Journal of Scientific Computing (2021) 86:6
https://doi.org/10.1007/s10915-020-01389-5

# Linear Convergence of a Rearrangement Method for the One-dimensional Poisson Equation

**Chiu-Yen Kao[1] · Seyyed Abbas Mohammadi[2] · Braxton Osting[3]**

Received: 10 June 2020 / Revised: 10 November 2020 / Accepted: 12 December 2020 /
Published online: 1 January 2021


**Abstract**

In this paper, we study a rearrangement method for solving a maximization problem associated with Poisson's equation with Dirichlet boundary conditions. The maximization problem is to find the forcing within a certain admissible set as to maximize the total displacement. The rearrangement method alternatively (i) solves the Poisson equation for a given forcing and (ii) defines a new forcing corresponding to a particular super-level-set of the solution. Rearrangement methods are frequently used for this problem and a wide variety of similar optimization problems due to their convergence guarantees and observed efficiency; however, the convergence rate for rearrangement methods has not generally been established. In this paper, for the one-dimensional problem, we establish linear convergence. We also discuss the higher dimensional problem and provide computational evidence for linear convergence of the rearrangement method in two dimensions.

**Keywords** Poisson equation · Rearrangement method · Linear convergence

**Mathematics Subject Classification** 35J05 · 49M05

Chiu-Yen Kao is supported in part by an NSF Grant DMS-1818948. Braxton Osting acknowledges partial support from NSF DMS 16-19755 and 17-52202.

✉ Braxton Osting
osting@math.utah.edu

Chiu-Yen Kao
ckao@cmc.edu

Seyyed Abbas Mohammadi
mohammadi@yu.ac.ir

1 Department of Mathematical Sciences, Claremont McKenna College, Claremont, CA 91711, USA

2 Department of Mathematics, College of Sciences, Yasouj University, Yasouj 75918-74934, Iran

3 Department of Mathematics, University of Utah, Salt Lake City, UT, USA

## 1 Introduction

Rearranging real-valued functions of one or more real variables is a powerful tool of mathematical analysis and plays an important role in various applications. Examples from mathematical analysis include the Schwarz rearrangement and Steiner symmetrization with respect to a hyperplane, which have each been used to prove a variety of isoperimetric inequalities [11,26]. One limitation of using rearrangement methods analytically is that the candidate optimizer must be known, typically requiring a relatively simple geometry, *e.g.*, a ball in $\mathbb{R}^d$. In the case when a candidate optimizer is not known, rearrangement methods can also be used computationally and have been applied to a wide variety of shape and operator coefficient optimization problems, including

1. extremal densities for the vibration of drums [9,18] and rods and plates [6,14,22],
2. extremal environments in a dispersal population dynamics model [7,12,15],
3. extremal conductivities for two-phase conductors [8,13,19,23],
4. extremal potentials in periodic Schrödinger's operator to maximize spectral gaps [17],
5. extremal problems for operators with other boundary conditions, *e.g.*, Robin [16], and
6. nonlinear eigenproblems, *e.g.*, extremal energies in quantum dots [1,20,23,24].

When rearrangement methods are used in each of these settings, it is commonly observed that they are very efficient, converging in far fewer iterations than other optimization approaches, including gradient-based methods. Rearrangement methods, like level-set and phase-field methods [25], are also capable of handling topological changes in the optimizer. However, as far as we are aware, there are no results on the convergence rate of rearrangement methods. We consider a relatively simple, prototypical optimization problem corresponding to the solution of the Poisson equation [3–5]. Recently a convergent rearrangement algorithm was developed to obtain the optimal solution for this problem [21]. The main contribution of this paper is to prove a convergence rate for this rearrangement method.

### 1.1 Mathematical Formulation

For a bounded domain $\Omega \subset \mathbb{R}^d$ and a given function $f : \Omega \to \mathbb{R}$, the *Poisson equation* is given by

$$-\Delta u(x) = f(x), \quad x \in \Omega, \tag{1a}$$

$$u(x) = 0, \quad x \in \partial\Omega. \tag{1b}$$

We consider the optimization problem

$$\max_{f \in \mathcal{A}} J(f), \qquad \text{where } J(f) := \int_\Omega u f \, dx \tag{2}$$

is the total displacement. Here, the function $u$ that appears in $J(f)$ is assumed to solve the Poisson Eq. (1) for the given $f$. The *admissible class*, $\mathcal{A} = \mathcal{A}(\alpha, \beta, \overline{f})$ is given by

$$\mathcal{A} = \left\{ f \in L^\infty(\Omega) : \alpha \le f(x) \le \beta \text{ a.e. } x \in \Omega \text{ and } \frac{1}{|\Omega|} \int_\Omega f \, dx = \overline{f} \right\}, \tag{3}$$

where $0 < \alpha < \overline{f} < \beta$ are given constants. Since for a solution $u$ of (1) we have $0 \le u(x) \le \sup_{x\in\Omega} f(x)$, see [2, Theorem 9.27], the objective is bounded above by $\beta^2|\Omega|$. Employing a compactness argument and rearrangement techniques, we can infer that there exists a

$f^\star(x) = \alpha + (\beta - \alpha)\chi_{D^\star}(x) \in \mathcal{A}$ solving (2). Also, the solution is unique in the case that $\Omega$ is a ball in $\mathbb{R}^d$ [3–5]. Furthermore, necessary optimality conditions require that every solution $f^\star$ satisfies $f^\star(x) = \alpha + (\beta - \alpha)\chi_D(x)$ for some measurable set $D \subset \Omega$ such that $\alpha(|\Omega| - |D|) + \beta|D| = \overline{f}|\Omega|$, *i.e.*,

$$|D| = \frac{\overline{f} - \alpha}{\beta - \alpha}|\Omega|.$$

Since $f^\star$ takes the pointwise bounds at almost every point of the domain, it is frequently referred to as a "bang-bang" optimizer. In the case where $\Omega$ is radial, it was proved that the maximizer is radially non-increasing [5]. Thus, the solution is explicitly given by $f^\star(x) = \alpha + (\beta - \alpha)\chi_B(x)$, where $B$ denotes the ball of measure $\frac{\overline{f}-\alpha}{\beta-\alpha}|\Omega|$.

To compute the solution to (2), a common and efficient method is referred to as a *rearrangement method*, as summarized in Algorithm 1. This method is iterative. For a given iterate $f_i$, the Poisson Eq. (1) is solved with right hand side given by $f_i$. A super-level-set of the solution $u_i$, $\{u_i \geq \gamma_i\}$, with volume $\frac{\overline{f}-\alpha}{\beta-\alpha}|\Omega|$, is identified and a new iterate is defined by $f_{i+1} = \alpha + (\beta - \alpha)\chi_{\{u_i \geq \gamma_i\}}, \in \mathcal{A}$. The process is repeated until convergence, typically either stationarity (in the discretized problem) or $\|f_i - f_{i+1}\| \leq \varepsilon$ for some convergence criterion $\varepsilon > 0$ and appropriate norm $\|\cdot\|$. Using the variational formulation for the Poisson equation,

$$\min_{u \in H_0^1(\Omega)} \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega f u dx,$$

and bathtub principle, it is not difficult to see that non-stationary iterates of the rearrangement method increase the objective function, *i.e.*, $J(f_{i+1}) > J(f_i)$ [21]. The sequence $\{f_i\}$ generated iteratively by the rearrangement method converges (along a subsequence) to a (local) maximizer in the sense of $L^2(\Omega)$ [21]. Thus, there is a particular sequence $\{f_i\}$ generated iteratively by the rearrangement method such that $f_i \to f^\star$ in $L^2(\Omega)$. *In this paper, we consider the convergence rate of the rearrangement method, i.e., we address the question: for what power $\gamma$ is there a constant $L > 0$, such that*

$$\|f_i - f^\star\| \leq L\|f_{i+1} - f^\star\|^\gamma,$$

*for sufficiently large $i$ and an appropriate norm, $\|\cdot\|$.*

---

**Algorithm 1:** The rearrangement algorithm for approximating solutions to (2).

---

**Input**: Let $\Omega \subset \mathbb{R}^d$ and $\beta > \overline{f} > \alpha > 0$. Choose an initial iteration, $f_0 \in \mathcal{A}$.
**Output**: Iterates of the rearrangement algorithm, $\{f_i\}$.
Set $i = 0$
**while** *not converged* **do**

**1. Solve the Poisson equation.** Solve the Poisson equation with right hand side given by $f_i$:

$$\begin{aligned} -\Delta u_i &= f_i & \Omega \\ u_i &= 0 & \partial\Omega. \end{aligned}$$

**2. Rearrangement Step.** Set

$$f_{i+1} = \alpha + (\beta - \alpha)\chi_{\{u_i \geq \gamma_i\}},$$

where $\gamma_i > 0$ is chosen so that $|\{u_i \geq \gamma_i\}| = \frac{\overline{f}-\alpha}{\beta-\alpha}|\Omega|$ and $f_{i+1} \in \mathcal{A}$;
Set $i = i + 1$

---

The general question of the convergence rate for the rearrangement method is difficult due to the fact that usually we do not have much information about the optimal function $f^\star$ and the associated solution to the Poisson equation $u^\star$ a priori, let alone the iterates $f_i$ and associated solutions, $u_i$. For instance, linear convergence of the method would require showing that there exists $L \in (0, 1)$ such that

$$\| f_{i+1} - f^\star\|_{L^2(\Omega)} \leq L \| f_i - f^\star\|_{L^2(\Omega)}, \tag{4}$$

for sufficiently large $i$. A simple calculation shows that $\| f_i - f^\star\|^2_{L^2(\Omega)} = (\beta-\alpha)^2 |D^\star \Delta D_i|$, where $\Delta$ denotes the symmetric difference of sets, $D^\star = \{x \in \Omega : u^\star(x) \geq \gamma^\star\}$ and $D_i = \{x \in \Omega : u_i(x) \geq \gamma_i\}$. Parameters $\gamma^\star$ and $\gamma_i$ are chosen in a way that $f^\star$ and $f_i$ belong to $\mathcal{A}$. The condition in (4) is equivalent to verifying that

$$|D^\star \Delta D_{i+1}| \leq L^2 |D^\star \Delta D_i|, \tag{5}$$

for sufficiently large $i$. To establish (5) for a general domain we need detailed information on the geometry and topology of sets $D^\star$ and $D_i$, $i > 0$, *e.g.*, connectivity, convexity, or symmetry. This, in turn, requires detailed information on the functions $u^\star$ and $u_i$, $i > 0$, *e.g.*, monotonicity and concavity.

**Algorithm 2:** The rearrangement algorithm in one dimension.

**Input**: Let $\Omega = (-1, 1)$, $\beta > \alpha > 0$, and $\delta \in (0, 1)$. Choose an initial iteration, $y_0 \in (0, 1)$, and a convergence criterion, $\varepsilon > 0$.
**Output**: Linearly convergent iterates of the rearrangement algorithm, $\{y_i\}$.
Set $i = 0$
**while** $|y_{i+1} - y_i| > \varepsilon$ **do**
  **if** $y_i \geq \lambda := \max\{3\delta - 1, \frac{\delta(\alpha+\beta)}{\alpha+\delta(\beta-\alpha)}\}$ **then**
    Set

$$y_{i+1} = \frac{\delta}{\alpha}(\beta - \alpha)(1 - y_i) \tag{11}$$

  **else**
    Set

$$y_{i+1} = y_i - \frac{2}{\beta - \alpha}\left[\beta\delta - \sqrt{\beta^2\delta^2 - y_i\delta(\beta - \alpha)(\alpha + \delta(\beta - \alpha))}\right] \tag{12}$$

  Set $i = i + 1$

## 1.2 Results

In this paper, as a first step to understanding the general question of the convergence rate for the rearrangement method, we consider the rearrangement method for the one-dimensional Poisson equation. This is a significant simplification, since the functions $u^\star$ and $u_i$, $i > 0$ are concave and the associated sets $D^\star$ and $D_i$, $i > 0$ are convex (hence, intervals). In this case, as we detail below, we prove that the rearrangement method is linearly convergent.

On the one-dimensional interval, $\Omega = (-1, 1)$, the Poisson Eq. (1) is written

$$-u''(x) = f(x) \qquad x \in (-1, 1), \tag{6a}$$

$$u(-1) = u(1) = 0. \tag{6b}$$

We know that the solution to (2) is given by the symmetric function,

$$f(x) = \alpha + (\beta - \alpha)\chi_{[-\delta,\delta]}. \tag{7}$$

for $\delta = \frac{\overline{f}-\alpha}{\beta-\alpha} \in (0, 1)$. In one dimension, since $-u'' = f > 0$, $u$ is a concave function, so every super-level set $\{x : u(x) \geq \gamma\}$ is a convex set, *i.e.*, an interval. Thus, the rearrangement algorithm in one dimension gives a sequence of intervals $[y_i - \delta, y_i + \delta]$ for $i \in \mathbb{N}$, with

$$y_i \to 0.$$

The map $y_i \mapsto y_{i+1} = g(y_i)$ is described by finding the appropriate super-level-set of the solution $u(x)$ to (6), which is a new interval, $[y_{i+1} - \delta, y_{i+1} + \delta]$. The rearrangement algorithm is summarized in Algorithm 2.

**Theorem 1** *Consider the rearrangement method on the one-dimensional interval, $\Omega = (-1, 1)$, for the optimization problem in (2) with parameters $\beta > \alpha > 0$ and $\delta \in (0, 1)$. The iterates are monotonic and converge linearly with constant $L := \left(1 - \frac{\alpha}{\beta}\right)(1 - \delta)$, i.e., there exists an integer $i_0 \geq 0$ such that*

$$|y_{i+1}| \leq L|y_i|, \qquad \forall i > i_0.$$

A Proof of Theorem 1, relying the Banach fixed-point theorem, is given in Sect. 2. In Fig. 1, the first few iterations of $u_i(x)$ for a choice of parameters $y_0 = 0.8$, $\alpha = 1$, $\beta = 2$, and $\delta = 0.2$ are shown. Denote the one-dimensional objective function by $j : [0, 1] \to \mathbb{R}$,

$$j(y) := \int_{-1}^{1} u(x; y) f(x; y)\, dx.$$

In Fig. 2, we plot $i$ versus $|y_i|$ and $|j(y_i) - j^\star|$ with logarithmic (base 10) scale used for the vertical axis. It only takes 35 iterations to reach the machine accuracy and a linear convergence is observed. Furthermore, the iterates lie on a line with slope $L = 0.4$, as expected theoretically. In Figs. 3 and 4, we demonstrate that it is possible to take many steps to reach the machine accuracy if the ratio $\frac{\beta}{\alpha}$ is high and $\delta$ is small. This is predicted theoretically as $L \approx 1$. We show this case by choosing $y_0 = 0.8$, $\alpha = 1$, $\beta = 100$, and $\delta = 0.01$. This gives $L = 0.9801$ and it takes 1625 iterations to reach machine accuracy.

We can explicitly evaluate

$$j(y) = \alpha \int_{-1}^{y-\delta} u_1(x; y)\, dx + \beta \int_{y-\delta}^{y+\delta} u_2(x; y)\, dx + \alpha \int_{y+\delta}^{1} u_3(x; y)\, dx \tag{8a}$$

$$= j^\star - 2\delta(\beta - \alpha)\,(\alpha + \delta(\beta - \alpha))\, y^2, \tag{8b}$$

where

$$j^\star = j(0) = \frac{2}{3}\alpha^2 - \frac{2}{3}\alpha\delta(\delta^2 - 3)(\beta - \alpha) + \frac{2}{3}\delta^2(3 - 2\delta)(\beta - \alpha)^2.$$

**Corollary 2** *The objective function values $j(y_k)$ converge linearly to the optimal value $j^\star = j(0)$ with rate $L^2$, where $L$ is the same constant as in Theorem 1, i.e.,*

$$|j(y_{k+1}) - j^\star| \leq L^2 |j(y_k) - j^\star|.$$

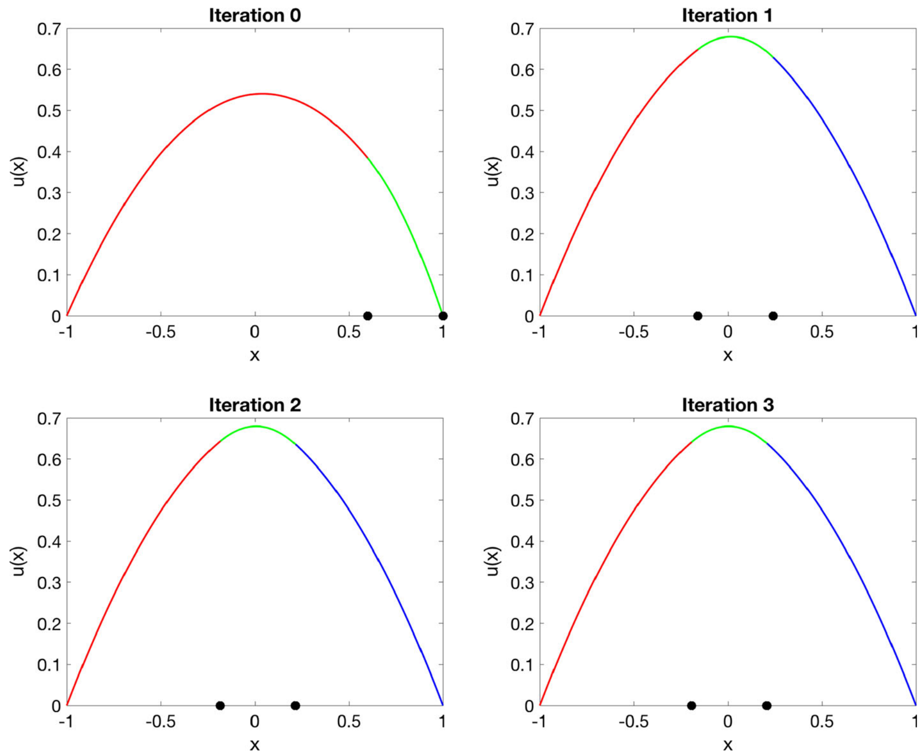


**Fig. 1** A plot of $u(x)$ for the initial $f$ and the first three iterations of the rearrangement method with $y_0 = 0.8$, $\alpha = 1$, $\beta = 2$, and $\delta = 0.2$. On the $x$-axis, the points $y_i - \delta$ and $y_i + \delta$ are indicated

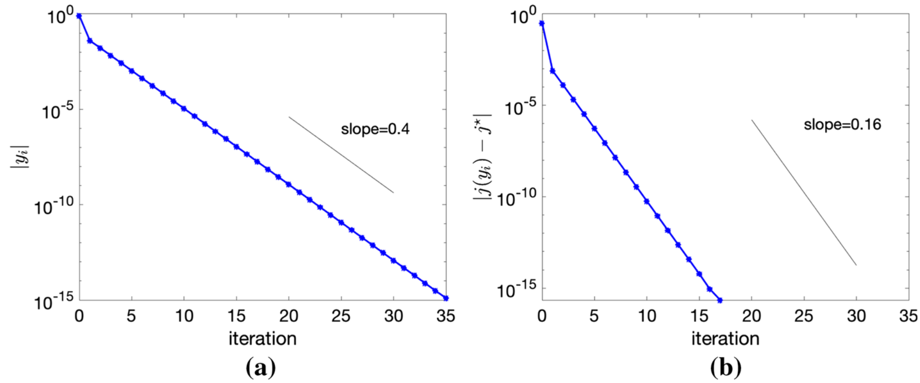


**Fig. 2** A plot of **a** the number of iteration $i$ versus $|y_i|$ and **b** $i$ versus $|j(y_i) - j^{\star}|$ with logarithmic (base 10) scale used for the vertical axis for the first 35 iterations of the rearrangement method with $y_0 = 0.8$, $\alpha = 1$, $\beta = 2$, and $\delta = 0.2$. Linear convergence is observed in both figures

A Proof of Corollary 2 is given in Sect. 2.

In Sect. 3 we give a more detailed discussion and some numerical evidence of linear convergence in higher dimensions. We show in Theorem 3 that in the case where $\Omega$ is a $d$-dimensional ball, $B(0, 1)$, and the initial condition, $f_0$, is radial, the rearrangement

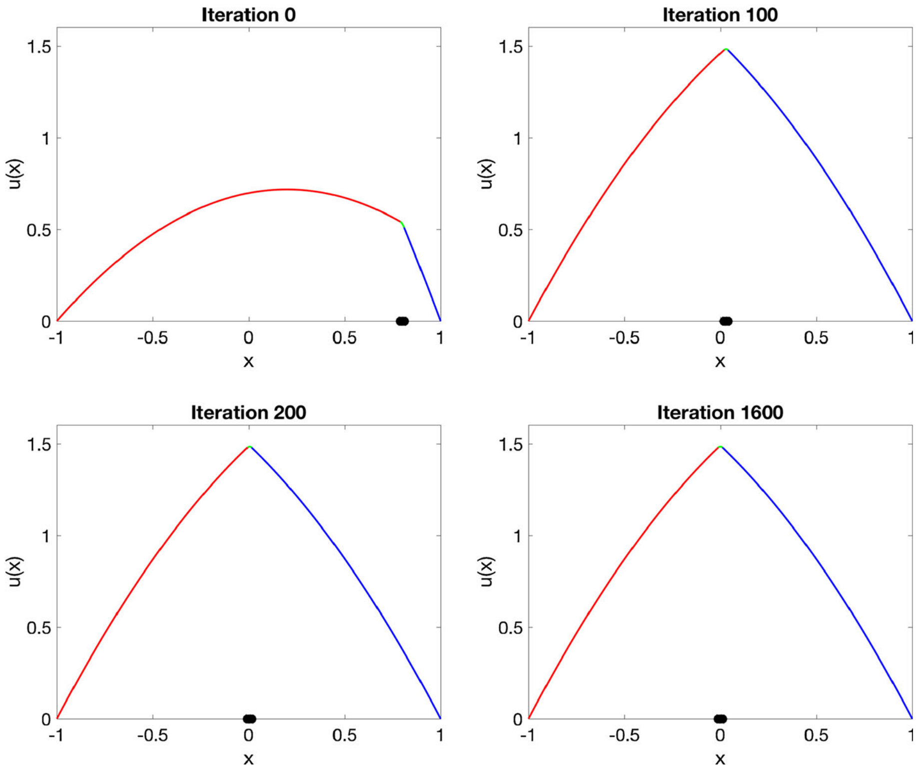


**Fig. 3** A plot of $u(x)$ for the initial $f$ and at 100, 200, and 1600 iterations of the rearrangement method with $y_0 = 0.8$, $\alpha = 1$, $\beta = 100$, and $\delta = 0.01$. On the $x$-axis, the points $y_i - \delta$ and $y_i + \delta$ are indicated

method converges to the optimal solution in just one iteration. In a variety of two-dimensional numerical experiments, we demonstrate linear convergence of the rearrangement method.

We conclude in Sect. 4 with a discussion.

## 2 Proof of Theorem 1, Linear Convergence in One Dimension

We consider a function $f$ of the form

$$f(x) = \alpha + (\beta - \alpha)\chi_{[y-\delta, y+\delta]}.$$

We can explicitly solve (6) as follows. We make the ansatz

$$\begin{aligned} u_1(x) &= -\frac{\alpha}{2}x^2 + c_1 x + c_2, \quad x \in [-1, y-\delta]; \\ u_2(x) &= -\frac{\beta}{2}x^2 + c_3 x + c_4, \quad x \in [y-\delta, y+\delta]; \\ u_3(x) &= -\frac{\alpha}{2}x^2 + c_5 x + c_6, \quad x \in [y+\delta, 1]. \end{aligned}$$

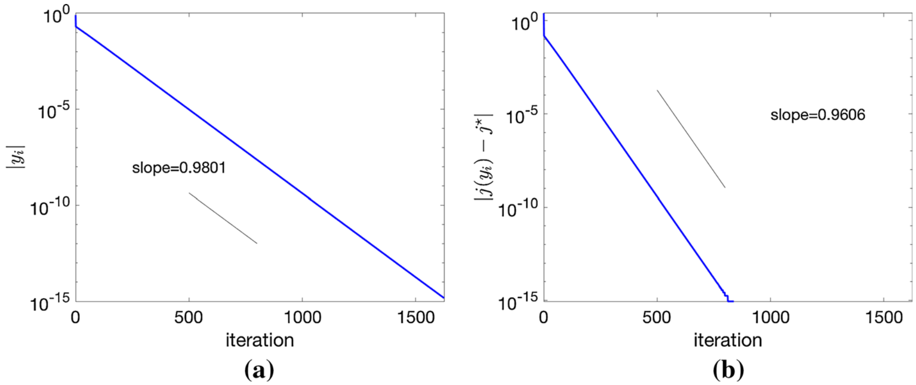


**Fig. 4** A plot of **a** the number of iteration $i$ versus $|y_i|$ and **b** $i$ versus $|j(y_i) - j^\star|$ with logarithmic (base 10) scale used for the vertical axis for the first 1625 iterations of the rearrangement method with $y_0 = 0.8$, $\alpha = 1$, $\beta = 100$, and $\delta = 0.01$. Linear convergence is observed in both figures

Clearly this ansatz satisfies (6a) on each subinterval. We find constants $\{c_i\}_{i=1}^6$ so that $u(x)$ is continuously differentiable at $x = y - \delta$ and $y + \delta$ (4 conditions) and satisfies the boundary conditions (6b) at $x = \pm 1$ (2 conditions). The constants are given by

$$\begin{aligned}
c_1 &= \delta(\beta - \alpha)\,(1 - y)\,, \\
c_2 &= \frac{\alpha}{2} + \delta(\beta - \alpha)\,(1 - y)\,, \\
c_3 &= (\beta - \alpha)(1 - \delta)y, \\
c_4 &= \frac{\alpha}{2} + (\beta - \alpha)\left(\delta - \frac{\delta^2}{2} - \frac{y^2}{2}\right), \\
c_5 &= -\delta(\beta - \alpha)\,(1 + y)\,, \\
c_6 &= \frac{\alpha}{2} + \delta(\beta - \alpha)\,(1 + y)\,.
\end{aligned}$$

The solution to the Poisson equation is then given by

$$\begin{aligned}
u_1(x) &= (1 + x)\left[\frac{\alpha}{2}(1 - x) + \delta(\beta - \alpha)\,(1 - y)\right], \quad x \in [-1, y - \delta]; \\
u_2(x) &= -\frac{\beta}{2}x^2 + (\beta - \alpha)(1 - \delta)yx + \frac{\alpha}{2} + (\beta - \alpha)\left(\delta - \frac{\delta^2}{2} - \frac{y^2}{2}\right), \quad x \in [y - \delta, y + \delta]; \\
u_3(x) &= (1 - x)\left[\frac{\alpha}{2}(1 + x) + \delta(\beta - \alpha)\,(1 + y)\right], \quad x \in [y + \delta, 1].
\end{aligned}$$

The rearrangement step is then to find the unique point $x = y_{i+1}$ satisfying:

$$u(y_{i+1} - \delta) = u(y_{i+1} + \delta). \tag{9}$$

We will refer to this mapping as $y_i \mapsto y_{i+1} = g(y_i)$. We identify four cases based on the interval: $[-1, y - \delta]$, $[y - \delta, y + \delta]$, or $[y + \delta, 1]$ in which the two points $y_{i+1} - \delta$ and $y_{i+1} + \delta$ in (9) are located:

| Case | Equation | Comment |
|---|---|---|
| 1 | $u_1(y_{i+1}-\delta) = u_1(y_{i+1}+\delta)$ | $y_{i+1} \le y_i - 2\delta$ |
| 2 | $u_1(y_{i+1}-\delta) = u_2(y_{i+1}+\delta)$ | $g$ is a contraction mapping on $[0, \phi]$ |
| 3 | $u_2(y_{i+1}-\delta) = u_3(y_{i+1}+\delta)$ | symmetry argument |
| 4 | $u_3(y_{i+1}-\delta) = u_3(y_{i+1}+\delta)$ | symmetry argument |

**Claim 1** Case 3 or Case 4 occurs if and only if $y_i < 0$.

For Case 3 or Case 4 to hold, we would require that $u_2(y-\delta) < u_2(y+\delta)$. But a short computation shows that this is equivalent to $y_i < 0$.

**Claim 2** Case 1 occurs only if $y_i \ge \lambda := \max\{3\delta - 1, \frac{\delta(\alpha+\beta)}{\alpha+\delta(\beta-\alpha)}\}$.

First, for Case 1 to occur, we need the interval of length $2\delta$ to be contained in the interval $[-1, y_i - \delta]$, which gives that $y_i \ge 3\delta - 1$. Second, for Case 1 to occur, *i.e.*, for there to exist a $y_{i+1} \in [-1, y_i - 2\delta]$ satisfying $u_1(y_{i+1}-\delta) = u_1(y_{i+1}+\delta)$, we need that $u_1(y_i - 3\delta) \ge u_1(y_i - \delta)$. This condition is equivalent to

$$y_i \ge \frac{\delta(\alpha+\beta)}{\alpha+\delta(\beta-\alpha)}. \tag{10}$$

These two conditions together give $y_i \ge \lambda$.

**Claim 3** In Case 1, we have that

$$y_{i+1} = \frac{\delta}{\alpha}(\beta-\alpha)(1-y_i). \tag{11}$$

and satisfies $y_{i+1} \in [0, y_i - 2\delta]$.

The solution $y_{i+1}$ is obtained by solving $u_1(y_{i+1}-\delta) = u_1(y_{i+1}+\delta)$. From (11), we immediately have that $y_{i+1} \ge 0$. We now use (10) and (11) to compute

$$\begin{aligned} y_i - y_{i+1} &= \left(1 + \frac{\delta}{\alpha}(\beta-\alpha)\right) y_i - \frac{\delta}{\alpha}(\beta-\alpha) \\ &\ge \left(1 + \frac{\delta}{\alpha}(\beta-\alpha)\right) \frac{\delta(\alpha+\beta)}{\alpha+\delta(\beta-\alpha)} - \frac{\delta}{\alpha}(\beta-\alpha) \\ &= 2\delta. \end{aligned}$$

**Claim 4** We cannot stay in Case 1 for more than $\left\lceil \frac{1-\lambda}{2\delta} \right\rceil$ iterations. At the first iteration in which Case 1 doesn't hold, we are in Case 2.

Case 1 only occurs if $y_i \ge \lambda$ and we have that $y_k \le y_0 - 2k\delta \le 1 - 2k\delta$. After $k = \left\lceil \frac{1-\lambda}{2\delta} \right\rceil$ iterations, we have $y_i < \lambda$. Since Case 3 and Case 4 only occur when $y_i < 0$ (see Claim 1), we conclude that the first iteration for which Case 1 doesn't hold, we are in Case 2.

**Claim 5** In Case 2, the solution, $y_{i+1}$, is given by

$$y_{i+1} = y_i - \frac{2}{\beta-\alpha}\left[\beta\delta - \sqrt{\beta^2\delta^2 - y_i\delta(\beta-\alpha)\left(\alpha+\delta(\beta-\alpha)\right)}\right] \tag{12}$$

and satisfies $y_{i+1} \in [0, y_i]$.

The solution, $y_{i+1}$, is obtained by solving the equation $u_1(y_{i+1}-\delta) = u_2(y_{i+1}+\delta)$. The following argument shows that the solution is real since the quantity in the square root is positive. In Case 2, we have either

$$y_i < \frac{\delta(\alpha+\beta)}{\alpha+\delta(\beta-\alpha)} \quad \text{or} \quad y_i < 3\delta - 1.$$

If $y_i < \frac{\delta(\alpha+\beta)}{\alpha+\delta(\beta-\alpha)}$, we then have that

$$\begin{aligned}
\beta^2\delta^2 - y_i\delta(\beta-\alpha)\left(\alpha+\delta(\beta-\alpha)\right) &\geq \beta^2\delta^2 - \frac{\delta^2(\alpha+\beta)}{\alpha+\delta(\beta-\alpha)}(\beta-\alpha)\left(\alpha+\delta(\beta-\alpha)\right)\\
&= \beta^2\delta^2 - (\beta^2-\alpha^2)\delta^2\\
&= \alpha^2\delta^2 > 0,
\end{aligned}$$

so the quantity in the square root is positive.

Now suppose that $0 \leq y_i \leq 3\delta - 1$, so that $\delta \geq \frac{1}{3}$. We also know that $y_i \leq 1-\delta$. We estimate the lower bound of the term in the square root

$$\begin{aligned}
\beta^2\delta^2 - y_i\delta(\beta-\alpha)\left(\alpha+\delta(\beta-\alpha)\right) &\geq \beta^2\delta^2 - \delta(1-\delta)(\beta-\alpha)\left(\alpha+\delta(\beta-\alpha)\right)\\
&= (\beta-\alpha)^2\delta^3 + \alpha\delta^2(3\beta-2\alpha) - \alpha\delta(\beta-\alpha)\\
&\geq (\beta-\alpha)^2\delta^3 + \alpha\frac{1}{3}\delta(3\beta-2\alpha) - \alpha\delta(\beta-\alpha)\\
&= (\beta-\alpha)^2\delta^3 + \frac{\alpha^2}{3}\delta > 0,
\end{aligned}$$

so the quantity in the square root is positive.

Inspecting (12), since the term in the square root is in the interval $(0, \beta\delta)$, the term in square brackets is in the interval $(0, 1)$. This shows that $y_{i+1} < y_i$.

We now show that $y_{i+1} > 0$. This is equivalent to showing that

$$u(-\delta) \leq u(\delta).$$

There are two conditions that must be independently checked:

$$u_1(-\delta) \leq u_1(\delta), \quad \text{if } 2\delta \leq y_i; \tag{13a}$$

$$u_1(-\delta) \leq u_2(\delta), \quad \text{if } y_i \leq 2\delta. \tag{13b}$$

To check the condition in (13a), we compute

$$u_1(-\delta) - u_1(\delta) = -2\delta^2(\beta-\alpha)(1-y_i) < 0,$$

which verifies (13a).

To check the condition in (13b), we compute

$$u_1(-\delta) - u_2(\delta) = (\beta-\alpha)y_i\left(\frac{y_i}{2} - 2\delta(1-\delta)\right).$$

If $\delta < \frac{1}{2}$, we use $y_i \leq 2\delta$ and compute

$$u_1(-\delta) - u_2(\delta) \leq (\beta-\alpha)y_i\left(\delta - 2\delta(1-\delta)\right) = -(\beta-\alpha)y_i\delta\left(1-2\delta\right) \leq 0.$$

If $\delta > \frac{1}{4}$, we use $y_i \leq 1-\delta$ and compute

$$u_1(-\delta) - u_2(\delta) \leq (\beta-\alpha)y_i\left(\frac{1-\delta}{2} - 2\delta(1-\delta)\right) = -\frac{1}{2}(\beta-\alpha)y_i\delta(1-\delta)(4\delta-1) \leq 0.$$

These two conditions verify (13), so $y_{i+1} > 0$.

**Claim 6** In Case 2, the mapping defining the iterates in (12) is a contraction map with constant $L = \left(1 - \frac{\alpha}{\beta}\right)(1-\delta)$ on the interval $[0, \phi]$, where

$$\phi = \frac{4\beta^3\delta(1-\delta)}{(\alpha+\delta(\beta-\alpha))(\beta+(\beta-\alpha)(1-\delta))^2} \tag{14a}$$

$$= \frac{\beta^2\delta}{(\beta-\alpha)(\alpha+\delta(\beta-\alpha))}\left(1 - \frac{(\alpha+\delta(\beta-\alpha))^2}{(\beta+(\beta-\alpha)(1-\delta))^2}\right). \tag{14b}$$

From (12), we have $y_{i+1} = g(y_i)$ where

$$g(y) = y - \frac{2}{\beta-\alpha}\left[\beta\delta - \sqrt{\beta^2\delta^2 - y\delta(\beta-\alpha)\left(\alpha+\delta(\beta-\alpha)\right)}\right]. \tag{15}$$

We will show that $g$ is a contraction mapping on $[0, \phi]$.

We compute the derivatives:

$$g'(y) = 1 - \frac{\delta\left(\alpha+\delta(\beta-\alpha)\right)}{\sqrt{\beta^2\delta^2 - y\delta(\beta-\alpha)\left(\alpha+\delta(\beta-\alpha)\right)}} \tag{16}$$

and

$$g''(y) = -\frac{1}{2}\frac{\delta^2(\beta-\alpha)\left(\alpha+\delta(\beta-\alpha)\right)^2}{\left(\beta^2\delta^2 - y\delta(\beta-\alpha)\left(\alpha+\delta(\beta-\alpha)\right)\right)^{\frac{3}{2}}}. \tag{17}$$

We observe that $g''(y) < 0$ for all $y \in [0, 1]$, which implies that $g'(y)$ is decreasing on $[0, 1]$. Computing $g'(0) = \frac{(\beta-\alpha)(1-\delta)}{\beta} = L \in (0, 1)$, we conclude that

$$g'(y) \le L < 1, \qquad \forall y \in [0, 1].$$

We next show that $g'(y) > -L$ for all $y \in [0, \phi]$. From the expression for $\phi$ in (14a), we immediately have that $\phi > 0$. To see that the two expressions in (14a) and (14b) are the same, we simply compute

$$(\beta+(\beta-\alpha)(1-\delta))^2 - (\alpha+\delta(\beta-\alpha))^2 = 4\beta(\beta-\alpha)(1-\delta).$$

Using the expression for $\phi$ in (14b), we compute, for $y \in [0, \phi]$,

$$\begin{aligned} g'(y) &= 1 - \frac{\delta\left(\alpha+(\beta-\alpha)\delta\right)}{\sqrt{\beta^2\delta^2 - y\delta(\beta-\alpha)\left(\alpha+(\beta-\alpha)\delta\right)}} \\ &\ge 1 - \frac{\delta\left(\alpha+(\beta-\alpha)\delta\right)}{\sqrt{\beta^2\delta^2 - \phi\delta(\beta-\alpha)\left(\alpha+(\beta-\alpha)\delta\right)}} \\ &= 1 - \frac{(\beta+(\beta-\alpha)(1-\delta))}{\beta} \\ &= -L. \end{aligned}$$

**Claim 7** In Case 2, if $y_i > \phi$, the iterates satisfy

$$y_i - y_{i+1} \ge \frac{4\beta\delta(1-\delta)}{\beta+(\beta-\alpha)(1-\delta)} > 0,$$

Using (12) and (14b), we have that

$$\begin{aligned} y_i - y_{i+1} &= \frac{2}{\beta-\alpha}\left[\beta\delta - \sqrt{\beta^2\delta^2 - y_i\delta(\beta-\alpha)\,(\alpha+(\beta-\alpha)\delta)}\right] \\ &\geq \frac{2}{\beta-\alpha}\left[\beta\delta - \sqrt{\beta^2\delta^2 - \phi\delta(\beta-\alpha)\,(\alpha+(\beta-\alpha)\delta)}\right] \\ &= \frac{2\beta\delta}{\beta-\alpha}\left(1 - \frac{(\alpha+(\beta-\alpha)\delta)}{(\beta+(\beta-\alpha)(1-\delta))}\right). \end{aligned}$$

Computing

$$(\beta+(\beta-\alpha)(1-\delta)) - (\alpha+(\beta-\alpha)\delta) = 2(\beta-\alpha)(1-\delta),$$

we have

$$y_i - y_{i+1} \geq \frac{4\beta\delta(1-\delta)}{\beta+(\beta-\alpha)(1-\delta)} > 0,$$

as desired.

### Completing the proof

Without loss of generality, we may assume that $y_0 > 0$. By Claims 1, 3, and 5, we can assume that $y_{i+1} \in [0, y_i]$ for all $i \geq 0$ and that only Cases 1 and 2 are possible. By Claim 4, we have that we cannot stay in Case 1 for more than a finite number of iterations, after which we are in Case 2. By Claim 5, once the iterates are in Case 2, they remain in Case 2. By Claim 7, after a finite number of iterations, we obtain $y_i \in [0, \phi]$. By Claim 6, $g$ is a contractive map on $[0, \phi]$, so by the Banach fixed-point theorem, we obtain linear convergence. This completes the Proof of Theorem 1. □

***Proof of Corollary 2*** Using (8), we compute

$$\frac{|j(y_{k+1}) - j^\star|}{|j(y_k) - j^\star|} = \frac{|-2\delta(\beta-\alpha)\,(\alpha+\delta(\beta-\alpha))\,y_{k+1}^2|}{|-2\delta(\beta-\alpha)\,(\alpha+\delta(\beta-\alpha))\,y_k^2|} = \frac{|y_{k+1}^2|}{|y_k^2|} \leq L^2,$$

as desired. □

## 3 Convergence in Higher Dimensions

In arbitrary dimension $d$, we can prove the following convergence result for the radially symmetric case.

**Theorem 3** *Consider the case where $\Omega$ is a $d$-dimensional ball, $B(0, 1)$, and the initial condition, $f_0$, is radial, i.e., in polar coordinates, $f_0 = f_0(r)$. The rearrangement method converges to the optimal solution in just one iteration.*

***Proof*** In this case, the solution, $u$, to the Poisson Eq. (1) is radial, satisfying

$$-\frac{1}{r^{d-1}}\partial_r\left(r^{d-1}\partial_r u\right) = f_0(r) \implies u'(r) = -\frac{1}{r^{d-1}}\int_0^r s^{d-1} f_0(s)\,ds < 0.$$

Since the solution is radially decreasing, all super-level-sets are balls centered at the origin. In particular, the first iteration of the rearrangement method will be the optimal solution. □

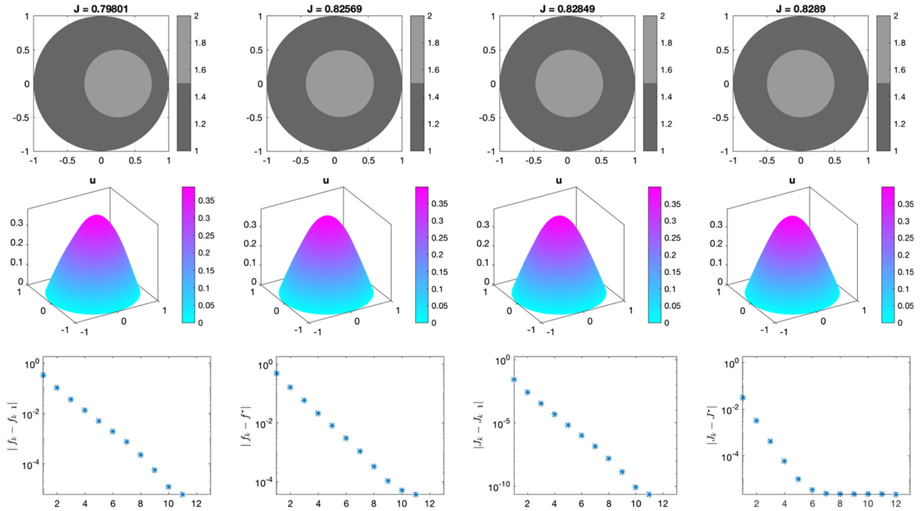


**Fig. 5** The functions $f_k$ and $u_k$ are shown for $k = 0, 1, 2, 11$ in the first and second row, respectively. The log plots of $\|f_k - f_{k+1}\|$, $\|f_k - f^\star\|$, $|J_k - J_{k+1}|$, and $|J_k - J^\star|$ versus $k$ are shown in the third row

However, as discussed in the introduction, in the case where $\Omega$ is not a ball, we consider a proof of the convergence rate for (2) in higher-dimensions to be a difficult problem. In the remainder of this section, we provide some numerical evidence for linear convergence in two dimensions. The results of these computational experiments are reported in Figs. 5, 6, 7, 8, and 9. In the following numerical simulations, we use $L_2$ norm to measure the successive difference $f_k - f_{k+1}$ and the error $f - f^\star$. The algorithm is terminated when the successive difference in $f$ is zero, *i.e.*, $f_k = f_{k+1}$. The mesh size and the number of iterations for each simulation is reported. Except for the experiment associated with Fig. 7, we choose $\alpha = 1$ and $\beta = 2$.

In Fig. 5, we show the results of a computational experiment on a disk $\Omega = \{(x, y) : x^2 + y^2 \leq 1\}$ with $|D| = \frac{\pi}{4}$. Since we proved that the radial initial $f$ only requires one iteration to reach the maximum, we choose a non-radial initial $f$ in order to demonstrate linear convergence. The calculation is done on a triangular mesh with 2, 097, 152 elements and $f_k$ and $u_k$ are shown for $k = 0, 1, 2, 11$. The theoretical optimal value of $J^\star \approx 0.828904036113266$ was derived in [16]. The function $f_k$ converges to the exact solution $f^\star = \chi_{[0,\frac{1}{2}]}(r) + 1$ which is a radially non-increasing function. We see that the log plots of $\|f_k - f_{k+1}\|$, $\|f_k - f^\star\|$, $|J_k - J_{k+1}|$, and $|J_k - J^\star|$ versus $k$ demonstrate first order convergence. The error $\|f_k - f^\star\|$ saturates at the level of $10^{-4} \sim 10^{-5}$ and the error $\|J_k - J^\star\|$ saturates at the level of $10^{-5} \sim 10^{-6}$ due to the expected numerical error coming from the finite element discretization. This error can be reduced when a finer mesh is used. This phenomenon is observed in the following simulations as well and will not be further discussed.

Figure 6 shows the results of a computational experiment on a unit square domain, $\Omega$, with $|D| = 0.25$. The calculation is done on a triangular mesh with 1, 048, 576 elements. The set $D$ is initialized as the strip, $D = \{(x, y) \in \Omega : x \geq 0.75\}$, and converges to a circular like shape in the middle of the unit square. As the exact solution is not available for this case, we show the log plots of $\|f_k - f_{k+1}\|$ and $|J_k - J_{k+1}|$ versus $k$ only, where we observe first order convergence.

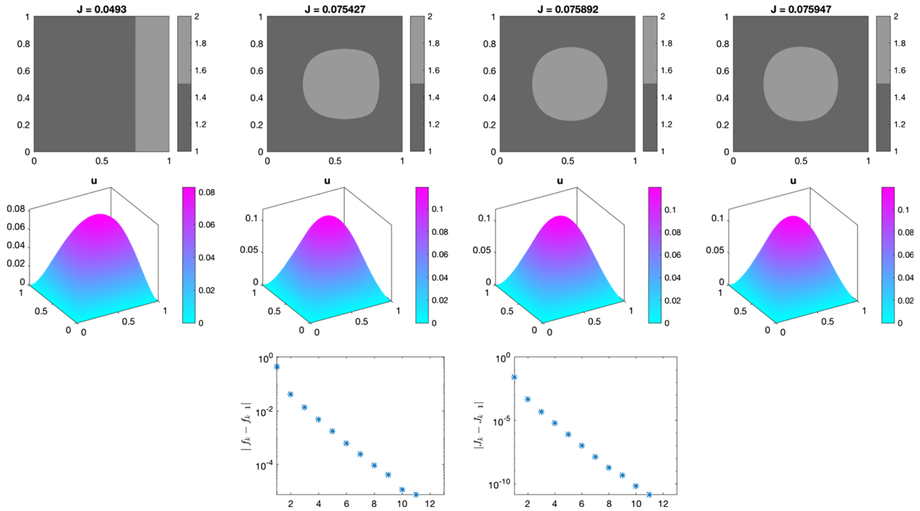


**Fig. 6** The functions $f_k$ and $u_k$ are shown for $k = 0, 1, 2, 11$ in the first and second row, respectively. The log plots of $\| f_k - f_{k+1} \|$ and $|J_k - J_{k+1}|$ versus $k$ are shown in the third row

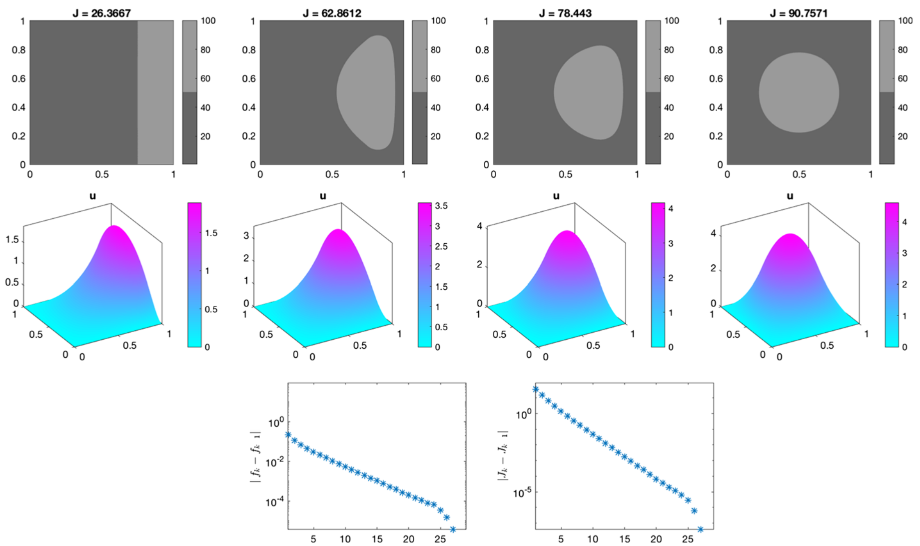


**Fig. 7** The functions $f_k$ and $u_k$ are shown for $k = 0, 1, 2, 27$ in the first and second row, respectively. The log plots of $\| f_k - f_{k+1} \|$ and $|J_k - J_{k+1}|$ versus $k$ are shown in the third row

Figure 7 shows the results of a numerical experiment that is the same as the previous one, except now, $\beta = 100$. When $\beta - \alpha$ is larger, we expect that it takes more iterations to achieve the maximizer. Instead of 11 iterations need for convergence in Fig. 6, this high-contrast simulation requires 27 iterations to reach the maximizer. Again we observe linear convergence, but the constant of convergence is larger. Similar behavior is observed for high-contrast simulations for other domains $\Omega$ below, but we we omit the report of those calculations for brevity.

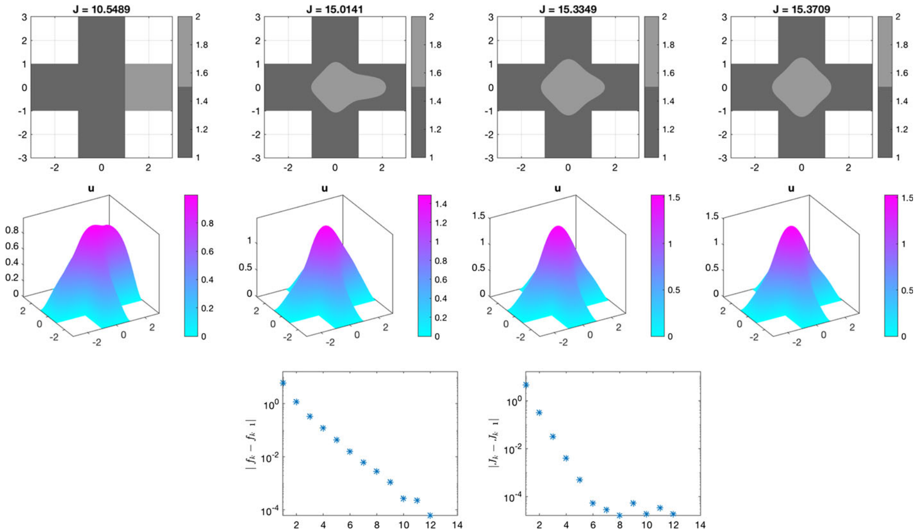

**Fig. 8** The functions $f_k$ and $u_k$ are shown for $k = 0, 1, 2, 13$ in the first and second row, respectively. The log plots of $\| f_k - f_{k+1} \|$ and $|J_k - J_{k+1}|$ versus $k$ are shown in the third row

In Fig. 8, we report numerical results on a cross-shaped domain which is Steiner symmetric with respect to $x$- and $y$-axis and $|D| = 0.25|\Omega|$. The calculation is done on a triangular mesh with 671, 744 elements. We initialize the simulation with $D = \{(x, y) \in \Omega : x \geq 1\}$. The maximizer has $D^*$ form a star-shaped domain and the function $u$ reaches it maximum at $(0, 0)$ which is the intersection point of $x$- and $y$-axis. It is noteworthy that the optimal domains obtained from the rearrangement algorithm in Figs. 5, 6, 7, and 8 inherit Steiner symmetry of $\Omega$ as it has been proved in [10].

Figure 9 shows the results of a numerical experiment on an annulus with the inner radius 0.5 and the outer radius 1. In this simulation, $|D| = 0.5|\Omega|$. The calculation is done on a triangular mesh with 2, 752, 512 elements. Even if the initial $D$ is simply-connected, it evolves to a domain which is not simply-connected in one iteration and then gradually converges to the optimal shape, which is an annulus. Linear convergence is observed both in log plots of $\| f_k - f_{k+1} \|$ and $|J_k - J_{k+1}|$ versus $k$.

Figure 10 shows the results of a numerical experiment on a L-shaped domain with a triangular mesh with 2, 621, 440 elements. In this simulation, $|D| = 0.5|\Omega|$. We observe linear convergence in both log plots of $\| f_k - f_{k+1} \|$ and $|J_k - J_{k+1}|$ versus $k$ on a domain without any symmetry.

## 4 Discussion

In this paper, we have studied a rearrangement method for the problem of maximizing the total displacement in Poisson's equation over forcings within an admissible class. In this paper, for the one-dimensional problem, we establish linear convergence; see Theorem 1 and Corollary 2. The proof, given in Sect. 1, relies on an explicit solution to the Poisson equation on an interval for forcings of the form $f(x) = \alpha + \beta \chi_D$, for a subinterval $D$. In Sect. 3, we provide computational evidence for linear convergence of the rearrangement method for

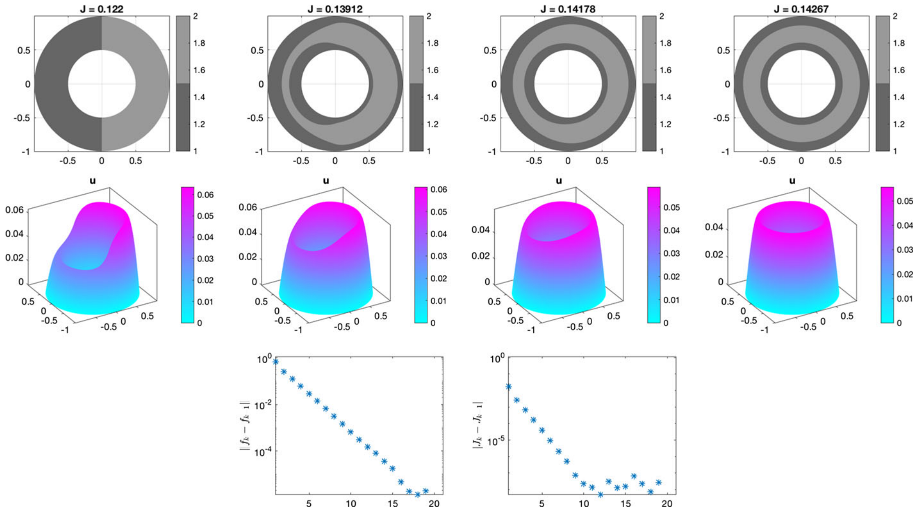

**Fig. 9** The functions $f_k$ and $u_k$ are shown for $k = 0, 1, 2, 19$ in the first and second row, respectively. The log plots of $\| f_k - f_{k+1} \|$ and $|J_k - J_{k+1}|$ versus $k$ are shown in the third row

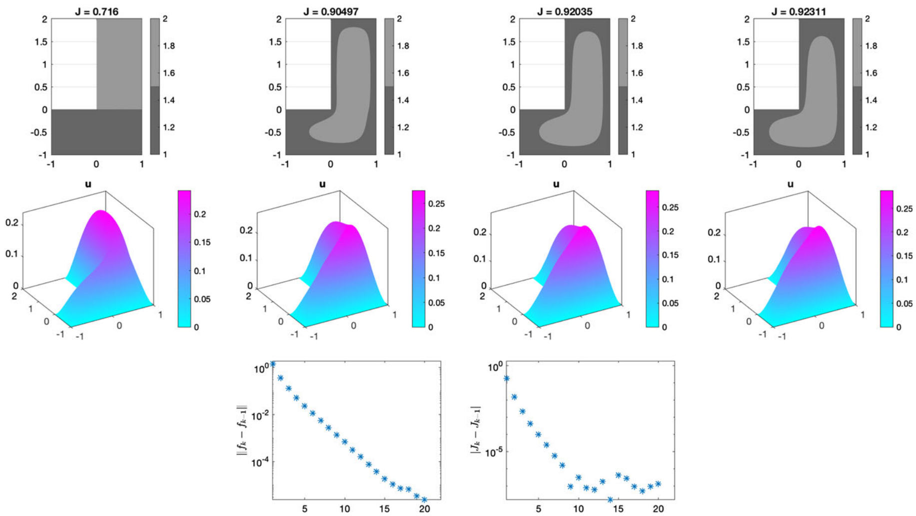

**Fig. 10** The functions $f_k$ and $u_k$ are shown for $k = 0, 1, 2, 20$ in the first and second row, respectively. The log plots of $\| f_k - f_{k+1} \|$ and $|J_k - J_{k+1}|$ versus $k$ are shown in the third row

various two dimensional domains. Here, our method of proof does not extend since an explicit solution is unavailable.

There are a variety of extensions and future directions of this work. We have numerically observed linear convergence in two dimensions. It remains to prove linear convergence, and also to establish the dependence of the constant in the linear convergence on (i) the shape of the domain $\Omega$, (ii) the ratio of volumes, $\frac{|D|}{|\Omega|}$, and (iii) the admissible class constants, $\alpha$, $\beta$, and $\overline{f}$. Numerical results suggest that the constant is decreasing in $\frac{|D|}{|\Omega|}$ and increasing in

$\beta - \alpha$, as in one dimension. It might be possible to prove a partial result in the low contrast regime ($\beta \approx \alpha$); see [21] for such results in another context.

One important direction would be to develop computational methods that are superlinear. However, it is not straightforward to apply Anderson's and Steffensen's acceleration schemes as the solution updates are not admissible.

In the introduction, we list several other problems in which rearrangement methods can be applied. It would be interesting to study the rate of convergence in these problems as well. One simple geometric example, and the only other example we are aware of where a convergence rate can be established, is for the Steiner symmetrization for triangles [11, p.50]. Here, the ratio of the triangle's height to base length, $x_n$, can be shown to satisfy the iterative equation

$$x_{n+1} = \frac{4x_n}{1 + 4x_n^2}.$$

It isn't difficult to show that $x_n \to \frac{\sqrt{3}}{2}$ at a linear rate.

**Acknowledgements** The authors would like to thank the Mathematics Division, National Center of Theoretical Sciences, Taipei, Taiwan for hosting a research pair program during June 15-June 30, 2019 to support this project.

## Compliance with ethical standards

**Conflict of interest** The authors declare that they have no conflict of interest.

## References

1. Antunes, P.R., Mohammadi, S.A., Voss, H.: A nonlinear eigenvalue optimization problem: Optimal potential functions. Nonlinear Anal.: Real World Appl. **40**, 307–327 (2018). https://doi.org/10.1016/j.nonrwa.2017.09.003
2. Brezis, H.: Functional analysis, Sobolev spaces and partial differential equations. Springer Science & Business Media (2010). https://doi.org/10.1007/978-0-387-70914-7
3. Burton, G.R.: Rearrangements of functions, maximization of convex functionals, and vortex rings. Math. Ann. **276**(2), 225–253 (1987). https://doi.org/10.1007/bf01450739
4. Burton, G.R.: Variational problems on classes of rearrangements and multiple configurations for steady vortices. Annal. de l'Inst. Henri Poincare (C) Non Linear Anal. **6**(4), 295–319 (1989). https://doi.org/10.1016/s0294-1449(16)30320-1
5. Burton, G.R., McLeod, J.B.: Maximisation and minimisation on classes of rearrangements. Proc. Royal Soc. Edinburgh Sect. A: Math. **119**(3–4), 287–300 (1991). https://doi.org/10.1017/s0308210500014840
6. Chen, W., Chou, C.S., Kao, C.Y.: Minimizing eigenvalues for inhomogeneous rods and plates. J. Sci. Comput. **69**(3), 983–1013 (2016). https://doi.org/10.1007/s10915-016-0222-9
7. Chugunova, M., Jadamba, B., Kao, C.Y., Klymko, C., Thomas, E., Zhao, B.: Study of a mixed dispersal population dynamics model. In: Topics in Numerical Partial Differential Equations and Scientific Computing, pp. 51–77. Springer (2016). https://doi.org/10.1007/978-1-4939-6399-7
8. Conca, C., Laurain, A., Mahadevan, R.: Minimization of the ground state for two phase conductors in low contrast regime. SIAM J. Appl. Math. **72**(4), 1238–1259 (2012). https://doi.org/10.1137/110847822
9. Cox, S.J.: The two phase drum with the deepest bass note. Japan J. Indus. Appl. Math. **8**(3), 345–355 (1991). https://doi.org/10.1007/BF03167141
10. Cuccu, F., Porru, G.: Symmetry of solutions to optimization problems related to partial differential equations. Proc. Royal Soc. Edinburgh Sect. A: Math. **136**(5), 921–934 (2006). https://doi.org/10.1017/s0308210500004807
11. Henrot, A.: Extremum problems for eigenvalues of elliptic operators. Springer Science & Business Media (2006). https://doi.org/10.1007/3-7643-7706-2

12. Hintermüller, M., Kao, C.Y., Laurain, A.: Principal eigenvalue minimization for an elliptic problem with indefinite weight and Robin boundary conditions. Appl. Math. Optim. **65**(1), 111–146 (2012). https://doi.org/10.1007/s00245-011-9153-x
13. Kang, D., Choi, P., Kao, C.Y.: Minimization of the first non-zero eigenvalue problem for two-phase conductors with Neumann boundary conditions. SIAM J. Appl. Math. **80**(4), 1607–1628 (2020). https://doi.org/10.1137/19m1251709
14. Kang, D., Kao, C.Y.: Minimization of inhomogeneous biharmonic eigenvalue problems. Appl. Math. Model. **51**, 587–604 (2017). https://doi.org/10.1016/j.apm.2017.07.015
15. Kao, C.Y., Lou, Y., Yanagida, E.: Principal eigenvalue for an elliptic problem with indefinite weight on cylindrical domains. Math. Biosci. Eng. **5**(2), 315–335 (2008). https://doi.org/10.3934/mbe.2008.5.315
16. Kao, C.Y., Mohammadi, S.A.: Extremal rearrangement problems involving Poisson's equation with Robin boundary conditions. J. Sci. Comput. submitted (2019)
17. Kao, C.Y., Osting, B.: Extremal spectral gaps for periodic Schrödinger operators. ESAIM: Control, Optimisation and Calculus of Variations (2018). https://doi.org/10.1051/cocv/2018029
18. Kao, C.Y., Su, S.: Efficient rearrangement algorithms for shape optimization on elliptic eigenvalue problems. J. Sci. Comput. **54**(2–3), 492–512 (2013). https://doi.org/10.1007/s10915-012-9629-0
19. Laurain, A.: Global minimizer of the ground state for two phase conductors in low contrast regime. ESAIM Control, Optim. Calculus Var. **20**(2), 362–388 (2014). https://doi.org/10.1051/cocv/2013067
20. Mohammadi, A., Bahrami, F.: A nonlinear eigenvalue problem arising in a nanostructured quantum dot. Commun. Nonlinear Sci. Numer. Simul. **19**(9), 3053–3062 (2014). https://doi.org/10.1016/j.cnsns.2013.11.017
21. Mohammadi, S.A.: Extremal energies of Laplacian operator: Different configurations for steady vortices. J. Math. Anal. Appl. **448**(1), 140–155 (2017). https://doi.org/10.1016/j.jmaa.2016.09.011
22. Mohammadi, S.A., Bahrami, F.: Extremal principal eigenvalue of the bi-Laplacian operator. Appl. Math. Model. **40**(3), 2291–2300 (2016). https://doi.org/10.1016/j.apm.2015.09.058
23. Mohammadi, S.A., Bozorgnia, F., Voss, H.: Optimal shape design for the p-laplacian eigenvalue problem. J. Sci. Comput. **78**(2), 1231–1249 (2019). https://doi.org/10.1007/s10915-018-0806-7
24. Mohammadi, S.A., Voss, H.: A minimization problem for an elliptic eigenvalue problem with nonlinear dependence on the eigenparameter. Nonlinear Anal.: Real World Appl. **31**, 119–131 (2016). https://doi.org/10.1016/j.nonrwa.2016.01.015
25. Osher, S., Sethian, J.A.: Fronts propagating with curvature-dependent speed: algorithms based on Hamilton-Jacobi formulations. J. Comput. Phys. **79**(1), 12–49 (1988). https://doi.org/10.1016/0021-9991(88)90002-2
26. Polya, G., Szegö, G.: Isoperimetric Inequalities in Mathematical Physics. (AM-27). Princeton University Press (1951). https://doi.org/10.1515/9781400882663