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Small C^1 actions of semidirect products on compact manifolds

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Let T be a compact fibered 3–manifold, presented as a mapping torus of a compact, orientable surface S with monodromy ψ, and let M be a compact Riemannian manifold. Our main result is that if the induced action ψ^* on $H^1(S,\mathbb{R})$ has no eigenvalues on the unit circle, then there exists a neighborhood $\mathcal{U}$ of the trivial action in the space of C^1 actions of $\pi_1(T)$ on M such that any action in $\mathcal{U}$ is abelian. We will prove that the same result holds in the generality of an infinite cyclic extension of an arbitrary finitely generated group H provided that the conjugation action of the cyclic group on $H^1(H,\mathbb{R}) \neq 0$ has no eigenvalues of modulus one. We thus generalize a result of A McCarthy, which addressed the case of abelian-by-cyclic groups acting on compact manifolds.

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1 Introduction

We consider smooth actions of finitely generated-by-cyclic groups on compact manifolds, motivated by the study of fibered hyperbolic 3–manifold groups. We let S be a compact, orientable surface of negative Euler characteristic, possibly with boundary. Thus, the fundamental group $\pi_1(S)$ is either a finitely generated free group or the fundamental group of a closed surface of genus g for some $g \geq 2$. If $\phi \in \mathrm{Homeo}(S)$ is a (possibly orientation-reversing) homeomorphism, then we may form $T = T_\phi$, the mapping torus of ϕ. We have that the fundamental group $\pi_1(T)$ fits into a short exact sequence of the form

$$1 \to \pi_1(S) \to \pi_1(T) \to \mathbb{Z} \to 1,$$

where the conjugation action of $\mathbb{Z}$ on $\pi_1(S)$ is by the induced action of ϕ. It is well known that, up to an inner automorphism of $\pi_1(S)$, this action depends only on the homotopy class of ϕ, and is therefore an invariant of the (extended) mapping class

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of ϕ. It follows that the isomorphism type of $\pi_1(T)$ depends only on the mapping class of ϕ.

We will be particularly interested in the action ϕ^* on the real cohomology of the fiber, and especially in the case where the induced map $\phi^*\colon H^1(S,\mathbb{R}) \to H^1(S,\mathbb{R})$ is hyperbolic. Here, the induced automorphism ϕ^* is said to be *hyperbolic* if $H^1(S,\mathbb{R}) \neq 0$ and if every eigenvalue of ψ^* has modulus different from one. More generally, an automorphism of a nonzero, finite-dimensional real vector space is hyperbolic if it has no eigenvalues of modulus one.

Examples of fibered 3–manifolds with hyperbolic monodromies include all compact 3–manifolds admitting sol geometry, as well as many fibered hyperbolic manifolds. For example, the figure eight knot complement fibers over the circle with a punctured torus as the fiber, and the monodromy given by an automorphism acting hyperbolically on the homology of the torus.

Fibered 3–manifold groups arising from mapping classes acting hyperbolically on the homology of the fiber fall into a much larger class of groups which we will be able to investigate with our methods. Here and throughout, we will let M be a compact Riemannian manifold. Recall that a short exact sequence of finitely generated groups

$$1 \to H \to G \to \mathbb{Z} \to 1$$

naturally determines $\psi \in \mathrm{Out}(H)$, and hence induces a unique linear automorphism ψ^* of $H^1(H,\mathbb{R})$. Abstractly as groups, we have that G is isomorphic to the semidirect product

$$G \cong H \rtimes_\psi \mathbb{Z},$$

where the outer automorphism ψ is given by the conjugation action of $\mathbb{Z} \cong G/H$ on H. We will study $\mathrm{Hom}(G, \mathrm{Diff}^1(M))$, the space of C^1 actions of G on M, in the case that ψ^* is hyperbolic.

1.1 Main result

We will use the symbol 1 to mean the identity map, the trivial group, the identity group element or the real number 1 depending on the context, as this will not cause confusion. The principal result of this paper is the following:

Theorem 1.1 *Suppose we have a short exact sequence of finitely generated groups*

$$1 \to H \to G \to \mathbb{Z} \to 1$$

which induces a hyperbolic automorphism ψ^* of $H^1(H,\mathbb{R})$. Then there exists a neighborhood $\mathcal{U} \subseteq \mathrm{Hom}(G, \mathrm{Diff}^1(M))$ of the trivial representation such that $\rho(H) = 1$ for all representations $\rho \in \mathcal{U}$.

Thus, sufficiently small actions of G on compact manifolds necessarily factor through cyclic groups provided the automorphism defining the extension G is hyperbolic on cohomology. The reader is directed to Section 2.1 for a discussion of the topology on $\mathrm{Hom}(G, \mathrm{Diff}^1(M))$. Theorem 1.1 may be viewed as an analogue of a result of A McCarthy [37], who proved a statement with the same conclusion for abelian-by-cyclic groups (the fundamental groups of compact 3–manifolds admitting sol geometry fall in this class).

For certain manifolds and with certain natural hypotheses, abelian-by-cyclic group actions by diffeomorphisms enjoy rather strong rigidity properties; see Hurtado and Xue [28].

Note that if H is a left-orderable group then it is not difficult to find faithful actions of G by homeomorphisms of the interval $[0,1]$ which are arbitrarily C^0–close to the identity, so the C^1 regularity assumption in Theorem 1.1 is essential.

By applying Theorem 1.1 to the above short exact sequence for a fibered 3–manifold group, we obtain the following result.

Corollary 1.2 *Let S, ϕ and T be as above. If ϕ induces a hyperbolic automorphism of $H^1(S,\mathbb{R})$, then there exists a neighborhood $\mathcal{U}$ of the trivial representation in $\mathrm{Hom}(\pi_1(T), \mathrm{Diff}^1(M))$ such that $\rho(H) = 1$ for all representations $\rho \in \mathcal{U}$.*

The hypotheses in Theorem 1.1 may be contrasted with the following result of Bonatti and Rezaei [8], which generalizes some work of Farb and Franks [19] and Jorquera [29], and is closely related to results of Navas [40] and Parkhe [41].

Theorem 1.3 (Bonatti and Rezaei) *Every finitely generated, residually torsion-free nilpotent group G admits a faithful representation $\rho\colon G \to \mathrm{Diff}^1([0,1])$ that is C^1–close to the identity.*

Here, a group is *residually torsion-free nilpotent* if every nontrivial element $g \in G$ survives in a torsion-free nilpotent quotient of G. A representation $\rho \in \mathrm{Hom}(G, \mathrm{Diff}^1(M))$ is said to be *C^1–close to the identity* if for every $\epsilon > 0$, there is an element $h = h_\epsilon \in$

$\mathrm{Diff}^1(M)$ such that h conjugates ρ into an ϵ–neighborhood of the trivial representation of G, in the C^1–topology on $\mathrm{Hom}(G, \mathrm{Diff}^1(M))$.

It is not difficult to check that if the group G is as in the statement of Theorem 1.1 then the only torsion-free nilpotent quotient admitted by G is $\mathbb{Z} = G/H$. Thus, the hyperbolicity of the map ψ^* plays a crucial role in the dynamics of the group G.

1.2 Unipotent monodromy maps and virtually special groups

An essential feature of Theorem 1.1 is its "unstable" nature, in the sense that it does not remain true after passing to finite-index subgroups of G. Indeed, we have the following fact, which follows fairly easily from known results:

Proposition 1.4 *Let N be a hyperbolic 3–manifold with finite volume. Then a finite-index subgroup of the fundamental group $\pi_1(N)$ admits a faithful representation ρ into $\mathrm{Diff}^1([0,1])$ such that ρ is C^1–close to the identity.*

Proof The essential point is that, combining rather deep results of Agol and Wise with some combinatorial group theory arguments of Duchamp and Krob, one sees that the fundamental group $\pi_1(N)$ contains a finite-index subgroup which is residually torsion-free nilpotent, and hence admits a faithful representation into $\mathrm{Diff}^1([0,1])$ that is C^1–close to the identity.

In more detail, by the work of Agol [1] and Wise [47], there is a finite-index subgroup $G_0 < \pi_1(N)$ such that G_0 is *special*. In particular, G_0 embeds in a *right-angled Artin group*; such a group is always residually torsion-free nilpotent by Duchamp and Krob [18]. See also the discussion in Aschenbrenner, Friedl and Wilton [2, Chapter 5]. Thus, the proposition follows from Theorem 1.3. □

Thus, if G is a group satisfying the hypotheses of Theorem 1.1, then, passing to a finite-index subgroup G_0, one often obtains a group satisfying the hypotheses of Theorem 1.3. In such a case, one can build a C^1 action of G on the disjoint union of n copies of $[0,1]$, where $n = [G : G_0]$, by an analogue of the induced representation of a finite-index subgroup. Such an action will permute the components of this manifold transitively. This does not contradict Theorem 1.1, since any such action will be outside of a fixed neighborhood of the trivial representation of G.

One can produce many fibered 3–manifold groups, even hyperbolic ones, which are residually torsion-free nilpotent, without using the deep results of Agol and Wise. Indeed, it suffices to use monodromy maps ϕ such that ϕ^* is unipotent (ie has all eigenvalues equal to one). In this case, the resulting G will always be residually torsion-free nilpotent; see Koberda [33]. In fact, a semidirect product of $\mathbb{Z}$ with a finitely generated, residually torsion-free nilpotent group H will again be residually torsion-free nilpotent if the induced $\mathbb{Z}$ action on $H^1(H,\mathbb{R})$ is unipotent.

It is not true that if a semidirect product $H \rtimes_\psi \mathbb{Z}$ is residually torsion-free nilpotent then ψ^* is unipotent. Indeed, considering a fibered hyperbolic 3–manifold group $\pi_1(T)$ satisfying the hypotheses of Theorem 1.1 and passing to a finite-index subgroup which is special (as in Proposition 1.4), we can obtain a new mapping torus structure on a finite cover T_0 of T with monodromy ϕ_0, and with a fiber S_0 which covers S. The action of ϕ_0^* on $H^1(S_0,\mathbb{R})$ will not be unipotent, since $H^1(S,\mathbb{R})$ will naturally sit inside $H^1(S_0,\mathbb{R})$ via pullback and will be invariant under ϕ_0^*.

The regularity assumption in Theorem 1.3 is subtle. Nonabelian nilpotent groups cannot admit faithful C^2 actions on any compact 1–manifold; see Plante and Thurston [43]. Right-angled Artin groups and specialness do not provide any help in producing higher regularity actions, since in dimension one they almost never admit faithful C^2 actions on compact manifolds; see Baik, Kim and Koberda [4; 32]. The compactness of the manifold acted upon here is also essential; see Baik, Kim and Koberda [3].

1.3 General group actions on compact manifolds

A robust trend in the theory of group actions on manifolds is that "large" groups should not act on "small" manifolds. Among the striking results in this area are the facts that irreducible lattices in higher-rank semisimple Lie groups do not admit infinite image C^1 actions (and often even C^0 actions) on compact 1–manifolds; see Burger and Monod [13], Ghys [25] and Witte [48]. For higher-dimensional manifolds, the work of Brown, Fisher and Hurtado shows that for $n \geq 3$, groups commensurable with $\mathrm{SL}_n(\mathbb{Z})$ do not admit faithful C^1 actions on m–dimensional compact manifolds for $m < n-2$, and for $m < n-1$ if the actions preserve a volume form; see Brown, Fisher and Hurtado [11; 12]. They obtain similar results for cocompact lattices in simple Lie groups.

Lattices in rank one Lie groups often do admit faithful smooth actions on compact 1–manifolds. By Bergeron, Haglund and Wise [5], many arithmetic lattices in $\mathrm{SO}(n,1)$ are virtually special and hence virtually residually torsion-free nilpotent, which by Theorem 1.3 furnishes many faithful C^1 actions of such lattices.

McCarthy's result [37] furnishes a class of solvable groups which admit no faithful, small C^1 actions on compact manifolds whatsoever. Topologically, her groups arise as fundamental groups of torus bundles over the circle, with no restrictions on the dimension. Our main result identifies a larger class of such groups, including ones within the much more dimensionally restricted and algebraically different class of compact 3–manifolds groups. For fibered 3–manifolds groups acting without at least some smallness assumptions, we can only make much weaker statements:

Proposition 1.5 *If T is a closed, hyperbolic, fibered 3–manifold, then the universal circle action of the fundamental group $\pi_1(T)$ on S^1 is not topologically conjugate to a C^3 action.*

Proposition 1.5 follows immediately from the work of Miyoshi [38]. We will deduce Proposition 1.5 from a stronger fact (Proposition 4.3) in Section 4 for the convenience of the reader.

There is no hope of establishing a result as sweeping as the Brown–Fisher–Hurtado resolution of many cases of the Zimmer conjecture for 3–manifold groups acting on the circle, even with maximal regularity assumptions:

Proposition 1.6 (eg Calegari [14]) *There exist finite-volume hyperbolic 3–manifold subgroups of $\mathrm{PSL}_2(\mathbb{R})$.*

Any such groups act by projective (and hence analytic) diffeomorphisms on S^1. We remark that Proposition 1.6 seems well known to experts. We refer the reader to Section 4.2 for a further discussion of analytic hyperbolic 3–manifold group actions on the circle.

1.4 Uniqueness of the presentation of *G*

We remark briefly that if $G = \pi_1(T)$ satisfies the hypotheses of Corollary 1.2 then there is an essentially unique homomorphism $G \to \mathbb{Z}$ whose kernel is isomorphic to a finitely generated group, and in particular the fibered 3–manifold structure on T is unique (see Thurston [46] and Stallings [44]). Thus, the induced map ψ^* is canonically defined, and one may therefore speak of *the* monodromy action. For fibered 3–manifold groups with first Betti number $b_1 > 1$ this is no longer the case.

2 Preliminaries

In this section, we gather the tools we will need to establish the principal result of this paper.

2.1 The space of C^1 actions of G

Recall that in our notation, M denotes a fixed compact Riemannian manifold. We denote by $\mathrm{Diff}^1(M)$ the group of C^1–diffeomorphisms of M. For $f \in \mathrm{Diff}^1(M)$, we will write

$$D_x f\colon T_x M \to T_{f(x)} M$$

for the Jacobian of f.

It will be convenient for us to assume that M is C^1 embedded in a Euclidean space $\mathbb{R}^N$ for some $N \gg 0$. For our purposes, we only require an embedding, though in principle one could require an isometric embedding by the Nash embedding theorem [39], for example. We reiterate that an isometric embedding is not necessary for the sequel.

For brevity, we let $\|X\|$ denote the ℓ^∞ norm when X is a function, a vector, a matrix or a tensor. We replace distances in M by distances in $\mathbb{R}^N$, and we equip the Jacobian of a diffeomorphism f of M with the ℓ^∞ norm arising from $\mathbb{R}^N$, which we denote by $\|D_x f\|$. Note that if $V \cong \mathbb{R}^N$ is a vector space equipped with the ℓ^∞ norm and $T \in \mathrm{End}(V)$, then we have the estimate

$$\|Tv\| \le N\|T\|\|v\|$$

for all $v \in V$. This estimate is in fact more general. Indeed, sometimes, we will consider linear maps which are defined on subspaces of $\mathbb{R}^N$ and which have values in $\mathbb{R}^N$ (for example, the Jacobian of a diffeomorphism of $M \subset \mathbb{R}^N$ as above). In this case, we are still able to write down a matrix (which will no longer be square) that represents this linear map. We will define the supremum norm of such a matrix by taking the maximum of the absolute values of the entries, in which case the same norm estimate holds as for an endomorphism of V. We will make essential use of this estimate in the sequel.

We define the C^1*–metric* on $\mathrm{Diff}^1(M)$ by

$$d(f,g) = \|f-g\| + \sup_{x\in M} \|D_x f - D_x g\|,$$

where all these distances and norms are now interpreted in the ambient Euclidean space.

If G is generated by a finite set S, we may define a metric d_S on $\mathrm{Hom}(G, \mathrm{Diff}^1(M))$ via

$$d_S(\rho,\rho') = \max_{s\in S} d(\rho(s), \rho'(s)).$$

This metric d_S determines the C^1*–topology* of $\mathrm{Hom}(G, \mathrm{Diff}^1(M))$, and this topology is independent of the choice of the generating set S.

For an arbitrary group G, we will write $\rho_0 \in \mathrm{Hom}(G, \mathrm{Diff}^1(M))$ for the trivial representation of G. We see that in order to prove Theorem 1.1, it suffices to find some $\epsilon > 0$ such that every representation $\rho \in \mathrm{Hom}(G, \mathrm{Diff}^1(M))$ satisfying $d_S(\rho, \rho_0) < \epsilon$ maps H to the identity 1.

2.2 Hyperbolic monodromies

Here, we recall some basic facts from linear algebra of hyperbolic automorphisms of a real vector space. Let V be a d–dimensional vector space over $\mathbb{R}$, and let $\|\cdot\|_d$ be a fixed norm on V. When $A \in \mathrm{GL}(V)$, we say that A is *hyperbolic* if every eigenvalue of A has modulus different from one.

Lemma 2.1 *Let $A \in \mathrm{GL}(V)$ be a hyperbolic automorphism. Then there is an A–invariant splitting $V = E^- \oplus E^+$ and a positive integer p_0 such that the following conclusions hold for all $p \geq p_0$:*

(1) *If $v \in E^-$ then*
$$\|A^p v\|_d \leq \tfrac{1}{2}\|v\|_d.$$

(2) *If $v \in E^+$ then*
$$\|A^p v\|_d \geq 2\|v\|_d.$$

We omit the proof of the lemma, which is well known; see [30, Chapter 1] for instance. As is standard from dynamics, E^- and E^+ are the *stable* and *unstable* subspaces of V associated to A. In the sequel, we will use the notation π_+ and π_- to denote projections $V \to E^+$ and $V \to E^-$ with kernels E^- and E^+, respectively. Observe that invariance of the splitting implies that A commutes with each projection π_+ and π_-.

2.3 Approximate linearization

A fundamental tool for proving Theorem 1.1 is the following result of Bonatti [6; 7], which arose as an interpretation of Thurston stability [45], and which we refer to as *approximate linearization*.

Lemma 2.2 *Let M be a compact manifold, let $\eta > 0$ and let $k \in \mathbb{N}$. Then there exists a neighborhood of the identity $\mathcal{V} \subseteq \mathrm{Diff}^1(M)$ such that for all points $x \in M$, for all diffeomorphisms $f_1, \dots, f_k \in \mathcal{V}$ and for all $\epsilon_1, \dots, \epsilon_k \in \{-1, 1\}$, we have*
$$\left\| f_k^{\epsilon_k} \circ \cdots \circ f_1^{\epsilon_1}(x) - x - \sum_{i=1}^{k} \epsilon_i (f_i(x) - x) \right\| \leq \eta \max_{i=1,\dots,k} \|f_i(x) - x\|.$$

For the rest of this paper, we will often suppress the notation $\rho \in \mathrm{Hom}(G, \mathrm{Diff}^1(M))$ and just write $gx = g(x) = \rho(g)(x)$ for $g \in G$ and $x \in M$. We define a *displacement vector* for g at x as

$$\Delta_x^\rho(g) := \rho(g)(x) - x,$$

regarded as an N–dimensional row vector. Here, we remind the reader that M is embedded as a submanifold of $\mathbb{R}^N$, so that the displacement vector becomes a vector in $\mathbb{R}^N$. Admittedly, the displacement vector depends on the choice of embedding, though this does not matter since we will ultimately be interested in whether or not it vanishes.

More generally, if $B = \{b_1, \ldots, b_n\} \subseteq G$ is a finite subset then we define an $n \times N$ matrix

$$\Delta_x^\rho(B) := (\Delta_x^\rho(b_i))_{1 \le i \le n}.$$

We often write Δ_x for Δ_x^ρ when the meaning is clear. Then the above lemma asserts that

$$\left\| \Delta_x(g_k^{\epsilon_k} \circ \cdots \circ g_1^{\epsilon_1}) - \sum_{i=1}^{k} \epsilon_i \Delta_x(g_i) \right\| \le \eta \|\Delta_x(\{g_1, \ldots, g_k\})\|$$

in the case when $g_i \in G$ and $\rho(g_i) \in \mathcal{V}$. Here, we remind the reader that we always use the supremum norm.

2.4 First homology and cohomology groups

We briefly recall for the reader unfamiliar with group homology that the first homology group of a group H is given by the abelianization

$$H_1(H, \mathbb{Z}) = H/[H, H].$$

When $R \in \{\mathbb{Z}, \mathbb{R}\}$, the first cohomology group $H^1(H, R)$ coincides with the abelian group of homomorphisms from H to R. In particular, $H^1(H, \mathbb{Z})$ is a free abelian group of the same rank as $H_1(H, \mathbb{Z})$.

3 Proof of Theorem 1.1

We are now ready to give a proof of Theorem 1.1. For this, we will fix an automorphism $\psi \in \mathrm{Aut}(H)$ such that G can be written as

$$G = \langle H, t \mid tht^{-1} = \psi(h) \text{ for all } h \in H \rangle.$$

3.1 Reducing to homologically independent generators

We first establish Lemma 3.1 below, which will say that we may more or less assume that H is finitely generated and free abelian.

Let $d \geq 1$ be the rank of $H^1(H,\mathbb{Z})$. We can find a finite generating set

$$S = S_0 \sqcup S_1$$

of H such that all of the following hold:

- The image of S_0 in $H_1(H,\mathbb{Z}) = H/[H,H]$ is a basis for the free part.
- The image of each element in S_1 is torsion or trivial in $H_1(H,\mathbb{Z})$.

We pick $K \geq 2$ so that $\tau^K = 0$ for all

$$\tau \in \ker\{H_1(H,\mathbb{Z}) \to H_1(H,\mathbb{R}) = H_1(H,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}\},$$

where the map between the homology groups is the tensoring map. We enumerate $S_0 = \{s_1, s_2, \dots, s_d\}$, and regard S_0 as an ordered set. Let $A := (\alpha_{ij})$ be the matrix of the hyperbolic linear automorphism

$$\psi^*\colon H^1(H,\mathbb{Z}) \to H^1(H,\mathbb{Z})$$

with respect to the basis which is dual to S_0, viewed as real homology classes. The action ψ_* on $H_1(H,\mathbb{Z})$ is then given by the transpose (α_{ji}). In this case, we can write each $\psi(s_j)$ as

$$\psi(s_j) = t s_j t^{-1} = \prod_{i=1}^{d} s_i^{\alpha_{ji}} \tau_j \tag{3-1}$$

for some element $\tau_j \in H$ such that $\tau_j^K \in [H,H]$. It will be convenient for us to define the subset

$$S' := \{u^K : u \in S_1 \cup \{\tau_1, \dots, \tau_d\}\} \subseteq [H,H].$$

Observe that each element $h \in [H,H]$ can be expressed as a product of commutators in S. It follows that h can be expressed as a *balanced* word in S, which is to say that all generators in S occur with exponent sum zero. Since $S' \subseteq [H,H]$, we can find an integer $k_0 \geq K$ such that every element in S' is a balanced word of length at most k_0 in S. Recall our convention $\|A\| := \max_{i,j} |\alpha_{ij}|$. We set

$$k := k_0 + d\|A\|. \tag{3-2}$$

Lemma 3.1 *Let $0 < \eta < 1$. Then there exists a neighborhood $\mathcal{U} \subseteq \mathrm{Hom}(G, \mathrm{Diff}^1(M))$ of the trivial representation ρ_0 such that each of the following relations hold for all representations $\rho \in \mathcal{U}$ and points $x \in M$:*

(1) $\|\Delta_x^\rho(S')\| \le \eta \|\Delta_x^\rho(S)\|$.

(2) $\|\Delta_x^\rho(S_1 \cup \{\tau_1, \dots, \tau_d\})\| \le \eta \|\Delta_x^\rho(S)\|$.

(3) $\|\Delta_x^\rho(S)\| = \|\Delta_x^\rho(S_0)\|$.

(4) $\|\Delta_x^\rho(\psi(S_0)) - A\Delta_x^\rho(S_0)\| \le 2\eta \|\Delta_x^\rho(S_0)\|$.

Proof Let k be defined as in (3-2). We have an identity neighborhood $\mathcal{V} \subseteq \mathrm{Diff}^1(M)$ furnished by Lemma 2.2 for η and k. We define $\mathcal{U}$ by

$$\mathcal{U} = \{\rho \in \mathrm{Hom}(G, \mathrm{Diff}^1(M)) : \rho(S \cup \{\tau_1, \dots, \tau_d\}) \subseteq \mathcal{V}\}.$$

We now fix $\rho \in \mathcal{U}$, and we suppress ρ from the notation by writing $g(x) := \rho(g)(x)$. Similarly, we write $\Delta_x(g) := \Delta_x^\rho(g)$. So, $\Delta_x(g)$ will be thought of as a function of the group element g, and which depends on x as well.

(1) Let $u \in S'$, so that u can be expressed as a balanced word in S with length at most $k_0 < k$. We see from Lemma 2.2 that

$$\|\Delta_x(u)\| \le \eta \|\Delta_x(S)\|.$$

This proves part (1).

(2) Let $u \in S_1 \cup \{\tau_1, \dots, \tau_d\}$. Since $u \in \mathcal{V}$ by assumption, we again use Lemma 2.2 to see that

$$\|\Delta_x(u^K) - K\Delta_x(u)\| \le \eta \|\Delta_x(u)\|.$$

Using the triangle inequality and part (1), we see that

$$K\|\Delta_x(u)\| \le \|\Delta_x(u^K)\| + \eta\|\Delta_x(u)\| \le \eta\|\Delta_x(S)\| + \eta\|\Delta_x(u)\|.$$

Since $K \ge 2$, we obtain the desired conclusion as

$$\|\Delta_x(u)\| \le \frac{\eta}{K-\eta}\|\Delta_x(S)\| \le \eta\|\Delta_x(S)\|.$$

Part (3) is obvious from the previous parts. For part (4), let us pick an arbitrary $s_j \in S_0$. From the expression (3-1) for $\psi(s_j) = t s_j t^{-1}$ and from Lemma 2.2, we can deduce that

$$\left\| \Delta_x(\psi(s_j)) - \sum_{i=1}^{d} \alpha_{ji} \Delta_x(s_i) - \Delta_x(\tau_j) \right\| \le \eta \|\Delta_x(S \cup \{\tau_j\})\| = \eta \|\Delta_x(S_0)\|.$$

The triangle inequality and the second and third parts of the lemma imply the conclusion of part (4). □

3.2 McCarthy's lemma

Retaining previous notation, we have a group G presented as $H \rtimes_\psi \langle t \rangle$. Another ingredient for the proof of the main theorem is the following lemma, which was proved by McCarthy [37, Lemmas 4.1 and 4.2] in the case when H is abelian:

Lemma 3.2 (cf Lemmas 4.1 and 4.2 of [37]) *For all $\eta \in \left(0, \frac{1}{3}\right)$, there exists a neighborhood $\mathcal{U} \subseteq \mathrm{Hom}(G, \mathrm{Diff}^1(M))$ of the trivial representation ρ_0 such that whenever $\rho \in \mathcal{U}$ and $x \in M$, we have*

$$\|\Delta^\rho_{\rho(t^{-1})(x)}(S_0) - A\Delta^\rho_x(S_0)\| \le \eta \|\Delta^\rho_x(S_0)\|.$$

Roughly speaking, under the above hypothesis, if we denote the displacement matrix of S_0 at x by v, then Av will be near to the displacement matrix of S_0 at $t^{-1}x$. Thus, one can apply hyperbolic dynamics to estimate the change of displacement matrices as points are moved under iterations of t^{-1}:

$$x \mapsto t^{-1}x \mapsto t^{-2}x \mapsto \cdots \mapsto t^{-n}x \mapsto \cdots.$$

Since McCarthy's arguments concerned the case where H is abelian and hence do not apply in this situation, let us reproduce proofs here which work for general groups.

Proof of Lemma 3.2 Fix $\eta' \in (0, \eta)$, which will be made explicit later. We pick a sufficiently small neighborhood $\mathcal{U} \subseteq \mathrm{Hom}(G, \mathrm{Diff}^1(M))$ of ρ_0, which is at least as small as the neighborhood $\mathcal{U}$ in Lemma 3.1 for this choice of η'. We let $\rho \in \mathcal{U}$, and again suppress the notation ρ in expressions. We also fix $x \in M$, and set $y := t^{-1}x$.

Suppose we have $s \in S_0$. From the definition of the derivative, we have that

$$\Delta_x(\psi(s)) = \Delta_{ty}(tst^{-1}) = ts(y) - t(y) = D_y t(\Delta_y(s)) + o\big(\|\Delta_y(s)\|\big).$$

Replacing $\mathcal{U}$ by a smaller neighborhood if necessary, we may assume that (with a slight abuse of notation)

$$o\big(\|\Delta_y(s)\|\big) < \eta' \|\Delta_y(s)\|$$

in norm, and that $N\|D_y t - 1\| \le \eta'$, where 1 denotes the identity map, and N is the dimension of the Euclidean space where M is embedded. It then follows that

$$\|\Delta_x(\psi(s)) - \Delta_y(s)\| \le N\|D_y t - 1\| \cdot \|\Delta_y(s)\| + o\big(\|\Delta_y(s)\|\big) \le 2\eta' \|\Delta_y(s)\|. \tag{3-3}$$

Here, we are using the ℓ^∞ norm estimate

$$\|Tv\| \le N\|T\|\|v\|$$

for arbitrary vectors v and linear maps $T\colon \mathbb{R}^N \to \mathbb{R}^N$.

Applying the triangle inequality, Lemma 3.1(4) and (3-3), we deduce that

$$\begin{aligned}\|\Delta_y(S_0) - A\Delta_x(S_0)\| &\le 2\eta'\|\Delta_y(S_0)\| + \|\Delta_x(\psi(S_0)) - A\Delta_x(S_0)\| \\ &\le 2\eta'\big(\|\Delta_y(S_0)\| + \|\Delta_x(S_0)\|\big).\end{aligned} \tag{3-4}$$

From the inequality (3-4), we note that

$$(1-2\eta')\|\Delta_y(S_0)\| \le \|A\Delta_x(S_0)\| + 2\eta'\|\Delta_x(S_0)\| \le (d\|A\| + 2\eta')\|\Delta_x(S_0)\|. \tag{3-5}$$

We will now choose $\eta' \in (0,\eta)$ sufficiently small that

$$\left(\frac{d\|A\| + 2\eta'}{1-2\eta'} + 1\right)\cdot 2\eta' \le \left(\frac{d\|A\| + \frac{2}{3}}{\frac{1}{3}} + 1\right)\cdot 2\eta' \le \eta.$$

Combining inequalities (3-4) and (3-5), we obtain the desired conclusion as

$$\|\Delta_y(S_0) - A\Delta_x(S_0)\| \le 2\eta'\left(\frac{d\|A\| + 2\eta'}{1-2\eta'} + 1\right)\|\Delta_x(S_0)\| \le \eta\|\Delta_x(S_0)\|. \qquad \square$$

3.3 Finishing the proof

We can now complete the proof of the main result.

Proof of Theorem 1.1 Let ρ be sufficiently near to ρ_0. By Lemma 3.1(3), it suffices for us to prove that the $d \times N$ matrix $\Delta_x(S_0)$ is equal to 0 for all points $x \in M$.

The hyperbolic automorphism ψ^* on $H^1(H,\mathbb{Z})$ induces an invariant splitting

$$\mathbb{R}^d = \bigoplus_{i=1}^{d} \mathbb{R}s_i = E^+ \oplus E^-,$$

as in Lemma 2.1. We may assume $p_0 = 1$ in that lemma after replacing ψ^* by a sufficiently large power; this is the same as passing to the kernel of the natural map $G \to \mathbb{Z}/p\mathbb{Z}$ given by reducing G/H modulo p.

Let us pick a point $x \in M$ such that the quantity

$$\max\big(\|\pi_+\Delta_z(S_0)\|, \|\pi_-\Delta_z(S_0)\|\big)$$

is itself maximized at $z = x$. Here, $\pi_\pm$ is regarded as a map from $\bigoplus_{i=1}^N \mathbb{R}^d$ to $\bigoplus_{i=1}^N E^\pm$.

For a proof by contradiction, we will suppose that this maximum is nonzero. We may further assume the maximum occurs for the unstable direction. Since the stable and unstable subspaces of a hyperbolic matrix are symmetric under inversion, the case where the maximum is in the stable direction is analogous.

Let us choose $\eta \in \left(0, \frac{1}{3}\right)$ and a neighborhood $\mathcal{U} \subseteq \mathrm{Hom}(G, \mathrm{Diff}^1(M))$ so that the conclusion of Lemma 3.2 holds for $\rho \in \mathcal{U}$. With this choice, using also the contraction property of π_+, we estimate

$$\begin{aligned}\|\pi_+\Delta_{t^{-1}x}(S_0) - \pi_+ A\Delta_x(S_0)\| &\le \|\Delta_{t^{-1}x}(S_0) - A\Delta_x(S_0)\| \le \eta\|\Delta_x(S_0)\| \\ &\le \eta\bigl(\|\pi_+\Delta_x(S_0)\| + \|\pi_-\Delta_x(S_0)\|\bigr) \\ &\le 2\eta\|\pi_+\Delta_x(S_0)\|.\end{aligned}$$

On the other hand, applying the triangle inequality and Lemma 2.1(2), we have

$$\begin{aligned}\|\pi_+\Delta_{t^{-1}x}(S_0) - A\pi_+\Delta_x(S_0)\| &\ge \|A\pi_+\Delta_x(S_0)\| - \|\pi_+\Delta_{t^{-1}x}(S_0)\| \\ &\ge 2\|\pi_+\Delta_x(S_0)\| - \|\pi_+\Delta_{t^{-1}x}(S_0)\|.\end{aligned}$$

Combining the above chains of inequalities, and using that $A\pi_+ = \pi_+ A$, we obtain

$$\|\pi_+\Delta_{t^{-1}x}(S_0)\| \ge 2(1-\eta)\|\pi_+\Delta_x(S_0)\| > \|\pi_+\Delta_x(S_0)\|.$$

This contradicts the maximality of our choices. □

4 General group actions and questions

As remarked in the introduction, there is no hope of ruling out highly regular faithful actions of 3–manifold groups on low-dimensional manifolds. Thus, Theorem 1.1 can be viewed as a local rigidity phenomenon of $\mathrm{Hom}(G, \mathrm{Diff}^1(M))$ near the trivial representation ρ_0 rather than as a global statement about this space of actions. In this section we discuss actions of 3–manifold groups on the circle which are not small, and thus are much less constrained.

4.1 Universal circle actions

First, we show that for certain types of faithful actions of 3–manifold groups, some regularity constraints persist. Let T be a fibered 3–manifold with closed, orientable fiber S and monodromy $\psi \in \mathrm{Mod}(S, p)$. We assume that $\chi(S) < 0$. Here, we have equipped S with a basepoint p, and we assume that elements of $\mathrm{Mod}(S, p)$ preserve p, as do isotopies between them.

We have that the fundamental group $\pi_1(S)$ naturally sits in $\mathrm{Mod}(S, p)$ as the kernel of the homomorphism $\mathrm{Mod}(S, p) \to \mathrm{Mod}(S)$ which forgets the basepoint p [21]. The short exact sequence

$$1 \to \pi_1(S) \to \mathrm{Mod}(S, p) \to \mathrm{Mod}(S) \to 1$$

is known as the Birman exact sequence. The mapping class group $\mathrm{Mod}(S, p)$ has a natural faithful action on S^1 by homeomorphisms, known as Nielsen's action (see [16]). This action of $\mathrm{Mod}(S, p)$ is not conjugate to a C^1 action, and even after passing to finite-index subgroups it is known not to be conjugate to a C^2 action [20; 4; 32; 42; 35]. Moreover, this action is not absolutely continuous, as can be easily seen from Proposition 4.1 below. However, one can topologically conjugate Nielsen's action to a bi-Lipschitz one; this is a general fact for countable groups acting on the circle [17]. We remark that Nielsen's action, as it is constructed by extensions of quasi-isometries of $\mathbb{H}^2$ to S^1, enjoys a regularity property known as quasisymmetry. See [16; 26; 22].

If $\psi \in \mathrm{Mod}(S, p)$ then the conjugation action of ψ on the group

$$\pi_1(S) = \ker\{\mathrm{Mod}(S, p) \to \mathrm{Mod}(S)\}$$

makes the group $\langle \psi, \pi_1(S)\rangle$ isomorphic to $\pi_1(T)$. We thus obtain an action, which is called the *universal circle action* of $\pi_1(T)$ (see [15]). While it follows that $\pi_1(T)$ admits a natural faithful action on S^1 by absolutely continuous homeomorphisms, the higher regularity properties of this action are somewhat mysterious.

We now give a proof of Proposition 1.5, which asserts that this action is not topologically conjugate to a C^3 action. As stated in the introduction, this result is known from the work of Miyoshi. The proof of Proposition 1.5 given in [38] follows similar lines to the argument given here, and is easily implied by the following two results:

Proposition 4.1 *Let S be a closed surface and $\rho\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ be a faithful discrete representation. Then the normalizer of $\rho(\pi_1(S))$ in $\mathrm{Homeo}^{\mathrm{ac}}(S^1)$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ which contains $\rho(\pi_1(S))$ as a finite-index subgroup.*

Proof Let g be an absolutely continuous homeomorphism of the circle which normalizes $\rho(\pi_1(S))$. Then, by an argument originally due to Sullivan (see Proposition III.4.1

of [23]), we see that g is actually contained in $\mathrm{PSL}_2(\mathbb{R})$. Ghys gives a relatively simple argument under the assumption of C^1 conjugacy, which in turn suffices for Proposition 1.5. In this case, all derivatives of hyperbolic elements at their fixed points must be preserved by the conjugacy. In other words, the marked length spectrum associated with the Fuchsian group $\rho(\pi_1(S))$ is invariant, and thus the isometry class of the corresponding hyperbolic surface is preserved.

Now, it follows from standard facts about Zariski-dense subgroups of simple Lie groups that the normalizer of a Fuchsian group in $\mathrm{PSL}_2(\mathbb{R})$ is necessarily Fuchsian, and contains the original Fuchsian group with finite index. Indeed, suppose $\Gamma < \mathrm{PSL}_2(\mathbb{R})$ is discrete and let $\{g_i\}_{i\in\mathbb{N}} \subset \mathrm{PSL}_2(\mathbb{R})$ normalize Γ. Suppose furthermore that $g_i \to 1$ as $i \to \infty$. Then it is not difficult to show that g_i must centralize Γ for i sufficiently large. If Γ is Zariski dense then g_i is the identity for i sufficiently large, so that the normalizer of Γ is again discrete. If Γ is cocompact then its normalizer must contain Γ with finite index. See [31, Theorem 2.3.8] for more details. The conclusion of the proposition now follows. □

Proposition 4.1 implies the following: Let $\psi \in \mathrm{Mod}(S)$ be pseudo-Anosov, and let $\widetilde{\psi}$ be in the preimage of ψ under the canonical map $\mathrm{Mod}(S,p) \to \mathrm{Mod}(S)$ given by deleting the marked point. Then $\widetilde{\psi}$ fails to act by an absolutely continuous homeomorphism on S^1 under Nielsen's action of $\mathrm{Mod}(S,p)$ on S^1.

The following result is known as Ghys' differentiable rigidity of Fuchsian actions [24]:

Theorem 4.2 *Let S be a closed surface and let $\rho\colon \pi_1(S) \to \mathrm{Diff}^r(S^1)$ for $r \geq 3$ be a representation which is topologically conjugate to a Fuchsian subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Then ρ is conjugate to a Fuchsian subgroup of $\mathrm{PSL}_2(\mathbb{R})$ by a C^r diffeomorphism.*

Proposition 1.5 is an immediate consequence of the following, which in turn is an obvious corollary of Proposition 4.1 and Theorem 4.2:

Proposition 4.3 *Let T be a hyperbolic fibered 3–manifold with a closed fiber S. If an action*

$$\rho\colon \pi_1(T) = \pi_1(S) \rtimes \langle t\rangle \to \mathrm{Homeo}_+(S^1)$$

has the property that $\rho(\pi_1(S))$ is topologically conjugate to a Fuchsian subgroup of $\mathrm{PSL}_2(\mathbb{R})$, then either $\rho(\pi_1(S)) \not\leq \mathrm{Diff}^3_+(S^1)$ or $\rho(t)$ is not absolutely continuous.

We remark that universal circle actions enjoy a strong C^0 rigidity property, namely that actions in the same connected component of the representation variety of $\pi_1(T) \to \mathrm{Homeo}_+(S^1)$ are semiconjugate to the standard action [9].

4.2 Analytic actions

Finally, we discuss faithful analytic actions of fibered 3–manifold groups on S^1. By Agol's resolution of the virtual fibering conjecture [1], we have that every hyperbolic 3–manifold virtually fibers over the circle. Thus, if a subgroup $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ is discrete (ie a Kleinian group) with finite covolume, then Γ has a finite-index subgroup which is $\pi_1(T)$ for some fibered 3–manifold T. Now, if the matrix entries of Γ are contained in a number field $K \supset \mathbb{Q}$ such that K has a real place (ie a Galois embedding $\sigma\colon K \to \mathbb{C}$ such that $\sigma(K) \subseteq \mathbb{R}$), then Γ can be identified with a subgroup of $\mathrm{PSL}_2(\mathbb{R})$.

Therefore, in order to establish Proposition 1.6, it suffices to produce such a Kleinian group. If Γ has matrix entries in a field K of odd degree over $\mathbb{Q}$, then K has at least one real place, since the number of complex places is even. Many such arithmetic Kleinian groups of finite covolume (and even cocompact ones such as the fundamental group of the Weeks manifold) exist; see [34, Section 13.7], for example. Note that since any discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ is virtually free or a closed surface group, a finite-volume hyperbolic 3–manifold group cannot occur as a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$.

4.3 Questions

There are several natural questions which arise from the discussion in this paper.

Question 4.4 (J Souto) Let T be a fibered 3–manifold and let $G = \pi_1(T)$. Is there a finite-index subgroup $G_0 < G$ such that $G_0 < \mathrm{Diff}^2(I)$? What about $G_0 < \mathrm{Diff}^\infty(I)$?

In [36], Marquis and Souto constructed a faithful C^∞ action of closed orientable surface groups, for genus $g \geq 2$, on the unit interval.

Question 4.5 Is the universal circle action of a fibered 3–manifold group topologically conjugate to a C^1 action?

In other words, Question 4.5 asks if we can replace the C^3 conclusion in Proposition 1.5 with a C^1 conclusion. Observe that Ghys' differentiable rigidity of Fuchsian actions does not hold in lower regularity (at least less than C^2): for arbitrary $\alpha < 1$, there are

$C^{1+\alpha}$ actions of $\pi_1(S)$ that are C^0 conjugate to a Fuchsian action, but that are not conjugate to a Fuchsian action by an absolutely continuous homeomorphism; see [27]. Other instances of this phenomenon arise from the theory of Hitchin representations [10]. A first attempt to answer Question 4.5 would be to investigate if the analogue of Proposition 4.1 holds for these actions: do they admit a C^1 normalizer which is not a finite extension of the image of $\pi_1(S)$?

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