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On local Turán problems

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ABSTRACT

Since its formulation, Turán's hypergraph problems have been among the most challenging open problems in extremal combinatorics. One of them is the following: given a 3-uniform hypergraph $\mathcal{F}$ on n vertices in which any five vertices span at least one edge, prove that $|\mathcal{F}| \geq (1/4 - o(1))\binom{n}{3}$. The construction showing that this bound would be best possible is simply $\binom{X}{3} \cup \binom{Y}{3}$ where X and Y evenly partition the vertex set. This construction has the following more general $(2p+1, p+1)$-property: any set of $2p+1$ vertices spans a complete sub-hypergraph on $p+1$ vertices. One of our main results says that, quite surprisingly, for all $p > 2$ the $(2p+1, p+1)$-property implies the conjectured lower bound.

1. Introduction

Let X be a finite set and $\binom{X}{r}$ the collection of all its r-subsets. Subsets $\mathcal{H}$ of $\binom{X}{r}$ are called r-uniform hypergraphs. Members of $\mathcal{H}$ are called edges. If $\binom{Y}{r} \subset \mathcal{H}$, then Y is said

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to be a clique and $|Y|$ is its size. We denote by K_t^r the r-uniform t-vertex clique. Note that every edge is a clique of size r.

For integers $q \geq p \geq r \geq 2$, we say that $\mathcal{H}$ has property (q,p) if for every $Z \in \binom{X}{q}$ there exists $Y \in \binom{Z}{p}$ spanning a clique in $\mathcal{H}$, that is, $\binom{Y}{r} \subset \mathcal{H}$.

Definition 1.1. Let $T_r(n,q,p) = \min\{|\mathcal{H}| : \mathcal{H} \subset \binom{[n]}{r},\ \mathcal{H}$ has property $(q,p)\}$. Set also $t_r(n,q,p) = T_r(n,q,p)/\binom{n}{r}$.

Eighty years ago, Turán [10] determined $T_2(n,q,2)$ and this result served as the starting point for a lot of research that led to the creation of the field of extremal graph theory. About two decades later Turán [11] proposed two conjectures concerning $T_3(n,4,3)$ and $T_3(n,5,3)$. To state their asymptotic forms, let us mention that Katona, Nemetz and Simonovits [6] used a simple averaging argument to show that $t_r(n,q,p)$ is monotone increasing as a function of n. Consequently the limit

$$\lim_{n\to\infty} t_r(n,q,p) =: t_r(q,p)$$

exists.

Conjecture 1.2 *(Turán).*

$$t_3(4,3) = \frac{4}{9}. \tag{1}$$

$$t_3(5,3) = \frac{1}{4}. \tag{2}$$

Even though this conjecture has been around for quite a long time, neither statement was proved. For (1) the best known bound stands as $t_3(4,3) \geq 0.438334$ by Razborov [8] using flag algebra. As for (2), the construction providing the upper bound is very simple, namely $\mathcal{H} = \binom{X_1}{3} \cup \binom{X_2}{3}$, with $X_1 \sqcup X_2 = [n]$, $|X_1| = \lceil \frac{n}{2} \rceil$, $|X_2| = \lfloor \frac{n}{2} \rfloor$.

Let us mention that in [2] it was shown that for the graph case,

$$t_2(q,p) = 1/\left\lfloor \frac{q-1}{p-1} \right\rfloor. \tag{3}$$

For general r, Frankl and Stechkin [4] proved that

$$t_r(q,p) = 1 \quad \text{ if } q \leq \frac{r}{r-1}(p-1). \tag{4}$$

It is easy to check that $\mathcal{H} = \binom{X_1}{r} \cup \binom{X_2}{r}$ has property $(2p+1, p+1)$ for all $p \geq r-1$. Consequently,

$$t_r(2p+1, p+1) \leq \frac{1}{2^{r-1}}. \tag{5}$$

For the case $r = 3$, it was proved by the first author [3] that

$$\lim_{p\to\infty} t_3(2p+1, p+1) = \frac{1}{4}. \tag{6}$$

By developing the methods used in [3], in Section 2 we generalize (6) to the r-uniform case.

Theorem 1.3. *For integers $r \geq 2$ and $a \geq 2$,*

$$\lim_{p\to\infty} t_r(ap+1, p+1) = \frac{1}{a^{r-1}}.$$

In the 3-uniform case (when $r = 3$), we are able to determine the exact value of $t_3(2p+1, p+1)$, for all $p \geq 3$, which strengthens (6).

Theorem 1.4. *For every integer $p \geq 3$,*

$$t_3(2p+1, p+1) = \frac{1}{4}.$$

We should remark that the proof of this result is relying on earlier Turán-type results of Mubayi and Rödl [7], and Baber and Talbot [1]. We are going to state these results in Section 3 before proving Theorem 1.4. In Section 4 we mention some open problems.

2. Proof of Theorem 1.3

Throughout the proof of Theorem 1.3, we assume $r \geq 3$, and $a \geq 2$ to be fixed, since the $r = 2$ case is already covered by (3). With r fixed, we also set $t(q,p) = t_r(q,p)$. For the pair (q,p) with $q \leq ap$, we call $ap - q$ the *excess* $e(q,p)$ of the pair (q,p). Note that since $q \geq p$, we always have $e(q,p) \leq aq - q = (a-1)q$. For $\mathcal{F} \subset \binom{Y}{r}$, a set Z is a (w,v)-*hole* if $|Z| = w$, the clique number of $\mathcal{F}|_Z$ (the sub-hypergraph of $\mathcal{F}$ induced by Z) is v, and $w > av$. We first establish the following two lemmas.

Lemma 2.1. *Suppose $\mathcal{G} \subset \binom{Y}{r}$ has property (q,p), and Z is a (w,v)-hole of $\mathcal{G}$ with $w < q$, then $\mathcal{G}|_{Y\setminus Z}$ has property $(q-w, p-v)$.*

Proof. Take an arbitrary set $U \in \binom{Y\setminus Z}{q-w}$, then $U \cup Z \in \binom{Y}{q}$. Since $\mathcal{G}$ has property (q,p), $\mathcal{G}|_{U\cup Z}$ contains a clique of size p. Hence $\mathcal{G}|_U$ contains a clique of size $p - v$. □

Lemma 2.2. *Suppose an r-uniform hypergraph $\mathcal{F}$ has property (q,p) for all pairs (q,p) with $q \leq a\ell$ and $p = \lceil q/a \rceil$ (in other words $\mathcal{F}$ does not have a (w,v)-hole with $a\ell \geq w > av$). Then for all $Y \in \binom{X}{a\ell}$,*

$$\left|\mathcal{F} \cap \binom{Y}{r}\right| \geq a\binom{\ell}{r}.$$

Proof. Instead of this we prove the following stronger statement. Let $(r-1)a \le s \le a\ell$ and $Y \in \binom{X}{s}$. Suppose further that $s = (a-b)t + b(t-1)$ for some $0 \le b < a$, then

$$\left|\mathcal{F} \cap \binom{Y}{r}\right| \ge (a-b)\binom{t}{r} + b\binom{t-1}{r}.$$

Note that the right-hand side is 0 when $s \le (r-1)a$, so the inequality is trivially true in this range. To prove the general case, we use induction on s. Since $s = (a-b)t+b(t-1) \in \{at-a+1, \cdots, at\}$, $\mathcal{F}$ has the (s,t) property from the assumption. Let $R \in \binom{Y}{t}$ span a clique and fix $y \in R$. There are $\binom{t-1}{r-1}$ edges in $\binom{R}{r} \cap \mathcal{F}$ containing y. Remove y from $\mathcal{F}$ and apply the inductive hypothesis to $\mathcal{F} \setminus \{y\}$. We infer that

$$\left|\mathcal{F} \cap \binom{Y \setminus \{y\}}{r}\right| \ge (a-b-1)\binom{t}{r} + (b+1)\binom{t-1}{r}.$$

Considering the at least $\binom{t-1}{r-1}$ edges containing y, we have

$$\begin{aligned}\left|\mathcal{F} \cap \binom{Y}{r}\right| &\ge (a-b-1)\binom{t}{r} + (b+1)\binom{t-1}{r} + \binom{t-1}{r-1} \\ &= (a-b)\binom{t}{r} + b\binom{t-1}{r}. \quad \square\end{aligned}$$

Now we can proceed as follows to prove Theorem 1.3. The upper bound $\lim_{p\to\infty} t_r(ap+1, p+1) \le \frac{1}{a^{r-1}}$ is immediate, since $\mathcal{H}_{n,r,a} := \binom{X_1}{r} \cup \cdots \cup \binom{X_a}{r}$ with $X_1 \sqcup \cdots \sqcup X_a = [n]$, $|X_i| \in \{\lfloor n/a \rfloor, \lceil n/a \rceil\}$ has property $(ap+1, p+1)$ and edge density $1/a^{r-1} + o(1)$. For the remainder of this section we focus on proving the lower bound.

Given $\varepsilon > 0$, let us fix a large integer $\ell > \ell_0(a, r, \varepsilon)$, to be determined later. Then fix a much larger integer $L \ge 2a^3\ell^2$, and consider a sufficiently large r-uniform hypergraph $\mathcal{F}_0 \subset \binom{[n]}{r}$ having property (q,p) with $q = aL$, $p = L$. Our aim is to find a subset $X \subset [n]$ with $|\binom{X}{r}| > (1-\varepsilon/2)\binom{n}{r}$ such that $\mathcal{F}_0 \cap \binom{X}{r}$ has no (w,v)-hole with $w \le a\ell$ and $r-1 \le v$.

To this end, we start with $\mathcal{F}_0$ and define $\mathcal{F}_i$ inductively. Let $q_0 = q, p_0 = p, X_0 = [n]$. Suppose that $\mathcal{F}_i \subset \binom{X_i}{r}$ has property (q_i, p_i) and it still has a (w_i, v_i)-hole. Then we let $Z_i \subset X_i$ be such a (w_i, v_i)-hole, and set

$$X_{i+1} = X_i \setminus Z_i, \qquad \mathcal{F}_{i+1} = \mathcal{F}_i \cap \binom{X_{i+1}}{r}.$$

By Lemma 2.1, $\mathcal{F}_{i+1}$ has property $(q_i - w_i, p_i - v_i)$. Moreover, the new excess satisfies

$$e(q_i - w_i, p_i - v_i) = a(p_i - v_i) - (q_i - w_i) = (ap_i - q_i) - (av_i - w_i) \ge e(q_i, p_i) + 1.$$

Set $q_{i+1} = q_i - w_i$, $p_{i+1} = p_i - v_i$ and continue. At every step

$$a(r-1) \le av_i < |X_i| - |X_{i-1}| = w_i \le a\ell.$$

Furthermore, since $v_i \ge r-1$ for every i, we have $i \le p/(r-1)$. Suppose at step i, the hypergraph $\mathcal{F}_i$ no longer contains a (w, v)-hole with $w \le a\ell$. In this case, we choose a subset Q of size $a\ell$ of $V(\mathcal{F}_i)$ uniformly at random. Then by Lemma 2.2,

$$\frac{|\mathcal{F}_i|}{\binom{X_i}{r}} = \frac{\mathbb{E}|\mathcal{F}_i \cap \binom{Q}{r}|}{\binom{a\ell}{r}} \ge \frac{a\binom{\ell}{r}}{\binom{a\ell}{r}}.$$

For sufficiently large $\ell > \ell_0(a, r, \varepsilon)$, this quantity is greater than $(1-\varepsilon/2) \cdot \frac{1}{a^{r-1}}$. On the other hand, $|X_i| \ge n - ia\ell \ge n - pa\ell/(r-1)$. Therefore when n is sufficiently large, $|\binom{X_i}{r}| > (1-\varepsilon/2)\binom{n}{r}$ and therefore

$$|\mathcal{F}_0| \ge |\mathcal{F}_i| \ge (1-\varepsilon/2) \cdot \frac{1}{a^{r-1}}\binom{|X_i|}{r} \ge (1-\varepsilon) \cdot \frac{1}{a^{r-1}}\binom{n}{r}.$$

Otherwise suppose this process continues to produce (w, v)-holes. Let m be the first index such that $q_m < 2a\ell$. In view of $e(q_m, p_m) \le (a-1)q_m$ and that $e(q_i, p_i)$ strictly increases after each step, $m \le (a-1)q_m$ follows. Thus

$$aL = q_0 = q_m + \sum_{i=0}^{m-1} w_i \le 2a\ell + ma\ell \le 2a\ell + (a-1) \cdot 2a\ell \cdot a\ell < 2a^3\ell^2,$$

contradicting $L \ge 2a^3\ell^2$.

Summarizing the two cases above, we have that $\lim_{L\to\infty} t_r(aL, L) \ge 1/a^{r-1}$. Note that a hypergraph having property $(aL+1, L+1)$ must also have property (aL, L). Therefore,

$$\lim_{p\to\infty} t_r(ap+1, p+1) \ge 1/a^{r-1}.$$

Together with the construction in the introduction that gives $t_r(ap+1, p+1) \le 1/a^{r-1}$, we conclude the proof of Theorem 1.3.

Remark. Since $\mathcal{H}_{n,r,a}$ also has property (ap, p), we have actually proved a result slightly stronger than Theorem 1.3, namely for every $a, r \ge 2$,

$$\lim_{p\to\infty} t_r(ap, p) = \frac{1}{a^{r-1}}.$$

3. The 3-uniform case

Note that Theorem 1.3, when applied to $a = 2$, gives

$$\lim_{p\to\infty} t_r(2p+1, p+1) = \frac{1}{2^{r-1}}.$$

In this section, we determine the exact value of $t_r(2p+1, p+1)$ for $r = 3$ and all $p \geq 3$, establishing Theorem 1.4. Our proof is based on two previously known Turán-type results. To apply them, let us change to the complementary notion of excluded configuration.

Definition 3.1. For an r-uniform hypergraph $\mathcal{F} \subset \binom{[n]}{r}$. Let $\alpha(\mathcal{F})$ be its independence number, that is, $\alpha(\mathcal{F}) = \max\{|A| : A \subset [n], \mathcal{F} \cap \binom{A}{r} = \emptyset\}$.

Let $\mathcal{F}^c = \binom{[n]}{r} \setminus \mathcal{F}$ be the complementary r-uniform hypergraph. Now $\mathcal{F}$ has property (q, p) if and only if $\alpha(\mathcal{H}) \geq p$ for all induced sub-hypergraphs $\mathcal{H} = \mathcal{F}^c \cap \binom{Q}{p}$, $Q \subset [n]$, $|Q| = q$.

For a collection of $\mathcal{G}_1, \cdots, \mathcal{G}_s$ of r-uniform hypergraphs, let

$$t(n, \mathcal{G}_1, \cdots, \mathcal{G}_s) = \max\left\{|\mathcal{F}| : \mathcal{F} \subset \binom{[n]}{r},\ \mathcal{F} \text{ contains no copy of } \mathcal{G}_i, i = 1, \cdots, s\right\}.$$

It is easily seen that $t(n, \mathcal{G}_1, \cdots, \mathcal{G}_s)/\binom{n}{r}$ is a monotone decreasing function of n. Consequently $\lim_{n\to\infty} t(n, \mathcal{G}_1, \cdots, \mathcal{G}_s)/\binom{n}{r}$ exists. This limit is denoted by $\pi(\mathcal{G}_1, \cdots, \mathcal{G}_s)$, and it is usually called the Turán density of $\{\mathcal{G}_1, \cdots, \mathcal{G}_s\}$.

Consider the following three hypergraphs from [7]:

$$\mathcal{R}_0 = \binom{[4]}{3} \cup \{(a, x, y) : a \in [4], x, y \in \{5, 6, 7\}, x \neq y\},$$
$$\mathcal{R}_1 = \mathcal{R}_0 \setminus \{\{1, 5, 6\}, \{2, 5, 7\}, \{3, 6, 7\}\},$$
$$\mathcal{R}_2 = \mathcal{R}_0 \setminus \{\{1, 5, 6\}, \{1, 5, 7\}, \{3, 6, 7\}\}.$$

It is easy to check that $\alpha(\mathcal{R}_i) = 3$ for $i = 0, 1, 2$. To prove $t_3(7, 4) = 1/4$, it suffices to prove

$$\pi(\mathcal{R}_1, \mathcal{R}_2) = \frac{3}{4}. \tag{7}$$

Actually Mubayi and the third author [7] proved a considerably stronger statement. Set $\mathcal{R} = \mathcal{R}_0 \setminus \{1, 5, 6\}$. Then

Proposition 3.2. *([7])* $\pi(\mathcal{R}) = \frac{3}{4}$.

Since the proof of Proposition 3.2 is rather short let us include it. Suppose that $\varepsilon > 0, n > n_0(\varepsilon)$ and $\mathcal{H} \subset \binom{[n]}{3}$ satisfies $|\mathcal{H}| \geq (3/4+\varepsilon)\binom{n}{3}$. Then for a 4-element set $Y \subset [n]$ chosen uniformly at random, the expected size of $|\mathcal{H} \cap \binom{Y}{3}| = 4|\mathcal{H}|/\binom{n}{3} \geq 3+\varepsilon$. Consequently, $\mathcal{H}$ contains many complete 3-uniform hypergraphs on 4 vertices. (As a matter of fact, instead of 3/4 to ensure that, Razborov [8] proved that $0.516\cdots$ would be sufficient to ensure the existence of K_4^3.) By symmetry, suppose $\binom{[4]}{3} \subset \mathcal{H}$. For $i \in [4]$ define the link graphs $\mathcal{H}(i) = \{(x,y) \subset [5,n] : (i,x,y) \in \mathcal{H}\}$. Let $\mathcal{G}$ be the multigraph whose edge set is the union (with multiplicities) $\mathcal{H}(1) \cup \cdots \cup \mathcal{H}(4)$. Should $|\mathcal{G}| > 3\binom{n-4}{2} + n - 6$ hold, we can apply a result of Füredi and Kündgen [5] which guarantees that there are three vertices in $\mathcal{G}$ spanning at least 11 edges, which corresponds to a copy of $\mathcal{R}$ in $\mathcal{H}$. In the opposite case $|\mathcal{H}(i)| < (3/4+\varepsilon/2)\binom{n}{2}$ for some $i \in [4]$, then we remove the vertex i and iterate. Either we find $\mathcal{R}$ or we arrive at a contradiction with $|\mathcal{H}| > (3/4+\varepsilon)\binom{n}{3}$.

The following result was proved by Baber and Talbot [1] using flag algebra.

Proposition 3.3. *(Theorem 18 in [1]) Let $\mathcal{T}$ be the 6-vertex 3-uniform vertex hypergraph with*

$$\mathcal{T} = \binom{[6]}{3} \setminus \{\{1,5,6\},\{2,4,6\},\{2,5,6\},\{3,4,6\},\{3,4,5\}\}.$$

Then $\pi(\mathcal{T}) = 3/4$.

Now we are ready to prove Theorem 1.4. Observe that if $\mathcal{G}$ and $\mathcal{H}$ are two hypergraphs and $\mathcal{F}$ is their vertex-disjoint union, then $\pi(\mathcal{F}) = \max\{\pi(G),\pi(H)\}$.

Proof of Theorem 1.4. We have the upper bound $t_3(2p+1,p+1) \leq 1/4$ from (5). Therefore it suffices to establish a matching lower bound. By considering the complement of the host hypergraph, it boils down to showing that if the edge density of a 3-uniform hypergraph $\mathcal{G}$ is greater than $3/4+o(1)$, then $\mathcal{G}$ contains a sub-hypergraph $\mathcal{H}$ on $2p+1$ vertices with $\alpha(\mathcal{H}) \leq p$. In other words, we need $\pi(\mathcal{H}) \leq 3/4$.

For odd $p \geq 3$, we let $\mathcal{H}_1$ be the vertex-disjoint union of $\mathcal{R}$ and $(p-3)/2$ copies of K_4^3. It is straightforward to check that $\mathcal{H}_1$ has $7+4\cdot(p-3)/2 = 2p+1$ vertices, independence number $3+(p-3) = p$, and $\pi(\mathcal{H}_1) = \max\{\pi(\mathcal{R}),\pi(K_4^3)\} = 3/4$. This gives $t_3(2p+1,p+1) \geq 1/4$ for all odd $p \geq 3$.

For even $p \geq 4$, we take $\mathcal{T}$ from Proposition 3.3, and blow up its vertices $1,2,3$ twice, and vertices $4,5,6$ once to obtain a 9-vertex hypergraph $\mathcal{T}'$. Note that a blow-up could only have lower Turán density, therefore $\pi(\mathcal{T}') \leq \pi(\mathcal{T}) = 3/4$. Moreover the independence number of $\mathcal{T}'$ is 4, since all the five non-edges of $\mathcal{T}$ contain at most one vertex from $\{1,2,3\}$ and $\{4,5,6\}$ itself is an edge. We then let $\mathcal{H}_2$ be the vertex-disjoint union of $\mathcal{T}'$ with $(p-4)/2$ copies of K_4^3. Then $\mathcal{H}_2$ has $9+4\cdot(p-4)/2 = 2p+1$ vertices, $\alpha(\mathcal{H}_2) = 4+(p-4) = p$, and $\pi(\mathcal{H}_2) = \max\{\pi(\mathcal{T}'),\pi(K_4^3)\} \leq 3/4$. Therefore for all even $p \geq 4$, we also have $t_3(2p+1,p+1) \geq 1/4$. This completes the proof. □

4. Concluding remarks

In this paper, we showed that for 3-uniform hypergraphs and $p \geq 3$, the $(2p+1, p+1)$ property implies the edge density is at least $1/4 - o(1)$. Maybe this can be extended to r-uniform hypergraphs and we wonder if the following holds:

Conjecture 4.1. *For integers $r \geq 2$, and p sufficiently large,*

$$t_r(2p+1, p+1) = \frac{1}{2^{r-1}}.$$

Our Theorem 1.3 indicates this is true in the limit, and Theorem 1.4 settles the $r = 3$ case except for $p = 2$, which corresponds to Turán's famous open problem for K_5^3. As we were informed by Sasha Sidorenko [9], the $r = 4, p = 3$ case of Conjecture 4.1 fails to be true since $t_3(7, 4) \leq 113721/(2^{17} \cdot 10) = 0.08676 \cdots < 1/8$.

Here we remark that $\mathcal{T}$ in Proposition 3.3 with the edge $\{1, 4, 5\}$ removed still has all the properties needed for the proof of Theorem 1.4. Perhaps one could find a simpler proof that this new hypergraph, much more symmetric than $\mathcal{T}$, still has Turán density 3/4. Such proof might provide some new insights on the above conjecture.

To determine $t_r(q, p)$, we essentially seek r-uniform hypergraph $\mathcal{H}$ with low independence number $\alpha(\mathcal{H})$ relative to its number of vertices, and low Turán density $\pi(\mathcal{H})$. In light of this observation and the results (3) and (4), could it possibly be true that for every positive real number $\gamma > 0$,

$$\lim_{p \to \infty} t_r(\gamma p + 1, p + 1) = 1 - \min_{\mathcal{H} \in \mathcal{F}} \pi(\mathcal{H}) = 1/\lfloor \gamma \rfloor^r,$$

where $\mathcal{F}$ is family of all the r-uniform hypergraph satisfying $|V(\mathcal{H})| \geq \gamma\alpha(\mathcal{H})$?

Finally, motivated by the asymptotic result (7) we propose the following conjecture:

Conjecture 4.2. *There exists n_0 such that for all integers $n > n_0$,*

$$t(2n, \mathcal{R}_1, \mathcal{R}_2) = \binom{2n}{3} - 2\binom{n}{3}.$$

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