weight one modular forms*

$$\boldsymbol{\omega} = F^1\boldsymbol{V}_{dR}$$

is isotropic with respect to this bilinear form, and $F^0\boldsymbol{V}_{dR} = (F^1\boldsymbol{V}_{dR})^\perp$. These constructions extend those of §4.2 from the open Shimura variety to its compactification.

In order to define q-expansions of sections of $\boldsymbol{\omega}^{\otimes k}$ on (4.6.1), we need to make some additional choices. The first choice is a boundary stratum

$$Z_K^{(\Phi,\sigma)}(G, \mathcal{D}, \Sigma)_{/\mathbb{C}} \subset \mathrm{Sh}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}} \tag{4.6.2}$$

indexed by a toroidal stratum representative (Φ, σ) in which the parabolic subgroup appearing in the underlying cusp label representative

$$\Phi = (P, \mathcal{D}^\circ, h)$$

is the stabilizer of an isotropic line $I \subset V$. The second choice is a nonzero vector $\ell \in I$, which will determine a trivialization of $\boldsymbol{\omega}$ in a formal neighborhood of the stratum (4.6.2).

As $\mathcal{D}$ has two connected components, there are exactly two continuous surjections $\nu : \mathcal{D} \to \mathcal{H}_0$. Fix one of them. It, along with the spinor similitude $\nu : G \to \mathbb{G}_m$, induces a morphism of Shimura data

$$(G, \mathcal{D}) \xrightarrow{\nu} (\mathbb{G}_m, \mathcal{H}_0).$$

Denote by $2\pi\epsilon^\circ \in \mathcal{H}_0$ the image of the component $\mathcal{D}^\circ$. There is a unique continuous extension of $\nu : \mathcal{D} \to \mathcal{H}_0$ to $\nu_\Phi : \mathcal{D}_\Phi \to \mathcal{H}_0$, and this determines a morphism of mixed Shimura data

$$(Q_\Phi, \mathcal{D}_\Phi) \xrightarrow{\nu_\Phi} (\mathbb{G}_m, \mathcal{H}_0), \tag{4.6.3}$$

where $\nu_\Phi : Q_\Phi \to \mathbb{G}_m$ is the character of (4.4.1).

Applying the functor of Proposition 3.3.2 to the Q_Φ-representations $I \subset V$ determines vector bundles $\boldsymbol{I}_{dR} \subset \boldsymbol{V}_{dR}$ on $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)$. The vector bundle $\boldsymbol{V}_{dR}$ is endowed with a filtration and a symmetric bilinear pairing, exactly as in the discussion following (4.6.1), and restricting the bilinear pairing yields a homomorphism

$$[\cdot,\cdot] : \boldsymbol{I}_{dR} \otimes \boldsymbol{\omega} \to \mathcal{O}_{\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma)}. \tag{4.6.4}$$

The choice of nonzero vector $\ell \in I$ defines a section

$$\boldsymbol{\ell}^{an} \in H^0\big(\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi)(\mathbb{C}), \boldsymbol{I}_{dR}^{an}\big)$$

of the line bundle

$$\boldsymbol{I}_{dR}^{an} = Q_\Phi(\mathbb{Q})\backslash\big(\mathcal{D}_\Phi \times I_{\mathbb{C}} \times Q_\Phi(\mathbb{A}_f)/K_\Phi\big)$$

by sending

$$(z,g) \mapsto \left(z, \frac{\mathrm{rat}(\nu_\Phi(g))}{\nu_\Phi(z)} \cdot \ell, g\right).$$

Proposition 4.6.1. *The holomorphic section $\boldsymbol{\ell}^{an}$ extends uniquely to the partial compactification $\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma)(\mathbb{C})$. This extension is algebraic and defined over $\mathbb{Q}$, and so arises from a unique global section*

$$\boldsymbol{\ell} \in H^0\big(\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma), \boldsymbol{I}_{dR}\big). \tag{4.6.5}$$

Moreover, (4.6.4) is an isomorphism, and induces an isomorphism

$$\boldsymbol{\omega} \xrightarrow{\psi\mapsto[\boldsymbol{\ell},\psi]} \mathcal{O}_{\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma)}.$$

Proof. As the action of Q_Φ on I is via $\nu_\Phi : Q_\Phi/U_\Phi \to \mathbb{G}_m$, the discussion of §3.5 (see especially Remark 3.5.2) identifies $\boldsymbol{I}_{dR}$ with the pullback of the line bundle $I \otimes \mathrm{Lie}(\mathbb{G}_m) \cong I \otimes \mathcal{O}_{\mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m,\mathcal{H}_0)}$ via

$$\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma) \xrightarrow{(2.3.7)} \mathrm{Sh}_{\bar{K}_\Phi}(\bar{Q}_\Phi,\bar{\mathcal{D}}_\Phi) = \mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m,\mathcal{H}_0).$$

The section (4.6.5) is simply the pullback of the trivializing section

$$\ell \otimes 1 \in H^0\big(\mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m,\mathcal{H}_0), I \otimes \mathcal{O}_{\mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m,\mathcal{H}_0)}\big).$$

It now suffices to prove that (4.6.4) is an isomorphism. Recall from §4.4 that $\check{M}(Q_\Phi,\mathcal{D}_\Phi)$ has functor of points

$$\check{M}(Q_\Phi,\mathcal{D}_\Phi)(S) = \left\{ \begin{array}{c} \text{isotropic lines } z \subset V \otimes \mathcal{O}_S \text{ such that} \\ V \to V/I^\perp \text{ identifies } z \cong (V/I^\perp) \otimes \mathcal{O}_S \end{array} \right\}.$$

Let $\check{I}$ and $\check{V}$ be the (constant) Q_Φ-equivariant vector bundles on $\check{M}(Q_\Phi,\mathcal{D}_\Phi)$ determined by the representations I and V. In the notation of Lemma 3.3.1, the line bundle $\check{\omega} = F^1\check{V}$ is the tautological bundle, and the bilinear form on V determines a Q_Φ-equivariant isomorphism

$$\check{I} \otimes \check{\omega} \to \check{V} \otimes \check{V} \xrightarrow{[-,-]} \mathcal{O}_{\check{M}(Q_\Phi,\mathcal{D}_\Phi)}.$$

By examining the construction of the functor in Proposition 3.3.2, the induced morphism (4.6.4) is also an isomorphism. □

It follows from the analysis of §4.4 that the diagram (2.3.7) has the form

$$\begin{tikzcd} \mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) \arrow[r] \arrow[d, "\nu_\Phi"'] & \mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) \arrow[dl] \\ \mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m, \mathcal{H}_0), & \end{tikzcd}$$

in which the arrow labeled ν_Φ is a torsor for the n-dimensional torus

$$T_\Phi = \mathrm{Spec}\Big(\mathbb{Q}[q_\alpha]_{\alpha \in \Gamma_\Phi^\vee(1)}\Big)$$

over the 0-dimensional base $\mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m, \mathcal{H}_0)$. To define q-expansions we will trivialize this torsor over an étale extension of the base, effectively putting coordinates on the mixed Shimura variety $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)$.

Choose an auxiliary isotropic line $I_* \subset V$ with $[I, I_*] \neq 0$. This choice fixes a section

$$\begin{tikzcd} (Q_\Phi, \mathcal{D}_\Phi) \arrow[r, "\nu_\Phi"'] & (\mathbb{G}_m, \mathcal{H}_0). \arrow[l, bend right, "s"'] \end{tikzcd}$$

The underlying morphism of groups $s : \mathbb{G}_m \to Q_\Phi$ sends, for any $\mathbb{Q}$-algebra R, $a \in R^\times$ to the orthogonal transformation

$$s(a) \cdot x = \begin{cases} ax & \text{if } x \in I_R \\ a^{-1}x & \text{if } x \in I_{*,R} \\ x & \text{if } x \in (I \oplus I_*)^\perp_R. \end{cases} \tag{4.6.6}$$

To characterize $s : \mathcal{H}_0 \to \mathcal{D}_\Phi$, we first use (4.4.3) to view

$$I_{*\mathbb{C}} \in \check{M}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}).$$

Recalling the isomorphism (4.4.4), the preimage of $I_{*\mathbb{C}}$ under the projection

$$\mathcal{D}_\Phi \cong \pi_0(\mathcal{D}) \times \check{M}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C}) \to \check{M}(Q_\Phi, \mathcal{D}_\Phi)(\mathbb{C})$$

consists of two points, indexed by the two connected components of $\mathcal{D}$. The function $s : \mathcal{H}_0 \to \mathcal{D}_\Phi$ is defined by sending $2\pi\epsilon^\circ \in \mathcal{H}_0$ to the point indexed by $\mathcal{D}^\circ$, and the other element of $\mathcal{H}_0$ to the point indexed by the other connected component of $\mathcal{D}$.

The section s determines a Levi decomposition $Q_\Phi = \mathbb{G}_m \ltimes U_\Phi$. Choose a compact open subgroup $K_0 \subset \mathbb{G}_m(\mathbb{A}_f)$ small enough that its image under (4.6.6) is contained in K_Φ, and set

$$K_{\Phi 0} = K_0 \ltimes (U_\Phi(\mathbb{A}_f) \cap K_\Phi) \subset K_\Phi.$$

Our hypothesis that K is neat implies that $K_0 \subset K_{\Phi 0} \subset K_\Phi$ are also neat.

Proposition 4.6.2. *Assume that the rational polyhedral cone $\sigma \subset U_\Phi(\mathbb{R})(-1)$ has (top) dimension n. The above choices determine a commutative diagram*

$$\begin{tikzcd} \bigsqcup_{a \in \mathbb{Q}^\times_{>0} \backslash \mathbb{A}_f^\times / K_0} \widehat{T}_\Phi(\sigma)_{/\mathbb{C}} \arrow[r, "\cong"] \arrow[d] & \widehat{\mathrm{Sh}}_{K_{\Phi 0}}(Q_\Phi, \mathcal{D}_\Phi, \sigma)_{/\mathbb{C}} \arrow[d] \\ \widehat{\mathrm{Sh}}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}} \arrow[r, "\cong"] & \widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)_{/\mathbb{C}} \end{tikzcd}$$

of formal algebraic spaces, in which the vertical arrows are formally étale surjections. Here

$$\widehat{T}_\Phi(\sigma) \stackrel{\mathrm{def}}{=} \mathrm{Spf}\Big(\mathbb{Q}[[q_\alpha]]_{\substack{\alpha \in \Gamma_\Phi^\vee(1) \\ \langle \alpha, \sigma \rangle \geqslant 0}} \Big) \tag{4.6.7}$$

is the formal completion of (2.3.6) along its closed stratum, the lower left corner is the formal completion of $\mathrm{Sh}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}}$ along the 0-dimensional stratum (4.6.2), and the bottom isomorphism is (2.6.3).

Proof. Consider the diagram

$$\begin{tikzcd} \mathrm{Sh}_{K_0}(\mathbb{G}_m, \mathcal{H}_0) \times_{\mathrm{Spec}(\mathbb{Q})} T_\Phi \arrow[r, equal] \arrow[dr] & \mathrm{Sh}_{K_{\Phi 0}}(Q_\Phi, \mathcal{D}_\Phi) \arrow[r] \arrow[d, "\nu_\Phi"'] & \mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) \arrow[d, "\nu_\Phi"'] \\ & \mathrm{Sh}_{K_0}(\mathbb{G}_m, \mathcal{H}_0) \arrow[r] \arrow[u, bend right, "s"'] & \mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m, \mathcal{H}_0) \end{tikzcd}$$

in which the arrows labeled ν_Φ are the T_Φ-torsors of (2.3.1), and the isomorphism " $=$ " is the trivialization induced by the section s.

There is a canonical bijection

$$\mathbb{Q}^\times_{>0} \backslash \mathbb{A}_f^\times / K_0 \xrightarrow{\cong} \mathrm{Sh}_{K_0}(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C}) \tag{4.6.8}$$

defined by $a \mapsto [(2\pi\epsilon^\circ, a)]$. We remind the reader that $2\pi\epsilon^\circ \in \mathcal{H}_0$ was fixed in the discussion preceding (4.6.3).

Using this, the top row of the above diagram exhibits $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)_{/\mathbb{C}}$ as an étale quotient

$$\bigsqcup_{a \in \mathbb{Q}^\times_{>0} \backslash \mathbb{A}_f^\times / K_0} T_{\Phi/\mathbb{C}} \cong \mathrm{Sh}_{K_{\Phi 0}}(Q_\Phi, \mathcal{D}_\Phi)_{/\mathbb{C}} \to \mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)_{/\mathbb{C}}. \tag{4.6.9}$$

This morphism extends to partial compactifications, and formally completing along the closed stratum yields a formally étale morphism

$$\bigsqcup_{a \in \mathbb{Q}^\times_{>0} \backslash \mathbb{A}_f^\times / K_0} \widehat{T}_\Phi(\sigma)_{/\mathbb{C}} \cong \widehat{\mathrm{Sh}}_{K_{\Phi 0}}(Q_\Phi, \mathcal{D}_\Phi, \sigma)_{/\mathbb{C}} \to \widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)_{/\mathbb{C}}.$$

This defines the top horizontal arrow and the right vertical arrow in the diagram. The vertical arrow on the left is defined by the commutativity of the diagram. ∎

Propositions 4.6.2 and 4.6.1 now give us a working theory of q-expansions along the 0-dimensional boundary stratum (4.6.2) determined by a top dimensional cone $\sigma \subset U_\Phi(\mathbb{R})(-1)$. Taking tensor powers in Proposition 4.6.1 determines an isomorphism

$$[\boldsymbol{\ell}^{\otimes k}, \cdot\,] : \boldsymbol{\omega}^{\otimes k} \cong \mathcal{O}_{\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)},$$

and hence any global section

$$\psi \in H^0\big(\mathrm{Sh}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}}, \boldsymbol{\omega}^{\otimes k}\big)$$

determines a formal function $[\boldsymbol{\ell}^{\otimes k}, \psi]$ on

$$\widehat{\mathrm{Sh}}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}} \cong \widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)_{/\mathbb{C}}.$$

Now pull this formal function back via the formally étale surjection

$$\bigsqcup_{a \in \mathbb{Q}^\times_{>0} \backslash \mathbb{A}_f^\times / K_0} \widehat{T}_\Phi(\sigma)_{/\mathbb{C}} \to \widehat{\mathrm{Sh}}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}}$$

of Proposition 4.6.2. By restricting the pullback to the copy of $\widehat{T}_\Phi(\sigma)_{/\mathbb{C}}$ indexed by a, we obtain a formal q-expansion (*a.k.a.* Fourier Jacobi expansion)

$$\mathrm{FJ}^{(a)}(\psi) = \sum_{\substack{\alpha \in \Gamma_\Phi^\vee(1) \\ \langle \alpha, \sigma \rangle \geqslant 0}} \mathrm{FJ}^{(a)}_\alpha(\psi) \cdot q_\alpha \in \mathbb{C}[[q_\alpha]]_{\substack{\alpha \in \Gamma_\Phi^\vee(1) \\ \langle \alpha, \sigma \rangle \geqslant 0}}. \tag{4.6.10}$$

We emphasize that (4.6.10) depends on the choice of toroidal stratum representative (Φ, σ), as well as on the choices of $\nu : \mathcal{D} \to \mathcal{H}_0$, I_*, and $\ell \in I$. These will always be clear from context.

For each $\tau \in \mathrm{Aut}(\mathbb{C})$, denote by $a_\tau \in \mathbb{Q}^\times_{>0} \backslash \mathbb{A}_f^\times$ the unique element with

$$\mathrm{rec}(a_\tau) = \tau|_{\mathbb{Q}^{ab}}.$$

The following is our q-expansion principle; see also [Hör14, Theorem 2.8.7].

Proposition 4.6.3 (Rational q-expansion principle)**.** *For any $a \in \mathbb{A}_f^\times$ and $\tau \in \mathrm{Aut}(\mathbb{C})$, the q-expansion coefficients of ψ and ψ^τ are related by*

$$\mathrm{FJ}^{(aa_\tau)}_\alpha(\psi^\tau) = \tau\big(\mathrm{FJ}^{(a)}_\alpha(\psi)\big).$$

Moreover, ψ is defined over a subfield $L \subset \mathbb{C}$ if and only if

$$\mathrm{FJ}^{(aa_\tau)}_\alpha(\psi) = \tau\big(\mathrm{FJ}^{(a)}_\alpha(\psi)\big)$$

for all $a \in \mathbb{A}_f^\times$, all $\tau \in \mathrm{Aut}(\mathbb{C}/L)$, and all $\alpha \in \Gamma_\Phi^\vee(1)$.

Proof. The formal scheme $\widehat{T}_\Phi(\sigma)$ of (4.6.7) has a distinguished $\mathbb{Q}$-valued point defined by $q_\alpha = 0$ (*i.e.* the unique point of the underlying reduced $\mathbb{Q}$-scheme), and so has a distinguished $\mathbb{C}$-valued point. Hence, using the morphism

$$\bigsqcup_{a \in \mathbb{Q}^\times_{>0} \backslash \mathbb{A}_f^\times / K_0} \widehat{T}_\Phi(\sigma)_{/\mathbb{C}} \to \widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)(\mathbb{C})$$

of Proposition 4.6.2, each $a \in \mathbb{Q}^\times_{>0}\backslash\mathbb{A}_f^\times$ determines a distinguished point

$$\mathrm{cusp}^{(a)} \in \widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)(\mathbb{C}).$$

By examining the proof of Proposition 4.6.2, the reciprocity law (3.5.2) implies that

$$\mathrm{cusp}^{(aa_\tau)} = \tau(\mathrm{cusp}^{(a)})$$

for any $\tau \in \mathrm{Aut}(\mathbb{C})$, and the q-expansion (4.6.10) is, tautologically, the image of the formal function $[\boldsymbol{\ell}^{\otimes k}, \psi]$ in the completed local ring at $\mathrm{cusp}^{(a)}$. The first claim is now a consequence of the equality

$$[\boldsymbol{\ell}^{\otimes k}, \psi]^\tau = [\boldsymbol{\ell}^{\otimes k}, \psi^\tau]$$

of formal functions on

$$\widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) \cong \widehat{\mathrm{Sh}}_K(G, \mathcal{D}, \Sigma).$$

The second claim follows from the first, and the observation that two rational sections ψ_1 and ψ_2 are equal if and only if $\mathrm{FJ}^{(a)}(\psi_1) = \mathrm{FJ}^{(a)}(\psi_2)$ for all a. Indeed, to check that $\psi_1 = \psi_2$, it suffices to check this in a formal neighborhood of one point on each connected component of $\mathrm{Sh}_K(G, \mathcal{D}, \Sigma)_{/\mathbb{C}}$. Using strong approximation for the simply connected group

$$\mathrm{Spin}(V) = \ker\big(\nu : G \to \mathbb{G}_m\big),$$

one can show that the fibers of

$$\mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C}) \to \mathrm{Sh}_{\nu(K)}(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C})$$

are connected. This implies that the images of the points $\mathrm{cusp}^{(a)}$ under

$$\widehat{\mathrm{Sh}}_{K_{\Phi 0}}(Q_\Phi, \mathcal{D}_\Phi, \sigma)(\mathbb{C}) \to \widehat{\mathrm{Sh}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)(\mathbb{C}) \cong \widehat{\mathrm{Sh}}_K(G, \mathcal{D}, \Sigma)(\mathbb{C})$$

hit every connected component of $\mathrm{Sh}_K(G, \mathcal{D}, \Sigma)(\mathbb{C})$. □

5. Borcherds products

Once again, we work with a fixed $\mathbb{Q}$-quadratic space (V, Q) of signature $(n, 2)$ with $n \geqslant 1$, and denote by $(G, \mathcal{D})$ the associated GSpin Shimura datum of §4.1. Fix a $\mathbb{Z}$-lattice $V_\mathbb{Z} \subset V$ on which the quadratic form is $\mathbb{Z}$-valued, and let K be as in (1.1.2). Recalling the notation of Remark 2.1.2, fix a choice of

$$2\pi i \in \mathcal{H}_0.$$

We recall the analytic theory of Borcherds products [Bor98, Bru02] on $\mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C})$ using the adelic formulation as in [Kud03]. Assuming that V contains an isotropic line, we express their product expansions in the algebraic language of §4.6.

5.1. **Weakly holomorphic forms.** Let $S(V_{\mathbb{A}_f})$ be the Schwartz space of locally constant $\mathbb{C}$-valued compactly supported functions on $V_{\mathbb{A}_f} = V \otimes \mathbb{A}_f$. For any $g \in G(\mathbb{A}_f)$ abbreviate

$$gV_{\mathbb{Z}} = gV_{\widehat{\mathbb{Z}}} \cap V.$$

Denote by $S_{V_{\mathbb{Z}}} \subset S(V_{\mathbb{A}_f})$ the finite dimensional subspace of functions invariant under $V_{\widehat{\mathbb{Z}}}$, and supported on its dual lattice; we often identify it with the space

$$S_{V_{\mathbb{Z}}} = \mathbb{C}[V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}]$$

of functions on $V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. The metaplectic double cover $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{Z})$ acts via the Weil representation

$$\rho_{V_{\mathbb{Z}}} : \widetilde{\mathrm{SL}}_2(\mathbb{Z}) \to \mathrm{Aut}_{\mathbb{C}}(S_{V_{\mathbb{Z}}})$$

as in [Bor98, Bru02, BF04]. Define the complex conjugate representation by

$$\overline{\rho}_{V_{\mathbb{Z}}}(\gamma) \cdot \varphi = \overline{(\rho_{V_{\mathbb{Z}}}(\gamma) \cdot \overline{\varphi})},$$

for $\gamma \in \widetilde{\mathrm{SL}}_2(\mathbb{Z})$ and $\varphi \in S_{V_{\mathbb{Z}}}$.

Remark 5.1.1. There is a canonical basis

$$\{\phi_\mu : \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}\} \subset S_{V_{\mathbb{Z}}},$$

in which ϕ_μ is the characteristic function of $\mu + V_{\mathbb{Z}}$. This allows us to identify $S_{V_{\mathbb{Z}}}$ with its own $\mathbb{C}$-linear dual. Under this identification, the complex conjugate representation $\overline{\rho}_{V_{\mathbb{Z}}}$ agrees with with contragredient representation $\rho^{\vee}_{V_{\mathbb{Z}}}$. It also agrees with the representation denoted $\omega_{V_{\mathbb{Z}}}$ in [AGHMP17, AGHMP18].

Denote by $M^!_{1-n/2}(\overline{\rho}_{V_{\mathbb{Z}}})$ the space of weakly holomorphic forms for $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ of weight $1 - n/2$ and representation $\overline{\rho}_{V_{\mathbb{Z}}}$, as in [Bor98, Bru02, BF04]. In particular, any

$$f(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} c(m) \cdot q^m \in M^!_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}}) \tag{5.1.1}$$

is an $S_{V_{\mathbb{Z}}}$-valued holomorphic function on the complex upper half-plane $\mathcal{H}$. Each Fourier coefficient $c(m) \in S_{V_{\mathbb{Z}}}$ is determined by its values $c(m, \mu)$ at the various cosets $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. Moreover, $c(m, \mu) \neq 0$ implies $m \equiv Q(\mu)$ modulo $\mathbb{Z}$.

Definition 5.1.2. The weakly holomorphic form (5.1.1) is *integral* if

$$c(m, \mu) \in \mathbb{Z}$$

for all $m \in \mathbb{Q}$ and all $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$.

It is a theorem of McGraw [McG03] that the space of all forms (5.1.1) has a $\mathbb{C}$-basis of integral forms.

5.2. **Borcherds products and regularized theta lifts.** We now recall the meromorphic Borcherds products of [Bor98, Bru02, Kud03].

Write $\tau = u + iv \in \mathcal{H}$ for the variable on the complex upper half-plane. For each $\varphi \in S(V_{\mathbb{A}_f})$ there is a Siegel theta function

$$\vartheta(\tau, z, g; \varphi) : \mathcal{H} \times \mathcal{D} \times G(\mathbb{A}_f) \to \mathbb{C},$$

as in [Kud03, (1.37)], satisfying the transformation law

$$\vartheta(\tau, \gamma z, \gamma g h; \varphi) = \vartheta(\tau, z, g; \varphi \circ h^{-1})$$

for any $\gamma \in G(\mathbb{Q})$ and any $h \in G(\mathbb{A}_f)$. If we use the basis of Remark 5.1.1 to define

$$\vartheta(\tau, z, g) = \sum_{\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}} \vartheta(\tau, z, g, \phi_\mu) \cdot \phi_\mu,$$

we obtain a function

$$\vartheta(\tau, z, g) : \mathcal{H} \times \mathcal{D} \times G(\mathbb{A}_f) \to S_{V_{\mathbb{Z}}},$$

which transforms in the variable τ like a modular form of weight $\frac{n}{2} - 1$ and representation $\rho_{V_{\mathbb{Z}}}$.

Given a weakly holomorphic form (5.1.1) one can regularize the divergent integral

$$\Theta^{\mathrm{reg}}(f)(z, g) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} f(\tau)\vartheta(\tau, z, g)\, \frac{du\, dv}{v^2} \tag{5.2.1}$$

as in [Bor98, Bru02, Kud03]. Here we are using the map $S_{V_{\mathbb{Z}}} \otimes S_{V_{\mathbb{Z}}} \to \mathbb{C}$ defined by

$$\phi_\mu \otimes \phi_\nu \mapsto \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{otherwise} \end{cases}$$

to obtain an $\mathrm{SL}_2(\mathbb{Z})$-invariant scalar-valued integrand $f(\tau)\vartheta(\tau, z, g)$.

As the subgroup K acts trivially on the quotient (4.1.5), the subspace $S_{V_{\mathbb{Z}}} \subset S(V_{\mathbb{A}_f})$ is K-invariant. It follows that (5.2.1) satisfies

$$\Theta^{\mathrm{reg}}(f)(\gamma z, \gamma g k) = \Theta^{\mathrm{reg}}(f)(z, g)$$

for any $\gamma \in G(\mathbb{Q})$ and any $k \in K$. This allows us to view $\Theta^{\mathrm{reg}}(f)$ as a function on $\mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C})$, which we call the *regularized theta lift*. Our $\Theta^{\mathrm{reg}}(f)$ is usually denoted $\Phi(f)$ in the literature.

Remark 5.2.1. The following fundamental theorem of Borcherds implies that the regularized theta lift is real analytic away from a prescribed divisor, with logarithmic singularities along that divisor. Remarkably, the regularization process gives $\Theta^{\mathrm{reg}}(f)$ a meaningful value at *every* point of $\mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C})$, including along the singular divisor. In the context of unitary Shimura varieties, this is [BHY15, Theorem 4.1] and [BHY15, Corollary 4.2], and the proof for orthogonal Shimura varieties is identical. In other words, $\Theta^{\mathrm{reg}}(f)$ is a well-defined (but discontinuous) function on all of $\mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C})$. Its values along the singular divisor will be made more explicit in §9.2 when we use the embedding trick to complete the proof of Theorem 9.1.1.

Theorem 5.2.2 (Borcherds)**.** *Assume that f is integral. After multiplying f by any sufficiently divisible positive integer*[4]*, there is a meromorphic section $\Psi(f)$ of the analytic line bundle $(\boldsymbol{\omega}^{an})^{\otimes c(0,0)/2}$ on $\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})$ such that, away from the support of $\mathrm{div}(\Psi(f))$, we have*

$$-4\log\|\Psi(f)\|_{\mathrm{naive}} = \Theta^{\mathrm{reg}}(f) + c(0,0)\log(\pi) + c(0,0)\Gamma'(1). \tag{5.2.2}$$

Here $\Gamma'(s)$ is the derivative of the usual Gamma function, and $\|-\|_{\mathrm{naive}}$ is the metric of (4.2.2).

Proof. Choose a connected component $\mathcal{D}^\circ \subset \mathcal{D}$, let $G(\mathbb{R})^\circ \subset G(\mathbb{R})$ be its stabilzer (this is just the subgroup of elements on which the spinor similitude $\nu : G \to \mathbb{G}_m$ is positive) and define $G(\mathbb{Q})^\circ$ similarly. Denote by $\boldsymbol{\omega}_{\mathcal{D}^\circ}$ the restriction to $\mathcal{D}^\circ$ of the tautological line bundle on (4.1.1). It carries an action of $G(\mathbb{R})^\circ$ covering the action on the base, and a $G(\mathbb{R})^\circ$ invariant metric (4.2.2).

For any $g \in G(\mathbb{A}_f)$, denote by

$$\Theta^{\mathrm{reg}}_g(f)(z) \stackrel{\mathrm{def}}{=} \Theta^{\mathrm{reg}}(f)(z,g)$$

the restriction of the regularized theta lift to the connected component

$$(G(\mathbb{Q})^\circ \cap gKg^{-1})\backslash\mathcal{D}^\circ \xrightarrow{z\mapsto(z,g)} \mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C}). \tag{5.2.3}$$

Borcherds [Bor98] proves the existence of a meromorphic section $\Psi_g(f)$ of $\boldsymbol{\omega}_{\mathcal{D}^\circ}^{\otimes c(0,0)/2}$ satisfying

$$-4\log\|\Psi_g(f)\|_{\mathrm{naive}} = \Theta^{\mathrm{reg}}_g(f) + c(0,0)\log(\pi) + c(0,0)\Gamma'(1). \tag{5.2.4}$$

Note that Borcherds does not work adelically. Instead, for every input form (5.1.1) he constructs a single meromorphic section $\Psi_{\mathrm{classical}}(f)$ over $\mathcal{D}^\circ$. However, $g \in G(\mathbb{A}_f)$ determines an isomorphism $V_{\mathbb{Z}}^\vee/V_{\mathbb{Z}} \to gV_{\mathbb{Z}}^\vee/gV_{\mathbb{Z}}$, which induces an isomorphism

$$M^!_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}}) \xrightarrow{f\mapsto g\cdot f} M^!_{1-\frac{n}{2}}(\overline{\rho}_{gV_{\mathbb{Z}}}).$$

Replacing the pair $(V_{\mathbb{Z}}, f)$ by $(gV_{\mathbb{Z}}, gf)$ yields another meromorphic section $\Psi_{\mathrm{classical}}(gf)$ over $\mathcal{D}^\circ$, and

$$\Psi_g(f) = \Psi_{\mathrm{classical}}(gf).$$

The relation (5.2.4) determines $\Psi_g(f)$ up to scaling by a complex number of absolute value 1, and the linearity of $f \mapsto \Theta^{\mathrm{reg}}_g(f)$ implies the multiplicativity

$$\Psi_g(f_1+f_2) = \Psi_g(f_1)\otimes\Psi_g(f_2)$$

relation, up to the ambiguity just noted. As $\Theta^{\mathrm{reg}}_g(f)$ is invariant under translation by every $\gamma \in G(\mathbb{Q})^\circ \cap gKg^{-1}$, we must have

$$\gamma\cdot\Psi_g(f)(z) = \xi_g(\gamma)\cdot\Psi_g(f)(\gamma z)$$

[4]In particular, we may assume $c(0,0) \in 2\mathbb{Z}$.

for some unitary character

$$\xi_g : G(\mathbb{Q})^\circ \cap gKg^{-1} \to \mathbb{C}^\times. \tag{5.2.5}$$

The main result of [Bor00] asserts that the character ξ_g is of finite order, and so we may replace f by a positive integer multiple in order to make it trivial. The section $\Psi_g(f)$ now descends to the quotient (5.2.3).

Repeating this procedure on every connected component of $\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})$ yields a section $\Psi(f)$ satisfying (5.2.2). □

The meromorphic section $\Psi(f)$ of the theorem is what is usually called the *Borcherds product* (or Borcherds lift) of f. We will use the same terminology to refer to the meromorphic section

$$\psi(f) = (2\pi i)^{c(0,0)}\Psi(2f)$$

of $(\boldsymbol{\omega}^{an})^{\otimes c(0,0)}$, which has better arithmetic properties. We will see in §9 that, after rescaling by a constant of absolute value 1 on every connected component of $\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})$, the section $\psi(f)$ is algebraic and defined over the reflex field $\mathbb{Q}$.

Proposition 5.2.3. *Assume that either $n \geqslant 3$, or that $n = 2$ and V has Witt index 1. The Borcherds product $\Psi(f)$ of Theorem 5.2.2, a priori a meromorphic section on $\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})$, is the analytification of a rational section on $\mathrm{Sh}_K(G,\mathcal{D})_{/\mathbb{C}}$.*

Proof. It suffices to prove this after shrinking K, so we may assume that K is neat and $\mathrm{Sh}_K(G,\mathcal{D})$ is a quasi-projective variety. The hypotheses on n imply that the boundary of the (normal and projective) Baily-Borel compacification

$$\mathrm{Sh}_K(G,\mathcal{D}) \hookrightarrow \mathrm{Sh}_K(G,\mathcal{D})^{\mathrm{BB}}$$

lies in codimension $\geqslant 2$.

Let D be the polar part of the divisor of $\Psi(f)$, so that D is an effective analytic divisor on $\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})$ with $\mathrm{div}(\Psi(f)) + D$ effective. The proof of Levi's generalization of Hartogs' theorem [GR84, §9.5] shows that the topological closure of D in $\mathrm{Sh}_K(G,\mathcal{D})^{\mathrm{BB}}(\mathbb{C})$ is again an analytic divisor. By Chow's theorem on the algebraicity of analytic divisors on projective varieties, this closure is algebraic, and so D itself was algebraic.

Now view $\Psi(f)$ as a holomorphic section of the analytification of the line bundle $\boldsymbol{\omega}^{\otimes c(0,0)/2} \otimes \mathcal{O}(D)$ on $\mathrm{Sh}_K(G,\mathcal{D})_{/\mathbb{C}}$. By Hartshorne's extension of GAGA [Har70, Theorem VI.2.1] this section is algebraic, as desired. □

5.3. **The product expansion I.** As in the proof of Theorem 5.2.2, fix a connected component $\mathcal{D}^\circ \subset \mathcal{D}$ and an $h \in G(\mathbb{A}_f)$, and denote by $\Psi_h(f)$ the restriction of the Borcherds product to the connected component

$$(G(\mathbb{Q})^\circ \cap hKh^{-1})\backslash\mathcal{D}^\circ \xrightarrow{z\mapsto(z,h)} \mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C}).$$

In this subsection we recall the product expansion for $\Psi_h(f)$ due to Borcherds. Let $\boldsymbol{\omega}_{\mathcal{D}^\circ}$ be the restriction to $\mathcal{D}^\circ$ of the tautological bundle on (4.1.1).

Assume throughout §5.3 that there exists an isotropic line $I \subset V$. Choose a second isotropic line $I_* \subset V$ with $[I, I_*] \neq 0$, but do this in a particular way: first choose a $\mathbb{Z}$-module generator

$$\ell \in I \cap hV_{\mathbb{Z}},$$

and then choose a $k \in hV_{\mathbb{Z}}^{\vee}$ such that $[\ell, k] = 1$. Now take I_* be the span of the isotropic vector

$$\ell_* = k - Q(k)\ell. \tag{5.3.1}$$

Obviously $[\ell, \ell_*] = 1$, but we need not have $\ell_* \in hV_{\mathbb{Z}}^{\vee}$.

Abbreviate $V_0 = I^{\perp}/I$. This is a $\mathbb{Q}$-vector space endowed with a quadratic form of signature $(n-1, 1)$, and a $\mathbb{Z}$-lattice

$$V_{0\mathbb{Z}} = (I^{\perp} \cap hV_{\mathbb{Z}})/(I \cap hV_{\mathbb{Z}}) \subset V_0. \tag{5.3.2}$$

Denote by

$$\mathrm{LightCone}(V_{0\mathbb{R}}) = \{w \in V_{0\mathbb{R}} : Q(w) < 0\}$$

the light cone in $V_{0\mathbb{R}}$. It is a disjoint union of two open convex cones. Every $v \in I_{\mathbb{C}}^{\perp}$ determines an isotropic vector

$$\ell_* + v - [\ell_*, v]\ell - Q(v)\ell \in V_{\mathbb{C}},$$

depending only on the image $v \in V_{0\mathbb{C}}$. The resulting injection $V_{0\mathbb{C}} \to \mathbb{P}^1(V_{\mathbb{C}})$ restricts to an isomorphism

$$V_{0\mathbb{R}} + i \cdot \mathrm{LightCone}(V_{0\mathbb{R}}) \cong \mathcal{D},$$

and we let

$$\mathrm{LightCone}^{\circ}(V_{0\mathbb{R}}) \subset \mathrm{LightCone}(V_{0\mathbb{R}}) \tag{5.3.3}$$

be the connected component with

$$V_{0\mathbb{R}} + i \cdot \mathrm{LightCone}^{\circ}(V_{0\mathbb{R}}) \cong \mathcal{D}^{\circ}.$$

There is an action $\rho_{V_{0\mathbb{Z}}}$ of $\widetilde{\mathrm{SL}}_2(\mathbb{Z})$ on the finite dimensional $\mathbb{C}$-vector space $S_{V_{0\mathbb{Z}}}$, exactly as in §5.1, and a weakly holomorphic modular form

$$f_0(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} \sum_{\lambda \in V_{0\mathbb{Z}}^{\vee}/V_{0\mathbb{Z}}} c_0(m, \lambda) \cdot q^m \in M^!_{1-\frac{n}{2}}(\overline{\rho}_{V_{0\mathbb{Z}}})$$

whose coefficients are defined by

$$c_0(m, \lambda) = \sum_{\substack{\mu \in hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}} \\ \mu \sim \lambda}} c(m, h^{-1}\mu).$$

Here we understand $h^{-1}\mu$ to mean the image of μ under the isomorphism $hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}} \to V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$ defined by multiplication by h^{-1}. The notation $\mu \sim \lambda$ requires explanation: denoting by

$$p : (I^{\perp} \cap hV_{\mathbb{Z}}^{\vee})/(I^{\perp} \cap hV_{\mathbb{Z}}) \to V_{0\mathbb{Z}}^{\vee}/V_{0\mathbb{Z}}$$

the natural map, $\mu \sim \lambda$ means that there is a

$$\tilde{\mu} \in I^{\perp} \cap (\mu + hV_{\mathbb{Z}}) \tag{5.3.4}$$

such that $p(\tilde{\mu}) = \lambda$.

Every vector $x \in V_0$ of positive length determines a hyperplane $x^\perp \subset V_{0\mathbb{R}}$. For each $m \in \mathbb{Q}_{>0}$ and $\lambda \in V_{0\mathbb{Z}}^\vee / V_{0\mathbb{Z}}$ define a formal sum of hyperplanes

$$H(m,\lambda) = \sum_{\substack{x \in \lambda + V_{0\mathbb{Z}} \\ Q(x)=m}} x^\perp,$$

in $V_{0\mathbb{R}}$, and set

$$H(f_0) = \sum_{\substack{m \in \mathbb{Q}_{>0} \\ \lambda \in V_{0\mathbb{Z}}^\vee / V_{0\mathbb{Z}}}} c_0(-m,\lambda) \cdot H(m,\lambda).$$

Definition 5.3.1. A *Weyl chamber* for f_0 is a connected component

$$\mathscr{W} \subset \mathrm{LightCone}^\circ(V_{0\mathbb{R}}) \smallsetminus \mathrm{Support}(H(f_0)). \tag{5.3.5}$$

Let N be the positive integer determined by $N\mathbb{Z} = [hV_\mathbb{Z}, I \cap hV_\mathbb{Z}]$, and note that $\ell/N \in hV_\mathbb{Z}^\vee$. Set

$$A = \prod_{\substack{x \in \mathbb{Z}/N\mathbb{Z} \\ x \neq 0}} \left(1 - e^{2\pi i x/N}\right)^{c(0, xh^{-1}\ell/N)}. \tag{5.3.6}$$

Tautologically, every fiber of $\boldsymbol{\omega}_{\mathcal{D}^\circ}$ is a line in $V_\mathbb{C}$, and each such fiber pairs nontrivially with the isotropic line $I_\mathbb{C}$. Using the nondegenerate pairing

$$[\,\cdot\,,\,\cdot\,] : I_\mathbb{C} \otimes \boldsymbol{\omega}_{\mathcal{D}^\circ} \to \mathcal{O}_{\mathcal{D}^\circ},$$

the Borcherds product $\Psi_h(f)$ and the isotropic vector $\ell \in I$ determine a meromorphic function $[\ell^{\otimes c(0,0)/2}, \Psi_h(f)]$ on $\mathcal{D}^\circ$. It is this function that Borcherds expresses as an infinite product.

Theorem 5.3.2 (Borcherds [Bor98, Bru02]). *For each Weyl chamber $\mathscr{W}$ there is a vector $\varrho \in V_0$ with the following property: For all*

$$v \in V_{0\mathbb{R}} + i \cdot \mathscr{W} \subset V_{0\mathbb{C}}$$

with $|Q(\mathrm{Im}(v))| \gg 0$, the value of $[\ell^{\otimes c(0,0)/2}, \Psi_h(f)]$ at the isotropic line

$$\ell_* + v - [\ell_*, v]\ell - Q(v)\ell \in \mathcal{D}^\circ$$

is given by the (convergent) infinite product

$$\kappa A \cdot e^{2\pi i [\varrho, v]} \prod_{\substack{\lambda \in V_{0\mathbb{Z}}^\vee \\ [\lambda, \mathscr{W}] > 0}} \prod_{\substack{\mu \in hV_\mathbb{Z}^\vee / hV_\mathbb{Z} \\ \mu \sim \lambda}} \left(1 - \zeta_\mu \cdot e^{2\pi i [\lambda, v]}\right)^{c(-Q(\lambda), h^{-1}\mu)}$$

for some $\kappa \in \mathbb{C}$ of absolute value 1. Here, recalling the vector $k \in hV_\mathbb{Z}^\vee$ appearing in (5.3.1), we have set

$$\zeta_\mu = e^{2\pi i [\mu, k]}.$$

Remark 5.3.3. The vector $\varrho \in V_0$ of the theorem is the *Weyl vector*. It is completely determined by the weakly holomorphic form f_0 and the choice of Weyl chamber $\mathscr{W}$.

5.4. **The product expansion II.** We now connect the product expansion of Theorem 5.3.2 with the algebraic theory of q-expansions from §4.6. Throughout §5.4 we assume that K is neat.

The theory of §4.6 applies to sections of the algebraic line bundle $\boldsymbol{\omega}_{/\mathbb{C}}$ on $\mathrm{Sh}_K(G,\mathcal{D})_{/\mathbb{C}}$ and at the moment we only know the algebraicity of Borcherds products in special cases (Proposition 5.2.3). Throughout §5.4, we simply assume that our given Borcherds product $\Psi(f)$ is algebraic.

Begin by choosing a cusp label representative

$$\Phi = (P, \mathcal{D}^\circ, h)$$

for which P is the stabilizer of an isotropic line I. Let $\ell \in I \cap hV_\mathbb{Z}$ be a generator, let ℓ_* be as in (5.3.1), and let $I_* = \mathbb{Q}\ell_*$.

Recall from the discussion surrounding (4.6.6) that the choice of I_* determines morphisms of mixed Shimura data

$$\begin{tikzcd} (Q_\Phi, \mathcal{D}_\Phi) \arrow[r, "\nu_\Phi"'] & (\mathbb{G}_m, \mathcal{H}_0) \arrow[l, bend right, "s"'] \end{tikzcd},$$

where we specify that $\nu_\Phi : \mathcal{D}_\Phi \to \mathcal{H}_0$ sends the connected component $\mathcal{D}_\Phi^\circ \subset \mathcal{D}_\Phi$ containing $\mathcal{D}^\circ$ to the $2\pi i \in \mathcal{H}_0$ fixed at the beginning of §5.

Set $V_0 = I^\perp/I$ as before. The connected component (5.3.3) was chosen in such a way that the isomorphism

$$V_{0\mathbb{C}} \xrightarrow{\otimes\ell} V_{0\mathbb{C}} \otimes I \xrightarrow{(4.4.2)} U_\Phi(\mathbb{C})$$

identifies

$$V_{0\mathbb{R}} + i \cdot \mathrm{LightCone}^\circ(V_{0\mathbb{R}}) \cong U_\Phi(\mathbb{R}) + C_\Phi,$$

where $C_\Phi \subset U_\Phi(\mathbb{R})(-1)$ is the open convex cone (2.4.1). Equivalently, the isomorphism

$$V_{0\mathbb{R}} \xrightarrow{\otimes(-2\pi i)^{-1}\ell} V_{0\mathbb{R}} \otimes I(-1) \cong U_\Phi(\mathbb{R})(-1) \tag{5.4.1}$$

(note the minus sign!) identifies $\mathrm{LightCone}^\circ(V_{0\mathbb{R}}) \cong C_\Phi$.

Lemma 5.4.1. *Fix a Weyl chamber $\mathscr{W}$ as in (5.3.5). After possibly shrinking K, there exists a K-admissible, complete cone decomposition Σ of $(G,\mathcal{D})$ having the no self-intersection property, and such that the following holds: there is some top-dimensional rational polyhedral cone $\sigma \in \Sigma_\Phi$ whose interior is identified with an open subset of $\mathscr{W}$ under the above isomorphism*

$$C_\Phi \cong \mathrm{LightCone}^\circ(V_{0\mathbb{R}}).$$

Proof. This is an elementary exercise. Using Remark 2.6.1, we first shrink K in order to find some K-admissible, complete cone decomposition Σ of $(G,\mathcal{D})$ having the no self-intersection property. We may furthermore choose Σ to be smooth, and applying barycentric subdivision [Pin89, §5.24] finitely many times yields a refinement of Σ with the desired properties. $\square$

For the remainder of §5.4 we assume that K, Σ, $\mathscr{W}$, and $\sigma \subset U_\Phi(\mathbb{R})(-1)$ are as in Lemma 5.4.1. As in §4.6, the line bundle $\boldsymbol{\omega}$ on $\mathrm{Sh}_K(G,\mathcal{D})$ has a canonical extension to $\mathrm{Sh}_K(G,\mathcal{D},\Sigma)$, and we view $\Psi(f)$ as a rational section over $\mathrm{Sh}_K(G,\mathcal{D},\Sigma)_{/\mathbb{C}}$.

The top-dimensional cone σ singles out a 0-dimensional stratum

$$Z_K^{(\Phi,\sigma)}(G,\mathcal{D},\Sigma) \subset \mathrm{Sh}_K(G,\mathcal{D},\Sigma)$$

as in §2.6. Completing along this stratum, Proposition 4.6.2 provides us with a formally étale surjection

$$\bigsqcup_{a\in\mathbb{Q}_{>0}^\times\backslash\mathbb{A}_f^\times/K_0} \mathrm{Spf}\Big(\mathbb{C}[[q_\alpha]]_{\substack{\alpha\in\Gamma_\Phi^\vee(1)\\ \langle\alpha,\sigma\rangle\geqslant 0}}\Big) \to \widehat{\mathrm{Sh}}_K(G,\mathcal{D},\Sigma)_{/\mathbb{C}},$$

where $K_0 \subset \mathbb{A}_f^\times$ is chosen small enough that the section (4.6.6) satisfies $s(K_0) \subset K_\Phi$. As in (4.6.10), the Borcherds product $\Psi(f)$ and the isotropic vector ℓ determine a rational formal function $[\boldsymbol{\ell}^{\otimes c(0,0)/2}, \Psi(f)]$ on the target, which pulls back to a rational formal function

$$\mathrm{FJ}^{(a)}(\Psi(f)) \in \mathrm{Frac}\Big(\mathbb{C}[[q_\alpha]]_{\substack{\alpha\in\Gamma_\Phi^\vee(1)\\ \langle\alpha,\sigma\rangle\geqslant 0}}\Big) \tag{5.4.2}$$

for every index a. The following proposition explains how this formal q-expansion varies with a.

Proposition 5.4.2. *Let $F \subset \mathbb{C}$ be the abelian extension of $\mathbb{Q}$ determined by*

$$\mathrm{rec} : \mathbb{Q}_{>0}^\times\backslash\mathbb{A}_f^\times/K_0 \cong \mathrm{Gal}(F/\mathbb{Q}).$$

The rational formal function (5.4.2) has the form

$$(2\pi i)^{c(0,0)/2} \cdot \mathrm{FJ}^{(a)}(\Psi(f)) = \kappa^{(a)} A^{\mathrm{rec}(a)} q_{\alpha(\varrho)} \cdot \mathrm{BP}(f)^{\mathrm{rec}(a)}. \tag{5.4.3}$$

Here $\kappa^{(a)} \in \mathbb{C}$ is some constant of absolute value 1, and the power series

$$\mathrm{BP}(f) \in \mathcal{O}_F[[q_\alpha]]_{\substack{\alpha\in\Gamma_\Phi^\vee(1)\\ \langle\alpha,\sigma\rangle\geqslant 0}}$$

(Borcherds Product) is the infinite product

$$\mathrm{BP}(f) = \prod_{\substack{\lambda\in V_{0\mathbb{Z}}^\vee\\ [\lambda,\mathscr{W}]>0}} \prod_{\substack{\mu\in hV_{\mathbb{Z}}^\vee/hV_{\mathbb{Z}}\\ \mu\sim\lambda}} \Big(1-\zeta_\mu\cdot q_{\alpha(\lambda)}\Big)^{c(-Q(\lambda),h^{-1}\mu)}.$$

The constant A and the roots of unity ζ_μ have the same meaning as in Theorem 5.3.2, and these constants lie in $\mathcal{O}_F$. The meaning of $q_{\alpha(\lambda)}$ is as follows: dualizing the isomorphism (5.4.1) yields an isomorphism

$$V_{0\mathbb{R}} \xrightarrow{\lambda\mapsto\alpha(\lambda)} U_\Phi(\mathbb{R})^\vee(1), \tag{5.4.4}$$

and the image of each $\lambda \in V_{0\mathbb{R}}$ appearing in the product satisfies

$$\alpha(\lambda) \in \Gamma_\Phi^\vee(1).$$

The condition $[\lambda, \mathscr{W}] > 0$ *implies* $\langle \alpha(\lambda), \sigma\rangle > 0$. *Of course* $q_{\alpha(\varrho)}$ *has the same meaning, with* $\varrho \in V_0$ *the Weyl vector of Theorem 5.3.2. Again* $\alpha(\varrho) \in \Gamma^\vee_\Phi(1)$, *but need not satisfy the positivity condition with respect to* σ.

Proof. First we address the field of definition of the constants A and ζ_μ.

Lemma 5.4.3. *The constant* A *of (5.3.6) lies in* $\mathcal{O}_F$, *and* $\zeta_\mu \in \mathcal{O}_F$ *for every* μ *appearing in the above product.*

Proof. Suppose $a \in K_0$. It follows from the discussion preceeding (4.1.5) that $s(a) \in hKh^{-1}$ stabilizes the lattice $hV_{\mathbb{Z}}$, and acts trivially on the quotient $hV_{\mathbb{Z}}^\vee/hV_{\mathbb{Z}}$. In particular, $s(a)$ acts trivially on the vector $\ell/N \in hV_{\mathbb{Z}}^\vee/hV_{\mathbb{Z}}$. On the other hand, by its very definition (4.6.6) we know that $s(a)$ acts by a on this vector. It follows that $(a-1)\ell/N \in hV_{\mathbb{Z}}$, from which we deduce first $a - 1 \in N\widehat{\mathbb{Z}}$, and then $A^{\mathrm{rec}(a)} = A$.

Suppose $\mu \in hV_{\mathbb{Z}}^\vee/hV_{\mathbb{Z}}$ satisfies $\mu \sim \lambda$ for some $\lambda \in V_{0\mathbb{Z}}^\vee$. By (5.3.4) we may fix some $\tilde{\mu} \in I^\perp \cap (\mu + hV_{\mathbb{Z}})$. This allows us to compute, using (5.3.1),

$$\begin{aligned}\zeta_\mu^{\mathrm{rec}(a)} = e^{2\pi i[\tilde{\mu}, ak]} &= e^{2\pi i[\tilde{\mu}, a\ell_*]} e^{2\pi i Q(k)\cdot[\tilde{\mu}, a\ell]} \\ &= e^{2\pi i[\tilde{\mu}, s(a)^{-1}\ell_*]} e^{2\pi i Q(k)\cdot[\tilde{\mu}, s(a)\ell]}.\end{aligned}$$

As $[\tilde{\mu}, \ell] = 0$, we have $[\tilde{\mu}, s(a)\ell] = 0 = [\tilde{\mu}, s(a)^{-1}\ell]$. Thus

$$\zeta_\mu^{\mathrm{rec}(a)} = e^{2\pi i[\tilde{\mu}, s(a)^{-1}\ell_*]} e^{2\pi i Q(k)\cdot[\tilde{\mu}, s(a)^{-1}\ell]} = e^{2\pi i[\tilde{\mu}, s(a)^{-1}k]} = e^{2\pi i[s(a)\tilde{\mu}, k]}.$$

As above, $s(a)$ acts trivially on $hV_{\mathbb{Z}}^\vee/hV_{\mathbb{Z}}$, and we conclude that

$$\zeta_\mu^{\mathrm{rec}(a)} = e^{2\pi i[\tilde{\mu}, k]} = \zeta_\mu.$$

□

Suppose $a \in \mathbb{A}_f^\times$. The image of the discrete group

$$\Gamma_\Phi^{(a)} = s(a)K_\Phi s(a)^{-1} \cap Q_\Phi(\mathbb{Q})^\circ$$

under $\nu_\Phi : Q_\Phi \to \mathbb{G}_m$ is contained in $\widehat{\mathbb{Z}}^\times \cap \mathbb{Q}_{>0}^\times = \{1\}$, and hence $\Gamma_\Phi^{(a)}$ is contained in $\ker(\nu_\Phi) = U_\Phi$. Recalling that the conjugation action of Q_Φ on U_Φ is by ν_Φ, we find that

$$\Gamma_\Phi^{(a)} = \mathrm{rat}(\nu_\Phi(s(a))) \cdot \big(K_\Phi \cap U_\Phi(\mathbb{Q})\big) = \mathrm{rat}(a) \cdot \Gamma_\Phi$$

as lattices in $U_\Phi(\mathbb{Q})$.

Recalling (4.6.9) and (2.6.4), consider the following commutative diagram of complex analytic spaces

$$\begin{tikzcd}
\bigsqcup_a \Gamma_\Phi^{(a)}\backslash \mathcal{D}^\circ \arrow[r, dashed] \arrow[d, "\cong"', "z\mapsto(z,s(a))"] & \bigsqcup_a T_\Phi(\mathbb{C}) \arrow[r, equal] \arrow[d, "\cong"] & \bigsqcup_a \Gamma_\Phi(-1)\otimes\mathbb{C}^\times \\
\mathscr{U}_{K_{\Phi 0}}(Q_\Phi,\mathcal{D}_\Phi) \arrow[r] \arrow[d] & \mathrm{Sh}_{K_{\Phi 0}}(Q_\Phi,\mathcal{D}_\Phi)(\mathbb{C}) \arrow[d] & \\
\mathscr{U}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi) \arrow[r] \arrow[d, "(z,g)\mapsto(z,gh)"] & \mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi)(\mathbb{C}) & \\
\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C}), & &
\end{tikzcd} \tag{5.4.5}$$

in which all horizontal arrows are open immersions, all vertical arrows are local isomorphisms on the source, and the disjoint unions are over a set of coset representatives $a\in\mathbb{Q}_{>0}^\times\backslash\mathbb{A}_f^\times/K_0$. The dotted arrow is, by definition, the unique open immersion making the upper left square commute.

Lemma 5.4.4. *Fix a $\lambda\in V_{0\mathbb{R}}$ whose image under (5.4.4) satisfies*

$$\alpha(\lambda)\in\Gamma_\Phi^\vee(1),$$

and suppose

$$v\in V_{0\mathbb{R}}+i\cdot\mathrm{LightCone}^\circ(V_{0\mathbb{R}}).$$

If we restrict the character

$$q_{\alpha(\lambda)}:T_\Phi(\mathbb{C})\to\mathbb{C}^\times$$

to a function $\Gamma_\Phi^{(a)}\backslash\mathcal{D}^\circ\to\mathbb{C}^\times$ via the open immersion in the top row of (5.4.5), its value at the isotropic vector

$$\ell_*+v-[\ell_*,v]\ell-Q(v)\ell\in\mathcal{D}^\circ$$

is $e^{2\pi i[\lambda,v]/\mathrm{rat}(a)}$.

Proof. The proof is a (rather tedious) exercise in tracing through the definitions. The dotted arrow in the diagram above is induced by the open immersion $\mathcal{D}^\circ\subset U_\Phi(\mathbb{C})\mathcal{D}^\circ=\mathcal{D}_\Phi^\circ$ and the isomorphisms

$$\bigsqcup_a \Gamma_\Phi^{(a)}\backslash\mathcal{D}_\Phi^\circ\cong\mathrm{Sh}_{K_{\Phi 0}}(Q_\Phi,\mathcal{D}_\Phi)(\mathbb{C})\cong\bigsqcup_a T_\Phi(\mathbb{C}). \tag{5.4.6}$$

The second isomorphism is the trivialization of the $T_\Phi(\mathbb{C})$-torsor

$$\mathrm{Sh}_{K_{\Phi 0}}(Q_\Phi,\mathcal{D}_\Phi)(\mathbb{C})\to\mathrm{Sh}_{K_0}(\mathbb{G}_m,\mathcal{H}_0)(\mathbb{C}) \tag{5.4.7}$$

induced by the section $s:(\mathbb{G}_m,\mathcal{H}_0)\to(Q_\Phi,\mathcal{D}_\Phi)$, as in the proof of Proposition 4.6.2.

Tracing through the proof of Proposition 2.3.1, this isomorphism is obtained by combining the isomorphism

$$U_\Phi(\mathbb{C})/\Gamma_\Phi \cong \Gamma_\Phi(-1) \otimes \mathbb{C}/\mathbb{Z}(1) \xrightarrow{\mathrm{id}\otimes\exp} \Gamma_\Phi(-1) \otimes \mathbb{C}^\times = T_\Phi(\mathbb{C}) \tag{5.4.8}$$

with the isomorphism

$$U_\Phi(\mathbb{C})/\Gamma_\Phi \xrightarrow{-\mathrm{rat}(a)} U_\Phi(\mathbb{C})/\Gamma_\Phi^{(a)} \cong \Gamma_\Phi^{(a)}\backslash \mathcal{D}_\Phi^\circ \tag{5.4.9}$$

obtained by trivializing $\Gamma_\Phi^{(a)}\backslash \mathcal{D}_\Phi^\circ$ as a $U_\Phi(\mathbb{C})/\Gamma_\Phi^{(a)}$-torsor using the point $\ell_* \in \mathcal{D}_\Phi^\circ$. Note the minus sign in (5.4.9), which arises from the minus sign in the isomorphism (2.3.5) used to define the torsor structure on (5.4.7).

Denote by β the composition

$$V_{0\mathbb{R}} \xrightarrow{\otimes\ell} V_{0\mathbb{R}} \otimes I \xrightarrow{(4.4.2)} U_\Phi(\mathbb{R}).$$

It is related to $\alpha(\lambda) \in U_\Phi(\mathbb{R})^\vee(1)$ by

$$\langle \alpha(\lambda), \beta(v) \rangle = -2\pi i \cdot [\lambda, v],$$

for all $v \in V_{0\mathbb{R}}$. Extending β complex linearly yields a commutative diagram

$$\begin{tikzcd}
& & T_\Phi(\mathbb{C}) \arrow[dd, equal, "(5.4.6)"] \\
V_{0\mathbb{C}} \arrow[r, "\beta"] & U_\Phi(\mathbb{C}) \arrow[ur, "(5.4.8)"] \arrow[dr, "(5.4.9)"'] & \\
& & \Gamma_\Phi^{(a)}\backslash \mathcal{D}_\Phi^\circ,
\end{tikzcd}$$

and going all the way back to the definitions preceding (2.3.6), we find that the pullback of $q_{\alpha(\lambda)} : T_\Phi(\mathbb{C}) \to \mathbb{C}^\times$ to a function on $V_{0\mathbb{C}}$ is given by

$$q_{\alpha(\lambda)}(v) = e^{-2\pi i[\lambda, v]}.$$

On the other hand, the composition along the bottom row sends

$$\frac{v}{-\mathrm{rat}(a)} \in V_{0\mathbb{C}}$$

to the point obtained by translating $\ell_* \in \mathcal{D}_\Phi^\circ$ by the vector $v \otimes \ell \in V_{0\mathbb{C}} \otimes I$, viewed as an element of $U_\Phi(\mathbb{C})$ using (4.4.2). This translate is

$$\ell_* + v - [\ell_*, v]\ell - Q(v)\ell \in \mathcal{D}_\Phi^\circ,$$

and hence the value of $q_{\alpha(\lambda)}$ at this point is

$$q_{\alpha(\lambda)}\left(\frac{v}{-\mathrm{rat}(a)}\right) = e^{2\pi i[\lambda, v]/\mathrm{rat}(a)}. \qquad \square$$

Lemma 5.4.5. *Suppose $v \in V_{0\mathbb{R}} + i \cdot \mathscr{W}$ with $|Q(\mathrm{Im}(v))| \gg 0$. The value of the meromorphic function*

$$\mathrm{rat}(a)^{c(0,0)/2} \cdot [\ell^{c(0,0)/2}, \Psi_{s(a)h}(f)]$$

at the isotropic line $\ell_* + v - [\ell_*, v]\ell - Q(v)\ell \in \mathcal{D}^\circ$ *is*

$$A_\Phi^{\mathrm{rec}(a)} \cdot e^{2\pi i[\varrho,v]/\mathrm{rat}(a)} \times \prod_{\substack{\lambda \in V_{0\mathbb{Z}}^\vee \\ [\lambda,\mathscr{W}]>0}} \prod_{\substack{\mu \in hV_{\mathbb{Z}}^\vee/hV_{\mathbb{Z}} \\ \mu\sim\lambda}} \Big(1 - \zeta_\mu^{\mathrm{rec}(a)} \cdot e^{2\pi i[\lambda,v]/\mathrm{rat}(a)}\Big)^{c(-Q(\lambda),h^{-1}\mu)},$$

up to scaling by a complex number of absolute value 1*.*

Proof. The proof amounts to carefully keeping track of how Theorem 5.3.2 changes when $\Psi_h(f)$ is replaced by $\Psi_{s(a)h}(f)$. The main source of confusion is that the vectors ℓ and ℓ_* appearing in Theorem 5.3.2 were chosen to have nice properties with respect to the lattice $hV_{\mathbb{Z}}$, and so we must first pick new isotropic vectors $\ell^{(a)}$ and $\ell_*^{(a)}$ having similarly nice properties with respect to $s(a)hV_{\mathbb{Z}}$.

Set $\ell^{(a)} = \mathrm{rat}(a)\ell$. This is a generator of

$$I \cap s(a)hV_{\mathbb{Z}} = \mathrm{rat}(a) \cdot (I \cap hV_{\mathbb{Z}}).$$

Now choose a $k^{(a)} \in s(a)hV_{\mathbb{Z}}^\vee$ such that $[\ell^{(a)}, k^{(a)}] = 1$, and let $I_*^{(a)} \subset V$ be the span of the isotropic vector

$$\ell_*^{(a)} = k^{(a)} - Q(k^{(a)})\ell^{(a)}.$$

Using the fact that Q_Φ acts trivially on the quotient $I^\perp/I$, it is easy to see that the lattice

$$V_{0\mathbb{Z}}^{(a)} = (I^\perp \cap s(a)hV_{\mathbb{Z}})/(I \cap s(a)hV_{\mathbb{Z}}) \subset I^\perp/I$$

is equal, as a subset of $I^\perp/I$, to the lattice $V_{0\mathbb{Z}}$ of (5.3.2). Thus replacing $hV_{\mathbb{Z}}$ by $s(a)hV_{\mathbb{Z}}$ has no effect on the construction of the modular form f_0, or on the formation of Weyl chambers or their corresponding Weyl vectors.

Similarly, as Q_Φ stabilizes I, the ideal $N\mathbb{Z} = [hV_{\mathbb{Z}}, I \cap hV_{\mathbb{Z}}]$ is unchanged if h is replaced by $s(a)h$. Replacing h by $s(a)h$ in the definition of A now determines a new constant

$$\begin{aligned} A^{(a)} &= \prod_{\substack{x\in\mathbb{Z}/N\mathbb{Z} \\ x\neq 0}} \Big(1 - e^{2\pi i x/N}\Big)^{c(0,x\cdot h^{-1}s(a)^{-1}\ell^{(a)}/N)} \\ &= \prod_{\substack{x\in\mathbb{Z}/N\mathbb{Z} \\ x\neq 0}} \Big(1 - e^{2\pi i x/N}\Big)^{c(0,x\cdot \mathrm{unit}(a)^{-1}h^{-1}\ell/N)} \\ &= \prod_{\substack{x\in\mathbb{Z}/N\mathbb{Z} \\ x\neq 0}} \Big(1 - e^{2\pi i x\cdot\mathrm{unit}(a)/N}\Big)^{c(0,xh^{-1}\ell/N)} \\ &= A^{\mathrm{rec}(a)}. \end{aligned}$$

Citing Theorem 5.3.2 with h replaced by $s(a)h$ everywhere, and using the isomorphism

$$s(a)hV_{\mathbb{Z}}^{\vee}/s(a)hV_{\mathbb{Z}} \cong hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}}$$

induced by the action of $s(a)^{-1}$, we find that the value of

$$[(\ell^{(a)})^{c(0,0)/2}, \Psi_{s(a)h}(f)] = \mathrm{rat}(a)^{c(0,0)/2}[\ell^{c(0,0)/2}, \Psi_{s(a)h}(f)] \tag{5.4.10}$$

at the isotropic line

$$\ell_*^{(a)} + v - [\ell_*^{(a)}, v]\ell^{(a)} - Q(v)\ell^{(a)} \in \mathcal{D}^{\circ}$$

is given by the infinite product

$$A_{\Phi}^{(a)} \cdot e^{2\pi i[\varrho,v]} \prod_{\substack{\lambda\in V_{0\mathbb{Z}}^{\vee} \\ [\lambda,\mathscr{W}]>0}} \prod_{\substack{\mu\in hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}} \\ \mu\sim\lambda}} \big(1 - e^{2\pi i[s(a)\mu,k^{(a)}]} \cdot e^{2\pi i[\lambda,v]}\big)^{c(-Q(\lambda),h^{-1}\mu)}.$$

Now make a change of variables. If we set $v^{(a)} = \ell_* - \mathrm{rat}(a)\ell_*^{(a)} \in V_0$, we find that the value of (5.4.10) at the isotropic line

$$\begin{aligned} &\ell_* + v - [\ell_*, v]\ell - Q(v)\ell \\ &= \quad \ell_*^{(a)} + \Big(\frac{v + v^{(a)}}{\mathrm{rat}(a)}\Big) - \Big[\ell_*^{(a)}, \Big(\frac{v + v^{(a)}}{\mathrm{rat}(a)}\Big)\Big]\ell^{(a)} - Q\Big(\frac{v + v^{(a)}}{\mathrm{rat}(a)}\Big)\ell^{(a)} \end{aligned}$$

is

$$\begin{aligned} &A_{\Phi}^{(a)} \cdot e^{2\pi i[\varrho,v+v^{(a)}]/\mathrm{rat}(a)} \\ &\quad\times \prod_{\substack{\lambda\in V_{0\mathbb{Z}}^{\vee} \\ [\lambda,\mathscr{W}]>0}} \prod_{\substack{\mu\in hV_{\mathbb{Z}}^{\vee}/hV_{\mathbb{Z}} \\ \mu\sim\lambda}} \Big(1 - e^{2\pi i[s(a)\mu,k^{(a)}]}e^{2\pi i[\lambda,v+v^{(a)}]/\mathrm{rat}(a)}\Big)^{c(-Q(\lambda),h^{-1}\mu)}. \end{aligned}$$

Assuming that $\mu \sim \lambda$, we may lift $\lambda \in I^{\perp}/I$ to $\tilde{\mu} \in I^{\perp} \cap (\mu + hV_{\mathbb{Z}})$. As $s(a) \in Q_{\Phi}(\mathbb{A}_f)$ acts trivially on $(I^{\perp}/I) \otimes \mathbb{A}_f$, we have

$$[\lambda, v^{(a)}] = [\lambda, s(a)^{-1}v^{(a)}] = [\tilde{\mu}, s(a)^{-1}v^{(a)}].$$

Using (4.6.6) and the definition of $v^{(a)}$, we find

$$\mathrm{rat}(a)^{-1}s(a)^{-1}v^{(a)} = \mathrm{unit}(a)\ell_* - s(a)^{-1}\ell_*^{(a)}.$$

Combining these relations with $[\tilde{\mu}, \ell_*] = [\tilde{\mu}, k]$ and $[\tilde{\mu}, \ell_*^{(a)}] = [\tilde{\mu}, k^{(a)}]$ shows that

$$\frac{[\lambda, v^{(a)}]}{\mathrm{rat}(a)} = [\tilde{\mu}, \mathrm{unit}(a)k - s(a)^{-1}k^{(a)}].$$

As $\mathrm{unit}(a)k - s(a^{-1})k^{(a)} \in hV_{\widehat{\mathbb{Z}}}^{\vee}$ and $\tilde{\mu} - \mu \in hV_{\mathbb{Z}}$, we deduce the equality

$$\frac{[\lambda, v^{(a)}]}{\mathrm{rat}(a)} = [\mu, \mathrm{unit}(a)k - s(a)^{-1}k^{(a)}]$$

in $\mathbb{Q}/\mathbb{Z} \cong \widehat{\mathbb{Q}}/\widehat{\mathbb{Z}}$. Thus

$$\begin{aligned} e^{2\pi i[s(a)\mu,k^{(a)}]}e^{2\pi i[\lambda,v+v^{(a)}]/\mathrm{rat}(a)} &= e^{2\pi i[\mu,s(a)^{-1}k^{(a)}]}e^{2\pi i[\lambda,v+v^{(a)}]/\mathrm{rat}(a)} \\ &= e^{2\pi i[\mu,\mathrm{unit}(a)k]}e^{2\pi i[\lambda,v]/\mathrm{rat}(a)} \\ &= \zeta_\mu^{\mathrm{unit}(a)} \cdot e^{2\pi i[\lambda,v]/\mathrm{rat}(a)}. \end{aligned}$$

Finally, the equality

$$e^{2\pi i[\varrho,v+v^{(a)}]/\mathrm{rat}(a)} = e^{2\pi i[\varrho,v]/\mathrm{rat}(a)}$$

holds up to a root of unity, simply because $[\varrho, v^{(a)}] \in \mathbb{Q}$. $\square$

Working on one connected component

$$\Gamma_\Phi^{(a)}\backslash\mathcal{D}^\circ \hookrightarrow \mathscr{U}_{K_{\Phi 0}}(Q_\Phi,\mathcal{D}_\Phi),$$

the pullback of $\Psi(f)$ is $\Psi_{s(a)h}(f)$. The pullback of the section $\boldsymbol{\ell}^{an}$ of the constant vector bundle $\boldsymbol{I}_{dR}^{an}$ determined by $I_\mathbb{C}$ is, by the definition preceding Proposition 4.6.1, the constant section determined by

$$\frac{\mathrm{rat}(a)}{2\pi i}\cdot \ell \in I_\mathbb{C}.$$

Thus on $\Gamma_\Phi^{(a)}\backslash\mathcal{D}^\circ$ we have the equality of meromorphic functions

$$(2\pi i)^{c(0,0)/2}\cdot[\boldsymbol{\ell}^{\otimes c(0,0)/2},\Psi(f)] = \mathrm{rat}(a)^{c(0,0)/2}\cdot[\ell^{\otimes c(0,0)/2},\Psi_{s(a)h}(f)].$$

Combining the two lemmas above, we see that the value of this meromorphic function at the isotropic line $\ell_* + v - [\ell_*, v]\ell - Q(v)\ell \in \mathcal{D}^\circ$ is

$$A_\Phi^{\mathrm{rec}(a)}\cdot q_{\alpha(\varrho)}\cdot \prod_{\substack{\lambda\in V_{0\mathbb{Z}}^\vee \\ [\lambda,\mathscr{W}]>0}} \prod_{\substack{\mu\in hV_\mathbb{Z}^\vee/hV_\mathbb{Z} \\ \mu\sim\lambda}} \left(1-\zeta_\mu^{\mathrm{rec}(a)}\cdot q_{\alpha(\lambda)}\right)^{c(-Q(\lambda),h^{-1}\mu)},$$

up to scaling by a complex number of absolute value 1. The stated q-expansion (5.4.3) follows from this.

It remains to prove the integrality conditions $\alpha(\lambda) \in \Gamma_\Phi^\vee(1)$. A priori, every $\alpha(\lambda)$ appearing in the product above (including $\lambda = \varrho$) lies in $U_\Phi(\mathbb{Q})^\vee(1)$. However, as the product itself is invariant under the action of

$$\Gamma_\Phi^{(a)} = \mathrm{rat}(a)\cdot\Gamma_\Phi \subset U_\Phi(\mathbb{Q})$$

on $\mathcal{D}^\circ$, the uniqueness of the q-expansion implies that only those terms

$$q_{\alpha(\lambda)} = e^{2\pi i[\lambda,v]/\mathrm{rat}(a)} \tag{5.4.11}$$

that are themselves invariant under $\Gamma_\Phi^{(a)}$ can appear. Pullback by the action of $u \in U_\Phi(\mathbb{Q})$ sends

$$q_{\alpha(\lambda)} \mapsto q_{\alpha(\lambda)}\cdot e^{\langle\alpha(\lambda),u\rangle/\mathrm{rat}(a)},$$

where $\langle -,-\rangle : U_\Phi(\mathbb{Q})^\vee(1)\otimes U_\Phi(\mathbb{Q}) \to \mathbb{Q}(1)$ is the tautological pairing, and it follows that the invariance of (5.4.11) under $\Gamma_\Phi^{(a)}$ is equivalent to the integrality condition $\alpha(\lambda) \in \Gamma_\Phi^\vee(1)$.

This completes the proof of Proposition 5.4.2. □

6. Integral models

As in §5, we keep $V_{\mathbb{Z}} \subset V$ of signature $(n, 2)$ with $n \geqslant 1$. Fix a prime p at which $V_{\mathbb{Z}}$ is maximal in the sense of §1.1, and assume that the compact open subgroup (1.1.2) factors as $K = K_p K^p$ with p-component

$$K_p = G(\mathbb{Q}_p) \cap C(V_{\mathbb{Z}_p})^\times.$$

Under this assumption, we recall from [AGHMP17, AGHMP18] the construction of an integral model

$$\mathcal{S}_K(G, \mathcal{D}) \to \mathrm{Spec}(\mathbb{Z}_{(p)})$$

of $\mathrm{Sh}_K(G, \mathcal{D})$, and extensions to this model of the line bundle of weight one modular forms and the special divisors. In those references it is assumed that $V_{\mathbb{Z}}$ is maximal at *every* prime, but nearly everything extends verbatim to the more general case considered here. Indeed, one only has to be careful about the definitions of special divisors in §6.4. Once the correct definitions are formulated the proofs of [*loc. cit.*] go through without significant change, and we simply give the appropriate citations without further comment.

The main new result is the pullback formula of Proposition 6.6.3, which describes how special divisors restrict under embeddings between orthogonal Shimura varieties of different dimension. This will be a crucial ingredient in our algebraic variant of the embedding trick of Borcherds.

6.1. Almost self-dual lattices.

The motivation for the following definition will become clear in §6.3.

Definition 6.1.1. We say that $V_{\mathbb{Z}_p}$ is *almost self-dual* if it has one of the following (mutually exclusive) properties:

- $V_{\mathbb{Z}_p}$ is self-dual;
- $p = 2$, $\dim_{\mathbb{Q}}(V)$ is odd, and $[V_{\mathbb{Z}_2}^\vee : V_{\mathbb{Z}_2}]$ is not divisible by 4.

Remark 6.1.2. Almost self-duality is equivalent to the smoothness of the quadric over $\mathbb{Z}_p$ parameterizing isotropic lines in $V_{\mathbb{Z}_p}$. Here, an *isotropic line* in V_R for an $\mathbb{Z}_p$-algebra R is a local direct summand $I \subset V_R$ of rank 1 that is locally generated by an element $v \in I$ satisfying $Q(v) = 0$.

Recall from §4.3 that G acts on the $\mathbb{Q}$-vector space $H = C(V)$, and that one may choose a $\mathbb{Z}$-valued symplectic form ψ on

$$H_{\mathbb{Z}} = C(V_{\mathbb{Z}})$$

in such a way that the action of G induces a Hodge embedding into the Siegel Shimura datum determined by (H, ψ). The following lemma will be used in §8 to choose ψ in a particularly nice way.

Lemma 6.1.3. *Assume that V has Witt index 2 (this is automatic if $n \geqslant 5$). If $V_{\mathbb{Z}_p}$ is almost self-dual, then we may choose ψ as above in such a way that $H_{\mathbb{Z}_p}$ is self-dual.*

Proof. Choose any isotropic line $I \subset V$, and let $\ell \in I \cap V_{\mathbb{Z}}$ be a $\mathbb{Z}$-module generator. Let N be the positive integer defined by

$$N\mathbb{Z} = [V_{\mathbb{Z}}, \ell].$$

On the one hand, $\ell/N \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$ is isotropic under the $\mathbb{Q}/\mathbb{Z}$-valued quadratic form induced by Q. On the other hand, maximality of $V_{\mathbb{Z}_p}$ implies that $V_{\mathbb{Z}_p}^{\vee}/V_{\mathbb{Z}_p}$ has no nonzero isotropic vectors. Thus $\ell/N \in V_{\mathbb{Z}_p}$, and so $p \nmid N$.

It follows that there is some $k \in V_{\mathbb{Z}}$ such that $p \nmid [k, \ell]$, and from this it is easy to see that there exists a vector $v \in \mathbb{Z}k + \mathbb{Z}\ell$ such that $Q(v)$ is negative and prime to p.

The $\mathbb{Q}$-span of $k, \ell \in V$ is a hyperbolic plane over $\mathbb{Q}$, and the $\mathbb{Z}_p$-span of $k, \ell \in V_{\mathbb{Z}_p}$ is an integral hyperbolic plane over $\mathbb{Z}_p$. It follows that the orthogonal complement

$$W = (\mathbb{Q}k + \mathbb{Q}\ell)^{\perp} \subset V$$

has Witt index 1, and that the $\mathbb{Z}$-lattice $W_{\mathbb{Z}} = W \cap V_{\mathbb{Z}}$ satisfies

$$V_{\mathbb{Z}_p} = \mathbb{Z}_p k \oplus \mathbb{Z}_p \ell \oplus W_{\mathbb{Z}_p}.$$

In particular $W_{\mathbb{Z}_p}$ is again maximal. Repeating the argument above with $V_{\mathbb{Z}}$ replaced by $W_{\mathbb{Z}}$, we find another vector $w \in V_{\mathbb{Z}}$ with $Q(w)$ negative and prime to p, and $[v, w] = 0$.

We have now constructed an element $\delta = vw \in C(V_{\mathbb{Z}})$ such that

$$\delta^2 = -Q(v)Q(w) \in \mathbb{Z}_{(p)}^{\times}.$$

Set $\psi(x, y) = \mathrm{Trd}(x\delta y^*)$, exactly as in §4.3.

It remains to prove that $H_{\mathbb{Z}_p}$ is self-dual. We will use the decomposition

$$H_{\mathbb{Z}_p} = H_{\mathbb{Z}_p}^{+} \oplus H_{\mathbb{Z}_p}^{-}$$

induced by the decomposition $C(V_{\mathbb{Z}_p}) = C^{+}(V_{\mathbb{Z}_p}) \oplus C^{-}(V_{\mathbb{Z}_p})$ into even and odd parts. It is not hard to see that these direct summands of $H_{\mathbb{Z}_p}$ are orthogonal to each other under ψ, and so it suffices to prove the self-duality of each summand individually.

According to [Con14, §C.2], the almost self-duality of $V_{\mathbb{Z}_p}$ implies that the even Clifford algebra $C^{+}(V_{\mathbb{Z}_p})$ is an Azumaya algebra over its center, and this center is itself a finite étale $\mathbb{Z}_p$-algebra. Equivalently, $C^{+}(V_{\mathbb{Z}_p})$ is isomorphic étale locally on $\mathrm{Spec}(\mathbb{Z}_p)$ to a finite product of matrix algebras. It follows from this that

$$C^{+}(V_{\mathbb{Z}_p}) \otimes C^{+}(V_{\mathbb{Z}_p}) \xrightarrow{x \otimes y \mapsto \mathrm{Trd}(xy)} \mathbb{Z}_p$$

is a perfect bilinear pairing. The self-duality of $H_{\mathbb{Z}_p}^{+}$ under ψ follows easily from this. The self-duality of $H_{\mathbb{Z}_p}^{-}$ then follows using the isomorphism

$$H_{\mathbb{Z}_p}^{-} \cong H_{\mathbb{Z}_p}^{+}$$

given by right multiplication by the $v \in C(V_{\mathbb{Z}})$ chosen above, and the relation

$$\psi(xv, yv) = -Q(v) \cdot \psi(x, y)$$

for all $x, y \in H$. □

6.2. **Isometric embeddings.** We will repeatedly find ourselves in the following situation. Suppose we have another quadratic space $(V^\diamond, Q^\diamond)$ of signature $(n^\diamond, 2)$, and an isometric embedding $V \hookrightarrow V^\diamond$. This induces a morphism of Clifford algebras $C(V) \to C(V^\diamond)$, which induces a morphism of GSpin Shimura data

$$(G, \mathcal{D}) \to (G^\diamond, \mathcal{D}^\diamond).$$

Just as we assume for (V, Q), suppose we are given a $\mathbb{Z}$-lattice $V^\diamond_{\mathbb{Z}} \subset V^\diamond$ on which $Q^\diamond$ is integer valued, and which is maximal at p. Let

$$K^\diamond = K^\diamond_p \cdot K^{\diamond,p} \subset G^\diamond(\mathbb{A}_f) \cap C(V^\diamond_{\widehat{\mathbb{Z}}})^\times$$

be a compact open subgroup with p-component

$$K^\diamond_p = G^\diamond(\mathbb{Q}_p) \cap C(V^\diamond_{\mathbb{Z}_p})^\times.$$

Assume that $V_{\mathbb{Z}} \subset V^\diamond_{\mathbb{Z}}$ and $K \subset K^\diamond$, so that we have a finite and unramified morphism

$$j : \mathrm{Sh}_K(G, \mathcal{D}) \to \mathrm{Sh}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond) \tag{6.2.1}$$

of canonical models. Our choices imply (using the assumption that $V_{\mathbb{Z}_p}$ is maximal) that $V_{\mathbb{Z}_p} = V_{\mathbb{Q}_p} \cap V^\diamond_{\mathbb{Z}_p}$ and $K_p = K^\diamond_p \cap G(\mathbb{Q}_p)$.

Lemma 6.2.1. *It is possible to choose $(V^\diamond, Q^\diamond)$ and $V^\diamond_{\mathbb{Z}}$ as above in such a way that $V^\diamond_{\mathbb{Z}}$ is self-dual. Moreover, we can ensure that $V \subset V^\diamond$ has codimension at most 2 if n is even, and has codimension at most 3 if n is odd.*

Proof. An exercise in the classification of quadratic spaces over $\mathbb{Q}$ shows that we may choose a positive definite quadratic space W in such a way that the orthogonal direct sum $V^\diamond = V \oplus W$ admits a self-dual lattice locally at every finite prime (for example, we may arrange for $V^\diamond$ to be a sum of hyperbolic planes locally at every finite prime). From Eichler's theorem that any two maximal lattices in a $\mathbb{Q}_p$-quadratic space are isometric [Ger08, Theorem 8.8], it follows that any maximal lattice in $V^\diamond$ is self-dual. Enlarging $V_{\mathbb{Z}}$ to a maximal lattice $V^\diamond_{\mathbb{Z}} \subset V^\diamond$ proves the first claim.

A more careful analysis, once again using the classification of quadratic spaces, also yields the second claim. □

6.3. **Definition of the integral model.** We now define our integral model of the Shimura variety $\mathrm{Sh}_K(G, \mathcal{D})$.

Assume first that $V_{\mathbb{Z}_p}$ is almost self-dual. This implies, by [Con14, §C.4], that

$$\mathcal{G} = \mathrm{GSpin}(V_{\mathbb{Z}_{(p)}})$$

is a reductive group scheme over $\mathbb{Z}_{(p)}$, and hence that $K_p = \mathcal{G}(\mathbb{Z}_p)$ is a hyperspecial compact open subgroup of $G(\mathbb{Q}_p)$. Thus $\mathrm{Sh}_K(G, \mathcal{D})$ admits a canonical smooth integral model $\mathcal{S}_K(G, \mathcal{D})$ over $\mathbb{Z}_{(p)}$ by the results of Kisin [Kis10] (and [KM16] if $p = 2$).

Remark 6.3.1. The notion of almost self-duality does not appear anywhere in our main references [Mad16, AGHMP17, AGHMP18] on integral models of $\mathrm{Sh}_K(G,\mathcal{D})$. This is due to an oversight on the authors' part: we did not realize that one could obtain smooth integral models even if $V_{\mathbb{Z}_p}$ fails to be self-dual.

According to [KM16, Proposition 3.7] there is a functor

$$N \mapsto (\boldsymbol{N}_{dR}, F^\bullet \boldsymbol{N}_{dR}) \tag{6.3.1}$$

from representations $\mathcal{G} \to \mathrm{GL}(N)$ on free $\mathbb{Z}_{(p)}$-modules of finite rank to filtered vectors bundles on $\mathcal{S}_K(G,\mathcal{D})$, restricting to the functor (3.3.2) in the generic fiber.[5]

Applying this functor to the representation $V_{\mathbb{Z}_{(p)}}$ yields a filtered vector bundle $(\boldsymbol{V}_{dR}, F^\bullet \boldsymbol{V}_{dR})$. Applying the functor to the representation $H_{\mathbb{Z}_{(p)}} = C(V_{\mathbb{Z}_{(p)}})$ yields a filtered vector bundle $(\boldsymbol{H}_{dR}, F^\bullet \boldsymbol{H}_{dR})$. The inclusion (4.1.2) restricts to $V_{\mathbb{Z}_{(p)}} \to \mathrm{End}(H_{\mathbb{Z}_{(p)}})$, which determines an injection

$$\boldsymbol{V}_{dR} \to \underline{\mathrm{End}}(\boldsymbol{H}_{dR})$$

onto a local direct summand.

For any local section x of $\boldsymbol{V}_{dR}$, the composition $x \circ x$ is a local section of the subsheaf $\mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})} \subset \underline{\mathrm{End}}(\boldsymbol{H}_{dR})$. This defines a quadratic form

$$\boldsymbol{Q} : \boldsymbol{V}_{dR} \to \mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})},$$

with an associated bilinear form

$$[-,-] : \boldsymbol{V}_{dR} \otimes \boldsymbol{V}_{dR} \to \mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})}$$

related as in (1.1.1). The filtration on $\boldsymbol{V}_{dR}$ has the form

$$0 = F^2\boldsymbol{V}_{dR} \subset F^1\boldsymbol{V}_{dR} \subset F^0\boldsymbol{V}_{dR} \subset F^{-1}\boldsymbol{V}_{dR} = \boldsymbol{V}_{dR},$$

in which $F^1\boldsymbol{V}_{dR}$ is an isotropic line, and $F^0\boldsymbol{V}_{dR} = (F^1\boldsymbol{V}_{dR})^\perp$. As in §4.2, the *line bundle of weight one modular forms* on $\mathcal{S}_K(G,\mathcal{D})$ is

$$\boldsymbol{\omega} = F^1\boldsymbol{V}_{dR}.$$

If $V_{\mathbb{Z}_p}$ is not almost self-dual then choose auxiliary data $(V^\diamond, Q^\diamond)$ as in §6.2 in such a way that $V^\diamond_{\mathbb{Z}_p}$ is almost self-dual. This determines a commutative diagram

$$\begin{tikzcd} \mathcal{S}_K(G,\mathcal{D}) \arrow[d] & \mathrm{Sh}_K(G,\mathcal{D}) \arrow[l] \arrow[d] \\ \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond) & \mathrm{Sh}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond) \arrow[l] \end{tikzcd} \tag{6.3.2}$$

in which the lower left corner is the canonical integral model of $\mathrm{Sh}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)$, and the upper right corner is defined as its normalization in $\mathrm{Sh}_K(G,\mathcal{D})$, in

[5]There is also a weight filtration on $\boldsymbol{N}_{dR}$, but, as noted in Remark 3.4.2, it is not very interesting over the pure Shimura variety.

the sense of [AGHMP18, Definition 4.2.1]. By construction, $\mathcal{S}_K(G,\mathcal{D})$ is a normal Deligne-Mumford stack, flat and of finite type over $\mathbb{Z}_{(p)}$.

Define the *line bundle of weight one modular forms* on $\mathcal{S}_K(G,\mathcal{D})$ by

$$\boldsymbol{\omega} = \boldsymbol{\omega}^\diamond|_{\mathcal{S}_K(G,\mathcal{D})}, \tag{6.3.3}$$

where $\boldsymbol{\omega}^\diamond$ is the line bundle on $\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)$ constructed in the almost self-dual case above. The line bundle (6.3.3) extends the line bundle of the same name previously constructed on the generic fiber.

The following is [AGHMP18, Proposition 4.4.1].

Proposition 6.3.2. *The $\mathbb{Z}_{(p)}$-stack $\mathcal{S}_K(G,\mathcal{D})$ and the line bundle $\boldsymbol{\omega}$ are independent of the auxiliary choices of $(V^\diamond,Q^\diamond)$, $V^\diamond_{\widehat{\mathbb{Z}}}$, and $K^\diamond$ used in their construction, and the Kuga-Satake abelian scheme of §4.3 extends uniquely to an abelian scheme $\mathcal{A}\to\mathcal{S}_K(G,\mathcal{D})$.*

The following is a restatement of the main result of [Mad16].

Proposition 6.3.3. *If $p>2$ and p^2 does not divide $[V_{\mathbb{Z}}^\vee : V_{\mathbb{Z}}]$, then $\mathcal{S}_K(G,\mathcal{D})$ is regular.*

Remark 6.3.4. Our $\mathcal{S}_K(G,\mathcal{D})$ is not quite the same as the integral model of [AGHMP17]. That integral model is obtained from $\mathcal{S}_K(G,\mathcal{D})$ by deleting certain closed substacks supported in characteristics p for which p^2 divides $[V_{\mathbb{Z}}^\vee : V_{\mathbb{Z}}]$. The point of deleting such substacks is that the vector bundle $\boldsymbol{V}_{dR}$ on $\mathrm{Sh}_K(G,\mathcal{D})$ of §4.2 then extends canonically to the remaining open substack. In the present work, as in [AGHMP18], the only automorphic vector bundle required on $\mathcal{S}_K(G,\mathcal{D})$ is the line bundle of modular forms $\boldsymbol{\omega}$ just constructed; we have no need of an extension of $\boldsymbol{V}_{dR}$ to $\mathcal{S}_K(G,\mathcal{D})$.

6.4. **Special divisors.** For $m\in\mathbb{Q}_{>0}$ and $\mu\in V_{\mathbb{Z}}^\vee/V_{\mathbb{Z}}$ there is a Cartier divisor $\mathcal{Z}(m,\mu)$ on $\mathcal{S}_K(G,\mathcal{D})$, defined in [AGHMP17, AGHMP18] in the case where $V_{\mathbb{Z}}$ is maximal. As we now assume only the weaker hypothesis that $V_{\mathbb{Z}}$ is maximal at p, the definition requires minor adjustment.

We first define the divisors in the generic fiber, where they were originally constructed by Kudla [Kud97]. Our construction is different, and has a more moduli-theoretic flavor.

By the theory of automorphic vector bundles described in §3, the G-equivariant inclusion (4.1.2) determines an inclusion

$$\boldsymbol{V}_{dR}\subset\underline{\mathrm{End}}(\boldsymbol{H}_{dR}) \tag{6.4.1}$$

of vector bundles on $\mathrm{Sh}_K(G,\mathcal{D})$, respecting the Hodge filtrations. Recall from §4.3 that the filtered vector bundle $\boldsymbol{H}_{dR}$ is canonically identified with the first relative de Rham homology of the Kuga-Satake abelian scheme

$$\pi : A\to\mathrm{Sh}_K(G,\mathcal{D}).$$

The compact open subgroup $K\subset G(\mathbb{A}_f)$ appears as a quotient of the étale fundamental group of $\mathrm{Sh}_K(G,\mathcal{D})$, and hence representations of K give rise to étale local systems. In particular, for any prime ℓ the $\mathbb{Z}_\ell$-lattice $H_{\mathbb{Z}_\ell}$

determines an étale sheaf of $\mathbb{Z}_\ell$-modules $\boldsymbol{H}_\ell$ on $\mathrm{Sh}_K(G,\mathcal{D})$. This is just the relative ℓ-adic Tate module

$$\boldsymbol{H}_\ell \cong \underline{\mathrm{Hom}}(R^1\pi_{et,*}\underline{\mathbb{Z}_\ell}, \underline{\mathbb{Z}_\ell})$$

of the Kuga-Satake abelian scheme.

As in the discussion preceding (4.1.5), K also acts on both

$$V_{\mathbb{Z}_\ell} = V_{\mathbb{Z}} \otimes \mathbb{Z}_\ell \qquad \text{and} \qquad V_{\mathbb{Z}_\ell}^\vee = V_{\mathbb{Z}}^\vee \otimes \mathbb{Z}_\ell,$$

and the induced action on the quotient $V_{\mathbb{Z}_\ell}^\vee / V_{\mathbb{Z}_\ell}$ is trivial. These representations of K determine étale sheaves of $\mathbb{Z}_\ell$-modules $\boldsymbol{V}_\ell \subset \boldsymbol{V}_\ell^\vee$, along with an inclusion of étale $\mathbb{Z}_\ell$-sheaves

$$\boldsymbol{V}_\ell \subset \underline{\mathrm{End}}(\boldsymbol{H}_\ell). \tag{6.4.2}$$

and a canonical trivialization $\boldsymbol{V}_\ell^\vee / \boldsymbol{V}_\ell \cong V_{\mathbb{Z}_\ell}^\vee / V_{\mathbb{Z}_\ell}$. In particular, each $\mu_\ell \in V_{\mathbb{Z}_\ell}^\vee / V_{\mathbb{Z}_\ell}$ determines a subsheaf of sets

$$\mu_\ell + \boldsymbol{V}_\ell \subset \boldsymbol{V}_\ell \otimes \mathbb{Q}_\ell. \tag{6.4.3}$$

Suppose we are given a $\mathbb{Q}$-scheme S and a morphism $S \to \mathrm{Sh}_K(G,\mathcal{D})$. Denote by $A_S \to S$ the pullback of the Kuga-Satake abelian scheme. A quasi-endomorphism[6] $x \in \mathrm{End}(A_S) \otimes \mathbb{Q}$ is *special* if

- its de Rham realization

$$x_{dR} \in H^0(S, \underline{\mathrm{End}}(\boldsymbol{H}_{dR})|_S)$$

  lies in the subsheaf $\boldsymbol{V}_{dR}|_S$, and
- its ℓ-adic realization

$$x_\ell \in H^0(S, \underline{\mathrm{End}}(\boldsymbol{H}_\ell)|_S \otimes \mathbb{Q}_\ell)$$

  lies in the subsheaf $\boldsymbol{V}_\ell|_S \otimes \mathbb{Q}_\ell$ for every prime ℓ.

The space of all special quasi-endomorphisms of A_S is a $\mathbb{Q}$-subspace

$$V(A_S) \subset \mathrm{End}(A_S) \otimes \mathbb{Q}.$$

Under the inclusion $V \subset \mathrm{End}(H)$, the quadratic form on V becomes $Q(x) = x \circ x$. Similarly, the square of any $x \in V(A_S)$ lies in $\mathbb{Q} \subset \mathrm{End}(A_S) \otimes \mathbb{Q}$, and $V(A_S)$ is endowed with the positive definite quadratic form $Q(x) = x \circ x$. For each $\mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}$, we now define

$$V_\mu(A_S) \subset V(A_S) \tag{6.4.4}$$

to be the set of all special quasi-endomorphisms whose ℓ-adic realization lies in the subsheaf (6.4.3) for every prime ℓ, and set

$$Z(m,\mu)(S) \stackrel{\mathrm{def}}{=} \{x \in V_\mu(A_S) : Q(x) = m\}.$$

We now explain how to extend this definition to the integral model.

[6]A quasi-endomorphism should really be defined as global section of the Zariski sheaf $\underline{\mathrm{End}}(A_S) \otimes \mathbb{Q}$ on S. If S is not of finite type over $\mathbb{Q}$, the space of such global sections can be strictly larger than $\mathrm{End}(A_S) \otimes \mathbb{Q}$. For simplicity of notation, we ignore this minor technical point.

First assume that $V_{\mathbb{Z}}$ is self-dual at p. As in the discussion following (6.3.1), the inclusion of vector bundles (6.4.1) has a canonical extension to the integral model $\mathcal{S}_K(G,\mathcal{D})$. Directly from the definitions, so does the inclusion of étale $\mathbb{Q}_\ell$-sheaves (6.4.2) for any $\ell \neq p$. As a substitute for p-adic étale cohomology, we use the inclusion

$$\boldsymbol{V}_{crys} \subset \underline{\mathrm{End}}(\boldsymbol{H}_{crys}) \tag{6.4.5}$$

of locally free crystals on $\mathcal{S}_K(G,\mathcal{D})_{/\mathbb{F}_p}$ as in [AGHMP18, Proposition 4.2.5]. There is a canonical isomorphism

$$\boldsymbol{H}_{crys} \cong \underline{\mathrm{Hom}}(R^1\pi_{crys,*}\mathcal{O}^{crys}_{\mathcal{A}_{\mathbb{F}_p}/\mathbb{Z}_p}, \mathcal{O}^{crys}_{\mathcal{M}_{\mathbb{F}_p}/\mathbb{Z}_p})$$

between $\boldsymbol{H}_{crys}$ and the first relative crystalline homology of the reduction of the Kuga-Satake abelian scheme $\pi : \mathcal{A} \to \mathcal{S}_K(G,\mathcal{D})$ of Proposition 6.3.2.

Still assuming that $V_{\mathbb{Z}}$ is self-dual at p, suppose we are given a $\mathbb{Z}_{(p)}$-scheme S and a morphism $S \to \mathcal{S}_K(G,\mathcal{D})$, and let $\mathcal{A}_S$ be the pullback of the Kuga-Satake abelian scheme. We call $x \in \mathrm{End}(A_S) \otimes \mathbb{Z}_{(p)}$ *special* if

- its de Rham realization
$$x_{dR} \in H^0(S, \underline{\mathrm{End}}(\boldsymbol{H}_{dR})|_S)$$
lies in the subsheaf $\boldsymbol{V}_{dR}|_S$,
- its ℓ-adic realization
$$x_\ell \in H^0(S, \underline{\mathrm{End}}(\boldsymbol{H}_\ell)|_S \otimes \mathbb{Q}_\ell)$$
lies in the subsheaf $\boldsymbol{V}_\ell|_S \otimes \mathbb{Q}_\ell$ for every prime $\ell \neq p$,
- its p-adic realization
$$x_p \in H^0(S_{\mathbb{Q}}, \underline{\mathrm{End}}(\boldsymbol{H}_p)|_{S_{\mathbb{Q}}})$$
over the generic fiber $S_{\mathbb{Q}}$ lies in the subsheaf $\boldsymbol{V}_p|_{S_{\mathbb{Q}}}$, and
- its crystalline realization
$$x_{crys} \in H^0(S_{\mathbb{F}_p}, \underline{\mathrm{End}}(\boldsymbol{H}_{crys})|_{S_{\mathbb{F}_p}})$$
over the special fiber $S_{\mathbb{F}_p}$ lies in the subcrystal $\boldsymbol{V}_{crys}|_{S_{\mathbb{F}_p}}$.

The space of all such $x \in \mathrm{End}(A_S) \otimes \mathbb{Z}_{(p)}$ is denoted

$$V(\mathcal{A}_S)_{\mathbb{Z}_{(p)}} \subset \mathrm{End}(\mathcal{A}_S) \otimes \mathbb{Z}_{(p)},$$

and tensoring with $\mathbb{Q}$ defines the subspace of all special quasi-endomorphisms

$$V(\mathcal{A}_S) \subset \mathrm{End}(\mathcal{A}_S) \otimes \mathbb{Q}.$$

It endowed with a positive definite quadratic form $Q(x) = x \circ x$ exactly as above. For any $\mu \in V_{\mathbb{Z}}^\vee/V_{\mathbb{Z}}$ we define

$$V_\mu(\mathcal{A}_S) \subset V(\mathcal{A}_S)_{\mathbb{Z}_{(p)}} \tag{6.4.6}$$

as the subset of elements whose ℓ-adic realization lies in (6.4.3) for every prime $\ell \neq p$.

Now consider the general case in which $V_{\mathbb{Z}} \subset V$ is only assumed to be maximal at p. In this generality we still have the étale $\mathbb{Q}_\ell$-sheaves (6.4.2) for

$\ell \neq p$. However, there is no adequate theory of automorphic vector bundles or crystals on $\mathcal{S}_K(G, \mathcal{D})$; compare with Remark 6.3.4. In particular, we have no adequate substitute for the sheaves in (6.4.5).

So that we may apply the results of [AGHMP17, AGHMP18], enlarge $V_{\mathbb{Z}}$ to a lattice $V'_{\mathbb{Z}} \subset V$ that is maximal at every prime. This choice determines a second $\mathbb{Z}$-lattice $H'_{\mathbb{Z}} \subset H_{\mathbb{Q}}$, and hence a second Kuga-Satake abelian scheme

$$\mathcal{A}' \to \mathcal{S}_K(G, \mathcal{D})$$

endowed with an isogeny $\mathcal{A} \to \mathcal{A}'$ of degree prime to p. Choose a larger quadratic space $V^{\diamond}$ as in §6.2 admitting a maximal lattice $V^{\diamond}_{\mathbb{Z}} \subset V^{\diamond}$ that is self-dual at p, and an isometric embedding $V'_{\mathbb{Z}} \to V^{\diamond}_{\mathbb{Z}}$.

By the very construction of the integral model, there is a finite morphism

$$\mathcal{S}_K(G, \mathcal{D}) \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond}).$$

According to [AGHMP17, Proposition 2.5.1], the abelian schemes $\mathcal{A}'$ and

$$\mathcal{A}^{\diamond} \to \mathcal{S}_{K^{\diamond}}(G^{\diamond}, \mathcal{D}^{\diamond})$$

carry right actions of the integral Clifford algebras $C(V'_{\mathbb{Z}})$ and $C(V^{\diamond}_{\mathbb{Z}})$, respectively, and are related by a canonical isomorphism

$$\mathcal{A}' \otimes_{C(V'_{\mathbb{Z}})} C(V^{\diamond}_{\mathbb{Z}}) \cong \mathcal{A}^{\diamond}|_{\mathcal{S}_K(G,\mathcal{D})}. \tag{6.4.7}$$

Note that the Serre tensor construction on the left is defined because the maximality of $V'_{\mathbb{Z}}$ implies that $V'_{\mathbb{Z}} \subset V^{\diamond}_{\mathbb{Z}}$ as a $\mathbb{Z}$-module direct summand, which implies that the natural map $C(V'_{\mathbb{Z}}) \to C(V^{\diamond}_{\mathbb{Z}})$ makes $C(V^{\diamond}_{\mathbb{Z}})$ into a free $C(V'_{\mathbb{Z}})$-module.

Definition 6.4.1. Suppose we are given a morphism $S \to \mathcal{S}_K(G, \mathcal{D})$. A quasi-endomorphism

$$x \in \mathrm{End}(\mathcal{A}_S) \otimes \mathbb{Q}$$

is *special* if the induced quasi-endomorphism of $\mathcal{A}'_S$ commutes with the action of $C(V'_{\mathbb{Z}})$, and its image under the map

$$\mathrm{End}_{C(V'_{\mathbb{Z}})}(\mathcal{A}'_S) \otimes \mathbb{Q} \to \mathrm{End}_{C(V^{\diamond}_{\mathbb{Z}})}(\mathcal{A}^{\diamond}_S) \otimes \mathbb{Q}$$

induced by (6.4.7) is a special quasi-endomorphism of $\mathcal{A}^{\diamond}_S$ (in the sense already defined for the self-dual-at-p lattice $V^{\diamond}_{\mathbb{Z}}$).

The following is [AGHMP18, Proposition 4.3.4].

Proposition 6.4.2. *If S is connected, then $x \in \mathrm{End}(\mathcal{A}_S) \otimes \mathbb{Q}$ is special if and only if the restriction $x_s \in \mathrm{End}(\mathcal{A}_s) \otimes \mathbb{Q}$ is special for some (equivalently, every) geometric point $s \to S$.*

Once again, the space of all special quasi-endomorphisms

$$V(\mathcal{A}_S) \subset \mathrm{End}(\mathcal{A}_S) \otimes \mathbb{Q}$$

carries a positive definite quadratic form $Q(x) = x \circ x$. By construction it comes with an isometric embedding

$$V(\mathcal{A}_S) \subset V(\mathcal{A}^{\diamond}_S). \tag{6.4.8}$$

It remains to define a subset

$$V_\mu(\mathcal{A}_S) \subset V(\mathcal{A}_S) \tag{6.4.9}$$

for each coset $\mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}$. Let $\mu_\ell \in V_{\mathbb{Z}_\ell}^\vee / V_{\mathbb{Z}_\ell}$ be the ℓ-component. If $\ell \neq p$ let

$$V_{\mu_\ell}(\mathcal{A}_S) \subset V(\mathcal{A}_S)$$

be the subset of elements whose ℓ-adic realization lies in the subsheaf (6.4.3). To treat the p-part of μ, define

$$\Lambda = \{x \in V_{\mathbb{Z}}^\diamond : x \perp V_{\mathbb{Z}}\}.$$

The maximality of $V_{\mathbb{Z}}$ at p implies that $V_{\mathbb{Z}_p} \subset V_{\mathbb{Z}_p}^\diamond$ is a $\mathbb{Z}_p$-module direct summand. From this and the self-duality of $V_{\mathbb{Z}}^\diamond$ at p it is easy to see that the projections to the two factors in

$$V^\diamond = V \oplus \Lambda_{\mathbb{Q}}$$

induce bijections

$$V_{\mathbb{Z}_p}^\vee / V_{\mathbb{Z}_p} \cong (V_{\mathbb{Z}_p}^\diamond)^\vee / V_{\mathbb{Z}_p}^\diamond \cong \Lambda_{\mathbb{Z}_p}^\vee / \Lambda_{\mathbb{Z}_p}. \tag{6.4.10}$$

The image of μ_p under this bijection is denoted $\bar{\mu}_p \in \Lambda_{\mathbb{Z}_p}^\vee / \Lambda_{\mathbb{Z}_p}$. As in [AGHMP17, Proposition 2.5.1], there is a canonical isometric embedding

$$\Lambda \to V(\mathcal{A}_S^\diamond)_{\mathbb{Z}_{(p)}}$$

whose image is orthogonal to that of (6.4.8). In fact, we have an orthogonal decmposition

$$V(\mathcal{A}_S^\diamond) = V(\mathcal{A}_S) \oplus \Lambda_{\mathbb{Q}},$$

which allows us to define

$$V_{\mu_p}(\mathcal{A}_S) = \{x \in V(\mathcal{A}_S) : x + \bar{\mu}_p \in V(\mathcal{A}_S^\diamond)_{\mathbb{Z}_{(p)}}\}. \tag{6.4.11}$$

Finally, define (6.4.9) by

$$V_\mu(\mathcal{A}_S) = \bigcap_\ell V_{\mu_\ell}(\mathcal{A}_S).$$

This set is independent of the choice of auxiliary data $V'_{\mathbb{Z}} \subset V_{\mathbb{Z}}^\diamond \subset V^\diamond$ used in its definition, and agrees with the definition (6.4.4) if S is a $\mathbb{Q}$-scheme. See [AGHMP18, Proposition 4.5.3].

The following is [AGHMP17, Proposition 2.7.2].

Proposition 6.4.3. *Given a positive $m \in \mathbb{Q}$ and a $\mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}$, the functor sending an $\mathcal{S}_K(G, \mathcal{D})$-scheme S to*

$$\mathcal{Z}(m, \mu)(S) = \{x \in V_\mu(\mathcal{A}_S) : Q(x) = m\}$$

is represented by a finite, unramified, and relatively representable morphism of Deligne-Mumford stacks

$$\mathcal{Z}(m, \mu) \to \mathcal{S}_K(G, \mathcal{D}). \tag{6.4.12}$$

In the next subsection we will justify in what sense the morphisms (6.4.12), which are not even closed immersions, deserve the name *special divisors*.

We end this section by describing what the morphism (6.4.12) looks like in the complex fiber. For each $g \in G(\mathbb{A}_f)$, the pullback of (6.4.12) via the complex uniformization

$$\mathcal{D} \xrightarrow{z \mapsto (z,g)} \mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C})$$

can be made explicit. Each $x \in V$ with $Q(x) > 0$ determines an analytic subset

$$\mathcal{D}(x) = \{z \in \mathcal{D} : [z, x] = 0\}$$

of the hermitian domain (4.1.1).

From the discussion of §4.3, we see that the fiber of the Kuga-Satake abelian scheme at a point $z \in \mathcal{D}$ is the complex torus

$$A_z(\mathbb{C}) = gH_{\mathbb{Z}} \backslash H_{\mathbb{C}} / zH_{\mathbb{C}}.$$

The action of $x \in V \subset \mathrm{End}(H)$ by left multiplication in the Clifford algebra $C(V)$ defines a quasi-endomorphism of $A_z(\mathbb{C})$ if and only if it preserves the subspace $zH_{\mathbb{C}} \subset H_{\mathbb{C}}$, and a linear algebra exercise shows that this condition is equivalent to $z \in \mathcal{D}(x)$. Using this, one can check that the pullback of (6.4.12) via the above complex uniformization is

$$\bigsqcup_{\substack{x \in g\mu + gV_{\mathbb{Z}} \\ Q(x) = m}} \mathcal{D}(x) \to \mathcal{D}. \tag{6.4.13}$$

Here, by mild abuse of notation, $g\mu$ is the image of μ under the action-by-g isomorphism $V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}} \to gV_{\mathbb{Z}}^{\vee}/gV_{\mathbb{Z}}$

6.5. **Deformation theory.** We need to explain the sense in which the morphism (6.4.12) merits the name *special divisor*. This is closely tied up with the deformation theory of special endomorphisms, which will also be needed in the proof of the pullback formula of Proposition 6.6.3 below.

It is enlightening to first consider the complex analytic situation of (6.4.13). Each subset $\mathcal{D}(x) \subset \mathcal{D}$ is not only an analytic divisor, but arises as the 0-locus of a canonical section

$$\mathrm{obst}_x^{an} \in H^0(\mathcal{D}, \boldsymbol{\omega}_{\mathcal{D}}^{-1}) \tag{6.5.1}$$

of the inverse of the tautological bundle $\boldsymbol{\omega}_{\mathcal{D}}$ on (4.1.1). Indeed, recalling that the fiber of $\boldsymbol{\omega}_{\mathcal{D}}$ at $z \in \mathcal{D}$ is the isotropic line $\mathbb{C}z \subset V_{\mathbb{C}}$, we define (6.5.1) as the linear functional

$$\mathbb{C}z \xrightarrow{z \mapsto [z,x]} \mathbb{C}.$$

This is the *analytic obstruction to deforming* x.

Returning to the algebraic world, suppose

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathcal{Z}(m,\mu) \arrow[d] \\ \widetilde{S} \arrow[r] & \mathcal{S}_K(G,\mathcal{D}) \end{tikzcd} \tag{6.5.2}$$

is a commutative diagram of stacks in which $S \to \widetilde{S}$ is a closed immersion of schemes defined by an ideal sheaf $J \subset \mathcal{O}_{\widetilde{S}}$ with $J^2 = 0$. After pullback to S, the Kuga-Satake abelian scheme $\mathcal{A} \to \mathcal{S}_K(G,\mathcal{D})$ acquires a tautological special quasi-endomorphism

$$x \in V_\mu(\mathcal{A}_S),$$

and we want to know when this lies in the image of the (injective) restriction map

$$V_\mu(\mathcal{A}_{\widetilde{S}}) \to V_\mu(\mathcal{A}_S). \tag{6.5.3}$$

Equivalently, when there is a (necessarily unique) dotted arrow

$$\begin{tikzcd} S \arrow[r] \arrow[d] & \mathcal{Z}(m,\mu) \arrow[d] \\ \widetilde{S} \arrow[r] \arrow[ur, dotted] & \mathcal{S}_K(G,\mathcal{D}) \end{tikzcd}$$

making the diagram commute.

Proposition 6.5.1. *In the situation above, there is a canonical section*

$$\mathrm{obst}_x \in H^0\big(\widetilde{S}, \boldsymbol{\omega}|_{\widetilde{S}}^{-1}\big), \tag{6.5.4}$$

called the obstruction to deforming x*, such that x lies in the image of (6.5.3) if and only if* $\mathrm{obst}_x = 0$.

Proof. Suppose first that $V_{\mathbb{Z}}$ is self-dual at p, so that we have an inclusion

$$\boldsymbol{V}_{dR} \to \underline{\mathrm{End}}(\boldsymbol{H}_{dR})$$

as a local direct summand of vector bundles on $\mathcal{S}_K(G,\mathcal{D})$. The vector bundle $\boldsymbol{H}_{dR}$ is identified with the first relative de Rham homology of the Kuga-Satake abelian scheme $\mathcal{S}$. As such, it is is endowed with its Gauss-Manin connection, which restricts to a flat connection

$$\nabla : \boldsymbol{V}_{dR} \to \boldsymbol{V}_{dR} \otimes \Omega^1_{\mathcal{S}_K(G,\mathcal{D})/\mathbb{Z}_{(p)}}.$$

Indeed, one can check this in the complex fiber, over which the connection becomes identified, using (3.1.3), with

$$\boldsymbol{V}_{Be} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})} \xrightarrow{1 \otimes d} \boldsymbol{V}_{Be} \otimes_{\mathbb{Z}} \Omega^1_{\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})}.$$

The de Rham realization

$$x_{dR} \in H^0(S, \boldsymbol{V}_{dR}|_S) \tag{6.5.5}$$

is parallel, and therefore admits parallel transport (the algebraic theory of parallel transport can be extracted from [BO78, §2], for example) to $\widetilde{S}$: there is a unique parallel extension of x_{dR} to

$$\widetilde{x}_{dR} \in H^0(\widetilde{S}, \boldsymbol{V}_{dR}|_{\widetilde{S}}). \tag{6.5.6}$$

We now define obst_x be the image of $\widetilde{x}_{dR}$ under $\boldsymbol{V}_{dR} \to \boldsymbol{V}_{dR}/F^0\boldsymbol{V}_{dR}$, and use the perfect bilinear pairing (4.2.1) to identify

$$\boldsymbol{V}_{dR}/F^0\boldsymbol{V}_{dR} \cong (F^1\boldsymbol{V}_{dR})^{-1} = \boldsymbol{\omega}^{-1}.$$

The local sections of $F^0\boldsymbol{V}_{dR}$ are precisely those local sections of $\boldsymbol{V}_{dR} \subset \underline{\mathrm{End}}(\boldsymbol{H}_{dR})$ which preserve the Hodge filtration $F^0\boldsymbol{H}_{dR} \subset \boldsymbol{H}_{dR}$. The vanishing of obst_x is equivalent to

$$\widetilde{x}_{dR} \in H^0(\widetilde{S}, F^0\boldsymbol{V}_{dR}|_{\widetilde{S}}),$$

which is therefore equivalent to the endomorphism

$$\widetilde{x}_{dR} \in \mathrm{End}(\boldsymbol{H}_{dR}|_{\widetilde{S}})$$

respecting the Hodge filtration. Using the deformation theory of abelian schemes described in [Lan13, Chapter 2], this is equivalent to

$$x \in V_\mu(\mathcal{A}_S) \subset \mathrm{End}(\mathcal{A}_S) \otimes \mathbb{Z}_{(p)}$$

admitting an extension to

$$\widetilde{x} \in \mathrm{End}(\mathcal{A}_{\widetilde{S}}) \otimes \mathbb{Z}_{(p)}.$$

Using Proposition 6.4.2 it is easy to see that when such an extension exists it must lie in $V_\mu(\mathcal{A}_S)$. This proves the claim when $V_{\mathbb{Z}}$ is self-dual at p.

We now explain how to construct the section (6.5.4) in general. Fix an isometric embedding $V_{\mathbb{Z}} \subset V_{\mathbb{Z}}^\diamond$ as in §6.2, and assume that $V_{\mathbb{Z}}^\diamond$ is self-dual at p, so that we have morphisms of integral models

$$\mathcal{S}_K(G, \mathcal{D}) \to \mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond).$$

In the notation of (6.4.11), the special quasi-endomorphism $x \in V_\mu(\mathcal{A}_S)$ determines another special quasi-endomorphism

$$x^\diamond = x + \bar{\mu}_p \in V(\mathcal{A}_S)_{\mathbb{Z}_{(p)}},$$

and x extends to $V_\mu(\mathcal{A}_{\widetilde{S}})$ and only if $x^\diamond$ extends to $V(\mathcal{A}_{\widetilde{S}})_{\mathbb{Z}_{(p)}}$.

The self-dual-at-p case considered above determines an obstruction to deforming $x^\diamond$, denoted

$$\mathrm{obst}_{x^\diamond} \in H^0\big(\widetilde{S}, \boldsymbol{\omega}^\diamond|_{\widetilde{S}}^{-1}\big).$$

Recalling that $\boldsymbol{\omega}|_{\widetilde{S}} = \boldsymbol{\omega}^\diamond|_{\widetilde{S}}$ by definition, we now define (6.5.4) by

$$\mathrm{obst}_x = \mathrm{obst}_{x^\diamond}.$$

It is easy to check that this does not depend on the auxiliary choice of $V_{\mathbb{Z}}^\diamond$ used in its construction, and has the desired properties. $\square$

Proposition 6.5.2. *Every geometric point of $\mathcal{S}_K(G,\mathcal{D})$ admits an étale neighborhood $U \to \mathcal{S}_K(G,\mathcal{D})$ such that*

$$\mathcal{Z}(m,\mu)_{/U} \to U$$

restricts to a closed immersion on every connected component of its domain. Each such closed immersion is an effective Cartier divisor on U.

Proof. The first claim is a formal consequence of Proposition 6.4.3, and holds for any finite, unramified, relatively representable morphism of Deligne-Mumford stacks. Indeed, if $\mathcal{O}_s$ denotes the étale local ring at a geometric point $s \to \mathcal{S}_K(G,\mathcal{D})$, then finiteness and relative representability imply that

$$\mathrm{Spec}(\mathcal{O}_s) \times_{\mathcal{S}_K(G,\mathcal{D})} \mathcal{Z}(m,\mu) \cong \bigsqcup_t \mathrm{Spec}(\mathcal{O}_t)$$

where t runs over the geometric points $t \to \mathcal{Z}(m,\mu)$ above s, and unramifiedness implies that each morphism $\mathcal{O}_s \to \mathcal{O}_t$ is surjective.

Fix one such t, set $J = \ker(\mathcal{O}_s \to \mathcal{O}_t)$, and consider the nilpotent thickening

$$\mathrm{Spec}(\mathcal{O}_t) = \mathrm{Spec}(\mathcal{O}_s/J) \hookrightarrow \mathrm{Spec}(\mathcal{O}_s/J^2).$$

In particular, we have a diagram

$$\begin{tikzcd} \mathrm{Spec}(\mathcal{O}_t) \arrow[r] \arrow[d] & \mathcal{Z}(m,\mu) \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_s/J^2) \arrow[r] & \mathcal{S}_K(G,\mathcal{D}) \end{tikzcd}$$

exactly as in (6.5.2). The pullback of the Kuga-Satake abelian scheme to $\mathrm{Spec}(\mathcal{O}_t)$ acquires a tautological special quasi-endomorphism x. The obstruction to deforming x is, after choosing a trivialization of $\boldsymbol{\omega}|_{\mathrm{Spec}(\mathcal{O}_t)}$, an element

$$\mathrm{obst}_x \in \mathcal{O}_s/J^2$$

that generates J/J^2 as an $\mathcal{O}_s$-module. Nakayama's lemma now implies that $J \subset \mathcal{O}_s$ is a principal ideal, and so $\mathrm{Spec}(\mathcal{O}_t) \hookrightarrow \mathrm{Spec}(\mathcal{O}_s)$ is an effective Cartier divisor.

This proves the claim on the level of étale local rings, and the extension to étale neighborhoods is routine. □

Proposition 6.5.2 is what justifies referring to the morphisms (6.4.12) as divisors, even though they are not closed immersions. In the notation of that proposition, every connected component of the source of

$$\mathcal{Z}(m,\mu)_{/U} \to U$$

determines a Cartier divisor on U. Summing over all such components and then glueing as U varies over an étale cover defines an effective Cartier divisor on $\mathcal{S}_K(G,\mathcal{D})$ in the usual sense. When no confusion can arise (and perhaps even when it can), we denote this Cartier divisor again by $\mathcal{Z}(m,\mu)$.

We end this subsection by explaining the precise relation between the analytic obstruction (6.5.1) and the algebraic obstruction (6.5.4).

Fix a $g \in G(\mathbb{A}_f)$. If we pull back the diagram (6.5.2) via the morphism

$$\mathcal{D} \xrightarrow{z \mapsto (z,g)} \mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C}) \tag{6.5.7}$$

we obtain (at least if $\widetilde{S}$ is of finite type over $\mathbb{Q}$) a diagram

$$\begin{tikzcd} \mathcal{S} = S^{an} \times_{\mathrm{Sh}_K(G,\mathcal{D})^{an}} \mathcal{D} \arrow[r] \arrow[d] & \bigsqcup \mathcal{D}(x) \arrow[d] \\ \widetilde{\mathcal{S}} = \widetilde{S}^{an} \times_{\mathrm{Sh}_K(G,\mathcal{D})^{an}} \mathcal{D} \arrow[r] & \mathcal{D}, \end{tikzcd}$$

of complex analytic spaces, in which the disjoint union is as in (6.4.13), and the vertical arrow on the left is defined by a coherent sheaf of ideals whose square is 0. In particular $\mathcal{S} \to \widetilde{\mathcal{S}}$ induces an isomorphism of underlying topological spaces.

For a fixed x, let $\mathcal{S}(x) \subset \mathcal{S}$ be the union of those connected components of whose image under the top horizontal arrow lies in the factor $\mathcal{D}(x)$. This determines a union of connected components $\widetilde{\mathcal{S}}(x) \subset \widetilde{\mathcal{S}}$, and gives us a diagram of complex analytic spaces

$$\begin{tikzcd} S^{an} \arrow[d] & \mathcal{S}(x) \arrow[l] \arrow[r] \arrow[d] & \mathcal{D}(x) \arrow[d] \\ \widetilde{S}^{an} & \widetilde{\mathcal{S}}(x) \arrow[l] \arrow[r] & \mathcal{D}. \end{tikzcd}$$

Proposition 6.5.3. *There is an equality of sections*

$$\mathrm{obst}_x^{an}|_{\widetilde{\mathcal{S}}(x)} = \mathrm{obst}_x|_{\widetilde{\mathcal{S}}(x)},$$

where the left hand side is the pullback of (6.5.1) via $\widetilde{\mathcal{S}}(x) \to \mathcal{D}$ and the right hand side is the pullback of (6.5.4) via $\widetilde{\mathcal{S}}(x) \to \widetilde{S}^{an}$.

Proof. The pullback of $\boldsymbol{V}_{dR}$ via (6.5.7) is canonically identified with the constant vector bundle

$$\boldsymbol{V}_{dR}|_{\mathcal{D}} = V \otimes \mathcal{O}_{\mathcal{D}},$$

and under this identification the pullback of the connection ∇ is the induced by the usual $d : \mathcal{O}_{\mathcal{D}} \to \Omega^1_{\mathcal{D}/\mathbb{C}}$.

By the discussion leading to (6.4.13), the pullback of (6.5.5) via $\mathcal{S}(x) \to S^{an}$ is identified with the constant section

$$x \otimes 1 \in H^0\big(\mathcal{S}(x), \boldsymbol{V}_{dR}|_{\mathcal{S}(x)}\big),$$

and the pullback of (6.5.6) via $\tilde{\mathcal{S}}(x) \to \widetilde{S}^{an}$ is its unique parallel extension

$$x \otimes 1 \in H^0\big(\widetilde{\mathcal{S}}(x), \boldsymbol{V}_{dR}|_{\widetilde{\mathcal{S}}(x)}\big).$$

Thus $\mathrm{obst}_x|_{\widetilde{\mathcal{S}}(x)}$ is the image of $x \otimes 1$ under

$$V \otimes \mathcal{O}_{\widetilde{\mathcal{S}}(x)} \cong \boldsymbol{V}_{dR}|_{\widetilde{\mathcal{S}}(x)} \to \big(\boldsymbol{V}_{dR}/F^0\boldsymbol{V}_{dR}\big)|_{\widetilde{\mathcal{S}}(x)} \cong \boldsymbol{\omega}^{-1}_{\widetilde{\mathcal{S}}(x)}.$$

On the other hand, the analytically defined obstruction (6.5.1) is, essentially by construction, the image of the constant section $x \otimes 1$ under

$$V \otimes \mathcal{O}_{\mathcal{D}} \cong \boldsymbol{V}_{dR}|_{\mathcal{D}} \to \big(\boldsymbol{V}_{dR}/F^0\boldsymbol{V}_{dR}\big)|_{\mathcal{D}} \cong \boldsymbol{\omega}^{-1}_{\mathcal{D}}.$$

The stated equality of sections over $\widetilde{\mathcal{S}}(x)$ follows immediately. □

6.6. **The pullback formula for special divisors.** Suppose we are in the general situation of §6.2 (in particular, we impose no assumption of self-duality on $V^\diamond_{\mathbb{Z}}$), so that we have a morphism (6.2.1) of Shimura varieties

$$\mathrm{Sh}_K(G,\mathcal{D}) \to \mathrm{Sh}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond).$$

The larger Shimura variety $\mathrm{Sh}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)$ has its own integral model

$$\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond) \to \mathrm{Spec}(\mathbb{Z}_{(p)}),$$

obtained by repeating the construction of §6.3 with $(G,\mathcal{D})$ replaced by $(G^\diamond,\mathcal{D}^\diamond)$. That is, choose an isometric embedding $V^\diamond \subset V^{\diamond\diamond}$ into a larger quadratic space that admits an almost self-dual lattice at p, and define $\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)$ as a normalization. Of course $\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)$ has its own line bundle $\boldsymbol{\omega}^\diamond$, its own Kuga-Satake abelian scheme, and its own collection of special divisors $\mathcal{Z}^\diamond(m,\mu)$.

Proposition 6.6.1. *The above morphism of canonical models extends uniquely to a finite morphism*

$$\mathcal{S}_K(G,\mathcal{D}) \to \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond) \tag{6.6.1}$$

of integral models. The line bundles of weight one modular forms on the source and target of (6.6.1) are related by a canonical isomorphism

$$\boldsymbol{\omega}^\diamond|_{\mathcal{S}_K(G,\mathcal{D})} \cong \boldsymbol{\omega}. \tag{6.6.2}$$

Proof. The existence and uniqueness of (6.6.1) is proved in [AGHMP17, Proposition 2.5.1].

If $V^\diamond_{\mathbb{Z}}$ is almost self-dual at p then (6.6.2) is just a restatement of the definition of $\boldsymbol{\omega}$. For the general case, one embeds $V^\diamond$ into a quadratic space $V^{\diamond\diamond}$ admitting a lattice that is almost self-dual at p. This allows one to identify both sides of (6.6.2) with the pullback of $\boldsymbol{\omega}^{\diamond\diamond}$ for some morphisms

$$\mathcal{S}_K(G,\mathcal{D}) \to \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond) \to \mathcal{S}_{K^{\diamond\diamond}}(G^{\diamond\diamond},\mathcal{D}^{\diamond\diamond})$$

into the larger Shimura variety determined by $V^{\diamond\diamond}$. □

Define a quadratic space

$$\Lambda = \{x \in L^\diamond_{\mathbb{Z}} : x \perp L\}$$

over $\mathbb{Z}$ of signature $(n^\diamond - n, 0)$. There are natural inclusions

$$V_{\mathbb{Z}} \oplus \Lambda \subset V^\diamond_{\mathbb{Z}} \subset (V^\diamond_{\mathbb{Z}})^\vee \subset V^\vee_{\mathbb{Z}} \oplus \Lambda^\vee$$

all of finite index, from which it follows that the orthogonal decomposition

$$V^\diamond = V \oplus \Lambda_{\mathbb{Q}}$$

identifies

$$\mu + V^\diamond_{\mathbb{Z}} = \bigsqcup_{\mu_1+\mu_2\in\mu} (\mu_1 + V_{\mathbb{Z}}) \times (\mu_2 + \Lambda).$$

Here the disjoint union over $\mu_1 + \mu_2 \in \mu$ is understood to mean the union over all pairs

$$(\mu_1, \mu_2) \in (V^\vee_{\mathbb{Z}}/V_{\mathbb{Z}}) \oplus (\Lambda^\vee/\Lambda)$$

satisfying $\mu_1 + \mu_2 \in (\mu + V^\diamond_{\mathbb{Z}})/(V_{\mathbb{Z}} \oplus \Lambda)$.

The following lemma gives a corresponding decomposition of special quasi-endomorphisms. For the proof see [AGHMP17, Proposition 2.6.4].

Proposition 6.6.2. *For any scheme S and any morphism $S \to \mathcal{S}_K(G,\mathcal{D})$ there is a canonical isometry*

$$V(\mathcal{A}^\diamond_S) \cong V(\mathcal{A}_S) \oplus \Lambda_{\mathbb{Q}},$$

which restricts to a bijection

$$V_\mu(\mathcal{A}^\diamond_S) \cong \bigsqcup_{\mu_1+\mu_2\in\mu} V_{\mu_1}(\mathcal{A}_S) \times (\mu_2 + \Lambda). \tag{6.6.3}$$

The relation between special divisors on the source and target of (6.6.1) is most easily expressed in terms of the line bundles associated to the divisors, rather than the divisors themselves. By abuse of notation, we now use $\mathcal{Z}(m,\mu)$ to denote also the line bundle on $\mathcal{S}_K(G,\mathcal{D})$ determined by the Cartier divisor of the same name, extend the definition to $m \leqslant 0$ by

$$\mathcal{Z}(m,\mu) = \begin{cases} \boldsymbol{\omega}^{-1} & \text{if } (m,\mu) = (0,0) \\ \mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})} & \text{otherwise,} \end{cases}$$

and use similar conventions for $\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)$.

Proposition 6.6.3. *For any rational number $m \geqslant 0$ and any $\mu \in (V^\diamond_{\mathbb{Z}})^\vee/V^\diamond_{\mathbb{Z}}$, there is a canonical isomorphism of line bundles*

$$\mathcal{Z}^\diamond(m,\mu)|_{\mathcal{S}_K(G,\mathcal{D})} \cong \bigotimes_{\substack{m_1+m_2=m\\ \mu_1+\mu_2\in\mu}} \mathcal{Z}(m_1,\mu_1)^{\otimes r_\Lambda(m_2,\mu_2)}$$

on $\mathcal{S}_K(G,\mathcal{D})$. Here we have set

$$R_\Lambda(m,\mu) = \{\lambda \in \mu + \Lambda : Q(\lambda) = m\}$$

and $r_\Lambda(m,\mu) = \#R_\Lambda(m,\mu)$.

Proof. If $m < 0$, or if $m = 0$ and $\mu \neq 0$, the tensor product on the right is empty, and both sides of the desired isomorphism are canonically trivial. If $(m,\mu) = (0,0)$ the claim is just a restatement of Proposition 6.6.1. Thus we may assume that $m > 0$.

The decomposition (6.6.3) induces an isomorphism

$$\mathcal{Z}^\diamond(m,\mu)_{/\mathcal{S}_K(G,\mathcal{D})} \cong \bigsqcup_{\substack{m_1+m_2=m \\ \mu_1+\mu_2\in\mu \\ m_1>0 \\ \lambda\in R_\Lambda(m_2,\mu_2)}} \mathcal{Z}(m_1,\mu_1) \sqcup \bigsqcup_{\substack{\mu_2\in\mu \\ \lambda\in R_\Lambda(m,\mu_2)}} \mathcal{S}_K(G,\mathcal{D}) \tag{6.6.4}$$

of $\mathcal{S}_K(G,\mathcal{D})$-stacks, where the condition $\mu_2 \in \mu$ means that

$$0 + \mu_2 \in (V_\mathbb{Z}^\vee/V_\mathbb{Z}) \oplus (\Lambda^\vee/\Lambda)$$

lies in the subset $(\mu + V_\mathbb{Z}^\diamond)/(V_\mathbb{Z}\oplus\Lambda)$. Explicitly, given any connected scheme S and a morphism

$$S \to \mathcal{S}_K(G,\mathcal{D}),$$

a lift of the morphism to the first disjoint union on the right hand side of (6.6.4) determines a pair

$$(x,\lambda) \in V_{\mu_1}(\mathcal{A}_S) \times (\mu_2 + \Lambda)$$

satisfying $m_1 = Q(x)$ and $m_2 = Q(\lambda)$. Using (6.6.3) we obtain a special quasi-endomorphism

$$x^\diamond = x + \lambda \in V_\mu(\mathcal{A}_S^\diamond).$$

Similarly, a lift to the second disjoint union determines a vector $\lambda \in \mu_2 + \Lambda$ satisfying $m = Q(\lambda)$, which determines a special quasi-endomorphism

$$x^\diamond = 0 + \lambda \in V_\mu(\mathcal{A}_S^\diamond). \tag{6.6.5}$$

In either case $Q(x^\diamond) = m$, and so our lift determines an S-point of the left hand side of (6.6.4).

If $\Lambda^\vee$ does not represent m, then $R_\Lambda(m,\mu_2) = \varnothing$ for all choices of μ_2, and the desired isomorphism of line bundles

$$\begin{aligned} \mathcal{Z}^\diamond(m,\mu)|_{\mathcal{S}_K(G,\mathcal{D})} &\cong \bigotimes_{\substack{m_1+m_2=m \\ \mu_1+\mu_2\in\mu \\ m_1>0}} \mathcal{Z}(m_1,\mu_1)^{\otimes r_\Lambda(m_2,\mu_2)} \\ &\cong \bigotimes_{\substack{m_1+m_2=m \\ \mu_1+\mu_2\in\mu}} \mathcal{Z}(m_1,\mu_1)^{\otimes r_\Lambda(m_2,\mu_2)}, \end{aligned}$$

on $\mathcal{S}_K(G,\mathcal{D})$ follows immediately from (6.6.4). In general, the decomposition (6.6.4) shows that the support of $\mathcal{Z}^\diamond(m,\mu)$ contains the image of (6.6.1) as soon as there is some $\mu_2 \in \mu$ for which $R_\Lambda(m,\mu_2)$ is nonempty. Thus we must compute an improper intersection.

Fix a geometric point $s \to \mathcal{S}_K(G,\mathcal{D})$ and, as in Proposition 6.5.2, an étale neighborhood

$$U^\diamond \to \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)$$

of s small enough that the morphism

$$\mathcal{Z}^\diamond(m,\mu)_{/U^\diamond} \to U^\diamond$$

restricts to a closed immersion on every connected component of the domain. By shrinking $U^\diamond$ we may assume that these connected components are in bijection with the set of lifts

$$\begin{tikzcd} & \mathcal{Z}^\diamond(m,\mu)_{/U^\diamond} \arrow[d] \\ s \arrow[r] \arrow[ur, dotted] & U^\diamond. \end{tikzcd}$$

Having so chosen $U^\diamond$, we then choose a connected étale neighborhood

$$U \to \mathcal{S}_K(G,\mathcal{D})$$

of s small enough that there exists a lift

$$\begin{tikzcd} U \arrow[r, dotted] \arrow[d] & U^\diamond \arrow[d] \\ \mathcal{S}_K(G,\mathcal{D}) \arrow[r] & \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond), \end{tikzcd}$$

and so that in the cartesian diagram

$$\begin{tikzcd} \mathcal{Z}^\diamond(m,\mu)_{/U} \arrow[r] \arrow[d] & \mathcal{Z}^\diamond(m,\mu)_{/U^\diamond} \arrow[d] \\ U \arrow[r] & U^\diamond \end{tikzcd}$$

each of the vertical arrows restricts to a closed immersion on every connected component of its source, and the top horizontal arrow induces a bijection on connected components.

The decomposition (6.6.4) induces a decomposition of U-schemes

$$\mathcal{Z}^\diamond(m,\mu)_{/U} \cong \bigsqcup_{\substack{m_1+m_2=m \\ \mu_1+\mu_2\in\mu \\ m_1>0 \\ \lambda\in R_\Lambda(m_2,\mu_2)}} \mathcal{Z}(m_1,\mu_1)_{/U} \sqcup \bigsqcup_{\substack{\mu_2\in\mu \\ \lambda\in R_\Lambda(m,\mu_2)}} U.$$

The first disjoint union defines a Cartier divisor on U. In the second disjoint union, the copy of U indexed by $\lambda \in R_\Lambda(m,\mu_2)$ is the image of the open and closed immersion

$$f_\lambda : U \to \mathcal{Z}^\diamond(m,\mu)_{/U}$$

obtained by endowing the Kuga-Satake abelian scheme $\mathcal{A}^\diamond_U$ with the special quasi-endomorphism $0+\lambda \in V_\mu(\mathcal{A}^\diamond_U)$ of (6.6.5).

There is a corresponding canonical decomposition of $U^\diamond$-schemes

$$\mathcal{Z}^\diamond(m,\mu)_{/U^\diamond} = \mathcal{Z}^\diamond_{\mathrm{prop}} \sqcup \bigsqcup_{\substack{\mu_2\in\mu \\ \lambda\in R_\Lambda(m,\mu_2)}} \mathcal{Z}^\diamond_\lambda, \tag{6.6.6}$$

in which

$$\mathcal{Z}^\diamond_\lambda \subset \mathcal{Z}^\diamond(m,\mu)_{/U^\diamond} \tag{6.6.7}$$

is the connected component containing the image of

$$U \xrightarrow{f_\lambda} \mathcal{Z}^\diamond(m,\mu)_{/U} \to \mathcal{Z}^\diamond(m,\mu)_{/U^\diamond}$$

and

$$\mathcal{Z}^\diamond_{\mathrm{prop}} \subset \mathcal{Z}^\diamond(m,\mu)_{/U^\diamond}$$

is the union of all connected components not of this form.

It is clear from the definitions that the image of $U \to U^\diamond$ intersects the Cartier divisor $\mathcal{Z}^\diamond_{\mathrm{prop}} \to U^\diamond$ properly, and in fact

$$\mathcal{Z}^\diamond_{\mathrm{prop}/U} \cong \bigsqcup_{\substack{m_1+m_2=m \\ \mu_1+\mu_2\in\mu \\ m_1>0 \\ \lambda\in R_\Lambda(m_2,\mu_2)}} \mathcal{Z}(m_1,\mu_1)_{/U}. \tag{6.6.8}$$

On the other hand, the image of $U \to U^\diamond$ is completely contained within the support of every $\mathcal{Z}^\diamond_\lambda \to U^\diamond$.

By mild abuse of notation, we denote again by $\mathcal{Z}^\diamond_{\mathrm{prop}}$ and $\mathcal{Z}^\diamond_\lambda$ the line bundles on $U^\diamond$ determined by the Cartier divisors of the same name.

Lemma 6.6.4. *There is a canonical isomorphisms of line bundles*

$$\mathcal{Z}^\diamond_{\mathrm{prop}}|_U \cong \bigotimes_{\substack{m_1+m_2=m \\ \mu_1+\mu_2\in\mu \\ m_1>0}} \mathcal{Z}(m_1,\mu_1)|_U^{\otimes r_\Lambda(m_2,\mu_2)},$$

and a canonical isomorphism

$$\mathcal{Z}^\diamond_\lambda|_U \cong \boldsymbol{\omega}|_U^{-1}. \tag{6.6.9}$$

Proof. The first isomorphism is clear from the isomorphism of U-schemes (6.6.8). The second isomorphism is more subtle, and is based on similar calculations in the context of unitary Shimura varieties; see especially [BHY15, Theorem 7.10] and [How19].

Our étale neighborhood $U^\diamond \to \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)$ was chosen in such a way that $\mathcal{Z}^\diamond_\lambda \to U^\diamond$ is a closed immersion defined by a locally principal sheaf of ideals $J_\lambda \subset \mathcal{O}_{U^\diamond}$. The closed subscheme

$$\widetilde{\mathcal{Z}}^\diamond_\lambda \subset U^\diamond$$

defined by J_λ^2 is called the *first order tube* around $Z^\diamond_\lambda$. We now have morphisms

$$U \xrightarrow{f_\lambda} \mathcal{Z}^\diamond_\lambda \hookrightarrow \widetilde{\mathcal{Z}}^\diamond_\lambda \hookrightarrow U^\diamond \to \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond).$$

Tautologically, J_λ^{-1} is the line bundle on $U^\diamond$ determined by the Cartier divisor $\mathcal{Z}^\diamond_\lambda$. Denote by σ_λ the constant function 1, viewed as a section of $\mathcal{O}_{U^\diamond} \subset J_\lambda^{-1}$, so that $\mathrm{div}(\sigma_\lambda) = \mathcal{Z}^\diamond_\lambda$.

On the other hand, after restriction to the connected component (6.6.7) the Kuga-Satake abelian scheme $\mathcal{A}^\diamond$ acquires a tautological

$$x^\diamond \in V_\mu(\mathcal{A}^\diamond_{\mathcal{Z}^\diamond_\lambda}).$$

The discussion of §6.5 then provides us with a canonical section

$$\mathrm{obst}_{x^\diamond} \in H^0\big(\widetilde{\mathcal{Z}}^\diamond_\lambda, \boldsymbol{\omega}|^{-1}_{\widetilde{\mathcal{Z}}^\diamond_\lambda}\big)$$

whose zero locus is the closed subscheme $\mathcal{Z}^\diamond_\lambda$.

The idea is roughly that the equality of divisors

$$\mathrm{div}(\sigma_\lambda) = \mathrm{div}(\mathrm{obst}_{x^\diamond})$$

should imply that there is a unique isomorphism of line bundles (6.6.9) over the first order tube sending $\sigma_\lambda \mapsto \mathrm{obst}_{x^\diamond}$, which we would then pull back via f_λ. This is a bit too strong. Instead, we argue that such an isomorphism exists Zariski locally on the first order tube, and that any two such local isomorphisms restrict to the same isomorphism over U.

Indeed, working Zariski locally, we can assume that

$$U = \mathrm{Spec}(R), \quad U^\diamond = \mathrm{Spec}(R^\diamond)$$

for integral domains R and $R^\diamond$, and

$$\mathcal{Z}^\diamond_\lambda = \mathrm{Spec}(R^\diamond/J), \quad \widetilde{\mathcal{Z}}^\diamond_\lambda = \mathrm{Spec}(R^\diamond/J^2).$$

The morphisms $U \to \mathcal{Z}^\diamond_\lambda \to U^\diamond$ then correspond to homomorphisms

$$R^\diamond \to R^\diamond/J \to R.$$

Let $\mathfrak{p} \subset R^\diamond$ be the kernel of this composition, so that $J \subset \mathfrak{p}$. Note that $p \notin \mathfrak{p}$, as the flatness of $\mathcal{S}_K(G,\mathcal{D})$ over $\mathbb{Z}_{(p)}$ implies that R has no p-torsion.

Assume that we have chosen trivializations of the line bundles $\boldsymbol{\omega}|_{U^\diamond}$ and $Z^\diamond_\lambda$ on $U^\diamond$, so that our sections obst_λ and σ_λ are identified with elements

$$a \in R^\diamond/J^2 \qquad \text{and} \qquad b \in R^\diamond,$$

respectively. Each of these elements generates the ideal $J/J^2 \subset R^\diamond/J^2$.

Lemma 6.6.5. *There exists $u \in R^\diamond/J^2$ such that $ua = b$. The image of any such u in $R^\diamond/\mathfrak{p} \subset R$ is a unit. If also $u'a = b$, then $u = u'$ in $R^\diamond/\mathfrak{p} \subset R$.*

Proof. Suppose we are given any $x \in R^\diamond/J^2$ with $bx = 0$. We claim that

$$x \in \mathfrak{p}/J^2.$$

If not, then any lift $x \in R^\diamond$ becomes a unit in the localization $R^\diamond_\mathfrak{p}$. As $bx \in J^2$, we obtain

$$b \in \mathfrak{p}^2 R^\diamond_\mathfrak{p}. \tag{6.6.10}$$

We have noted above that $p \notin \mathfrak{p}$, and so $R^\diamond_\mathfrak{p}$ is a $\mathbb{Q}$-algebra. The source and target of

$$\mathcal{Z}^\diamond(m,\mu) \to \mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond)$$

have smooth generic fibers, and so $R^\diamond_\mathfrak{p} \to R^\diamond_\mathfrak{p}/bR^\diamond_\mathfrak{p}$ is a morphism of regular local rings. By (6.6.10), this morphism induces an isomorphism on tangent spaces, and so is itself an isomorphism. Thus $b = 0$ in $R^\diamond_\mathfrak{p}$, and hence also in $R^\diamond$. This contradicts the fact that b generates the ideal J.

As a and b generate both generate J/J^2, there exist $u, v \in R^\diamond/J^2$ such that $ua = b$ and $vb = a$. Obviously $b \cdot (1 - uv) = 0$, and taking $x = 1 - uv$ the paragraph above implies $1 - uv \in \mathfrak{p}/J^2$. Thus the image of u in $R^\diamond/\mathfrak{p}$ is a unit with inverse v. If also $u'a = b$, the same argument shows that the image of u' in $R^\diamond/\mathfrak{p}$ is a unit with inverse v, and hence $u = u'$ in $R^\diamond/\mathfrak{p}$. $\square$

The discussion above provides us with a canonical isomorphism

$$\mathcal{Z}^\diamond_\lambda|_U \cong \boldsymbol{\omega}|_U^{-1}$$

Zariski locally on U, and glueing over an open cover completes the proof of Lemma 6.6.4. $\square$

We now complete the proof of Proposition 6.6.3. If we interpret the isomorphism of $U^\diamond$-schemes (6.6.6) as an isomorphism

$$\mathcal{Z}^\diamond(m,\mu)|_{U^\diamond} \cong \mathcal{Z}^\diamond_{\mathrm{prop}} \otimes \bigotimes_{\substack{\mu_2 \in \mu \\ \lambda \in R_\Lambda(m,\mu_2)}} \mathcal{Z}^\diamond_\lambda,$$

of line bundles on $U^\diamond$, pull back via $U \to U^\diamond$, and use Lemma 6.6.4, we obtain canonical isomorphisms

$$\begin{aligned}
\mathcal{Z}^\diamond(m,\mu)|_U &\cong \Bigg(\bigotimes_{\substack{m_1+m_2=m \\ \mu_1+\mu_2 \in \mu \\ m_1>0}} \mathcal{Z}(m_1,\mu_1)|_U^{\otimes r_\Lambda(m_2,\mu_2)} \Bigg) \otimes \Bigg(\bigotimes_{\mu_2 \in \mu} \boldsymbol{\omega}|_U^{-r_\Lambda(m,\mu_2)} \Bigg) \\
&\cong \bigotimes_{\substack{m_1+m_2=m \\ \mu_1+\mu_2=\mu}} \mathcal{Z}(m_1,\mu_1)|_U^{\otimes r_\Lambda(m_2,\mu_2)}
\end{aligned}$$

of line bundles over the étale neighborhood U of $s \to \mathcal{S}_K(G,\mathcal{D})$. Now let U vary over an étale cover and apply descent. $\square$

7. Normality and flatness

Keep $V_\mathbb{Z} \subset V$ and $K \subset G(\mathbb{A}_f)$ as in §6, and once again fix a prime p at which $V_\mathbb{Z}$ is maximal. After some technical preliminaries in §7.1, we prove in §7.2 that the special fiber of the integral model

$$\mathcal{S}_K(G,\mathcal{D}) \to \mathrm{Spec}(\mathbb{Z}_{(p)})$$

is geometrically normal if $n \geqslant 6$, and that the special divisors are flat if $n \geqslant 4$. When $p \neq 2$ these results already appear[7] in [AGHMP18]. Here we use similar ideas, but employ the methods of Ogus [Ogu79] to control the dimension of the supersingular locus, as these apply even when $p = 2$.

[7] with the sharper bounds $n \geqslant 5$ and $n \geqslant 3$, respectively

7.1. **Local properties of special cycles.** Suppose in this subsection that $V_{\mathbb{Z}}$ is self-dual at p. As in the discussion of §6.3, the smooth integral model $\mathcal{S}_K(G,\mathcal{D})$ comes with filtered vector bundles $0 \subset F^0\boldsymbol{H}_{dR} \subset \boldsymbol{H}_{dR}$ and

$$0 \subset F^1\boldsymbol{V}_{dR} \subset F^0\boldsymbol{V}_{dR} \subset \boldsymbol{V}_{dR},$$

along with an injection $\boldsymbol{V}_{dR} \to \underline{\mathrm{End}}(\boldsymbol{H}_{dR})$ onto a local direct summand. Composition in $\underline{\mathrm{End}}(\boldsymbol{H}_{dR})$ endows $\boldsymbol{V}_{dR}$ with a quadratic form

$$\boldsymbol{Q} : \boldsymbol{V}_{dR} \to \mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})},$$

under which $F^1\boldsymbol{V}_{dR}$ is an isotropic line with orthogonal subsheaf $F^0\boldsymbol{V}_{dR}$.

Recall from (6.4.6) the $\mathbb{Z}$-module (6.4.6) of special quasi-endomoprhisms

$$V_0(\mathcal{A}_S) \subset \mathrm{End}(\mathcal{A}_S) \otimes \mathbb{Z}_{(p)}$$

determined by the trivial coset $0 \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. Any $x \in V_0(\mathcal{A}_S)$ has a *de Rham realization* $\boldsymbol{x}_{dR}$, which is a global section of the subsheaf

$$\boldsymbol{V}_{dR,S} \subset \underline{\mathrm{End}}(\boldsymbol{H}_{dR,S}).$$

In particular, de Rham realization defines a morphism of $\mathcal{O}_S$-modules

$$V_0(\mathcal{A}_S) \otimes \mathcal{O}_S \to \boldsymbol{V}_{dR,S}.$$

compatible with the quadratic forms on source and target. In fact, as is clear from the proof of Proposition 6.5.1, the image is contained in $F^0\boldsymbol{V}_{dR,S}$.

Fix a positive definite quadratic space Λ over $\mathbb{Z}$, and consider the stack

$$\mathcal{Z}(\Lambda) \to \mathcal{S}_K(G,\mathcal{D}) \tag{7.1.1}$$

with functor of points

$$\mathcal{Z}(\Lambda)(S) = \{\text{isometric embeddings } \iota : \Lambda \to V_0(\mathcal{A}_S)\}$$

for any morphism $S \to \mathcal{S}_K(G,\mathcal{D})$. As observed in [AGHMP18, §4.4] (see also Lemma 7.1.1 below), this is a Deligne-Mumford stack over $\mathbb{Z}_{(p)}$ whose generic fiber is smooth of dimension $n - \mathrm{rank}(\Lambda)$. Moreover, the morphism (7.1.1) is finite and unramified.

We now briefly recall the deformation theory of these stacks. As in the proof of Proposition 6.5.1, we have a canonical flat connection

$$\nabla : \boldsymbol{V}_{dR} \to \boldsymbol{V}_{dR} \otimes \Omega^1_{\mathcal{S}_K(G,\mathcal{D})/\mathbb{Z}_{(p)}}.$$

This connection satsfies Griffiths's transversality with respect to the Hodge filtration, and the Kodaira-Spencer map associated with it induces an isomorphism

$$F^1\boldsymbol{V}_{dR} \otimes \big(\Omega^1_{\mathcal{S}_K(G,\mathcal{D})/\mathbb{Z}_{(p)}}\big)^{\vee} \cong F^0\boldsymbol{V}_{dR}/F^1\boldsymbol{V}_{dR}.$$

Dualizing, and using the bilinear pairing on $\boldsymbol{V}_{dR}$, we obtain an isomorphism

$$F^0\boldsymbol{V}_{dR}/F^1\boldsymbol{V}_{dR} \cong (\boldsymbol{V}_{dR}/F^0\boldsymbol{V}_{dR}) \otimes \Omega^1_{\mathcal{S}_K(G,\mathcal{D})/\mathbb{Z}_{(p)}}.$$

This is [Mad16, Proposition 4.16], whose proof applies also when $p = 2$; one only has to replace appeals to results from [Kis10] with appeals to the analogous results from [KM16].

Now, suppose that we have a point s of $\mathcal{S}_K(G,\mathcal{D})$ valued in a field k. If $\tilde{s}$ is any lift of s to the ring of dual numbers $k[\epsilon]$, the connection ∇ induces a canonical isomorphism

$$\xi_{\tilde{s}} : \boldsymbol{V}_{dR,s} \otimes_k k[\epsilon] \cong \boldsymbol{V}_{dR,\tilde{s}},$$

and thus gives rise to an isotropic line

$$F^1_{\tilde{s}}(\boldsymbol{V}_{dR,s} \otimes_k k[\epsilon]) \stackrel{\text{def}}{=} \xi_{\tilde{s}}^{-1}(F^1\boldsymbol{V}_{dR,\tilde{s}}) \subset \boldsymbol{V}_{dR,s} \otimes_k k[\epsilon].$$

By construction, this line lifts $F^1\boldsymbol{V}_{dR,s}$.

The properties of the Kodaira-Spencer map mentioned above can now be reinterpreted as saying that the association

$$\tilde{s} \mapsto F^1_{\tilde{s}}(\boldsymbol{V}_{dR,s} \otimes_k k[\epsilon])$$

is a bijection from the tangent space of $\mathcal{S}_K(G,\mathcal{D})$ at s to the space of isotropic lines in $\boldsymbol{V}_{dR,s} \otimes_k k[\epsilon]$ lifting $F^1\boldsymbol{V}_{dR,s}$. This latter space can be canonically identified with the k-vector space

$$\mathrm{Hom}_k(F^1\boldsymbol{V}_{dR,s}, F^0\boldsymbol{V}_{dR,s}/F^1\boldsymbol{V}_{dR,s})$$

as follows: Any lift $F^1_{\tilde{s}}(\boldsymbol{V}_{dR,t} \otimes_k k[\epsilon])$ will be contained in $F^0\boldsymbol{V}_{dR,s} \otimes_k k[\epsilon]$, and so we can consider the associated map

$$F^1_{\tilde{s}}(\boldsymbol{V}_{dR,s} \otimes_k k[\epsilon]) \to (F^0\boldsymbol{V}_{dR,s}/F^1\boldsymbol{V}_{dR,s}) \otimes_k k[\epsilon],$$

which will factor as

$$\begin{tikzcd} F^1_{\tilde{s}}(\boldsymbol{V}_{dR,s} \otimes_k k[\epsilon]) \arrow[r] \arrow[d, "\epsilon\mapsto 0"'] & (F^0\boldsymbol{V}_{dR,s}/F^1\boldsymbol{V}_{dR,s}) \otimes_k k[\epsilon] \\ F^1\boldsymbol{V}_{dR,s} \arrow[r, "\varphi_{\tilde{s}}"'] & F^0\boldsymbol{V}_{dR,s}/F^1\boldsymbol{V}_{dR,s} \arrow[u, "1\otimes\epsilon"'] \end{tikzcd} .$$

The desired identification is now given by the assignment $F^1_{\tilde{s}} \mapsto \varphi_{\tilde{s}}$.

We can say more. Suppose that s lifts to a k-point of $\mathcal{Z}(\Lambda)$ corresponding to an embedding $\Lambda \hookrightarrow V(\mathcal{A}_s)$. We will continue to use s to denote this lift as well. The de Rham realization of the embedding gives a map

$$\Lambda \to F^0\boldsymbol{V}_{dR,s},$$

and we let

$$\boldsymbol{\Lambda}_{dR,s} \subset F^0\boldsymbol{V}_{dR,s}$$

be the k-subspace generated by its image. Now, the bijection from the previous paragraph identifies the tangent space of $\mathcal{Z}(\Lambda)$ at s with the space of isotropic lines in $\boldsymbol{V}_{dR,s} \otimes_k k[\epsilon]$ that lift $F^1\boldsymbol{V}_{dR,s}$ and are also orthogonal to $\boldsymbol{\Lambda}_{dR,s}$. This space in turn can be identified with the k-vector space

$$\mathrm{Hom}_k(F^1\boldsymbol{V}_{dR,s}, \overline{\boldsymbol{\Lambda}}^{\perp}_{dR,s}), \tag{7.1.2}$$

where

$$\overline{\boldsymbol{\Lambda}}_{dR,s} \subset F^0\boldsymbol{V}_{dR,s}/F^1\boldsymbol{V}_{dR,s}$$

is the the image of $\boldsymbol{\Lambda}_{dR,s}$ and $\overline{\boldsymbol{\Lambda}}^{\perp}_{dR,s}$ is its orthogonal complement.

For proofs of the above statements, which use the explicit description of the complete local rings of $\mathcal{S}_K(G,\mathcal{D})$, see [Mad16, Prop. 5.16]. As observed there, they also apply more generally to arbitrary nilpotent divided power thickenings. We record some immediate consequences.

Lemma 7.1.1. *Let the notation be as above, and set $r = \mathrm{rank}(\Lambda)$.*

1. *The completed étale local ring $\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}$ is a quotient of $\widehat{\mathcal{O}}_{\mathcal{S}_K(G,\mathcal{D}),s}$ by an ideal generated by $\mathrm{rank}(\Lambda)$ elements.*
2. *$\mathcal{Z}(\Lambda)$ is smooth at s if and only if $\overline{\mathbf{\Lambda}}_{dR,s}$ has k-dimension $\mathrm{rank}(\Lambda)$. In particular, the generic fiber of $\mathcal{Z}(\Lambda)$ is smooth.*
3. *Suppose that k has characteristic p, and that the Krull dimension of $\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}/(p)$ is $n - \mathrm{rank}(\Lambda)$. Then $\mathcal{Z}(\Lambda)$ and $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ are local complete intersections at s. Moreover, $\mathcal{Z}(\Lambda)$ is flat over $\mathbb{Z}_{(p)}$ at s.*

Proof. The first claim is a consequence of the deformation theory explained above (more precisely, of its generalization to arbitrary square-zero thickenings) and Nakayama's lemma. See [Mad16, Corollary 5.17].

For the second claim, note that $\mathcal{Z}(\Lambda)$ will be smooth at s if and only if its tangent space at s has dimension $n - \mathrm{rank}(\Lambda)$. As we have identified the tangent space with (7.1.2), this is equivalent to $\overline{\mathbf{\Lambda}}_{dR,s}$ having dimension $\mathrm{rank}(\Lambda)$. For the assertion about the generic fiber, it suffices to check the criterion for smoothness at every $\mathbb{C}$-valued point. Now, note that the de Rham realization

$$V_0(\mathcal{A}_s)\otimes_{\mathbb{Z}}\mathbb{C} \to \boldsymbol{V}_{dR,s},$$

is injective, and also that the image of this realization is precisely the weight $(0,0)$ part of the Hodge structure on $\boldsymbol{V}_{dR,s}$, and hence is complementary to $F^1\boldsymbol{V}_{dR,s}$. This implies that $\overline{\mathbf{\Lambda}}_{dR,s}$ has dimension r over $\mathbb{C}$, and hence that $\mathcal{Z}(\Lambda)$ is smooth at s.

Now we come to the third claim. Note that $\widehat{\mathcal{O}}_{\mathcal{S}_K(G,\mathcal{D}),s}$ is formally smooth over $W(k)$ of Krull dimension $n+1$. Hence, $\widehat{\mathcal{O}}_{\mathcal{S}_K(G,\mathcal{D}),s}/(p)$ is also formally smooth over k of Krull dimension n, and $\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}/(p)$, which is its quotient by an ideal generated by $\mathrm{rank}(\Lambda)$ element, is a complete intersection as soon as

$$\dim\big(\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}/(p)\big) = n - \mathrm{rank}(\Lambda).$$

This is precisely our hypothesis.

Now, note that we have

$$n - \mathrm{rank}(\Lambda) + 1 \leqslant \dim(\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}) \leqslant \dim\big(\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}/(p)\big) + 1 = n - \mathrm{rank}(\Lambda) + 1.$$

Here, the first two inequalities follow from Krull's Hauptidealsatz. This shows

$$\dim(\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}) = n - \mathrm{rank}(\Lambda) + 1,$$

and implies that $\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}$ is a complete intersection ring.

Finally, to see that $\mathcal{Z}(\Lambda)$ is flat over $\mathbb{Z}_{(p)}$ at s, note that p cannot be a zero divisor in $\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}$: Indeed, the equality

$$\dim\,(\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}/(p)) = n - \mathrm{rank}(\Lambda) = \dim\ \widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s} - 1$$

implies that p is not contained in any minimal prime of $\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}$. Since $\widehat{\mathcal{O}}_{\mathcal{Z}(\Lambda),s}$ is a complete intersection ring and hence Cohen-Macaulay, this implies that p is not a zero divisor. □

For any morphism $S \to \mathcal{Z}(\Lambda)$, de Rham realization defines a morphism

$$\Lambda \otimes \mathcal{O}_S \to \boldsymbol{V}_{dR,S}.$$

Let $\boldsymbol{\Lambda}_{dR,S} \subset \boldsymbol{V}_{dR,S}$ be the image of this morphism.

We will consider the canonical open substack

$$\mathcal{Z}^{\mathrm{pr}}(\Lambda) \hookrightarrow \mathcal{Z}(\Lambda) \tag{7.1.3}$$

characterized by the property that a morphism $S \to \mathcal{Z}(\Lambda)$ factors through $\mathcal{Z}^{\mathrm{pr}}(\Lambda)$ if and only if $\boldsymbol{\Lambda}_{dR,S} \subset \boldsymbol{V}_{dR,S}$ is a local direct summand of rank equal to $\mathrm{rank}(\Lambda)$.

Proposition 7.1.2. *Consider the following assertions:*

(1) For any generic geometric point η of $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$, the Kuga-Satake abelian scheme $\mathcal{A}_\eta$ is ordinary, and the tautological map $\Lambda \to V_0(\mathcal{A}_\eta)$ is an isomorphism.

(2) The special fiber $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ is a generically smooth local complete intersection of dimension $n - \mathrm{rank}(\Lambda)$.

(3) The special fiber $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ is smooth outside of a codimension 2 subspace.

Then (1) and (2) hold whenever $\mathrm{rank}(\Lambda) \leqslant n/2$, *and (3) holds whenever* $\mathrm{rank}(\Lambda) \leqslant (n-1)/2$.

Proof. We will prove the proposition by induction on the rank of Λ. For any integer $r \geqslant 0$ and $i \in \{1,2,3\}$, let $P_i(r)$ be the statement that assertion (i) is valid whenever $\mathrm{rank}(\Lambda) = r$. We claim

(i) if $0 \leqslant r \leqslant (n-1)/2$ then $P_2(r)$ implies $P_1(r)$,
(ii) if $r \leqslant (n-2)/2$ then $P_1(r)$ and $P_2(r)$ together imply $P_2(r+1)$,
(iii) if $r \leqslant (n-3)/2$ then $P_1(r)$ and $P_2(r)$ together imply $P_3(r+1)$.

Once the claims are proved, the lemma will follow by induction. Indeed, the base case $P_2(0)$ is implied by the smoothness of $\mathcal{S}_K(G,\mathcal{D})$.

The claims themselves follow from an argument derived from [Ogu79], which was used in [Mad16, Proposition 6.17], and exploits the following simple lemma.

Lemma 7.1.3. *Let Z be an $\mathbb{F}_p$-scheme admitting an unramified map $Z \to \mathcal{S}_K(G,\mathcal{D})$. Suppose that we have a local direct summand $\boldsymbol{N} \subset \boldsymbol{V}_{dR}|_Z$ that is horizontal for the integrable connection*

$$\boldsymbol{V}_{dR,Z} \to \boldsymbol{V}_{dR,Z} \otimes_{\mathcal{O}_Z} \Omega^1_{Z/\mathbb{F}_p}$$

induced from the one on $\boldsymbol{V}_{dR}$. *Suppose also that*

$$F^1\boldsymbol{V}_{dR,Z} \subset \boldsymbol{N}.$$

Then $\dim(Z) \leqslant \operatorname{rank}(\boldsymbol{N}) - 1$. *If, in addition* $\boldsymbol{N} \cap F^0\boldsymbol{V}_{dR,Z}$ *is a local direct summand of* $\boldsymbol{V}_{dR,Z}$, *then we in fact have*

$$\dim(Z) \leqslant \operatorname{rank}(\boldsymbol{N} \cap F^0\boldsymbol{V}_{dR,Z}) - 1.$$

Proof. For the first assertion, it is enough to show that, at any point $z \in Z(k)$ valued in a field k, the tangent space of Z at z has dimension at most $\operatorname{rank}(\boldsymbol{N}) - 1$. But our hypotheses imply that, if $\tilde{z} \in Z(k[\epsilon])$ is any lift of Z, then we must have

$$F^1_{\tilde{z}}(\boldsymbol{V}_{dR,Z} \otimes_k k[\epsilon]) \subset (\boldsymbol{N}_z \otimes_k k[\epsilon]) \cap (F^0\boldsymbol{V}_{dR,z} \otimes_k k[\epsilon]).$$

This, combined with the fact that Z is unramified over $\mathcal{S}_K(G,\mathcal{D})$, implies that the tangent space of Z at z can be identified with a subspace of

$$\operatorname{Hom}_k(F^1\boldsymbol{V}_{dR,z}, \overline{\boldsymbol{N}}_z) \subset \operatorname{Hom}_k(F^1\boldsymbol{V}_{dR,z}, F^0\boldsymbol{V}_{dR,z}/F^1\boldsymbol{V}_{dR,z}),$$

where $\overline{\boldsymbol{N}}_z$ is the image of $\boldsymbol{N}_z \cap F^0\boldsymbol{V}_{dR,z}$ in $F^0\boldsymbol{V}_{dR,z}/F^1\boldsymbol{V}_{dR,z}$. We are now done, since $\overline{\boldsymbol{N}}_z$ has dimension at most $\operatorname{rank}(\boldsymbol{N}) - 1$.

The second assertion is immediate from the proof of the first. □

We begin with claim (i). Assume $P_2(r)$, and suppose $\operatorname{rank}(\Lambda) = r$. Fix a geometric generic point η of $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$. Then $P_2(r)$ implies that there is a smooth $\mathbb{F}_p$-scheme U, equidimensional of dimension $n-r$, and an étale map $U \to \mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$, whose image contains η.

As explained in the proof of [Mad16, Proposition 6.17], there is a canonical isotropic line

$$\boldsymbol{C} \subset \boldsymbol{V}_{dR,U},$$

called the *conjugate filtration*, which is horizontal for the connection on $\boldsymbol{V}_{dR,U}$, is contained in $\boldsymbol{\Lambda}^{\perp}_{dR,U}$, and is such that a point $t \in U(k)$ is non-ordinary if and only if $\boldsymbol{C}_t \subset F^0\boldsymbol{V}_{dR,t}$, or, equivalently, if and only if

$$F^1\boldsymbol{V}_{dR,t} \subset \boldsymbol{C}_t^{\perp} \cap \boldsymbol{\Lambda}^{\perp}_{dR,t}.$$

Now, we have

$$\boldsymbol{C}_t \subset \boldsymbol{\Lambda}_{dR,t}$$

only if $F^1\boldsymbol{V}_{dR,t} \subset \boldsymbol{\Lambda}_{dR,t}$. See for instance [Mad16, Lemma 4.20]. Therefore, since we are assuming that U is smooth, the subsheaf

$$\boldsymbol{C}_U + \boldsymbol{\Lambda}_{dR,U} \subset \boldsymbol{V}_{dR,U}$$

is a horizontal local direct summand of rank $r+1$.

By Lemma 7.1.3, if $Z \subset U$ is a closed subscheme with

$$F^1\boldsymbol{V}_{dR,Z} \subset \boldsymbol{C}_Z + \boldsymbol{\Lambda}_{dR,Z},$$

then $\dim Z \leqslant r$. Using $r \leqslant (n-1)/2$, we see that $r = \dim Z < \dim U = n-r$. Therefore, after shrinking U if necessary, we can assume that

$$F^1\boldsymbol{V}_{dR,U} + \boldsymbol{C}_U + \boldsymbol{\Lambda}_{dR,U} \subset \boldsymbol{V}_{dR,U}$$

is a direct summand of rank $r+2$, or, equivalently, that

$$F^0\boldsymbol{V}_{dR,U} \cap \boldsymbol{C}_U^{\perp} \cap \boldsymbol{\Lambda}_{dR,U}^{\perp} \subset \boldsymbol{V}_{dR,U}$$

is a direct summand of rank $n-r$. Therefore, once again by Lemma 7.1.3, the locus in U where $F^1\boldsymbol{V}_{dR,U}$ is contained in this direct summand has dimension at most $n-r-1$. But this is precisely the non-ordinary locus in U. As $\dim(U) = n-r$, this shows the first part of $P_1(r)$.

Suppose now that the map $\Lambda \to V_0(\mathcal{A}_\eta)$ is not a bijection, so that there exists $x \in V_0(\mathcal{A}_\eta)$ such that

$$\widetilde{\Lambda} = \Lambda + \langle x \rangle \subset V_0(\mathcal{A}_\eta)$$

is a direct summand of rank $r+1$, and its de Rham realization

$$\widetilde{\boldsymbol{\Lambda}}_{dR,\eta} = \boldsymbol{\Lambda}_{dR,\eta} + \langle \boldsymbol{x}_{dR,\eta} \rangle \subset \boldsymbol{V}_{dR,\eta}$$

is a $k(\eta)$-vector subspace of dimension $r+1$.

After shrinking U if necessary, we can assume that $x \in V_0(\mathcal{A}_U)$, and that de Rham realization gives us a local direct summand

$$\widetilde{\boldsymbol{\Lambda}}_{dR,U} = \boldsymbol{\Lambda}_{dR,U} + \langle \boldsymbol{x}_{dR,U} \rangle \subset \boldsymbol{V}_{dR,U}$$

of rank $r+1$ that is horizontal for the connection. However, the discussion of the deformation theory above Lemma 7.1.1 implies that, over U, the Kodaira-Spencer map factors through an isomorphism

$$(F^0\boldsymbol{V}_{dR,U}/F^1\boldsymbol{V}_{dR,U})/\overline{\boldsymbol{\Lambda}}_{dR,U} \cong (\boldsymbol{V}_{dR,U}/F^1\boldsymbol{V}_{dR,U}) \otimes_{\mathcal{O}_U} \Omega^1_{U/\mathbb{F}_p}.$$

However, the horizontality of $\widetilde{\boldsymbol{\Lambda}}_{dR,U}$ guarantees that its (non-trivial) image on the left-hand side is in the kernel of the Kodaira-Spencer map. This contradiction finishes the proof of claim (i).

We will prove claims (ii) and (iii). Suppose that $P_1(r)$ and $P_2(r)$ hold and that $\mathrm{rank}(\Lambda) = r+1$. Write $\Lambda = \Lambda_1 \oplus \Lambda_0$, where $\mathrm{rank}(\Lambda_0) = 1$. Then we have an obvious factorization

$$\mathcal{Z}^{\mathrm{pr}}(\Lambda) \to \mathcal{Z}^{\mathrm{pr}}(\Lambda_1) \to \mathcal{S}_K(G,\mathcal{D}).$$

The first arrow exhibits $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ as a divisor on $\mathcal{Z}^{\mathrm{pr}}(\Lambda_1)_{\mathbb{F}_p}$ (étale locally on the source, in the sense of Proposition 6.5.2). Indeed, the complete local rings of the former are cut out by one equation in those of the latter, and $P_1(r)$ shows that $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ does not contain any generic points of $\mathcal{Z}^{\mathrm{pr}}(\Lambda_1)_{\mathbb{F}_p}$. Therefore, by Lemma 7.1.1 and $P_2(r)$, we find that $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ is a local complete intersection of dimension $n-(r+1)$.

Let $W \subset \mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ be the nonsmooth locus, with its reduced substack structure. We find from Lemma 7.1.1 that

$$F^1\boldsymbol{V}_{dR}|_W \subset \boldsymbol{\Lambda}_{dR}|_W.$$

By Lemma 7.1.3, this implies that $\dim(W) \leqslant r$. This is bounded by $n-r-2$ under the hypothesis $r \leqslant (n-2)/2$, and by $n-r-3$ if $r \leqslant (n-3)/2$. This proves (ii) and (iii), and completes the proof of Proposition 7.1.2. $\square$

It will be useful to recall some bounds on the dimension of the supersingular locus in the mod-p fiber of $\mathcal{S}_K(G,\mathcal{D})$ under the assumption that $V_{\mathbb{Z}_p}$ is almost self-dual.

Proposition 7.1.4. *Suppose that $V_{\mathbb{Z}_p}$ is almost self-dual of rank $n+2$, and suppose that $Z \to \mathcal{S}_K(G,\mathcal{D})$ is an unramified morphism from an $\mathbb{F}_p$-scheme Z such that, for all points $z \in Z(k)$ valued in a field k, the abelian variety $\mathcal{A}_z$ is supersingular. Then $\dim(Z) \leqslant n/2$. If $V_{\mathbb{Q}_p}$ is an orthogonal sum of hyperbolic planes, we have the sharper bound*

$$\dim(Z) \leqslant \frac{n}{2} - 1.$$

Proof. If $V_{\mathbb{Q}_p}$ is not an orthogonal sum of hyperbolic planes, then we can find an embedding

$$V_{\mathbb{Z}} \hookrightarrow V^{\diamond}_{\mathbb{Z}},$$

where $V^{\diamond}_{\mathbb{Q}_p}$ is of this form, and where the codimension of $V \subset V^{\diamond}$ is 1 if n is odd and 2 if n is even. Using such an embedding, the proposition can be reduced to proving the final assertion, and so we may assume that $V_{\mathbb{Q}_p}$ (and hence $V_{\mathbb{Z}_p}$) is an orthogonal sum of hyperbolic planes.

When $p > 2$, the proposition follows from the much finer results of [HP17], which give a complete description of the supersingular locus of $\mathcal{S}_K(G,\mathcal{D})_{\mathbb{F}_p}$. However, if one is only interested in upper bounds, one can appeal to the methods of [Ogu01], which apply even when $p = 2$ and $V_{\mathbb{Z}_p}$ is self-dual. See in particular Proposition 14 of [*loc. cit.*]

For the convenience of the reader, we sketch the basic idea here. First, we can replace Z with its underlying reduced scheme. Second, we can throw away its singular part, and assume that Z is smooth.

If $z \in Z(k)$ is a geometric point, then the *Artin invariant* of z is the k-codimension of the image of $V_0(\mathcal{A}_z)\otimes_{\mathbb{Z}} k \to \boldsymbol{V}_{dR,z}$. This is an integer between 1 and $n/2$. Ogus's argument shows that there is a canonical filtration of $F^0\boldsymbol{V}_{dR,Z}$ by coherent, isotropic, horizontal coherent subsheaves

$$\boldsymbol{E}_1 \subset \cdots \boldsymbol{E}_i \ \subset \cdots \subset \boldsymbol{E}_{n/2} \subset F^0\boldsymbol{V}_{dR,Z}$$

with the following properties:

- A geometric point $z \in Z(k)$ has Artin invariant $\leqslant j$ if and only if
$$F^1\boldsymbol{V}_{dR,z} \subset \boldsymbol{E}_{j,z}.$$
- If $Z_{\geqslant j} \subset Z$ is the open subscheme where the Artin invariant is $\geqslant j$, then $\boldsymbol{E}_{j,Z_{\geqslant j}}$ is a rank j local direct summand of $\boldsymbol{V}_{dR,Z_{\geqslant j}}$.

Note that the first condition ensures that locus where the Artin invariant is bounded below by j is indeed an open subscheme of Z.

Given these two properties, it is immediate from Lemma 7.1.3 that the dimension of Z is bounded above by $r-1$, where r is the maximal Artin invariant attained by a geometric point of Z. This proves the proposition.

The construction of $\boldsymbol{E}_j$ is as follows. For $j = 1$, $\boldsymbol{E}_1$ is just the conjugate filtration $\boldsymbol{C} \subset \boldsymbol{V}_{dR,Z}$ already encountered in the proof of Proposition 7.1.2.

The crystalline Frobenius on the crystalline realization of $\mathcal{A}_Z$ induces an isometry

$$\gamma : \mathrm{Fr}_Z^*(F^0\boldsymbol{V}_{dR,Z}/F^1\boldsymbol{V}_{dR,Z}) \cong \boldsymbol{C}^\perp/\boldsymbol{C},$$

where Fr_Z is the absolute Frobenius on Z. Now inductively define $\boldsymbol{E}_j \subset \boldsymbol{C}^\perp$ as the pre-image of the image of $\boldsymbol{E}_{j-1}$ under the composition

$$\mathrm{Fr}_Z^*\boldsymbol{E}_{j-1} \hookrightarrow \mathrm{Fr}_Z^* F^0\boldsymbol{V}_{dR,Z} \xrightarrow{\gamma} \boldsymbol{C}^\perp/\boldsymbol{C}.$$

It follows from the argument in [Ogu01, Lemma 5] that $\boldsymbol{E}_j$ is a subsheaf of $F^0\boldsymbol{V}_{dR,Z}$ for all j, so that the inductive procedure is well-defined. That it is isotropic, coherent and horizontal follows from the construction. That the filtration thus obtained has the desired properties follows from the arguments in Proposition 6 and Lemma 9 of [*loc. cit.*]. □

Lemma 7.1.5. *Suppose that Λ is maximal at p. The complement of $\mathcal{Z}^{\mathrm{pr}}(\Lambda)$ in $\mathcal{Z}(\Lambda)$ lies above the supersingular locus of $\mathcal{S}_K(G,\mathcal{D})_{\mathbb{F}_p}$. If we let*

$$m = \begin{cases} \frac{n}{2} & \textit{if } V_{\mathbb{Z}_p} \textit{ is an orthogonal sum of hyperbolic planes} \\ \lfloor\frac{n}{2}\rfloor & \textit{if } n \textit{ is odd} \\ \frac{n}{2}-1 & \textit{otherwise,} \end{cases}$$

then the following properties hold.

1. *If* $\mathrm{rank}(\Lambda) \leqslant m$ *then* $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ *is dense in* $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$.
2. *If* $\mathrm{rank}(\Lambda) \leqslant m-1$ *then the complement of* $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ *in* $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ *has codimension at least* 2.

Proof. Once we know that the complement is supported above the supersingular locus of the mod-p fiber, the rest will follow from the bounds in Proposition 7.1.4.

To prove the assertion on the complement, we first note that the open immersion (7.1.3) induces an isomorphism of the generic fibers; see [Mad16, Prop. 6.16]. Therefore, we only have to show that the mod-p fiber of the complement is supported on the supersingular locus. Equivalently, we must show that, for any non-supersingular point $s \in \mathcal{Z}(\Lambda)(k)$ valued in a field k of characteristic p, the subspace $\boldsymbol{\Lambda}_{dR,s} \subset \boldsymbol{V}_{dR,s}$ has k-dimension $\mathrm{rank}(\Lambda)$.

Arguing as in [Mad16, §6.27], we find that, for such a point s, the de Rham realization map

$$V_0(\mathcal{A}_s) \otimes k \to \boldsymbol{V}_{dR,s}$$

is injective. Moreover, by the maximality of Λ at p, the image of

$$\Lambda \otimes \mathbb{Z}_{(p)} \to V_0(\mathcal{A}_s) \otimes \mathbb{Z}_{(p)}$$

is a $\mathbb{Z}_{(p)}$-module direct summand of rank $\mathrm{rank}(\Lambda)$. Combining these two observations shows that the subspace $\boldsymbol{\Lambda}_{dR,s} \subset \boldsymbol{V}_{dR,s}$ has k-dimension $\mathrm{rank}(\Lambda)$, and completes the proof of the lemma. □

Proposition 7.1.6. *Suppose that Λ is maximal at p, and let m be defined as in Lemma 7.1.5.*

(1) *If* $\mathrm{rank}(\Lambda) \leqslant m$ *then* $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ *is a generically smooth local complete intersection of dimension* $n - \mathrm{rank}(\Lambda)$*. Moreover,* $\mathcal{Z}(\Lambda)$ *is normal and flat over* $\mathbb{Z}_{(p)}$*.*
(2) *If* $\mathrm{rank}(\Lambda) \leqslant m-1$ *then* $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ *is geometrically normal.*

Proof. Note that we always have

$$m \leqslant \frac{n}{2} \quad \text{and} \quad m-1 \leqslant \frac{n-1}{2}.$$

First suppose $\mathrm{rank}(\Lambda) \leqslant m$. Combining Proposition 7.1.2 and Lemma 7.1.5 shows that $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ is a generically smooth local complete intersection of dimension $n - \mathrm{rank}(\Lambda)$, and is dense in $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$. Hence $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ is itself generically smooth of dimension $n - \mathrm{rank}(\Lambda)$.

It now follows from claim (3) of Lemma 7.1.1 that $\mathcal{Z}(\Lambda)$ is a local complete intersection, flat over $\mathbb{Z}_{(p)}$. In particular, it is Cohen-Macaualy and so satisfies Serre's property (S_k) for all $k \geqslant 1$. Recall from claim (2) of Lemma 7.1.1 that the generic fiber of $\mathcal{Z}(\Lambda)$ is smooth over $\mathbb{Q}$. As we have already proved that the special fiber is generically smooth, $\mathcal{Z}(\Lambda)$ is regular in codimension one, and hence satisfies Serre's property (R_1). Claim (2) now follows from Serre's criterion for normality.

Now suppose $\mathrm{rank}(\Lambda) \leqslant m-1$. We have already shown that the geometric fiber of $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ is a local complete intersection. So, just as above, to show that it is normal it is enough to show that it is regular in codimension one. This follows by combining Proposition 7.1.2 and Lemma 7.1.5, which shows that $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p}$ is smooth outside of a codimension two subspace, and that its complement in $\mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ has codimension at least 2.

□

7.2. **Normality of the fibers, and flatness of divisors.** We return to the general setting in which $V_{\mathbb{Z}} \subset V$ is any maximal lattice, and deduce two important consequences from the results of §7.1.

Proposition 7.2.1. *If* $n \geqslant 6$*, the special fiber of* $\mathcal{S}_K(G,\mathcal{D})$ *is geometrically normal.*

Proof. When $p > 2$, this is part of [AGHMP18, Theorem 4.4.5]. The same idea of proof works in general, bolstered now by Proposition 7.1.6.

Using Lemma 6.2.1, we may choose an embedding $V_{\mathbb{Z}} \hookrightarrow V_{\mathbb{Z}}^{\diamond}$ as in §6.2 in such a way that $V_{\mathbb{Z}}^{\diamond}$ is self-dual at p, and

$$\Lambda = \{x \in V_{\mathbb{Z}}^{\diamond} : x \perp V_{\mathbb{Z}}\}$$

has rank at most r, where $r = 2$ if n is even and $r = 3$ otherwise.[8]

[8] If $p \neq 2$ we can choose $V_{\mathbb{Z}}^{\diamond}$ to be self-dual at p with $r = 2$. In this case, we can improve the bound to $n \geqslant 5$ as in [AGHMP18, Theorem 4.4.5].

There is a commutative diagram

$$\begin{tikzcd} & \mathcal{Z}^\diamond(\Lambda) \arrow[d] \\ \mathcal{S}_K(G,\mathcal{D}) \arrow[r] \arrow[ur] & \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond) \end{tikzcd}$$

in which the vertical morphism is defined as in (7.1.1), the horizontal morphism is (6.6.1), and the diagonal arrow is induced by the isometric embedding

$$\Lambda \to V_0(\mathcal{A}^\diamond_{\mathcal{S}_K(G,\mathcal{D})}).$$

determined by (6.6.3).

The self-duality of $V^\diamond_{\mathbb{Z}_p}$ gives us an isomorphism

$$V^\vee_{\mathbb{Z}_p}/V_{\mathbb{Z}_p} \cong \Lambda^\vee_{\mathbb{Z}_p}/\Lambda_{\mathbb{Z}_p}$$

of quadratic spaces over $\mathbb{Q}_p/\mathbb{Z}_p$, as in (6.4.10). The maximality of $V_\mathbb{Z}$ at p implies that the left hand side contains no nonzero isotropic vectors, and so neither does the right hand side. This implies the maximality of Λ at p. With this in hand, we may apply Proposition 7.1.6 and the inequality

$$r \leqslant \frac{n+r}{2} - 2,$$

which holds as $n \geqslant 6$, to see that $\mathcal{Z}^\diamond(\Lambda)$ has geometrically normal fibers.

Thus it suffices to show that the diagonal arrow is an open and closed immersion. This holds in the generic fiber by [Mad16, Lemma 7.1], and hence also on the level of integral models as the source and target are both normal. □

Proposition 7.2.2. *Assume that $n \geqslant 4$. For every positive $m \in \mathbb{Q}$ and $\mu \in V^\vee_\mathbb{Z}/V_\mathbb{Z}$, the special divisor $\mathcal{Z}(m,\mu)$ is flat over $\mathbb{Z}_{(p)}$.*

Proof. When $p > 2$ this is [AGHMP18, Proposition 4.5.8]. We explain how to extend the proof to the general case.

As in the proof of Proposition 7.2.1 fix an embedding $V_\mathbb{Z} \hookrightarrow V^\diamond_\mathbb{Z}$ with $V^\diamond_\mathbb{Z}$ self-dual at p, and so that

$$\Lambda = \{x \in V^\diamond_\mathbb{Z} : x \perp V_\mathbb{Z}\}$$

is maximal of rank at most r with $r = 2$ when n is even and $r = 3$ otherwise.[9]

Consider again the finite unramified morphism

$$\mathcal{Z}^\diamond(\Lambda) \to \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond).$$

By Proposition 7.1.6, this is normal and flat over $\mathbb{Z}_{(p)}$, as long as we have

$$2 \leqslant \frac{n+2}{2} - 1,$$

[9]Once again, if $p > 2$, then we can always take $r = 2$ and the result can be strengthened to only require $n \geqslant 3$.

for n even and

$$3 \leqslant \frac{n+3}{2} - 1,$$

for n odd. These inequalities hold for $n \geqslant 4$.

Using the decomposition (6.6.4), we may choose a positive $m^\diamond \in \mathbb{Q}$ and a $\mu^\diamond \in (V_\mathbb{Z}^\diamond)^\vee / V_\mathbb{Z}^\diamond$ in such a way that

$$\mathcal{Z}(m,\mu) \subset \mathcal{Z}^\diamond(m^\diamond,\mu^\diamond) \times_{\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)} \mathcal{S}_K(G,\mathcal{D})$$

as an open and closed substack. Now use the open and closed immersion

$$\mathcal{S}_K(G,\mathcal{D}) \to \mathcal{Z}^\diamond(\Lambda)$$

from the proof of Proposition 7.2.1 to identify

$$\mathcal{Z}(m,\mu) \subset \mathcal{Z}^\diamond(m^\diamond,\mu^\diamond) \times_{\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)} \mathcal{Z}^\diamond(\Lambda) \tag{7.2.1}$$

as a union of connected components. In particular, by Proposition 6.5.2, the projection

$$\mathcal{Z}(m,\mu) \to \mathcal{Z}^\diamond(\Lambda) \tag{7.2.2}$$

is, étale locally on the target, a disjoint union of closed immersions each defined by a single equation.

Lemma 7.2.3. *The image of (7.2.2) contains no irreducible component of $\mathcal{Z}^\diamond(\Lambda)_{\mathbb{F}_p}$.*

Proof. An S-point of $\mathcal{Z}(m,\mu)$ determines a special quasi-endomorphism $x \in V(\mathcal{A}_S)$ with $Q(x) = m$. The image of such an S-point under the inclusion (7.2.1) determines an $x^\diamond \in V(\mathcal{A}_S^\diamond)$, as well as an isometric embedding $\iota : \Lambda \to V_0(\mathcal{A}_S)$. Unpacking the construction of the inclusion (7.2.1), we find that the orthogonal decomposition

$$V(\mathcal{A}_S^\diamond) = V(\mathcal{A}_S) \oplus \Lambda_\mathbb{Q},$$

of Proposition 6.6.2 identifies $x^\diamond = x + \iota(\lambda)$ for some $\lambda \in \Lambda_\mathbb{Q}$. In particular, x determines a nonzero element of $V(\mathcal{A}_S^\diamond)$ orthogonal to $\iota(\Lambda_\mathbb{Q})$, and

$$\iota : \Lambda_\mathbb{Q} \to V(\mathcal{A}_S^\diamond)$$

is not surjective.

In contrast, for every generic point η of $\mathcal{Z}^\diamond(\Lambda)_{\mathbb{F}_p}$ we have

$$\iota_\eta : \Lambda \cong V_0(\mathcal{A}_\eta^\diamond).$$

Indeed, this follows from the density $\mathcal{Z}^{\mathrm{pr}}(\Lambda)_{\mathbb{F}_p} \subset \mathcal{Z}(\Lambda)_{\mathbb{F}_p}$ proved in Lemma 7.1.5, and assertion (1) of Proposition 7.1.2. It can be checked that the numerical hypotheses hold under our hypothesis $n \geqslant 4$.

Thus the image of (7.2.2) cannot contain the generic point of any irreducible component of $\mathcal{Z}^\diamond(\Lambda)_{\mathbb{F}_p}$, completing the proof of the lemma. $\square$

To complete the proof of Proposition 7.2.2, we apply the following lemma to the complete local ring of the local complete intersection (and hence Cohen-Macaulay) stack $\mathcal{Z}^\diamond(\Lambda)$ at a point in the image of (7.2.2), and taking a to be the equation defining the complete local ring of $\mathcal{Z}(m,\mu)$ at a point in the pre-image.

Lemma 7.2.4. *Let R be a complete local flat $\mathbb{Z}_{(p)}$-algebra that is Cohen-Macaulay. Suppose that $a \in R$ is such that $\mathrm{Spec}(R/aR) \subset \mathrm{Spec}\, R$ does not contain any irreducible component of $\mathrm{Spec}(R \otimes_{\mathbb{Z}_{(p)}} \mathbb{F}_p)$. Then R/aR is also flat over $\mathbb{Z}_{(p)}$.*

Proof. Since R is $\mathbb{Z}_{(p)}$-flat, R/pR is once again Cohen-Macaulay. Our hypotheses imply that the image $\overline{a} \in R/pR$ of a is not contained in any minimal prime of R/pR, which means that $\overline{a}$ is a non-zero divisor in R/pR. Since R is local, this is equivalent to saying that p is a non-zero divisor in R/aR, which shows that R/aR is $\mathbb{Z}_{(p)}$-flat. □

This completes the proof of Proposition 7.2.2 □

8. Integral theory of q-expansions

Keep the hypotheses and notation of §6 and §7. In particular, we fix a prime p at which $V_{\mathbb{Z}} \subset V$ is maximal. We now consider toroidal compactifications of the integral model

$$\mathcal{S}_K(G,\mathcal{D}) \to \mathrm{Spec}(\mathbb{Z}_{(p)})$$

If V is anisotropic then [Mad19, Corollary 4.1.7] shows that the integral model is already proper. Therefore, in this subsection, we assume that V admits an isotropic vector.

8.1. **Toroidal compactification.** Fix auxiliary data $V^\diamond_{\mathbb{Z}} \subset V^\diamond$ and $K^\diamond$ as in §6.2, and choose this in such a way that $V^\diamond_{\mathbb{Z}}$ is almost self-dual at p. In particular, from (6.3.2) we have the finite morphism

$$\mathcal{S}_K(G,\mathcal{D}) \to \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)$$

of integral models, under which $\boldsymbol{\omega}^\diamond$ pulls back to $\boldsymbol{\omega}$.

We may choose the auxiliary $V^\diamond$ to have signature $(n^\diamond, 2)$ with $n^\diamond \geqslant 5$. By Lemma 6.1.3, this allows us to choose a symplectic form $\psi^\diamond$ on

$$H^\diamond = C(V^\diamond)$$

in such a way that the $\mathbb{Z}$-lattice $H^\diamond_{\mathbb{Z}} = C(V^\diamond_{\mathbb{Z}})$ is self-dual at p. As in §4.3 we obtain an embedding

$$(G^\diamond,\mathcal{D}^\diamond) \to (G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}})$$

into the Siegel Shimura datum determined by $(H^\diamond,\psi^\diamond)$. Recalling the Shimura datum $(\mathbb{G}_m,\mathcal{H}_0)$ of §3.5, this also fixes a morphism $(G^\diamond,\mathcal{D}^\diamond) \to (\mathbb{G}_m,\mathcal{H}_0)$.

Define reductive groups over $\mathbb{Z}_{(p)}$by

$$\mathcal{G}^\diamond = \mathrm{GSpin}(V^\diamond_{\mathbb{Z}_{(p)}}), \quad \mathcal{G}^{\mathrm{Sg}} = \mathrm{GSp}(H^\diamond_{\mathbb{Z}_{(p)}}),$$

so that $G^\diamond \to G^{\mathrm{Sg}}$ extends to a closed immersion $\mathcal{G}^\diamond \to \mathcal{G}^{\mathrm{Sg}}$. Fix a compact open subgroup

$$K^{\mathrm{Sg}} = K_p^{\mathrm{Sg}} K^{\mathrm{Sg},p} \subset G^{\mathrm{Sg}}(\mathbb{A}_f)$$

containing $K^\diamond$ and satisfying $K_p^{\mathrm{Sg}} = \mathcal{G}^{\mathrm{Sg}}(\mathbb{Z}_p)$. After shrinking the prime-to-p parts of

$$K \subset K^\diamond \subset K^{\mathrm{Sg}},$$

we assume that all three are neat.

We can construct a toroidal compactification of $\mathcal{S}_K(G,\mathcal{D})$ as follows. Fix a finite, complete K^{Sg}-admissible cone decomposition Σ^{Sg} for $(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}})$. As explained in §2.5, it pulls back to a finite, complete, $K^\diamond$-admissible polyhedral cone decomposition $\Sigma^\diamond$ for $(G^\diamond,\mathcal{D}^\diamond)$, and a finite, complete, K-admissible polyhedral cone decomposition Σ for $(G,\mathcal{D})$. If Σ^{Sg} has the no self-intersection property, then so do the decompositions induced from it.

Assume that K^{Sg} and Σ^{Sg} are chosen so that Σ^{Sg} is smooth and satisfies the no self-intersection property. We obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{S}_K(G,\mathcal{D},\Sigma) \arrow[d] & \mathrm{Sh}_K(G,\mathcal{D},\Sigma) \arrow[l] \arrow[d] \\
\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond) \arrow[d] & \mathrm{Sh}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond) \arrow[l] \arrow[d] \\
\mathcal{S}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}},\Sigma^{\mathrm{Sg}}) & \mathrm{Sh}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}},\Sigma^{\mathrm{Sg}}), \arrow[l]
\end{tikzcd} \tag{8.1.1}$$

where $\mathcal{S}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}},\Sigma^{\mathrm{Sg}})$ is the toroidal compactification of $\mathcal{S}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}})$ constructed by Faltings-Chai. Note that the neatness of K^{Sg} implies that it is an algebraic space, rather than a stack, but does not guarantee that it is a scheme. The two algebraic spaces above it are defined by normalization, exactly as in (6.3.2).

According to [Mad19, Theorem 4.1.5], the algebraic space $\mathcal{S}_K(G,\mathcal{D},\Sigma)$ is proper over $\mathbb{Z}_{(p)}$ and admits a stratification

$$\mathcal{S}_K(G,\mathcal{D},\Sigma) = \bigsqcup_{(\Phi,\sigma)\in \mathrm{Strat}_K(G,\mathcal{D},\Sigma)} \mathcal{Z}_K^{(\Phi,\sigma)}(G,\mathcal{D},\Sigma) \tag{8.1.2}$$

by locally closed subspaces, extending (2.6.1), in which every stratum is flat over $\mathbb{Z}_{(p)}$. The unique open stratum is $\mathcal{S}_K(G,\mathcal{D})$, and its complement is a Cartier divisor.

Fix a toroidal stratum representative $(\Phi,\sigma) \in \mathrm{Strat}_K(G,\mathcal{D},\Sigma)$ in such a way that the parabolic subgroup underlying Φ is the stabilizer of an isotropic line. As in §2.3, the cusp label representative Φ determines a T_Φ-torsor

$$\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi) \to \mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m,\mathcal{H}_0),$$

and the rational polyhedral cone σ determines a partial compactification

$$\mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi) \hookrightarrow \mathrm{Sh}_{K_\Phi}(Q_\Phi,\mathcal{D}_\Phi,\sigma).$$

The base $\mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m, \mathcal{H}_0)$ of the T_Φ-torsor, being a zero dimensional étale scheme over $\mathbb{Q}$, has a canonical finite normal integral model defined as the normalization of $\mathrm{Spec}(\mathbb{Z}_{(p)})$. The picture is

$$\begin{tikzcd} \mathcal{S}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m, \mathcal{H}_0) \arrow[d] & \mathrm{Sh}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m, \mathcal{H}_0) \arrow[l] \arrow[d] \\ \mathrm{Spec}(\mathbb{Z}_{(p)}) & \mathrm{Spec}(\mathbb{Q}). \arrow[l] \end{tikzcd}$$

Proposition 8.1.1. *Define an integral model*

$$\mathcal{T}_\Phi = \mathrm{Spec}\Big(\mathbb{Z}_{(p)}[q_\alpha]_{\alpha \in \Gamma_\Phi^\vee(1)}\Big)$$

of the torus T_Φ of §2.3.

1. *The $\mathbb{Q}$-scheme $\mathrm{Sh}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi)$ admits a canonical integral model*
$$\mathcal{S}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) \to \mathrm{Spec}(\mathbb{Z}_{(p)}),$$
endowed with the structure of a relative $\mathcal{T}_\Phi$-torsor
$$\mathcal{S}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) \to \mathcal{S}_{\nu_\Phi(K_\Phi)}(\mathbb{G}_m, \mathcal{H}_0)$$
compatible with the torsor structure (2.3.1) in the generic fiber.
2. *There is a canonical isomorphism*
$$\widehat{\mathcal{S}}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma) \cong \widehat{\mathcal{S}}_K(G, \mathcal{D}, \Sigma)$$
of formal algebraic spaces extending (2.6.3).

Here $\mathcal{S}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi) \hookrightarrow \mathcal{S}_{K_\Phi}(Q_\Phi, \mathcal{D}_\Phi, \sigma)$ is the partial compactification determined by the rational polyhedral cone

$$\sigma \subset U_\Phi(\mathbb{R})(-1) = \mathrm{Hom}(\mathbb{G}_m, \mathcal{T}_\Phi)_\mathbb{R}$$

and the formal scheme on the left hand side is its completion along its unique closed stratum. On the right,

$$\widehat{\mathcal{S}}_K(G, \mathcal{D}, \Sigma) = \mathcal{S}_K(G, \mathcal{D}, \Sigma)^\wedge_{\mathcal{Z}_K^{(\Phi,\sigma)}(G,\mathcal{D},\Sigma)}$$

is the formal completion along the stratum indexed by (Φ, σ).

Proof. This is a consequence of [Mad19, Theorem 4.1.5]. □

By [Mad19, Theorem 2] and [FC90], both $\mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond)$ and the Faltings-Chai compactification are proper. They admit stratifications

$$\mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond) = \bigsqcup_{(\Phi^\diamond, \sigma^\diamond) \in \mathrm{Strat}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond)} \mathcal{Z}_{K^\diamond}^{(\Phi^\diamond, \sigma^\diamond)}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond),$$

and

$$\mathcal{S}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}}, \Sigma^{\mathrm{Sg}}) = \bigsqcup_{(\Phi^{\mathrm{Sg}}, \sigma^{\mathrm{Sg}}) \in \mathrm{Strat}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}})} \mathcal{Z}_{K^{\mathrm{Sg}}}^{(\Phi^{\mathrm{Sg}}, \sigma^{\mathrm{Sg}})}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}}, \Sigma^{\mathrm{Sg}}).$$

analogous to (8.1.2). By [Mad19, (4.1.13)], these stratifications satisfy a natural compatibility: if

$$(\Phi, \sigma) \in \mathrm{Strat}_K(G, \mathcal{D}, \Sigma)$$

has images $(\Phi^\diamond, \sigma^\diamond)$ and $(\Phi^{\mathrm{Sg}}, \sigma^{\mathrm{Sg}})$, in the sense of §2.5, then the maps in (8.1.1) induce maps on strata

$$\mathcal{Z}_K^{(\Phi,\sigma)}(G, \mathcal{D}, \Sigma) \to \mathcal{Z}_{K^\diamond}^{(\Phi^\diamond,\sigma^\diamond)}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond) \to \mathcal{Z}_{K^{\mathrm{Sg}}}^{(\Phi^{\mathrm{Sg}},\sigma^{\mathrm{Sg}})}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}}, \Sigma^{\mathrm{Sg}}).$$

Applying the functor of Proposition 8.1.2 below to the $\mathcal{G}^\diamond$-representation $V^\diamond_{\mathbb{Z}_{(p)}}$ yields a line bundle $\boldsymbol{\omega}^\diamond = F^1 \boldsymbol{V}^\diamond_{dR}$ on $\mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond)$, which we pull back to a line bundle $\boldsymbol{\omega}$ on $\mathcal{S}_K(G, \mathcal{D}, \Sigma)$. This gives an extension of (6.3.3) to the toroidal compactification.

Proposition 8.1.2. *There is a functor*

$$N \mapsto (\boldsymbol{N}_{dR}, F^\bullet \boldsymbol{N}_{dR})$$

from representations $\mathcal{G}^\diamond \to \mathrm{GL}(N)$ on free $\mathbb{Z}_{(p)}$-modules of finite rank to filtered vectors bundles on $\mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond)$, extending the functor (6.3.1) on the open stratum, and the functor of Theorem 3.4.1 in the generic fiber.

Proof. Consider the filtered vector bundle $(\boldsymbol{H}^\diamond_{dR}, F^\bullet \boldsymbol{H}^\diamond_{dR})$ over $\mathrm{Sh}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond)$ obtained by applying the functor (3.3.2) to the representation

$$G^\diamond \to G^{\mathrm{Sg}} = \mathrm{GSp}(H^\diamond).$$

Now let $\nu^\diamond : G^\diamond \to \mathbb{G}_m$ be the spinor similitude, and let $\mathbb{Q}(\nu^\diamond)$ denote the corresponding one-dimensional representation of $G^\diamond$. It determines a line bundle on $\mathrm{Sh}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond)$, which is canonically a pullback via the morphism

$$\mathrm{Sh}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond) \xrightarrow{\nu^\diamond} \mathrm{Sh}_{\nu^\diamond(K^\diamond)}(\mathbb{G}_m, \mathcal{H}_0).$$

Combining this with Remark 3.5.2, we see that the line bundle determined by $\mathbb{Q}(\nu^\diamond)$ is canonically identified with $\mathrm{Lie}(\mathbb{G}_m)$, and hence the $G^\diamond$-equivariant morphism $\psi^\diamond : H \otimes H \to \mathbb{Q}(\nu^\diamond)$ induces an alternating form

$$\psi^\diamond : \boldsymbol{H}^\diamond_{dR} \otimes \boldsymbol{H}^\diamond_{dR} \to \mathrm{Lie}(\mathbb{G}_m).$$

The nontrivial step $F^0 \boldsymbol{H}^\diamond_{dR}$ in the filtration is a Lagrangian subsheaf with respect to this pairing.

The vector bundle $\boldsymbol{H}^\diamond_{dR}$ is canonically identified with the pullback via

$$\mathrm{Sh}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond) \to \mathrm{Sh}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}})$$

of the first relative homology

$$\boldsymbol{H}^{\mathrm{Sg}}_{dR} = \underline{\mathrm{Hom}}\big(R^1\pi_* \Omega^\bullet_{A^{\mathrm{Sg}}/\mathrm{Sh}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}})}, \mathcal{O}_{\mathrm{Sh}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}})}\big)$$

of the universal polarized abelian scheme $\pi : A^{\mathrm{Sg}} \to \mathrm{Sh}_{K^{\mathrm{Sg}}}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}})$. As the universal abelian scheme extends canonically to the integral model, so does $\boldsymbol{H}^{\mathrm{Sg}}_{dR}$. Its pullback defines an extension of $\boldsymbol{H}^\diamond_{dR}$, along with its filtration and alternating form, to the integral model $\mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond)$.

Now fix a family of tensors

$$\{s_\alpha\} \subset H^{\diamond,\otimes}_{\mathbb{Z}_{(p)}}$$

that cut out the reductive subgroup $\mathcal{G}^\diamond \subset \mathcal{G}^{\mathrm{Sg}}$. The functoriality of (3.3.2) implies that these tensors define global sections $\{\boldsymbol{s}_{\alpha,dR}\}$ of $\boldsymbol{H}^{\diamond,\otimes}_{dR}$ over the generic fiber. By [Kis10, Corollary 2.3.9], they extend (necessarily uniquely) to sections over the integral model $\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond)$.

By [Mad19, Proposition 4.3.7], the filtered vector bundle $(\boldsymbol{H}^\diamond_{dR}, F^\bullet\boldsymbol{H}^\diamond_{dR})$ admits a canonical extension to $\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$. The alternating form $\psi^\diamond$ and the sections $\boldsymbol{s}_{\alpha,dR}$ also extend (necessarily uniquely).

This allows us to define a $\mathcal{G}^\diamond$-torsor

$$\mathcal{J}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond) \xrightarrow{a} \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$$

whose functor of points assigns to a scheme $S \to \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$ the set of all pairs (f,f_0) of isomorphisms

$$f : \boldsymbol{H}^\diamond_{dR/S} \cong H^\diamond_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_S, \quad f_0 : \mathrm{Lie}(\mathbb{G}_m)_{/S} \cong \mathcal{O}_S \tag{8.1.3}$$

satisfying $f(\boldsymbol{s}_{\alpha,dR}) = s_\alpha \otimes 1$ for all α, and making the diagram

$$\begin{tikzcd} \boldsymbol{H}^\diamond_{dR/S} \otimes \boldsymbol{H}^\diamond_{dR/S} \arrow[r, "\psi^\diamond"] \arrow[d, "f\otimes f"'] & \mathrm{Lie}(\mathbb{G}_m) \arrow[d, "f_0"] \\ (H^\diamond_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_S) \otimes (H^\diamond_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_S) \arrow[r, "\psi^\diamond"] & \mathcal{O}_S \end{tikzcd}$$

commute.

Define smooth $\mathbb{Z}_{(p)}$-schemes $\check{\mathcal{M}}^\diamond$ and $\check{\mathcal{M}}^{\mathrm{Sg}}$ with functors of points

$$\check{\mathcal{M}}(G^\diamond,\mathcal{D}^\diamond)(S) = \{\text{isotropic lines } z \subset V^\diamond_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_S\}$$
$$\check{\mathcal{M}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}})(S) = \{\text{Lagrangian subsheaves } F^0 \subset H^\diamond_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_S\}.$$

These are integral models of the compact duals $\check{M}(G^\diamond,\mathcal{D}^\diamond)$ and $\check{M}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}})$ of §4.3, and are related, using (4.1.2), by a closed immersion

$$\check{\mathcal{M}}(G^\diamond,\mathcal{D}^\diamond) \to \check{\mathcal{M}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}}) \tag{8.1.4}$$

sending the isotropic line $z \subset V^\diamond_{\mathbb{Z}_{(p)}}$ to the Lagrangian $zH^\diamond_{\mathbb{Z}_{(p)}} \subset H^\diamond_{\mathbb{Z}_{(p)}}$.

We now have a diagram

$$\begin{tikzcd} \mathcal{J}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond) \arrow[r, "b"] \arrow[d, "a"'] & \check{\mathcal{M}}(G^\diamond,\mathcal{D}^\diamond) \\ \mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond) & \end{tikzcd} \tag{8.1.5}$$

in which a is a $\mathcal{G}^\diamond$-torsor and b is $\mathcal{G}^\diamond$-equivariant, extending the diagram (3.4.1) already constructed in the generic fiber. To define the morphism b we first define a morphism

$$\mathcal{J}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond) \to \check{\mathcal{M}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}})$$

by sending an S-point (f, f_0) to the Lagrangian subsheaf

$$f(F^0\boldsymbol{H}_{dR/S}) \subset H_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_S.$$

This morphism factors through (8.1.4). Indeed, as (8.1.4) is a closed immersion, this is a formal consequence of the fact that we have such a factorization in the generic fiber, as can be checked using the analogous complex analytic construction.

With the diagram (8.1.5) in hand, the construction of the desired functor proceeds by simply imitating the construction (3.3.2) used in the generic fiber. $\square$

8.2. **Integral q-expansions.** Continue with the assumptions of §8.1, and now fix a toroidal stratum representative

$$(\Phi, \sigma) \in \mathrm{Strat}_K(G, \mathcal{D}, \Sigma)$$

as in §4.6. Thus $\Phi = (P, \mathcal{D}^\circ, h)$ with P the stabilizer of an isotropic line $I \subset V$, and $\sigma \in \Sigma_\Phi$ is a top dimensional rational polyhedral cone. Let

$$(\Phi^\diamond, \sigma^\diamond) \in \mathrm{Strat}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond)$$

be the image of (Φ, σ), in the sense of §2.5.

The formal completions along the corresponding strata

$$\begin{aligned} \mathcal{Z}_K^{(\Phi,\sigma)}(G, \mathcal{D}, \Sigma) &\subset \mathcal{S}_K(G, \mathcal{D}, \Sigma) \\ \mathcal{Z}_{K^\diamond}^{(\Phi^\diamond,\sigma^\diamond)}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond) &\subset \mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond), \end{aligned} \tag{8.2.1}$$

are denoted

$$\begin{aligned} \widehat{\mathcal{S}}_K(G, \mathcal{D}, \Sigma) &= \mathcal{S}_K(G, \mathcal{D}, \Sigma)^\wedge_{\mathcal{Z}_K^{(\Phi,\sigma)}(G,\mathcal{D},\Sigma)} \\ \widehat{\mathcal{S}}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond) &= \mathcal{S}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond)^\wedge_{\mathcal{Z}_{K^\diamond}^{(\Phi^\diamond,\sigma^\diamond)}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)}. \end{aligned}$$

These are formal algebraic spaces over $\mathbb{Z}_{(p)}$ related by a finite morphism

$$\widehat{\mathcal{S}}_K(G, \mathcal{D}, \Sigma) \to \widehat{\mathcal{S}}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond). \tag{8.2.2}$$

Fix a $\mathbb{Z}_{(p)}$-module generator $\ell \in I \cap V_{\mathbb{Z}_{(p)}}$. Recall from the discussion leading to (4.6.10) that such an ℓ determines an isomorphism

$$[\boldsymbol{\ell}^{\otimes k}, -] : \boldsymbol{\omega}^{\otimes k} \to \mathcal{O}_{\widehat{\mathrm{Sh}}_K(G,\mathcal{D},\Sigma)}$$

of line bundles on $\widehat{\mathrm{Sh}}_K(G, \mathcal{D}, \Sigma)$.

Proposition 8.2.1. *The above isomorphism extends uniquely to an isomorphism*

$$[\boldsymbol{\ell}^{\otimes k}, -] : \boldsymbol{\omega}^{\otimes k} \to \mathcal{O}_{\widehat{\mathcal{S}}_K(G,\mathcal{D},\Sigma)}$$

of line bundles on the integral model $\widehat{\mathcal{S}}_K(G, \mathcal{D}, \Sigma)$.

Proof. The maximality of $V_{\mathbb{Z}_p}$ implies that $V_{\mathbb{Z}_p} \subset V^\diamond_{\mathbb{Z}_p}$ is a $\mathbb{Z}_p$-module direct summand. In particular,

$$I_{\mathbb{Z}_{(p)}} = I \cap V_{\mathbb{Z}_{(p)}} = I \cap V^\diamond_{\mathbb{Z}_{(p)}} \subset V^\diamond_{\mathbb{Z}_{(p)}}$$

is a $\mathbb{Z}_{(p)}$-module direct summand generated by ℓ. Because $\boldsymbol{\omega}$ is defined as the pullback of $\boldsymbol{\omega}^\diamond$, and because the uniqueness part of the claim is obvious, it suffices to construct an isomorphism

$$[\boldsymbol{\ell}, -] : \boldsymbol{\omega}^\diamond \to \mathcal{O}_{\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond)} \tag{8.2.3}$$

extending the one in the generic fiber, and then pull back along (8.2.2).

We return to the notation of the proof of Proposition 8.1.2. Let $\mathcal{P}^\diamond \subset \mathcal{G}^\diamond$ be the stabilizer of the isotropic line $I_{\mathbb{Z}_{(p)}} \subset V^\diamond_{\mathbb{Z}_{(p)}}$, define a $\mathcal{P}^\diamond$-stable weight filtration

$$\mathrm{wt}_{-3} H^\diamond_{\mathbb{Z}_{(p)}} = 0, \quad \mathrm{wt}_{-2} H^\diamond_{\mathbb{Z}_{(p)}} = \mathrm{wt}_{-1} H^\diamond_{\mathbb{Z}_{(p)}} = I_{\mathbb{Z}_{(p)}} H^\diamond_{\mathbb{Z}_{(p)}}, \quad \mathrm{wt}_0 H^\diamond_{\mathbb{Z}_{(p)}} = H^\diamond_{\mathbb{Z}_{(p)}},$$

and set

$$\mathcal{Q}^\diamond_\Phi = \ker\big(\mathcal{P}^\diamond \to \mathrm{GL}(\mathrm{gr}_0(H^\diamond_{\mathbb{Z}_{(p)}}))\big).$$

Compare with the discussion of §4.4.

The $\mathbb{Z}_{(p)}$-schemes of (8.1.4) sit in a commutative diagram

$$\begin{tikzcd} \check{\mathcal{M}}^\diamond_\Phi \arrow[r] \arrow[d] & \check{\mathcal{M}}^{\mathrm{Sg}}_\Phi \arrow[d] \\ \check{\mathcal{M}}(G^\diamond, \mathcal{D}^\diamond) \arrow[r] & \check{\mathcal{M}}(G^{\mathrm{Sg}}, \mathcal{D}^{\mathrm{Sg}}) \end{tikzcd}$$

in which the horizontal arrows are closed immersions, and the vertical arrows are open immersions. The $\mathbb{Z}_{(p)}$-schemes in the top row are defined by their functors of points, which are

$$\check{\mathcal{M}}^\diamond_\Phi(S) = \left\{ \begin{array}{c} \text{isotropic lines } z \subset V^\diamond_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_S \text{ such that} \\ V^\diamond_{\mathbb{Z}_{(p)}} \to V^\diamond_{\mathbb{Z}_{(p)}} / I^\perp_{\mathbb{Z}_{(p)}} \\ \text{identifies } z \cong (V^\diamond_{\mathbb{Z}_{(p)}} / I^\perp_{\mathbb{Z}_{(p)}}) \otimes \mathcal{O}_S \end{array} \right\}$$

and

$$\check{\mathcal{M}}^{\mathrm{Sg}}_\Phi(S) = \left\{ \begin{array}{c} \text{Lagrangian subsheaves } F^0 \subset H^\diamond_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_S \text{ such that} \\ H^\diamond_{\mathbb{Z}_{(p)}} \to \mathrm{gr}_0(H^\diamond_{\mathbb{Z}_{(p)}}) \\ \text{identifies } F^0 \cong \mathrm{gr}_0(H^\diamond_{\mathbb{Z}_{(p)}}) \otimes \mathcal{O}_S \end{array} \right\}.$$

Passing to formal completions, the diagram (8.1.5) determines a diagram

$$\begin{tikzcd} \widehat{\mathcal{J}}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond) \arrow[r, "b"] \arrow[d, "a"'] & \check{\mathcal{M}}(G^\diamond, \mathcal{D}^\diamond) \\ \widehat{\mathcal{S}}_{K^\diamond}(G^\diamond, \mathcal{D}^\diamond, \Sigma^\diamond) & \end{tikzcd} \tag{8.2.4}$$

of formal algebraic spaces over $\mathbb{Z}_{(p)}$, in which a is a $\mathcal{G}^\diamond$-torsor and b is $\mathcal{G}^\diamond$-equivariant, and $\widehat{\mathcal{J}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$ is the formal completion of $\mathcal{J}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$ along the fiber over the stratum (8.2.1).

Lemma 8.2.2. *The $\mathcal{G}^\diamond$-torsor in (8.2.4) admits a canonical reduction of structure to a $\mathcal{Q}^\diamond_\Phi$-torsor $\mathcal{J}^\diamond_\Phi$, sitting in a diagram*

$$\begin{tikzcd} \mathcal{J}^\diamond_\Phi \arrow[r, "b"] \arrow[d, "a"'] & \check{\mathcal{M}}^\diamond_\Phi \\ \widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond). & \end{tikzcd}$$

Proof. The essential point is that the filtered vector bundle $(\boldsymbol{H}^\diamond_{dR}, F^\bullet\boldsymbol{H}^\diamond_{dR})$ on $\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$ used in the construction of the $\mathcal{G}^\diamond$-torsor

$$\mathcal{J}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)\to\mathcal{S}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$$

acquires extra structure after restriction to $\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$. Namely, it acquires a weight filtration

$$\mathrm{wt}_{-3}\boldsymbol{H}^\diamond_{dR}=0,\quad \mathrm{wt}_{-2}\boldsymbol{H}^\diamond_{dR}=\mathrm{wt}_{-1}\boldsymbol{H}^\diamond_{dR},\quad \mathrm{wt}_0\boldsymbol{H}^\diamond_{dR}=\boldsymbol{H}^\diamond_{dR},$$

along with distinguished isomorphisms

$$\begin{aligned}\mathrm{gr}_{-2}(\boldsymbol{H}^\diamond_{dR})&\cong\mathrm{gr}_{-2}(H^\diamond_{\mathbb{Z}_{(p)}})\otimes\mathrm{Lie}(\mathbb{G}_m)\\ \mathrm{gr}_0(\boldsymbol{H}^\diamond_{dR})&\cong\mathrm{gr}_0(H^\diamond_{\mathbb{Z}_{(p)}})\otimes\mathcal{O}_{\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)}.\end{aligned}$$

This follows from the discussion of [Mad19, (4.3.1)]. The essential point is that over the formal completion $\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$ there is a canonical degenerating abelian scheme, and the desired extension of $\boldsymbol{H}^\diamond_{dR}$ is its de Rham realization. The extension of the weight and Hodge filtrations is also a consequence of this observation; see [Mad19, §1], and in particular [Mad19, Proposition 1.3.5].

The desired reduction of structure $\mathcal{J}^\diamond_\Phi\subset\widehat{\mathcal{J}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$ is now defined as the closed formal algebraic subspace parametrizing pairs of isomorphisms (f,f_0) as in (8.1.3) that respect this additional structure.

Moreover, after restricting $\boldsymbol{H}^\diamond_{dR}$ to $\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$, the surjection $\boldsymbol{H}^\diamond_{dR}\to\mathrm{gr}_0\boldsymbol{H}^\diamond_{dR}$ identifies $F^0\boldsymbol{H}^\diamond_{dR}\cong\mathrm{gr}_0\boldsymbol{H}^\diamond_{dR}$. Indeed, in the language of [Mad19, §1], this just amounts to the observation that the de Rham realization of a 1-motive with trivial abelian part has trivial weight and Hodge filtrations.

As the composition

$$\mathcal{J}^\diamond_\Phi\subset\widehat{\mathcal{J}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)\xrightarrow{b}\check{\mathcal{M}}(G^\diamond,\mathcal{D}^\diamond)\subset\check{\mathcal{M}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}})$$

sends $(f,f_0)\mapsto f(F^0\boldsymbol{H}^\diamond_{dR})$, it takes values in the open subscheme

$$\check{\mathcal{M}}^{\mathrm{Sg}}_\Phi\subset\check{\mathcal{M}}(G^{\mathrm{Sg}},\mathcal{D}^{\mathrm{Sg}}).$$

It therefore take values in the closed subscheme $\check{\mathcal{M}}^\diamond_\Phi \subset \check{\mathcal{M}}^{\mathrm{Sg}}_\Phi$, as this can be checked in the generic fiber, where it follows from the analogous complex analytic constructions. $\square$

Returning to the main proof, let $\check{I} \subset \check{V}^\diamond$ be the constant $\mathcal{Q}^\diamond_\Phi$-equivariant line bundles on $\check{\mathcal{M}}^\diamond_\Phi$ determined by the representations $I_{\mathbb{Z}_{(p)}} \subset V^\diamond_{\mathbb{Z}_{(p)}}$, and let $\check{\omega}^\diamond \subset \check{V}^\diamond$ be the tautological line bundle. The self-duality of $V^\diamond_{\mathbb{Z}_{(p)}}$ guarantees that the bilinear pairing on $\check{V}^\diamond$ restricts to an isomorphism

$$[-,-] : \check{I} \otimes \check{\omega}^\diamond \to \mathcal{O}_{\check{\mathcal{M}}^\diamond_\Phi}.$$

Pulling back these line bundles to $\mathcal{J}^\diamond_\Phi$ and taking the quotient by $\mathcal{Q}^\diamond_\Phi$, we obtain an isomorphism

$$[-,-] : \boldsymbol{I}_{dR} \otimes \boldsymbol{\omega}^\diamond \to \mathcal{O}_{\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)}$$

of line bundles on $\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$.

On the other hand, the action of $\mathcal{Q}^\diamond_\Phi$ on $I_{\mathbb{Z}_{(p)}}$ is through the character $\nu^\diamond_\Phi$, which agrees with the restriction of $\nu^\diamond : \mathcal{G}^\diamond \to \mathbb{G}_m$ to $\mathcal{Q}^\diamond_\Phi$. The canonical morphism

$$\mathcal{J}^\diamond_\Phi \to \widehat{\mathcal{J}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond) \xrightarrow{(f,f_0)\mapsto f_0} \underline{\mathrm{Iso}}\big(\mathrm{Lie}(\mathbb{G}_m), \mathcal{O}_{\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)}\big)$$

of formal algebraic spaces over $\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$ identifies $\ker(\nu^\diamond_\Phi)\backslash\mathcal{J}^\diamond_\Phi$ with the trivial $\mathbb{G}_m$-torsor

$$\underline{\mathrm{Iso}}\big(\mathrm{Lie}(\mathbb{G}_m), \mathcal{O}_{\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)}\big) \cong \underline{\mathrm{Aut}}\big(\mathcal{O}_{\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)}\big)$$

over $\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)$. As the action of $\mathcal{G}^\diamond$ on $I_{\mathbb{Z}_{(p)}}$ is via $\nu^\diamond_\Phi$, this trivialization fixes an isomorphism

$$\begin{aligned} \boldsymbol{I}_{dR} &= \mathcal{Q}^\diamond_\Phi\backslash\big(I_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_{\mathcal{J}^\diamond_\Phi}\big) \\ &= \mathbb{G}_m\backslash\big(I_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_{\ker(\nu^\diamond_\Phi)\backslash\mathcal{J}^\diamond_\Phi}\big) \\ &\cong I_{\mathbb{Z}_{(p)}} \otimes \mathcal{O}_{\widehat{\mathcal{S}}_{K^\diamond}(G^\diamond,\mathcal{D}^\diamond,\Sigma^\diamond)}. \end{aligned}$$

The generator $\ell \in I_{\mathbb{Z}_{(p)}}$ now determines a trivializing section $\boldsymbol{\ell} = \ell \otimes 1$ of $\boldsymbol{I}_{dR}$, defining the desired isomorphism (8.2.3). This completes the proof of Proposition 8.2.1. $\square$

Let $I_* \subset V$ and

$$\begin{tikzcd} (Q_\Phi,\mathcal{D}_\Phi) \arrow[r, "\nu_\Phi"'] & (\mathbb{G}_m,\mathcal{H}_0) \arrow[l, bend right, "s"'] \end{tikzcd}.$$

be as in the discussion preceding Proposition 4.6.2. Choose a compact open subgroup $K_0 \subset \mathbb{A}_f^\times$ small enough that $s(K_0) \subset K_\Phi$, and assume that K_0 factors as

$$K_0 = \mathbb{Z}_p^\times \cdot K_0^p.$$

Let $F/\mathbb{Q}$ be the abelian extension of $\mathbb{Q}$ determined by

$$\mathrm{rec} : \mathbb{Q}^\times_{>0}\backslash\mathbb{A}_f^\times/K_0 \cong \mathrm{Gal}(F/\mathbb{Q}).$$

Fix a prime $\mathfrak{p} \subset \mathcal{O}_F$ above p, and let $R \subset F$ be the localization of $\mathcal{O}_F$ at $\mathfrak{p}$. Note that the above assumption on K implies that p is unramified in F.

Proposition 8.2.3. *If we set*

$$\widehat{\mathcal{T}}_\Phi(\sigma) = \mathrm{Spf}\Big(\mathbb{Z}_{(p)}[[q_\alpha]]_{\substack{\alpha \in \Gamma_\Phi^\vee(1) \\ \langle \alpha, \sigma\rangle \geq 0}}\Big),$$

there is a unique morphism

$$\bigsqcup_{a \in \mathbb{Q}^\times_{>0} \backslash \mathbb{A}_f^\times / K_0} \widehat{\mathcal{T}}_\Phi(\sigma)_{/R} \to \widehat{\mathcal{S}}_K(G, \mathcal{D}, \Sigma)_{/R}$$

of formal algebraic spaces over R whose base change to $\mathbb{C}$ agrees with the morphism of Proposition 4.6.2. Moreover, if t is any point of the source and s is its image in $\widehat{\mathcal{S}}_K(G, \mathcal{D}, \Sigma)_{/R}$, the induced map on étale local rings $\mathcal{O}_s^{et} \to \mathcal{O}_t^{et}$ is faithfully flat.

Proof. The uniqueness of such a morphism is clear. We have to show existence. The proof of this proceeds just as that of Proposition 4.6.2, except that it uses Proposition 8.1.1 as input. The only additional observation required is that we have an isomorphism

$$\bigsqcup_{a \in \mathbb{Q}^\times_{>0} \backslash \mathbb{A}_f^\times / K_0} \mathrm{Spec}(R) \cong \mathcal{S}_{K_0}(\mathbb{G}_m, \mathcal{H}_0)_{/R} \tag{8.2.5}$$

of R-schemes, which realizes (4.6.8) on $\mathbb{C}$-points. Here $\mathcal{S}_{K_0}(\mathbb{G}_m, \mathcal{H}_0)$ is defined as the normalization of $\mathrm{Spec}(\mathbb{Z}_{(p)})$ in $\mathrm{Sh}_{K_0}(\mathbb{G}_m, \mathcal{H}_0)$.

To see this, note that the defining property of canonical models provides an isomorphism

$$\mathrm{Spec}(F) \cong \mathrm{Sh}_{K_0}(\mathbb{G}_m, \mathcal{H}_0)$$

of $\mathbb{Q}$-schemes, and hence an isomorphism F-schemes

$$\bigsqcup_{a \in \mathbb{Q}^\times_{>0} \backslash \mathbb{A}_f^\times / K_0} \mathrm{Spec}(F) \cong \mathrm{Sh}_{K_0}(\mathbb{G}_m, \mathcal{H}_0)_{/F}.$$

Using the fact that p is unramified in F, one can see that this isomorphism extends to (8.2.5). □

Suppose ψ is a section of the line bundle $\boldsymbol{\omega}^{\otimes k}$ on $\mathrm{Sh}_K(G, \mathcal{D})_{/F}$. It follows from Proposition 4.6.3 that the q-expansion (4.6.10) of ψ has coefficients in F for every $a \in \mathbb{A}_f^\times$. If we view ψ as a rational section on $\mathcal{S}_K(G, \mathcal{D}, \Sigma)_{/R}$, the following result gives a criterion for testing flatness of its divisor.

Corollary 8.2.4. *Assume that the special fiber of $\mathcal{S}_K(G, \mathcal{D})_{/R}$ is geometrically normal, and for every $a \in \mathbb{A}_f^\times$ the q-expansion (4.6.10) satisfies*

$$\mathrm{FJ}^{(a)}(\psi) \in R[[q_\alpha]]_{\substack{\alpha \in \Gamma_\Phi^\vee(1) \\ \langle \alpha, \sigma\rangle \geq 0}}.$$

If this q-expansion is nonzero modulo $\mathfrak{p}$ for all a, then $\mathrm{div}(\psi)$ is R-flat.

Proof. As $\mathcal{S}_K(G,\mathcal{D},\Sigma)_{/R}$ is flat over R, to show that $\mathrm{div}(\psi)$ is R-flat it is enough to show that its support does not contain any irreducible components of the special fiber of $\mathcal{S}_K(G,\mathcal{D},\Sigma)_{/R}$.

Every connected component

$$C \subset \mathcal{S}_K(G,\mathcal{D},\Sigma)_{/R}.$$

has irreducible special fiber. Indeed, we have assumed that the special fiber of $\mathcal{S}_K(G,\mathcal{D})_{/R}$ is geometrically normal. It therefore follows from [Mad19, Theorem 1] that the special fiber of $\mathcal{S}_K(G,\mathcal{D},\Sigma)_{/R}$ is also geometrically normal. On the other hand, [Mad19, Corollary 4.1.11] shows that C has geometrically connected special fiber. Therefore the special fiber of C is both connected and normal, and hence is irreducible.

As in the proof of Proposition 4.6.3, the closed stratum

$$\mathcal{Z}_K^{(\Phi,\sigma)}(G,\mathcal{D},\Sigma)_{/R} \subset \mathcal{S}_K(G,\mathcal{D},\Sigma)_{/R}$$

meets every connected component. Pick a closed point s of this stratum lying on the connected component C. By the definition of $\mathrm{FJ}^{(a)}(\psi)$, and from Proposition 8.2.3, our hypothesis on the q-expansion implies that the restriction of ψ to the completed local ring $\mathcal{O}_s$ of s defines a rational section of $\boldsymbol{\omega}^{\otimes k}$ whose divisor is an R-flat Cartier divisor on $\mathrm{Spf}(\mathcal{O}_s)$.

It follows that $\mathrm{div}(\psi)$ does not contain the special fiber of C, and varying C shows that $\mathrm{div}(\psi)$ contains no irreducible components of the special fiber of $\mathcal{S}_K(G,\mathcal{D},\Sigma)_{/R}$. □

Remark 8.2.5. If $V_{\mathbb{Z}_p}$ is almost self-dual, then $\mathcal{S}_K(G,\mathcal{D})$ is smooth over $\mathbb{Z}_{(p)}$, and hence has geometrically normal special fiber. Without the assumption of almost self-duality, Proposition 7.2.1 tells us that the special fiber is geometrically normal whenever $n \geqslant 6$.

9. Borcherds products on integral models

Keep $V_{\mathbb{Z}} \subset V$ of signature $(n,2)$ with $n \geqslant 1$, and let $(G,\mathcal{D})$ be the associated GSpin Shimura datum. As in the introduction, let Ω be a finite set of prime numbers containing all primes at which $V_{\mathbb{Z}}$ is not maximal, and choose (1.1.2) to be factorizable $K = \prod_p K_p$ with

$$K_p = G(\mathbb{Q}_p) \cap C(V_{\mathbb{Z}_p})^\times$$

for all $p \notin \Omega$. Set $\mathbb{Z}_\Omega = \mathbb{Z}[1/p : p \in \Omega]$.

9.1. **Statement of the main result.** In §6.3 and §6.4 we constructed, for every prime $p \notin \Omega$, an integral model over $\mathbb{Z}_{(p)}$ of the Shimura variety $\mathrm{Sh}_K(G,\mathcal{D})$, along with a family of special divisors and a line bundle of weight one modular forms. As explained in [AGHMP17, §2.4] and [AGHMP18, §4.5], as p varies these models arise as the localizations of a flat and normal integral model

$$\mathcal{S}_K(G,\mathcal{D}) \to \mathrm{Spec}(\mathbb{Z}_\Omega),$$

endowed with a family of special divisors $\mathcal{Z}(m,\mu)$ indexed by positive $m \in \mathbb{Q}$ and $\mu \in L^\vee/L$, and a line bundle of weight one modular forms $\boldsymbol{\omega}$.

Theorem 9.1.1. *Suppose*

$$f(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} c(m) \cdot q^m \in M^!_{1-\frac{n}{2}}(\overline{\rho}_{V_\mathbb{Z}})$$

is a weakly holomorphic form as in (5.1.1), and assume f is integral in the sense of Definition 5.1.2. After multiplying f by any sufficiently divisible positive integer, there is a rational section $\psi(f)$ of $\boldsymbol{\omega}^{\otimes c(0,0)}$ over $\mathcal{S}_K(G,\mathcal{D})$ whose norm under the metric (4.2.3) is related to the regularized theta lift of §5.2 by

$$-2\log\|\psi(f)\| = \Theta^{\mathrm{reg}}(f), \tag{9.1.1}$$

and whose divisor is

$$\mathrm{div}(\psi(f)) = \sum_{\substack{m>0 \\ \mu \in V_\mathbb{Z}^\vee/V_\mathbb{Z}}} c(-m,\mu) \cdot \mathcal{Z}(m,\mu). \tag{9.1.2}$$

The remainder of this subsection is devoted to proving Theorem 9.1.1 under some restrictive hypotheses on the pair $V_\mathbb{Z} \subset V$. These will allow us to deduce algebraicity of the Borcherds product from Proposition 5.2.3, prove its descent to $\mathbb{Q}$ using the q-expansion principle of Proposition 4.6.3, and deduce the equality of divisors (9.1.2) from the flatness of both sides over $\mathbb{Z}_\Omega$.

Proposition 9.1.2. *If $n \geqslant 6$, and if there exists an $h \in G(\mathbb{A}_f)$ and isotropic vectors $\ell, \ell_* \in hV_\mathbb{Z}$ such that $[\ell,\ell_*] = 1$, then Theorem 9.1.1 holds.*

Proof. It suffices to treat the case where

$$K = G(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}})^\times,$$

for then we can pull back $\psi(f)$ to any smaller level structure.

The vectors $\ell, \ell_* \in V$ satisfy the relation (5.3.1) with $k = \ell_*$. Let I and I_* be the isotropic lines in V spanned by ℓ and ℓ_*, respectively. Let P be the stabilizer of I, and let $\mathcal{D}^\circ \subset \mathcal{D}$ be a connected component. This determines a cusp label representative

$$\Phi = (P, \mathcal{D}^\circ, h).$$

Although we will not use this fact explicitly, the following lemma implies that the 0-dimensional stratum of the Baily-Borel compactification $\mathrm{Sh}_K(G,\mathcal{D})^{\mathrm{BB}}$ indexed by Φ is geometrically connected. In other words, Baily-Borel compactification has a cusp defined over $\mathbb{Q}$.

Lemma 9.1.3. *The complex orbifold $\mathrm{Sh}_K(G,\mathcal{D})(\mathbb{C})$ is connected, and the section (4.6.6) determined by I_* satisfies*

$$s(\widehat{\mathbb{Z}}^\times) \subset K_\Phi. \tag{9.1.3}$$

Proof. We first prove (9.1.3). Consider the hyperbolic place

$$W = \mathbb{Q}\ell + \mathbb{Q}\ell_* \subset V.$$

Its corresponding spinor similitude group $\mathrm{GSpin}(W)$ is just the unit group of the even Clifford algebra $C^+(W)$. The natural inclusion $\mathrm{GSpin}(W) \to G$ takes values in the subgroup Q_Φ, and the cocharacter (4.6.6) factors as

$$\mathbb{G}_m \xrightarrow{s} \mathrm{GSpin}(W) \to Q_\Phi$$

where the first arrow sends $a \in \mathbb{Q}^\times$ to

$$s(a) = a^{-1}\ell_*\ell + \ell\ell_* \in C^+(W)^\times.$$

From this explicit formula and the inclusion

$$H_{\widehat{\mathbb{Z}}} = \widehat{\mathbb{Z}}\ell \oplus \widehat{\mathbb{Z}}\ell_* \subset hV_{\widehat{\mathbb{Z}}},$$

it is clear that (4.6.6) satisfies

$$s(\widehat{\mathbb{Z}}^\times) \subset C^+(W_{\widehat{\mathbb{Z}}})^\times \subset Q_\Phi(\mathbb{A}_f) \cap C(hV_{\widehat{\mathbb{Z}}})^\times = K_\Phi.$$

Now we prove the connectedness claim. From (9.1.3) it follows that

$$\widehat{\mathbb{Z}}^\times = \nu_\Phi(s(\widehat{\mathbb{Z}}^\times)) \subset \nu_\Phi(K_\Phi) \subset \nu(K),$$

and hence the 0-dimensional Shimura variety

$$\mathrm{Sh}_{\nu(K)}(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C}) = \mathbb{Q}^\times \backslash \mathcal{H}_0 \times \mathbb{A}_f^\times / \nu(K)$$

consists of a single point. The proof of Proposition 4.6.3 shows that the fibers of

$$\mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C}) \to \mathrm{Sh}_{\nu(K)}(\mathbb{G}_m, \mathcal{H}_0)(\mathbb{C})$$

are connected, completing the proof. □

Applying Theorem 5.2.2 and Proposition 5.2.3 to the form $2f$ gives us a rational section

$$\psi(f) = (2\pi i)^{c(0,0)}\Psi(2f) \tag{9.1.4}$$

of $\boldsymbol{\omega}^{\otimes c(0,0)}$ over $\mathrm{Sh}_K(G, \mathcal{D})_{/\mathbb{C}}$. We first prove that $\psi(f)$ can be rescaled by a constant of absolute value 1 to make it defined over $\mathbb{Q}$.

Fix a neat compact open subgroup $\tilde{K} \subset K$ small enough that there is a $\tilde{K}$-admissible complete cone decomposition Σ for $(G, \mathcal{D})$ satisfying the conclusion of Lemma 5.4.1. In particular, we have a top-dimensional rational polyhedral cone $\sigma \in \Sigma_\Phi$ whose interior is contained in a fixed Weyl chamber

$$\mathscr{W} \subset \mathrm{LightCone}^\circ(V_{0\mathbb{R}}) \cong C_\Phi.$$

Let $\tilde{\psi}(f)$ denote the pullback of $\psi(f)$ to $\mathrm{Sh}_{\tilde{K}}(G, \mathcal{D}, \Sigma)_{/\mathbb{C}}$. Recalling the construction of q-expansions of (4.6.10), the toroidal stratum representative

$$(\Phi, \sigma) \in \mathrm{Strat}_{\tilde{K}}(G, \mathcal{D}, \Sigma)$$

determines a collection of formal q-expansions

$$\mathrm{FJ}^{(a)}(\tilde{\psi}(f)) \in \mathbb{C}[[q_\alpha]]_{\substack{\alpha \in \Gamma_\Phi^\vee(1) \\ \langle \alpha, \sigma \rangle \geqslant 0}} \tag{9.1.5}$$

indexed by $a \in \mathbb{Q}_{>0}^\times \backslash \mathbb{A}_f^\times / \tilde{K}_0$, where $\tilde{K}_0 \subset \mathbb{G}_m(\mathbb{A}_f)$ is chosen small enough that its image under (4.6.6) is contained in $\tilde{K}_\Phi$.

We can read off these q-expansions from Proposition 5.4.2, which implies

$$\mathrm{FJ}^{(a)}(\tilde{\psi}(f)) = \big(\kappa^{(a)} \cdot q_{\alpha(\varrho)} \cdot \mathrm{BP}(f)\big)^2, \tag{9.1.6}$$

for an explicit

$$\mathrm{BP}(f) \in \mathbb{Z}[[q_\alpha]]_{\substack{\alpha \in \Gamma_\Phi^\vee(1),\\ \langle \alpha, \sigma\rangle \geqslant 0}}, \tag{9.1.7}$$

and some constants $\kappa^{(a)} \in \mathbb{C}$ of absolute value 1. Indeed, the hypotheses on $\ell, \ell_* \in V_{\mathbb{Z}}$ imply that the constants N and A appearing in (5.3.6) are equal to 1, and our choice of $k = \ell_* \in hV_{\mathbb{Z}}$ implies that $\zeta_\mu = 1$ for all $\mu \in hV_{\mathbb{Z}}^\vee / hV_{\mathbb{Z}}$.

Moreover, it is clear from the presentation of $\mathrm{BP}(f)$ as a product that its constant term is equal to 1.

The q-expansion (9.1.5) is actually independent of a. Indeed, using the notation of (5.4.5), with K replaced by $\tilde{K}$ throughout, these q-expansions can be computed in terms of the pullback of $\psi(f)$ to the upper left corner in

$$\begin{array}{ccc} \displaystyle\bigsqcup_{a \in \mathbb{Q}_{>0}^\times \backslash \mathbb{A}_f^\times / \tilde{K}_0} \tilde{\Gamma}_\Phi^{(a)} \backslash \mathcal{D}^\circ & \xrightarrow{z \mapsto (z, s(a)h)} & \mathrm{Sh}_{\tilde{K}}(G, \mathcal{D})(\mathbb{C}) \\ \downarrow & & \downarrow \\ (K_\Phi \cap U_\Phi(\mathbb{Q})) \backslash \mathcal{D}^\circ & \xrightarrow{z \mapsto (z, h)} & \mathrm{Sh}_K(G, \mathcal{D})(\mathbb{C}). \end{array}$$

Here we have chosen our coset representatives $a \in \widehat{\mathbb{Z}}^\times$. This implies, by Lemma 9.1.3, that $s(a) \in K_\Phi \subset hKh^{-1}$, and so

$$\tilde{\Gamma}_\Phi^{(a)} = s(a) \tilde{K}_\Phi s(a)^{-1} \cap U_\Phi(\mathbb{Q}) \subset K_\Phi \cap U_\Phi(\mathbb{Q})$$

and $s(a)hK = hK$. It follows that the pullback of $\psi(f)$ to the upper left corner is the same on every copy of $\mathcal{D}^\circ$.

Having proved that all of the $\kappa^{(a)}$ are equal, we may rescale $\psi(f)$ by a constant of absolute value 1 to make all of them equal to 1. The q-expansion principle of Proposition 4.6.3 now implies that $\tilde{\psi}(f)$ is defined over $\mathbb{Q}$, and the same is therefore true of $\psi(f)$. The equality (9.1.1) follows from the equality (5.2.2).

It only remains to prove the equality of divisors (9.1.2). In the generic fiber, this follows from (9.1.1) and the analysis of the singularities of $\Theta^{\mathrm{reg}}(f)$ found in [Bor98] or [Bru02]. To prove equality on the integral model, it therefore suffices to prove that both sides of the desired equality are flat over $\mathbb{Z}_\Omega$. Flatness of the special divisors $\mathcal{Z}(m, \mu)$ is Proposition 7.2.2.

To prove the flatness of $\mathrm{div}(\psi(f))$ it suffices to show, for every prime $p \notin \Omega$, that $\mathrm{div}(\psi(f))$ has no irreducible components supported in characteristic p. This follows from Corollary 8.2.4 and the observation made above that (9.1.7) has nonzero reduction at p.

The only technical point is that to apply Corollary 8.2.4 to the integral model of $\mathrm{Sh}_{\tilde{K}}(G,\mathcal{D},\Sigma)$ over $\mathbb{Z}_{(p)}$, we must choose $\tilde{K}$ to have p-component

$$\tilde{K}_p = G(\mathbb{Q}_p) \cap C(V_{\mathbb{Z}_p})^\times,$$

and similarly choose $\tilde{K}_0$ to have p-component $\mathbb{Z}_p^\times$. As p varies, this forces us to vary $\tilde{K}$. As we need $\tilde{K}$ to satisfy the conclusion of Lemma 5.4.1, this may require us to also vary both Σ and the rational polyhedral cone $\sigma \in \Sigma_\Phi$. Thus, having rescaled the Borcherds product to eliminate the constants $\kappa^{(a)}$ at one boundary stratum, we may be forced to apply Corollary 8.2.4 at a different boundary stratum of a different toroidal compactification at different level structure, at which we must deal with new constants $\kappa^{(a)}$.

This is not really a problem. For a given p, one can check using Remark 2.4.9 that it is possible to choose $\tilde{K}$ (and hence Σ and $\sigma \in \Sigma_\Phi$) as in Lemma 5.4.1 by shrinking only the prime-to-p part of K. Using Lemma 9.1.3, we may then choose $\tilde{K}_0$ to have p-component $\mathbb{Z}_p^\times$. Now pull back $\psi(f)$ via the resulting étale cover

$$\mathcal{S}_{\tilde{K}}(G,\mathcal{D})_{/\mathbb{Z}_{(p)}} \to \mathcal{S}_K(G,\mathcal{D})_{/\mathbb{Z}_{(p)}}$$

over integral models over $\mathbb{Z}_{(p)}$ to obtain a section $\tilde{\psi}(f)$ whose q-expansion again has the form (9.1.6) for some constants $\kappa^{(a)}$ of absolute value 1.

The point is simply that our $\psi(f)$, hence also $\tilde{\psi}(f)$, has been rescaled so that it is defined over $\mathbb{Q}$. This allows us to use the q-expansion principle of Proposition 4.6.3 to deduce that each $\kappa^{(a)}$ is rational, hence is ± 1. Thus the power series (9.1.6) has integer coefficients and nonzero reduction at p. Corollary 8.2.4 implies that the divisor of $\tilde{\psi}(f)$ has no irreducible components in characteristic p, so the same holds for $\psi(f)$. □

9.2. **Proof of Theorem 9.1.1.** In this subsection we complete the proof of Theorem 9.1.1 by developing a purely algebraic analogue of the embedding trick of Borcherds. This allows us to deduce the general case from the special case proved in Proposition 9.1.2.

According to [Bor98, Lemma 8.1] there exist self-dual $\mathbb{Z}$-quadratic spaces $\Lambda^{[1]}$ and $\Lambda^{[2]}$ of signature $(24,0)$ whose corresponding theta series

$$\vartheta^{[i]}(\tau) = \sum_{x \in \Lambda^{[i]}} q^{Q(x)} \in M_{12}(\mathrm{SL}_2(\mathbb{Z}),\mathbb{C})$$

are related by

$$\vartheta^{[2]} - \vartheta^{[1]} = 24\Delta. \tag{9.2.1}$$

Here Δ is Ramanujan's modular discriminant, and Q is the quadratic form on $\Lambda^{[i]}$. Denote by

$$r^{[i]}(m) = \#\{x \in \Lambda^{[i]} : Q(x) = m\}$$

the m^{th} Fourier coefficient of $\vartheta^{[i]}$. Set

$$V_{\mathbb{Z}}^{[i]} = V_{\mathbb{Z}} \oplus \Lambda^{[i]} \qquad \text{and} \qquad V^{[i]} = V \oplus \Lambda_{\mathbb{Q}}^{[i]}. \tag{9.2.2}$$

In the notation of §5.1, the inclusion $V_{\mathbb{Z}} \hookrightarrow V_{\mathbb{Z}}^{[i]}$ identifies

$$V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}} \cong (V_{\mathbb{Z}}^{[i]})^{\vee}/V_{\mathbb{Z}}^{[i]},$$

and the induced isomorphism

$$S_{V_{\mathbb{Z}}} \cong S_{V_{\mathbb{Z}}^{[i]}}$$

is compatible with the Weil representations on source and target. The fixed weakly holomorphic form f of (5.1.1) therefore determines a form

$$f^{[i]}(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ m \gg -\infty}} c^{[i]}(m) \cdot q^m \in M^!_{-11-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}^{[i]}})$$

by setting $f^{[i]} = f/(24\Delta)$. The relation

$$f = \vartheta^{[2]} f^{[2]} - \vartheta^{[1]} f^{[1]}$$

implies the equality of Fourier coefficients

$$c(m,\mu) = \sum_{k \geqslant 0} r^{[2]}(k) \cdot c^{[2]}(m-k,\mu) - \sum_{k \geqslant 0} r^{[1]}(k) \cdot c^{[1]}(m-k,\mu). \tag{9.2.3}$$

Each $V^{[i]}$ determines a GSpin Shimura datum $(G^{[i]}, \mathcal{D}^{[i]})$. By choosing

$$K^{[i]} = G^{[i]}(\mathbb{A}_f) \cap C(V_{\widehat{\mathbb{Z}}}^{[i]})^{\times}$$

for our compact open subgroups, we put ourselves in the situation of §6.6. Note that in §6.6 the integral models were over $\mathbb{Z}_{(p)}$, but everything extends verbatim to $\mathbb{Z}_{\Omega}$. In particular, we have finite morphisms of integral models

$$\begin{tikzcd} & \mathcal{S}_K(G,\mathcal{D}) \arrow[dl, "j^{[1]}"'] \arrow[dr, "j^{[2]}"] & \\ \mathcal{S}^{[1]} & & \mathcal{S}^{[2]}, \end{tikzcd}$$

over $\mathbb{Z}_{\Omega}$, where we abbreviate

$$\mathcal{S}^{[i]} = \mathcal{S}_{K^{[i]}}(G^{[i]}, \mathcal{D}^{[i]}).$$

Each $\mathcal{S}^{[i]}$ has its own line bundle of weight one modular forms $\boldsymbol{\omega}^{[i]}$ and its own family $\mathcal{Z}^{[i]}(m,\mu)$ of special divisors.

The following lemma shows that each $V_{\mathbb{Z}}^{[i]} \subset V^{[i]}$ satisfies the hypotheses of Proposition 9.1.2. Thus, after replacing f (and hence both $f^{[1]}$ and $f^{[2]}$) by a positive integer multiple, we obtain a Borcherds product $\psi(f^{[i]})$ on $\mathcal{S}^{[i]}$ with divisor

$$\operatorname{div}(\psi(f^{[i]}) = \sum_{\substack{m>0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c^{[i]}(-m,\mu) \cdot \mathcal{Z}^{[i]}(m,\mu). \tag{9.2.4}$$

Lemma 9.2.1. *There exist isotropic vectors* $\ell, \ell_* \in V_{\mathbb{Z}}^{[i]}$ *with* $[\ell, \ell_*] = 1$.

Proof. Let $\mathbb{H} = \mathbb{Z}\ell \oplus \mathbb{Z}\ell_*$ be the integral hyperbolic plane, so that ℓ and ℓ_* are isotropic with $[\ell, \ell_*] = 1$. To prove the existence of an isometric embedding $\mathbb{H} \to V_{\mathbb{Z}}^{[i]}$, we first prove the existence everywhere locally.

At the archimedean place this is clear from the signature, so fix a prime p. The $\mathbb{Q}_p$-quadratic space $\Lambda^{[i]} \otimes_{\mathbb{Z}} \mathbb{Q}_p$ has dimension $\geqslant 5$, so admits an isometric embedding

$$\mathbb{H} \otimes \mathbb{Q}_p \to \Lambda^{[i]} \otimes \mathbb{Q}_p.$$

Enlarging the image of $\mathbb{H} \otimes \mathbb{Z}_p$ to a maximal lattice, and invoking Eichler's theorem that all maximal lattices in a $\mathbb{Q}_p$-quadratic space are isometric [Ger08, Theorem 8.8], we find that $\mathbb{H} \otimes \mathbb{Z}_p$ embeds into the (self-dual, hence maximal) lattice $\Lambda^{[i]} \otimes \mathbb{Z}_p$. A fortiori, it embeds into $V_{\mathbb{Z}}^{[i]} \otimes_{\mathbb{Z}} \mathbb{Z}_p$.

The existence of the desired embeddings everywhere locally implies that there exist isometric embeddings

$$a : \mathbb{H} \otimes \mathbb{Q} \to V_{\mathbb{Z}}^{[i]} \otimes \mathbb{Q}, \tag{9.2.5}$$

and

$$\alpha : \mathbb{H} \otimes \widehat{\mathbb{Z}} \to V_{\mathbb{Z}}^{[i]} \otimes \widehat{\mathbb{Z}}.$$

We may choose these in such a way that a and α induce the same embedding of $\mathbb{Q}_p$-quadratic spaces at all but finitely many primes p. All embeddings

$$\mathbb{H} \otimes \mathbb{Q}_p \to V_{\mathbb{Z}}^{[i]} \otimes \mathbb{Q}_p$$

lie in a single $\mathrm{SO}(V^{[i]})(\mathbb{Q}_p)$-orbit, and so there exists a

$$g \in \mathrm{SO}(V^{[i]})(\mathbb{A}_f)$$

such that

$$ga(\mathbb{H} \otimes \widehat{\mathbb{Z}}) = \alpha(\mathbb{H} \otimes \widehat{\mathbb{Z}}). \tag{9.2.6}$$

Fix a subspace $W \subset V_{\mathbb{Z}}^{[i]} \otimes \mathbb{Q}$ of signature $(2, 1)$ perpendicular to the image of (9.2.5). There exists an isomorphism $\mathrm{SO}(W) \cong \mathrm{PGL}_2$ identifying the spinor norm

$$\mathrm{SO}(W)(\mathbb{A}_f) \to \mathbb{A}_f^\times / (\mathbb{A}_f^\times)^2$$

with the determinant, and hence the spinor norm is surjective. This allows us to modify g by an element of $\mathrm{SO}(W)(\mathbb{A}_f)$, which does not change the relation (9.2.6), in order to arrange that g has trivial spinor norm. Now choose any lift

$$g \in \mathrm{Spin}(V^{[i]})(\mathbb{A}_f),$$

and note that (9.2.6) implies

$$ga(\mathbb{H} \otimes \widehat{\mathbb{Z}}) \subset V_{\mathbb{Z}}^{[i]} \otimes \widehat{\mathbb{Z}}.$$

As the spin group is simply connected, it satisfies strong approximation. By choosing $\gamma \in \mathrm{Spin}(V^{[i]})(\mathbb{Q})$ sufficiently close to g, we find an isometric embedding $\gamma a : \mathbb{H} \to V_{\mathbb{Z}}^{[i]}$. $\square$

At least formally, we wish to define

$$\psi(f) = \frac{(j^{[2]})^*\psi(f^{[2]})}{(j^{[1]})^*\psi(f^{[1]})}.$$

As noted in §1.4, the image of $j^{[i]}$ will typically be contained in the support of the divisor of $\psi(f^{[i]})$, and so the quotient on the right will typically be either $0/0$ or ∞/∞.

The key to making sense of this quotient is to combine the following lemma, which is really just a restatement of (9.2.4), with the pullback formula of Proposition 6.6.3. As in the pullback formula, we use $\mathcal{Z}^{[i]}(m,\mu)$ to denote both the special divisor and its corresponding line bundle, and extend the definition to $m \leqslant 0$ by

$$\mathcal{Z}^{[i]}(m,\mu) = \begin{cases} (\boldsymbol{\omega}^{[i]})^{-1} & \text{if } (m,\mu)=(0,0) \\ \mathcal{O}_{\mathcal{S}^{[i]}} & \text{otherwise.} \end{cases}$$

Lemma 9.2.2. *The Borcherds product $\psi(f^{[i]})$ determines an isomorphism of line bundles*

$$\mathcal{O}_{\mathcal{S}^{[i]}} \cong \bigotimes_{\substack{m\geqslant 0 \\ \mu\in V_{\mathbb{Z}}^\vee/V_{\mathbb{Z}}}} \mathcal{Z}^{[i]}(m,\mu)^{\otimes c^{[i]}(-m,\mu)}.$$

Proof. If $m>0$ there is a canonical section

$$s^{[i]}(m,\mu) \in H^0\big(\mathcal{S}^{[i]}, \mathcal{Z}^{[i]}(m,\mu)\big)$$

with divisor the Cartier divisor $\mathcal{Z}^{[i]}(m,\mu)$ of the same name. This is just the constant function 1, viewed as a section of

$$\mathcal{O}_{\mathcal{S}^{[i]}} \subset \mathcal{Z}^{[i]}(m,\mu).$$

The equality of divisors (9.2.4) implies that there is a unique isomorphism

$$(\boldsymbol{\omega}^{[i]})^{\otimes c^{[i]}(0,0)} \cong \bigotimes_{\substack{m>0 \\ \mu\in V_{\mathbb{Z}}^\vee/V_{\mathbb{Z}}}} \mathcal{Z}^{[i]}(m,\mu)^{\otimes c^{[i]}(-m,\mu)}$$

sending

$$\psi(f^{[i]}) \mapsto \bigotimes_{\substack{m>0 \\ \mu\in V_{\mathbb{Z}}^\vee/V_{\mathbb{Z}}}} s^{[i]}(m,\mu)^{\otimes c^{[i]}(-m,\mu)},$$

and so the claim is immediate from the definition of $\mathcal{Z}^{[i]}(0,\mu)$. □

Proof of Theorem 9.1.1. If we pull back the isomorphism of Lemma 9.2.2 via $j^{[i]}$ and use the pullback formula of Proposition 6.6.3, we obtain isomorphisms of line bundles

$$\mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})} \cong \bigotimes_{\substack{m_1,m_2\geqslant 0 \\ \mu\in V_{\mathbb{Z}}^\vee/V_{\mathbb{Z}}}} \mathcal{Z}(m_1,\mu)^{\otimes r^{[i]}(m_2)\cdot c^{[i]}(-m_1-m_2,\mu)}$$

for $i \in \{1,2\}$. These two isomorphisms, along with (9.2.3), determine an isomorphism

$$\mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})} \cong \bigotimes_{\substack{m \geqslant 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} \mathcal{Z}(m,\mu)^{\otimes c(-m,\mu)}.$$

Now simply reverse the reasoning in the proof of Lemma 9.2.2. Rewrite the isomorphism above as

$$\boldsymbol{\omega}^{c(0,0)} \cong \bigotimes_{\substack{m > 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} \mathcal{Z}(m,\mu)^{\otimes c(-m,\mu)}.$$

Each line bundle on the right admits a canonical section $s(m,\mu)$ whose divisor is the Cartier divisor $\mathcal{Z}(m,\mu)$ of the same name, and so the rational section of $\boldsymbol{\omega}^{c(0,0)}$ defined by

$$\psi(f) = \bigotimes_{\substack{m > 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} s(m,\mu)^{\otimes c(-m,\mu)} \tag{9.2.7}$$

has divisor

$$\operatorname{div}(\psi(f)) = \sum_{\substack{m > 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu) \cdot \mathcal{Z}(m,\mu).$$

To complete the proof of Theorem 9.1.1, we need only prove that the section defined by (9.2.7) satisfies the norm relation (9.1.1).

Fix a $g \in G(\mathbb{A}_f)$, and consider the complex uniformizations

$$\begin{tikzcd}[column sep=small] & \mathcal{D}^{[1]} \arrow[rrr] & & & \mathcal{S}^{[1]}(\mathbb{C}) \\ \mathcal{D} \arrow[ur, "j^{[1]}"] \arrow[rr] \arrow[dr, "j^{[2]}"'] & & \mathcal{S}_K(G,\mathcal{D})(\mathbb{C}) \arrow[urr, "j^{[1]}"] \arrow[drr, "j^{[2]}"'] & & \\ & \mathcal{D}^{[2]} \arrow[rrr] & & & \mathcal{S}^{[2]}(\mathbb{C}) \end{tikzcd}$$

in which all horizontal arrows send $z \mapsto (z,g)$.

Denote by $\psi_g(f)$ the pullback of $\psi(f)$ to $\mathcal{D}$. The similarly defined meromorphic sections $\psi_g(f^{[i]})$ on $\mathcal{D}^{[i]}$ are already assumed to satisfy the norm relation

$$-2\log\|\psi_g(f^{[i]})\| = \Theta_g^{\mathrm{reg}}(f^{[i]})$$

on $\mathcal{D}^{[i]}$, where

$$\Theta_g^{\mathrm{reg}}(f^{[i]}) = \Theta^{\mathrm{reg}}(f^{[i]}, g)$$

is the regularized theta lift of §5.2.

Recall from §6.5 that every $x \in V^{[i]}$ with $Q(x) > 0$ determines a global section

$$\mathrm{obst}_x^{an} \in H^0\big(\mathcal{D}^{[i]}, \boldsymbol{\omega}_{\mathcal{D}^{[i]}}^{-1}\big),$$

with zero locus the analytic divisor

$$\mathcal{D}^{[i]}(x) = \{z \in \mathcal{D}^{[i]} : [z, x] = 0\}.$$

The pullback of $\mathcal{Z}^{[i]}(m,\mu)(\mathbb{C})$ to $\mathcal{D}^{[i]}$ is given by the locally finite sum of analytic divisors

$$\sum_{\substack{x \in g\mu + gV_{\mathbb{Z}}^{[i]} \\ Q(x)=m}} \mathcal{D}^{[i]}(x).$$

Define the *renormalized* Borcherds product

$$\tilde{\psi}_g(f^{[i]}) = \psi_g(f^{[i]}) \otimes \bigotimes_{m>0} \bigotimes_{\substack{\lambda \in \Lambda^{[i]} \\ Q(\lambda)=m}} (\mathrm{obst}_\lambda^{an})^{\otimes - c^{[i]}(-m,0)}.$$

This is a meromorphic section of $\bigotimes_{m \geqslant 0} \big(\boldsymbol{\omega}_{\mathcal{D}^{[i]}}\big)^{\otimes r^{[i]}(m) c^{[i]}(-m,0)}$ with divisor

$$\mathrm{div}\big(\tilde{\psi}_g(f^{[i]})\big) = \sum_{\substack{m>0 \\ \mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}}} c^{[i]}(-m,\mu) \sum_{\substack{x \in g\mu + gV_{\mathbb{Z}}^{[i]} \\ Q(x)=m \\ x \notin \Lambda^{[i]}}} \mathcal{D}^{[i]}(x).$$

Note that each divisor $\mathcal{D}^{[i]}(x)$ appearing on the right hand side intersects the subspace $\mathcal{D} \subset \mathcal{D}^{[i]}$ properly. Indeed, If we decompose $x = y + \lambda$ with $y \in g\mu + gV_{\mathbb{Z}}$ and $\lambda \in \Lambda$, then

$$\mathcal{D}^{[i]}(x) \cap \mathcal{D} = \begin{cases} \mathcal{D}(y) & \text{if } Q(y) > 0 \\ \varnothing & \text{otherwise,} \end{cases}$$

where $\mathcal{D}(y) = \{z \in \mathcal{D} : [z, y] = 0\}$.

This is the point: by renormalizing the Borcherds product we have removed precisely the part of its divisor that intersects $\mathcal{D}$ improperly, and so the renormalized Borcherds product has a well-defined pullback to $\mathcal{D}$. Indeed, using the relation (9.2.3), we see that

$$\psi_g(f) = \frac{(j^{[2]})^* \tilde{\psi}_g(f^{[2]})}{(j^{[1]})^* \tilde{\psi}_g(f^{[1]})} \tag{9.2.8}$$

is a section of the line bundle $\boldsymbol{\omega}_{\mathcal{D}}^{\otimes c(0,0)}$. By directly comparing the algebraic and analytic constructions, which ultimately boils down to the comparison of algebraic and analytic obstructions found in Proposition 6.5.3, one can check that it agrees with the $\psi_g(f)$ defined at the beginning of the proof.

Define the *renormalized* regularized theta lift

$$\widetilde{\Theta}_g^{\mathrm{reg}}(f^{[i]}) = \Theta_g^{\mathrm{reg}}(f^{[i]}) + 2 \sum_{m>0} c^{[i]}(-m,0) \sum_{\substack{\lambda \in \Lambda^{[i]} \\ Q(\lambda)=m}} \log \|\mathrm{obst}_\lambda^{an}\|$$

so that

$$-2 \log \|\tilde{\psi}_g(f^{[i]})\| = \widetilde{\Theta}_g^{\mathrm{reg}}(f^{[i]}). \tag{9.2.9}$$

Combining this with (9.2.8) yields

$$-2\log\|\psi_g(f)\| = (j^{[2]})^*\widetilde{\Theta}^{\mathrm{reg}}_g(f^{[2]}) - (j^{[1]})^*\widetilde{\Theta}^{\mathrm{reg}}_g(f^{[1]}).$$

As was noted in Remark 5.2.1, the regularized theta lift $\Theta^{\mathrm{reg}}_g(f^{[i]})$ is *over-regularized*, in the sense that its definition makes sense at every point of $\mathcal{D}^{[i]}$, even at points of the divisor along which $\Theta^{\mathrm{reg}}_g(f^{[i]})$ has its logarithmic singularities. As in [AGHMP17, Proposition 5.5.1], its values along the discontinuity agree with the values of $\widetilde{\Theta}^{\mathrm{reg}}_g(f^{[i]})$, and in fact we have

$$(j^{[i]})^*\Theta^{\mathrm{reg}}_g(f^{[i]}) = (j^{[i]})^*\widetilde{\Theta}^{\mathrm{reg}}_g(f^{[i]})$$

as functions on $\mathcal{D}$.

On the other hand, for each $i \in \{1,2\}$, the regularized theta lift has the form

$$\Theta^{\mathrm{reg}}_g(f^{[i]})(z) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} f^{[i]}(\tau)\vartheta^{[i]}(\tau,z,g)\ \frac{du\,dv}{v^2}$$

as in (5.2.1). As in [BY09, (4.16)], when we restrict the variable z to $\mathcal{D}\subset\mathcal{D}^{[i]}$ the theta kernel factors as

$$\vartheta^{[i]}(\tau,z,g) = \vartheta(\tau,z,g)\cdot\vartheta^{[i]}(\tau),$$

where $\vartheta(\tau,z,g)$ is the theta kernel defining $\Theta^{\mathrm{reg}}_g(f)$. Thus

$$(j^{[i]})^*\Theta^{\mathrm{reg}}_g(f^{[i]}) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} f(\tau)\vartheta(\tau,z,g)\cdot\frac{\vartheta^{[i]}(\tau)}{24\Delta}\ \frac{du\,dv}{v^2}$$

Combining this last equality with (9.2.1) proves the first equality in

$$\begin{aligned}\Theta^{\mathrm{reg}}_g(f) &= (j^{[2]})^*\Theta^{\mathrm{reg}}_g(f^{[2]}) - (j^{[1]})^*\Theta^{\mathrm{reg}}_g(f^{[1]})\\ &= (j^{[2]})^*\widetilde{\Theta}^{\mathrm{reg}}_g(f^{[2]}) - (j^{[1]})^*\widetilde{\Theta}^{\mathrm{reg}}_g(f^{[1]}),\end{aligned}$$

which is just a more explicit statement of [Bor98, Lemma 8.1]. Combining this with (9.2.8) and (9.2.9) shows that $\psi(f)$ satisfies the norm relation (9.1.1), and completes the proof of Theorem 9.1.1. □

9.3. **A remark on sufficient divisibility.** In order to obtain a Borcherds product $\psi(f)$ on the integral model $\mathcal{S}_K(G,\mathcal{D})\to\mathrm{Spec}(\mathbb{Z}_\Omega)$, Theorem 9.1.1 requires that we first multiply the integral form

$$f(\tau) = \sum_{\substack{m\in\mathbb{Q}\\ m\gg-\infty}} c(m)\cdot q^m \in M^!_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}})$$

by some unspecified positive integer N. In fact, examination of the proof shows that $N = N(V_{\mathbb{Z}})$ depends only on the quadratic lattice $V_{\mathbb{Z}}$, and not on the choice of f, the finite set of primes Ω, or the level subgroup K.

Indeed, one first checks this in the situation of Proposition 9.1.2. Thus we assume that $n\geqslant 6$, and that there exists an $h\in G(\mathbb{A}_f)$ and isotropic vectors $\ell,\ell_*\in hV_{\mathbb{Z}}$ with $[\ell,\ell_*]=1$. As in the proof of that proposition, one can reduce to the case

$$K = G(\mathbb{A}_f)\cap C(V_{\widehat{\mathbb{Z}}})^\times.$$

The only point in the proof of Proposition 9.1.2 where one must replace f by Nf is when Theorem 5.2.2 and Proposition 5.2.3 are invoked to obtain the Borcherds product (9.1.4) over the complex fiber $\mathrm{Sh}_K(G,\mathcal{D})_{/\mathbb{C}}$. Thus we only need to require that N be chosen divisible enough that the multipliers

$$\xi_g(f) : G(\mathbb{Q})^\circ \cap gKg^{-1} \to \mathbb{C}^\times$$

of (5.2.5) satisfy $\xi_g(f)^N = 1$, as f varies over all integral forms as above and $g \in G(\mathbb{A}_f)$ runs over the finite set of indices in

$$\bigsqcup_g (G(\mathbb{Q})^\circ \cap gKg^{-1})\backslash \mathcal{D}^\circ \cong \mathrm{Sh}_K(G,\mathcal{D})_{/\mathbb{C}}.$$

This is possible, as the natural map

$$G(\mathbb{Q})^\circ \cap gKg^{-1} \to \mathrm{SO}(gV_\mathbb{Z})$$

has kernel $\{\pm 1\}$, and its image has finite abelianization; see [Bor00].

The general case follows by examining the constructions of §9.2. Applying the special case above to the lattices in (9.2.2) yields positive integers $N(V_\mathbb{Z}^{[1]})$ and $N(V_\mathbb{Z}^{[2]})$. Any multiple of

$$N(V_\mathbb{Z}) = N(V_\mathbb{Z}^{[1]}) \cdot N(V_\mathbb{Z}^{[2]})$$

is then "sufficiently divisible" for the purposes of Theorem 9.1.1.

9.4. **Modularity of the generating series.** For any positive $m \in \mathbb{Q}$ and any $\mu \in V_\mathbb{Z}^\vee/V_\mathbb{Z}$, we denote again by

$$\mathcal{Z}(m,\mu) \in \mathrm{Pic}(\mathcal{S}_K(G,\mathcal{D}))$$

the line bundle defined by the Cartier divisor of the same name. Extend the definition to $m = 0$ by

$$\mathcal{Z}(0,\mu) = \begin{cases} \boldsymbol{\omega}^{-1} & \text{if } \mu = 0 \\ \mathcal{O}_{\mathcal{S}_K(G,\mathcal{D})} & \text{otherwise.} \end{cases}$$

Recall from §5.1 the Weil representation

$$\rho_{V_\mathbb{Z}} : \widetilde{\mathrm{SL}}_2(\mathbb{Z}) \to \mathrm{Aut}_\mathbb{C}(S_{V_\mathbb{Z}})$$

on $S_{V_\mathbb{Z}} = \mathbb{C}[V_\mathbb{Z}^\vee/V_\mathbb{Z}]$.

Theorem 9.4.1. *Let $\phi_\mu \in S_{V_\mathbb{Z}}$ be the characteristic function of the coset $\mu \in V_\mathbb{Z}^\vee/V_\mathbb{Z}$. For any $\mathbb{Z}$-linear map $\alpha : \mathrm{Pic}(\mathcal{S}_K(G,\mathcal{D})) \to \mathbb{C}$ we have*

$$\sum_{\substack{m \geqslant 0 \\ \mu \in V_\mathbb{Z}^\vee/V_\mathbb{Z}}} \alpha(\mathcal{Z}(m,\mu)) \cdot \phi_\mu \cdot q^m \in M_{1+\frac{n}{2}}(\rho_{V_\mathbb{Z}}).$$

Proof. According to the modularity criterion of [Bor99, Theorem 3.1], a formal q-expansion

$$\sum_{\substack{m \geqslant 0 \\ \mu \in V_\mathbb{Z}^\vee/V_\mathbb{Z}}} a(m,\mu) \cdot \phi_\mu \cdot q^m$$

with coefficients in $S_{V_{\mathbb{Z}}}$ defines an element of $M_{1+\frac{n}{2}}(\rho_{V_{\mathbb{Z}}})$ if and only if

$$0 = \sum_{\substack{m \geqslant 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu) \cdot a(m,\mu) \tag{9.4.1}$$

for every

$$f(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(m,\mu) \cdot q^m \in M^!_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}}).$$

By the main result of [McG03], it suffices to verify (9.4.1) only for $f(\tau)$ that are integral, in the sense of Definition 5.1.2.

For any integral $f(\tau)$, Theorem 9.1.1 implies that

$$\boldsymbol{\omega}^{c(0,0)} = \sum_{\substack{m>0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu) \cdot \mathcal{Z}(m,\mu)$$

up to a torsion element in $\mathrm{Pic}(\mathcal{S}_K(G,\mathcal{D}))$, and hence

$$\sum_{\substack{m \geqslant 0 \\ \mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}}} c(-m,\mu) \cdot \mathcal{Z}(m,\mu) \in \mathrm{Pic}(\mathcal{S}_K(G,\mathcal{D}))$$

is killed by any $\mathbb{Z}$-linear map $\alpha : \mathrm{Pic}(\mathcal{S}_K(G,\mathcal{D})) \to \mathbb{C}$. Thus the claimed modularity follows from the result of Borcherds cited above. $\square$

9.5. **Modularity of the arithmetic generating series.** Bruinier [Bru02] has defined a Green function $\Theta^{\mathrm{reg}}(F_{m,\mu})$ for the divisor $\mathcal{Z}(m,\mu)$. This Green function is constructed, as in (5.2.1), as the regularized theta lift of a harmonic Hejhal-Poincare series

$$F_{m,\mu} \in H_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}})$$

whose holomorphic part, in the sense of [BF04, §3], has the form

$$F^+_{m,\mu}(\tau) = \left(\frac{\phi_\mu + \phi_{-\mu}}{2} \right) \cdot q^{-m} + O(1),$$

where $\phi_\mu \in S_{V_{\mathbb{Z}}}$ is the characteristic function of the coset $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. See [AGHMP17, §3.2] and the references therein.

This Green function determines a metric on the corresponding line bundle, and so determines a class

$$\widehat{\mathcal{Z}}(m,\mu) = (\mathcal{Z}(m,\mu), \Theta^{\mathrm{reg}}(F_{m,\mu})) \in \widehat{\mathrm{Pic}}(\mathcal{S}_K(G,\mathcal{D}))$$

for every positive $m \in \mathbb{Q}$ and $\mu \in V_{\mathbb{Z}}^{\vee}/V_{\mathbb{Z}}$. Recall that that we have defined a metric (4.2.3) on the line bundle $\boldsymbol{\omega}$, and so obtain a class

$$\widehat{\boldsymbol{\omega}} \in \widehat{\mathrm{Pic}}(\mathcal{S}_K(G,\mathcal{D})).$$

We define

$$\widehat{\mathcal{Z}}(0,\mu) = \begin{cases} \widehat{\boldsymbol{\omega}}^{-1} & \text{if } \mu = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 9.5.1. *Suppose $n \geqslant 3$. For any $\mathbb{Z}$-linear functional*

$$\alpha : \widehat{\mathrm{Pic}}(\mathcal{S}_K(G,\mathcal{D})) \to \mathbb{C}$$

we have

$$\sum_{\substack{m \geqslant 0 \\ \mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}}} \alpha\big(\widehat{\mathcal{Z}}(m,\mu)\big) \cdot \phi_\mu \cdot q^m \in M_{1+\frac{n}{2}}(\rho_{V_{\mathbb{Z}}}).$$

Proof. The assumption that $n \geqslant 3$ implies that any form

$$f(\tau) = \sum_{\substack{m \in \mathbb{Q} \\ \mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}}} c(m,\mu) \cdot q^m \in M^!_{1-\frac{n}{2}}(\overline{\rho}_{V_{\mathbb{Z}}}).$$

has negative weight. As in [BF04, Remark 3.10], this implies that any such f is a linear combination of the Hejhal-Poincare series $F_{m,\mu}$, and in fact

$$f = \sum_{\substack{m > 0 \\ \mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}}} c(-m,\mu) \cdot F_{m,\mu}.$$

This last equality follows, as in the proof of [BHY15, Lemma 3.10], from the fact that the difference between the two sides is a harmonic weak Maass form whose holomorphic part is $O(1)$. In particular, we have the equality of regularized theta lifts

$$\Theta^{\mathrm{reg}}(f) = \sum_{\substack{m > 0 \\ \mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}}} c(-m,\mu) \cdot \Theta^{\mathrm{reg}}(F_{m,\mu}).$$

Now assume that f is integral. After replacing f by a positive integer multiple, Theorem 9.1.1 provides us with a Borcherds product $\psi(f)$ with arithmetic divisor

$$\widehat{\mathrm{div}}(\psi(f)) = (\mathrm{div}(\psi(f)), -\log\|\psi(f)\|^2) = \sum_{\substack{m > 0 \\ \mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}}} c(-m,\mu) \cdot \widehat{\mathcal{Z}}(m,\mu).$$

On the other hand, in the group of metrized line bundles we have

$$\widehat{\mathrm{div}}(\psi(f)) = \widehat{\boldsymbol{\omega}}^{\otimes c(0,0)} = -c(0,0) \cdot \widehat{\mathcal{Z}}(0,0).$$

The above relations show that

$$\sum_{\substack{m \geqslant 0 \\ \mu \in V_{\mathbb{Z}}^\vee / V_{\mathbb{Z}}}} c(-m,\mu) \cdot \widehat{\mathcal{Z}}(m,\mu) \in \widehat{\mathrm{Pic}}(\mathcal{S}_K(G,\mathcal{D}))$$

is a torsion element for any integral f. Exactly as in the proof of Theorem 9.4.1, the claim follows from the modularity criterion of Borcherds. □

References

[AGHMP17] F. Andreatta, E. Z. Goren, B. Howard, and K. Madapusi Pera, *Height pairings on orthogonal Shimura varieties*, Compos. Math. **153** (2017), no. 3, 474–534.

[AGHMP18] ——— *Faltings heights of abelian varieties with complex multiplication*, *Ann. of Math.* **187** (2018), no. 2, 391-531.

[BO78] P. Berthelot and A. Ogus, *Notes on Crystalline Cohomology*, Princeton University Press,1978.

[Bor95] R. Borcherds, *Automorphic forms on* $\mathrm{O}_{s+2,2}(\mathbf{R})$ *and infinite products*, Invent. Math. **120** (1995), no. 1, 161–213.

[Bor98] ——— *Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998), no. 3, 491–562.

[Bor99] ———, *The Gross-Kohnen-Zagier theorem in higher dimensions*, Duke Math. J. **97** (1999), no. 2, 219–233.

[Bor00] ———, *Correction to: "The Gross-Kohnen-Zagier theorem in higher dimensions" [Duke Math. J.* **97** *(1999), no. 2, 219–233]*, Duke Math. J. **105** (2000), no. 1, 183–184.

[Bru02] J.H. Bruinier, *Borcherds products on* $O(2,l)$ *and Chern classes of Heegner divisors*, Lecture Notes in Mathematics, Springer, 2002.

[BBK] J.H. Bruinier, J.I. Burgos Gil, and Ulf Kühn, *Borcherds products and arithmetic intersection theory on Hilbert modular surfaces*, Duke Math. J. **139** (2007), no. 1, 1–88.

[BF04] J.H. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. **125** (2004), no. 1, 45–90.

[BHY15] J.H. Bruinier, B. Howard, and T.H. Yang, *Heights of Kudla-Rapoport divisors and derivatives of L-functions*, Invent. Math. **201** (2015), no. 1, 1–95.

[BY09] J.H. Bruinier and T.H. Yang, *Faltings heights of CM cycles and derivatives of L-functions*, Invent. Math. **177** (2009), no. 3, 631–681.

[Con14] B. Conrad, *Reductive group schemes*, Autour des schémas en groupes. Vol. I, Panor. Synthèses, vol. 42/43, Soc. Math. France, Paris, 2014, pp. 93–444.

[Del79] P. Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., ZZZIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 247–289.

[FC90] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 22, Springer-Verlag, Berlin, 1990, With an appendix by David Mumford.

[Ger08] L. Gerstein, *Basic quadratic forms*, Graduate Studies in Mathematics, vol. 90, American Mathematical Society, Providence, RI, 2008. MR 2396246 (2009e:11064)

[GR84] H. Grauert and R. Remmert, *Coherent analytic sheaves*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 265, Springer-Verlag, Berlin, 1984.

[Har84] M. Harris, *Arithmetic vector bundles on Shimura varieties*, Automorphic forms of several variables (Katata, 1983), Progr. Math., vol. 46, Birkhäuser Boston, Boston, MA, 1984, pp. 138–159.

[Har85] ———, *Arithmetic vector bundles and automorphic forms on Shimura varieties. I*, Invent. Math. **82** (1985), no. 1, 151–189.

[Har86] ———, *Arithmetic vector bundles and automorphic forms on Shimura varieties. II*, Compositio Math. **60** (1986), no. 3, 323–378.

[Har89] ———, *Functorial properties of toroidal compactifications of locally symmetric varieties*, Proc. London Math. Soc. (3) **59** (1989), no. 1, 1–22.

[HZ94a] M. Harris and S. Zucker, *Boundary cohomology of Shimura varieties. I. Coherent cohomology on toroidal compactifications*, Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 3, 249–344.

[HZ94b] ———, *Boundary cohomology of Shimura varieties. II. Hodge theory at the boundary*, Invent. Math. **116** (1994), no. 1-3, 243–308.

[HZ01] ———, *Boundary cohomology of Shimura varieties. III. Coherent cohomology on higher-rank boundary strata and applications to Hodge theory*, Mém. Soc. Math. Fr. (N.S.) (2001), no. 85, vi+116.

[Har70] R. Hartshorne. *Ample subvarieties of algebraic varieties*, Lecture Notes in Mathematics, Vol. 156, Springer-Verlag, Berlin-New York, 1970.

[Hör10] F. Hörmann, *The arithmetic volume of Shimura varieties of orthogonal type*, Ph.D. thesis, Humbolt-Universität zu Berlin, 2010.

[Hör14] ———, *The geometric and arithmetic volume of Shimura varieties of orthogonal type*, CRM Monograph Series, vol. 35, American Mathematical Society, Providence, RI, 2014.

[How19] B. Howard *Linear invariance of intersections on unitary Rapoport-Zink spaces*, Forum. Math. **31** (2019), no. 5, 1265–1281.

[HP17] B. Howard and G. Pappas, *Rapoport-Zink spaces for spinor groups*, Compos. Math. **154** (2017), 1050–1118.

[KM16] W. Kim and K. Madapusi Pera, *2-adic integral canonical models*, Forum Math. Sigma **4** (2016), e28, 34.

[Kis10] M. Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. **23** (2010), no. 4, 967–1012.

[Kud97] S. S. Kudla, *Algebraic cycles on Shimura varieties of orthogonal type*, Duke Math. J. **86** (1997), no. 1, 39–78.

[Kud03] ——— *Integrals of Borcherds forms*, Compositio Math. **137** (2003), no. 3, 293–349.

[Kud04] ———, *Special cycles and derivatives of Eisenstein series*, Heegner points and Rankin L-series, Math. Sci. Res. Inst. Publ., vol. 49, Cambridge Univ. Press, Cambridge, 2004, pp. 243–270.

[KR99] S. S. Kudla and M. Rapoport, *Arithmetic Hirzebruch-Zagier cycles*, J. Reine Angew. Math. **515** (1999), 155–244.

[KR00a] ———, *Cycles on Siegel threefolds and derivatives of Eisenstein series*, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 5, 695–756.

[KR00b] ———, *Height pairings on Shimura curves and p-adic uniformization*, Invent. Math. **142** (2000), no. 1, 153–223.

[Lan13] K.-W. Lan. *Arithmetic compactifications of PEL-type Shimura varieties*, vol. 36 of *London Mathematical Society Monographs,* Princeton University Press, 2013

[Mad16] K. Madapusi Pera, *Integral canonical models for Spin Shimura varieties*, Compos. Math. **152** (2016), no. 4, 769–824.

[Mad19] ———, *Toroidal compactifications of integral models of Shimura varieties of Hodge type.*, Ann. Sci. Éc. Norm. Supér. **52** (2019), no. 1, 393–514

[McG03] W. J. McGraw, *The rationality of vector valued modular forms associated with the Weil representation*, Math. Ann. **326** (2003), no. 1, 105–122.

[Mil88] J. S. Milne, *Automorphic vector bundles on connected Shimura varieties*, Invent. Math. **92** (1988), no. 1, 91–128.

[Mil90] ———, *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*, Automorphic forms, Shimura varieties, and L-functions, Vol.

I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 283–414.

[Ogu79] A. Ogus, *Supersingular $K3$ crystals in Journes de Gomtrie Algbrique de Rennes (Rennes, 1978)*, Asterisque **64** (1979), 3–86

[Ogu01] ——— *Singularities of the height strata in the moduli of $K3$ surfaces*, Moduli of abelian varieties, Progr. Math., vol. 195, Birkhäuser, Basel, 2001, pp. 325–343.

[Pin89] R. Pink, *Arithmetical compactification of mixed Shimura varieties*, Ph.D. thesis, Bonn, 1989.

[PS08] C. Peters and J. Steenbrink, *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 52, Springer-Verlag, Berlin, 2008.

[Wil00] J. Wildeshaus, *Mixed sheaves on Shimura varieties and their higher direct images in toroidal compactifications*, J. Algebraic Geom. **9** (2000), no. 2, 323–353.

Department of Mathematics, Boston College, 140 Commonwealth Ave, Chestnut Hill, MA 02467, USA

E-mail address: `howardbe@bc.edu`

Department of Mathematics, Boston College, 140 Commonwealth Ave, Chestnut Hill, MA 02467, USA

E-mail address: `madapusi@bc.edu`