The Design of Tasks that Address Applications to Teaching Secondary Mathematics for Use in Undergraduate Mathematics Courses

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Abstract
This paper describes theoretical design principles emerging from the development of tasks for standard undergraduate mathematics courses that address applications to teaching secondary mathematics. While researchers recognize that mathematical knowledge for teaching is a form of applied mathematics, applications to teaching remain largely absent from curriculum resources for courses for mathematics majors. We developed various materials that contain applications to teaching that have been integrated into four standard undergraduate mathematics courses. Three primary principles influenced the design of the tasks that prepare future teachers to learn and apply mathematics in a manner central to their future work. Additionally, this paper provides guidance for instructors desiring to develop or implement similar applications. The process of developing these tasks underscores the importance of key features regarding the roles of human beings in the tasks, the intentional focus on advanced content connected to school mathematics, and the integration of active engagement strategies.

Keywords: Design principles; Applications to teaching; Secondary mathematics; Mathematical knowledge for teaching

1. Introduction

Because the population of undergraduates enrolled in courses for mathematics majors includes prospective secondary mathematics teachers, these courses can play a large role in providing future teachers with opportunities for developing mathematical knowledge for teaching (MKT). However, research has shown that many prospective secondary mathematics teachers complete these courses without having gained sufficiently deep understanding of knowledge central for teaching (Speer, King, & Howell, 2015; Zazkis & Leikin, 2010) or having made connections between what they are studying and what they will be teaching (Wasserman, 2018). One way to address prospective teachers' MKT is to embed applications to teaching into these courses, which mirrors the common inclusion of other science applications in undergraduate mathematics courses. Yet, Lai and Patterson (2017) document that applications to teaching remain largely absent from textbooks for courses for mathematics majors.

Common textbooks contain many applications relevant for traditional applied mathematics majors, such as ballistic motion in calculus (see Fig. 1) or statistical analysis used to interpret the effectiveness of clinical trials. Whereas emerging physicists or engineers see many mathematics applications linked to their future work, prospective secondary mathematics teachers in traditional mathematics courses do not. Researchers (e.g., Bass, 2005; Cuoco, 2018) posit that mathematical knowledge for teaching is a form of applied mathematics. Thus, applications of mathematics to teaching should be brought on par with other, already-ubiquitous applications of mathematics.

8. If a ball is thrown vertically upward with a velocity of 80 ft/s, then its height after t seconds is $s = 80t - 16t^2$.

   a. What is the maximum height reached by the ball?

   b. What is the velocity of the ball when it is 96 ft above the ground on its way up? On its way down?

Fig. 1. Example of a common application to applied mathematics (Stewart, 2016).

The Association of Mathematics Teacher Educators (AMTE) released the Standards for Preparing Teachers of Mathematics (AMTE, 2017) in which they recommend that prospective teachers need more than strong content knowledge to effectively teach. Teaching involves interacting with students. Teachers must attend to students' mathematical reasoning during instruction by explaining mathematical concepts, responding to questions, linking ideas to past and future courses, posing meaningful questions to assess and advance students' understanding, and using evidence of student thinking (AMTE, 2017; Ball, Thames, & Phelps, 2008). Future teachers need opportunities to engage in these practices throughout their teacher preparation programs. Grossman et al. (2009) refers to these opportunities as *approximations of practice*, or "opportunities to engage in practices that are more or less proximal to the practices of a profession" (p. 2058). Thus, by developing applications to teaching and incorporating them into undergraduate mathematics major courses, the mathematics education community can provide prospective secondary teachers opportunities in content courses to simultaneously develop a deep understanding of content central to teaching secondary mathematics alongside skills and practices fundamental for teaching. Providing support while prospective teachers learn to navigate a student's work and ways of thinking, for instance, may better prepare them to use this skill when they begin teaching in a secondary classroom.

We take the stance that while approximations of practice are especially important for prospective teachers as they situate mathematics in the practice of their future careers, they are equally beneficial for undergraduates who are not obtaining a teaching credential. Many undergraduate mathematics majors see themselves in a teaching role in the future, whether that be tutoring or attending graduate school and becoming a teaching assistant. In addition, "learning by teaching" is effective and strengthens everyone's understanding and retention of the concept (Fiorella & Mayer, 2013). Including applications of mathematics to teaching does not detract from the learning of mathematics, in the same way that including applications to engineering does not detract from the learning of mathematics.

Through our work on the META Math project, we have developed and field tested various materials that contain applications to teaching that have been integrated into four standard undergraduate courses: Calculus I, Abstract Algebra, Discrete Mathematics, and Statistics. The classrooms in which these materials have been field tested have included student populations composed of (1) mostly prospective teachers, (2) a mixture of both prospective teachers and undergraduates not intending to become teachers, and (3) mostly undergraduates not intending to become teachers. Through this field testing, we have learned about how to develop materials that meet instructors' expectations for preparing future teachers and engage undergraduates in a way that helps them understand the connections between the mathematics they will teach and the mathematics they are learning.

In this paper, we describe our efforts in developing materials that embed applications to teaching in undergraduate mathematics courses, establishing theoretical design principles as a way to provide guidance for instructors who want to develop and implement similar applications to teaching in their own courses. Rather than presenting empirical data from field testing, what we share herein is the set of design principles at which we have arrived based on our work with instructors and undergraduates over the past two years. We further highlight different features of these materials, concluding with insights we have gained in the process of development.

1.1 Connections to Teaching Secondary Mathematics

The Mathematical Education of Teachers II (MET II) Report of the Conference Board of the Mathematical Sciences (CBMS) recommends that future secondary mathematics teachers encounter opportunities to explicitly connect the advanced mathematics concepts they learn as part of their continuing education to the primary or secondary school mathematics concepts they will eventually teach to their own students (CBMS, 2012). They provide examples of such connections, such as: "Linear equations and functions are prominent in secondary school mathematics, and geometric interpretations of them in higher dimensions can deepen teachers' understanding of these notions" (CBMS, 2012, p. 58). While this observation is mathematically valuable, the MET II Report does not explicitly define *connections* or give practical insight into how such connections can be effectively integrated into traditional undergraduate mathematics major courses. Mathematics instructors who do not specialize in K-12 mathematics education may find it particularly difficult to adapt their existing lecture notes or lesson plans to include such connections (Lai, 2016; Álvarez & Burroughs, 2018; Álvarez & White, 2018).

To better identify *connections to teaching*, we developed a framework which delineates five different conceptualizations of how a connection to teaching might manifest in the context of a lesson (Table 1; Arnold et al., 2020). The five connections are grounded in existing literature on MKT, proposed by Ball et al. (2008), where it is suggested that teachers need not only advanced content knowledge but also a kind of pedagogical knowledge that is enhanced by that content knowledge. For example, it is important that teachers "be able to hear and interpret students' emerging and incomplete thinking" (Ball et al., 2008, p. 401), a pedagogical activity only possible with a deep understanding of the content being taught.

Table 1
Five connections to teaching and their descriptions.

Connection	Description
Content Knowledge	Undergraduates use course content in applied contexts or to answer mathematical questions in the course.
Explaining Mathematical Content	Undergraduates justify mathematical procedures or theorems and use of related mathematical concepts.
Looking Back/ Looking Forward	Undergraduates explain how mathematics topics are related over a span of K-12 curriculum through undergraduate mathematics.
School Student Thinking	Undergraduates evaluate the mathematics underlying a student's work and explain what that student may understand.

Guiding School Students' Understanding	Undergraduates pose or evaluate guiding questions to help a hypothetical student understand a mathematical concept and explain how the questions may guide the student's learning.

We have developed nine lessons that can be integrated into Calculus I, Discrete Mathematics, Abstract Algebra, or Statistics curricula. Our development team consists of four mathematicians who are mathematics education researchers and two mathematics education graduate students. All six members have experience in high school teaching settings, and the mathematics education researchers specialize in mathematics teacher preparation and professional development. Each lesson is textbook-independent, designed to involve active learning, and is intended to span approximately two 50-minute class periods. Each lesson consists of a class activity, along with suggested homework and assessment questions. We have inserted tasks that embed applications to teaching throughout these lessons to make our five connections to teaching school mathematics explicit.

1.2. Approximations of Practice

Teaching is a human activity. As such, mathematics teachers' interactions with learners interweave their content knowledge with their capacity to respond to student thinking and with their perceptions of students as learners. Teachers are called on to engage in interpersonal interactions that require both mathematical expertise and skills for probing student thinking or finding meaning in learners' perspectives. Prospective teachers need to gain experience with the variety of questions, conjectures, and ideas arising from learners. The uncertainty about what K-12 students may ask or how they may respond to teachers' questions contributes to the complexity and challenge of teacher preparation.

Grossman and colleagues (2009) propose a framework for thinking about "the teaching of practice" and preparing future teachers for conditions of uncertainty while in a university setting rather than a clinical setting. One of three concepts they have identified in their framework is "approximations of practice," which refers to opportunities given to beginning teachers to engage in high-leverage practices fundamental to teaching — in other words, practices that are important for deepening students' understanding and also practices that advance their knowledge for teaching. Specific examples in mathematics may include reading a vignette of a teacher facilitating a class discussion, watching a video of a class engaged in a number talk, interpreting a student's thinking, designing a lesson, and playing the role of a teacher by implementing a portion of a lesson to their peers. These approximations of practice lie on a continuum of authenticity, and Grossman et al. (2009) state that even though these approximations are not "entirely authentic in terms of their audience or execution, they can provide opportunities for students to experiment with new skills, roles, and ways of thinking with more support and feedback than actual practice in the field allows" (p. 2077). Grossman et al. posit that approximations of practice play a critical role in teacher preparation and may be a way to bridge the gap between what prospective teachers do in their teacher preparation courses and what they actually experience in their own classrooms.

The framework for approximations of practice is intended to span across subject areas, grade levels, and context (Grossman et al., 2009). In this work, we focus on approximations of practice as applied to secondary mathematics, with a broad focus on teaching practices that advance a prospective teacher's mathematical knowledge in analyzing and interpreting students' thinking and guiding students' understanding. Certainly, there are approximations of practice that are not applications of mathematics to teaching. For example, a prospective teacher might write a letter to hypothetical parents to introduce a hypothetical classroom; it is an approximation of a real teaching task that mathematics teachers will do in their professional work, but it is not an application of mathematics to teaching. In this paper, we provide examples of applications of mathematics to teaching, some of which incorporate approximations of practice, while others address mathematical knowledge for teaching without incorporating approximations of practice.

1.3 Mathematics in its Human Context

Mathematics and teaching are human activities embedded in social and cultural contexts; effective teachers understand their students as people and humanize their classrooms. A humanizing classroom is one that places the ideas of students at its center, celebrates different approaches and understandings, and encourages students to interpret and understand each other's mathematical thinking. Bishop (1988) states that to humanize a mathematics classroom, teachers must "create a particular kind of social environment" while learners "construct ideas and modify them in interaction with that environment" and describes how curriculum can support such a structure.

Secondary mathematics teachers can have difficulty in recognizing the cultural context of a mathematics classroom, or to see mathematics as part of a social and political space (Parker, Bartell, & Novak, 2017). Mathematics education researchers are called on to address cultural, social, and political aspects of this work, including the work of teacher preparation, as part of participation in a discipline that holds equity at the forefront (Aguirre et al., 2017; Gutierrez, 2018). In this paper, we focus on applications of mathematics to secondary teaching in an explicitly human context, so that the prospective teachers who engage with our materials will see that the human context of mathematics is held on par with the mathematics content.

1.4 Designing Applications of Mathematics to Teaching

The development of tasks that address applications of mathematics to teaching derives from the theory and research on task design. Research on what makes a mathematical task "effective" along with frameworks for task design has received growing attention from the mathematics education community (e.g., Smith & Stein, 1998; Watson & Ohtani, 2015; Liljedahl, Chernoff, & Zazkis, 2007). The use of different mathematical tasks in the classroom can lead to different kinds of learning opportunities for students. Watson and Ohtani articulate that attention to task design in research and in classroom practice is important from a cognitive, cultural, and practical perspective. Cognitively demanding tasks (see Smith & Stein, 1998) have a substantial impact on students' learning and conceptual understanding by providing students with the opportunity to "do" mathematics. Watson and Ohtani (2015) discuss that from a cultural perspective, mathematical tasks shape the students' experience with mathematics. Culture, among other aspects, plays a large role in one's learning experience. Practically speaking, mathematical tasks are a staple of the mathematics classroom; they are the "things to do" (Watson & Ohtani, 2015, p. 3).

There are many features to consider when designing a mathematical task. The design process begins with considerations regarding the purpose and curricular aim of the task. Is the task focused on helping students learn a mathematical concept, a pedagogical concept, or some specialized or practical aspect of mathematics (see Watson & Ohtani, 2015)? It is also important to incorporate knowledge of mathematics and pedagogy when designing tasks, especially when considering how the tasks will be used. Multiple researchers tie the work of task design to Ball et al.’s (2008) theory of MKT. Sullivan, Knott, and Yang (2015), for instance, describe how knowledge for teaching mathematics and knowledge of pedagogical practices are two key components of task design. Liljedahl et al. (2007) expand on the interaction between these knowledge bases. They specify four ways tasks are used in teacher education: the use of mathematics to promote understanding of mathematics; the use of pedagogy to promote understanding of mathematics; the use of mathematics to promote understanding of pedagogy; and the use pedagogy to promote understanding of pedagogy. Additionally, Thanheiser et al. (2015) offer three design aspects to consider when creating tasks for prospective teachers: cognitive demand level, authenticity in terms of connections to the K-12 classroom, and the extent to which tasks provide opportunities for prospective teachers to develop their MKT.

Other researchers, drawing inspiration from the MKT framework, have designed tasks and frameworks specifically for prospective secondary mathematics teachers. Heid, Wilson, and Blume (2015) focused solely on mathematical understandings, and their Mathematical Understanding for Secondary Teaching (MUST) framework is grounded in classroom practice. They designed teaching prompts that allow prospective teachers to analyze secondary mathematics content and describe what mathematical understanding a teacher would use. Moreover, Lai et al. (2019) focused on improving prospective teachers’ experiences in undergraduate mathematics courses by relying on pedagogical contexts. They used approximations of practice (Grossman et al., 2009) embedded in pedagogical contexts and designed a framework to analyze prospective teachers’ development of MKT. Wasserman et al. (2019) developed an instructional model that examines how advanced mathematics in standard undergraduate mathematics courses is related to school mathematics. They focused on both mathematical and pedagogical contexts and used pedagogical teaching situations, such as how a teacher responded to a student, to motivate the advanced mathematics undergraduates would learn.

We focus on designing materials for prospective secondary mathematics teachers that will be used by a variety of undergraduates. Like Wasserman et al. (2019) and Lai et al. (2019), we consider both mathematical and pedagogical contexts in the design of our tasks. To draw attention to the practice of teaching, we embed our five connections between undergraduate mathematics and school mathematics in these tasks.

2. Design Principles

We refer to *applications of mathematics to teaching* as *tasks* that situate undergraduate mathematics topics in the context of teaching secondary mathematics. The tasks we developed are influenced by the five connections to teaching we defined in section 1.1. Because learning to teach requires interpersonal interactions with other people, in this paper we focus on tasks that include hypothetical situations that feature human beings. We created these tasks with two objectives in mind: (1) to scaffold undergraduates’ advancement of content learning goals or (2) to provide undergraduates opportunities to engage in practices necessary for mathematics teaching. Our three design principles, *Habit of Respect*, *Active Engagement*, and *Recognition of Mathematics as a Human Activity*, guided the overall development of these tasks.

How did we arrive at these three design principles? Liljedahl et al. (2007) describe a recursive process for developing good tasks: predictive analysis, trial, reflective analysis, and adjustment. This aligns with the process we used in designing the tasks and arriving at our design principles. That is, our initial efforts at designing and predicting the affordances of tasks resulted in some mathematically rich tasks, but when implemented, often left the connections to teaching that we were trying to promote too easy to miss. We reflected throughout the length of the initial semester of field testing on what aspects of our tasks were effective in highlighting our five types of connections and enriched those components. The interviews with instructors and undergraduates, together with examinations of how undergraduates responded to the tasks, gave us valuable insight into revisions that were necessary to make. We also drew inspiration from our own experiences in the K-12 classroom, reflecting on our roles as teachers, mathematics teacher educators, and researchers. After another semester of field testing, we refined even further. Now, the benefit of looking back and reflecting on this process enables us to articulate our three design principles. We did not have a sophisticated enough view to articulate them at the beginning, and only came across them through a process of intense reflection and discussion. What we present here illustrates some of the progression we made and is the final result. We expect that describing our process will help others begin their design process with the benefit of the discoveries we made.

Habit of Respect. Effective teachers validate students' thinking and recognize that when students make errors, they are often basing their reasoning on justifications that make sense to them. The *Habit of Respect* design principle reflects our aim to promote practices that nurture students' assets and understandings and to offer alternatives to deficit-perspectives that focus on lack of understanding. The practice of evaluating student work requires more than the content knowledge needed to assess whether the student's mathematical work is correct or incorrect; it also requires knowledge to assess what a student does and does not understand. This principle centers on helping future teachers to address different perspectives in a manner that conveys respect for student thinking and reasoning and for the students as members of the classroom community, both when a student's work is correct and when it is incorrect.

Active Engagement. All undergraduates should build deep conceptual understanding of the mathematics they are learning, and research has shown that this understanding is fostered when students are actively engaged with these concepts (CBMS, 2016; Freeman et al., 2014). As K-12 mathematics instruction continues to shift to a more student-centered approach, it is necessary for prospective teachers themselves to learn mathematics in such an environment so that they are better equipped to teach in a student-centered manner. Classroom experiences that allow prospective teachers to validate conjectures, justify their reasoning, or reflect on meaningful questions all provide valuable models for their future teaching. They learn to pause to allow students to make sense of new ideas, to invite students to be co-discoverers of mathematical concepts, and to establish mathematical norms that encourage students to take ownership for the creation of mathematics. The *Active Engagement* design principle encompasses the understanding that undergraduates can actively construct knowledge rather than simply be told, for example, a method to use, though there are times when it is unreasonable to expect undergraduates to "invent" a clever strategy or method on their own. At the heart of this design principle is actively engaging undergraduates in developing understanding and deriving meaning underlying the methods, theorems, or ideas relevant for a mathematical task.

Recognition of Mathematics as a Human Activity. Mathematics teachers require more than a deep underlying understanding of the mathematics they will teach because the practice of teaching involves interacting with people, the learners of the mathematics. Thus, prospective teachers need to be prepared to communicate mathematical content to other human beings. During any given day, teachers have numerous interactions with their students: teachers listen and respond to students' questions; teachers ask students questions; teachers probe student thinking; teachers evaluate students' work and decide how to help move students' mathematical ideas forward when their ideas are incomplete; teachers consider students' posed conjectures. For all of these human interactions, teachers' fluent understanding of the mathematical content they are teaching must be coupled with an understanding of how to interact with students and their mathematical work. The *Recognition of Mathematics as a Human Activity* design principle simultaneously addresses content knowledge and the types of interpersonal interactions that are valuable to future teachers.

3. Illustrations of the Design Principles

Embedded throughout the META Math lessons are various tasks that feature mathematics in a human context. The examples we provide in this section were developed for the purpose of inserting approximations of practice into the curriculum or to advance content learning goals during the lesson. In the subsections below, we illustrate our design principles with these examples. While the examples often include more than one of the three design principles, tasks focusing on approximations of practice aligned closely with the *Recognition of Mathematics as a Human Activity* design principle, and tasks intended to advance content learning goals aligned closely with the *Active Engagement* design principle. In all of our tasks, we strive to incorporate the *Habit of Respect* design principle to provide more experiences with appropriate language, non-deficit thinking, and positive dispositions in mathematics teaching that reflect this essential habit for teachers.

3.1. Examples of Approximations of Practice

Approximations of practice provide future teachers opportunities to practice and learn high-leverage skills necessary for teaching while in the relatively simplified setting of a university classroom, without the complexities and distractions of a clinical setting. We use the following categories presented in Table 2 to label the various types of approximations of practice tasks we developed, though many tasks encompass more than one of these categories.

Table 2
Categories of the "approximations of practice" tasks embedded in our lessons

Category	Description
Analyzing Mathematical Reasoning	Undergraduates are presented with a student's mathematical work or conjecture. Undergraduates may be asked to (1) speculate on the reasoning a student may have used to make their conjecture or to carry out their work, (2) identify if the student's reasoning is flawed, (3) explain when the student's reasoning works, or (4) explain what mathematical understanding may underlie the student's reasoning.
Examining Overgeneralization	Hypothetical students make a conjecture or use mathematical reasoning based on methods or theorems that students typically may memorize, and their work or conjecture inappropriately overgeneralizes or applies the method or theorem. Undergraduates are asked to consider when the students' methods are inconsistent, incomplete, or when the method fails. Overgeneralization is a common source of student mathematical misconceptions (Van Dooren, De Bock, Janssens, & Verschaffel, 2008), so while the *Examining Overgeneralization* tasks might be viewed as a subset of

	Analyzing Mathematical Reasoning, we choose to set them apart to emphasize their importance and prevalence.
Encountering Multiple Perspectives	Undergraduates are presented with two or more students' mathematical work on the same problem. Each student uses different reasoning, but the mathematical work is valid. Undergraduates are prompted to explain why the student reasoning is correct and how to help students understand different perspectives.
Highlighting Teaching Decisions	Undergraduates are presented with a teaching decision made by a hypothetical teacher and are asked to explain the value of that decision.
Posing and Evaluating Questions	Undergraduates are presented with student work that contains flawed reasoning. Often paired with an *Analyzing Mathematical Reasoning* task wherein undergraduates are asked to first describe what the student does and does not understand, these tasks then further prompt undergraduates to write or evaluate questions they can ask the student to guide their understanding

3.1.1. Analyzing Mathematical Reasoning

A daily teaching practice entails analyzing the mathematical reasoning evidenced in student work. Teachers need to be able to assess whether a student's mathematical work is correct. If so, why is the work valid? If not, where are there flaws in the student's reasoning, when did they occur, and what understanding does the work convey? *Analyzing Mathematical Reasoning* tasks approximate this teaching practice by presenting hypothetical student work, both correct and incorrect, and prompting undergraduates to determine what mathematical ideas the student does or does not understand and to identify, if appropriate, where an error occurred. These tasks may explicitly state whether a student's work is correct or leave the undergraduate to establish the accuracy of the student's work. The provision of this information depends on whether the task intends to focus undergraduate discourse on *if* the work is accurate or on *why* the work is flawed. Consider the example presented in Fig. 2 from a lesson on foundations of divisibility used in a discrete mathematics class, focused on a hypothetical student, Adam. The problem explicitly tells undergraduates that Adam's conjecture is incorrect and prompts undergraduates to identify where the error occurs.

2. Adam incorrectly claims that if a number is divisible by both 2 and 10, then it is divisible by 20. Adam's proof to his conjecture is shown below. Identify the error in his proof and explain why it is an error.

   Let n be any integer that is divisible by 2 and 10. By the definition of divisibility, since n is divisible by 2, there is an integer k where $n=2k$. Since n is also divisible by 10, that means that k must be divisible by 10. By definition of divisibility, there is an integer l where $k=10l$. Using substitution, $n=2k = 2(10l)=20l$. Since $n=20l$ for some integer l, n is divisible by 20.

Fig. 2. Adam's task: An example of an Analyzing Mathematical Reasoning task from the Foundations of Divisibility lesson in Discrete Mathematics.

Alternatively, Fig. 3 presents a task from a lesson in abstract algebra focused on solving equations in $\mathbb{Z}_n$. Students are prompted to analyze Thuy's work, which contains an error, but the undergraduate is not told whether her work is correct. Undergraduates must first assess the accuracy of Thuy's work themselves and describe what underlying assumptions she makes.

2. Thuy's work for finding solutions to $x^2 - x = 0$ in $\mathbb{Z}_4$ is shown below.

   $x^2 - x = 0$
   $x(x-1) = 0$
   Therefore, either
   $x = 0$ or $x - 1 = 0$
   The solution set is $\{0, 1\}$

   a. From her work, what assumption does Thuy seem to be making about $\mathbb{Z}_4$? Is this assumption correct?
   b. Thuy checks each element of $\mathbb{Z}_4$ and verifies that her solution set is correct. Her teacher asks her to attempt to solve the same equation, this time in $\mathbb{Z}_6$. What is the teacher hoping Thuy will understand about her approach by working in $\mathbb{Z}_6$?

Fig. 3. Thuy's task: An example of an Analyzing Mathematical Reasoning task from the Solving Equations in $\mathbb{Z}_n$ lesson in Abstract Algebra.

3.1.2 Examining Overgeneralization

Examining Overgeneralization tasks can illustrate instances where hypothetical students apply a rule or method incorrectly and ask undergraduates to explain why the students might have made this error. In the following example from the foundations of divisibility lesson in discrete mathematics, Olivia first learns the divisibility rule for six and then incorrectly applies similar logic to develop a divisibility rule for 60 (Fig. 4). Here, Olivia may not recognize that the divisibility rule for six holds because the factors 2 and 3 are relatively prime. These types of tasks provide opportunities for prospective teachers to consider how and why students can overgeneralize a rule and to think about how they would respond to a student's conjecture in a way that considers and respects the mathematical knowledge the student demonstrated. We often drew inspiration from our own experiences in the K-12 classroom and with common student mistakes to develop these types of tasks.

2. Olivia says that "because 60 is divisible by 6 and 10, if a number is divisible by both 6 and 10, then the number is divisible by 60."
   a. Provide two reasons why you think Olivia made this conjecture?
   b. Olivia's conjecture is false. Provide a counterexample showing that it is false.
   c. Write two questions you can ask Olivia to help her understand that her conjecture is not true. Explain how your questions might help Olivia.

Fig. 4. Olivia's task: An example of an Examining Overgeneralization task from the Foundations of Divisibility lesson in Discrete Mathematics.

Other *Examining Overgeneralization* tasks feature hypothetical students applying commonly taught methods that expire (Dougherty, Bush, & Karp, 2017). For example, in a Calculus I lesson about the derivatives of inverse functions, we prompt undergraduates to consider the mathematical ideas and properties underlying a method for finding inverses commonly taught in high school that leads to difficulties in many contexts. Undergraduates are first presented with Alex's work in finding the inverse of *f(x)=(2/3)x+1*, as shown in Fig. 5. Alex employs the method of "switching *x* and *y* and solving for *y,*" a method undergraduates often use themselves. Later, in problem 2 of the lesson (see Fig. 6), undergraduates are presented with contexts in which Alex's method introduces mathematical inconsistencies and are asked to describe why this common method is mathematically incomplete. In working through this sequence of problems, undergraduates will learn that the "switch *x* and *y* and solve for *y*" method presents difficulties when the variables in question have associated units or different domains. The goal is to help undergraduates learn that this common method used for quick computation, when overgeneralized or used without sufficient attention to the context, detracts from important considerations related to the properties of inverses. This highlights the limitations or expiration of the method for building mathematical meanings.

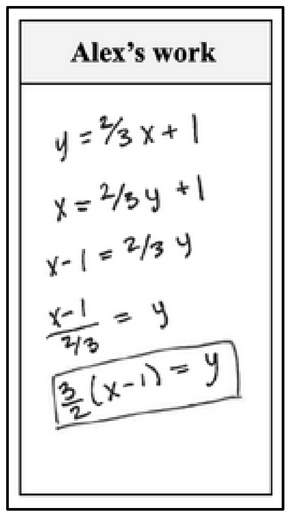


Fig. 5. Alex's work for finding the inverse function for *f(x)=(2/3)x+1*, an excerpt from a task in the Derivative of Inverse Functions lesson in Calculus I.

2. Now consider two problems where a high school student used Alex's method of switching the variables and solving for the dependent variable to find the inverse function.

Find the inverse function of $T(C) = \frac{9}{5}C + 32$ where C is the temperature in Celsius and $F = T(C)$ gives the temperature in Fahrenheit.	Find the inverse function of $f(x) = \frac{2x+1}{x-1}$ for $x \neq 1$.
Let $F = \frac{9}{5}C + 32$ $C = \frac{9}{5}F + 32$ $C - 32 = \frac{9}{5}F$ $\frac{5}{9}(C-32) = F$	$y = \frac{2x+1}{x-1}$, $x \neq 1$ $x = \frac{2y+1}{y-1}$ $(y-1)x = 2y+1$ $yx - x = 2y+1$ $yx - 2y = 1+x$ $y(x-2) = x+1$ $y = \frac{x+1}{x-2}$, $x \neq 1$

a. Describe why the student's work for the temperature function is problematic.
b. Describe why the student's work for the rational function is problematic.
c. Why is it problematic to use Alex's method of switching the variables and solving for the dependent variable to find an inverse function?

Fig. 6. Problematic situations that arise in an Examining Overgeneralization task from the Derivatives of Inverse Functions lesson in Calculus I.

3.1.3. Encountering Multiple Perspectives

In the classroom, students often share their ideas and ways of thinking and commonly present different ways to solve a problem. Different approaches can highlight distinct mathematical aspects of a concept or a method. An important teaching practice is to recognize that there are many ways to view and approach a mathematics problem and to acknowledge and link these perspectives to the underlying mathematical ideas. *Encountering Multiple Perspectives* tasks highlight different ways students may solve a problem and provide undergraduates an opportunity to not only see these different approaches, but also investigate the mathematics behind each perspective.

The inverse function task mentioned above includes two other hypothetical students, Jordan and Kelly, as shown in Fig. 7. Undergraduates are asked to examine Alex, Jordan, and Kelly's work, where Jordan and Kelly's work reflects a reliance on the definition of an inverse function, the composition of a function with its inverse, and function notation to arrive at an expression of the inverse that avoids the same difficulties as Alex's method. Examining multiple perspectives in this problem allows undergraduates to examine how a common method, which might be the method they rely on, suppresses important mathematical aspects of inverse functions, and to study other methods that highlight those important aspects.

1. Consider how Alex, Jordan and Kelly found the inverse function of $f(x) = \frac{2}{3}x + 1$.

Alex's work	Jordan's work	Kelly's work
$y = 2/3\,x + 1$ $x = 2/3\,y + 1$ $x - 1 = 2/3\,y$ $\frac{x-1}{2/3} = y$ $\boxed{\frac{3}{2}(x-1) = y}$	$f \circ f^{-1}(y) = y$ $f(x) = \frac{2}{3}x + 1$ So: $f(f^{-1}(y)) = \frac{2}{3}f^{-1}(y) + 1 = y$ $\frac{2}{3}f^{-1}(y) = y - 1$ $\frac{3}{2}\cdot\frac{2}{3}f^{-1}(y) = \frac{3}{2}(y-1)$ $\boxed{f^{-1}(y) = \frac{3}{2}(y-1)}$	$y = \frac{2x}{3} + 1$ $y - 1 = \frac{2x}{3}$ $3(y-1) = 2x$ $\frac{3(y-1)}{2} = x$ But $f(x) = y \Rightarrow$ $f^{-1}(f(x)) = f^{-1}(y) \Rightarrow$ $x = f^{-1}(y)$ So $\boxed{f^{-1}(y) = \frac{3(y-1)}{2}}$

Compare and contrast the key mathematical ideas used by Alex, Jordan, and Kelly to find the inverse function of $f(x) = \frac{2}{3}x + 1$. Make sure to identify which properties of inverse functions each student uses, if any.

Fig. 7. Alex, Jordan, and Kelly's task from the Derivatives of Inverse Functions lesson in Calculus I: Alex's method (from Fig. 6) alongside two other hypothetical students' methods.

Figure 8 presents another example of an *Encountering Multiple Perspectives* task. In a binomial theorem lesson used in discrete mathematics, Phoebe and Anita both correctly state that the coefficient of a^3b^2 in a binomial expansion is 10, but they arrive at their answers differently. Phoebe counts the "*a*'s" and Anita counts the "*b*'s," thus leading to two different combinations that produce the same result. Teachers need to be prepared to help their students recognize that both approaches are correct and to be able to explain *why*. By incorporating these types of tasks into the curriculum, prospective teachers can practice interacting with different students and responding to their different ways of thinking. A teacher who sees the power of understanding multiple paths to a solution may be more open to encouraging their students to look at a problem in multiple ways.

6. Phoebe and Anita were working on the following problem.

Suppose you were to expand

$$(a+b)^5 = (a+b)(a+b)(a+b)(a+b)(a+b)$$

Fill in the coefficients in the expansion:

$$(a+b)^5 = \_\_a^5 + \_\_a^4b + \_\_a^3b^2 + \_\_a^2b^3 + \_\_ab^4 + \_\_b^5$$

Explain how you determined your coefficients.

Phoebe says that the coefficient of a^3b^2 is $\binom{5}{3} = 10$ because she was counting the a's. Anita claims that the coefficient is $\binom{5}{2} = 10$ since she was counting the b's. Write two questions you could ask Phoebe and Anita to help them resolve this. Explain how your questions might help them.

Fig. 8. Phoebe and Anita's task: An example of an Encountering Multiple Perspectives task from the Binomial Theorem lesson in Discrete Mathematics.

3.1.4. Highlighting Teaching Decisions

Teaching involves making many decisions. Teachers decide, for instance, how to structure a classroom, how to sequence a course of lessons, how to elicit students' ideas, and how to use assessments to enhance learning. Prospective teachers may not be aware of all of these decisions until they are teaching in their own classrooms. *Highlighting Teaching Decisions* tasks bring relevant instructional decisions to light and give undergraduates the opportunity to think about these decisions before they begin teaching in a K-12 classroom.

One instance that illuminates an important instructional decision occurs in part b of Thuy's task in section 3.1.1 (see Fig. 3). Thuy's instructor prompts her to work the same problem in an integral domain that is less conducive to her approach. Undergraduates describe why challenging a student's misconceptions with counterexamples may be more effective than telling them they did a problem wrong, especially when they can see that the student has still gotten the correct answer. In this part of the problem, the undergraduate is analyzing a teaching decision.

Another example focuses on the sequence of topics taught in a curriculum (Fig. 9). In a statistics lesson on variability, undergraduates learn about different measures of spread, including mean absolute deviation (MAD) and standard deviation (SD). Teachers are often called on to "look back" and make connections to concepts previously taught. Middle school students typically calculate and use the MAD as a measure of variability and this measure of variability is then built upon in high school as students learn about SD. Thus, this task prompts undergraduates to consider why it is important for students to first learn MAD before SD.

2. Consider different measures of variability.
    a. Why might it be helpful for students to learn MAD (mean absolute deviation) before SD (standard deviation)?

Fig. 9. A Highlighting Teaching Decisions task from the Variability lesson in Statistics.

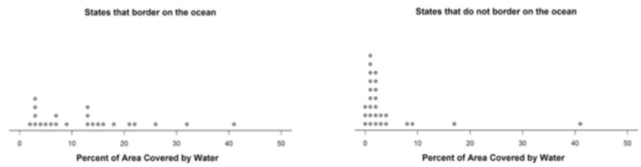
3.1.5. Posing and Evaluating Questions

NCTM (2014) states that "Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships" (p. 10). Thus, a common practice of teaching is to ask students mathematical questions about their work, which can be challenging for novice teachers. *Posing and Evaluating Questions* tasks are intended to help undergraduates understand different types of questions that can be asked and how those questions may or may not be helpful to the student. This practice of posing questions often occurs after the teacher has evaluated student work in some way, meaning that these types of tasks occur after undergraduates complete a corresponding *Analyzing Mathematical Reasoning* task.

In some instances, as demonstrated by Zayn's task in a lesson on variability used in Statistics (Fig. 10), undergraduates are asked to generate questions on their own that they could ask a student. Then, undergraduates are prompted to explain how the questions may be helpful. In this task, undergraduates apply their content knowledge and develop skills to respond to incorrect student work in a manner that conveys respect for student thinking. Undergraduates evaluate the student work to not only determine whether the solution is correct, but also to determine what that solution reveals about student thinking. This serves the purpose of allowing undergraduates to respectfully consider what the student understands when thinking of questions to pose that help guide student learning. Undergraduates can recognize that Zayn's answer has some validity to it: while the median and IQR may be easier measures to compute, that is not a *statistical* reason to use these measures over the mean and standard deviation. In this task, undergraduates gain experience in posing questions that help pinpoint what students understand about the problem, a valuable practice for deepening students' conceptual understanding.

4. Zayn, a high school student, is working on the following question.

Carlton found data on the percent of area that is covered by water for each of the 50 states in the U.S. He made the dotplots below to compare the distributions for states that border an ocean and states that do not border an ocean.



What is the best statistical reason for using the median and IQR, rather than the mean and standard deviation, to compare the centers and spreads of these distributions?

Zayn gives the following incorrect answer:
"*The median and IQR are easier to calculate than the mean and standard deviation.*"

a. Why is he incorrect?
b. Write a question that you could ask Zayn to help him revise his work. Explain how your question could help his understanding.

Fig. 10. Zayn's task (adapted from the LOCUS Project, 2020): An example of a Posing and Evaluating Questions task from the Variability lesson in Statistics.

In other instances, undergraduates are prompted to evaluate pre-written questions and explain how they may or may not help guide a student's understanding. For instance, we could alter part b in Zayn's task above to include an assortment of pre-written questions (Fig. 11). Here, undergraduates are asked to explain how these questions (1) assess understanding, (2) advance understanding, and (3) overlook student work or strategies that may be leveraged to help the student reflect on or build their understanding. Assessing questions are meant to gather information about what the student does or does not understand. Advancing questions are those that "build on, but do not take over or funnel, student thinking" (NCTM, 2014, p. 41). Questions that are not helpful may, for instance, ignore the work students have completed or directly inform students on the next step to complete.

b. Consider the following questions that one might ask Zayn to help him revise his work.
   i. Explain how the following question may help a teacher assess what Zayn understands.
      "Can you explain why your answer is a statistical reason?"
   ii. Explain how the following question may help Zayn to advance in his understanding of summary measures?
      "How do you calculate all of those summary measures? How is each data point used when calculating each measure?"
   iii. Explain why the following question may not help Zayn in revising his work.
      "What about using the mean and standard deviation instead?"

Fig. 11. An example of including pre-written questions in Zayn's task in a lesson on variability in Statistics.

3.2. Examples of Advancing Content Learning Goals

All of the examples in this paper involve interacting with human beings about mathematics; yet, that alone suffices neither for classifying them as approximations of practice nor for ensuring that they provide future teachers with opportunities to engage in teaching practices. Some tasks that involve people scaffold the delivery of formal mathematical content at the heart of a course. These tasks remain situated in the context of teaching but in such a way that the undergraduates focus on the *mathematical content* rather than teaching practices. We categorized the tasks we developed for the purpose of advancing content learning goals as shown in Table 3.

Table 3

Categories of the "advancing content learning goals" tasks embedded in our lessons

Category	**Description**
Introducing Ideas	A goal in several of the tasks is to introduce a method or proof technique that is new to undergraduates. Following our *Active Engagement* design principle, we created these tasks to introduce new mathematical ideas in a manner that allows undergraduates to construct their own knowledge, rather than simply being presented with a method or technique.
Applying Content Knowledge	Some tasks draw upon undergraduates' (possibly recently acquired) advanced content knowledge. The decisions they make and the responses to and observations of the work of the human beings in the tasks requires the undergraduates to apply their content knowledge to resolve mathematical questions.

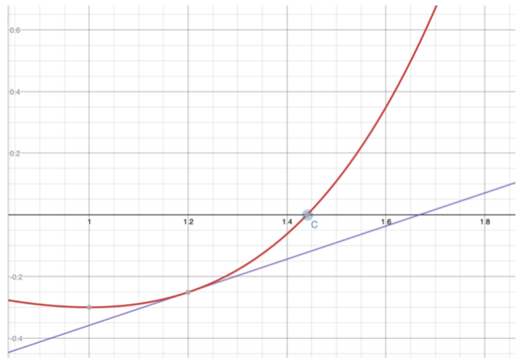
3.2.1. Introducing Ideas

Some of our materials introduce or scaffold important mathematical ideas mostly grounded in relatively direct methods of instruction; we made this choice with classroom time constraints in mind or because it was unreasonable to expect undergraduates to "invent" a clever strategy or procedure on their own. In these cases, we situate the instruction in an unfolding narrative of hypothetical students working through a mathematical problem, using the hypothetical students' discoveries and sticking points to focus undergraduates' attention on important mathematical concepts and on forming mathematical meanings (or, on sense making and reasoning in problem solving).

The sequence of questions in a Newton's Method lesson in Calculus I (excerpt from lesson shown in Fig. 12) illustrates our use of tasks that introduce mathematical ideas in the context of human beings engaging in thinking and reasoning. These two questions graphically introduce the first few steps of Newton's Method by incorporating a collaborative discussion that arises in a hypothetical group of students. Undergraduates follow Nnamdi and Mari's thought processes, and in doing so, they perform the necessary first steps of Newton's Method while also considering how these steps are beneficial for estimating zeros of a function. The process is not presented as fact and Nnamdi and Mari are not authorities, so undergraduates are more likely to engage in examining the appropriateness of their work.

Consider the function, $f(x) = \frac{1}{10}x^5 - \frac{1}{2}x + \frac{1}{10}$, for which Nnamdi wanted to find zeros. Nnamdi initially thinks that $x = 1.2$ is a good estimate of the zero, C.

1. Nnamdi zooms in on the graph and sketches the tangent line at $x_0 = 1.2$ (see graph below).



    a. Label the x-intercept of Nnamdi's tangent line as x_1.
    b. Write the equation of Nnamdi's tangent line in point-slope form and find the value of x_1.

2. Mari continues Nnamdi's idea of using a tangent line and she sketches the tangent line to the graph of $f(x)$ at the point $(x_1, f(x_1))$. She claims that the x-intercept of her tangent line seems to be closer to the zero C than x_1.
    a. Sketch in Mari's tangent line. Label the x-intercept of her tangent line as x_2.
    b. Do you agree with Mari's claim? Explain why or why not.
    c. Write the equation of Mari's tangent line in point-slope form and find the value of x_2.

Fig. 12. Nnamdi, Mari, and Amy's task: An example of Introducing Ideas tasks from the Newton's Method lesson in Calculus.

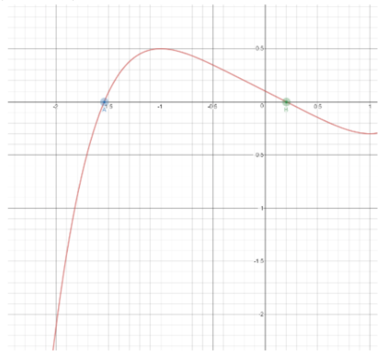
3.2.2. Applying Content Knowledge

After learning a new theorem or method, undergraduates should be able to apply this knowledge to solve mathematical questions. This next sequence of questions from the Newton's Method lesson (see Fig. 13) illustrates tasks where the focus is on undergraduates applying their content knowledge to both new and similar mathematical situations.

3. Amy said she used both Mari's and Nnamdi's ideas to find a point, x_3, even closer to the zero C.
   a. What do you think she did? Explain.
   b. Find x_3.

⋮

6. Reconsider $f(x) = \frac{1}{10}x^5 - \frac{1}{2}x + \frac{1}{10}$. Nnamdi now wants to use Newton's method to approximate the zero, A. He wonders what will happen if he uses the following initial guesses: -0.5, -1, -1.5, and -2.



   a. Without doing any calculations, what zero of f do you expect each of these initial guesses to lead to and why? Use the graph above to graphically show (by drawing tangent lines) what happens when you apply Newton's method using these initial guesses.
      i. $x = -0.5$
      ii. $x = -1$
      iii. $x = -1.5$
      iv. $x = -2$
   b. Use Newton's method with all four initial guesses to calculate a zero of f. Give your answer to three decimal places, when applicable.
   c. Summarize to Nnamdi what you observe in the graph of f that indicates what zero you will approximate given your initial guess.

Fig. 13. Nnamdi, Mari, and Amy's task: Examples of Applying Content Knowledge tasks from the Newton's Method lesson in Calculus.

The work that undergraduates are asked to do in problem 3 is to apply what they learned from Nnamdi and Mari's reasoning in problems 1 and 2 (Fig. 12) to describe and justify the process Amy used. To answer problem 6a (Fig. 13), undergraduates need to generalize the procedures from problems 1 and 2, which involves them in the process of creating a method for determining zeros of a function based upon their understanding of Nnamdi's and Mari's reasoning. This requires them to engage at a deeper level than if they are simply following steps that have been outlined for them. In problem 6b, undergraduates are asked to apply Newton's Method (which they just formalized) with different initial guesses to calculate a zero of a function. Finally, in problem 6c, undergraduates return to the context of Nnamdi to summarize what they have learned graphically about Newton's Method. The setup for this task involves Nnamdi's attempts to estimate the zero using tangent lines, and the sequence of questions a, b, and c engages the undergraduates in circling back to Nnamdi's initial question. By asking undergraduates to summarize their ideas to Nnamdi, the task utilizes the narrative device of Nnamdi as a peer learner in calculus; the undergraduates learn alongside him, with the end result that they are building experience with explaining what they have learned to another mathematics learner.

4. Guidance for Creating Tasks that Address Applications of Mathematics to Teaching

The design principles and types of tasks described in this paper can guide the development of tasks that include applications of mathematics to teaching. Figure 14 depicts the relationship between our design principles and the tasks we designed to enhance prospective teachers' MKT by embedding mathematics in a human context. These tasks were guided by our three design principles and each task includes one or more of our five connections to teaching. We can further categorize tasks based on whether we intended the task to help undergraduates advance content learning goals or to engage undergraduates in practices fundamental to teaching secondary school mathematics.

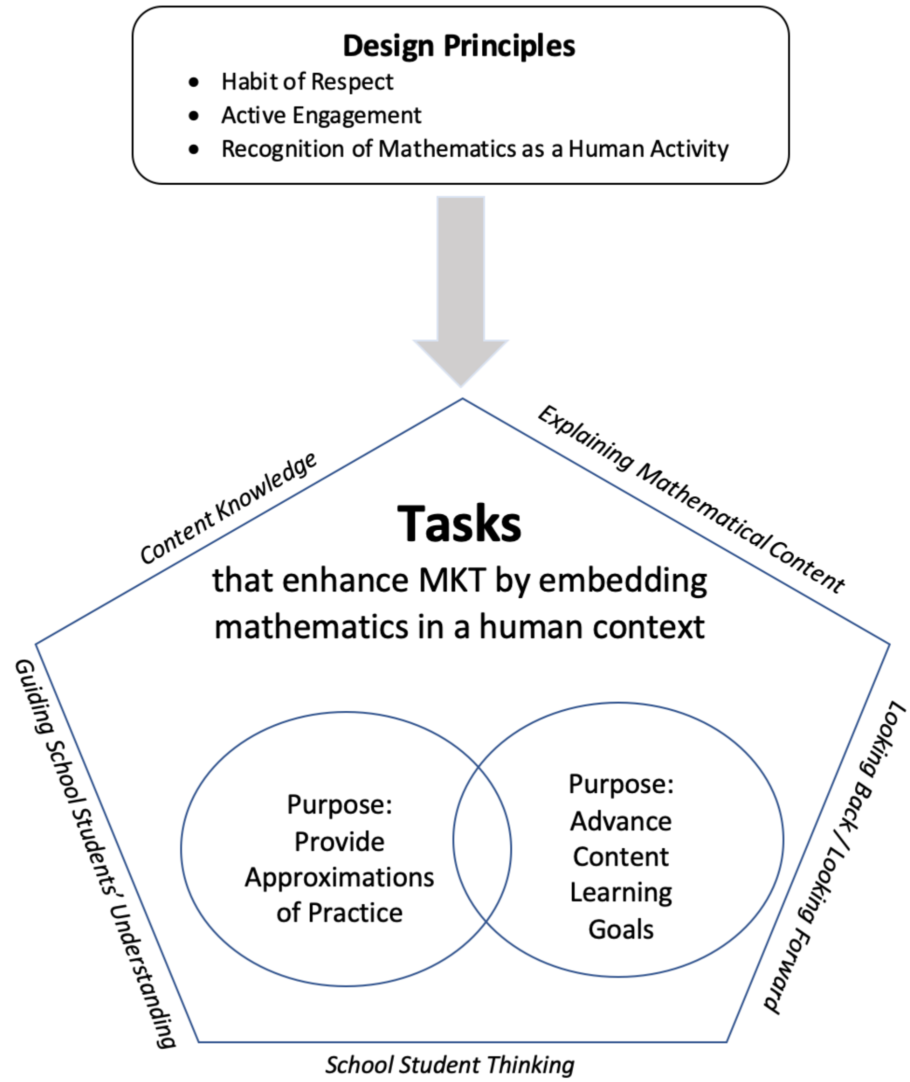


Fig. 14. Depiction of the relationship between our design principles and tasks that embed connections to teaching

The tasks we developed for the META Math lessons have been field tested in several university classes across the United States in the last two years. By analyzing how instructors used these tasks and how undergraduates responded to them, we gained valuable insight that influenced key revisions. For instance, instructors and undergraduates provided useful feedback about what worked well in the tasks and what could be improved. In their interviews, many expressed positive impressions of encountering hypothetical student work in the tasks. Instructors, for instance, liked how the hypothetical students make similar mistakes as their own undergraduates. One instructor pointed out that an early iteration of a task that involved a hypothetical student did not require the hypothetical student context in any meaningful way. This feedback led us to enhance certain attributes of our hypothetical student work. We also learned how much time instructors spent on each task and whether more guidance on tasks was needed. Examining undergraduates' written responses to our tasks also afforded us the opportunity to reflect on the quality of their responses. We used this information to revise tasks to address discrepancies between the type of response we expected and those received.

In an effort to help instructors build from our experience when creating these tasks, we describe what we learned and offer three important recommendations to consider when developing them: (1) give human beings meaningful roles in tasks; (2) focus undergraduates' attention on central ideas; and (3) provide undergraduates sufficient scaffolding. We further elaborate on these recommendations in the following sections.

4.1 Give Human Beings Meaningful Roles in Tasks

We became more clear about how we included the hypothetical human beings in our tasks and identified ways that the role of a human being contributed to the learning of mathematics. There was a purpose for the inclusion of humans, and they were not just "window dressing" to a typical mathematics problem. We looked for ways to incorporate the human context in a meaningful way; if the task could be posed without the human context and the meaning of the task did not change, we refined the human's role to ensure that it was more than ancillary.

In our first drafts of the materials, all of the tasks that included a human were in the category of *approximations of practice*. For example, the Phoebe and Anita task (Fig. 8) provided undergraduates in discrete mathematics an opportunity to first view two students' differing (and correct) approaches to find the coefficient of a term in a binomial expansion. The task prompted undergraduates to consider how they, as teachers, would help Phoebe and Anita understand that both methods were appropriate. But, as we continued developing applications to teaching, we added hypothetical students to tasks meant to advance content learning goals. For instance, the Newton's Method task from section 3.2 (Figs. 12 and 13) did not originally include the human beings Nnamdi, Mari, and Amy. Rather, the task contained a set of instructions for undergraduates to follow from which they were to uncover Newton's Method. We found that the task took too long, did not achieve our active engagement design principle, and undergraduates were not taking away the insights and connections to teaching we intended. We adjusted by shifting the focus of this task to follow three students' reasoning. Inserting Nnamdi, Mari, and Amy enhanced the way undergraduates engaged with the Newton's Method task. We found that this inclusion humanized the mathematics in the lesson and encouraged undergraduates to consider mathematical ideas rather than taking a presented idea as "true" because an authority stated it. These humans provided a pacing device for the task and using them lets us "pause" the task in places where we want undergraduates to deeply think about content before moving on to the next part of the task.

When incorporating human beings into tasks, we also became more intentional about the names of the hypothetical students we used. To better reflect gender, cultural, and ethnic diversity, we strove to include names that allowed a variety of genders and represented a range of possible cultural ties and ethnicities (e.g., Thuy, Zayn, Nnamdi, Amy). We also chose to include realistic names. For example, the Alex, Kelly, and Jordan task from the inverse lesson in calculus initially used the names Tom, Jerry, and Felix, and we found we were unsatisfied that these evoked images of cartoon characters rather than people.

4.2 Focus Undergraduates' Attention on Central Ideas

Undergraduates can be easily distracted by or drawn toward examining computations, and we became intentional about refining tasks to reduce computations and to provide guidance about whether the work we are presenting is error free. In all of our tasks, regardless of mathematical content, we found that when undergraduates analyzed student work, they were more likely to check the accuracy of computations than to think about the deeper mathematical ideas underlying the computations or methods shown in the work. As an example, one early version of Alex, Jordan, and Kelly's task from the Calculus I lesson on derivatives of inverse functions (see Fig. 7) instead featured Tom, Jerry, and Felix finding the inverse of a rational function (Fig. 15).

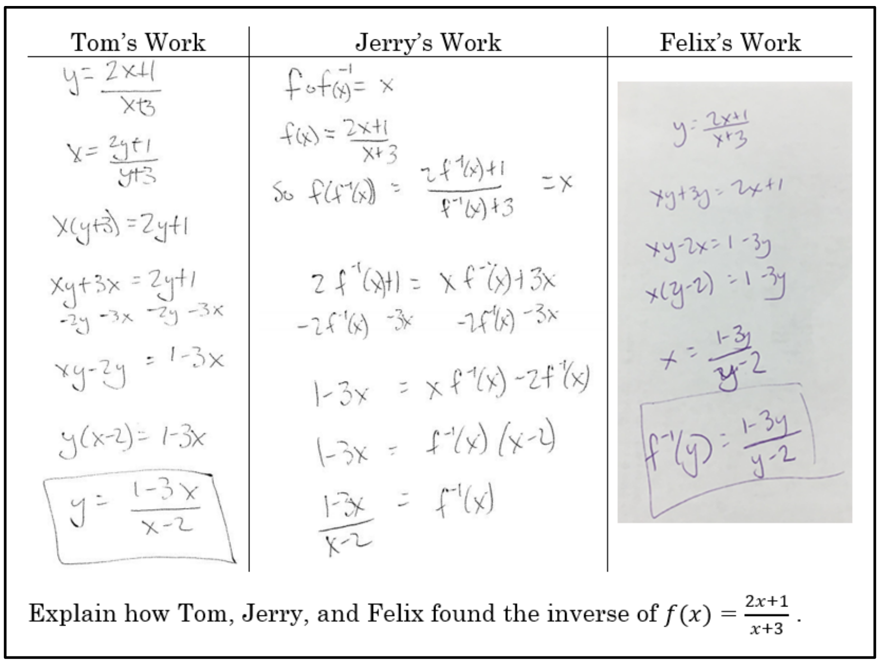


Fig. 15. Version 1 of three students finding the inverse of a function.

Because the rational function involves relatively more complex computations for the undergraduates, instructors reported that too much class time was used investigating the computational work of Tom, Jerry, and Felix. To focus undergraduates' attention on the conceptual ideas we had intended for this task to highlight, we changed the function from a rational function to a polynomial function of degree 1 (referred to as linear functions in school mathematics) (Fig. 16) and encouraged instructors to let their undergraduates know that all computations were correct. The simplicity of the linear function minimized opportunities for undergraduates to get bogged down in checking computations.

1. Consider how Alex, Jordan and Kelly found the inverse function of $f(x) = \frac{2}{3}x + 1$.

Alex's work	Jordan's work	Kelly's work
$y = 2/3\, x + 1$ $x = 2/3\, y + 1$ $x - 1 = 2/3\, y$ $\frac{x-1}{2/3} = y$ $\boxed{\frac{3}{2}(x-1) = y}$	$f \circ f^{-1}(y) = y$ $f(x) = \frac{2}{3}x + 1$ So: $f(f^{-1}(y)) = \frac{2}{3}f^{-1}(y) + 1 = y$ $\frac{2}{3}f^{-1}(y) = y - 1$ $\frac{3}{2} \cdot \frac{2}{3} f^{-1}(y) = \frac{3}{2}(y-1)$ $\boxed{f^{-1}(y) = \frac{3}{2}(y-1)}$	$y = \frac{2x}{3} + 1$ $y - 1 = \frac{2x}{3}$ $3(y-1) = 2x$ $\frac{3(y-1)}{2} = x$ But $f(x) = y \Rightarrow$ $f^{-1}(f(x)) = f^{-1}(y) \Rightarrow$ $x = f^{-1}(y)$ So $\boxed{f^{-1}(y) = \frac{3(y-1)}{2}}$

Compare and contrast the key mathematical ideas used by Alex, Jordan, and Kelly to find the inverse function of $f(x) = \frac{2}{3}x + 1$. Make sure to identify which properties of inverse functions each student uses, if any.

Fig. 16. Alex, Jordan, and Kelly's task: Version 2 of three students finding the inverse of a function.

We also provided more structure in the questions we wanted undergraduates to consider in an attempt to focus on the conceptual understanding of each approach. When prompted to "Explain how…" in the first version of this task, undergraduates felt compelled to list the sequence of algebraic manipulations of each student rather than attend to the mathematical reasoning that gave rise to their method. The second version prompts undergraduates for a comparison, which leaves less room for surface-level commentary, and reminds undergraduates to identify important properties of inverse functions. Finally, we found the first version of Felix's work needlessly opaque about how the student used the composition property of inverses. By making the mathematical reasoning explicit, we gave undergraduates a physical place to point to in discussions about how Felix (now named Kelly) used the properties of inverse functions.

4.3 Provide Undergraduates Sufficient Scaffolding

The *Active Engagement* design principle inspired us to think of ways we could incorporate the right balance of openness and structure in our tasks. We wanted to present undergraduates with opportunities to consider new ideas and ask students questions after analyzing their hypothetical work. Initially, some of our tasks were too open, leaving undergraduates unsure of how to respond, as well as leaving instructors unsure of how to help undergraduates proceed with the problems. This became readily apparent in tasks where undergraduates were prompted to first analyze student work and then write questions and justify how they would help guide a student's understanding. Figure 17 illustrates an early draft of such a task in a discrete mathematics lesson on the binomial theorem. In this problem undergraduates were presented with Henry, a high school student, and his incorrect work expanding a binomial. In part c, we asked undergraduates to write questions they would ask Henry to help him correct his work.

1. Henry, a high school student, expanded $(2x-y)^4$ using the Binomial Theorem and made some errors. Below is his work.

$$(2x-y)^4 = 2x^4 - 8x^3y - 12x^2y^2 - 8xy^3 - y^4$$

   a. What errors did Henry make?
   b. What does Henry understand about the Binomial Theorem?
   c. What questions can you ask Henry to help him correct his work? Explain how your questions might help Henry.

Fig. 17. Version 1 of Henry's task in a Binomial Theorem lesson in Discrete Mathematics.

Undergraduates found it difficult to come up with questions and were often unsure of where to start. Instructors also needed more guidance to help their undergraduates with this type of task. While our intention was to provide undergraduates the opportunity to create any type of question, it became clear that more guidance was needed, particularly direction on what questions would be helpful to ask a student who is trying to learn the content.

Teachers ask many types of questions of students, and some questions are more beneficial than others. For instance, certain questions advance a student's learning, others assess a student's learning, some clarify what the student did, and some are not as effective in helping the student. Based on the feedback we received and the kinds of questions undergraduates wrote in the initial round of pilot testing, we revised part c of Henry's task to include more guidance (see Fig. 18). Instead of asking undergraduates to generate questions on their own, a skill that takes time to master, we posed pre-written questions and then prompted undergraduates to explain the benefit (or lack thereof) in asking each question. This structure tended to be more productive than the open-ended prompts and helped undergraduates understand that there are a variety of questions a teacher could pose to a student and some are more helpful than others.

1. Henry, a high school student, expanded $(2x-y)^4$ using the Binomial Theorem and made some errors. Below is his work.

$$(2x-y)^4 = 2x^4 - 8x^3y - 12x^2y^2 - 8xy^3 - y^4$$

   a. What errors did Henry make?
   b. What does Henry understand about the Binomial Theorem?
   c. Consider the following questions that one might ask Henry about his work.
      i. Explain how the following question could help Henry to advance in his understanding of the binomial theorem:
         How is $(2x-y)^4$ similar to $(a+b)^4$ and how is it different?
      ii. Explain how the following question can help you assess what Henry understands about the binomial theorem:
         Why doesn't $(-3y)^2 = -3y^2$?
      iii. Explain why the following question would not help Henry:
         Do the exponents look right?

Fig. 18. Version 2 of Henry's task in a Binomial Theorem lesson in Discrete Mathematics.

4.4 Considerations for Instructors before Creating Applications of Mathematics to Teaching

After establishing which undergraduate mathematics course to target for including applications to teaching, instructors should identify topics in the course that build upon core concepts from school mathematics or relate to critical understandings prospective teachers should develop for deeper insight into the mathematics they will teach. While we have integrated applications to teaching into Calculus I, Abstract Algebra, Statistics, and Discrete Mathematics, there are many other mathematics courses that allow appropriate inclusion of such applications. At the core of these tasks is mathematics content that is central to secondary mathematics. Thus, gaining familiarity with the standards and expectations in school mathematics becomes an important component in this process for instructors. Such familiarity may come from studying relevant state standards for mathematics or the Common Core State Standards for Mathematics (NGA, 2010), or from speaking with colleagues with experience in mathematics education or who have recent direct experience teaching K-12 mathematics.

Instructors should consider the purpose of the task. It might be to advance content learning goals during the lesson, or it might be to provide undergraduates with an opportunity to engage in approximations of teaching practices, or it could be a combination of the two. If the intent is to advance content learning goals, consider writing tasks with the *Active Engagement* design principle in mind, as it helps guide and focus undergraduates on learning the underlying mathematical content. If the intent is to focus on teaching practices, consider how to meaningfully incorporate human beings in the problems and consider which teaching practices to address.

In our materials, we have integrated tasks that address applications of mathematics to teaching throughout class activities, homework questions and assessment items. Incorporating tasks into class activities provides a supportive environment for undergraduates to first encounter and discuss these ideas with their peers. Homework tasks allow undergraduates to develop their skills and mathematical independence in situations that are analogous to those of the class activity. Finally, the inclusion of these tasks on assessments aligns assessment with in-class and homework activities, reinforces the value placed on using mathematical knowledge in this way, and provides the opportunity for undergraduates to demonstrate their understanding. Full lessons and reports of implementation are available with or without an MAA membership in the *META Lessons on the Mathematical Knowledge for Teaching* community at MAA Connect (MAA, 2020).

4.5 Considerations for Researchers Concerning the Design of Tasks Addressing MKT for Secondary Mathematics Teachers

Our work aims to embed applications to secondary teaching in undergraduate mathematics courses using lessons that include a class activity, homework set, and associated assessment items. Similar to Wasserman et al. (2019), we intend for our curriculum materials to highlight explicit connections to teaching in undergraduate mathematics major courses. In addressing the development of MKT for secondary teachers, we focus on five specific connections, acknowledging that there are other aspects of teaching that are not incorporated into our tasks. Unique to our tasks is the implicit attention to habit of respect and interacting with other human beings, both of which are central to the work of teaching. Furthermore, both Wasserman et al. (2019) and Heid et al. (2015) use secondary mathematics teaching situations as a prompt at the outset of the lesson to motivate the learning of advanced mathematics content. In our work, the teaching application is used both as a vehicle for bridging undergraduates' advanced mathematical knowledge to secondary school mathematics and for strengthening undergraduates' understanding of the advanced mathematics from an encounter with school mathematics. While Heid et al.'s (2015) situations focus on the mathematical understandings teachers need to address the situation, we use our tasks to probe undergraduates' understandings and to lay the foundation for questioning strategies that place value on analyzing student thinking. Our focus on the process of communication involves the mathematics and gives leverage to the ways in which ideas are communicated by placing value on student thinking and highlighting aspects of human interactions.

5. Conclusion

Our three design principles, *Habit of Respect*, *Active Engagement*, and *Recognition of Mathematics as a Human Activity*, are embedded throughout our tasks to help prepare future teachers to learn and apply mathematics in a way that is central to their future work. We arrived at these principles after having begun the process of designing tasks and reflecting on the features that were common to our tasks. The *Habit of Respect* design principle recognizes that students do present work or offer solutions that might be incorrect, and teachers must learn to address these still-forming notions in their students in an affirming manner that conveys placing value on student thinking. Our *Active Engagement* design principle relies on having undergraduates construct their own knowledge rather than expecting the instructor to be responsible for imparting all information. The *Recognition of Mathematics as a Human Activity* places value on how teachers and students interact and communicate in the process of their mathematical learning.

Our tasks targeted opportunities to deepen undergraduates' reasoning about key concepts and methods while also planting seeds for ways in which mathematical interactions can be respectful of others' thinking. They provide examples of how teachers can be facilitators of cognitive restructuring by attempting to understand a student's developing notions rather than to "correct." Encouraging instructors in mathematics departments to implement these tasks in active learning environments also provides prospective secondary mathematics teachers with experiences in their mathematics courses that support the value of student discourse and interaction in the learning of mathematics.

Including applications to teaching in mathematics content courses that prospective secondary mathematics teachers take can address mathematical content in a robust manner. Such applications can advance content learning goals and meet the needs of prospective secondary mathematics teachers as they make connections between the advanced mathematics they are learning, the mathematics they will teach, and the complex human context that is central in the work of teaching.

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