A VANISHING EXT-BRANCHING THEOREM FOR $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$

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Abstract. We prove a conjecture of Dipendra Prasad on the Ext-branching from $\mathrm{GL}_{n+1}(F)$ to $\mathrm{GL}_n(F)$, where F is a p-adic field, give a projectivity criterion, and derive some interesting consequences.

1. Introduction

Decomposing a smooth representation of $\mathrm{GL}_{n+1}(F)$, when restricted to $\mathrm{GL}_n(F)$, is a well known and well studied problem that was initiated in a paper of Prasad [Pr1]. Today, this problem is a part of a large family of Gan-Gross-Prasad restriction problems [GGP] that are at the center of much research in Representation Theory and Number Theory. In order to describe what is known, and what is new in this paper, let $G_n = \mathrm{GL}_n(F)$ and let $\mathrm{Alg}(G_n)$ be the category of smooth representations of G_n. For every $\pi \in \mathrm{Alg}(G_n)$, let $\mathrm{Wh}(\pi)$ be the space of Whittaker functionals on π. If π is irreducible then $\mathrm{Wh}(\pi)$ is one or zero dimensional. We say that π is generic or degenerate, respectively. Let π_1 be an irreducible representation of G_{n+1}. One of the most significant results in the subject is that the restriction of π_1 to G_n is multiplicity free [AGRS], [AG], [SZ], that is, for every irreducible representation π_2 of G_n,

$$\dim \mathrm{Hom}_{G_n}(\pi_1, \pi_2) \leq 1$$

and it is one if both representations are generic. On the other hand, Dipendra Prasad proved in [Pr2] the following beautiful formula:

$$\mathrm{EP}(\pi_1, \pi_2) := \sum (-1)^i \dim \mathrm{Ext}^i_{G_n}(\pi_1, \pi_2) = \dim \mathrm{Wh}(\pi_1) \cdot \dim \mathrm{Wh}(\pi_2).$$

In particular, the formula implies that $\mathrm{EP}(\pi_1, \pi_2) = 1$ if both representations are generic. Since $\dim \mathrm{Hom}_{G_n}(\pi_1, \pi_2) = 1$, Prasad had conjectured that $\mathrm{Ext}^i_{G_n}(\pi_1, \pi_2)$ vanish for $i > 0$ if both representations are generic.

The first main result in this paper is a proof of this conjecture. In [Ch3], we shall generalize the result to other Bessel and Fourier-Jacobi models (in the sense of [GGP]). The proof is based on the theory of Bernstein-Zelevinsky derivatives [BZ1], [BZ2] with the following, additional ingredient. The theory of derivatives describes how a smooth representation of G_{n+1} restricts to the mirabolic subgroup M_{n+1}. However, instead of M_{n+1} one can consider the transpose $M^\top_{n+1}$ of M_{n+1}, and develop a theory of derivatives with respect to $M^\top_{n+1}$. Thus we have two notions of derivatives: those with respect to M_{n+1} are called right derivatives and those with respect to $M^\top_{n+1}$ are called left derivatives. These two derivatives are related by the outer automorphism of G_{n+1} defined by $\theta_{n+1}(g) = (g^{-1})^\top$. Since

$M_{n+1}^{\top}$ is not conjugated to M_{n+1} in G_{n+1}, the information provided by left and right derivatives taken together is stronger, and is essential to our combinatorial arguments. Let us illustrate the argument when π_1 is the Steinberg representation of $\mathrm{GL}_2(F)$. Let $\nu(g) = |g|$ be a character of GL_1. The theory of derivatives implies that the restriction of π_1 to $\mathrm{GL}_1(F)$ is given by the following Bernstein-Zelevinsky filtration

$$0 \to C_c(F^\times) \to \pi_1 \to \mathbb{C} \to 0$$

where $C_c(F^\times)$ is the space of locally constant, compactly supported functions on $F^\times$, and $\mathrm{GL}_1(F)$ acts on $\mathbb{C}$ by the character ν or ν^{-1}, depending whether we use right or left derivatives, respectively. Thus, for a given character π_2 of $\mathrm{GL}_1(F)$, one can clearly arrange that the character on the quotient $\mathbb{C}$ in the above sequence is different from π_2. Now higher extension spaces vanish since $C_c(F^\times)$ is projective. Even multiplicity one statement is clear since it holds for $C_c(F^\times)$. The general case, restricting from G_{n+1} to G_n, follows this strategy. The bottom piece of the Bernstein-Zelevinsky filtration of π_1 is the Gelfand-Graev representation of G_n, thus vanishing of higher extensions, and multiplicity one for generic representations, follow from projectivity [CS], and multiplicity one for the Gelfand-Graev representation of G_n, respectively.

The theory of left and right derivatives is expected to have more applications on restiction problems. In [Ch2], we further prove that there are no isomorphic irreducible quotients (and submodules) for the i-th left and i-th right shifted derivatives of an irreducible representation of G_n unless the derivatives are the highest one. Such result has consequences on the indecomposability of a restricted representation as well as to the submodule restriction problem.

Let K_r be the r-th principal congruence subgroup in G_n. Let $\pi \in \mathrm{Alg}(G_n)$ be generated by the subspace π^{K_r} of K_r-fixed vectors (so π is contained in finitely many Bernstein components). Then the left ${}^{(i)}\pi$ and the right $\pi^{(i)}$ derivative are related by the isomorphism $(\pi^{(i)})^\vee \cong {}^{(i)}(\pi^\vee)$. We establish this as a consequence of a "second adjointness isomorphism" for Bernstein-Zelevinsky derivatives, that naturally involves the left derivative, proved in the appendix. This result is of independent interest.

The second main result is a projectivity criterion for the representation π_1 of G_{n+1}, when restricted to G_n, formulated in [Ch2]. In [Ch2], we shall use the criteria to classify all irreducible representations which are projective when restricted from G_{n+1} to G_n. Assume that π_1 is projective as G_n-module. Then higher extension spaces vanish without assuming that π_2 is generic. Now assume that π_1 or π_2 is degenerate. Then $\mathrm{EP}(\pi_1, \pi_2) = 0$ by the Prasad's formula. If π_2 is a quotient of π_1 then $\mathrm{Ext}^i_{G_n}(\pi_1, \pi_2) \neq 0$ for some $i > 0$, and this contradicts projectivity of π_1. Thus a necessary condition for π_1 to be G_n-projective is that it is generic and all its irreducible quotients are generic. In this paper we show that this is also a sufficient condition. The proof relies heavily on the Hecke algebra methods from our earlier paper [CS]. Moreover, if π_1 is projective, we identify each Bernstein component of π_1 with an explicit projective Hecke algebra module, independent of π_1. We also show that the necessary condition is satisfied if π_1 is an essentially square integrable representation. Therefore essentially square integrable representations of G_{n+1} are

projective G_n-modules, and any two such representations are isomorphic as G_n-modules. This result generalizes the classical result of Bernstein and Zelevinsky which says that any two cuspidal representations of G_{n+1} are isomorphic when restricted to the mirabolic subgroup M_{n+1} of G_{n+1}.

Finally we would like to point the following consequence of our results to the submodule restriction problem: $\mathrm{Hom}_{G_n}(\pi_2, \pi_1)$ (note that π_1 and π_2 have switched the places). The two restriction problems are related by a cohomological duality, due to Nori and Prasad [NP] (also see [Ch]),

$$\mathrm{Ext}^i_{G_n}(\pi_2, \pi_1)^\vee \cong \mathrm{Ext}^{d(\pi_2)-i}_{G_n}(\pi_1, D(\pi_2)),$$

where $d(\pi_2)$ is the cohomological dimension of π_2 and $D(\pi_2)$ is the Aubert involute of π_2. This duality gives an additional importance to the cohomological restriction problem studied in this paper. Since $d(\pi_2) > 0$, due to the presence of one-dimensional center in G_n, it follows that $\mathrm{Hom}_{G_n}(\pi_2, \pi_1) = 0$ for all irreducible π_2 if π_1 is projective, in particular, this is true if π_1 is an essentially square integrable representation.

2. Bernstein-Zelevinsky derivatives

In this section we study Bernstein-Zelevinsky derivatives, or simply derivatives, as functors from $\mathrm{Alg}(G_n)$ to $\mathrm{Alg}(G_{n-i})$. We state a "second adjointness isomorphism" for these functors, as well as an Ext version of the formula. Mirabolic group will appear in the next section.

2.1. **Notation.** Let $G_n = \mathrm{GL}_n(F)$, where F is p-adic field. Let $\nu(g) = |\det(g)|$ be the character of G_n, where $|\cdot|$ is the absolute value on F. Let B_n be the Borel subgroup of G_n consisting of upper triangular matrices and let U_n be the unipotent radical of B_n. Let

$$R_{n-i} = \left\{ \begin{pmatrix} g & x \\ 0 & u \end{pmatrix} : g \in G_{n-i}, u \in U_i, x \in \mathrm{Mat}_{n-i,i}(F) \right\}.$$

We have an obvious Levi decomposition $R_{n-i} = G_{n-i}E_{n-i}$, where E_{n-i} is the unipotent radical of R_{n-i}. Moreover, $E_{n-i} = N_{n-i}U_i$ where N_{n-i} is the unipotent radical of the maximal parabolic subgroup P_{n-i} consisting of block upper triangular matrices and Levi factor $G_{n-i} \times G_i$. Fix a non-zero additive character ψ of F. Let ψ_i be the character of E_{n-i} defined by

$$\psi_i \left(\begin{smallmatrix} 1 & x \\ 0 & u \end{smallmatrix} \right) = \psi(u_{1,2} + \ldots + u_{i-1,i})$$

where $u_{1,2}, \ldots, u_{i-1,i}$ are the entries of u above the diagonal. Let δ_{R_i} be the modular character of R_{n-i}. The modular character is trivial on the unipotent radical E_{n-i}, and it is equal to ν^i on the Levi factor G_{n-i}. Let π be a smooth representation of G_n on a vector space V. The right i-th Bernstein-Zelevinsky derivative of π is a smooth representation $\pi^{(i)}$ of G_{n-i} on the vector space $V^{(i)}$ defined by

$$V^{(i)} = V/\langle \pi(e)v - \psi_i(e)v : e \in E_{n-i}, v \in V \rangle.$$

The representation $\pi^{(i)}$ is the natural action of the Levi factor G_{n-i} on $V^{(i)}$ twisted by $\delta_{R_{n-i}}^{-1/2}$, that is, Bernstein-Zelevinsky derivatives in this paper are normalized, as is parabolic induction and corresponding Jacquet functors.

From the definition of derivatives, the factorization $R_{n-i} = G_{n-i}E_{n-i}$, and the Frobenius reciprocity, one can easily prove the "first adjointness" isomorphism for right Bernstein-Zelevinsky derivatives: For any smooth representation π of G_n and smooth representation σ of G_{n-i}:

$$\mathrm{Hom}_{G_n}(\pi, \mathrm{Ind}_{R_{n-i}}^{G_n}(\sigma \otimes \psi_i)) \cong \mathrm{Hom}_{G_{n-i}}(\pi^{(i)}, \sigma).$$

We called the derivative right because there is also a left derivative, which is taken with respect to the transpose of the groups used to define right derivatives. More precisely, the underlying vector space for the left derivative ${}^{(i)}\pi$ is

$${}^{(i)}V = V/\langle \pi(e)v - \psi_i^\top(e)v : e \in E_{n-i}^\top, v \in V \rangle,$$

where $\psi_i^\top$ is the character of $E_{n-i}^\top$ defined by

$$\psi_i^\top \left(\begin{smallmatrix} 1 & 0 \\ x & u \end{smallmatrix} \right) = \bar{\psi}(u_{2,1} + \ldots + u_{i,i-1}).$$

Let $\theta_n(g) = (g^{-1})^\top$ be the outer automorphism of G_n where $g^\top$ is the transpose of g. Then the left derivative of π is related to the right derivative by the identity

$${}^{(i)}\pi = \theta_{n-i}(\theta_n(\pi)^{(i)}).$$

Let K_r be the r-th principal congruence subgroup in G_n. Let π be a representation of G_n generated by π^{K_r}, the space of K_r-fixed vectors in π. By 4.2 Theorem in [BZ1], any submodule of π is also generated by its subspace of K_r-fixed vectors. Thus representations of G_n generated by K_r-fixed vectors form a categorical direct summand. The following is the "second adjointness" isomorphism for left Bernstein-Zelevinsky derivatives, proved in the appendix.

Lemma 2.1. *Let K_r be the r-th principal congruence subgroup in G_n. For any representation π of G_n generated by π^{K_r}, and any smooth representation σ of G_{n-i},*

$$\mathrm{Hom}_{G_n}(\mathrm{ind}_{R_{n-i}}^{G_n}(\sigma \otimes \bar{\psi}_i), \pi) \cong \mathrm{Hom}_{G_{n-i}}(\sigma, {}^{(i)}\pi).$$

This isomorphism is functorial in both σ and π.

We now derive some consequences of the two adjointness isomorphisms. The first consequence is a relationship between right and left derivatives via the contragredient:

Lemma 2.2. *Let K_r be the r-th principal congruence subgroup in G_n. Let π be a representation of G_n generated by π^{K_r}. Then $(\pi^{(i)})^\vee \cong {}^{(i)}(\pi^\vee)$.*

Proof. If we insert $\pi^\vee$ in Lemma 2.1 then, for every smooth representation σ of G_{n-i},

$$\mathrm{Hom}_{G_n}(\mathrm{ind}_{R_{n-i}}^{G_n}(\sigma \otimes \bar{\psi}_i), \pi^\vee) \cong \mathrm{Hom}_{G_{n-i}}(\sigma, {}^{(i)}(\pi^\vee)).$$

On the other hand, by Proposition 4.2 in [Pr2],

$$\mathrm{Hom}_{G_n}(\mathrm{ind}_{R_{n-i}}^{G_n}(\sigma \otimes \bar{\psi}_i), \pi^\vee) \cong \mathrm{Hom}_{G_{n-i}}(\sigma, (\pi^{(i)})^\vee)$$

Thus we have an isomorphism

$$\mathrm{Hom}_{G_{n-i}}(\sigma, (\pi^{(i)})^\vee) \cong \mathrm{Hom}_{G_{n-i}}(\sigma, {}^{(i)}(\pi^\vee))$$

functorial in σ. Now we regard the functors

$$h_1 := \mathrm{Hom}_{G_{n-i}}(., (\pi^{(i)})^\vee), \quad h_2 := \mathrm{Hom}_{G_{n-i}}(., {}^{(i)}(\pi^\vee))$$

as objects in the functor category $\mathcal{F}$ which contains contravariant functors from the category of representations of G_{n-i} to the category of abelain groups. The Yoneda lemma (see e.g. Lemma 4.3.5 [St]) asserts that there are natural isomorphisms

$$\mathrm{Hom}_{\mathcal{F}}(h_1, h_2) \cong \mathrm{Hom}_{G_{n-i}}((\pi^{(i)})^\vee, {}^{(i)}(\pi^\vee))$$

and

$$\mathrm{Hom}_{\mathcal{F}}(h_2, h_1) \cong \mathrm{Hom}_{G_{n-i}}({}^{(i)}(\pi^\vee), (\pi^{(i)})^\vee).$$

The naturality is in the sense of loc. cit. and so $h_1 \cong h_2$ in $\mathcal{F}$ implies ${}^{(i)}(\pi^\vee) \cong (\pi^{(i)})^\vee$.

□

Remark: The statement of Lemma 2.1 is optimal in the sense that it cannot be extended to all smooth representations π. To that end, observe that Lemma 2.2, the case $i = n$, says that we have an isomorphism of vector spaces $(\pi^\vee)_{U_n,\psi_n} \cong (\pi_{U_n,\psi_n})^\vee$ for every G_n-module π generated by π^{K_r}. Let π be any smooth G_n-module. It can be written as a direct sum

$$\pi \cong \oplus_{r=1}^{\infty} \pi_r$$

where π_r is generated by $\pi_r^{K_r}$ and $\pi_r^{K_{r-1}} = 0$. Then

$$\pi^\vee \cong \oplus_{r=1}^{\infty} \pi_r^\vee$$

hence $\pi^\vee_{U_n,\psi_n}$ is a direct sum of $(\pi_r^\vee)_{U_n,\psi_n} \cong ((\pi_r)_{U_n,\psi_n})^\vee$. But $(\pi_{U_n,\psi_n})^\vee$ is a product of $((\pi_r)_{U_n,\psi_n})^\vee$, hence much larger unless the sum over r is finite.

The following is not needed in this work, however, it is used in the sequel of this paper [Ch2] to prove that both, socle and co-socle, of derivatives of irreducible representations are multiplicity free, see Proposition 2.5 in loc. cit..

Lemma 2.3. *Let π be an irreducible representation of G_n. The socle of ${}^{(i)}\pi$ is isomorphic to the co-socle of ${}^{(i)}\pi$. In particular, if the irreducible subquotients of ${}^{(i)}\pi$ are multiplicity free, then ${}^{(i)}\pi$ is a direct sum of its irreducible subquotients.*

Proof. The key observation is that, in view of Lemma 2.2, we have two ways to compute ${}^{(i)}\pi$:

$${}^{(i)}\pi = \theta_{n-i}(\theta_n(\pi)^{(i)}) = ((\pi^\vee)^{(i)})^\vee.$$

Since π is irreducible, we have $\theta_n(\pi) \cong \pi^\vee$, and if we denote by σ either of two isomorphic representations $\theta_n(\pi)^{(i)}$ and $(\pi^\vee)^{(i)}$, we see that on one hand ${}^{(i)}\pi$ is obtained from σ by applying a co-variant functor θ, and on the other hand by applying the contra-variant functor taking the contragradient. Since these two functors coincide on irreducible representations, the corollary follows. □

Lemma 2.4. *For any representation π of G_n generated by π^{K_r} and smooth representation σ of G_{n-i},*

$$\mathrm{Ext}^j_{G_n}(\mathrm{ind}^{G_n}_{R_{n-i}}(\sigma\otimes\bar{\psi}_i),\pi)\cong\mathrm{Ext}^j_{G_{n-i}}(\sigma,{}^{(i)}\pi).$$

Proof. In order to compute the right hand side we need to use a projective resolution of σ. Using the induction in stages,

$$\mathrm{ind}^{G_n}_{R_{n-i}}(\sigma\otimes\bar{\psi}_i)\cong\mathrm{Ind}^{G_n}_{P_{n-i}}(\sigma\boxtimes\mathrm{ind}^{G_i}_{U_i}(\bar{\psi}_i)).$$

The Gelfand-Graev representation $\mathrm{ind}^{G_i}_{U_i}(\bar{\psi}_i)$ is projective by Corollary A.6 [CS]. Thus, if σ is projective it follows that $\mathrm{ind}^{G_n}_{R_{n-i}}(\sigma\otimes\bar{\psi}_i)$ is projective, since the parabolic induction takes projective modules into projective modules. So we have shown that taking a projective resolution of σ also gives a projective resolution of $\mathrm{ind}^{G_n}_{R_{n-i}}(\sigma\otimes\bar{\psi}_i)$. Hence the lemma follows from Lemma 2.1.

□

2.2. **Zelevinsky segments.** Here we follow [Ze]. Let ρ be a cuspidal representation of G_r. For any $a,b\in\mathbb{C}$ with $b-a\in\mathbb{Z}_{\geq 0}$, we have a Zelevinsky segment let $\Delta=[\nu^a\rho,\nu^{a+1}\rho,\dots,\nu^b\rho]$. The absolute length of Δ is defined to be $r(b-a+1)$, and the relative $b-a+1$. We can truncate Δ form each side to obtain two segments of absolute length $r(b-a)$:

$${}^-\Delta=[\nu^{a+1}\rho,\dots,\nu^b\rho]\text{ and }\Delta^-=[\nu^a\rho,\dots,\nu^{b-1}\rho].$$

Moreover, if we perform the truncation k-times, the resulting segments will be denoted by ${}^{(k)}\Delta$ and $\Delta^{(k)}$. The induced representation $\nu^a\rho\times\nu^{a+1}\rho\times\ldots\times\nu^b\rho$ contains a unique irreducible submodule denoted by $\langle\Delta\rangle$.

Proposition 2.5. *Let $i>0$ be an integer. The i-th left and right derivatives of $\langle\Delta\rangle$ vanish unless $i=r$ when*

$${}^{(r)}\langle\Delta\rangle=\langle{}^-\Delta\rangle\text{ and }\langle\Delta\rangle^{(r)}=\langle\Delta^-\rangle.$$

Corollary 2.6. *Let $\Delta_1,\dots,\Delta_k$ be segments. Let π be an irreducible subquotient of $\langle\Delta_1\rangle\times\ldots\times\langle\Delta_k\rangle$. If a right derivative of π is generic, then every Δ_j is of the relative length one or two, and if the relative length is two, then Δ_j^- contributes to the cuspidal support of the right derivative of π. Similarly, if a left derivative of π is generic, then ${}^-\Delta_j$ contributes to the cuspidal support of the left derivative.*

Proof. Observe that $\langle\Delta\rangle$ is generic if and only if the relative length of Δ is one. By the Leibniz rule, a right derivative of $\langle\Delta_1\rangle\times\ldots\times\langle\Delta_k\rangle$ has a filtration whose subquotients are $\langle\Delta_1'\rangle\times\ldots\times\langle\Delta_k'\rangle$ where Δ_j' is Δ_j or Δ_j^-. This representation is generic if and only if the relative length of every Δ_j is one or two, and if it is two then $\Delta_j'=\Delta_j^-$. □

We summarize some other results from [Ze] that we shall need. The induced representation $\nu^a\rho\times\nu^{a+1}\rho\times\ldots\times\nu^b\rho$ also contains a unique irreducible quotient denoted by $\mathrm{St}(\Delta)$. This representation is an essentially square integrable representation i.e. its matrix coefficients are square integrable when restricted to the derived subgroup. Every essentially square integrable representation is isomorphic to $\mathrm{St}(\Delta)$ for some segment Δ.

Proposition 2.7. *Let $i > 0$ be an integer. The i-th left and right derivatives of $\mathrm{St}(\Delta)$ vanish unless $i = jr$, for some integer j, when*

$$^{(i)}\mathrm{St}(\Delta) = \mathrm{St}(\Delta^{(j)}) \text{ and } \mathrm{St}(\Delta)^{(i)} = \mathrm{St}(^{(j)}\Delta).$$

Finally, let $\mathfrak{m} = \{\Delta_1, \dots, \Delta_k\}$ be a multisegment, that is, a multiset of segments. Let

$$\mathrm{St}(\mathfrak{m}) = \mathrm{St}(\Delta_1) \times \ldots \times \mathrm{St}(\Delta_k).$$

We observe that this representation depends on the ordering of the segments, but its semi-simplification does not. One say that $\mathfrak{m}$ is generic if no two segments are linked, see page 184 in [Ze]. Then, by Theorems 4.2 and 9.7 in [Ze], $\mathrm{St}(\mathfrak{m})$ is an irreducible generic representation, and every such representation arises in this way.

3. Bernstein-Zelevinsky filtration

In this section we begin our study of the restriction problem from G_{n+1} to G_n. Using the second adjointness formula, for both left and right derivatives, we prove that degenerate representations of G_n cannot be quotients of essentially square integrable representations of G_{n+1}.

3.1. **Bernstein-Zelevinsky functors.** Let $M_{n+1} \subseteq G_{n+1}$ be the mirabolic subgroup

$$M_{n+1} = \left\{ \begin{pmatrix} g & u \\ 0 & 1 \end{pmatrix} : g \in G_n, u \in \mathrm{Mat}_{n,1}(F) \right\}.$$

be the mirabolic subgroup of G_{n+1}. We have an obvious Levi decomposition $M_{n+1} = G_n E_n$. Abusing notation, let ψ be the character of E_n defined by $\psi(u) = \psi(u_n)$ where u_n is the bottom entry of the column vector u. Note that the stabilizer of ψ in G_n is M_n. We have a pair of functors

$$\Phi^- : \mathrm{Alg}(M_{n+1}) \to \mathrm{Alg}(M_n) \text{ and } \Phi^+ : \mathrm{Alg}(M_n) \to \mathrm{Alg}(M_{n+1})$$

defined by $\Phi^-(\tau) = \tau_{E_n,\psi}$ and $\Phi^+(\tau) = \mathrm{ind}_{M_n E_n}^{M_{n+1}}(\tau \boxtimes \psi)$. We also have a pair of functors

$$\Psi^- : \mathrm{Alg}(M_{n+1}) \to \mathrm{Alg}(G_n) \text{ and } \Psi^+ : \mathrm{Alg}(G_n) \to \mathrm{Alg}(M_{n+1})$$

where $\Psi^-(\tau) = \tau_{E_n}$ and Ψ^+ is simply the inflation. All functors are normalized as in [BZ2]. Any $\tau \in \mathrm{Alg}(M_{n+1})$ has an M_{n+1}-filtration

$$\tau_n \subset \ldots \subset \tau_0 = \tau$$

where, $\tau_i = (\Phi^+)^i(\Phi^-)^i(\tau)$, and

$$\tau_i/\tau_{i+1} = (\Phi^+)^i \Psi^+ \Psi^- (\Phi^-)^i(\tau).$$

Observe that $\Psi^-(\Phi^-)^i(\tau) = \tau^{(i+1)}$, is the $(i+1)$-th derivative, and the subquotients of the filtration, considered as G_n-modules, are

$$\tau_i/\tau_{i+1} \cong \mathrm{ind}_{R_{n-i}}^{G_n}(\nu^{1/2} \cdot \tau^{(i+1)} \boxtimes \psi_i),$$

where we have used notation from the previous section. In particular, τ_n is a multiple of the Gelfand-Graev representation. We derive some consequences of this filtration that we shall need later.

Lemma 3.1. *Let $\tau \in \mathrm{Alg}(M_{n+1})$ such that its derivatives are all finitely generated. When τ is considered a G_n-module, its Bernstein components are finitely generated.*

Proof. Recall that $P_{n-i} \supseteq R_{n-i}$ is the maximal parabolic subgroup of G_n with the Levi factor $G_i \times G_{n-i}$. Using induction in stages, the i-th subquotient in the Bernstein-Zelevinsky filtration of τ can be written as

$$\mathrm{Ind}_{P_{n-i}}^{G_n}(\nu^{1/2} \cdot \tau^{(i+1)} \boxtimes \mathrm{ind}_{U_i}^{G_i}(\psi_i)).$$

By the assumption $\tau^{(i+1)}$ is a finitely generated G_{n-i}-module and the Bernstein components of the Gelfand-Graev representation $\mathrm{ind}_{U_i}^{G_i}(\psi_i)$ are finitely generated [BH]. Lemma follows since parabolic induction sends finitely generated modules to finitely generated modules by 3.11 in [BD]. □

Lemma 3.2. *Let $\pi_1 \in \mathrm{Alg}(G_{n+1})$ and π_2 an admissible representation of G_n. If π_2 is a quotient of π_1 then, for some $i, j \geq 0$.*

$$\mathrm{Hom}_{G_{n-i}}(\nu^{1/2} \cdot \pi_1^{(i+1)}, {}^{(i)}\pi_2) \neq 0 \text{ and } \mathrm{Hom}_{G_{n-j}}(\nu^{-1/2} \cdot {}^{(j+1)}\pi_1, \pi_2^{(j)}) \neq 0.$$

Proof. In order to prove the first isomorphism, we restrict π_1 to G_n, by way of M_{n+1}, and use the second adjointness formula. For the second we restrict to G_n, by way of $M_{n+1}^{\top}$, i.e. we reverse the roles of left and right derivatives. □

3.2. **Essentially square integrable representations.**

Theorem 3.3. *Let $\Delta = [\nu^a\rho, \dots, \nu^b\rho]$ be a segment of absolute length $n+1$, where ρ is a cuspidal representation of G_r. Let π be an irreducible G_n-module. If π is a quotient of $\mathrm{St}(\Delta)$ then π is generic.*

Proof. Let $l = b - a + 1$, in particular, $n + 1 = lr$. Assume that π is degenerate. Let $\mathfrak{m} = \{\Delta_1, \dots, \Delta_k\}$ be a multi-segment, from the Zelevinsky classification, such that π is the unique submodule of $\langle\Delta_1\rangle \times \dots \times \langle\Delta_k\rangle$. Since π is degenerate, by 9.10 in [Ze] one segment in $\mathfrak{m}$ has the relative length at least two. If π is a quotient of $\mathrm{St}(\Delta)$ then, by Lemma 3.2, ${}^{(i)}\pi$ contains $\nu^{1/2} \cdot \mathrm{St}(\Delta)^{(i+1)}$ as a generic submodule for some i. Now we can apply Corollary 2.6 : the relative length of each segment in $\mathfrak{m}$ is one or two, and one of them is $[\nu^{c-1/2}\rho, \nu^{c+1/2}\rho]$ where $\nu^{c+1/2}\rho$ contributes to the cuspidal support of $\nu^{1/2} \cdot \mathrm{St}(\Delta)^{(i+1)}$. It follows that $\nu^{1/2} \cdot \mathrm{St}(\Delta)^{(i+1)}$ is a generalized Steinberg representation corresponding to a segment ending in $\nu^{b+1/2}$, and containing $\nu^{c+1/2}\rho$. Thus, for every $d = c, \dots, b$, $\nu^{d+1/2}\rho$ contributes to the cuspidal support of ${}^{(i)}\pi$ as well as to the cuspidal support of π. Similarly, if we use the second identity in Lemma 3.2, then for every $d = a, \dots, c$, $\nu^{d-1/2}\rho$ contributes to the cuspidal support of π. We see that $\mathfrak{m}$ contains segments of total relative length $\geq l$ and absolute length $(l+1)r = n+1+r > n$. This is a contradiction. □

4. Vanishing of Ext's

The purpose of this section is to prove:

Theorem 4.1. *Let π_1 be an irreducible generic representation of G_{n+1} and π_2 an irreducible generic representation of G_n. Then*

$$\mathrm{Ext}_{G_n}^i(\pi_1, \pi_2) = 0 \text{ if } i > 0 \text{ and } \mathrm{Hom}_{G_n}(\pi_1, \pi_2) = \mathbb{C}.$$

Let us explain the strategy of the proof. Fix π_2, and assume that π_2 is a subquotient of $\rho_1 \times \ldots \times \rho_k$ where ρ_i are cuspidal representations. Let $m(\pi_1)$ be the integer that counts the number of cuspidal representations ρ in the support of π_1 such that ρ is an unramified twist of a ρ_i, for some $1 \leq i \leq k$. The proof is by induction on $m(\pi_1)$. The base case $m(\pi_1) = 0$ is easy. It is deduced from the Bernstein-Zelevinsky filtration of π_1 where the bottom piece is the Gelfand-Graev representation of G_n. Assume now that $\pi_1 = \mathrm{St}(\mathfrak{m}_1)$ and $\pi_2 = \mathrm{St}(\mathfrak{m}_2)$ for a pair of generic multisegments $\mathfrak{m}_1$ and $\mathfrak{m}_2$ i.e. no two segments in $\mathfrak{m}_i$ are linked. Let $\Delta = [\nu^a\rho, \ldots, \nu^b\rho]$ be a segment in $\mathfrak{m}_1$ such that ρ contributes to $m(\pi_2)$. Assume that Δ is also a shortest such segment. Write $\pi_1 = \mathrm{St}(\Delta) \times \pi$ where $\pi = \mathrm{St}(\mathfrak{m})$ and $\mathfrak{m} = \mathfrak{m}_1 \backslash \Delta$. Let r be the integer such that $\rho \in \mathrm{Alg}(G_r)$. Let $\rho' \in \mathrm{Alg}(G_r)$ be another cuspidal representation such that no unramified twist of ρ' appears in the cuspidal supports of π_1 and π_2. Now both $\rho' \times \mathrm{St}({}^-\Delta) \times \pi$ and $\rho' \times \mathrm{St}(\Delta^-) \times \pi \in \mathrm{Alg}(G_{n+1})$ are irreducible and satisfy the induction assumption. We shall use this information to prove the theorem for π_1.

4.1. **Transfer.** Let $l = s + r$. Recall that P_s is the maximal parabolic of G_l with the Levi $G_s \times G_r$. Starting with $\sigma \in \mathrm{Alg}(G_s)$ and $\tau \in \mathrm{Alg}(M_r)$ one can manufacture two representations of M_l. The first one is obtained by the (normalized) induction from $P_s \cap M_l$ and, abusing notation, denoted by $\sigma \times \tau$. The second is obtained by the normalized induction from $P_s^\top \cap M_l$ but only after σ is multiplied by $\nu^{-1/2}$, see [BZ2] page 457, where the definition uses a different subgroup, but conjugated in M_l. This representation is denoted by $\tau \times \sigma$.

Our interest in these representations comes from the following, 4.13 Proposition in [BZ2].

Proposition 4.2. *Let $\rho \in \mathrm{Alg}(G_r)$, $\sigma \in \mathrm{Alg}(G_s)$ and $\tau \in \mathrm{Alg}(M_r)$. Let $\rho|_M$ and $\sigma|_M$ denote restrictions to M_r and M_s, respectively.*

(1) *There exists an exact sequence in* $\mathrm{Alg}(M_l)$
$$0 \to (\rho|_M) \times \sigma \to \rho \times \sigma \to \rho \times (\sigma|_M) \to 0$$
(2) *If Ω is any of the four functors $\Phi^\pm$ and $\Psi^\pm$, then*
$$\Omega(\sigma \times \tau) = \sigma \times \Omega(\tau).$$
(3) *$\Psi^-(\tau \times \sigma) = \Psi^-(\tau) \times \sigma$, and there exists an exact sequence in* $\mathrm{Alg}(M_{l-1})$
$$0 \to \Phi^-(\tau) \times \sigma \to \Phi^-(\tau \times \sigma) \to \Psi^-(\tau) \times (\sigma|_M) \to 0$$

Proposition 4.3. *Let $\Delta = [\nu^a\rho, \ldots, \nu^b\rho]$ be a segment where $\rho \in \mathrm{Alg}(G_r)$. Let $\tau_r = (\Phi^+)^{r-1}(1) \in \mathrm{Alg}(M_r)$, the Gelfand-Graev module. Then $\mathrm{St}(\Delta)|_M$ is isomorphic to $\tau_r \times \mathrm{St}({}^-\Delta)$.*

Proof. Recall that $\rho|_M \cong \tau_r$ (this is true for every cuspidal representation). Note that $\mathrm{St}(\Delta)$ is a quotient of $\nu^a\rho \times \mathrm{St}({}^-\Delta)$. By Proposition 4.2 (1), we have an exact sequence of mirabolic subgroup modules
$$0 \to \tau_r \times \mathrm{St}({}^-\Delta) \to \nu^a\rho \times \mathrm{St}({}^-\Delta) \to \nu^a\rho \times (\mathrm{St}({}^-\Delta)|_M) \to 0$$
By Proposition 4.2 (2), any derivative of the quotient in the above sequence is equal to $\nu^a\rho \times \mathrm{St}({}^{(k)}\Delta)$ with $k > 1$. Since $\nu^a\rho$ and ${}^{(k)}\Delta$ are not linked, the corresponding

subquotients in the Bernstein-Zelevinsky filtration are irreducible as mirabolic subgroup modules. Observe that they are non-isomorphic to the subquotients of the Bernstein-Zelevinsky filtration of $\mathrm{St}(\Delta)$. Hence the projection from $\nu^a\rho \times \mathrm{St}(^-\Delta)$ onto $\mathrm{St}(\Delta)$, restricted to $\tau_r \times \mathrm{St}(^-\Delta)$ gives the desired isomorphism. □

Now we arrive to a key result:

Corollary 4.4. *Let $\rho, \rho' \in \mathrm{Alg}(G_r)$ be any two irreducible cuspidal representations. Let $\Delta = [\nu^a\rho, \dots, \nu^b\rho]$, and $\pi \in \mathrm{Alg}(G_s)$. Then we have an isomorphism of mirabolic modules*

$$\mathrm{St}(\Delta)|_M \times \pi \cong \rho'|_M \times (\mathrm{St}(^-\Delta) \times \pi).$$

Proof. By Proposition 4.3, we can substitute $\mathrm{St}(\Delta)|_M = \tau_r \times \mathrm{St}(^-\Delta)$. Furthermore, we have a natural isomorphism

$$(\tau_r \times \mathrm{St}(^-\Delta)) \times \pi \cong \tau_r \times (\mathrm{St}(^-\Delta)) \times \pi)$$

given by the induction in stages in two different orders. We finish by observing that $\tau_r = \rho'|_M$. □

Now we continue with the proof of vanishing for $\pi_1 = \mathrm{St}(\Delta) \times \pi$, notation as in the start of the section. By Proposition 4.2 (1) there is an exact sequence in $\mathrm{Alg}(M_{n+1})$

$$0 \to (\mathrm{St}(\Delta)|_M) \times \pi \to \mathrm{St}(\Delta) \times \pi \to \mathrm{St}(\Delta) \times (\pi|_M) \to 0.$$

Likewise, there is an exact sequence in $\mathrm{Alg}(M_{n+1})$

$$0 \to \rho'|_M \times (\mathrm{St}(^-\Delta) \times \pi) \to \rho' \times (\mathrm{St}(^-\Delta) \times \pi) \to \rho' \times (\mathrm{St}(^-\Delta) \times \pi)|_M \to 0.$$

Note that the submodules in the two sequences are isomorphic by Corollary 4.4. Furthermore, by the choice of ρ',

$$\mathrm{Ext}^i_{G_n}(\rho' \times (\mathrm{St}(^-\Delta) \times \pi)|_M, \pi_2) = 0 \text{ if } i \geq 0.$$

Now we can apply the induction assumption to $\rho' \times \mathrm{St}(^-\Delta) \times \pi$ and conclude that

$$\mathrm{Ext}^i_{G_n}((\mathrm{St}(\Delta)|_M) \times \pi, \pi_2) = 0 \text{ if } i > 0 \text{ and } \cong \mathbb{C} \text{ if } i = 0.$$

Hence, in order to establish the conjecture for the pair (π_1, π_2), it suffices to show that

$$\mathrm{Ext}^i_{G_n}(\mathrm{St}(\Delta) \times (\pi|_M), \pi_2) = 0 \text{ if } i \geq 0,$$

and to do this it suffices to show vanishing for each subquotient in the Bernstein-Zelevinsky filtration of $\mathrm{St}(\Delta) \times (\pi|_M)$. By Proposition 4.2 part (2), the derivatives of $\mathrm{St}(\Delta) \times (\pi|_M)$ are computed on the second factor. Thus, combining with the second adjointness formula, it suffices to show that

- $\mathrm{Ext}^j_{G_n}(\nu^{1/2}\mathrm{St}(\Delta) \times \pi^{(i+1)}, {}^{(i)}\pi_2) = 0$ for $i, j \geq 0$.

Alternatively, by reversing the roles of left and right derivatives, it suffices to show that

- $\mathrm{Ext}^j_{G_n}(\nu^{-1/2}\mathrm{St}(\Delta) \times {}^{(i+1)}\pi, \pi_2^{(i)}) = 0$ for $i, j \geq 0$.

Hence it suffices to show that the cuspidal support of $\nu^{1/2}(\mathrm{St}(\Delta) \times \pi^{(i+1)})$ and of ${}^{(i)}\pi_2$ are different for all i, or, they are different for $\nu^{-1/2}(\mathrm{St}(\Delta) \times {}^{(i+1)}\pi)$ and $\pi_2^{(i)}$ for all i. The strategy is to show that, if both statements fail, then $\mathfrak{m}_2$ contains linked segments.

4.2. **Combinatorics.** Let $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_k\}$ be a multisegment. Then $\mathrm{St}(\mathfrak{m})$ is generic but reducible if some segments are linked. However, if Δ_i and Δ_j are linked, then they can be replaced by $\Delta_i \cap \Delta_j$ and $\Delta_i \cup \Delta_j$. This process (called recombination henceforth) eventually leads to a generic segment such that the corresponding irreducible generic representation is the unique generic subquotient in $\mathrm{St}(\mathfrak{m})$. Important observation is that the recombination does not change the cuspidal support. The following is a key lemma.

Lemma 4.5. *Let $\mathfrak{m}$ be a generic multisegment and $\mathfrak{m}'$ a multisegment obtained by truncating $\mathfrak{m}$ from the right. Then the generic segment corresponding to $\mathfrak{m}'$ by recombination is also obtained from $\mathfrak{m}$ by truncating from the right.*

Proof. This is proved by induction on the number of steps in the recombination process. If that number is 0 there is nothing to prove. Otherwise there is a pair of linked segments Δ' and Δ'' in $\mathfrak{m}'$ such that the first step in the recombination is replacing Δ' and Δ'' by $\Delta' \cap \Delta''$ and $\Delta' \cup \Delta''$. It is trivial to see that the resulting multisegment is also obtained by right truncation from $\mathfrak{m}$. The proof follows by induction. □

4.3. **Finishing the proof.** Let $l = b - a + 1$ be the relative length of Δ. We note that ${}^{(i)}\pi_2$ is glued from $\mathrm{St}(\mathfrak{m}_2')$ where $\mathfrak{m}_2'$ runs over all multisegments obtained from $\mathfrak{m}_2$ by truncating from the right i-times (in the sense of absolute length). By the previous lemma the cuspidal support of ${}^{(i)}\pi_2$ is given by such generic multisegments. Likewise, $\mathrm{St}(\Delta) \times \pi^{(i+1)}$ is glued from $\mathrm{St}(\Delta) \times \mathrm{St}({}'\mathfrak{m})$ where ${}'\mathfrak{m}$ runs over all multisegments obtained from $\mathfrak{m}$ by truncating from the left $i+1$-times, and to determine the cuspidal support we need to consider only generic ${}'\mathfrak{m}$. However, $\{\Delta\} \cup {}'\mathfrak{m}$ needs not be generic. There could be segments in ${}'\mathfrak{m}$ linked to Δ. Since Δ is not linked to any segment in $\mathfrak{m}$ and ${}'\mathfrak{m}$ is obtained from $\mathfrak{m}$ by left truncation, it follows that linking occurs over the right end point of Δ. Let Δ_0 be the longest segment in ${}'\mathfrak{m}$ linked to Δ. It is easy to see that $\Delta \cup \Delta_0$ is a segment in the generic multisegment corresponding to $\{\Delta\} \cup {}'\mathfrak{m}$ by the recombination process. Note that $\Delta \cup \Delta_0$ starts with $\nu^a\rho$ and is of relative length at least l. Thus the cuspidal supports of $\nu^{1/2}(\mathrm{St}(\Delta) \times \pi^{(i+1)})$ and ${}^{(i)}\pi_2$ can have a point in common only if $\mathfrak{m}_2$ contains a segment starting with $\nu^{a+1/2}\rho$ and of relative length at least l. Similarly, the cuspidal supports of $\nu^{-1/2}(\mathrm{St}(\Delta) \times {}^{(i+1)}\pi)$ and $\pi_2^{(i)}$ can have a point in common only if $\mathfrak{m}_2$ contains a segment ending with $\nu^{b-1/2}\rho$ and of length at least l. In other words we have constructed a pair of linked segments in $\mathfrak{m}_2$, a contradiction. This completes the proof of the Ext-vanishing theorem.

5. Hecke algebra methods

The main goal of this section is to prove that an irreducible representation π_1 of G_{n+1} when restricted to G_n is projective if π_1 is generic and all its irreducible

G_n-quotients are generic. The proof uses Hecke algebras and identifies all Bernstein components of a projective π_1 with the sign-projective module of the Hecke algebra corresponding to the Bushnell-Kutzko type [BK1], [BK2], [BK3]. As a consequence, any two projective representations of G_{n+1} are isomorphic when restricted to G_n.

5.1. **Hecke algebras.** Let $\Delta = [\nu^a\rho, \dots, \nu^b\rho]$ be a Zelevinsky segment. Let $m = b-a+1$. The Bernstein component of $\mathrm{St}(\Delta)$ is equivalent to the category of representations of a Hecke algebra $\mathcal{H}_m$ arising from a simple Bushnell-Kutzko type τ_Δ, that is, if π is a smooth representation in the Bernstein component, then $\mathrm{Hom}(\tau_\Delta, \pi)$ is the corresponding $\mathcal{H}_m$-module. The algebra $\mathcal{H}_m$ is isomorphic to the Iwahori Hecke algebra of $\mathrm{GL}_m(E)$, for some field E. Thus, as an abstract algebra, $\mathcal{H}_m$ is generated by $\theta_1, \dots, \theta_m$, and T_w ($w \in S_m$) satisfying the following relations (see, e.g. [Ho, (50) and (57)]):

(1) $\theta_k\theta_l = \theta_l\theta_k$ for any $k, l = 1, \dots, m$;
(2) $T_{s_k}\theta_k - \theta_{k+1}T_{s_k} = (q-1)\theta_k$, where q is a prime power depending on τ_Δ and s_k is the transposition of numbers k and $k+1$;
(3) $T_{s_k}\theta_l = \theta_l T_{s_k}$ if $l \neq k, k+1$;
(4) $(T_{s_k} - q)(T_{s_k} + 1) = 0$, where s_k is as in (2), and T_w satisfies a braid relation.

Let $\mathcal{A}_m = \mathbb{C}[\theta_1^{\pm 1}, \dots, \theta_m^{\pm 1}]$ and $\mathcal{H}_{S_m}$ be the span of T_w, $w \in S_m$. Then $\mathcal{H}_m \cong \mathcal{A}_m \otimes \mathcal{H}_{S_m}$. The finite dimensional algebra $\mathcal{H}_{S_m}$ has a one dimensional sign representation $\mathrm{sgn}(T_w) = (-1)^{\ell(w)}$, where ℓ is the length function on S_m. An irreducible representation π in the component is Whittaker generic if and only if $\mathrm{Hom}(\tau_\Delta, \pi)$ contains the sign type as $\mathcal{H}_{S_m}$-module [CS].

Let $\Delta_1, \dots, \Delta_r$ be segments such that for $i \neq j$, the cuspidal representations ρ_i and ρ_j are not unramified twists of each other. The Bernstein component of $\mathrm{St}(\Delta_1) \times \cdots \times \mathrm{St}(\Delta_r)$ is equivalent to the category of representations of a Hecke algebra $\mathcal{H}$ arising from a semi-simple Bushnell-Kutzko type τ. We have $\mathcal{H} \cong \mathcal{H}_{m_1} \otimes \cdots \otimes \mathcal{H}_{m_r}$ and $\mathcal{H} \cong \mathcal{A} \otimes \mathcal{H}_S$ where $\mathcal{A} \cong \mathcal{A}_{m_1} \otimes \cdots \otimes \mathcal{A}_{m_r}$ and $\mathcal{H}_S \cong \mathcal{H}_{S_{m_1}} \otimes \cdots \otimes \mathcal{H}_{S_{m_r}}$. The subalgebra $\mathcal{A}$ is isomorphic to the ring of Laurent polynomials in $m = m_1 + \ldots + m_r$ variables, while $\mathcal{H}_S$ is spanned by T_w, $w \in S = S_{m_1} \times \cdots \times S_{m_r}$. An irreducible representation π in the component can be written as $\pi_1 \times \cdots \times \pi_r$ where π_i is in the component of $\mathrm{St}(\Delta_i)$, thus it clear that π is Whittaker generic if and only if $\mathrm{Hom}(\tau, \pi)$ contains the sign type of $\mathcal{H}_S$.

5.2. **Some projective modules.** Let χ be a character of $\mathcal{A}$. The $\mathcal{H}$-module $\mathcal{H} \otimes_{\mathcal{A}} \chi$ is called a principal series representation of $\mathcal{H}$. A twisted Steinberg representation of $\mathcal{H}$ is any one-dimensional $\mathcal{H}$-module such that the restriction to $\mathcal{H}_S$ is the sign type. For example, if $\pi = \mathrm{St}(\Delta_1) \times \cdots \times \mathrm{St}(\Delta_r)$, then $\mathrm{Hom}(\tau, \pi)$ is a twisted Steinberg representation.

The following is Theorem 2.1 in [CS]. (It is stated there for $\mathcal{H}$ arising from the singleton partition (m) but the proof is applicable to a general partition $(m_1, \dots, m_r)$).

Theorem 5.1. *Let Π be an $\mathcal{H}$-module. Assume that*

(1) *Π is projective and finitely generated.*
(2) *$\dim \mathrm{Hom}_{\mathcal{H}}(\Pi, \pi) \leq 1$ for an irreducible principal series representation π.*
(3) *A twisted Steinberg representation is a quotient of Π.*

Then $\Pi \cong \mathcal{H} \otimes_{\mathcal{H}_S} \mathrm{sgn}$. *Conversely,* $\mathcal{H} \otimes_{\mathcal{H}_S} \mathrm{sgn}$ *satisfies the above properties.*

As in [CS], we have the following corollary.

Corollary 5.2. *Let* Γ *be the summand of the Gelfand-Graev representation corresponding to the Bernstein component of* $\mathrm{St}(\Delta_1) \times \cdots \times \mathrm{St}(\Delta_r)$. *Then we have an isomorphism* $\mathrm{Hom}(\tau, \Gamma) \cong \mathcal{H} \otimes_{\mathcal{H}_S} \mathrm{sgn}$ *of* $\mathcal{H}$*-modules.*

5.3. **Projectivity for Hecke algebras.** Let $\mathcal{Z}$ be the center of $\mathcal{H}$. Recall that $\mathcal{Z} = \mathcal{A}^S$, in particular, $\mathcal{H}$ is a finitely generated $\mathcal{Z}$-module. Let $\mathcal{J}$ be a maximal ideal in $\mathcal{Z}$. Let $\hat{\mathcal{H}}$ denote the $\mathcal{J}$-adic completion of $\mathcal{H}$ [AM]. For every $\mathcal{H}$-module Π, let $\hat{\Pi}$ denote the $\mathcal{J}$-adic completion of Π. If Π is finitely generated, then $\hat{\Pi} \cong \hat{\mathcal{H}} \otimes_{\mathcal{H}} \Pi$.

Theorem 5.3. *Let* Π *be a finitely generated* $\mathcal{H}$*-module and* $\mathcal{J}$ *a maximal ideal in* $\mathcal{Z}$. *Let* π *be the unique irreducible* $\mathcal{H}$*-module annihilated by* $\mathcal{J}$ *and containing the sign type. Assume that*

(1) $\dim \mathrm{Hom}_{\mathcal{H}}(\Pi, \pi) = 1$
(2) Π *has no other irreducible quotients annihilated by* $\mathcal{J}$.
(3) Π *contains a torsion free element for* $\mathcal{A}$.

Then $\hat{\Pi} \cong \hat{\mathcal{H}} \otimes_{\mathcal{H}_S} \mathrm{sgn}$.

Proof. In order to simplify notation, write $\Sigma = \mathcal{H} \otimes_{\mathcal{H}_S} \mathrm{sgn}$. Since Π is finitely generated, $\hat{\Pi}/\mathcal{J}\hat{\Pi} \cong \Pi/\mathcal{J}\Pi$ is a finite dimensional $\mathcal{H}$-module, annihilated by $\mathcal{J}$. By (2) it must be generated by the sign type subspace. Let r be the dimension of the sign type in $\Pi/\mathcal{J}\Pi$. By Frobenius reciprocity, we have a surjection $f : \Sigma^{\oplus r} \to \Pi/\mathcal{J}\Pi$ which descends to a surjection $\bar{f} : (\Sigma/\mathcal{J}\Sigma)^{\oplus r} \to \Pi/\mathcal{J}\Pi$. Observe that $\bar{f}$ is bijective on the sign type, since the sign type in $\Sigma/\mathcal{J}\Sigma$ is one-dimensional. Since π is the unique irreducible quotient of $\Sigma/\mathcal{J}\Sigma$ and $\bar{f}$ is bijective on the sign type, it follows that π^r is a quotient of $\Pi/\mathcal{J}\Pi$. This forces $r = 1$ by (1) and, by Nakayama lemma, we have a surjection $\hat{f} : \hat{\Sigma} \to \hat{\Pi}$. Since $\hat{\Sigma} \cong \hat{\mathcal{A}}$, as $\hat{\mathcal{A}}$-modules, (3) implies that the surjection is in fact an isomorphism.

□

Corollary 5.4. *Let* Π *be a finitely generated* $\mathcal{H}$*-module and* $\mathcal{J}$ *a maximal ideal in* $\mathcal{Z}$. *Assume that the conditions of Theorem 5.3 are satisfied. Then, for all* $\mathcal{H}$*-modules* σ *annihilated by* $\mathcal{J}$ *and for all* $i > 0$,

$$\mathrm{Ext}^i_{\mathcal{H}}(\Pi, \sigma) = 0.$$

Proof. To compute $\mathrm{Ext}^i_{\mathcal{H}}(\Pi, \sigma)$ we take a sufficiently long free resolution of Π

$$\cdots \to \mathcal{H}^r \to \mathcal{H}^s \to \Pi \to 0.$$

Let $\hat{\mathcal{Z}}$ be the $\mathcal{J}$-adic completion of $\mathcal{Z}$. By Proposition 10.13 in [AM], the completion of finitely generated $\mathcal{Z}$-modules is isomorphic to tensoring by $\hat{\mathcal{Z}}$. Since $\hat{\mathcal{Z}}$ is a flat $\mathcal{Z}$-module, by Proposition 10.14 in [AM] , it follows that

$$\cdots \to \hat{\mathcal{H}}^r \to \hat{\mathcal{H}}^s \to \hat{\Pi} \to 0$$

is also exact. Now, since σ is annihilated by $\mathcal{J}$, it is easy to check that

$$\mathrm{Ext}^i_{\mathcal{H}}(\Pi, \sigma) \cong \mathrm{Ext}^i_{\hat{\mathcal{H}}}(\hat{\Pi}, \sigma).$$

The latter spaces are trivial for $i > 0$ by projectivity of $\hat{\mathcal{H}} \otimes_{\mathcal{H}_S} \mathrm{sgn}$. □

Corollary 5.5. *Let Π be a finitely generated $\mathcal{H}$-module. Assume that the conditions of Theorem 5.3 are satisfied for every maximal ideal in $\mathcal{Z}$. Then $\Pi \cong \mathcal{H} \otimes_{\mathcal{H}_S} \mathrm{sgn}$.*

Proof. Corollary 5.4 implies that $\mathrm{Ext}^i_{\mathcal{H}}(\Pi, \sigma) = 0$, $i > 0$, for all finite length modules σ. Since Π is also finitely generated, it is projective by Theorem A.1 in the appendix of [CS]. Now we can apply Theorem 5.1. □

5.4. **Projectivity for groups.** Now we can apply the Hecke-module results to the restriction problem, one Bernstein component at the time. Let π_1 be an irreducible generic representation of G_{n+1} and fix a Bushnell-Kutzko type τ for G_n. Let $\Pi = \mathrm{Hom}(\tau, \pi_1)$ be the corresponding $\mathcal{H}$-module. Note that the conditions (1) and (3) in Theorem 5.3 are satisfied for every maximal ideal $\mathcal{J}$. Indeed, the condition (1) because all irreducible generic G_n-representations are quotients of π_1 with multiplicity one and (3) because π_1, restricted to G_n, contains the Gelfand-Graev representation whose τ component is a free $\mathcal{A}$-module. Theorem 5.3 implies the following local Ext vanishing result for groups.

Theorem 5.6. *Let π_1 be an irreducible generic representation of G_{n+1}. Let $\mathcal{J}$ be a maximal ideal of the Bernstein center of G_n. Assume that no degenerate irreducible representation of G_n annihilated by $\mathcal{J}$ is a quotient of π_1. Then $\mathrm{Ext}^i_{G_n}(\pi_1, \pi_2) = 0$, $i > 0$, for all irreducible representation π_2 of G_n annihilated by $\mathcal{J}$.*

Finally, we have [Ch2]:

Theorem 5.7. *Let π_1 be an irreducible generic representation of G_{n+1} whose irreducible G_n-quotients are all generic. Then π_1, considered a G_n-module, is projective. Moreover, any two such representations of G_{n+1} are isomorphic as G_n-modules. This holds for all essentially square integrable representation of G_{n+1}.*

Proof. Indeed, by Corollary 5.5, $\mathrm{Hom}(\tau, \pi_1) \cong \mathcal{H} \otimes_{\mathcal{H}_S} \mathrm{sgn}$ for any Bernstein component of π_1. Thus every component of π_1 is a projective G_n-module independent of π_1, as long as π_1 has no degenerate quotients. And these conditions are satisfied for essentially square integrable representations by Theorem 3.3.

□

6. Appendix

In this appendix we prove Lemma 2.1, that is, the second adjointness isomorphism for Bernstein-Zelevisky derivatives. The key ingredient is Rodier's approximation [Ro] of the Whittaker character by characters of compact pro-p groups.

6.1. **Groups.** Let F be a p-adic field, R its ring of integers and P the maximal ideal generated by a prime ϖ. Let ψ be the character of F of conductor R. Let $G = \mathrm{GL}_n(F)$ and let U be the group of unipotent upper triangular matrices in G. Let $\psi_U : U \to \mathbb{C}$ be a Whittaker character defined by

$$\psi_U(u) = \psi(u_{1,2} + \cdots + u_{n-1,n})$$

where $u_{i,j}$ denote the entries of the matrix u.

For every natural number r, let L_r be the lattice in $M_n(F)$ consisting of all matrices whose entries are in P^r. Then

$$K_r = 1 + L_r$$

is a principal congruence subgroup of G. Let $t = (t_i) \in G$ be a diagonal matrix such that $t_i/t_{i+1} = \varpi^2$ for $i = 1, \ldots, n-1$. Let $H_r = t^{-r} K_r t^r$. Let $B^\top$ be the Borel subgroup of lower-triangular matrices. Then we have a parhoric decomposition

$$H_r = (H_r \cap B^\top)(H_r \cap U).$$

The sequence of groups $H_r \cap B^\top$ is decreasing with trivial intersection, while the sequence of groups $H_r \cap U$ is increasing with union U. Let ψ_r be a character of H_r defined by

$$\psi_r(g) = \psi(g_{1,2} + \cdots + g_{n-1,n}).$$

Observe that

$$\psi_r|_{H_r \cap U} = \psi_U|_{H_r \cap U}.$$

6.2. **Representations.** Let π be a smooth G-module. For every non-negative integer r we have a projection map $P_r : \pi \to \pi^{H_r, \psi_r}$ defined by

$$P_r(v) = \mathrm{vol}(H_r)^{-1} \int_{H_r} \bar{\psi}_r(u)\pi(g)v \; dg.$$

For $r \leq s$ we have maps $i_r^s : \pi^{H_r,\psi_r} \to \pi^{H_s,\psi_s}$ defined by restricting P_s to π^{H_r,ψ_r}. From the parahoric decomposition of H_r, it is easy to see that

$$i_r^s(v) = \mathrm{vol}(H_s \cap U)^{-1} \int_{H_s \cap U} \bar{\psi}_s(u)\pi(u)v \; du.$$

This formula, in turn, implies that these maps form a direct system i.e. $i_s^t \circ i_r^s = i_r^t$, for $r \leq s \leq t$. We have natural maps $i_r : \pi^{H_r,\psi_r} \to \pi_{U,\psi_U}$. Observe that $i_s \circ i_r^s = i_r$. Hence we have a map from a direct limit

$$i_\pi : \lim_r \pi^{H_r,\psi_r} \to \pi_{U,\psi_U}.$$

Proposition 6.1. *For every smooth G-module π the map i_π is an isomorphism of vector spaces.*

Proof. Surjectivity: Let $v \in \pi$. Since $H_r \cap B^\top \to \{1\}$ there exists r such that v is $H_r \cap B^\top$-invariant. Let

$$w = \mathrm{vol}(H_r \cap U)^{-1} \int_{H_r \cap U} \bar{\psi}_r(u)\pi(u)v \; du \in \pi^{H_r,\psi_r}.$$

Then v and w have the same projection on π_{U,ψ_U}. Injectivity: Let $v \in \pi^{H_r,\psi_r}$ that projects to 0 in π_{U,ψ_U}. Then there exists an open compact subgroup $U_c \subset U$ such that

$$\int_{U_c} \bar{\psi}_s(u)\pi(u)v \; du = 0.$$

Since $H_s \cap U \to U$ there exists $s \geq r$ such that $H_s \cap U \supset U_c$. Then the above integral, with U_c substituted by $H_s \cap U$, vanishes. In other words, $i_r^s(v) = 0$ and hence $v = 0$, viewed as an element of the direct limit. $\square$

For $r \leq s$ we have maps $p_r^s : \pi^{H_s,\psi_s} \to \pi^{H_r,\psi_r}$, going in the opposite direction, defined by restricting P_r to $\pi^{H_s\psi_s}$. From the parahoric decomposition of H_r, it is easy to see that

$$p_r^s(v) = \mathrm{vol}(H_r \cap B^\top)^{-1} \int_{H_r \cap B^\top} \pi(g)v \; dg$$

and this implies that these maps form an inverse system i.e. $p_r^s \circ p_s^t = p_r^t$, for $r \leq s \leq t$.

By Proposition 4 in [Ro] (see also VI, page 169), there exists an integer r_0, independent of π, such that $p_r^s \circ i_r^s$ is a non-zero multiple of the identity on π^{H_r,ψ_r}, if $r_0 \leq r \leq s$. Thus i_r^s is an injection and p_s^r is a surjection. It follows, from Proposition 6.1, that the maps $i_r : \pi^{H_r,\psi_r} \to \pi_{U,\psi_U}$ are injections, for all $r \geq r_0$.

We shall use surjectivity of the maps p_r^s to construct a natural complement of π^{H_r,ψ_r} in π_{U,ψ_U}. So fix $r \geq r_0$, and for every $s \geq r$, let τ_s be the kernel of p_r^s. Observe that τ_s is a complement of π^{H_r,ψ_r} in π^{H_s,ψ_s}, where we have identified π^{H_r,ψ_r} with its image in π^{H_s,ψ_s}. We claim that τ_s, for $s \geq r$ form an injective subsystem. To that end, let $t \geq s$. We need to prove if $v \in \tau_s$ then $i_s^t(v) \in \tau_t$, that is, $p_r^t(i_s^t(v)) = 0$. Write $p_r^t = p_r^s \circ p_s^t$. Then

$$p_r^t(i_s^t(v)) = p_r^s \circ p_s^t((i_s^t(v)) = p_r^s(p_s^t \circ i_s^t(v)) = 0$$

where for the last equality we used that $p_s^t \circ i_s^t(v)$ is a multiple of v. Hence the direct limit

$$\pi_c^{H_r,\psi_r} := \lim_{s \geq r} \tau_s$$

is a complement of π^{H_r,ψ_r} in $\lim_{s\geq r} \pi^{H_r,\psi_r} \cong \pi_{U,\psi_U}$.

We apply the above considerations to $\pi = S(G)$, the space of locally constant, compactly supported functions on G, considered a G-module with respect to the action by left translations. In this case, the vector spaces π^{H_r,ψ_r} and π_{U,ψ_U} are naturally G-modules, coming from the right translation action of G on $S(G)$ and the maps i_r^s, p_r^s and i_r are G-morphisms. Observe that $S(G)^{H_r,\psi_r} = \mathrm{ind}_{H_r}^G(\psi_r)$ and $S(G)_{U,\psi_U} \cong \mathrm{ind}_U^G(\psi)$, the Gelfand-Graev representation. Hence $\lim_r \mathrm{ind}_{H_r}^G(\psi_r) \cong \mathrm{ind}_U^G(\psi)$, as G-modules. Moreover, if $r \geq r_0$, then $\mathrm{ind}_{H_r}^G(\psi_r)$ is a direct summand of $\mathrm{ind}_U^G(\psi)$. We record this in the following:

Proposition 6.2. *For every $r \geq r_0$, $\mathrm{ind}_{H_r}^G(\psi_r)$ is a direct G-invariant summand of $\mathrm{ind}_U^G(\psi)$:*

$$\mathrm{ind}_U^G(\psi) \cong \mathrm{ind}_{H_r}^G(\psi_r) \oplus \mathrm{ind}_{H_r}^G(\psi_r)_c.$$

Proposition 6.3. *Fix $r \geq r_0$. For almost all $s \geq r$, $(\mathrm{ind}_{H_s}^G(\psi_s)_c)^{K_r}$ is trivial.*

Proof. The key is the following lemma:

Lemma 6.4. *Let $r \geq r_0$. Let π be an irreducible Whittaker generic G-module such that $\pi^{K_r} \neq 0$. There exists a positive integer m, independent of π, such that $\pi^{H_{mr}\psi_{mr}} \neq 0$.*

Proof. The first step in the proof is a reduction to supercuspidal representations. Let $P = MN$ be a standard parabolic subgroup of block upper-triangular matrices. Assume that π is a Whittaker generic subquotient of $\mathrm{Ind}_{P^\top}^G \sigma$, where $P^\top$ is the transpose of P. Let $K = \mathrm{GL}_n(R)$. Using $G = P^\top K$ and normality of K_r in K, it is easy to see that $\pi^{K_r} \neq 0$ implies that $\sigma^{K_r^M} \neq 0$ where $K_r^M = K_r \cap M$. Now assume that $\sigma^{H_s^M, \psi_s^M} \neq 0$ where $H_s^M = H_s \cap M$ and ψ_s^M is the restriction of ψ_s to H_s^M. Let $v \in \sigma^{H_s^M, \psi_s^M}$, and define $f \in \mathrm{Ind}_{P^\top}^G \sigma$, supported on $P^\top(H_s \cap N)$, such that $f(1) = v$ and that it is right $(\psi_s)|_{H_s \cap N}$-invariant. Then $f \in (\mathrm{Ind}_{P^\top}^G \sigma)^{H_s, \psi_s}$. This type must belong to the Whittaker generic subquotient of the induced representation by injectivity of the map i_s.

It remains to deal with supercuspidal π. Let ℓ be a Whittaker functional on π, and for every $v \in \pi$ we have a Whittaker function $f_v(g) = \ell(\pi(g)v)$. Let $T(r) \subset T$ be the subset of $t = (t_1, \ldots, t_n)$ such $1/q^{(2m-2)r} \leq |t_i/t_{i+1}| \leq q^{(2m-2)r}$, for all i. By Theorem 2.1 in [La], there exists m, independent of π, such that f_v is supported on $UT(r)K$ for all $v \in \pi^{K_r}$. Since f_v is non-zero, for a non-zero v, there exists $t \in T(r)$ and $k \in K$ such that $\ell(\pi(tk)v) \neq 0$. Since K normalizes K_r, $\pi(k)v \in \pi^{K_r}$. It follows that $\pi(tk)v$ is fixed by tK_rt^{-1}. Observe that this group contains $H_{mr} \cap B^\top$, by the definition of $T(r)$, hence

$$w = \mathrm{vol}(H_{mr} \cap U)^{-1} \int_{H_{mr} \cap U} \bar{\psi}_U(u)\pi(u)\pi(tk)v \in \pi^{H_{mr}, \psi_{mr}}$$

and it is non-zero since $\ell(w) = \ell(\pi(tk)v) \neq 0$. The lemma is proved. □

Take $s \geq mr$, where m is as in the lemma. Recall that, by [BH], Bernstein's components of $\mathrm{ind}_U^G \psi_U$ are finitely generated and hence admit irreducible quotients. Thus, if $(\mathrm{ind}_{H_s}^G (\psi_s)_c)^{K_r} \neq 0$ then $\mathrm{ind}_{H_s}^G (\psi_s)_c$ has an irreducible quotient π such that $\pi^{K_r} \neq 0$. Then $\pi^{H_s, \psi_s} \neq 0$ by the lemma, and hence $\dim_G(\mathrm{ind}_U^G \psi_U, \pi) \geq 2$, by Proposition 6.2, a contradiction. □

Proposition 6.5. *For every G-module π generated by π^{K_r}, and every vector space σ*

$$\mathrm{Hom}_G(\sigma \otimes \mathrm{ind}_U^G \psi_U, \pi) \cong \mathrm{Hom}(\sigma, \pi_{U, \psi_U}).$$

Proof. By Propositions 6.2 and 6.3, for almost all $s \geq r$,

$$\mathrm{Hom}_G(\sigma \otimes \mathrm{ind}_U^G \psi_U, \pi) \cong \mathrm{Hom}_G(\sigma \otimes \mathrm{ind}_{H_s}^G (\psi_s), \pi).$$

Let $\mathbb{C}[G]$ denote the group algebra of G. Then we can write

$$\sigma \otimes \mathrm{ind}_{H_s}^G (\psi_s) \cong \mathrm{ind}_{H_s}^G (\sigma \otimes \psi_s) \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H_s]} (\sigma \otimes \psi_s).$$

Hence, by the Frobenius reciprocity,

$$\mathrm{Hom}_G(\sigma \otimes \mathrm{ind}_{H_s}^G (\psi_s), \pi) \cong \mathrm{Hom}(\sigma, \pi^{H_s, \psi_s}).$$

Now observe that the starting space $\mathrm{Hom}_G(\sigma \otimes \mathrm{ind}_U^G \psi_U, \pi)$ does not depend on s. It follows that the spaces π^{H_s, ψ_s} are isomorphic for almost all s. In particular, $\pi^{H_s, \psi_s} \cong \pi_{U, \psi_U}$ for such s. Hence

$$\mathrm{Hom}_G(\sigma \otimes \mathrm{ind}_U^G \psi_U, \pi) \cong \mathrm{Hom}(\sigma, (\pi)_{U, \psi_U}).$$

□

6.3. **Second adjointness.** Now we are ready to prove Lemma 2.1. We resume using the notation from the main body of the paper, in particular, $G_n = \mathrm{GL}_n(F)$, U_n is the group of upper-triangular unipotent matrices, and $P_{n-i} = M_{n-i}N_{n-i}$ the standard maximal parabolic subgroup of block upper-triangular matrices with the Levi $G_{n-i} \times G_i$. Let π be a smooth representation of G_n generated by vectors fixed by the r-th principal congruence subgroup in G_n, and σ a smooth representation of G_{n-i}, as in the statement of the lemma. Using the induction in stages,

$$\mathrm{ind}_{R_{n-i}}^{G_n}(\sigma \otimes \bar{\psi}_i) \cong \mathrm{Ind}_{P_{n-i}}^{G_n}(\sigma \boxtimes \mathrm{ind}_{U_i}^{G_i}(\bar{\psi}_i)).$$

By the second adjointness isomorphism for parabolic induction, due to Bernstein,

$$\mathrm{Hom}_{G_n}(\mathrm{Ind}_{P_{n-i}}^{G_n}(\sigma \boxtimes \mathrm{ind}_{U_i}^{G_i}(\bar{\psi}_i)), \pi) \cong \mathrm{Hom}_{G_{n-i}\times G_i}(\sigma \boxtimes \mathrm{ind}_{U_i}^{G_i}(\bar{\psi}_i), \pi_{N_{n-i}^\top}).$$

It is easy to see that $\pi_{N_{n-i}^\top}$, as a G_i-module, is also generated by vectors fixed by the r-th principal congruence subgroup in G_i. Thus we can apply Proposition 6.5 to G_i to derive

$$\mathrm{Hom}_{G_{n-i}\times G_i}(\sigma \boxtimes \mathrm{ind}_{U_i}^{G_i}(\bar{\psi}_i), \pi_{N_{n-i}^\top}) \cong \mathrm{Hom}_{G_{n-i}}(\sigma, {}^{(i)}\pi).$$

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