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Effective generation of right-angled artin groups in mapping class groups

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Abstract
We show that given a collection $X = \{f_1, \ldots, f_m\}$ of pure mapping classes on a surface S, there is an explicit constant N, depending only on X, such that their Nth powers $\{f_1^N, \ldots, f_m^N\}$ generate the expected right-angled Artin subgroup of MCG(S). Moreover, we show that these subgroups are undistorted, and that each element is pseudo-Anosov on the largest possible subsurface.

Keywords Right-angled Artin groups · Mapping class groups · Curve complex

Mathematics Subject Classification 57K20

1 Introduction

Let S be a finite-type orientable surface satisfying $\chi(S) < 0$. By the *mapping class group* of S we mean

$$MCG(S) := Homeo^+(S)/homotopy.$$

The study of free subgroups of $MCG(S)$ dates back to Klein [13], who classically showed that the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

generate a free subgroup of $SL(2, \mathbb{Z})$. Indeed, we may identify $SL(2, \mathbb{Z})$ with $MCG(T^2)$, the mapping class of the torus, and these matrices correspond to the squares of the Dehn twists about the standard meridian and longitude curves. More generally, it follows from Ivanov's [10] and McCarthy's [19] proof of the Tits alternative for $MCG(S)$ that there is in fact an abundance of free subgroups.

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We wish to broaden our view to the larger class of *right-angled Artin groups* (RAAGs). Recall that a RAAG has a presentation determined by a finite simplicial graph Γ:

$$A(\Gamma) = \langle v_i \in V(\Gamma) \mid [v_i, v_j] = 1 \iff (v_i, v_j) \in E(\Gamma)\rangle.$$

Regarding subgroups of $MCG(S)$ of this form, Koberda [14] showed that they too can be found in abundance. In the statement below, we say that a mapping class is *pure* if it is either pseudo-Anosov or else fixes a multi-curve C component-wise and restricts to either a pseudo-Anosov mapping class or the identity on the complementary components $S\backslash C$. We call a mapping class of the latter type a *partial pseudo-Anosov* mapping class, and we call the components of $S\backslash C$ where its action is non-trivial its *support*; in the case of a Dehn twist we define its support to be an annular neighborhood of the twisting curve, and the support of a pseudo-Anosov mapping class is all of S.

Theorem 1 (Koberda, [14] Theorem 1.1) *Let $\{f_1, \ldots, f_m\}$ be an irredundant collection of pure mapping classes supported on connected subsurfaces $S_1, \ldots, S_m \subseteq S$. There exists some $N \neq 0$ such that for all $n \geq N$,*

$$\langle f_1^n, \ldots, f_m^n\rangle \cong A(\Gamma),$$

where Γ is the co-intersection graph of the subsurfaces $\{S_i\}$.

Here, *irredundancy* means that no pair of mapping classes have a common power, and the *co-intersection graph* has as vertices the subsurfaces S_i and edges between vertices whose corresponding subsurfaces can be realized disjointly. Koberda's proof goes by playing ping-pong on the space of geodesic laminations on S, and it is not clear how the number N depends on S or on the given mapping classes.

The goal of this paper is to effectivize and strengthen Koberda's theorem. The constant in the statement of the theorem below is explicitly computed in Sect. 4.

Theorem 2 *Let $\{f_1, \ldots, f_m\}$ be an irredundant collection of pure mapping classes supported on connected subsurfaces $S_1, \ldots, S_m \subseteq S$. There exists an explicit constant $N = N(\{f_i\})$, depending only on certain geometric data extracted from the given collection of mapping classes, such that for all $n \geq N$,*

$$H = \langle f_1^n, \ldots, f_m^n\rangle \cong A(\Gamma),$$

where Γ is the co-intersection graph of the subsurfaces $\{S_i\}$. Moreover, increasing N in a controlled way, we can guarantee that H is undistorted in $MCG(S)$.

We remark that a similar statement was claimed via different means by Sun [22], though there are some gaps in their arguments.

Computing the constant explicitly in the case that all mapping classes in question are Dehn twists, we have the following corollary.

Corollary 1 *Let $\{t_1, \ldots, t_m\}$ be a collection of Dehn twists about distinct curves $\{\beta_1, \ldots, \beta_m\}$, and let*

$$N = 18 + \max_{i,j} i(\beta_i, \beta_j),$$

where $i(\cdot, \cdot)$ denotes geometric intersecion number. Then for all $n \geq N$, we have

$$\langle t_1^n, \ldots, t_m^n\rangle \cong A(\Gamma),$$

where Γ is the subgraph of the curve graph $\mathcal{C}(S)$ spanned by $\{\beta_1, \ldots, \beta_m\}$.

A similar bound has been found by Seo [21] using methods from hyperbolic and coarse geometry, and Bass-Serre theory. That these subgroups are undistorted follows from a careful study of the construction of "admissible" embeddings of RAAGs into mapping class groups due to Clay–Leininger–Mangahas [8].

It is worth mentioning that if there are more than two mapping classes involved, N *necessarily* depends on the given mapping classes, as the following example illustrates. Let β_1 and β_2 be two non-trivially intersecting simple closed curves, and consider the Dehn twists

$$t_1 = t_{\beta_1}, \;\; t_2 = t_{\beta_2}, \;\; \text{and} \;\; t_3 = t_1^{2^K} t_2 t_1^{-2^K}$$

for some $K > 0$. Then for no $1 \leq k \leq K$ is $\langle t_1^{2^k}, t_2^{2^k}, t_3^{2^k} \rangle$ isomorphic to a free group of rank 3, even though the corresponding subgraph of $\mathcal{C}(S)$ is disconnected.

Using similar methods, we are also able to determine the Nielsen-Thurston type for all elements of these subgroups.

Theorem 3 *Let H be as in Theorem 2. Then every $h \in H$ is pseudo-Anosov on its support. In particular, if the support of h is all of S, then h is pseudo-Anosov.*

The support of an element $h \in H$ is the union of the supports of the given generators of H appearing in a cyclically reduced representative of the conjugacy class of h. The method of proof we employ again closely resembles that of Clay–Leininger–Mangahas.

In Sect. 2 we establish the relevant terminology and some basic notions we will need from coarse geometry, geometric group theory (including a proof of a new ping-pong lemma for RAAGs), and the theory of surfaces and their mapping class groups. In Sect. 3 we recall the construction of subsurface projections and other relevant results from [18], which we use to build our ping-pong table. The essential result in this section is a modification of the well-known Behrstock inequality. In Sect. 4 we carry out the proofs.

2 Background

2.1 Coarse geometry

If (X_1, d_{X_1}) and (X_2, d_{X_2}) are metric spaces, we say a (not-necessarily-continuous) map $f : X_1 \to X_2$ is a (A, B)-*quasi-isometric embedding* if there are constants $A \geq 1$ and $B \geq 0$ such that for all $x, y \in X_1$,

$$\frac{1}{A} d_{X_1}(x, y) - B \leq d_{X_2}(f(x), f(y)) \leq A d_{X_1}(x, y) + B.$$

If there is a constant $D > 0$ such that any $x_2 \in X_2$ is within D of $f(X_1)$, we further say f is a *quasi-isometry*, and that X_1 and X_2 are *quasi-isometric*.

Recall that to a group G with generating set Y we may associate the Cayley graph $Cay(G, Y)$, and that equipped with the graph metric $Cay(G, Y)$ is a metric space. Moreover, if G is finitely generated, then any two metrics coming from different finite generating sets Y and Y' yield quasi-isometric Cayley graphs. We may then put a (left-invariant) metric d_G on G, the *word metric*, defined by

$$d_G(g, h) := d_{Cay(G,Y)}(g, h) = d_{Cay(G,Y)}(1, h^{-1}g).$$

Regarding the statement of Theorem 2, we say that a finitely generated subgroup $H < G$ is *undistorted* if the inclusion of H into G is a quasi-isometric embedding with respect to their respective word metrics.

We say a metric space X is δ-*hyperbolic* if there exists a constant δ so that all geodesic triangles are "δ-slim", *i.e.* each side is contained in the union of the δ-neighborhoods of the other two.

2.2 Right-angled artin groups

Given a finite simplicial graph Γ with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, the *right-angled Artin group on* Γ is the group with the presentation

$$A(\Gamma) := \langle v_i \in V(\Gamma) \mid [v_i, v_j] = 1 \iff (v_i, v_j) \in E(\Gamma)\rangle.$$

We call the v_i the *vertex generators* of $A(\Gamma)$. The standard examples of such groups are free groups (when Γ has no edges), free abelian groups (when Γ is a complete graph), and free and direct products of such groups (corresponding to disjoint union of graphs and join of graphs, respectively). The following is a modification of the ping-pong lemma for RAAGs found in [14], itself a generalization of the classical ping-pong lemma for free groups. It is the main tool used in proving Theorem 2.

Lemma 1 (Ping-Pong) *Let $A(\Gamma)$ be a right-angled Artin group acting on a set X such that there exist non-empty subsets $X_i' \subseteq X_i \subset X$ for each vertex generator v_i satsifying*

1. *For $i \neq j$, if $X_i \cap X_j \neq \varnothing$, then there exists $x_i \in X_i$ which does not belong to X_j, and vice versa*
2. *If u is a word in the vertex generators not containing a power of v_j, wherein every vertex generator commutes with v_j, then $u(X_j') \subseteq X_j$*
3. *If v_i and v_j do not commute, then X_i and X_j are disjoint and $v_i^r(X_j) \subset X_i'$ for all $r \neq 0$, and vice versa*

Then the $A(\Gamma)$ action on X is faithful.

Proof If Γ splits as a join, then $A(\Gamma)$ splits as a direct product, and we can play ping-pong on each factor. Hence, we will assume that Γ is not a join, so that, in particular, for each vertex generator v_i there is at least one other vertex generator v_j which does not commute with it. Let $w \neq 1 \in A(\Gamma)$ be a word in the vertex generators. We begin by putting w into a normal form called *central form*, due to M. Kapovich (*cf*. [12] in the proof of Lemma 2.3,[14] in the second proof of Lemma 3.1). Given a representative of w written in the vertex generators, we can perform two operations which do not change the equivalence class of w: a shuffle, where we replace a subword $v_i^r v_j^s$ with $v_j^s v_i^r$ if v_i and v_j commute, and a deletion, where we remove subwords $v_i^r v_i^{-r}$. Starting with any representative of w (in the vertex generators), we can perform these two operations until w may be written as

$$w = u_k v_{i_k}^{r_k} u_{k-1} v_{i_{k-1}}^{r_{k-1}} \cdots u_1 v_{i_1}^{r_1}$$

where

- each u_j is a word in the vertex generators of $A(\Gamma)$, such that each generator appearing in u_j commutes with each other generator appearing in u_j
- v_{i_j} commutes with each generator appearing in u_j
- v_{i_j} does not commute with $v_{i_{j+1}}$ for all $1 \leq j < k$.

We call k the *central-word length* of w.

We now show that w acts non-trivially on X. First suppose that $k = 1$, so that $w = u_1 v_{i_1}^{r_1}$. By assumption there is some generator v_j which does not commute with v_{i_1}, and we choose

$x_j \in X_j$. Applying (3), we have $v_{i_1}^{r_1} x_j \in X'_{i_1}$, then applying (2) we have $u_1 v_{i_1}^{r_1} x_j \in X_{i_1}$. Again by (3), since $X_{i_1} \cap X_j = \varnothing$, we see that $wx_j \neq x_j$ and we are done. Now suppose $k \geq 2$ and that w is written in central form. If v_{i_2} and v_{i_k} are distinct, then either by (1) or (3) we can choose $x_{i_2} \in X_{i_2}$ which does not belong to X_{i_k}; note that since v_{i_2} and v_{i_1} don't commute by assumption, x_{i_2} also does not belong to X_{i_1}. Repeatedly applying the argument above to this word, we have that $wx_{i_2} \in X_{i_k}$, so in particular $wx_{i_2} \neq x_{i_2}$. Finally, if $v_{i_2} = v_{i_k}$, then we can conjugate w by $v_{i_2}^{r_k}$, choose $x_{i_1} \in X_{i_1}$, and apply the same process to

$$v_{i_2}^{r_k} w v_{i_2}^{-r_k} = u_k v_{i_2}^{2r_k} u_{k-1} v_{i_{k-1}}^{r_{k-1}} \cdots u_1 v_{i_1}^{r_1} v_{i_2}^{-r_k},$$

which is indeed in central form. □

2.3 Surfaces and their mapping class groups

Let S be a connected, oriented finite-type surface, possibly with punctures, satisfying $\chi(S) < 0$. The *mapping class group* of S, which we denote by $MCG(S)$, is the group of homotopy classes of orientation-preserving homeomorphisms of S. We call elements of $MCG(S)$ *mapping classes*. An *essential simple closed curve* is the homotopy class of a non-nullhomotopic and non-peripheral simple closed curve on S, and an *essential subsurface* $S' \subseteq S$ is either a regular neighborhood of an essential simple closed curve (*i.e.* an annulus), or a component of the complement of a collection of pairwise disjoint essential simple closed curves (*i.e.* the complement of a multi-curve). For both essential simple closed curves and essential subsurfaces, we will not distinguish between a representative and its homotopy class.

We frequently study mapping class groups of surfaces via their action on essential simple closed curves and subsurfaces. With respect to this action, there is a trichotomy of mapping classes, due to work of Nielsen and Thurston (*cf*. [6]): given $f \in MCG(S)$, f is either

1. finite order,
2. reducible, *i.e.* infinite order and preserves a non-empty multi-curve C set-wise, or
3. pseudo-Anosov, *i.e.* infinite order and no power of f preserves any multi-curve

For a reducible mapping class f, it follows from Birman–Lubotzky–McCarthy [4] that some power f fixes a multi-curve C component-wise, and restricts to a pseudo-Anosov mapping class or the identity on each component of $S \backslash C$. We call such partial pseudo-Anosov mapping classes, as well as pseudo-Anosov mapping classes, *pure* (this definition is slightly different than the original one concerning mapping classes in the kernel of the action of $MCG(S)$ on the $\mathbb{Z}/3\mathbb{Z}$-homology of S, but every infinite order mapping class has a power which is pure in either sense). The *support* of a pure mapping class f is all of S if f is pseudo-Anosov, an annulus about the twisting curve if f is a Dehn twist, or the components of $S \backslash C$ where the action of f is non-trivial if f is a partial pseudo-Anosov mapping class. A partial pseudo-Anosov mapping class f could also multi-twist about its fixed multicurve - in this case we define the support to be the components of $S \backslash C$ where the action of f is non-trivial together with the annular neighborhoods of those curves in C where f is twisting. In particular, if a partial pseudo-Anosov mapping class f exhibits such "boundary twisting", its support is disconnected by definition.

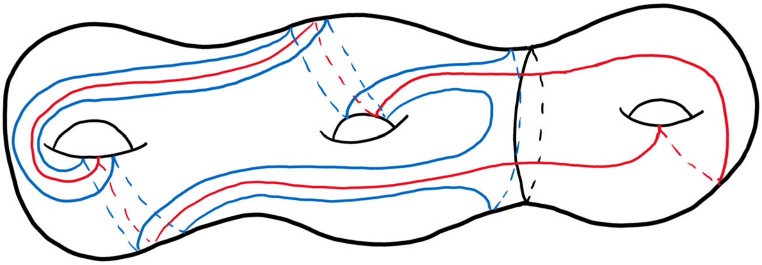

Fig. 1 The projection of the red curve to the left genus two subsurface consists of the blue curves

3 Subsurface projections and the Masur-Minsky Machinery

Our ping-pong sets will be given in terms of Ivanov–Masur–Minsky's *subsurface projections* of essential simple closed curves to essential subsurfaces of S. Recall that the *curve graph* of S, denoted $\mathcal{C}(S)$, is the graph whose vertices are essential simple closed curves, and whose edges are spanned by vertices corresponding to pairs of essential simple closed curves which can be realized disjointly. We equip $\mathcal{C}(S)$ with the graph metric. A celebrated theorem of Masur-Minsky [17] says that with this metric, $\mathcal{C}(S)$ is δ-hyperbolic. Moreover, Aougab [1], Bowditch [5], and Clay et al. [7] have shown that the hyperbolicity constant δ can be made independent of S, and Hensel et al. [9] have shown that $\delta = 17$ suffices. In the sequel, we will use the notation δ instead of its explicit value to make clear the dependence on the hyperbolic geometry of $\mathcal{C}(S)$.

3.1 Constructing subsurface projections

Fix a hyperbolic metric on S, and for each essential simple closed curve, take its unique geodesic representative. Given an essential, non-annular subsurface $S' \subset S$, we define a coarse "projection" map $\pi_{S'} : \mathcal{C}(S) \to \mathcal{C}(S')$ as follows. Let γ be an essential simple closed curve on S. If γ is disjoint from S' entirely, then $\pi_{S'}(\gamma) = \varnothing$, and if γ is properly contained is S' then $\pi_{S'}(\gamma) = \gamma$. Otherwise, γ non-trivially intersects $\partial S'$, and we define $\pi_{S'}(\gamma)$ to be the set of essential simple closed curves obtained by considering each arc α of $\gamma \cap S'$ and taking the boundary of a regular neighborhood of $\alpha \cup \partial S'$, see Fig. 1. Note that geodesic simple closed curves are always in minimal position, so that each arc of this intersection cannot be homotoped out of S', and thus each simple closed curve obtained this way is essential.

Given two essential simple closed curves β and γ with non-trivial projection to S', we define their *projection distance* $d_{S'}(\beta, \gamma)$ to be

$$d_{S'}(\beta, \gamma) := diam_{\mathcal{C}(S')}\{\pi_{S'}(\beta) \cup \pi_{S'}(\gamma)\}. \tag{1}$$

The projection of essential simple closed curves to essential annuli is defined differently: again fix a hyperbolic metric on S, and let β and γ be (the unique geodesic representatives of) intersecting essential simple closed curves. Consider the (compactified) cover S_β of S corresponding to β. We define the projection $\pi_\beta(\gamma)$ to be the collection of lifts c of γ to the cover S_β which connect the two boundary components, see Fig. 2. We can assemble the set of all homotopy (rel. boundary) classes of such arcs in S_β into a graph $\mathcal{A}(\beta)$, the *arc complex* of β, with edges representing pairs of vertices corresponding to homotopy (rel. boundary) classes of arcs which admit representatives with disjoint interiors, and equipped with the

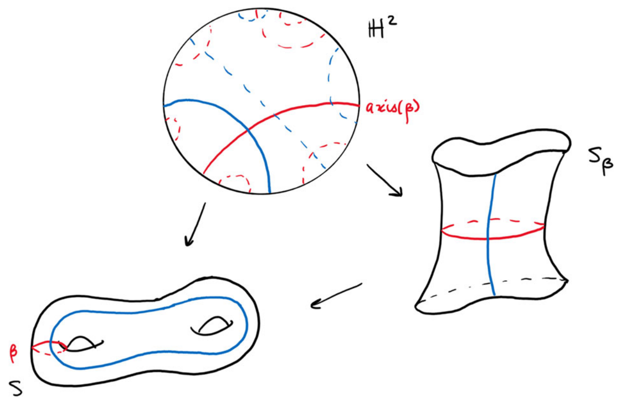

Fig. 2 The annular cover of S corresponding to the red curve, with a lift of the blue curve

graph metric. Given another essential simple closed curve δ which intersects β, we define the projection distance $d_\beta(\gamma, \delta)$ exactly as above; roughly, this distance measures how much γ "twists" around δ relative to β. Though we chose a hyperbolic metric, it is not hard to see that $d_\beta(\delta, \gamma) \leq i(\delta, \gamma) + 1$.

A useful fact about these projections is that they are coarsely invariant under mapping classes supported away from the essential subsurface we're projecting to.

Lemma 2 (Mousley [20] Lemma 3.1) *Let $f \in MCG(S)$ be a pure mapping class supported on an essential subsurface $S_j \subset S$ which is disjoint from an essential subsurface S_i. If S_i is an annulus about a curve β, we also require that ∂S_j does not contain β. Let γ and δ be essential simple closed curves on S. Then*

$$|d_{S_i}(\gamma, \delta) - d_{S_i}(\gamma, f(\delta))| \leq 4.$$

We note that it is precisely because of this coarse invariance that in the statement of the ping-pong lemma, we required the existence of "coarsely preserved" subsets $X_i' \subset X_i$.

3.2 The distance formula

To show that the RAAGs we generate are undistorted in $MCG(S)$, we will need a way to relate word length in $MCG(S)$ to the only other available data we will have, namely projection distances. This relationship is captured by the following "distance formula" of Masur-Minsky [18]. Before stating it, we establish notation. A (complete clean) *marking* μ on S consists of a pants decomposition $\{\beta_i\}$, called the *base* of μ, together with a *transversal* for each β_i satisfying certain properties which are unnecessary for the discussion at hand. Masur-Minsky build a graph $\widetilde{\mathcal{M}}(S)$, called the *marking graph* of S, whose vertices correspond to markings and whose edges are spanned by vertices corresponding to markings related by certain *elementary moves*. Equipped with the graph metric, the graph $\widetilde{\mathcal{M}}(S)$ is locally finite and admits a cocompact action of $MCG(S)$ by isometries, so that $MCG(S)$ and $\widetilde{\mathcal{M}}(S)$ are quasi-isometric. We define the projection of a marking μ to an essential non-annular subsurface $S' \subseteq S$ to be $\pi_{S'}(base(\mu))$ and we define the projection of μ to an essential annulus to be either $\pi_{S'}(base(\mu))$ if the core curve of the annulus is not in base(μ), and the projection of the corresponding transversal otherwise.

Theorem 4 (Masur-Minsky, [18] Theorem 6.10 and Theorem 7.1) *There exists $K_0 = K_0(S) > 0$ with the following property: for all $K \geq K_0$, there exist constants $A \geq 1$ and $B \geq 0$ such that for all pairs of markings $\mu, \mu' \in \widetilde{\mathcal{M}}(S)$ we have*

$$\frac{1}{A} \sum_{S' \subseteq S} [[d_{S'}(\mu, \mu')]]_K - B \leq d_{\widetilde{\mathcal{M}}(S)}(\mu, \mu') \leq A \sum_{S' \subseteq S} [[d_{S'}(\mu, \mu')]]_K + B,$$

where the sums are taken over all essential subsurfaces (including S itself) and where $[[x]]_K = x$ if $x \geq K$ and is 0 otherwise.

In particular, we can approximate the word length of a mapping class f by looking at the subsurface projections distances between μ and $f(\mu)$.

3.3 A multi-scale behrstock inequality

A key idea in the proof of Theorem 2 is a modification of the following inequality due to Behrstock [2]. A constructive proof due to Leininger can be found in [16].

Lemma 3 (Behrstock Inequality) *Let S_i, S_j, and S_k be three pairwise intersecting essential subsurfaces or simple closed curves. Then*

$$d_{S_i}(\partial S_j, \partial S_k) \geq 10 \implies d_{S_j}(\partial S_i, \partial S_k) \leq 4.$$

If S_i (or S_j or S_k) is an annulus, we replace ∂S_i with the core curve β_i. If all three are annuli, we may further replace 4 with 3.

The modification we will make will allow us to not only consider subsurface projections, but also nearest-point projections to geodesics in $\mathcal{C}(S)$ and its subgraphs $\mathcal{C}(S')$ for essential subsurfaces $S' \subset S$. For the proof, we will need the following results concerning distance bounds in the curve graph. The first two are straightforward computations in δ-hyperbolic geometry.

Proposition 1 *Let $\alpha \subset \mathcal{C}(S)$ be a geodesic, and $x, y \in \mathcal{C}(S)$. Then*

$$d_{\mathcal{C}(S)}(\pi_\alpha(x), \pi_\alpha(y)) \leq d_{\mathcal{C}(S)}(x, y) + 24\delta,$$

where π_α is a coarse nearest-point projection map.

Lemma 4 *Let $x \in \mathcal{C}(S)$ and let $\alpha \subset \mathcal{C}(S)$ be a geodesic. Then*

$$diam\{\pi_\alpha(x)\} \leq 4\delta \tag{2}$$

We also need a theorem of Masur-Minsky [18], known as the Bounded Geodesic Image Theorem. The uniform statement below is due to Webb [23], and the constant was recently shown by Jin [11] to bounded above by 44.

Theorem 5 (Bounded Geodesic Image Theorem) *There exists a constant K_{BGIT} with the following property: if $S' \subset S$ is a subsurface and α is a geodesic in $\mathcal{C}(S)$ with the property that $\pi_{S'}(z) \neq \varnothing$ for all $z \in \alpha$, then*

$$diam_{\mathcal{C}(S')}\{\pi_{S'}(\alpha)\} \leq K_{BGIT}.$$

A different version of the lemma below was observed by Sun [22]. We provide an original proof for the reader's convenience and to clarify certain points of Sun's argument.

Lemma 5 (Multi-Scale Behrstock Inequality) *Let β be an essential simple closed curve on S, and let α_1 and α_2 be either essential simple closed curves or geodesics in $\mathcal{C}(S_1) \subseteq \mathcal{C}(S)$ and $\mathcal{C}(S_1) \subseteq \mathcal{C}(S)$ respectively, where S_1 and S_2 are either proper essential subsurfaces of S or S itself. Then*

$$\min d_{\alpha_1}(b_1, a_2) \geq K_{BGIT} + 48\delta \implies \min d_{\alpha_2}(b_2, a_1) < K_{BGIT} + 48\delta$$

The minima are taken over $b_i \in \pi_{\alpha_i}(\beta)$ and $a_i \in \pi_{\alpha_j}(\alpha_i)$, where by π_{α_i} we mean either the previously defined projection to essential annuli if α_i is an essential simple closed curve, or the composition of subsurface projection to S_i followed by nearest-point projection to α_i if α_i is a geodesic.

Proof We break the proof into cases depending on the type of each α_i and the configuration of the S_i within S. The game will be to show that if one of the quantities is suitably large, the other is bounded. In the arguments below there is repeated implicit use of Proposition 1 and Lemma 4.

<u>**Case 1**</u>: α_1 and α_2 are both essential simple closed curves.

In this case, we can use the Behrstock Inequality.

<u>**Case 2**</u>: α_1 is an essential simple closed curve and α_2 is a geodesic in $\mathcal{C}(S_2)$.

We first consider the case that $S_2 = S$. If $\min d_{\alpha_1}(b_1, a_2) \geq K_{BGIT}$, then by the contrapositive of the Bounded Geodesic Image Theorem, a geodesic connecting β to $\pi_{\alpha_2}(\beta)$ passes through the 1-neighborhood of α_1. If z is the vertex on this geodesic which is adjacent to α_1, then we have

$$\begin{aligned} \min d_{\alpha_2}(b_2, a_1) &\leq d_{\alpha_2}(\beta, \alpha_1) \\ &\leq d_{\alpha_2}(\beta, z) + d_{\alpha_2}(z, \alpha_1) \end{aligned}$$

By construction, the nearest-point projections of β and z to α_2 overlap. Also, by Proposition 1 either $d_{\alpha_2}(z, \alpha_1) < 8\delta + 2$ or else $d_{\alpha_2}(z, \alpha_1) \leq 1 + 24$. Hence,

$$\begin{aligned} \min d_{\alpha_2}(b_2, a_1) &\leq 8\delta + (1 + 24\delta) \\ &= 1 + 32\delta \end{aligned}$$

We now suppose that S_2 is a proper essential subsurface of S and that $\min d_{\alpha_1}(b_1, a_2) \geq 11$. Since each vertex in α_2 represents a curve which is disjoint from ∂S_2, we then have $d_{\alpha_1}(\beta, \partial S_2) \geq 10$. Applying the Behrstock inequality yields $d_{S_2}(\beta, \alpha_1) \leq 4$, and so

$$\begin{aligned} \min d_{\alpha_2}(b_2, a_1) &\leq d_{\alpha_2}(\beta, \alpha_1) \\ &\leq 4 + 24\delta \end{aligned}$$

<u>**Case 3**</u>: α_1 and α_2 are both geodesics in their respective curve complexes

We first consider the case that $S_1 = S_2$. Assume $\min d_{\alpha_2}(b_2, a_1) \geq 8\delta + 2$, so that in particular

$$d_{\alpha_2}(\beta, \pi_{\alpha_2}(\pi_{\alpha_1}(\beta))) \geq 8\delta + 2$$

By hyperbolicity (*cf*. [18], Lemma 7.5), a geodesic in S_2 between β and $\pi_{\alpha_1}(\beta)$ passes within 2δ of the geodesic subsegment of α_2 connecting their projections. Let z be a point on the geodesic segment between β and $\pi_{\alpha_1}(\beta)$ that is at most a distance 2δ from a point y on α_2. Then we have

$$\min d_{\alpha_1}(b_1, a_2) \leq d_{\alpha_1}(\beta, y)$$

$$
\begin{aligned}
&\le d_{\alpha_1}(\beta, z) + d_{\alpha_1}(z, y)\\
&\le 8\delta + (2\delta + 24\delta)\\
&= 34\delta
\end{aligned}
$$

Next, if S_1 is nested in S_2 and we assume $\min d_{\alpha_2}(b_2, a_1) \ge 34\delta$, then each vertex on the geodesic between β and $\pi_{\alpha_2}(\beta)$ has distance at least 2 from ∂S_1. Hence, $d_{S_1}(\gamma, \pi_{\alpha_2}(\gamma)) \le K_{BGIT}$, and thus

$$
\begin{aligned}
\min d_{\alpha_1}(b_1, a_2) &\le d_{\alpha_1}(\beta, \pi_{\alpha_2}(\beta))\\
&\le K_{BGIT} + 24\delta
\end{aligned}
$$

Finally, we consider the case that ∂S_1 and ∂S_2 intersect. Suppose that $\min d_{\alpha_1}(b_1, a_2) \ge 11 + 48\delta$, and let z be any vertex on α_2. Then

$$
\begin{aligned}
d_{S_1}(\beta, \partial S_2) &\ge d_{\alpha_1}(\beta, \partial S_2) - 24\delta\\
&\ge d_{\alpha_1}(\beta, z) - d_{\alpha_1}(z, \partial S_2) - 24\delta\\
&\ge (11 + 48\delta) - (1 + 24\delta) - 24\delta\\
&= 10
\end{aligned}
$$

Hence, by the Behrstock inequality, we have $d_{S_2}(\beta, \partial S_1) \le 4$, and so, choosing y on α_1,

$$
\begin{aligned}
\min d_{\alpha_2}(b_2, a_1) &\le d_{\alpha_2}(\beta, y)\\
&\le d_{\alpha_2}(\beta, \partial S_1) + d_{\alpha_2}(\partial S_1, y)\\
&\le (d_{S_2}(\beta, \partial S_1) + 24\delta) + (1 + 24\delta)\\
&= 5 + 48\delta
\end{aligned}
$$

Thus $K_{BGIT} + 48\delta$ suffices for all cases. □

3.4 The action on the curve graph

The following are a set of results of Masur-Minsky from [17] and [18] concerning the action on the curve graph of pseudo-Anosov mapping classes. The first tells us that they act on $\mathcal{C}(S)$ like hyperbolic isometries.

Proposition 2 ([17], Prop. 3.6) *There exists a constant $c = c(S) > 0$ such that, for any pseudo-Anosov mapping class $f \in MCG(S)$, any simple closed curve γ, and any $n \in \mathbb{Z}\backslash\{0\}$, we have*

$$
d_S(f^n(\gamma), \gamma) \ge c|n|.
$$

Masur-Minsky proved the above for the "non-sporadic" surfaces. For sporadic cases, namely $S_{1,1}$ and $S_{0,4}$, we redefine the curve graph in such a way that we obtain the Farey graph, where it is noted by Mangahas [15] that the same result follows by considering the action of hyperbolic isometries on the Farey graph embedded in $\mathbb{H}^2$. It is easy to show that Proposition 2 implies that for any essential simple closed curve γ and any pseudo-Anosov mapping class f, the bi-infinite sequence of curves $\{f^n(\gamma) | n \in \mathbb{Z}\}$ is an f-invariant quasi-geodesic. By restricting a pure mapping class to a pseudo-Anosov component $S' \subset S$, we obtain such a lower bound for the action of f on $\mathcal{C}(S') \subset \mathcal{C}(S)$, and for a power of a Dehn twist acting on its corresponding arc complex, the quantity c can be taken to be 1.

As noted above, any pseudo-Anosov mapping class f preserves many *quasi-geodesics* in $\mathcal{C}(S)$. However, the Multi-scale Behrstock Inequality was stated in terms of projections to *geodesics*. In order to apply the Multi-scale Behrstock Inequality, we will need the following proposition of Masur-Minsky.

Proposition 3 ([18], Prop. 7.6) *Let $f \in MCG(S)$ be pseudo-Anosov. There exists a bi-infinite geodesic α in $\mathcal{C}(S)$ such that for all j, α and $f^j(\alpha)$ are 2δ fellow travelers.*

The geodesic α and its f-translates are referred to as a *quasi-axis* for f. A straightforward computation shows that the nearest point projections of any vertex x in $\mathcal{C}(S)$ to any two geodesics in a quasi-axis are at most 10δ apart. Applying Proposition 2 to the action of f on its quasi-axis, we have

Lemma 6 ([18], Lemma 7.7) *Given $A > 0$, let N be the smallest integer such that $c(S)N > A + 10\delta$, where $c(S)$ is the constant from Proposition 2. Then for all $n \geq N$,*

$$d_{\mathcal{C}(S)}(\pi(x), \pi(f^n(x))) \geq A.$$

where π denotes a coarse nearest-point projection to the quasi-axis of f.

4 The proofs

The goal of this section is to prove the following, which is the statements of Theorems 2 and 3 combined.

Theorem 6 *Let $\{f_1, \ldots, f_m\}$ be an irredundant collection of pure mapping classes supported on connected subsurfaces $S_1, \ldots, S_m \subseteq S$. Let*

$$N = \frac{5K_{BGIT} + 200\delta + M_2 + M_1 + 4}{\min\limits_{1 \leq i \leq m} c(S_i)},$$

where $c(S_i)$ is as in Proposition 2 and M_1 and M_2 are defined below. Then for all $n \geq N$,

$$H = \langle f_1^n, \ldots, f_m^n \rangle \cong A(\Gamma),$$

where Γ is the co-intersection graph of the subsurfaces $\{S_i\}$. After increasing N in a controlled way, we can guarantee that H is undistorted in $MCG(S)$. Moreover, each $h \in H$ is pseudo-Anosov on its support.

We break the proof into three parts, first proving that H is indeed the desired RAAG, then proving that H is undistorted in $MCG(S)$, and finally proving that each element is pseudo-Anosov on its support.

4.1 Theorem 6 Part 1: generation

We first show that the group generated by $\{f_1^n, \ldots, f_m^n\}$ is the expected RAAG.

Proof Let $\{f_1, \ldots, f_m\} \in$ MCG(S) be an irredundant collection of pure mapping classes with connected supporting subsurfaces $\{S_1, \ldots, S_m\}$. For each $1 \leq i \leq m$, let α_i be a geodesic in the quasi-axis for f_i in $\mathcal{C}(S_i) \subseteq \mathcal{C}(S)$ or the core curve of S_i if S_i is an essential annulus. As in the proof of the ping-pong lemma, we assume that the co-intersection graph of the collection

$\{S_i\}$ is not a non-trivial join, so that for each f_i there is some f_j which does not commute with it. We will explicitly construct a constant N and a group action so that for all $n \geq N$, $\{f_1^n, \ldots, f_m^n\}$ satisfy the criteria for ping-pong. To this end, set

$$X = \{\beta \mid \beta \text{ an essential simple closed curve in } S\},$$

and for each $1 \leq i \leq m$, set

$$X_i = \{\beta \mid \min d_{\alpha_i}(b_i, a_j) > K_{BGIT} + 48\delta \ \text{ for all } j \text{ such that } S_j \cap S_i \neq \emptyset\},$$
$$X_i' = \{\beta \mid \min d_{\alpha_i}(b_i, a_j) > K_{BGIT} + 48\delta + 4 \ \text{ for all } j \text{ such that } S_j \cap S_i \neq \emptyset\},$$

where the minima are taken over $b_i \in \pi_{\alpha_i}(\beta)$ and $a_j \in \pi_{\alpha_i}(\alpha_j)$. Observe that if S_i and S_j intersect, then by the Multi-scale Behrstock Inequality their corresponding sets X_i and X_j are disjoint. Moreover, since we assumed the mapping classes were irredundant, *i.e.* no two have a common power, no two preserve the same ending lamination in the Gromov boundary $\partial\mathcal{C}(S)$. Hence, no chosen α_i fellow travels another chosen α_j, and so these geodesics have bounded diameter projections to one another..

Let w be a word in the abstract RAAG generated by $\{f_1^n, \ldots, f_m^n\}$. We begin by putting w into central form as in the proof of the ping-pong lemma: using only shuffles and deletions, we may write w as

$$w = u_k g_k u_{k-1} g_{k-1} \cdots u_1 g_1,$$

where each g_j represents some power of some f_i^n, and each u_j is a word in the generators satisfying the necessary properties of the central form. We possibly make one further modification to this representative. For each g_j which is a power of a Dehn twist, if a power of a generator appearing in the corresponding u_j is supported on a subsurface containing the twisting curve as a boundary component, we may shuffle $u_j g_j$ to $u_j' g_j'$, where g_j' is the aforementioned power of a generator and u_j' contains the original g_j instead. To see that this modification does not violate the central form, note that since g_{j-1} and g_{j+1} don't commute with g_j, their supports intersect the twisting curve of g_j, which is the boundary of the support of g_j'. Hence the supports of g_{j-1} and g_{j+1} both intersect that of g_j'.

We may now play ping-pong. Up to relabelling, we assume $g_1 = f_1^{nr_1}$,

$g_2 = f_2^{nr_2}$, and $g_k = f_j^{nr_j}$ for some j. Choose $\beta \in X_2 \setminus (X_2 \cap X_j)$; either g_2 and g_k don't commute, so their corresponding sets X_2 and X_j are disjoint, or they commute and their supports are disjoint, and we can choose a β which intersects S_2 but not S_j. If g_k is also a power of f_2^n, conjugate w by g_k, choose $\beta \in X_1$, and run the same argument below. Since g_1 and g_2 don't commute, their corresponding sets X_1 and X_2 are disjoint. In particular, since $\beta \in X_2$, it satisfies

$$\min d_{\alpha_1}(b_1, a_2) \leq K_{BGIT} + 48\delta.$$

For each ℓ such that $S_\ell \cap S_1 \neq \varnothing$, we have

$$\begin{aligned}
\min d_{\alpha_1}(b_1, a_\ell) &\leq d_{\alpha_1}(\beta, \alpha_\ell) \\
&\leq d_{\alpha_1}(\beta, \alpha_2) + d_{\alpha_1}(\alpha_2, \alpha_\ell) \\
&\leq (K_{BGIT} + 48\delta + diam\{\pi_{\alpha_1}(\beta)\} + diam\{\pi_{\alpha_1}(\alpha_2)\}) + M_1 \\
&= K_{BGIT} + 48\delta + 4\delta + M_2 + M_1 \\
&= K_{BGIT} + 52\delta + M_2 + M_1,
\end{aligned}$$

where

$$M_1 = \max_{1\le i,\ell,s\le m} d_{\alpha_i}(\alpha_\ell, \alpha_s)$$
$$M_2 = \max_{1\le i,j\le m} diam\{\pi_{\alpha_i}(\alpha_j)\}.$$

Choosing $b' \in \pi_{\alpha_1}(\beta)$ and $a'_\ell \in \pi_{\alpha_1}(\alpha_\ell)$ which realize $\min d_{\alpha_1}(b_1, a_\ell)$, we have

$$\begin{aligned} d_{\alpha_1}(f_1^N(b'), a_{\ell'}) &\ge d_{\alpha_1}(f_1^N(b'), b') - d_{\alpha_1}(b', a'_\ell) \\ &\ge d_{\alpha_1}(f_1^N(b'), b') - (K_{BGIT} + 52\delta + M_2 + M_1) \end{aligned}$$

Hence, if

$$\begin{aligned} d_{\alpha_1}(f_1^N(b'), b') \ge\ & 2K_{BGIT} + 110\delta + M_2 + M_1 + 4 \\ & + diam\{\pi_{\alpha_1}(f_1^N(b'))\} + diam\{\pi_{\alpha_1}(b')\} \\ =\ & 2K_{BGIT} + 118\delta + M_2 + M_1 + 4, \end{aligned}$$

we will have $f_1^N(b') \in X'_1$. Invoking Lemma 6, we set

$$N = \frac{5K_{BGIT} + 200\delta + M_2 + M_1 + 4}{\min\limits_{1\le i\le m} c(S_i)},$$

which is in fact much larger than we need here, but will be useful later. Thus, $g_1(\beta) \in X_1$, and by Lemma 2, $u_1 g_1(\beta) \in X_1$. Running this process until it terminates after the application of $u_k g_k$, we see that $w(\beta) \in X_j$, and we are done. □

If we restrict Theorem 6 to the case where all the f_i are Dehn twists, the constant N simplifies quite a bit.

Corollary 2 *1 Let $\{t_1, \ldots, t_m\}$ be a collection of Dehn twists about distinct essential simple closed curves $\{\beta_1, \ldots, \beta_m\}$ on S, and let*

$$N = 18 + \max_{i,j} i(\beta_i, \beta_j).$$

Then for all $n \ge N$, we have

$$\langle t_1^n, \ldots, t_m^n \rangle \cong A(\Gamma),$$

where Γ is the subgraph of $\mathcal{C}(S)$ spanned by the curves $\{\beta_i\}$.

Proof As we are dealing only with essential annuli, we don't need to account for the constant $c(S_i)$ from Proposition 2 (since for Dehn twists, $c = 1$), and we can use the original Behrstock inequality. Following the proof of Theorem 2, for each $1 \le i \le m$ we set

$$X_i = \{\gamma \mid d_{\beta_i}(\gamma, \beta_j) \ge 10 \text{ for all } j \text{ such that } \beta_j \cap \beta_i \ne \emptyset\},$$
$$X'_i = \{\gamma \mid d_{\beta_i}(\gamma, \beta_j) \ge 14 \text{ for all } j \text{ such that } \beta_j \cap \beta_i \ne \emptyset\},$$

and we write $w = u_k g_k \cdots u_1 g_1$, where each g_j is a power of some t_i^n, in central form; relabelling, we assume $g_1 = t_1^{nr_i}$, $g_2 = t_2^{nr_2}$, and $g_k = t_j^{nr_k}$ for some j. Choose $\beta \in X_2 \backslash (X_2 \cap X_j)$, or in the case that g_k is also a power of t_2^n, conjugate w by g_k, choose $\gamma \in X_1$, and run the same argument below.

Since $\gamma \in X_2$, we have $d_{\beta_1}(\gamma, \beta_2) \leq 3$. For any ℓ such that $i(\beta_1, \beta_\ell) \neq 0$, we then have

$$\begin{aligned} d_{\beta_1}(\gamma, \beta_\ell) &\leq d_{\beta_1}(\gamma, \beta_2) + d_{\beta_1}(\beta_2, \beta_\ell) \\ &\leq 3 + M_1. \end{aligned}$$

Then

$$\begin{aligned} d_{\beta_1}(t_1^N(\gamma), \beta_\ell) &\geq d_{\beta_1}(t_1^N(\gamma), \gamma) - d_{\beta_1}(\gamma, \beta_\ell) \\ &\geq N - 3 - M_1. \end{aligned}$$

Hence, setting $N = 17 + M_1$ suffices to finish the proof. But we previously noted that

$$\begin{aligned} M_1 &= \max_{1 \leq i, \ell, s \leq m} d_{\beta_i}(\beta_\ell, \beta_s) \\ &\leq \max_{1 \leq \ell, s \leq m} i(\beta_\ell, \beta_s) + 1, \end{aligned}$$

so we set $N = 18 + \max_{1 \leq \ell, s \leq m} i(\beta_\ell, \beta_s)$ so that the constant is independent of any choice of hyperbolic metric. □

This should be compared to the main theorem of [21], where a similar (quadratic) bound was computed. As an easy application, we state the following.

Corollary 3 *Let $\{\beta_1, \ldots, \beta_m\}$ be a collection of essential simple closed curves such that no three curves pairwise intersect. Then the 19th powers of the corresponding Dehn twists generate a RAAG.*

Proof Since no three curve pairwise intersect, the projection distances $d_{\beta_i}(\beta_j, \beta_k)$ are uniformly bounded above by 1.

4.2 Theorem 6 Part 2: undistortion

We now show that the subgroups generated in the previous section are undistorted in $MCG(S)$, after increasing the power N by a controlled amount. To do this, we will borrow the following theorem from [8]. Though we use projections to geodesics instead of just subsurface projections, the proof is nearly identical, so we only provide a sketch highlighting the necessary modifications that need to be made.

Theorem 7 *Let H be as above, $\mu \in \widetilde{\mathcal{M}}(S)$ be a marking on S, and let*

$$N = \frac{5K_{BGIT} + K_0 + 200\delta + 2M_3 + M_2 + M_1 + 4}{\min_{1 \leq i \leq m} c(S_i)},$$

where

$$M_3 = \max_{1 \leq i, j \leq m} d_{\alpha_i}(\mu, \alpha_j),$$

and where K_0 is as in the Masur-Minsky distance formula. Let $w = g_1 \cdots g_k \in H$, where $g_i = (f_j^n)^{e_i}$ for $n \geq N$. Then

$$d_{g_1 \cdots g_{i-1}\alpha_j}(\mu, w\mu) \geq (K_0 + K_{BGIT} + 48\delta)|e_i|.$$

Proof (sketch) First, we remark that while the set of mapping classes considered in [8] explicitly excludes Dehn twists, pseudo-Anosovs, and mapping classes with the same or nested supports, the Multi-scale Behrstock Inequality allows *us* to consider them. The proof goes by induction on k, the (minimal) number of "syllables" of w (note that we are *not* using central form). The base case is simply the claim that

$$\begin{aligned} d_{\alpha_j}(\mu, g_1(\mu)) &= d_{\alpha_j}(\mu, (f_j^n)^{e_1}(\mu)) \\ &\geq (K_0 + K_{BGIT} + 48\delta)|e_i|, \end{aligned}$$

which is true by construction of N. For the induction, we break w into subwords:

$$\begin{aligned} w &= (g_1 \cdots g_\ell)(g_{\ell+1} \cdots g_{i-1})g_i(g_{i+_1} \cdots g_k) \\ &= abg_ic. \end{aligned}$$

Via repeated applications of the triangle inequality, using Lemma 2 where necessary, the claim reduces to the statement that the distances

$$d_{\alpha_j}(a^{-1}(\mu), \mu) \ , \ d_{\alpha_j}(c(\mu), \mu)$$

are both bounded in terms of the constants appearing in the numerator of N. This is also shown via the triangle inequality, using Lemma 2 where necessary, as well as the Multi-scale Behrstock Inequality. □

The proof of undistortion below is nearly identical to that of [8].

Proof Via the quasi-isometry between $MCG(S)$ and $\widetilde{\mathcal{M}}(S)$, it suffices to show that there are constants $A \geq 1$ and $B \geq 0$ such that for all $w \in H$

$$\frac{1}{A} d_{\widetilde{\mathcal{M}}(S)}(\mu, w\mu) - B \leq d_H(1, w) \leq A d_{\widetilde{\mathcal{M}}(S)}(\mu, w\mu) + B.$$

For any group G acting by isometries on a metric space (X, d_X), we always have

$$d_X(x, gx) \leq A d_G(1, g),$$

where $A \geq \max d_X(x, s_i x)$, and s_i is a generator for G. Hence, we need only to find A and B so that for all $w \in H$

$$d_H(1, w) \leq A d_{\widetilde{\mathcal{M}}(S)}(\mu, w\mu) + B.$$

Let H be as above and let N be as in Theorem 5. Let $w = (f_{\ell_k}^n)^{e_k} \cdots (f_{\ell_1}^n)^{e_1}$, $n \geq N$ and set $g_j = (f_{\ell_j}^n)^{e_i}$. Then

$$\begin{aligned} d_H(1, w) &= \sum_{i=1}^k |e_i| \\ &\leq \sum_{i=1}^k K_0 |e_i| \\ &\leq \sum_{i=1}^k d_{g_1 \cdots g_{i-1} \alpha_{\ell_i}}(\mu, w\mu). \end{aligned}$$

By Proposition 1, each term in the last sum satisfies

$$d_{g_1\cdots g_{i-1}\alpha_{\ell_i}}(\mu, w\mu) \leq d_{g_1\cdots g_{i-1}S_{\ell_i}}(\mu, w\mu) + 24\delta.$$

Thus,

$$\begin{aligned}\sum_{i=1}^{k} d_{g_1\cdots g_{i-1}\alpha_{\ell_i}}(\mu, w\mu) &\leq \sum_{i=1}^{k}(d_{g_1\cdots g_{i-1}S_{\ell_i}}(\mu, w\mu) + 24\delta)\\ &\leq \sum_{S'\subseteq S}[[d_{S'}(\mu, w\mu)]]_K\\ &\leq A d_{\widetilde{\mathcal{M}}(S)}(\mu, w\mu) + B,\end{aligned}$$

where $K \geq K_0$, and the last inequality follows from the Masur-Minsky distance formula. □

4.3 Theorem 6 Part 3: Nielsen-Thurston type

Finally, we show that each $w \in H$ is pseudo-Anosov on its support. This will follow from showing that for any w, we can find an essential simple closed curve whose orbit goes off to infinity in $\mathcal{C}(S)$. We begin by stating a lemma of Bestvina–Bromberg–Fujiwara [3].

Lemma 7 ([3], Lemma 4.20) *Let $\{\beta_i\}_{i=0}^k$ be a sequence of essential simple closed curves in $\mathcal{C}(S)$ such that each consecutive triple of curves satisfies*

$$d_{S_i}(\beta_{i-1}, \beta_{i+1}) \geq 3K_{BGIT},$$

where S_i is an essential subsurface with $\beta_i \in \partial S_i$. Then

$$d_{\mathcal{C}(S)}(\beta_0, \beta_k) = \sum_{i=1}^{k} d_{\mathcal{C}(S)}(\beta_{i-1}, \beta_i) - 2k$$

We will construct such a sequence so that consecutive curves are distance at least 3 apart, which by the above lemma must go off to infinity.

Proof Let H be as in Theorem 6 with N as in Theorem 7. Without loss of generality, we assume that the support of $w \in H$ is all of S (the same argument holds restricting to the curve graph of the support in the case that the support is a proper subsurface). Write $w = u_1 g_1 \cdots u_k g_k$ in central form, where each g_i is a power of some generator f_j^n, $n \geq N$, of H.

If each g_i is a pseudo-Anosov mapping class, then by Theorem 7, there is a generator such that the appropriate translate of its axis "witnesses" a large distance between any essential simple closed curve β and its image $w\beta$, *i.e.* for some j,

$$d_{u_1 g_1\cdots u_{j-1} g_{j-1}\alpha_j}(\beta, w\beta) \geq K_0 + K_{BGIT} + 48\delta.$$

In this case, since w and β were arbitrary, we have that no *power* of w fixes any essential simple closed curve, *i.e.* w is pseudo-Anosov.

Now assume that at least one g_i is reducible with support S'; up to conjugation, we may assume that g_1 is a power of this reducible. We first claim that $\beta \in \partial S'$ and $w\beta$ fill S, *i.e.* have distance at least 3 in $\mathcal{C}(S)$. As is noted in ([8], Lemma 6.2), the subsurfaces supporting the g_i fill S if and only if the subsurfaces $u_1 g_1 \cdots u_{j-1} g_{j-1} S_j$, where $1 \leq j \leq k$ and S_j is the support of g_j, also fill S. This implies that β and $w\beta$ fill S. Indeed, suppose γ is another essential simple closed curve. As the subsurfaces $u_1 g_1 \cdots u_{j-1} g_{j-1} S_j$ fill S, γ has non-trivial projection to at least one of them. But in this subsurface, β and $w\beta$ have large projection, so γ cannot be disjoint from both simultaneously. Hence, β and $w\beta$ fill S, *i.e.* $d_{\mathcal{C}(S)}(\beta, w\beta) \geq 3$, and the same is true of $w^{\ell}\beta$ and $w^{\ell+1}\beta$ for all $\ell \in \mathbb{Z}$.

It remains to show that the sequence $\{w^{\ell}\beta\}$ satisfies

$$d_{w^{\ell}S'}(w^{\ell-1}\beta, w^{\ell+1}\beta) \geq 3K_{BGIT}$$

which by equivariance of projections is equivalent to

$$d_{S'}(w^{-1}\beta, w\beta) \geq 3K_{BGIT}$$

Using the given expression for w and the triangle inequality, we have

$$d_{S'}(u_1g_1\cdots u_kg_k\beta, u_k^{-1}g_k^{-1}\cdots u_1^{-1}g_1^{-1}\beta) \geq d_{S'}(u_1g_1\cdots u_kg_k\beta, u_2g_2\cdots u_kg_k\beta) - d_{S'}(u_2g_2\cdots u_kg_k\beta, u_k^{-1}g_k^{-1}\cdots u_2^{-1}g_2^{-1}\beta).$$

The subtracted term on the right-hand side satisfies

$$d_{S'}(u_2g_2\cdots u_kg_k\beta, u_k^{-1}g_k^{-1}\cdots u_2^{-1}g_2^{-1}\beta) \leq d_{S'}(u_2g_2\cdots u_kg_k\beta, \alpha_j) + d_{S'}(\alpha_j, \alpha_i) + d_{S'}(\alpha_i, u_k^{-1}g_k^{-1}\cdots u_2^{-1}g_2^{-1}\beta),$$

where $g_2 = (f_j^n)^{e_2}$ and $g_k = (f_i^n)^{e_k}$. Setting

$$R = d_{S'}(u_1g_1\cdots u_kg_k\beta, u_2g_2\cdots u_kg_k\beta),$$

what we are trying to show reduces to

$$R \geq d_{S'}(u_2g_2\cdots u_kg_k\beta, \alpha_j) + M_1 + d_{S'}(\alpha_i, u_k^{-1}g_k^{-1}\cdots u_2^{-1}g_2^{-1}\beta) + 3K_{BGIT}$$

By the construction of N, R is at least {numerator of N} − 4. Moreover, the first and third terms on the right-hand side are both bounded above by

$K_{BGIT} + 48\delta$ (the bound from the Multi-scale Behrstock Inequality)—indeed, we have by Theorem 7 that

$$d_{\alpha_j}(u_2g_2\cdots u_kg_k\beta, \beta) \geq K_{BGIT} + 48\delta$$

and so

$$d_{S'}(u_2g_2\cdots u_kg_k\beta, \alpha_j) < K_{BGIT} + 48\delta$$

by the Multi-scale Behrstock Inequality; the same argument holds for the other term. Thus the inequality we are trying to show is

$$\{\text{numerator of N}\} - 4 \geq 5K_{BGIT} + 96\delta + M_1$$

which is true by construction. □

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