



On a conjecture of Lynch

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ABSTRACT

We comment on a conjecture of Lynch on annihilators of local cohomology.

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The following conjecture has recently attracted attention, e.g. [2–5].

Conjecture 1 [6, conjecture 1.2]. Let R be a local ring, and I an ideal of R . If the cohomological dimension of I is a positive integer c , then

$$\dim R/\operatorname{ann}_R H_I^c(R) = \dim R/H_I^0(R).$$

The conjecture is known to be false: the first counterexamples were constructed in [1]; these are nonequidimensional, with $\dim R \geq 5$. We present here a modification—with a short, elementary proof—that serves as a counterexample with $\dim R = 3$. This is a counterexample, as well, to [6, proposition 4.3] and to [6, theorem 4.4]; the error there is in the chain of inequalities in the proof of Proposition 4.3, [6, p. 550], in the reduction from a complete local ring to a complete local unmixed ring: the cohomological dimension may change under the reduction step.

Example 2. Let k be a field, and set $R := k[x, y, z_1, z_2]/(xyz_1, xyz_2)$. Consider the local cohomology module $H_{(x,y)}^2(R)$. Using a Čech complex on x and y , one sees that

$$H_{(x,y)}^2(R) = R_{xy}/\operatorname{image}(R_x + R_y).$$

The images of z_1 and z_2 are zero in R_{xy} , so the local cohomology module above agrees with $S_{xy}/\operatorname{image}(S_x + S_y)$, where $S := R/(z_1, z_2)$ is isomorphic to the polynomial ring $k[x, y]$. Hence, $H_{(x,y)}^2(R)$ is a nonzero R -module, with annihilator (z_1, z_2) . On the other hand, since the ideal (x, y) contains nonzerodivisors, $H_{(x,y)}^0(R) = 0$. Hence, one has

$$\dim R/\operatorname{ann}_R H_{(x,y)}^2(R) = 2 \quad \text{whereas} \quad \dim R/H_{(x,y)}^0(R) = 3.$$

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