AN EXCEPTIONAL SIEGEL-WEIL FORMULA AND POLES OF THE SPIN L-FUNCTION OF PGSp_6

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Abstract. We show a Siegel-Weil formula in the setting of exceptional theta correspondence. Using this, together with a new Rankin-Selberg integral for the Spin L-function of PGSp_6 discovered by A. Pollack, we prove that a cuspidal representation of PGSp_6 is a (weak) functorial lift from the exceptional group G_2 if its (partial) Spin L-function has a pole at $s = 1$.

1. Introduction

Let F be a totally real number field, and $\mathbb{A}$ its ring of adèles. Let $\pi \cong \otimes_v \pi_v$ be an irreducible cuspidal automorphic representation of the group $\mathrm{PGSp}_6(\mathbb{A})$, which is unramified outside a finite set S of places (including all real places). Since the Langlands dual group of PGSp_6 is $\mathrm{Spin}_7(\mathbb{C})$, there is an associated semi-simple conjugacy class s_v in $\mathrm{Spin}_7(\mathbb{C})$ for $v \notin S$; this is the Satake parameter of the local component π_v. If r denotes the 8-dimensional spin representation of $\mathrm{Spin}_7(\mathbb{C})$, the partial spin L-function corresponding to π is defined to be the product

$$L^S(s, \pi, \mathrm{Spin}) = \prod_{v \notin S} \frac{1}{\det(1 - r(s_v) q_v^{-s})}$$

where q_v is the order of the residual field of the local field F_v.

It is well known that the stabilizer in $\mathrm{Spin}_7(\mathbb{C})$ of a generic vector in the spin representation is the exceptional group $G_2(\mathbb{C})$, giving a well-defined conjugacy class of embedding

$$\iota : G_2(\mathbb{C}) \longrightarrow \mathrm{Spin}_7(\mathbb{C}).$$

Therefore, as a special case of the Langlands functoriality principle, if $L^S(s, \pi, \mathrm{Spin})$ has a simple pole at $s = 1$, then one expects π to be a functorial lift from an exceptional group of absolute type $\mathbf{G}_2$ defined over F. We note that every such group is given as the automorphism group of an octonion algebra $\mathbb{O}$ over F, and by the Hasse principle, the number of isomorphism classes of such groups is 2^n where n is the number of real places of F.

As explained in a recent paper of Chenevier [C, §6.12], if π is a tempered cuspidal representation of PGSp_6 such that for almost all places v, the Satake parameter s_v of π_v belongs to $\iota(G_2(\mathbb{C}))$ (or more accurately, the conjugacy class s_v meets $\iota(G_2(\mathbb{C}))$), then $L^S(s, \pi, \mathrm{Spin})$ will have a pole at $s = 1$ and so one expects such a tempered π to be a functorial lift from G_2. In this paper we also prove a slightly weaker version of this expectation:

2010 *Mathematics Subject Classification.* 11F27, 11F70, 22E50.
Key words and phrases. Siegel-Weil formula, Spin L-function, theta correspondence.

Theorem 1.1. *In the above setting, suppose that π is a cuspidal automorphic representation of PGSp_6 such that $L^S(s,\pi,\mathrm{Spin})$ has a pole at $s=1$. Then there exists an octonion algebra $\mathbb{O}$ over F and a cuspidal automorphic representation π' of $\mathrm{Aut}(\mathbb{O})$ such that the Satake parameters of π' are mapped by ι to those of π (i.e. π is a weak functorial lift of π').*

If the cuspidal representation π of PGSp_6 is tempered, then the following are equivalent:

(a) *For almost all places v, the Satake parameter s_v of π_v is contained in $\iota(G_2(\mathbb{C}))$.*
(b) *There exists an octonion algebra $\mathbb{O}$ over F and a cuspidal automorphic representation π' of $\mathrm{Aut}(\mathbb{O})$ such that π is a weak functorial lift of π'.*

Since the local Langlands classification is not known for $\mathbf{G}_2$ or for PGSp_6, this is essentially the best possible result one can expect at the moment. However, if π is unramified everywhere or if it corresponds to a classical Siegel modular form, then π is a functorial lift. Special cases of this result were previously obtained by Ginzburg and Jiang [GJ], Gan and Gurevich [GG09] and Pollack and Shah [PS].

Our proof of Theorem 1.1 is based on the following three ingredients:

(1) An exceptional theta correspondence for the dual pair $\mathrm{Aut}(\mathbb{O}) \times \mathrm{PGSp}_6$ arising from the minimal representation Π of a group of absolute type $\mathbf{E}_7$.
(2) A Siegel-Weil formula proved in this paper; see Theorem 1.2 below.
(3) An integral representation of the spin L-function of π recently discovered by A. Pollack [P].

In greater detail, let J be the exceptional Jordan algebra of 3×3 hermitian symmetric matrices with coefficients in an octonion algebra $\mathbb{O}$. By the Koecher-Tits construction, the algebra J gives rise to an adjoint group G of absolute type $\mathbf{E}_7$, with a maximal parabolic subgroup $P = MN$, such that the unipotent radical N is commutative and isomorphic to J. Since G is adjoint, the conjugation action of M on N is faithful, and M is isomorphic to the similitude group of the natural cubic norm form on J. Thus the natural action of $\mathrm{Aut}(\mathbb{O})$ on J gives an embedding of $\mathrm{Aut}(\mathbb{O})$ into M. The centralizer of $\mathrm{Aut}(\mathbb{O})$ is PGSp_6. To see this, observe that the centralizer of $\mathrm{Aut}(\mathbb{O})$ in J is the Jordan subalgebra J_F of 3×3 symmetric matrices with coefficients in F. The group PGSp_6 arises from J_F by the Koecher-Tits construction. This gives the dual pair

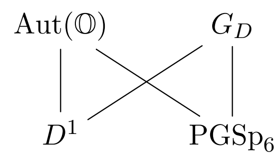
$$\mathrm{Aut}(\mathbb{O}) \times \mathrm{PGSp}_6 \subset G$$

alluded to in (1) above.

We can now describe another dual pair in G. Let D be a quaternion algebra over F, and assume that we have an embedding $i : D \to \mathbb{O}$. The centralizer of D in $\mathrm{Aut}(\mathbb{O})$ is isomorphic to D^1, the group of norm one elements in D. Conversely, the centralizer (i.e. the pointwise stabilizer) of D^1 in $\mathbb{O}$ is $i(D) \subset \mathbb{O}$. Thus the centralizer of D^1 in J is the Jordan subalgebra J_D of 3×3 hermitian symmetric matrices with coefficients in D, and the centralizer of D^1 in G is a group G_D of absolute type $\mathbf{D}_6$ arising from J_D by the Koecher-Tits construction. Thus we have a dual pair

$$D^1 \times G_D \longrightarrow G.$$

Indeed, the two dual pairs we have described fit into the following see-saw diagram, where the vertical lines represent inclusions of groups:



The Siegel-Weil formula mentioned in (2) above concerns the global theta lift $\Theta(1)$ of the trivial representation of D^1 to G_D, obtained by restricting the minimal representation Π of G to the dual pair $D^1 \times G_D$. Roughly speaking, $\Theta(1)$ is the space of automorphic functions on G_D obtained by averaging the functions in Π over $D^1(F)\backslash D^1(\mathbb{A})$. We prove that $\Theta(1)$ is an irreducible automorphic representation of G_D and determine its local components (as abstract representations) by computing the corresponding local theta lifts. We have not computed the local theta lift for complex groups, and this is the source of the restriction in the paper to totally real fields F. The Siegel-Weil formula identifies the functions in $\Theta(1)$ as residues of certain Siegel-Eisenstein series.

More precisely, since G_D arises from J_D by the Koecher-Tits construction, it contains a maximal parabolic subgroup with abelian unipotent radical isomorphic to J_D. Let $E_D(s,f)$ be the degenerate Eisenstein series attached to this maximal parabolic subgroup, where $s \in \mathbb{R}$ and f varies over all standard sections of the corresponding degenerate principal series representation $I_D(s)$. In [HS], it was proved that $E_D(s,f)$ has at most a simple pole at $s=1$, and the residual representation

$$\mathcal{E}_D := \{\mathrm{Res}_{s=1} E_D(s,f) : f \in I_D(s)\}$$

was completely determined. Our main result is the following Siegel-Weil identity in the space of automorphic forms of G_D:

Theorem 1.2. *For fixed quaternion F-algebra D, we have:*

$$\mathcal{E}_D = \oplus_{i:D\to\mathbb{O}} \Theta(1).$$

Here the sum is taken over all isomorphism classes of embeddings $i : D \to \mathbb{O}$ into octonion algebras over F.

We emphasize that D is fixed here but $\mathbb{O}$ vary. If D is split, i.e. a matrix algebra, then $\mathbb{O}$ is also split, and there is only one term on the right. In general the number of summands on the right is equal to 2^m where m is the number of real places v of F such that D_v is a division algebra.

At this point, we need the result of A. Pollack [P]: there exists a quaternion algebra D such that the partial spin-L-function $L^S(\pi,s)$ is given as an integral, over PGSp_6, of a function $h \in \pi$ against the Eisenstein series $E_D(s,f)$. Thus, if the L-function has a pole at $s=1$, then the integral of h against the elements of $\mathcal{E}_D$ is non-zero. The Siegel-Weil identity (i.e. Theorem 1.2) then implies that π appears in the exceptional theta correspondence for the dual pair $\mathrm{Aut}(\mathbb{O}) \times \mathrm{PGSp}_6$, for some $\mathbb{O}$ containing D. Since this exceptional theta correspondence is known to be functorial for spherical representations (see [LS] and [SW15]), this completes the proof that π is a weak lift from a group of absolute type $\mathbf{G}_2$.

2. Groups

2.1. **Octonion algebra.** Let F be a field of characteristic 0, and D be a quaternion algebra over F. It is a 4-dimensional associative and non-commutative algebra over F which comes equipped with a conjugation map $x \mapsto \overline{x}$ with associated norm $N(x) = x\overline{x} = \overline{x}x$ and trace $\mathrm{tr}\,(x) = x + \overline{x}$. Moreover, $N : \mathbb{O} \to F$ is a nondegenerate quadratic form.

An octonion algebra $\mathbb{O}$ over F is obtained by doubling the quaternion algebra D. More precisely, fix a non-zero element λ in F. As a vector space over F, $\mathbb{O}$ is a set of pairs (a, b) of elements in D. The multiplication is defined by the formula

$$(a, b) \cdot (c, d) = (ac + \lambda d\bar{b}, \bar{a}d + cb).$$

If $x = (a, b)$, then the conjugation map is $\bar{x} = (\bar{a}, -b)$, so that $N(x) = x \cdot \bar{x} = N(a) - \lambda N(b)$ is the norm and $\mathrm{tr}\,(x) = x + \bar{x} = \mathrm{tr}\,(a)$ the trace on $\mathbb{O}$. In particular, $\mathbb{O}$ is split if λ is a norm of an element in D. Every element x of $\mathbb{O}$ satisfies its characteristic polynomial $t^2 - \mathrm{tr}\,(x)t + N(x)$. The automorphism group $\mathrm{Aut}(\mathbb{O})$ of the F-algebra $\mathbb{O}$ is an exceptional group of the Lie type $\mathbf{G}_2$. It is a simple linear algebraic group of rank 2 which is both simply connected and adjoint. The algebra D is naturally a subalgebra of $\mathbb{O}$, consisting of all $x = (a, 0)$. Let D^1 be the group of norm one elements in D. Then any $g \in D^1$ acts as an automorphism of $\mathbb{O}$ by $g \cdot (a, b) = (a, b\bar{g})$ for all $(a, b) \in \mathbb{O}$. The subgroup $D^1 \subset \mathrm{Aut}(\mathbb{O})$ is precisely the pointwise stabilizer of the subalgebra $D \subset \mathbb{O}$.

2.2. **Albert algebra.** An Albert algebra is an exceptional 27-dimensional Jordan algebra J over F. It can be realized as the set of matrices

$$A = \begin{pmatrix} \alpha & x & \bar{z} \\ \bar{x} & \beta & y \\ z & \bar{y} & \gamma \end{pmatrix}$$

where $\alpha, \beta, \gamma \in F$ and $x, y, z \in \mathbb{O}$. The determinant $A \mapsto \det A$ defines a natural cubic form on J. Let M be the similitude group of this cubic form. It is a reductive group of semisimple type E_6. The M-orbits in J are classified by the rank of the matrix A. Without going into a general definition of the rank, we say that $A \neq 0$ has rank one, if $A^2 = \mathrm{tr}\,(A) \cdot A$. Explicitly, this means that the entries of A satisfy the equalities

$$N(x) = \alpha\beta,\ N(y) = \beta\gamma,\ N(z) = \gamma\alpha,\ \gamma\bar{x} = yz,\ \alpha\bar{y} = zx,\ \beta\bar{z} = xy.$$

2.3. **Dual pairs.** Assume that G is a reductive group over F, adjoint and of absolute type E_7, arising from the Albert algebra J via the Koecher-Tits construction. For our purposes it will be more convenient to realize G as a quotient, modulo one dimensional center $C \cong F^\times$, of a reductive group $\tilde{G}$ acting on the 56-dimensional representation $W = F + J + J + F$. In particular, G acts on the projective space $\mathbb{P}(W)$. Let P be a maximal parabolic and $\bar{P}$ its opposite, defined as fixing the points $(1, 0, 0, 0)$ and $(0, 0, 0, 1)$ in $\mathbb{P}(W)$. Then $P = MN$ where N is the unipotent radical and $M = P \cap \bar{P}$ a Levi. Then M is isomorphic to the similitude group of the cubic form det on J, and $N \cong J$, as M-modules.

Recall that we have constructed $\mathbb{O}$ by doubling a quaternion subalgebra D. Let J_F and J_D be the subalgebras consisting of all elements in J with off-diagonal entries in F and D, respectively. Let $J_0 = F$ be the scalar subalgebra of J. Consider a sequence of simple, simply connected groups

$$D^1 \subset \mathrm{Aut}(\mathbb{O}) \subset \mathrm{Aut}(J)$$

where an element in $\mathrm{Aut}(\mathbb{O})$ acts on the off-diagonal entries of elements in J. The pointwise stabilizers in J of these three groups are, respectively,

$$J_D \supset J_F \supset J_0 = F.$$

Observe that $\mathrm{Aut}(J)$ naturally acts on W, giving an embedding $\mathrm{Aut}(J) \subset \tilde{G}$. The centralizers in $\tilde{G}$ of the three groups in the sequence are, respectively,

$$\tilde{G}_D \supset \mathrm{GSp}_6(F) \supset \mathrm{GL}_2(F)$$

These three groups act on 32, 14 and 4-dimensional subspaces of W obtained by replacing J by J_D, J_F and J_0, respectively. It is worth mentioning that the 4-dimensional representation of $\mathrm{GL}_2(F)$ is the symmetric cube of the standard 2-dimensional representation, twisted by $\det^{-1}$. The group $\tilde{G}_D$ acts on

$$W_D = F + J_D + J_D + F.$$

It is worth noting that the action of $\tilde{G}_D$ on W_D is not faithful (it has $\mu_2 \subset D^1$ as its kernel). A detailed description of $\tilde{G}_D/\mu_2$ and its action on W_D can be found in Pollack's paper [P]. Let G_D be the quotient of $\tilde{G}_D$ by the center $C \cong F^\times$ of $\tilde{G}$. Then $D^1 \times G_D$ is a dual pair in G, as mentioned in the introduction.

Let $P_D = M_D N_D = G_D \cap P$. With the identification $N \cong J$ fixed, we have $N_D \cong J_D$. The group P_D is a maximal parabolic subgroup of type $\mathbf{A}_5$.

3. Minimal representation

Let F be a real or p-adic field. Let $I(s)$ be the degenerate principal series representation of G attached to P where $s \in \mathbb{R}$. We normalize s as in [We] so that the trivial representation is a quotient and a submodule at $s = 9$ and $s = -9$ respectively, whereas the minimal representation Π is a quotient and a sub-module at $s = 5$ and $s = -5$, respectively. Note however that the group G is simply-connected in [We] whereas our G is adjoint here.

3.1. **Unitary model.**

Fix $\psi : F \to \mathbb{C}^\times$, a non-trivial additive character, unitary if $F = \mathbb{R}$. After identifying $N \cong J$ and $\bar{N} \cong J$ (note that the resulting actions of M on J are dual to each other) any $A \in J \cong \bar{N}$ defines a character of N given by

$$\psi_A(B) = \psi(\mathrm{tr}\,(A \circ B)) = \psi_B(A)$$

for $B \in J \cong N$, where $A \circ B$ denotes the Jordan multiplication. Every unitary character of N is equal to ψ_A for some A. Let $\Omega \subseteq J \cong \bar{N}$ be the set of rank one elements in J. A unitary model of the minimal representation is $\mathcal{H} = L^2(\Omega)$. Here only the acton of the maximal parabolic $P = MN$ is obvious: the group M acts geometrically,

$$\pi(m)(f)(A) = \chi(m) f(m^{-1}A),$$

for $f \in \Pi$, where for some character $\chi : M \to \mathbb{R}^\times$, while $B \in J \cong N$ acts on f by multiplying it by ψ_B.

3.2. **Smooth model.** We have the following [KS15].

Theorem 3.1. *Let Π be the subspace of G-smooth vectors in the unitary minimal representation $\mathcal{H}$. Then*

$$C_c^\infty(\Omega) \subset \Pi \subset C^\infty(\Omega).$$

If F is p-adic, then

$$\Pi_N \cong \Pi / C_c^\infty(\Omega) \quad \text{as } M\text{-modules.}$$

If $A \in J$ is nonzero, then any continuous functional ℓ on Π such that $\ell(B \cdot f) = \psi_A(B) \cdot \ell(f)$ for all $B \in N$ and $f \in \Pi$ is equal to a multiple of the evaluation map $\delta_A(f) = f(A)$. In particular, $\ell = 0$ if A is not of rank one.

3.3. **Spherical vector.** It is not so easy to characterise the subspace $\Pi \subset C^\infty(\Omega)$. However, we can describe a spherical vector in Π in the split case. The algebra $\mathbb{O}$ is obtained by doubling the matrix algebra $D = M_2(F)$ with $\lambda = 1$. Assume firstly that F is a p-adic field. Let O be the ring of integers in F and ϖ a uniformizing element. We have an obvious integral structure on D (the lattice of integral matrices), and hence on $\mathbb{O}$, the integral lattice being the set of pairs (a, b) where $a, b \in M_2(O)$. This lattice is a maximal order in $\mathbb{O}$. Now we have an integral structure on J so that $J(O)$ is the set of elements $A \in J$ such that the diagonal entries are integral, and off diagonal contained in the maximal order in $\mathbb{O}$. The greatest common divisor of entries of $A \in J(O)$, is simply the largest power ϖ^n dividing A i.e. such that A/ϖ^n is in $J(O)$. We have the following [SW07]:

Theorem 3.2. *Assume G is split and F a p-adic field. Assume the conductor of ψ is O. Then the spherical vector in Π is a function $f^\circ \in C^\infty(\Omega)$ supported in $J(O)$. Its value at $A \in \Omega$ depends on the GCD of entries of A. More precisely, if the GCD of the entries of A is ϖ^n, and q is the order of the residual field, then*

$$f^\circ(A) = 1 + q^3 + \ldots + q^{3n}.$$

Since Π is generated by f° as a P-module, and the action of P on Π is easy to describe, this theorem gives us a good handle on Π.

Assume now that $F = \mathbb{R}$; in this case, one has a similar result due to Dvorsky-Sahi [DS99]. For every $a \in M_2(\mathbb{R})$, let $||a||^2$ is the sum of squares of its entries. For $x = (a, b) \in \mathbb{O}$, let $||x||^2 = ||a||^2 + ||b||^2$. Extend this to $A \in J$ by

$$||A||^2 = \alpha^2 + \beta^2 + \gamma^2 + ||x||^2 + ||y||^2 + ||z||^2.$$

Let $K_{3/2}(u)$ denote the modified Bessel function of the second kind. Recall that $K_{3/2}(u) > 0$, for $u > 0$, and is rapidly decreasing as $u \to +\infty$. Then [DS99, Theorem 0.1]:

Theorem 3.3. *Assume G is split and $F = \mathbb{R}$. Then the spherical vector in Π is a function $f^\circ \in C^\infty(\Omega)$ given by*

$$f^\circ(A) = ||A||^{-3/2} K_{3/2}(||A||).$$

4. Local Theta Lifts: p-adic case

In this section, let F be a p-adic field, so that the octonion algebra $\mathbb{O}$ is split. We are interested in understanding the theta lift of the trivial representation of D^1 to the group G_D.

4.1. **N_D-spectrum.** A crucial step is to understand the N_D-spectrum of the minimal representation Π. In this case we have an exact sequence of P-modules

$$0 \to C_c^\infty(\Omega) \to \Pi \to \Pi_N \to 0.$$

The characters of $N_D \cong J_D$ are identified with the elements in J_D using the trace paring, as we did for J. We shall only need three characters, denoted by ψ_1, ψ_2 and ψ_3, corresponding to the elements

$$\begin{pmatrix} \pm1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \end{pmatrix}$$

of rank 1, 2 and 3, respectively. We need to allow signs to capture all possible rank 1, 2 and 3 orbits in the real case. The following lemma is one of the keys in this paper, and we emphasize that we do not assume that D is split here.

Lemma 4.1. *Let Π be the minimal representation of G. Then:*

(i) $\Pi_{N_D,\psi_3} = 0$.
(ii) $\Pi_{N_D,\psi_2} \cong C_c^\infty(D^1)$, *as D^1-modules.*
(iii) *If D is a division algebra, then $\Pi_{N_D,\psi_1} \cong \mathbb{C}$, as D^1-modules.*

Proof. Let $\omega_i \subseteq \Omega$ be the set of all $A \in \Omega$ such that the restriction of ψ_A to N_D is equal to ψ_i. Because ψ_i is not the trivial character, the set ω_i is (Zariski) closed in Ω. Hence,

$$\Pi_{N_D,\psi_i} \cong C_c^\infty(\omega_i).$$

It remains to determine each ω_i. Let's start with $i = 3$. Then ω_3 consists of all $A \in \Omega$ such that

$$A = \begin{pmatrix} \pm1 & x & -z \\ -x & \pm1 & y \\ z & -y & \pm1 \end{pmatrix}$$

where $x = (0, a), y = (0, b)$ and $z = (0, c)$ for some $a, b, c \in D$. Since $A \in \Omega$, we further have $A^2 = \mathrm{tr}\,(A)A$. Looking at the off-diagonal terms, we get the equations

$$yx = \pm z,\ zy = \pm x \text{ and } xz = \pm y.$$

But the products yx, zy and xz have the second coordinate equal to 0. Hence $z = x = y = 0$. But then A cannot be a rank 1 matrix. Hence ω_3 is empty, and this proves (i).

For (ii) we see analogously that $y = z = 0$. Now A has the rank 1 if and only if the first 2×2 minor is 0. This gives $x^2 = \pm1$. Writing this out, with $x = (0, a)$ we see that $\lambda a\bar{a} = \pm1$. Hence ω_2 is identified with the set of all elements in D with a fixed non-zero norm. This is a principal homogeneous space for D^1. This establishes (ii). In the last case it is easy to see that $x = y = z = 0$. □

We now derive a consequence. Let $\Theta(1)$ be the maximal quotient of Π on which D^1 acts trivially; it is naturally a G_D-module. Lemma 4.1 implies that

$$\Theta(1)_{N_D,\psi_3} = 0 \text{ and } \Theta(1)_{N_D,\psi_2} = \mathbb{C}.$$

Let $I_D(s)$ be the degenerate principal series representation of G_D attached to P_D normalized as in [We]. In particular, the trivial representation is a quotient for $s = 5$ and a submodule for

$s = -5$. The inclusion $\Pi \to I(-5)$ composed with the restriction of functions from G to G_D gives a non-zero D^1-invariant map $\Pi \to I_D(-1)$, which clearly factors through $\Theta(1)$. By [We] and [HS], $I_D(-1)$ has a composition series of length 2. The unique irreducible submodule Σ has N_D-rank 2. We have:

Corollary 4.2. *The above construction gives a surjective G_D-equivariant map*

$$\Theta(1) \to \Sigma \subset I_D(-1)$$

whose kernel has N_D-rank no greater than one. If D is a division algebra, then $\Theta(1) \cong \Sigma$.

Proof. It remains to prove the last statement. The spherical, rank 2 representation Σ is the classical theta lift of the trivial representation of the quaternionic form of Sp(4) [Y]. Using the theta correspondence, it is easy to check that $\Sigma_{N_D,\psi_1} \cong \mathbb{C}$. Thus, from Lemma 4.1 (iii) it follows that the kernel of the map $\Theta(1) \to \Sigma$ has N_D-rank 0, i.e. N_D acts trivially. Since D^1 is compact, $\Theta(1)$ is a summand of the minimal representation. By the classical result of Howe-Moore the minimal representation cannot contain non-zero vectors fixed by N_D. Thus the kernel is trivial. □

4.2. **Local lifts for split D.** We shall strengthen here the result of Corollary 4.2 by showing that $\Theta(1) \cong \Sigma$ even when D is split, in which case G is also split.

Let $T \subset G$ be a maximal split torus, so we have the associated root groups. Furthermore, $D^1 \cong \mathrm{SL}_2$ and it is conjugated to a root SL_2. Without loss of generality, we can assume that SL_2 corresponds to the highest root for some choice of positive roots. Let $T_1 = \mathrm{SL}_2 \cap T$. Then the centralizer of T_1 in G is a Levi subgroup L of semisimple type $\mathbf{D}_6$. The Levi subgroup L is contained in two maximal parabolic subgroups: $Q = LU$ and its opposite $\bar{Q} = L\bar{U}$. The unipotent radical U is a two-step unipotent group with the center U_1 given by the root group corresponding to the highest root. Similarly, the center of $\bar{U}$ is the root subgroup $\bar{U}_1$ corresponding to the lowest root. These two root groups U_1 and $\bar{U}_1$ generate SL_2. We identify $T_1 \cong \mathrm{GL}_1$ so that $x \in \mathrm{GL}_1$ acts on U/U_1 as multiplication by x.

The conjugation action of L on U_1 and $\bar{U}_1$ is given by a character and its inverse; this character is given by $x \mapsto x^2$ when restricted to $T_1 \subset L$. Hence G_D is the kernel of this character, which is the derived group of L. Since G is of adjoint type, G_D acts faithfully on U/U_1 (a 32-dimensional spin representation). Note that the representation U/U_1 is not W_D, the 32-dimensional representation of $\tilde{G}_D$, from §2.3.

More precisely, recall that the center of Spin_{12} can be identified with $\mu_2 \times \mu_2$ in such a way that the outer automorphism exchanges the two μ_2's, and fixes the diagonal μ_2^{Δ}. The quotient of Spin_{12} by μ_2^{Δ} is the special orthogonal group SO_{12}. On the other hand, the quotient of Spin_{12} by $\mu_2 = \mu_2 \times \{1\}$ and $\mu_2' = \{1\} \times \mu_2$ are isomorphic (being isomorphic via the outer automorphism). Then one has:

$$G_D \cong \mathrm{Spin}_{12}/\mu_2 \quad \text{and} \quad L \cong T_1 \times_{\mu_2} G_D \cong \mathrm{GL}_1 \times_{\mu_2'} (\mathrm{Spin}_{12}/\mu_2),$$

so that L has connected center. On the other hand, the group $\tilde{G}_D$ from §2.3 is given by

$$\tilde{G}_D \cong \mathrm{GL}_1 \times_{\mu_2} \mathrm{Spin}_{12}.$$

As we mentioned in §2.3, the action of $\tilde{G}_D$ on W_D is not faithful.

We now need a result on the restriction of Π to the maximal parabolic subgroup $Q = LU$. By [MS97, Theorem 6.1], the space of U_1-coinvariants of Π, an L-module, sits in an exact sequence

$$0 \to C_c^\infty(\omega) \to \Pi_{U_1} \to \Pi_U \to 0$$

where ω is the L-orbit of highest weight vectors in $\bar{U}/\bar{U}_1$. The action of L on $C_c^\infty(\omega)$ arises from the natural action of L on ω twisted by an unramified character.

Let $Q_D = L_D U_D$ be a maximal parabolic subgroup in G_D stabilizing the line through a point $v \in \omega$. Note that the Levi factor L_D of Q_D is also of type $\mathbf{A}_5$ (like that of P_D). The action of Q_D on the line gives a homomorphism $\chi : Q_D \to \mathrm{GL}_1$. Thus the stabilizer in $G_D \times \mathrm{GL}_1$ of v consists of all pairs (g, x) such that $g \in Q_D$ and $\chi(g) = x$. Since $G_D \times \mathrm{GL}_1$ acts transitively on ω, it is easy to see that the following holds:

Theorem 4.3. *The normalized Jacquet functor Π_{U_1}, as a $G_D \times \mathrm{GL}_1$-module, has a 2-step filtration with the following quotient and submodule respectively:*

- $\Pi_U = \Pi(G_D) \otimes |\cdot|^2 \oplus |\cdot|^3$ *where $\Pi(G_D)$ is the minimal representation of G_D, and $|\cdot|$ is the absolute value character of GL_1.*
- $\mathrm{Ind}_{Q_D}^{G_D} C_c^\infty(\mathrm{GL}_1)$ *where $C_c^\infty(\mathrm{GL}_1)$ is the regular representation of GL_1 (and the induction is normalized).*

Now we can prove the following result which strengthens Corollary 4.2 and which is needed later.

Proposition 4.4. *Assume that we are in the p-adic case with D split. Then $\Theta(1)$ is irreducible and isomorphic to Σ, the N_D-rank 2 representation of G_D that appears as the unique irreducible quotient of $I_D(1)$.*

Proof. Let π be an irreducible representation of SL_2 and $\Theta(\pi)$ the corresponding big theta lift. We first note that $\Theta(\pi)$ is always non-trivial, as a simple consequence of Lemma 4.1. Moreover, $\Theta(\pi)_{N_D,\psi_2}$ is isomorphic to $\pi^\vee$, so that it is infinite dimensional if and only if π is.

Let $J(s)$ be the principal series for SL_2 normalized so that the trivial representation is a quotient for $s = 1$ and a submodule for $s = -1$. Likewise, let $J_D(s)$ denote the degenerate principal series associated to Q_D, normalized so that the trivial representation occurs at $J_D(\pm 5)$.

If $-s \neq 2, 3$, then Theorem 4.3 implies by way of the Frobenius reciprocity that

$$\mathrm{Hom}(\Theta(J(-s)), \mathbb{C}) \cong \mathrm{Hom}_{\mathrm{SL}_2}(\Pi, J(-s)) \cong \mathrm{Hom}(J_D(s), \mathbb{C})$$

as G_D-modules. For generic s, both $J(-s)$ and $J_D(s)$ are irreducible and the above identity implies that

$$\Theta(J(-s)) \cong J_D(s)$$

for such s. It follows from Lemma 4.1 that $J_D(s)_{N_D,\psi_2}$ is infinite dimensional for such s. However, since the restriction of $J_D(s)$ to N_D is independent of s, it follows that $J_D(s)_{N_D,\psi_2}$ is in fact infinite dimensional for all s.

Now if π is a submodule of $J(-s)$ with $-s \neq 2, 3$, then it follows that

$$\mathrm{Hom}(\Theta(\pi), \mathbb{C}) \cong \mathrm{Hom}_{\mathrm{SL}_2}(\Pi, \pi) \subset \mathrm{Hom}_{\mathrm{SL}_2}(\Pi, J(-s)) \cong \mathrm{Hom}(J_D(s), \mathbb{C}),$$

so that $\Theta(\pi)$ is a quotient of $J_D(s)$. In particular, for $\pi = 1$ the trivial representation, we may take $s = 1$ to deduce that $\Theta(1)$ is a quotient of $J_D(1)$. Since we know that $\Theta(1)_{N_D,\psi_2}$ is one-dimensional whereas $J_D(1)_{N_D,\psi_2}$ is infinite-dimensional, we conclude that $\Theta(1)$ is isomorphic to the unique irreducible quotient of $J_D(1)$ which has N_D-rank 2. In particular, $\Theta(1)$ is irreducible and isomorphic to Σ, the unique quotient of $I_D(1)$. □

As a side remark, the representations $J_D(s)$ have U_D-rank 3. However, since Π has N_D-rank 2, it follows that the two parabolic subgroups P_D and Q_D are not conjugate in G_D. But the two principal series $I_D(s)$ and $J_D(s)$ share all small rank subquotients: the trivial representation, the minimal representation and the rank 2 representation Σ, as the above argument shows.

5. Global lifting

Assume now that F is a global field, with its local completions denoted by F_v, and let $\mathbb{A}$ be the ring of adèles over F.

5.1. **Global theta lifting.** Let $\Pi = \otimes \Pi_v$ be the restricted tensor product of minimal representations over all local places v of F, where $\Pi_v \subset C^\infty(\Omega_v)$, as in Theorem 3.1. Every element in Π is a finite linear combination of pure tensors $f = \otimes f_v$, where $f_v = f_v^\circ$ for almost all places v. There is a unique (up to a non-zero scalar) embedding $\theta : \Pi \to \mathcal{A}(G(F)\backslash G(\mathbb{A}))$ of Π into the space of automorphic functions of uniform moderate growth.

We restrict $\theta(f)$ to the dual pair $D^1 \times G_D$ and for every $h \in \mathcal{A}(D^1(F)\backslash D^1(\mathbb{A}))$, consider the function $\Theta(f,h)$ on G_D defined by

$$\Theta(f,h)(g_D) = \int_{D^1(F)\backslash D^1(\mathbb{A})} \theta(f)(g_D g) \cdot \bar{h}(g)\ dg.$$

If this is to be of any use, we require the function $\theta(f)(g_D g) \cdot \bar{h}(g)$ to be of rapid decay on $D^1(F)\backslash D^1(\mathbb{A})$ and of moderate growth on $G_D(F)\backslash G_D(\mathbb{A})$. This condition is clearly satisfied if D^1 is anisotropic or if h is a cusp form. It is also satisfied for a regularized theta lift, to be constructed in the next section. Namely, for any finite place v, we will construct an element z in the Bernstein center of $\mathrm{SL}_2(F_v)$, such that for any $f \in \Pi$, the function $\theta(z \cdot f)(g_1 g)$ is of rapid decay on $D^1(F)\backslash D^1(\mathbb{A})$ and of moderate growth on $G_D(F)\backslash G_D(\mathbb{A})$. (See Proposition 6.1, and the discussion of this particular dual pair thereafter.) In particular, in all these cases, the following integral is convergent:

$$\int_{N_D(F)\backslash N_D(\mathbb{A})} \int_{D^1(F)\backslash D^1(\mathbb{A})} |\theta(z \cdot f)(ng) \cdot \bar{h}(g)|\ dg dn.$$

5.2. **Fourier expansion.** Let $\psi : \mathbb{A}/F \to \mathbb{C}^\times$ be a non-trivial character. Then any $A \in J(F)$ defines a character ψ_A of $N(F)\backslash N(\mathbb{A})$ by $\psi_A(B) = \psi(\mathrm{tr}\,(A \circ B))$ for all $B \in N(\mathbb{A}) \cong J(\mathbb{A})$. For every $\varphi \in \mathcal{A}(G(F)\backslash G(\mathbb{A}))$, let

$$\varphi_A(g) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng) \cdot \overline{\psi_A(n)}\ dn$$

be the Fourier coefficient corresponding to A. We have a Fourier expansion

$$\theta(f)(g) = \theta(f)_0(g) + \sum_{A\in\Omega(F)} \theta(f)_A(g).$$

By uniqueness of local functionals, Theorem 3.1, for every $A \in \Omega(F)$ there exists a non-zero scalar c_A such that

$$\theta(f)_A(g) = c_A \prod_v (g_v \cdot f_v)(A).$$

This formula is particularly useful if $g_v \in M(F_v)$, for then $(g_v \cdot f_v)(A) = \chi_v(g_v) \cdot f_v(g_v^{-1} \cdot A)$ for the character χ_v of $M(F_v)$.

Let ψ_2 and ψ_3 be the rank 2 and 3 characters of $N_D(\mathbb{A})$, as in the local case. Recall that $x \in \mathbb{O}$ is a pair $x = (y, z)$ of elements in D, and $N(x) = N(y) - \lambda N(z)$ for some $\lambda \in F^\times$. Let φ_{N_D,ψ_i} denote the global Fourier coefficient with respect to these two characters. Let $\omega_2(F)$ be the set of all rank one matrices

$$\begin{pmatrix} \pm 1 & x & 0 \\ -x & \pm 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in J(F)$$

such that $x = (0, a)$ and $\lambda N(a) = \pm 1$ (for only one choice of sign, depending on ψ_2) i.e. the 2×2 minor is 0. Then we have a global version of Lemma 4.1.

Lemma 5.1. *For every $f \in \Pi$, $\theta(f)_{N_D,\psi_3} = 0$ and*

$$\theta(f)_{N_D,\psi_2}(g) = \sum_{B\in\omega_2(F)} \theta(f)_B(g).$$

5.3. **Non-vanishing of the theta lift.** We shall prove non-vanishing of the (regularized) theta lift by computing the Fourier coefficient

$$\Theta(f, h)_{N_D,\psi_2}(1) = \int_{N_D(F)\backslash N_D(\mathbb{A})} \int_{D^1(F)\backslash D^1(\mathbb{A})} \theta(f)(ng) \cdot \bar{h}(g) \cdot \bar{\psi}_2(n)\ dgdn.$$

Since this integral is absolutely convergent, we can reverse the order of integration. Then, using Lemma 5.1, we obtain

$$\Theta(f, h)_{N_D,\psi_2}(1) = \int_{D^1(F)\backslash D^1(\mathbb{A})} \sum_{B\in\omega_2(F)} \theta(f)_B(g) \cdot \bar{h}(g)\ dg.$$

Lemma 5.2. *Fix*

$$A = \begin{pmatrix} \pm 1 & x & 0 \\ -x & \pm 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \omega_2(F)$$

where $x = (0, a)$, $a \in D$ satisfies $\lambda N(a) = \pm 1$.

For every automorphic form h and every $f \in \Pi$ we have

$$\int_{D^1(F)\backslash D^1(\mathbb{A})} \sum_{B\in\omega_2(F)} \theta(f)_B(g)\bar{h}(g)\ dg = c_A \int_{D^1(\mathbb{A})} f(g^{-1}A)\bar{h}(g)\ dg$$

where the second integral is absolutely convergent.

Proof. Since $\omega_2(F)$ is a principal homogeneous $D^1(F)$-space, the identity formally follows by unfolding the left hand side and using the formula for $\theta(f)_A(g)$ as a product of local functionals given above. Hence it remains to discuss the issue of absolute convergence.

We may assume $f = \otimes_v f_v$ is a pure tensor. For each place v, observe that if $g \in \mathrm{SL}_2(F_v)$, then $g^{-1}A$ is obtained from A by replacing x by xg. Hence, $g \mapsto g^{-1}A$ gives a closed embedding of $\mathrm{SL}_2(F_v)$ into $J(F_v)$, with image contained in Ω_v. In particular, this image is bounded away from the vertex 0 of the cone Ω_v. Since h is of moderate growth on $\mathrm{SL}_2(F_v)$, to show that the integral in question is absolutely convergent, we need to show that $g \mapsto f_v(g^{-1}A)$ is a Schwartz function on $\mathrm{SL}_2(F_v)$. For this, it suffices to show that as a function on the cone Ω_v, f_v is rapidly decreasing towards infinity, as we shall explain below.

In greater detail, assume first that v is a finite place. Due to N_v-smoothness, $f_v \in \Pi_v$ is supported on a lattice in J_v (and thus vanishes towards infinity). It follows that $g \mapsto f_v(g^{-1}A)$ is a compactly supported function on $\mathrm{SL}_2(F_v)$. Moreover, let S be a finite set of places containing all archimedean places such that for $v \notin S$, all data is unramified: $D(F_v)$ is split, $\lambda \in O_v^\times$, $a \in \mathrm{GL}_2(O_v)$, ψ_v has the conductor O_v, $f_v = f_v^\circ$, and h is right $\mathrm{SL}_2(O_v)$-invariant. Here O_v is the maximal order in F_v. It follows from Theorem 3.2 that $g \mapsto f_v^\circ(g^{-1}A)$ is the characteristic function of $\mathrm{SL}_2(O_v)$ for all $v \notin S$. Thus if we normalize the local measures so that $\mathrm{vol}(\mathrm{SL}_2(O_v)) = 1$ for all $v \notin S$, then

$$\int_{D^1(\mathbb{A})} |f(g^{-1}A)\bar{h}(g)|\, dg = \int_{D^1(\mathbb{A}_S)} |f_S(g^{-1}A)\bar{h}(g)|\, dg$$

where the subscript S denotes the product of the local data over all places $v \in S$.

Consider now the case where v is a real place. We need to show that $C \mapsto f_v(C)$ is of rapid decay in $||C||$, where $C \in \Omega(\mathbb{R})$. To that end, let $m_v \in M(\mathbb{R})$ such that $C = m_v^{-1} \cdot A$. Then, up to a non-zero constant c, independent of C,

$$f_v(C) = c \cdot \chi_v(m_v)^{-1} \cdot \theta(f)_A(m_v)$$

for the character χ_v of $M(\mathbb{R})$. Now observe that m_v can be taken a product of an element k_v in a maximal compact subgroup of $M(\mathbb{R})$ and an element z_v in Z_v, the identity component of the center of $M(\mathbb{R})$. We fix an isomorphism $\nu : Z_v \to \mathbb{R}^+$ such that the conjugation action of $z_v \in Z_v$ on $N(\mathbb{R})$ is given by multiplication by $\nu(z_v)$. Now, in order to prove that f_v is rapidly decreasing towards infinity, we shall give a global argument exploiting the automorphic form $\theta(f)$ (though a local proof is also possible). Namely, it suffices to show that $z_v \mapsto \theta(f)_A(z_v k_v)$ is rapidly decreasing as $\nu(z_v) \to \infty$, with bounds independent of k_v. This can be proved using the usual method of integration by parts, as in [MW, Pg. 30, Lemma].

More precisely, if $X \in J \cong \mathfrak{n}$, then the X-derivative of the character ψ_A is a multiple of ψ_A. Using the definition of the Fourier coefficient and integration by parts, one obtains that

$$\theta(f)_A(z_v) \text{ is a multiple of } (R_Y \cdot \theta(f))_A(z_v k_v) \cdot \nu(z_v)^{-1},$$

where $Y = k_v^{-1} X k_v$ and R_Y denotes the right Y-derivative of the automorphic form $\theta(f)$. We can repeat this procedure to get any negative power of $\nu(z_v)$. The rapid decay follows from the fact that $\theta(f)$ is of uniform moderate growth, and the fact that $Y = k_v^{-1} X k_v$ is a linear combination of vectors in any fixed basis of $\mathfrak{n}$, with bounded coeffecients, as k_v runs over the maximal compact subgroup in $M(\mathbb{R})$.

Finally, suppose that $g \in \mathrm{SL}_2(\mathbb{R})$ belongs to the double coset of the diagonal matrix $\begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}$, $t > 0$, in the Cartan decomposition of $\mathrm{SL}_2(\mathbb{R})$. If we assume for simplicity that $\lambda = 1$, so that a in $x = (0, a)$ can be taken the identity matrix, then $||xg||^2 = t^2 + 1/t^2$ (on the nose) and $||g^{-1}A|| = t + 1/t$. In particular, $t < ||g^{-1}A|| < t + 1$ for $t > 1$. Hence, the rapid decay towards infinity of f_v (as a function on Ω_v) implies that $g \mapsto f_v(g^{-1}A)$ has rapid decay on $\mathrm{SL}_2(\mathbb{R})$, as desired.

□

We are now ready to prove the non-vanishing of the global theta lift. Assume firstly that h is a cusp form. Then we have shown that

$$\Theta(f,h)_{N_D,\psi_2}(1) = \int_{D^1(\mathbb{A}_S)} f_S(g^{-1}A)\bar{h}(g)\ dg$$

for some large finite set of places. Since for every $v \in S$ the local f_v can be an arbitrary compactly supported smooth function on Ω_v the integral will not vanish for some choice of data. Now consider the regularized theta integral $\Theta(z \cdot f, h)$, where h is in an automorphic form, not necessarily cuspidal, and z is an element of the Bernstein center of $\mathrm{SL}_2(F_v)$ (see the next section for the construction of z). The corresponding Fourier coefficient is

$$\Theta(z \cdot f,h)_{N_D,\psi_2}(1) = \int_{D^1(\mathbb{A})} (z \cdot f)(g^{-1}A)\bar{h}(g)\ dg.$$

Let K_v be a sufficiently small open compact subgroup of $\mathrm{SL}_2(F_v)$ such that f_v is K_v-invariant. Then $z \cdot f_v = \alpha \cdot f_v$ where α is a K_v bi-invariant, compactly supported function on $\mathrm{SL}_2(F_v)$. Let $\alpha^\vee(g) = \bar{\alpha}(g^{-1})$ and define $z^\vee \cdot h = \alpha^\vee \cdot h$. Using the convergence guaranteed by Lemma 5.2,

$$\int_{D^1(\mathbb{A})} (z \cdot f)(g^{-1}A)\bar{h}(g)\ dg = \int_{D^1(\mathbb{A})} f(g^{-1}A)(\overline{z^\vee \cdot h})(g)\ dg,$$

and this can again be arranged to be non-zero, provided $z^\vee \cdot h \neq 0$. Hence we have proved the following:

Theorem 5.3. *If h is a non-zero cusp form on $D^1(\mathbb{A})$, then $\Theta(f,h) \neq 0$ for some $f \in \Pi$. If h is an (not necessarily cuspidal) automorphic form such that $z^\vee \cdot h \neq 0$, then $\Theta(z \cdot f, h) \neq 0$ for some $f \in \Pi$.*

Remark: The main reason for introduction of the regularized theta lift is to be able to handle the lift of $h = 1$ in the case when D is split. In this case we can take all data to be simplest possible, i.e. $\lambda = 1$, the matrix A with $a = (0, x)$ with x identity matrix, etc. Then non-vanishing of the theta lift is achieved with the spherical vector f_∞° at any real place. Indeed, if $g \in \mathrm{SL}_2(\mathbb{R})$ belongs to the double coset of the diagonal matrix $\begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}$, with $t > 0$, in the Cartan decomposition of $\mathrm{SL}_2(\mathbb{R})$, then $||xg||^2 = t^2 + 1/t^2$ and $||g^{-1}A|| = t + 1/t$. Write $u = t + 1/t$ so that

$$du = (t - \frac{1}{t})\frac{dt}{t}.$$

Using the formula for the spherical vector given by Theorem 3.3 and the formula for the Haar measure on $\mathrm{SL}_2(\mathbb{R})$ with respect to the Cartan decomposition, we have

$$\int_{\mathrm{SL}_2(\mathbb{R})} f^\circ_\infty(g^{-1}A)\ dg = \int_2^\infty \frac{1}{2}\cdot u^{-1/2}K_{3/2}(u)\ du > 0.$$

It will be interesting to compute the value of this integral.

6. Regularizing Theta

Following some ideas of Kudla and Rallis [KR], the first author introduced in [G] a regularized theta integral for a particular exceptional dual pair. We simplify the arguments so that regularization is now available for a wider class of examples. The notations used in this largely self-contained section will differ from those of the other sections of this paper. We first recall some basic facts about the notions of uniform moderate growth and rapid decay.

6.1. **Moderate growth and rapid decay.** Let k be a number field and let $\mathbb{A}$ denote the corresponding ring of adelés. Let G be a reductive group over k. In order to keep notation simple, we shall assume that G is split with a finite center. Fix a maximal split torus T and a minimal parabolic subgroup P containing T. Let N be the unipotent radical of P. We have a root system Φ, obtained by T acting on the Lie algebra $\mathfrak{g}$ of G and a set of simple roots in Φ corresponding to the choice of P.

If we fix a place v of k, then G_v will denote the group of k_v-points of G. Similarly, we shall use the subscript v to denote various other subgroups of G_v. A smooth function f on $G(\mathbb{A})$ is of uniform moderate growth if there exists an integer m such that for every X in the enveloping algebra of $\mathfrak{g}$ there exists a constant c_X such that

$$|R_X f(g)| \leq c_X ||g||^m$$

where R_X denotes the action of the enveloping algebra on smooth functions obtained by the differentiation from the right and $||g||$ is a height function on G defined in [MW, page 20]. Since there exists a constant c such that $||gh|| \leq c||g||\cdot||h||$ for all $g, h \in G(\mathbb{A})$, it is easy to see that the constants c_X for the right-translates $R_h f$ of f are of moderate growth in h, more precisely, of growth $||h||^{m+d}$ where d is the degree of X.

Now assume that v is a real or complex place of k. Let $P_v = M_v A_v N_v$ be the Langlands decomposition of P_v. For $\epsilon > 0$, let $A_{v,\epsilon}$ be a cone in A_v consisting of $a \in A_v$ such that $\alpha(a) > \epsilon$ for all simple roots α. Let A be the product of the A_v's and let A_ϵ be the product of the $A_{v,\epsilon}$'s over all real and complex places v. Let ω_N be a compact set in $N(\mathbb{A})$ containing the identity element. Let K be a product of maximal compact subgroups K_v of G_v where we have taken K_v to be hyperspecial for all p-adic places. Then

$$S = \omega_N A_\epsilon K$$

is a Siegel domain in $G(\mathbb{A})$. If ω_N is sufficiently large, and ϵ is sufficiently small, then $G(\mathbb{A}) = G(k)S$.

Let Π be an automorphic representation of G. Then any smooth $f \in \Pi$ is of uniform moderate growth. In terms of the Siegel domain S, this means the following. Let $\rho_P : A \to \mathbb{R}^+$ be the modular character. There exists an integer m such that for every X in the enveloping algebra of $\mathfrak{g}$, there exists a constant c_X such that

$$|R_X f(nak)| \leq c_X \cdot \rho_P(a)^m$$

on S, where the constants m and c_X are not necessarily the same, but related to those above.

Now let $Q \supseteq P$ be a maximal parabolic with a unipotent radical $U \subseteq N$, corresponding to a simple root α. We have a standard Levi factor L of Q defined as the centralizer of a fundamental co-character $\chi : \mathbb{G}_m \to T$ (or a power of it). In any case, any element in A_v is uniquely written as a product $\prod_\chi \chi(t_\chi)$, over all fundamental co-characters χ, where $t_\chi \in \mathbb{R}^+$. The element $\prod_\chi \chi(t_\chi)$ is contained in the cone $A_{v,\epsilon}$ if $t_\chi > \epsilon$ for all χ. Let f_U be the constant term of f along U. Then, if f has a uniform moderate growth, by [MW, page 30, Lemma] for every positive integer i, there is a constant c_i such that

$$|(f - f_U)(nak)| \leq c_i \cdot \rho_P(a)^m \cdot \alpha^{-i}(a)$$

on S. In particular, if $f_U = 0$, then f is rapidly decreasing in the variable t_χ. If $f_U = 0$ for all maximal parabolic subgroups, then f is rapidly decreasing on S, and that's how the rapid decrease of cusp forms is established. The proof of [MW, page 30, Lemma] involves integration by parts, so it is easy to see that the constants c_i for the right-translates $R_h f$ of f are of moderate growth in h, more precisely, of the growth $||h||^{m'}$ where m' depends on i: a larger i will demand a larger m'.

We highlight another important issue here. Assume that f belongs to an automorphic representation π. Then a Frechét space topology on π is given by the family of semi-norms

$$||f||_X = \sup_{nak \in S} |R_X f(nak)| \cdot \rho_P(a)^{-m}$$

where m depends on π and works for all X in the enveloping algebra. Then [MW, page 30, Lemma] says that convergence in these seminorms implies convergence in the seminorm

$$\sup_{nak \in S} |(f - f_U)(nak)| \cdot \rho_P(a)^{-m} \cdot \alpha^i(a).$$

This observation will later imply that the regularized theta integral gives a continuous pairing.

6.2. **Restricting to a subgroup.** Let $G_1 \times G_2 \subseteq G$ a dual pair in G. Let T_1 be a maximal split torus in G_1 and fix a minimal parabolic subgroup P_1 containing T_1. Without loss of generality, we can assume that $T_1 \subseteq T$ and $P_1 \subseteq P$. Let $Q_1 \supseteq P_1$ be a maximal parabolic subgroup of G_1. Let $\chi_1 : G_m \to T_1$ be the corresponding fundamental cocharacter (or a multiple of which) so that the centralizer of χ_1 in G_1 is a Levi factor L_1 of Q_1. Assume that:

Hypothesis: *For every fundamental cocharacter χ_1 of G_1, there is a fundamental cocharacter χ of G such that χ_1 is a multiple of χ.*

This hypothesis holds in the following examples:

- the dual pair $G_1 \times G_2 = D^1 \times G_D = \mathrm{SL}_2 \times G_D$ studied in this paper; here $G_1 = \mathrm{SL}_2$ corresponds to the highest root and the highest weight is also a fundamental weight for $\mathbf{E}_7$ (the ambient group G).
- the split exceptional dual pairs in G of type $\mathbf{E}_n$ where one member of the dual pair is the type $\mathbf{G}_2$, see [LS]. In particular, this includes the case $\mathrm{PGL}_3 \times G_2$ treated in [G].

The above hypothesis has the following consequences:

- It implies that the cone $A_{1,\epsilon}$ sits as a subcone of A_ϵ; in fact, it is a direct factor in the above cases. In particular, we have an inclusion of Siegel domains $S_1 \subset S$.
- Given a fundamental cocharacter χ_1 of G_1, the associated fundamental cocharacter χ of G given by the hypothesis corresponds to a simple root and so determines a maximal parabolic subgroup $Q_{\chi_1} = L_{\chi_1} U_{\chi_1}$ of G. In the following, we will sometimes write $U = U_{\chi_1}$ to simplify notation.

Now let v be a p-adic place and z an element of the Bernstein's center of $G_1(k_v)$. Then $z \cdot \Pi$ is naturally a $G_1(\mathbb{A}) \times G_2(\mathbb{A})$-submodule of Π. For a fixed cocharacter χ_1 of G_1, with associated maximal parabolic $Q = LU$, assume that

$$z \cdot \Pi_v \subset \operatorname{Ker}\left(\Pi_v \longrightarrow (\Pi_v)_{U(k_v)}\right).$$

We claim that this implies that $(z \cdot f)_U = 0$ on $G_1(\mathbb{A}) \times G_2(\mathbb{A})$. Indeed, if $g \in G_1(\mathbb{A}) \times G_2(\mathbb{A})$, then

$$(z \cdot f)_U(g) = (R_g(z \cdot f))_U(1) = (z \cdot R_g(f))_U(1) = 0$$

where R_g denotes the right translation by g. Here, the second equality holds since z and R_g commute, and the third equality holds since the projection of $z \cdot \Pi$ on Π_U vanishes. Write $g = g_1 \times g_2 \in G_1(\mathbb{A}) \times G_2(\mathbb{A})$ and assume that $g_1 \in S_1$. Using the hypothesis that $S_1 \subseteq S$ and the estimates for $|R_{g_2}(z \cdot f) - (R_{g_2}(z \cdot f))_U|$ on S from the last subsection, it follows that

$$(z \cdot f)(g_1 \times g_2) = R_{g_2}(z \cdot f)(g_1)$$

is of moderate growth in both variables and in the variable $g_1 \in S_1$, it is rapidly decreasing in the direction of the fundamental co-character χ_1. More precisely, we summarise the discussion in this subsection in the following proposition.

Proposition 6.1. *Assume that:*

(i) *For every fundamental cocharacter χ_1 of G_1, there is a fundamental cocharacter χ of G such that χ_1 is a multiple of χ, which in turn determines a maximal parabolic subgroup $Q_{\chi_1} = L_{\chi_1} U_{\chi_1}$;*

(ii) *One can find an element z in the Bernstein center of $G_1(k_v)$ such that for every fundamental cocharacter χ_1 of G_1, the natural projection of Π_v to $(\Pi_v)_{U_{\chi_1}(k_v)}$ vanishes on $z \cdot \Pi_v$ for every fundamental co-character χ_1 of G_1.*

Then for every integer n, there exists an integer m and a constant c such that

$$|(z \cdot f)(g_1 \times g_2)| \leq c||g_1||^{-n}||g_2||^m$$

for all $g_1 \in S_1$ and $g_2 \in G_2(\mathbb{A})$.

In the context of the above proposition, a small trade-off here is that increasing n can be obtained only by increasing m at the same time. But this is still good enough to define regularized theta lift which produces functions of moderate growth as output. To exploit the proposition, it remains then to construct an appropriate z. We also need to assure that $z \cdot \Pi_v \neq 0$ and this may not be always possible, as will be discussed in the next subsection.

6.3. **Bernstein's center.** We work here locally over a p-adic field. Thus all our groups are local and we drop the subscript v. For simplicity, we shall discuss only the Bernstein center for the Bernstein component containing the trivial representation of G_1.

To that end, let $\hat{T}_1$ be the complex torus dual to T_1, and let $W(G_1)$ be the Weyl group of G_1. The Bernstein's center $Z(G_1)$ of the said component is isomorphic to the algebra of $W(G_1)$-invariant regular functions on $\hat{T}_1$. Similarly, the Bernstein's center $Z(L_1)$ of the Levi factor L_1 is isomorphic to the algebra of $W(L_1)$-invariant regular functions on $\hat{T}_1$. In particular, we have a natural map $j : Z(G_1) \to Z(L_1)$. Let π be a smooth representation of G_1, and let $p : \pi \to \pi_{U_1}$ be the natural projection onto the normalised Jacquet module π_{U_1}. Then, for every $z \in Z(G_1)$ and $v \in \pi$, we have

$$p(z \cdot v) = j(z) \cdot (p(v)).$$

Now let Π be a smooth representation of G. Recall that we want to find a non-zero $z \in Z(G_1)$ such that $z \cdot \Pi$ is in the kernel of the projection of Π onto Π_U. Since Π_U is a quotient of Π_{U_1}, and $j(z)$ is acting on Π_U, we need to find z such that $j(z) = 0$ on Π_U. This is always possible if Π is a finite length G-module, in which case Π_U is a finite length L-module. In particular, the center of L acts finitely on Π_U. Hence, the center of L_1 (= the center of L) acts finitely on Π_U and the $Z(L_1)$-spectrum of Π_U is contained in a proper subvariety of $\hat{T}_1$. In particular, any non-zero $W(G_1)$-invariant function z vanishing on the subvariety will have the desired property that $j(z)$ vanishes on Π_U. Hence, a non-trivial z with the desired property always exists.

A potential trouble is that such a z may kill the whole Π. However, if G is split, G_1 is the smaller member of the dual pair (also split) and Π the minimal representation, then the spherical matrix coefficient Φ of Π, when restricted to G_1, is typically contained in $L^{2-\epsilon}(G_1)$ for some $\epsilon > 0$. (This is easy to check in any given situation, see [LS]). Thus, in such situations, it makes sense to integrate Φ against spherical tempered functions of G_1, i.e. to consider the spherical transform of Φ on $G_1(k_v)$. This integral will be non-zero for almost all tempered spherical functions, hence almost all spherical tempered representations of $G_1(k_v)$ will appear as a quotient of Π. (This argument for the nonvanishing of the theta lift of almost all irreducible spherical tempered representations holds over archimedean fields as well, as we shall exploit in Lemma 7.6 below). Hence, if z kills Π, then z kills all spherical tempered representations of $G_1(k_v)$ and hence z must be equal to 0. Therefore the desired regularization can be carried out in this case.

Let's look at our dual pair $G_1 \times G_2 = \mathrm{SL}_2 \times G_D$ in G, and Π is the minimal representation. The Bernstein's center is

$$Z(G_1) = \mathbb{C}[x^{\pm 1}]^{S_2}$$

where S_2 acts by permuting x and x^{-1}. Let

$$z = (x - q^2)(x^{-1} - q^2)(x - q^3)(x^{-1} - q^3)$$

where q is the order of the residual field. This element satisfies our requirement, since $j(z)$ vanishes on Π_U by Theorem 4.3, and the spherical matrix coefficient of Π is integrable when restricted to SL_2.

6.4. **Global** $\Theta(1)$**.** Let z be the element in the Bernstein center of $G_1 = \mathrm{SL}_2$, as in the previous subsection. We define $\Theta(1)$ as the space of automorphic functions, $g_D \in G_D(\mathbb{A})$,

$$\Theta(f)(g_D) = \int_{D^1(F)\backslash D^1(\mathbb{A})} \theta(z \cdot f)(g_D g)\ dg.$$

where we assume that f_∞ is K_∞-finite. (We assume this finiteness since in the next section we will determine the local lift at real places in the language of $(\mathfrak{g}, K)$-modules.) We want to show that $\Theta(1) \neq 0$, using Theorem 5.3. The input in the theta kernel is $h = 1$, so the first thing is to show that $z^\vee \cdot 1 \neq 0$. In the case at hand, $z^\vee$ is obtained from z by replacing x by x^{-1} in the above expression of z. In particular, $z = z^\vee$. Moreover, z acts on the trivial representation by the scalar obtained by substituting $x = q$, and this is non-zero. It remains to argue that we can arrange f_∞ to be K_∞-finite. This follows by the continuity of the regularized theta integral, which ensures that the non-vanishing for smooth f implies the non-vanishing for K_∞-finite vectors. Alternatively, by the remark following Theorem 5.3, non-vanishing can be achieved with $f_\infty = f_\infty^\circ$ the spherical vector.

7. Correspondence for real groups

In this section, we work over the field $\mathbb{R}$ of real numbers. The goal of this section is to determine $\Theta(1)$ explicitly. For this, we need to consider various cases separately. Indeed, recall that G is arising from an Albert algebra via the Koecher-Tits construction. There are two real forms of octonion algebra, the classical Graves algebra and its split form, and these two algebras can be used to define two Albert algebras of 3×3-hermitian symmetric matrices with coefficients in the octonion algebra. The group G is split or of the relative rank 3 depending on whether the octonion algebra is split or not. Moreover, it will be convenient to work with the simply connected G and $(\mathfrak{g}, K)$-modules corresponding to minimal representations. Then the centralizer of D^1 in G is G_1, a simply-connected cover of G_D in the sense of algebraic groups. An advantage of working with G_1 is that its maximal compact subgroup K_1 is a connected Lie group, so its irreducible representations are parameterized by highest weights. Observe that the Lie group G_D has two topological connected components and G_1 is a two-fold cover of the identity component of G_D. The maximal compact subgroup K_D has two connected components meeting the connected components of G_D. Hence there is a natural bijection between irreducible spherical representation of G_D and G_1, via the pullback by the natural map $G_1 \to G_D$. We shall use this observation, in the case when G is split, to prove that $\Theta(1)$ is an irreducible, spherical $(\mathfrak{g}_D, K_D)$-module by computing its K_1-types.

7.1. **Non-split** $\mathbb{O}$**.** Assume first that $\mathbb{O}$ and hence G is not split. Then the minimal representation of the adjoint group, when restricted to the simply connected G, breaks up as $\Pi = \Pi_{1,0} \oplus \Pi_{0,1}$, a sum of a holomorphic and an anti-holomorphic irreducible representation. This sum is the socle of the degenerate principal series $I(-5)$. The maximal compact subgroup K is of type E_6, and has one dimensional center $\mathrm{U}(1)$ that acts on the Lie algebra $\mathfrak{g}$ with weights -2, 0 and 2. The weight 2 space is a 27 dimensional representation of K. Let ω be its highest weight. Then

$$\Pi_{1,0} = \oplus_{n\geq 0} V_{n\omega}(12)$$

where 12 denotes a twist of the irreducible K-module $V_{n\omega}$ such that $\mathrm{U}(1)$ acts with the weight $2n + 12$ on it.

In this case, D is necessarily non split. Let $K_1 \cong \mathrm{U}_6$ be a maximal compact subgroup of G_1. The socle of $I_D(-1)$, considered a representation of G_1 is a direct sum of three representations $\Sigma_{2,0} \oplus \Sigma_{1,1} \oplus \Sigma_{0,2}$, a holomorphic, a spherical and an anti-holomorphic representation, respectively, [Sa93, Theorem C]. The lowest and the highest K_1-types of $\Sigma_{2,0}$ and $\Sigma_{0,2}$ are one-dimensional with U(1)-weights 12 and -12, respectively. Observe that $\Sigma_{2,0} \oplus \Sigma_{0,2}$ has, up to an isomorphism, unique structure of $(\mathfrak{g}_D, K_D)$-module. One has:

Theorem 7.1. *If $\mathbb{O}$ is non-split (so G is not split), we have*

$$\Pi_{1,0}^{D^1} \cong \Sigma_{2,0} \quad \text{and} \quad \Pi_{0,1}^{D^1} \cong \Sigma_{0,2}.$$

In particular, $\Theta(1) = \Sigma_{2,0} \oplus \Sigma_{0,2}$, as $(\mathfrak{g}_D, K_D)$-modules.

Proof. Since $\Pi_{1,0}$ is unitarizable and U(1)-admissible, the restriction to $\mathfrak{g}_1$ is a direct sum of irreducible lowest weight representations. The minimal type of $\Pi_{1,0}$ generates an irreducible lowest weight $(\mathfrak{g}_1, K_1)$-module, with the minimal U_6-type $\det^2$, i.e. U(1)-weight 12. Thus $\Sigma_{2,0} \subseteq \Pi_{1,0}^{D^1}$. The infinitesimal character of $\Sigma_{2,0}$ is $(3, 2, 1, 0, -1, -2)$, in terms of the standard realization of the D_6 root system. If the inclusion $\Sigma_{2,0} \subseteq \Pi_{1,0}^{D^1}$ is strict, then $\Pi_{1,0}^{D^1}$ contains another lowest weight representation with the same infinitesimal character. There is precisely one other irreducible lowest weight $(\mathfrak{g}_1, K_1)$-module with this infinitesimal character, with the minimal U_6-type $\det^3$, i.e. U(1)-weight 18. Thus the number of irreducible summands in $\Pi_{1,0}^{D^1}$ is bounded by the dimension of $\mathrm{SU}_6 \times D^1$-invariants in $\Pi_{1,0}$. By the Cartan-Helgason theorem, a finite dimensional irreducible representation of E_6 has a line fixed by $A_5 \times A_1$ if and only if it is self-dual. It follows that the space of $\mathrm{SU}_6 \times D^1$-invariants in $\Pi_{1,0}$ is one dimensional, the only contribution coming from the trivial K-type.

□

7.2. **Split $\mathbb{O}$ but nonsplit D.** We move on to the case when $\mathbb{O}$ and hence G is split. Let K be a maximal compact subgroup of G, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition of the complexification of the Lie algebra of G. Then $\mathfrak{k}$ is isomorphic to $\mathfrak{sl}_8$. Fixing this isomorphism, we see that as a $K \cong \mathrm{SU}_8/\mu_2$-module, $\mathfrak{p}$ is isomorphic to V_{ω_4}, where ω_4 is the 4-th fundamental weight. The minimal representation Π is a direct sum of K-types $V_{n\omega_4}$, where $n = 0, 1, 2, \ldots$.

We have two cases depending on D. Assume in this subsection that D is a division algebra. In this case $D^1 \cong \mathrm{SU}_2$ is compact, and embeds into SU_8 as a 2×2 block. The centralizer of SU_2 in $K = \mathrm{SU}_8/\mu_2$ is $K_1 \cong \mathrm{U}_6$. The minimal representation Π decomposes discretely when restricted to this dual pair. A simple application of the Gelfand-Zetlin rule shows that the K_1-types of $\Theta(1)$ are multiplicity free and the highest weights of the K_1-types which occur are

$$(x, x, 0, 0, y, y)$$

where $x \geq 0 \geq y$ are any two integers. Here we are using the standard description of highest weights for U_6 by 6-tuples of non-increasing integers. But these are precisely the K_1-types of the spherical submodule of $I_D(-1)$, i.e. the G_1-constituent $\Sigma_{1,1}$ in [Sa93, Theorem C]. In view of the map $\Theta(1) \to I_D(-1)$ of $(\mathfrak{g}_D, K_D)$-modules, this proves

Theorem 7.2. *When $\mathbb{O}$ is split but D is non-split, one has an isomorphism of spherical $(\mathfrak{g}_D, K_D)$-modules*

$$\Theta(1) = \Pi^{D^1} \cong \Sigma_{1,1}.$$

7.3. **Split $\mathbb{O}$ and split D.** This is the most involved case. Let (e,h,f) be an $\mathfrak{sl}_2$-triple spanning the complexified Lie algebra of $D^1 = \mathrm{SL}_2$. After conjugating by G, if necessary, we can assume that the triple is stable under the Cartan involution. Then $e \in \mathfrak{p}$ is a highest weight vector for the action of K, and $h \in \mathfrak{k}$. Let $\Theta(1)$ be the maximal quotient of the $(\mathfrak{g}, K)$-module of the minimal representation such that the $\mathfrak{sl}_2$ triple acts trivially.

Theorem 7.3. $\Theta(1)$ *is irreducible and isomorphic to the unique submodule Σ of $I_D(-1)$, which is a spherical representation.*

The proof of this result will take the rest of this section. After conjugating by K, if necessary, we can assume that

$$h = \frac{1}{2}\begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& -1 &&& \\ &&&&& -1 && \\ &&&&&& -1 & \\ &&&&&&& -1 \end{pmatrix} \in \mathfrak{sl}_8.$$

Let G_1 be the centralizer of the $\mathfrak{sl}_2$-triple in G. It is a group isomorphic to $\mathrm{Spin}(6,6)$ (recall that we are assuming G is simply-connected in this section, unlike the discussion in Section 4.2). Let $\mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}_1$ be the corresponding Cartan decomposition. Then $\mathfrak{k}_1 \cong \mathfrak{sl}_4 \oplus \mathfrak{sl}_4$ sitting block diagonally in $\mathfrak{sl}_8$. The centralizer of h in SU_8/μ_2 is

$$K_1 = \mathrm{SU}_4 \times \mathrm{SU}_4/\Delta\mu_2$$

and this confirms that G_1 is simply connected (as an algebraic group in a given (non-hermitian) isogeny class is determined by its maximal compact subgroup).

Let Π be the $(\mathfrak{g}, K)$-module corresponding to the minimal representation of G. Then, as a K-module,

$$\Pi = \oplus_{n \geq 0} V_{n\omega_4}.$$

We shall also need the following facts about the action of e on Π. From the formula for the tensor product $V_{\omega_4} \otimes V_{\mathbf{n}\omega_4}$ it follows that

$$e \cdot V_{n\omega_4} \subseteq V_{(n-1)\omega_4} \oplus V_{(n+1)\omega_4}.$$

Since Π is not a highest weight module, by [V, Lemma 3.4], e is injective on Π. The same results hold for f.

Let π be an irreducible $\mathfrak{sl}_2$-module such that h acts semi-simply and integrally. Let $\Theta(\pi)$ be the big theta lift of π; it is a $(\mathfrak{g}_1, K_1)$-module. We shall now partially determine the structure of K_1-types of $\Theta(\pi)$. In order to state the result, we need some additional notation.

A highest weight μ for SU_4 is represented by a quadruple (x,y,z,u) of integers, such that $x \geq y \geq z \geq u$, and it is determined by the triple

$$\alpha = x-y,\ \beta = y-z,\ \gamma = z-u$$

of non-negative integers.

Proposition 7.4. *Let $V \otimes U$ be a $K_1 \cong \mathrm{SU}_4 \times \mathrm{SU}_4/\Delta\mu_2$-type of $\Theta(\pi)$. Then $U \cong V^*$, the dual representation of V, and the multiplicity of $V \otimes V^*$ in $\Theta(\pi)$ is at most one. If $\pi = 1$, the trivial representation, and μ is the highest weight of V, then $\alpha = \gamma$.*

Proof. We need the following lemma which can be easily deduced from the Gelfand-Zetlin branching rule.

Lemma 7.5. *The restriction of $V_{n\omega_4}$ to $\mathfrak{sl}_4 \oplus \mathfrak{sl}_4 \oplus \mathbb{C}h$ is multiplicity free and given by*

$$V_{n\omega_4} = \oplus_{n\geq x\geq y\geq z\geq u\geq 0} V_\mu \otimes V_\mu^* \otimes \mathbb{C}(m)$$

where μ is represented by the quadruple (x,y,z,u) and h acts on $\mathbb{C}(m)$ by the integer $m = x+y+z+u-2n$.

It follows from the lemma that the only K_1-types appearing in the restriction of Π are isomorphic to $V \otimes V^*$, as claimed. In order to prove multiplicity one in $\Theta(\pi)$, we proceed as follows.

Let m be an integer appearing as an h-type in π. Let Ω be the Casimir element for $\mathfrak{sl}_2$ and let $\chi : \mathbb{C}[\Omega] \to \mathbb{C}$ be the central character of π. Let $\Pi(\mu,m)$ be the maximal subspace of Π such that h acts as the integer m and $\mathfrak{sl}_4 \oplus \mathfrak{sl}_4$ as a multiple of $V_\mu \otimes V_\mu^*$. Note that $\Pi(\mu,m)$ is naturally a $\mathbb{C}[\Omega]$-module, and it suffices to show that the maximal quotient of $\Pi(\mu,m)$ such that $\mathbb{C}[\Omega]$ acts on it by χ is isomorphic to $V_\mu \otimes V_\mu^*$ as an $\mathfrak{sl}_4 \oplus \mathfrak{sl}_4$-module. We have a canonical isomorphism

$$\Pi(\mu,m) \cong (V_\mu \otimes V_\mu^*) \otimes \mathrm{Hom}_{\mathfrak{sl}_4\oplus\mathfrak{sl}_4}(V_\mu \otimes V_\mu^*, \Pi(m))$$

and $\mathbb{C}[\Omega]$ acts on

$$\mathrm{Hom}_{\mathfrak{sl}_4\oplus\mathfrak{sl}_4}(V_\mu \otimes V_\mu^*, \Pi(m)) = \oplus_{n\geq 0} \mathrm{Hom}_{\mathfrak{sl}_4\oplus\mathfrak{sl}_4}(V_\mu \otimes V_\mu^*, V_{n\omega_4}(m))$$

Now notice that, given μ and m, $\mathrm{Hom}_{\mathfrak{sl}_4\oplus\mathfrak{sl}_4}(V_\mu \otimes V_\mu^*, V_{n\omega_4}(m)) \neq 0$ for only one parity of n. Furthermore, if this space is non-zero for some n, then it is non-zero for $n+2$, as μ is also represented by $(x+1,y+1,z+1,u+1)$ and

$$m = x+y+z+u-2n = x+1+y+1+z+1+u+1-2(n+2).$$

Let n_0 be the first integer such that $\mathrm{Hom}_{\mathfrak{sl}_4\oplus\mathfrak{sl}_4}(V_\mu \otimes V_\mu^*, V_{n_0\omega_4}(m)) \neq 0$ and let T_0 be a generator of this one-dimensional space. We then have a natural map

$$A : \mathbb{C}[\Omega] \cdot T_0 \to \mathrm{Hom}_{\mathfrak{sl}_4\oplus\mathfrak{sl}_4}(V_\mu \otimes V_\mu^*, \Pi(m)).$$

Lemma 7.6. *The map A is an isomorphism.*

Proof. Let i be a non-negative integer. Let $\mathbb{C}[\Omega]_i$ be the space of polynomials of degree $\leq i$, and let

$$\mathrm{Hom}_{\mathfrak{sl}_4\oplus\mathfrak{sl}_4}(V_\mu \otimes V_\mu^*, \Pi(m))_i = \oplus_{j=0}^{i}\mathrm{Hom}_{\mathfrak{sl}_4\oplus\mathfrak{sl}_4}(V_\mu \otimes V_\mu^*, V_{(n_0+2j)\omega_4}(m)).$$

These two spaces have dimension $i+1$ and define filtrations of $\mathbb{C}[\Omega]$ and $\mathrm{Hom}_{\mathfrak{sl}_4\oplus\mathfrak{sl}_4}(V_\mu \otimes V_\mu^*, \Pi(m))$ as i increases. Since Ω has degree 2, as an element of the enveloping algebra of $\mathfrak{g}$, the map A preserves the two filtrations. Thus, in oder to prove the claim, it suffices to show that A is injective.

But if it is not, then there would be a polynomial $p(\Omega)$ acting trivially on $V_\mu \otimes V_\mu^* \subseteq V_{n_0\omega_4}(m)$. Under the action of $\mathfrak{sl}_2$, this subspace would generate a finite length representation $F_0 \subset \Pi$ of $\mathfrak{sl}_2$. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, and let $U_n(\mathfrak{g}) \subset U(\mathfrak{g})$ be the PBW filtration. We have $\Pi = U(\mathfrak{g}) \cdot F_0$, since Π is irreducible $\mathfrak{g}$-module. Hence Π is a union of $F_n = U_n(\mathfrak{g}) \cdot F_0$. Now observe that each F_n is a finite length $\mathfrak{sl}_2$-module. Hence there could be only countably many irreducible $\mathfrak{sl}_2$-modules appearing as quotients of Π. But this contradicts the fact that almost all spherical tempered representations of $\mathfrak{sl}_2$ are quotients, as discussed in Section 6.3. The lemma is proved. □

Lemma 7.6 implies that

$$\Pi(\mu, m) \cong (V_\mu \otimes V_\mu^*) \otimes \mathbb{C}[\Omega]$$

as $\mathbb{C}[\Omega]$-modules. Hence, if we fix a character χ of $\mathbb{C}[\Omega]$, the maximal quotient of $\Pi(\mu, m)$ such that $\mathbb{C}[\Omega]$ acts by χ is isomorphic to $V_\mu \otimes V_\mu^*$. This proves that $\Theta(\pi)$ has multiplicity free K_1-types.

We proceed to narrow down the K_1-types appearing in $\Theta(1)$. For every μ, the action of e on Π gives an injective map

$$e : \Pi(\mu, -2) \to \Pi(\mu, 0).$$

Lemma 7.7. *If $e : \Pi(\mu, -2) \to \Pi(\mu, 0)$ is bijective, then $V_\mu \otimes V_\mu^*$ is not a K_1-type of $\Theta(1)$.*

Proof. The image of e is necessarily contained in the kernel of the natural surjective map $\Pi(\mu, 0) \to \Theta(1)(\mu)$. Hence the lemma follows. □

Consider the filtration $\Pi(\mu, m)_i = \oplus_{n\leq i} V_{n\omega_4}(\mu, m)$ of $\Pi(\mu, m)$. Then we have an injective map

$$e : \Pi(\mu, -2)_i \to \Pi(\mu, 0)_{i+1}$$

for all i. Hence, if the dimensions of the two spaces are equal for all i, then e is bijective. This will happen precisely when $V_\mu \otimes V_\mu^*$ occurs in $V_{n\omega_4}(-2)$ but not in $V_{(n-1)\omega_4}(0)$, for some n. The occurrence in $V_{n\omega_4}(-2)$ implies that there exists a unique quadruple (x, y, z, u) representing μ such that

$$n \geq x \geq y \geq z \geq u \geq 0 \text{ and } x + y + z + u - 2n = -2$$

Then $V_\mu \otimes V_\mu^*$ occurs in $V_{(n-1)\omega_4}(0)$ if and only if

$$n - 1 \geq x \geq y \geq z \geq u \geq 0$$

i.e. $n > x$. Thus, if $n = x$, then $V_\mu \otimes V_\mu^*$ does not appear in $\Theta(1)$.

Let's see what this means in terms of α, β and γ. We have to find n such that μ is represented by

$$(x, y, z, u) = (n, n - \alpha, n - \alpha - \beta, n - \alpha - \beta - \gamma).$$

Since the last entry must be non-negative, we have $n \geq \alpha + \beta + \gamma$. On the other hand, h has to act as -2, hence $x + y + z + u - 2n = -2$ and this is equivalent to $2n = 3\alpha + 2\beta + \gamma - 2$. Combining with the previous inequality, we obtain $\alpha \geq \gamma + 2$. Hence the types with μ such that $\alpha \geq \gamma + 2$ do not appear in $\Theta(1)$. Replacing the role of e with f, a similar argument shows that the types such that $\gamma \geq \alpha + 2$ do not appear either. Hence $|\alpha - \gamma| \leq 1$ for all types that appear in $\Theta(1)$. Since $\alpha \equiv \gamma \pmod 2$ for any type in $\Pi(0)$, $\alpha = \gamma$ for all types that appear in $\Theta(1)$. This completes the proof of Proposition 7.4. □

The types of $\Theta(1)$, as described in Proposition 7.4, are the same as the types of the spherical rank-2 submodule in $I_D(-1)$ by [Sa95, Theorem 4B]. This proves Theorem 7.3.

8. Siegel-Weil formula

We are now ready to prove the desired Siegel-Weil formula (Theorem 1.2 in the introduction). Assume that F is a totally real global field and D a quaternion algebra over F.

8.1. **The representation $\Theta(1)$.** We have shown that the global (regularized) theta lift $\Theta(1)$ is a non-zero automorphic representation of $G_D(\mathbb{A})$. We have also studied the abstract local theta lift of the trivial representation of D^1 to G_D. The following summarizes what we have shown:

Proposition 8.1. *(i) The automorphic representation $\Theta(1)$ is irreducible and occurs with multiplicity one in the space of automorphic forms of G_D;*

(ii) For every p-adic place v of F, the local component $\Theta(1)_v$ is isomorphic to the unique irreducible quotient of the local degenerate principal series $I_D(1)$.

(iii) For every real place v of F, the local component $\Theta(1)_v$ is an irreducible quotient of $I_D(1)$ as described in Theorems 7.1, 7.2 and 7.3.

Proof. Indeed, we have shown that the abstract local theta lift $\Theta(1_v)$ is irreducible. Hence the global $\Theta(1)$ is an irreducible automorphic representation. The fact that $\Theta(1)$ has multiplicity one in the space of automorphic forms follows by [KS15, Theorem 1.1]. Note that the required conditions, as spelled out in the introduction of [KS15], are satisfied by the recent work of Möllers and Schwarz [MS17]. □

8.2. **A Siegel-Weil formula.** For a flat section $\Phi \in I_D(s)$, let $E_D(s, \Phi)$ be the associated Eisenstein series. Then $E_D(s, \Phi)$ has at most simple poles at $s = 1, 3$ or 5 and the corresponding residual representations are completely described in [HS, Theorem 6.4]. Set

$$\mathcal{E} = \{\mathrm{Res}_{s=1} E_D(s, \Phi) : \Phi \in I_D(s)\},$$

We can now prove Theorem 1.2 in the introduction (which we restate here):

Theorem 8.2. *Let F be a totally real global field and D a quaternion algebra over F. Then we have the following identity in the space of automorphic representations $G_D(\mathbb{A})$,*

$$\mathcal{E} = \oplus_{i:D\to\mathbb{O}}\Theta(1),$$

where the sum is taken over all isomorphism classes of embeddings $i : D \to \mathbb{O}$ into octonion algebras over F.

Proof. Comparing Proposition 8.1 with [HS, Theorem 6.4], one sees that $\Theta(1)$ is isomorphic, as an abstract representation, to a summand of $\mathcal{E}$. In view of the multiplicity one result in Proposition 8.1(i), it follows that $\Theta(1)$ is equal to that irreducible summand, as a subspace of the space of automorphic forms.

Now recall that the dual pair $D^1 \times G_D$ arises from an embedding of D into an octonion algebra $\mathbb{O}$. Every such embedding is unique up to conjugacy by $\mathrm{Aut}(\mathbb{O})$. However, given D there are multiple octonion algebras over F containing D. An isomorphism class of octonion algebras $\mathbb{O}$ over F is specified by the isomorphism class of its local completions $\mathbb{O}_v$ for real places v. At each real place, we have two choices: the classical octonion algebra and its split form. But D_v embeds into both if and only if it is a quaternion division algebra. Hence the number of octonion algebras over F containing D is is 2^m where m is the number of real places v such that D_v is the quaternion algebra. Now, by an easy check left to the reader, non-isomorphic $\mathbb{O}$ give non-isomorphic $\Theta(1)$. Moreover, using our description of $\Theta(1)$ in Proposition 8.1 and [HS, Theorem 6.4], one sees that all those possible $\Theta(1)$ sum up to $\mathcal{E}$. This proves the theorem. □

9. Spin L-function

To complete the proof of the main result of this paper, i.e. Theorem 1.1 in the introduction, the remaining ingredient we need is a Rankin-Selberg integral for the degree 8 Spin L-function for cuspidal representations of PGSp_6 which was discovered by A. Pollack [P]. However, since the paper [P] works over $\mathbb{Q}$ whereas we are working over a general number field F, we recall some details here for the sake of completeness.

9.1. **Global zeta integrals.** Suppose π is a cuspidal automorphic representation of $\mathrm{PGSp}_6(\mathbb{A})$. Let U be the unipotent radical of the Siegel maximal parabolic subgroup of $\mathrm{PGSp}_6(F)$. Let J_F be the Jordan algebra of 3×3 symmetric matrices with coefficients in F. Then $U \cong J_F$ and any $T \in J_F \cong \bar{U}_F$ defines an additive character $\phi_T : U(\mathbb{A})/U(F) \to \mathbb{C}^\times$. Since π is cuspidal, there exists a non-degenerate T (i.e. $\det(T) \neq 0$) such that the global Fourier coefficient ϕ_T is a non-zero function on $\mathrm{PGSp}_6(\mathbb{A})$ for any $\phi \in \pi$. We fix such a T (which depends on π) in the following.

The non-degenerate orbits on $U(F) \cong J_F$, under the action of the Siegel Levi factor in $\mathrm{PGSp}_6(F)$, are parameterized by quaternion algebras over F. So let D be the algebra corresponding to T. Let $\tilde{G}_D$ be the reductive group of type $\mathbf{D}_6$ acting on W_D, as in Subsection 2.3. We shall assume that $\tilde{G}_D$ acts from the right on W_D. Let $\omega \subset W_D$ be the $\tilde{G}_D$-orbit of $(1, 0, 0, 0)$, i.e. the orbit consisting of highest weight vectors. Following Pollack [P], for every

Schwartz function $\Phi = \otimes_v \Phi_v$ on $W_D(\mathbb{A})$ define an Eisenstein series on $\tilde{G}_D(\mathbb{A})$ by

$$E_\Phi(g) = \sum_{x \in \omega(F)} \Phi(xg).$$

Recall that $\mathrm{GSp}_6 \subseteq \tilde{G}_D$ and let ν be the isogeny homomorphism of GSp_6. Define a global zeta integral

$$Z(\phi, \Phi, s) = \int_{\mathrm{GSp}_6(F)\backslash \mathrm{GSp}_6(\mathbb{A})} \phi(g) \cdot E_\Phi(g) \cdot |\nu(g)|^s \, dg$$

for $\phi \in \pi$ and Φ as above. This integral is absolutely convergent for $s \in \mathbb{C}$ with sufficiently large real part. After integrating over the center of $\mathrm{GSp}_6(\mathbb{A})$, we see that

$$Z(\phi, \varphi, s) = \int_{\mathrm{PGSp}_6(F)\backslash P\mathrm{GSp}_6(\mathbb{A})} \phi(g) \cdot E(\Phi_s, g) \, dg$$

where $\Phi_s \in I_D(2s-5)$, and $s \mapsto \Phi_s$ is a holomorphic section for $s > 0$. Observe that the meromorphic continuation of $E(\Phi_s, g)$ gives a meromorphic continuation of $Z(\phi, \Phi, s)$.

9.2. **Unfolding.** Let $V \subset U$ be the codimension one subgroup such that the character ψ_T is trivial on $V(\mathbb{A})$. There is a $w_T \in \omega(F)$, contained in the third summand of W_D, such that the stabilizer of w_T in Sp_6 is $V \subset U$ (see [P, Proposition 5.5]). The integral unfolds into

$$Z(\phi, \Phi, s) = \int_{V(\mathbb{A})\backslash \mathrm{GSp}_6(\mathbb{A})} \phi_T(g) \Phi(w_T g) |\nu(g)|^s \, dg.$$

Furthermore, by Therem 9.4 in the first arXiv version of the paper [P] (more precisely in arXiv:1506.03406v1), for a sufficiently large set of places S, including the set S_∞ of all real places,

$$Z(\phi, \varphi, s) = L^S(s-2, \pi, \mathrm{Spin}) \cdot c(s) \cdot \int_{V(\mathbb{A}_S)\backslash \mathrm{GSp}_6(\mathbb{A}_S)} \phi_T(g) \Phi_S(w_T g) |\nu(g)|^s \, dg.$$

Here $c(s)$ denotes a product of partial Dedekind zeta functions that we have omitted writing down since they do not affect analytic properties of $Z(\phi, \varphi, s)$ at our point of interest $s = 3$. We note that Pollack works over $F = \mathbb{Q}$. However, he has kindly informed us that

- The unfolding of the integral representation works over any number field.
- The unramified computation is valid for any non-archimedean place v away from 2 such that D_v is split. The proof goes through line by line, if one makes the following changes of notation: every time p is used as a uniformizer, replace p with ϖ; every time p is used as a magnitude, replace p with q.

In other words, the above identity holds for S containing all real places, places of even residual characteristic and places where D is ramified.

9.3. **Nonvanishing.** The following technical result was contained in an earlier version of Pollack's paper. However, as this particular version is no longer publicly available (even on the arXiv), we reproduce the proof for the sake of completeness.

Lemma 9.1. *Let $s_0 \in \mathbb{C}$. For some data ϕ and Φ_S,*

$$Z(\phi, \Phi_S, s) = \int_{V(\mathbb{A}_S)\backslash \mathrm{GSp}_6(\mathbb{A}_S)} \phi_T(g)\Phi_S(w_T g)|\nu(g)|^s \ dg$$

extends to a meromorphic function on $\mathbb{C}$ which is non-vanishing at s_0.

Proof. Observe that the meromorphic continuation is clear, since the global zeta integral has a meromorphic continuation, and so does the partial L-function, since it appears in the constant term of an Eisenstein series on the exceptional group F_4 (á la Langlands-Shahidi theory). It remains to deal with non-vanishing.

Let $v \in S$. Consider the local version of the zeta integral:

$$Z(\phi, \Phi_v, s) = \int_{V_v\backslash \mathrm{GSp}_6(F_v)} \phi_T(g) \cdot \Phi_v(w_T g) \cdot |\nu(g)|^s \ dg.$$

The stabilizer of w_T in GSp_6 is $V \rtimes A$, where A is a one-dimensional torus that we can identify with GL_1 using the isogeny character ν. Thus GSp_6 is a semi-direct product of A and Sp_6, and we can write

$$Z(\phi, \Phi_v, s) = \int_{V_v\backslash \mathrm{Sp}_6(F_v)} \Phi_v(w_T g) \int_{A_v} \phi_T(ag) \cdot |\nu(a)|^{s-5} \ da\, dg$$

for an invariant measure da on A_v. Let us set

$$H(\phi, s) = \int_{A_v} \phi_T(a) \cdot |\nu(a)|^{s-5} \ da.$$

If h is a Schwartz function on U_v, let

$$\hat{h}(a) = \int_{U_v} h(u)\psi_T(aua^{-1}) \ du$$

where $a \in A_v$. Since $\phi_T(aug) = \psi_T(aua^{-1}) \cdot \phi_T(ag)$ for $u \in U_v$, it follows that

$$H(h * \phi, s) = \int_{A_v} \hat{h}(a) \cdot \phi_T(a) \cdot |\nu(a)|^{s-5} \ da.$$

Since $\hat{h}$ can be any compactly supported function on A_v, the integral can be arranged to be non-zero for any s_0. In fact, if v is a finite place, then $\hat{h}$ can be picked so that the integral is 1 for all s. Thus, if v is a finite place, we can assume that ϕ has been chosen so that $H(\phi, s) = 1$, for all s.

Next, there exists a compactly supported function φ on $\mathrm{Sp}_6(F_v)$ such that $\varphi * \phi = \phi$. Observe that

$$H(\varphi * \phi, s) = \int_{V_v\backslash \mathrm{Sp}_6(F_v)} \varphi'(g) \int_{A_v} \phi_T(ag) \cdot |\nu(a)|^{s-5} \ da\, dg$$

where φ' is a smooth compactly supported function on $V(F_v)\backslash \mathrm{Sp}_6(F_v)$ defined by

$$\varphi'(g) = \int_{V(F_v)} \varphi(ug) \ du.$$

Using the Iwasawa decompostion of $\mathrm{Sp}_6(F_v)$, it is not difficult to see that the map $g \mapsto w_T g$ gives a locally closed embedding (i.e. an immersion) of $V_v \backslash \mathrm{Sp}_6(F_v)$ into $W_D(F_v)$, with the closure of the image containing the extra point 0. Hence, any smooth compactly supported function on $V_v \backslash \mathrm{Sp}_6(F_v)$ is the restriction of a smooth compactly supported function on $W_D(F_v)$. In particular, we can pick Φ_v such that $\Phi_v(w_T g) = \varphi'(g)$, for all $g \in \mathrm{Sp}_6(F_v)$. Then, with this choice of Φ_v, one has

$$Z(\phi, \Phi_v, s) = H(\phi, s) = 1.$$

It follows that

$$Z(\phi, \Phi_S, \varphi, s) = Z(\phi, \Phi_\infty, s)$$

for some choice of data.

Let $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$. Assume as above that ϕ has been chosen so that $H(\phi, s_0) \neq 0$. While we perhaps cannot write $\phi = \varphi * \phi$, for a compactly supported function on $\mathrm{Sp}_6(F_\infty)$, by the well known theorem of Dixmier-Malliavin, there exist finitely many compactly supported functions φ_i on $\mathrm{Sp}_6(F_\infty)$ such that $\phi = \sum_i \varphi_i * \phi_i$ for some ϕ_i. Then, as in the finite place case, there exists compactly supported Φ^i_∞ such that

$$\sum_i Z(\phi, \Phi^i_\infty, s) = H(\phi, s).$$

Since $H(\phi, s_0) \neq 0$, we see that $Z(\phi, \Phi^i_\infty, s_0) \neq 0$ for some i. This completes the proof of the lemma. □

10. Applications to Functoriality

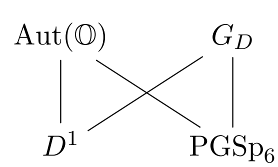
Finally, we are ready to assemble the various ingredients and complete the proof of Theorem 1.1 (which we reproduce here):

Theorem 10.1. *Suppose that π is a cuspidal automorphic representation of PGSp_6 such that $L^S(s, \pi, \mathrm{Spin})$ has a pole at $s = 1$. Then there exists an octonion algebra $\mathbb{O}$ over F and a cuspidal automorphic representation π' of $\mathrm{Aut}(\mathbb{O})$ such that the Satake parameters of π' are mapped by ι to those of π (i.e. π is a weak functorial lift of π').*

If the cuspidal representation π of PGSp_6 is tempered, then the following are equivalent:

(a) *For almost all places v, the Satake parameter s_v of π_v is contained in $\iota(G_2(\mathbb{C}))$.*
(b) *There exists an octonion algebra $\mathbb{O}$ over F and a cuspidal automorphic representation π' of $\mathrm{Aut}(\mathbb{O})$ such that π is a weak functorial lift of π'.*

Proof. As explained in the introduction, we shall make use of the following see-saw dual pair in G:



Let π be an irreducible cuspidal automorphic representation of PGSp_6 and consider its global theta lift π' on G_2. It can be shown (by a standard computation of the constant term of the global theta lift) that π' is contained in the space of cusp forms on G_2. This was explained in [GJ, Theorem 3.1], noting that the genericity assumption on π was not needed there. See also [GG09, Proposition 5.2] (note though that there is a typo in the first paragraph of the proof of [GG09, Proposition 5.2]: the word "nonzero" should be "zero").

Now suppose that the partial (degree 8) spin L-function $L^S(s,\pi,\mathrm{Spin})$ of π has a pole at $s=1$. Then, by Lemma 9.1, it follows that $\mathrm{Res}_{s=3}Z(\phi,\Phi,s)$ is non-zero, for some $\phi\in\pi$. At this point we note that Pollack has a slightly different choice of the parameter of the Eisenstein series: his parameter s' and our s are related by $s=2s'-5$. Hence the integral of ϕ against some residue $\mathrm{Res}_{s=1}E_D(s,\Phi)$ is non-zero. Since the space of residues at $s=1$ is invariant under the complex conjugation, it follows that the integral of $\bar\phi$ against some residue $\mathrm{Res}_{s=1}E_D(s,\Phi)$ is non-zero. By the Siegel-Weil formula (Theorem 8.2), it follows that

$$\int_{\mathrm{PGSp}_6(F)\backslash\mathrm{PGSp}_6(\mathbb{A})}\bar\phi(g)\cdot\left(\int_{D^1(F)\backslash D^1(\mathbb{A})}\theta(f)(gh)\,dh\right)\,dg\neq 0$$

for some $\mathbb{O}\supset D$, $f\in\Pi_{\mathbb{O}}$ and $\phi\in\pi$, where $\theta(f)$ is rapidly decreasing on $D^1(F)\backslash D^1(\mathbb{A})$ and of moderate growth on $\mathrm{PGSp}_6(\mathbb{A})$. Exchanging the order of integration, we deduce that the global theta lift of π to $\mathrm{Aut}(\mathbb{O})$ is nonzero, i.e.

$$\phi'(h)=\int_{\mathrm{PGSp}_6(F)\backslash\mathrm{PGSp}_6(\mathbb{A})}\theta(f)(gh)\bar\phi(g)\;dg$$

is a non-zero function of uniform moderate growth on $\mathrm{Aut}(\mathbb{O})\backslash\mathrm{Aut}(\mathbb{O}\otimes_F\mathbb{A})$.

It is given that ϕ is an eigenfunction for the center of the enveloping algebra of $\mathrm{PGSp}_6(F_v)$ for every real place v of F. By [HPS] and [Li99], for every element z in the center of the enveloping algebra of $\mathrm{PGSp}_6(F_v)$, there exists an element z' in the center of the enveloping algebra of $\mathrm{Aut}(\mathbb{O}_v)$ such that $z=z'$ when acting on the minimal representation. In particular, $z'\cdot f=z\cdot f$. Thus ϕ' is an eigenfunction for the center of the enveloping algebra of $\mathrm{Aut}(\mathbb{O}_v)$ for every real place v of F. (At this point we use that ϕ has rapid decrease to justify that differentiation of f can be moved over to differentiation of ϕ.)

Similarly, it is given that ϕ is an eigenfunction for the Hecke algebra for almost all finite places. But so is ϕ' by matching of Hecke operators under the exceptional theta correspondences [SW15]. Moreover, by [SW15, Theorem 1.1], if s_v' are the Satake conjugacy classes in $G_2(\mathbb{C})$ corresponding to ϕ' and s_v are the Satake conjugacy classes in $\mathrm{Spin}_7(\mathbb{C})$ corresponding to ϕ, then $s_v=\iota(s_v')$ where $\iota:G_2(\mathbb{C})\to\mathrm{Spin}_7(\mathbb{C})$ is the natural inclusion. Hence, the submodule generated by all such global theta lifts ϕ' gives an automorphic representation π' which weakly lifts to π. This proves the first assertion of the theorem.

For the second part of the theorem, it is clear that (b) implies (a). Conversely, as observed by Chenevier [C, Thm. 6.18, equation (6.6)], the hypothesis (a) in the theorem implies that

$$L^S(s,\pi,\mathrm{Spin})=\zeta^S(s)\cdot L^S(s,\pi,\mathrm{Std})$$

where the last L-function on the right is the degree 7 (partial) standard L-function of π. Since we are assuming that π is tempered, it follows that $L^S(1,\pi,\mathrm{Std})$ is finite and nonzero. Hence $L^S(s,\pi,\mathrm{Spin})$ has a pole at $s=1$ and the results we have shown above imply that (b) holds, with π' the global theta lift of π to $\mathrm{Aut}(\mathbb{O})$.

This completes the proof of the theorem. □

Remark: Let us comment on the relation of Theorem 10.1 with [C, Thm. 6.18]:

- In [C, Thm. 6.18], Chenevier showed the second part of Theorem 10.1 for globally generic cuspidal representations, by reducing it to the first part of Theorem 10.1, which is a result of Ginzburg-Jiang for globally generic cuspidal representations. As Chenevier remarked in [C, Remark 6.19], if one has an analog of the endoscopic classification of Arthur for PGSp_6, one would know that any tempered cuspidal representation of PGSp_6 is nearly equivalent to a globally generic cuspidal representation, in which case the second part of the theorem will follow for tempered cuspidal representations by reduction to the globally generic case.
- In our proof of Theorem 10.1, our argument reducing the second part of the theorem to the first follows Chenevier's. Thus, the main innovation of Theorem 10.1 is a direct proof of the first part of the theorem for all cuspidal representations, regardless of whether they are globally generic or tempered. In particular, this gives the second part of the theorem without resort to an Arthur type classification for PGSp_6.

We can strengthen our results in the case when $F = \mathbb{Q}$ and π is a cuspidal representation of $\mathrm{PGSp}_6(\mathbb{A})$ that corresponds to a classical Siegel holomorphic form of positive weight. Recall that there are two isomorphism classes of octonion algebras over $\mathbb{Q}$: the classical octonion algebra $\mathbb{O}^c$ and its split form $\mathbb{O}^s$. Then $\mathrm{Aut}(\mathbb{O}^c_\infty)$ is an anisotropic group, while $\mathrm{Aut}(\mathbb{O}^s_\infty)$ is split.

Theorem 10.2. *Let $F = \mathbb{Q}$, and π a cuspidal representation of $\mathrm{PGSp}_6(\mathbb{A})$ that corresponds to a classical Siegel holomorphic form ϕ_{2r} of weight $2r > 0$. If $L^S(s, \pi, \mathrm{Spin})$ has a pole at $s = 1$, then π is a lift from $\mathrm{Aut}(\mathbb{O}^c)$. Moreover, if the level of ϕ_{2r} is one, then π is a strong functorial lift from $\mathrm{Aut}(\mathbb{O}^c)$.*

Proof. Let $\mathrm{U}_3(\mathbb{R})$ be the maximal compact subgroup of $\mathrm{Sp}_6(\mathbb{R})$. By our assumption, π_∞ is a lowest weight module, with the minimal $\mathrm{U}_3(\mathbb{R})$-type $\det^{2r}$, $r > 0$. We need the following:

Lemma 10.3. *Let σ is a lowest weight module of $\mathrm{Sp}_6(\mathbb{R})$, with the minimal $\mathrm{U}_3(\mathbb{R})$-type $\det^{2r}$, $r > 0$. Then σ does not occur in the exceptional theta correspondence with split $G_2(\mathbb{R})$.*

Proof. Adopting the notation from [LS], let $G' = G_2(\mathbb{R})$, $\mathfrak{g}'$ the Lie algebra of G', K' a maximal compact subgroup of G', and $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ the corresponding Cartan decomposition. Let

$$\Pi = \oplus_{n=0}^{\infty} V_n$$

be the decomposition of the minimal representation of the split real E_7 into its K-types. Let $V_n^{\det^{2r}}$ be the maximal subspace of V_n on which $\mathrm{U}_3(\mathbb{R})$ acts by the character $\det^{2r}$. If $r = 0$, by [LS, Proposition 5.2], the dimension of this space is equal to the dimension of $S_n(\mathfrak{p}')$, the space of the n-th symmetric tensor power of $\mathfrak{p}'$. But this result can be easily generalized to any r: $V_n^{\det^{2r}}$ is non-trivial only for $n \geq 3r$, and the dimension of $V_{n+3r}^{\det^{2r}}$ is is equal to the dimension of $S_n(\mathfrak{p}')$. In particular, $V_{3r}^{\det^{2r}}$ is one-dimensional. Let v_r be a vector spanning this line. The group K' acts on this line and the vector v_r is fixed by K', since K' is semisimple. By [LS, Lemma 3.1], the matrix coefficient of v_r, when restricted to G', is contained in $L^{3/2+\epsilon}(G')$. This fact, combined with the dimension of $\det^{2r}$-invariants in the types of Π, implies that

$$\Pi^{\det^{2r}} = U(\mathfrak{g}') \cdot v_r \cong U(\mathfrak{g}') \otimes_{U(k')} \mathbb{C}$$

as explained in the introduction of [LS], where the case $r = 0$ is discussed. After taking $\det^{2r}$-invariants in $\Pi \to \Theta(\sigma) \boxtimes \sigma$, it follows that $\Theta(\sigma)$ is a quotient of $U(\mathfrak{g}') \otimes_{U(k')} \mathbb{C}$. Thus any irreducible quotient σ' of $\Theta(\sigma)$ is spherical. It was also shown in [LS] that $\Theta(\sigma')$ has unique irreducible quotient, and it is spherical. This is a contradiction, since σ is not spherical, and hence it cannot appear in this correspondence. □

The correspondence for the dual pair $\mathrm{Aut}(\mathbb{O}^c_\infty) \times \mathrm{Sp}_6(\mathbb{R})$ was completely determined in [GrS] and is functorial. Thus, if ϕ_{2r} is of level one, i.e. spherical at all primes, then π is indeed a (strong) functorial lift from $\mathrm{Aut}(\mathbb{O}^c)$.

□

Acknowledgments: The authors would like to thank Aaron Pollack for clarifying a few points about his Rankin-Selberg integral for the Spin L-function, especially with regards to the extent the unfolding and unramified computations hold over a more general field. Thanks are also due to the referee for his/her meticulous work, pertinent comments and useful suggestions. W.T. Gan is partially supported by an MOE Tier 2 grant R146-000-233-112. G. Savin is partially supported by Simons Foundation grant 579347, and a grant from the National Science Foundation, DMS-1901745.

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