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More concordance homomorphisms from knot Floer homology

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We define an infinite family of linearly independent, integer-valued smooth concordance homomorphisms. Our homomorphisms are explicitly computable and rely on local equivalence classes of knot Floer complexes over the ring $\mathbb{F}[U,V]/(UV=0)$. We compare our invariants to other concordance homomorphisms coming from knot Floer homology, and discuss applications to topologically slice knots, concordance genus and concordance unknotting number.

57M25, 57N70, 57R58

1 Introduction

Beginning with the τ–invariant [19], the knot Floer homology package of Ozsváth and Szabó [21] and independently J Rasmussen [26] has had numerous applications to the study of smooth knot concordance. See Hom [12] for a survey of such applications.

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The goal of this paper is to add to the (already infinite) list of explicitly computable homomorphisms from the smooth knot concordance group $\mathcal{C}$ to $\mathbb{Z}$:

Theorem 1.1 *For each $j \in \mathbb{N}$, there is a surjective homomorphism*

$$\varphi_j : \mathcal{C} \to \mathbb{Z}.$$

Moreover,

$$\bigoplus_{j=1}^{\infty} \varphi_j : \mathcal{C} \to \bigoplus_{j=1}^{\infty} \mathbb{Z}$$

is surjective. In particular, the φ_j are linearly independent.

Our homomorphisms are similar in spirit to Ozsváth, Stipsicz and Szabó's Υ–invariant, which gives a homomorphism

$$\Upsilon_K : \mathcal{C} \to \mathrm{Cont}([0,2]),$$

where $\mathrm{Cont}([0,2])$ denotes the vector space of piecewise-linear functions on $[0,2]$. Indeed, Υ is defined using t–modified knot Floer homology and can be thought of as a generalization of τ to the t–modified knot Floer homology setting. A slight repackaging (by considering the slopes of $\Upsilon_K(t)$) yields a $\mathbb{Z}$–valued homomorphism for each rational value of t. Similarly, our invariants can be thought of as a generalization of τ to a shifted version of knot Floer homology. The homomorphisms φ_j are then certain linear combinations of τ associated to shifted knot Floer homology. Just as τ can be recovered from $\Upsilon(t)$, it can also be recovered from φ_j:

Proposition 7.6 *Let K be a knot in S^3. Then we have the following equality relating the Ozsváth-Szabó τ–invariant with φ_j:*

$$\tau(K) = \sum_{j \in \mathbb{N}} j\varphi_j(K).$$

Both $\Upsilon(t)$ and φ_j factor through the local equivalence group of knot Floer complexes (see Zemke [27, Theorem 1.5], forgetting the involutive part; equivalently, stable equivalence from [12, Theorem 1]; equivalently, ν^+–equivalence of Kim and Park [13]). Following [27, Section 3], the knot Floer complex can be viewed as a module over $\mathbb{F}[U,V]$; local equivalence is then an equivalence relation between certain such complexes. In our setting, the invariants φ_j actually factor through the local equivalence group defined over the ring $\mathbb{F}[U,V]/(UV=0)$, which is the same as the group constructed using ε–equivalence in Hom [8, Definition 1]. The advantage of quotienting by $UV = 0$ is that the resulting local equivalence group is totally ordered; this total order is the same as the order induced by ε; see Hom [9]. Using this order, we have the following characterization result:

Theorem 1.2 *Every knot Floer complex coming from a knot in S^3 is locally equivalent mod UV to a standard complex* (defined in Section 4.1) *and can be completely described by a finite* (*symmetric*) *sequence of nonzero integers* $(a_i)_{i=1}^{2n}$. *Moreover, if we endow the integers with the unusual order*

$$-1 <^! -2 <^! -3 <^! \cdots <^! 0 <^! \cdots <^! 3 <^! 2 <^! 1,$$

then local equivalence classes mod UV are ordered lexicographically with respect to their standard representatives.

1.1 Properties of φ_j

The homomorphisms φ_j have many properties in common with Υ: both invariants take a particularly simple form on homologically thin knots and L–space knots. We use the convention that K is an *L–space knot* if K admits a positive L–space surgery.

Proposition 8.1 *If K is homologically thin, then*

$$\varphi_j(K) = \begin{cases} \tau(K) & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 8.2 *Let K be an L–space knot with Alexander polynomial*

$$\Delta_K(t) = \sum_{i=0}^{n} (-1)^i t^{b_i},$$

where $(b_i)_{i=0}^{n}$ is a decreasing sequence of integers and n is even. Define

$$c_i = b_{2i-2} - b_{2i-1} \quad \text{for } 1 \le i \le \tfrac{1}{2}n.$$

Then

$$\varphi_j(C) = \#\{c_i \mid c_i = j\}.$$

Example 1.3 Consider the torus knot $T_{3,4}$. We have that $\Delta_{T_{3,4}}(t) = t^6 - t^5 + t^3 - t + 1$, and so, by Proposition 8.2, we have

$$\varphi_j(T_{3,4}) = \begin{cases} 1 & \text{if } j = 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

See Figure 1 for a visual depiction of $\mathrm{CFK}^\infty(T_{3,4})$.

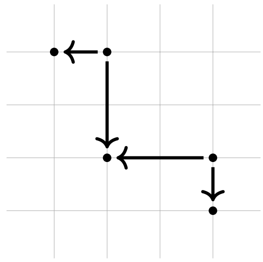

Figure 1: The knot Floer complex of $T_{3,4}$.

Example 1.4 More generally, the torus knot $T_{n,n+1}$ has Alexander polynomial

$$\Delta_{T_{n,n+1}}(t) = \sum_{i=0}^{n-1} t^{ni} - \sum_{i=0}^{n-2} t^{ni+i+1},$$

which yields $(c_i)_{i=1}^{n-1} = (1, 2, \dots, n-1)$. Thus

$$\varphi_j(T_{n,n+1}) = \begin{cases} 1 & \text{if } j = 1, 2, \dots, n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1.5 If K is an L–space knot, then, by Proposition 8.2, $\varphi_j(K) \geq 0$ for all j. This provides an easy (although fairly weak) method for showing that a linear combination of knots is not concordant to any L–space knot.

Remark 1.6 In Propositions 8.1 and 8.2 (as well as in the above examples), φ_j is the (signed) count of the number of horizontal arrows of length j. We will see in Definition 7.1 that φ_j is equal to the signed count of horizontal arrows in the standard complex representative of K (in the sense of Theorem 1.2).

While $\Upsilon(t)$ and φ_j have many properties in common, there do exist knots K for which $\Upsilon_K(t) \equiv 0$ while $\varphi_j(K)$ is nontrivial. Let $K_{p,q}$ denote the (p,q)–cable of K, where p denotes the longitudinal winding.

Proposition 1.7 *Let* $K = T_{2,5} \# -T_{4,5} \# T_{2,3;2,5}$. *Then* $\Upsilon_K(t) \equiv 0$, *while*

$$\varphi_j(K) = \begin{cases} 2 & \text{if } j = 1, \\ -1 & \text{if } j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

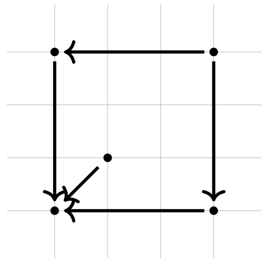

Figure 2: The complex from Ozsváth, Stipsicz and Szabó [18, Figure 6].

Proof The fact that $\Upsilon_K(t) \equiv 0$ follows from the proof of Hom [11, Theorem 2]. The computation of $\varphi_j(K)$ follows from Proposition 8.2 and the fact that the φ_j are homomorphisms. (Note that $T_{2,3;2,5}$ is an L–space knot; see the proof of [11, Lemma 2.1] for the relevant Alexander polynomial.) □

Conversely, while we do not have an explicit topological example, there is no algebraic obstruction to the existence of knots with $\varphi_j(K)$ trivial and $\Upsilon_K(t)$ nontrivial.

Proposition 1.8 *Suppose there exists a knot K whose knot Floer complex is given by Figure 2. Then $\Upsilon_K(t)$ is nontrivial, while $\varphi_j(K) = 0$ for all j.*

Proof The computation of $\Upsilon_K(t)$ is given in Proposition 9.4 of Ozsváth, Stipsicz and Szabó [18]. Since diagonal arrows vanish modulo UV, it is easily checked that the above complex is trivial in local equivalence (see Section 3). This implies that $\varphi_j(K) = 0$ for all j. □

1.2 Topological applications of φ_j

The homomorphisms φ_j have applications to $\mathcal{C}_{\mathrm{TS}}$, the subgroup of $\mathcal{C}$ generated by topologically slice knots. (That is, $\mathcal{C}_{\mathrm{TS}}$ is generated by knots bounding locally flat disks in B^4.) Let D denote the positively clasped, untwisted Whitehead double of $T_{2,3}$, and let $K_n = D_{n,n+1} \# -T_{n,n+1}$.

Theorem 1.9 *Consider the topologically slice knots K_n described above. For each index n, we have $\varphi_n(K_n) = 1$ and $\varphi_j(K_n) = 0$ for all $j > n$. In particular, the homomorphisms*

$$\bigoplus_{j=1}^{\infty} \varphi_j : \mathcal{C}_{\mathrm{TS}} \to \bigoplus_{j=1}^{\infty} \mathbb{Z}$$

map the span of the K_n isomorphically onto $\bigoplus_{j=1}^{\infty} \mathbb{Z}$.

Remark 1.10 The knots K_n are the same knots as considered in [9]. However, there is an error in the proof of the main result of [9]. Fortunately, the above theorem shows that the knots K_n do in fact generate an infinite-rank summand of $\mathcal{C}_{\mathrm{TS}}$. Moreover, they show this in a way that preserves the spirit of [9], namely by considering knot Floer complexes modulo ε–equivalence and extracting numerical invariants based on the lengths of vertical and horizontal arrows.

We also have applications of φ_j to concordance genus and concordance unknotting number. Recall that the *concordance genus* of K is defined to be

$$g_c(K) = \min\{g(K') \mid K \text{ and } K' \text{ are smoothly concordant}\},$$

where $g(K')$ denotes the Seifert genus of K'. Note that

$$g_c(K) \geq g_4(K),$$

where $g_4(K)$ denotes the smooth four-ball genus of K. The *concordance unknotting number* of K is defined to be

$$u_c(K) = \min\{u(K') \mid K \text{ and } K' \text{ are smoothly concordant}\},$$

where $u(K')$ denotes the unknotting number of K'. Note that, again,

$$u_c(K) \geq g_4(K).$$

Since $g_4(K) \geq |\tau(K)|$, the knot Floer homology of K provides lower bounds on both $g_c(K)$ and $u_c(K)$. Here, we show that the invariants φ_j bound concordance genus and concordance unknotting number as follows:

Theorem 1.11 *Let*

$$N(K) = \begin{cases} 0 & \text{if } \varphi_j(K) = 0 \text{ for all } j, \\ \max\{j \mid \varphi_j(K) \neq 0\} & \text{otherwise.} \end{cases}$$

Then

(1) $g_c(K) \geq \frac{1}{2}N(K)$, *and*

(2) $u_c(K) \geq N(K)$.

Let $\mathrm{Tors}_U\, M$ denotes the U–torsion submodule of an $\mathbb{F}[U]$–module M. The quantity $N(K)$ is bounded above by the maximal order of an element in $\mathrm{Tors}_U\, \mathrm{HFK}^-(K)$, as follows:

Proposition 7.27 *If $U^M \cdot \operatorname{Tors}_U \mathrm{HFK}^-(K) = 0$, then $\varphi_j(K) = 0$ for all $j > M$. In particular, $N(K) \leq M$.*

The bounds in Theorem 1.11(2) are sharp (eg for the trefoil); it is unknown to the authors whether the bound in Theorem 1.11(1) is sharp. Note that in many cases, the bounds are rather weak; for example, $N(T_{n,n+1}) = n-1$, while $g_4(T_{n,n+1}) = \tau(T_{n,n+1}) = \frac{1}{2}n(n-1)$. The proof of the concordance genus bound in Theorem 1.11(1) is similar to the proof of Hom [10, Theorem 2], and indeed is strong enough to recover [10, Theorem 3]. The proof of Theorem 1.11(2) relies on unknotting number bounds from Alishahi and Eftekhary [1].

We have the following application of Theorem 1.11(2):

Theorem 1.12 *There exist topologically slice knots $\{K_n\}_{n=1}^{\infty}$ such that $g_4(K_n) = 1$ for all n, while $u_c(K_n) \geq n$.*

The knots used to prove Theorem 1.12 are the same knots appearing in [10, Theorem 3]. In [17], Owens and Strle give examples of knots for which $u_c(K) - g_4(K) = 1$. As far as the authors know, Theorem 1.12 gives the first known examples of knots for which $u_c(K) - g_4(K)$ is arbitrarily large.

1.3 Remarks

We conclude with a few remarks relating the present work with other results. In [24], Ozsváth and Szabó define a bordered-algebraic knot invariant which is isomorphic to the knot Floer complex over the ring $\mathbb{F}[U,V]/(UV=0)$. Their bordered-algebraic knot invariant is particularly amenable to computer computation. It should thus be possible to implement an effective computer program to calculate the homomorphisms φ_j.

Theorem 6.1 is closely related to horizontally and vertically simplified bases for the knot Floer complex, defined in Lipshitz, Ozsváth and Thurston [15, Section 11.5]. Indeed, Corollary 6.2 states every knot Floer complex over $\mathbb{F}[U,V]/(UV=0)$ contains a direct summand with a simultaneously vertically and horizontally simplified basis, and that this summand supports $\mathrm{HF}^\infty(S^3)$. This is closely related to the notion of loop-type modules, defined in Hanselman and Watson [3, Definition 3.1]. (Note that over the ring $\mathbb{F}[U,V]$, not every complex admits a simultaneously vertically and horizontally simplified basis; see [9, Figure 3].)

Lastly, we point out that the techniques in this paper are the knot Floer analogues of the techniques used in Dai, Hom, Stoffregen and Truong [2] to study the three-dimensional homology cobordism group.

Organization

In Section 2, we briefly recall the definition of the knot Floer complex, working over the ring $\mathcal{R} = \mathbb{F}[U, V]/(UV = 0)$. In Section 3, we introduce the notion of a knot-like complex, and define the local equivalence group $\mathfrak{K}$ of knot-like complexes. In Section 4, we define a particularly simple family of knot-like complexes, which we call standard complexes. We use these to construct a sequence of numerical invariants associated to any knot-like complex in Section 5. This is used in Section 6 to show that every knot-like complex is locally equivalent to a standard complex. In Section 7, we apply our characterization of knot-like complexes to define the homomorphisms φ_j. In Section 8, we prove Propositions 8.1 and 8.2 (computing φ_j for thin and L–space knots). In Section 9, we prove Theorem 1.9 (on an infinite-rank summand of $\mathcal{C}_{\mathrm{TS}}$), and in Section 10, we prove Theorems 1.11 and 1.12 (on applications of φ_j to g_c and u_c). Finally, we conclude with some further remarks and open questions in Section 11.

Throughout, we work over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. We use the convention that $\mathbb{N} = \mathbb{Z}_{>0}$.

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2 Background on knot Floer homology

In this section, we give a brief overview of knot Floer homology, primarily to establish notation. We assume that the reader is familiar with knot Floer homology as in [21; 26]; see [16; 12] for survey articles on this subject. Our conventions mostly follow those in [28]; see in particular [28, Section 1.5].

Definition 2.1 Let $\mathcal{R} = \mathbb{F}[U, V]/(UV = 0)$, endowed with a relative bigrading $\mathrm{gr} = (\mathrm{gr}_U, \mathrm{gr}_V)$, where $\mathrm{gr}(U) = (-2, 0)$ and $\mathrm{gr}(V) = (0, -2)$. We call gr_U the *U–grading* and gr_V the *V–grading*.

Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ be a doubly pointed Heegaard diagram compatible with (S^3, K). Define $\mathrm{CFK}_{\mathcal{R}}(\mathcal{H})$ to be the chain complex freely generated over $\mathcal{R}$ by $\boldsymbol{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ with differential

$$\partial \boldsymbol{x} = \sum_{\boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \sum_{\substack{\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) \\ \mu(\phi)=1}} U^{n_w(\phi)} V^{n_z(\phi)} \boldsymbol{y},$$

where, as usual, $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ denotes homotopy classes of disks in $\mathrm{Sym}^g(\Sigma)$ connecting $\boldsymbol{x}$ to $\boldsymbol{y}$, and $\mu(\phi)$ denotes the Maslov index of ϕ. The chain complex $\mathrm{CFK}_{\mathcal{R}}(\mathcal{H})$ comes equipped with a relative bigrading $\mathrm{gr} = (\mathrm{gr}_U, \mathrm{gr}_V)$, defined as follows. Given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$, let the relative grading shifts be given by

$$\mathrm{gr}_U(\boldsymbol{x}, \boldsymbol{y}) = \mu(\phi) - 2n_w(\phi), \quad \mathrm{gr}_V(\boldsymbol{x}, \boldsymbol{y}) = \mu(\phi) - 2n_z(\phi).$$

It follows that the differential has degree $(-1, -1)$. (In the literature, gr_U is usually referred to as *Maslov grading*.) We define a relative *Alexander grading* by

$$A(\boldsymbol{x}, \boldsymbol{y}) = \tfrac{1}{2}(\mathrm{gr}_U(\boldsymbol{x}, \boldsymbol{y}) - \mathrm{gr}_V(\boldsymbol{x}, \boldsymbol{y})) = n_z(\phi) - n_w(\phi).$$

Note that the variable U lowers gr_U by 2, preserves gr_V and lowers A by 1. The variable V preserves gr_U, lowers gr_V by 2 and increases A by 1. The differential preserves the Alexander grading.

Up to chain homotopy over $\mathcal{R}$, the chain complex $\mathrm{CFK}_{\mathcal{R}}(\mathcal{H})$ is an invariant of $K \subset S^3$, and so we will typically write $\mathrm{CFK}_{\mathcal{R}}(K)$ rather than $\mathrm{CFK}_{\mathcal{R}}(\mathcal{H})$. We now recall some facts from [21]. The complex $\mathrm{CFK}_{\mathcal{R}}(K)$ has the following symmetry property. Let $\overline{\mathrm{CFK}}_{\mathcal{R}}(K)$ denote the complex obtained by interchanging the roles of U and V. (Note that we thus also interchange the values of gr_U and gr_V.) Then

$$\mathrm{CFK}_{\mathcal{R}}(K) \simeq \overline{\mathrm{CFK}}_{\mathcal{R}}(K).$$

The knot Floer complex behaves nicely with respect to connected sums. Indeed, we have that

$$\mathrm{CFK}_{\mathcal{R}}(K_1 \# K_2) \simeq \mathrm{CFK}_{\mathcal{R}}(K_1) \otimes_{\mathcal{R}} \mathrm{CFK}_{\mathcal{R}}(K_2).$$

We also have that

$$\mathrm{CFK}_{\mathcal{R}}(-K) \simeq \mathrm{CFK}_{\mathcal{R}}(K)^{\vee},$$

where $\mathrm{CFK}_{\mathcal{R}}(K)^{\vee} = \mathrm{Hom}_{\mathcal{R}}(\mathrm{CFK}_{\mathcal{R}}(K), \mathcal{R})$.

Remark 2.2 Since the differential preserves the Alexander grading, the complex $\mathrm{CFK}_{\mathcal{R}}$ splits — as a chain complex over $\mathbb{F}$, but *not* as an $\mathcal{R}$–module — as a direct sum over the Alexander grading:

$$\mathrm{CFK}_{\mathcal{R}}(K) = \bigoplus_{s \in \mathbb{Z}} \mathrm{CFK}_{\mathcal{R}}(K, s),$$

where

$$U : \mathrm{CFK}_{\mathcal{R}}(K, s) \to \mathrm{CFK}_{\mathcal{R}}(K, s-1), \quad V : \mathrm{CFK}_{\mathcal{R}}(K, s) \to \mathrm{CFK}_{\mathcal{R}}(K, s+1).$$

The chain complex $\mathrm{CFK}_{\mathcal{R}}(K, s)$ is isomorphic to the complex $\widehat{A}_s$ from [23]; that is, $H_*(\mathrm{CFK}_{\mathcal{R}}(K, s))$ is isomorphic (as a relatively graded vector space) to $\widehat{\mathrm{HF}}(S^3_N(K), \mathfrak{s}_s)$, the Heegaard Floer homology of large surgery on K in the spin^c structure corresponding to s.

The version of knot Floer homology we have constructed here follows slightly different conventions than the usual definition in eg [21]. For the convenience of the reader, we recall some of the most salient features of the standard knot Floer homology package, and explicitly translate them into our setting. For further discussion, see Section 1.5 of [28].

First, consider the $\mathbb{F}$–vector space $\widehat{\mathrm{HFK}}(K)$, which is defined by not allowing holomorphic disks in the definition of ∂ to cross either the w or the z basepoint. In our context, this is isomorphic to $H_*(\mathrm{CFK}_{\mathcal{R}}(K)/(U, V))$, where (U, V) denotes the ideal generated by U and V. The Alexander grading is given by $A = \frac{1}{2}(\mathrm{gr}_U - \mathrm{gr}_V)$ and the Maslov grading is given by $M = \mathrm{gr}_U$.

Next, consider the $\mathbb{F}[U]$–module $\mathrm{HFK}^-(K)$, which is defined by taking the homology of the associated graded complex of $\mathrm{CFK}^-(K)$ with respect to the Alexander filtration. This is equivalent to allowing holomorphic disks to cross the w but not the z basepoint. In our context, this yields $H_*(\mathrm{CFK}_{\mathcal{R}}(K)/V)$, where again the Alexander grading is given by $A = \frac{1}{2}(\mathrm{gr}_U - \mathrm{gr}_V)$ and the Maslov grading is given by $M = \mathrm{gr}_U$. It is a standard fact that for knots in S^3, the $\mathbb{F}[U]$–module $\mathrm{HFK}^-(K) \cong H_*(\mathrm{CFK}_{\mathcal{R}}(K)/V)$ has a single U–nontorsion tower.[1] By symmetry, it follows that $H_*(\mathrm{CFK}_{\mathcal{R}}(K)/U)$ has a single V–nontorsion tower.

We now claim that these two nontorsion towers satisfy the following grading normalizations:

(1) The U–gradings of all V–nontorsion classes in $H_*(\mathrm{CFK}_{\mathcal{R}}(K)/U)$ are zero.

(2) The V–gradings of all U–nontorsion classes in $H_*(\mathrm{CFK}_{\mathcal{R}}(K)/V)$ are zero.

[1] By this, we mean that $H_*(\mathrm{CFK}_{\mathcal{R}}(K)/V)/U\text{–torsion} \cong \mathbb{F}[U]$. Note, however, that this copy of $\mathbb{F}[U]$ is not required to be generated by an element with $\mathrm{gr}_U = 0$.

Note that all V–nontorsion classes in $H_*(\mathrm{CFK}_{\mathcal{R}}(K)/U)$ have the same U–grading, since multiplication by V does not change gr_U. Similarly, all U–nontorsion classes in $H_*(\mathrm{CFK}_{\mathcal{R}}(K)/V)$ have the same V–grading. To see the claim, consider the complex $\mathrm{CFK}_{\mathcal{R}}(K)$ and set $U=0$ and $V=1$. This means that we allow holomorphic disks to cross the z but not the w basepoint, and we disregard the Alexander filtration. This yields a complex whose homology computes $\widehat{\mathrm{HF}}(S^3)\cong\mathbb{F}$, which is concentrated in Maslov grading zero. Using the fact that the Maslov grading is equal to gr_U, some thought shows that the V–nontorsion tower of $H_*(\mathrm{CFK}_{\mathcal{R}}(K)/U)$ is thus generated by an element with $\mathrm{gr}_U=0$. By symmetry, we likewise have that any U–nontorsion element in $H_*(\mathrm{CFK}_{\mathcal{R}}(K)/V)$ has $\mathrm{gr}_V=0$.

Finally, recall that the concordance invariant $\tau(K)$ is defined to be the negative of the maximal Alexander grading of any U–nontorsion element in $\mathrm{HFK}^-(K)\cong H_*(\mathrm{CFK}_{\mathcal{R}}(K)/V)$. By the previous two paragraphs, this means that

$$\tau(K)=-\max\bigl\{\tfrac{1}{2}\,\mathrm{gr}_U(x)\mid x\in H_*(\mathrm{CFK}_{\mathcal{R}}(K)/V)\text{ is not }U\text{–torsion}\bigr\}.$$

By symmetry, we conclude that, similarly,

$$\tau(K)=-\max\bigl\{\tfrac{1}{2}\,\mathrm{gr}_V(x)\mid x\in H_*(\mathrm{CFK}_{\mathcal{R}}(K)/U)\text{ is not }V\text{–torsion}\bigr\}.$$

The reader should think of the complexes $\mathrm{CFK}_{\mathcal{R}}(K)/U$ and $\mathrm{CFK}_{\mathcal{R}}(K)/V$ as deleting horizontal and vertical arrows (respectively) in the pictorial representation of $\mathrm{CFK}_{\mathcal{R}}$. It may be helpful to keep in mind Figure 1. There, the V–nontorsion tower of $H_*(\mathrm{CFK}_{\mathcal{R}}(K)/U)$ is generated by the top-left basis element, while the U–nontorsion tower of $H_*(\mathrm{CFK}_{\mathcal{R}}(K)/V)$ is generated by the bottom-right basis element.

The following definition is particularly useful in applications of knot Floer homology to concordance:

Definition 2.3 Let K_1 and K_2 be knots in S^3. We say $\mathrm{CFK}_{\mathcal{R}}(K_1)$ and $\mathrm{CFK}_{\mathcal{R}}(K_2)$ are *locally equivalent* if there exist absolutely U–graded, absolutely V–graded $\mathcal{R}$–equivariant chain maps

$$f\colon \mathrm{CFK}_{\mathcal{R}}(K_1)\to\mathrm{CFK}_{\mathcal{R}}(K_2)\quad\text{and}\quad g\colon \mathrm{CFK}_{\mathcal{R}}(K_2)\to\mathrm{CFK}_{\mathcal{R}}(K_1)$$

such that f and g induce isomorphisms on $H_*(\mathrm{CFK}_{\mathcal{R}}(K_i)/U)/V\text{–torsion}$. Roughly speaking, this means that f maps the top of the V–tower in $H_*(\mathrm{CFK}_{\mathcal{R}}(K_1)/U)$ to the top of the V–tower in $H_*(\mathrm{CFK}_{\mathcal{R}}(K_2)/U)$, and vice versa for g.

Local equivalence is considered in the involutive setting in [27, Section 2.3].

Remark 2.4 $\mathrm{CFK}_{\mathcal{R}}(K)$ is locally equivalent to $\mathrm{CFK}_{\mathcal{R}}(\mathrm{O})$, where O denotes the unknot, if and only if $\mathrm{CFK}_{\mathcal{R}}(K) \simeq \mathrm{CFK}_{\mathcal{R}}(\mathrm{O}) \oplus A$, where A is a chain complex over $\mathcal{R}$ with $U^{-1}H_*(A) = V^{-1}H_*(A) = 0$. It is straightforward to verify that local equivalence over $\mathcal{R}$ and ε–equivalence (see [9, Section 2]) are the same (after translating between $\mathcal{R}$–modules and bifiltered chain complexes over $\mathbb{F}[U, U^{-1}]$).

Theorem 2.5 [27, Theorem 1.5; 8, Theorem 2] *If K_1 and K_2 are concordant, then $\mathrm{CFK}_{\mathcal{R}}(K_1)$ and $\mathrm{CFK}_{\mathcal{R}}(K_2)$ are locally equivalent.*

Theorem 2.5 follows from [27, Theorem 1.5] by forgetting the involutive component and quotienting by UV, or from [8, Theorem 2] by translating from ε–equivalence and bifiltered chain complexes to local equivalence and $\mathcal{R}$–modules.

3 Knot-like complexes and their properties

In this section, we consider abstract $\mathcal{R}$–complexes satisfying many of the same formal properties as $\mathrm{CFK}_{\mathcal{R}}(K)$. We show that modulo local equivalence, the set of such complexes forms a group, with the operation induced by tensor product. Moreover, we show that this group is totally ordered.

3.1 Knot-like complexes

We begin by defining knot-like complexes, so named because they are $\mathcal{R}$–complexes satisfying many of the properties of $\mathrm{CFK}_{\mathcal{R}}$ from the previous section.

Definition 3.1 A *knot-like complex C* is a free, finitely generated, bigraded chain complex over $\mathcal{R}$ such that:

(1) $H_*(C/U)$ has a single V–nontorsion tower, lying in $\mathrm{gr}_U = 0$.

(2) $H_*(C/V)$ has a single U–nontorsion tower, lying in $\mathrm{gr}_V = 0$.

Again, we mean by this that $H_*(C/U)/V$–torsion is isomorphic to $\mathbb{F}[V]$, and that all of the V–nontorsion elements in $H_*(C/U)$ have U–grading zero. A similar statement holds for $H_*(C/V)$. The differential ∂ is required to have degree $(-1,-1)$.

Remark 3.2 We do *not* in general require any symmetry with respect to interchanging U and V.

Definition 3.3 Let C_1 and C_2 be two knot-like complexes. We say that $C_1 \leq C_2$ if there exists an absolutely U–graded, relatively V–graded $\mathcal{R}$–equivariant chain map

$$f : C_1 \to C_2$$

such that f induces an isomorphism on $H_*(C_i/U)/V\text{–torsion}$. We call f a *local map*. We say that two knot-like complexes C_1 and C_2 are *locally equivalent*, denoted by $C_1 \sim C_2$, if $C_1 \leq C_2$ and $C_2 \leq C_1$.

We will also occasionally use the terminology:

Definition 3.4 Let C be a knot-like complex and let $x \in C$. We say that x is a *V–tower class* if $[x]$ is a maximally V–graded, V–nontorsion cycle in $H_*(C/U)$. Similarly, we say that x is a *U–tower class* if $[x]$ is a maximally U–graded, U–nontorsion cycle in $H_*(C/V)$. Thus f (as defined above) sends V–tower classes to V–tower classes.

Remark 3.5 The f in Definition 3.3 is *not* required to be absolutely V–graded, but rather only relatively V–graded. Thus, a priori the notion of local equivalence in Definition 3.3 is strictly weaker than the notion of local equivalence presented in Definition 2.3; ie we might have two knot-like complexes C_1 and C_2 which are locally equivalent via maps f and g that introduce complementary V–grading shifts. However, we will show in Lemma 6.9 that if C_1 and C_2 are locally equivalent (in the sense of Definition 3.3) via f and g, then f and g induce isomorphisms on $H_*(C_i/V)/U$–torsion (ie send U–tower classes to U–tower classes), even without any symmetry requirements on the C_i. Combined with the normalization conventions of Definition 3.1, this shows that f and g are absolutely V–graded.

It is straightforward to verify that $\leq$ is a partial order on the set of local equivalence classes of knot-like complexes.

Remark 3.6 Our notion of local equivalence agrees with [27, Definition 2.4] after forgetting ι_K and modding out by the ideal generated by UV. This definition of local equivalence also agrees with the equivalence relation defined using ε from [8, Section 4.1]; for this, see Theorem 6.1 and Corollary 6.2.

Let (U, V) denote the ideal generated by U and V. If C is a free, finitely generated chain complex over $\mathcal{R}$, then every element x in (U, V) can be uniquely expressed as $x_U + x_V$, where $x_U \in \operatorname{im} U$ and $x_V \in \operatorname{im} V$.

Definition 3.7 We say a chain complex over $\mathcal{R}$ is *reduced* if $\partial \equiv 0 \bmod (U, V)$. In a reduced complex, we can write ∂ as the sum $\partial = \partial_U + \partial_V$, where if $\partial x = y$, then $\partial_U x = y_U$ and $\partial_V x = y_V$. Note that $\partial_U^2 = \partial_V^2 = 0$. We call ∂_U the *U–differential* and refer to elements with $\partial_U x = 0$ as *U–cycles*; similarly, we call ∂_V the *V–differential* and refer to elements with $\partial_V x = 0$ as *V–cycles*.

Lemma 3.8 *Every knot-like complex C is locally equivalent to a reduced knot-like complex C'.*

Proof Suppose that C is not reduced. Then there exists $x \in C$ such that ∂x is not in the ideal generated by U and V. We claim that we may complete $\{x, \partial x\}$ to a basis $\{x, \partial x, y_1, \dots, y_n\}$ for C such that the y_i generate a subcomplex C' of C. To see this, first complete $\{x, \partial x\}$ to an $\mathcal{R}$–basis $\{x, \partial x, y_1, \dots, y_n\}$ for C, where ∂ does not necessarily preserve the span of the y_i. Here, we are using the fact that if N is a (free) submodule of a free module M, then a basis for N can be extended to a basis for M if and only if M/N is also free. To apply this in our case, note that x and ∂x do not lie in the image of (U, V). A grading argument then shows that no linear combination of x and ∂x lies in the image of (U, V).

For each y_i, we then write ∂y_i as a linear combination of x, ∂x and the other basis elements y_j. By adding multiples of x to y_i, we may assume that ∂x does not appear in any differential ∂y_i. This also shows that x does not appear in ∂y_i, since then we would have

$$0 = \partial^2 y_i = \partial\Big(P(U, V)x + \sum P_j(U, V)y_j\Big)$$

for some polynomials $P(U, V)$ and $P_j(U, V)$, which would imply that ∂x appears in some ∂y_j.

It follows that

$$0 \to \langle x, \partial x \rangle \to C \xrightarrow{p} C' \to 0$$

is a split short exact sequence of freely generated $\mathcal{R}$–complexes. Since $\langle x, \partial x \rangle$ is acyclic by construction, the projection $p\colon C \to C'$ and section $s\colon C' \to C$ both induce isomorphisms on homology. Hence C and C' are locally equivalent. Since C is finitely generated, we may iterate this procedure to arrive at a reduced complex. □

From now on, we will assume that all of our knot-like complexes are reduced.

3.2 The local equivalence group of knot-like complexes

We now show that knot-like complexes modulo local equivalence form a group, with the operation induced by tensor product. Moreover, we will show that the partial order $\leq$ is in fact a total order. We begin with some routine formalism:

Definition 3.9 The *product* of two knot-like complexes C_1 and C_2 is $C_1 \otimes_{\mathcal{R}} C_2$.

Lemma 3.10 *The product of two knot-like complexes is a knot-like complex.*

Proof This is straightforward. □

Definition 3.11 Let $\mathfrak{K}$ denote the set of local equivalence classes of knot-like complexes, with the operation induced by $\otimes$.

Proposition 3.12 *The pair* $(\mathfrak{K}, \otimes)$ *forms an abelian group.*

Proof This is straightforward to verify. The identity is given by $\mathcal{R}$ with trivial differential, and the inverse of $[C]$ is $[C^\vee]$, where $C^\vee = \operatorname{Hom}_{\mathcal{R}}(C, \mathcal{R})$. □

Remark 3.13 See [27, Proposition 2.6] for the analogous result in the involutive setting over the ring $\mathbb{F}[U, V]$.

We now come to the significantly more interesting proposition.

Proposition 3.14 *The relation* $\leq$ *defines a total order on* $\mathfrak{K}$.

Proposition 3.14 is a consequence of the following lemma:

Lemma 3.15 *Let C be a knot-like complex. If there does not exist a local map $f: \mathcal{R} \to C$, then there exists a local map $g: C \to \mathcal{R}$.*

Proof The idea of the proof is we build a basis $\{x, t_i\}$ for C such that quotienting by the span of $\{t_i\}$ gives the desired local map. Roughly, we first find a basis for the subcomplex A generated by elements w such that some U–power of w is in the image of ∂_U or some V–power of w is in the image of ∂_V. We then extend this basis by an element x representing a V–nontorsion class in $H_*(C/U)$. We use the absence of a local map from $\mathcal{R}$ to C in order to guarantee that x is not in A. Finally, we complete this to a basis for all of C. We describe this argument more precisely below.

We begin by finding a "vertically simplified" basis for C which is especially nice with respect to ∂_V. Since $\mathbb{F}[V] \cong \mathcal{R}/U$ is a PID, the complex C/U admits a basis $\mathcal{B} = \{x, y_i, z_i\}$ over $\mathbb{F}[V]$ such that

$$\partial_V x = 0, \quad \partial_V y_i = V^{\eta_i} z_i \quad \text{and} \quad \partial_V z_i = 0$$

for some set of positive integers η_i. Since C is a free $\mathcal{R}$–module, it is easily checked that choosing any lift of $\mathcal{B}$ from C/U to C yields an $\mathcal{R}$–basis for C, which (by abuse of notation) we also denote by $\mathcal{B} = \{x, y_i, z_i\}$. Moreover, since $UV = 0$, these elements also satisfy the equalities $\partial_V x = 0$, $\partial_V y_i = V^{\eta_i} z_i$ and $\partial_V z_i = 0$. We will henceforth think of C as a free module over this basis, so that

$$C = \mathrm{Span}_{\mathbb{F}}\{x, y_i, z_i\} \otimes_{\mathbb{F}} \mathcal{R}.$$

Note that $\operatorname{im} \partial_V$ is contained in $\mathrm{Span}_{\mathbb{F}[V]}\{z_i\}$. We will also have cause to consider the $\mathbb{F}[U]$–module C/V, which we identify with

$$C/V = \mathrm{Span}_{\mathbb{F}}\{x, y_i, z_i\} \otimes_{\mathbb{F}} \mathbb{F}[U],$$

as well as the $\mathbb{F}$–vector space $C/(U, V)$, which we identify with

$$C/(U, V) = \mathrm{Span}_{\mathbb{F}}\{x, y_i, z_i\}.$$

These identifications allow us to view elements of $C/(U, V)$ as elements of C/V (and elements of C/V as elements of C) in the obvious way — an $\mathbb{F}$–linear combination of basis elements in $C/(U, V)$ may be viewed as the same linear combination in C/V, and so on. That is, they specify lifts from $C/(U, V)$ to C/V and from C/V to C.

Now let P be the submodule of C/V consisting of elements w such that some U–power of w lies in the image of ∂_U:

$$P = \{w \in C/V : U^n w \in \operatorname{im} \partial_U \text{ for some } n \geq 0\}.$$

Note that P has the property that if $Uw \in P$, then $w \in P$. Moreover, by the fact that $\partial_U^2 = 0$, we have that every element $w \in P$ is a ∂_U–cycle, that is, $\partial_U w = 0$. Choose an $\mathbb{F}[U]$–basis $p_1, \dots, p_r$ for P. Let $\overline{p}_i$ denote the reduction of p_i modulo U in $C/(U, V)$. Explicitly, if p_i is a linear combination (over $\mathbb{F}[U]$) of the basis elements $\{x, y_i, z_i\}$, then $\overline{p}_i$ consists of those terms which are not decorated by any powers of U. Note that p_i differs from the canonical lift of $\overline{p}_i$ by an element in $\operatorname{im} U$.

We claim that the $\overline{p}_i$ are linearly independent as elements of $C/(U, V)$. Suppose not. Then we have some linear combination

$$\overline{p}_{i_1} + \cdots + \overline{p}_{i_k} = 0.$$

Lifting this to C/V, this implies that $p_{i_1}+\cdots+p_{i_k}=Uw$ for some w. However, this means that $w\in P$. Writing w as a linear combination of the p_i gives a contradiction.

Consider the subspaces of $C/(U,V)$ given by $\overline{P}=\mathrm{Span}_{\mathbb{F}}\{\overline{p}_1,\ldots,\overline{p}_r\}$ and $\overline{Z}=\mathrm{Span}_{\mathbb{F}}\{z_i\}$. Extend the linearly independent set $\{\overline{p}_1,\ldots,\overline{p}_r\}$ to a basis

$$\{\overline{p}_1,\ldots,\overline{p}_r,z_{i_1},\ldots,z_{i_s}\}$$

for $\overline{P}+\overline{Z}$ in $C/(U,V)$. We claim that x (viewed as an element of $C/(U,V)$) does not lie in $\overline{P}+\overline{Z}$. Indeed, if it did, we would have $x=\overline{p}+\sum z_{i_j}$ for some $\overline{p}\in\overline{P}$ and sum of the z_{i_j}. Lifting this to C/V shows that

$$x+\sum z_{i_j}+Uw\in P$$

for some $w\in C/V$. By construction of P, we have that the above expression is a ∂_U–cycle. Viewing it as an element of C, we also see that it is a ∂_V–cycle, since $\partial_V x=\partial_V z_i=0$ and $\partial_V(Uw)=U\partial_V w=0$. This means that we can specify a local map from $\mathcal{R}$ to C by sending the generator of $\mathcal{R}$ to $x+\sum z_{i_j}+Uw$, which generates the V–tower in C/U (by definition of x and the z_i). This would contradict the hypothesis of the lemma. Thus, $x\notin\overline{P}+\overline{Z}$.

Now consider the set of generators $S=\{x,p_1,\ldots,p_r,z_{i_1},\ldots,z_{i_s}\}$ in C/V. It is straightforward to check that this is linearly independent by reducing any putative linear relation modulo U. We also claim that if $Uw\in\mathrm{Span}_{\mathbb{F}[U]}S$, then $w\in\mathrm{Span}_{\mathbb{F}[U]}S$. Indeed, suppose not. Then we have

$$Uw=U^*x+\sum U^*p_i+\sum U^*z_{i_j},$$

where at least one term on the right-hand side appears with a U–exponent of zero. Reducing both sides modulo U, we obtain a nontrivial linear relation among the generators $\{x,\overline{p}_1,\ldots,\overline{p}_r,z_{i_1},\ldots,z_{i_s}\}$, a contradiction. It follows that we may extend S to an $\mathbb{F}[U]$–basis

$$\{x,p_1,\ldots,p_r,z_{i_1},\ldots,z_{i_s},w_1,\ldots,w_t\}$$

for all of C/V.[2] This then gives an $\mathcal{R}$–basis for all of C.

By construction,

$$D=\mathrm{Span}_{\mathcal{R}}\{p_1,\ldots,p_r,z_{i_1},\ldots,z_{i_s},w_1,\ldots,w_t\}$$

[2]As in the proof of Lemma 3.8, we are using the fact that if N is a (free) submodule of a free module M, then a basis for N can be extended to a basis for M if and only if M/N is also free.

is a subcomplex of C. Indeed, the image of ∂_U is contained in the span of the p_i. Similarly, the image of ∂_V is contained in the span of the p_i and z_{i_j}. To see this, note that any z_k is an $\mathbb{F}$–linear combination of the $\bar{p}_i$ and the z_{i_j}. Hence (viewing these as elements of C), we have

$$z_k = \sum p_i + \sum z_{i_j} + Uw$$

for some element w, since $\bar{p}_i = p_i \bmod U$. Thus, for any y_k, we have

$$\partial_V y_k = V^{\eta_k} z_k = V^{\eta_k}\Big(\sum p_i + \sum z_{i_j} + Uw\Big) = V^{\eta_k}\Big(\sum p_i + \sum z_{i_j}\Big).$$

Hence $\partial D \subset D$. Then the quotient map

$$C \to C/D \cong \mathcal{R}$$

is a local map from C to $\mathcal{R}$. □

Proof of Proposition 3.14 We need to show totality of $\le$. Let C_1 and C_2 be two knot-like complexes. Consider $C_1 \otimes C_2^\vee$. By Lemma 3.15, we have that either $C_1 \otimes C_2^\vee \ge \mathcal{R}$ or $C_1 \otimes C_2^\vee \le \mathcal{R}$. By tensoring with C_2, either $C_1 \ge C_2$ or $C_1 \le C_2$, as desired. □

Remark 3.16 The group $\mathfrak{K}$ should be compared to the group $\mathcal{CFK}$ defined in [9] using ε–equivalence. Indeed, $\mathcal{CFK}$ is isomorphic (as an ordered group) to the subgroup of $\mathfrak{K}$ generated by $\{\mathrm{CFK}_{\mathcal{R}}(K) \mid K \text{ a knot in } S^3\}$. In particular, the order $\le$ defined in Definition 3.3 agrees with the order given by ε.

4 Standard complexes and their properties

In this section, we define a convenient family of knot-like complexes called standard complexes.

Remark 4.1 The reader should compare with [2, Section 4], which carries out the analogous construction in the setting of almost ι–complexes. Indeed, an almost ι–complex may be viewed as a complex over the ring $\mathbb{F}[U,Q]/(Q^2 = QU = 0)$. In our case, this corresponds (roughly) to passing to the ring $\mathbb{F}[U,V]/(UV = V^2 = 0)$.

4.1 Standard complexes

Let C be a knot-like complex generated by $x_0, \dots, x_n$. We say there is a *U^m–arrow* between x_i and x_j for $m \in \mathbb{N}$ if one of the following occurs:

(1) $\partial_U x_i = U^m x_j$, or

(2) $\partial_U x_j = U^m x_i$.

The arrow goes from x_i to x_j in (1) and from x_j to x_i in (2). We define V^m*–arrows* analogously by replacing U with V.

Remark 4.2 In the traditional depiction of CFK^∞ as a bifiltered complex in the (i,j)–plane, each generator (over $\mathbb{F}$) is placed in its appropriate bigrading and is decorated with a power of U. An arrow between two generators indicates that one (with its U–power decoration) appears in the differential of the other. This is not quite the same as the pictorial description we use here. Instead, we suppress writing the decorations of our generators and use their spatial placement in the plane to determine the appropriate U– or V–powers appearing in the differential. That is, a horizontal arrow of length m from x_k to x_l indicates that x_l appears in the differential of x_k with a coefficient of U^m, and similarly for vertical arrows and powers of V. It can be shown, however, that (modulo an infinite number of translations) this produces the same shape as in the previous picture.

Definition 4.3 Let $n \in 2\mathbb{N}$, and let $(b_1,\dots,b_n)$ be a sequence of nonzero integers. A *standard complex of type* $(b_1,\dots,b_n)$, denoted by $C(b_1,\dots,b_n)$, is the knot-like complex freely generated over $\mathcal{R}$ by

$$\{x_0, x_1, \dots, x_n\}.$$

Each pair of generators x_i and x_{i-1} for i odd are connected by $U^{|b_i|}$–arrows, and each pair of generators x_i and x_{i-1} for i even are connected by $V^{|b_i|}$–arrows. The direction is determined by the sign of b_i, as follows. If b_i is positive, then the arrow goes from x_i to x_{i-1}, and if b_i is negative, then the arrow goes from x_{i-1} to x_i. We call n the *length* of the standard complex and $\{x_i\}_{i=1}^n$ the *preferred basis*. Explicitly, the differential on $C(b_1,\dots,b_n)$ is as follows. For i odd,

$$\partial_U x_{i-1} = U^{|b_i|} x_i \quad \text{if } b_i < 0, \qquad \partial_U x_i = U^{b_i} x_{i-1} \quad \text{if } b_i > 0,$$

while for i even,

$$\partial_V x_{i-1} = V^{|b_i|} x_i \quad \text{if } b_i < 0, \qquad \partial_V x_i = V^{b_i} x_{i-1} \quad \text{if } b_i > 0.$$

All other differentials are zero.

Note that x_0 generates $H_*(C(b_1,\dots,b_n)/U)/V\text{–torsion}$. Similarly, x_n generates $H_*(C(b_1,\dots,b_n)/V)/U\text{–torsion}$. There is thus a unique grading on $C(b_1,\dots,b_n)$ which makes it into a knot-like complex: namely, $\mathrm{gr}_U(x_0) = 0$ and $\mathrm{gr}_V(x_n) = 0$. The fact that the differential has degree $(-1,-1)$ then determines the rest of the gradings.

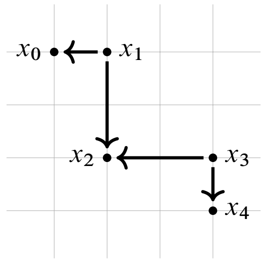

Figure 3: The standard complex $C(1,-2,2,-1)$. A horizontal (respectively, vertical) arrow of length m from x_i to x_j means that $\partial_U x_i = U^m x_j$ (respectively, $\partial_V x_i = V^m x_j$).

Note that $\mathrm{gr}_U(x_i) \equiv \mathrm{gr}_V(x_i) \equiv i \mod 2$; we refer to this as the *parity* of (the grading of) a generator of $C(b_1,\dots,b_n)$.

Definition 4.4 A standard complex $C(b_1,\dots,b_n)$ is *symmetric* if $b_i = -b_{n+1-i}$.

Example 4.5 We define the trivial standard complex $C(0) \cong \mathcal{R}$ to be the complex generated over $\mathcal{R}$ by a single element with U– and V–grading zero.

Example 4.6 The standard complex $C(1,-2,2,-1)$ is generated over $\mathcal{R}$ by

$$x_0, x_1, x_2, x_3, x_4$$

with

$$\partial x_0 = \partial x_2 = \partial x_4 = 0, \quad \partial x_1 = Ux_0 + V^2 x_2, \quad \partial x_3 = U^2 x_2 + V x_4.$$

The gradings of the generators are

$$\begin{aligned} \mathrm{gr}(x_0) &= (0,-6), \\ \mathrm{gr}(x_1) &= (-1,-5), \\ \mathrm{gr}(x_2) &= (-2,-2), \end{aligned} \qquad \begin{aligned} \mathrm{gr}(x_3) &= (-5,-1), \\ \mathrm{gr}(x_4) &= (-6,0). \end{aligned}$$

See Figure 3 for a visual depiction of $C(1,-2,2,-1)$, where a horizontal (resp. vertical) arrow of length m from x_i to x_j represents a U^m–arrow (resp. V^m–arrow). Note that to read off the standard complex from the figure, we start at x_0 and follow the unique path to x_4, recording the direction and length of each arrow that we traverse. Namely, traversing an arrow of length m against the direction of the arrow yields a $+m$, while traversing an arrow of length m in the direction of the arrow yields a $-m$.

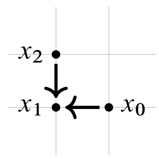

Figure 4: The standard complex $C(-1,1)$.

Example 4.7 The standard complex $C(-1,1)$ is generated over $\mathcal{R}$ by

$$x_0, x_1, x_2$$

with

$$\partial x_0 = U x_1, \quad \partial x_1 = 0, \quad \partial x_2 = V x_1$$

and gradings

$$\mathrm{gr}(x_0) = (0,2), \quad \mathrm{gr}(x_1) = (1,1), \quad \mathrm{gr}(x_2) = (2,0).$$

See Figure 4 for a visual depiction.

Example 4.8 The standard complex $C(1,-2,-1,1,2,-1)$ is generated over $\mathcal{R}$ by

$$x_0, x_1, x_2, x_3, x_4, x_5, x_6$$

with nonzero differentials

$$\partial x_1 = U x_0 + V^2 x_2, \quad \partial x_2 = U x_3, \quad \partial x_4 = V x_3, \quad \partial x_5 = U^2 x_4 + V x_6$$

with gradings

$$\begin{aligned} \mathrm{gr}(x_0) &= (0,-4), & \mathrm{gr}(x_4) &= (0,-2), \\ \mathrm{gr}(x_1) &= (-1,-3), & \mathrm{gr}(x_5) &= (-3,-1), \\ \mathrm{gr}(x_2) &= (-2,0), & \mathrm{gr}(x_6) &= (-4,0). \\ \mathrm{gr}(x_3) &= (-1,-1), \end{aligned}$$

See Figure 5 for a visual depiction.

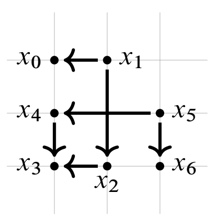

Figure 5: The standard complex $C(1,-2,-1,1,2,-1)$.

Lemma 4.9 *The dual of* $C(b_1,\dots,b_n)$ *is* $C(-b_1,\dots,-b_n)$.

Proof This is a straightforward consequence of the definitions. □

4.2 An unusual order on the integers

Let $\mathbb{Z}^! = (\mathbb{Z}, \leq^!)$ denote the integers with the unusual order

$$-1 <^! -2 <^! -3 <^! \cdots <^! 0 <^! \cdots <^! 3 <^! 2 <^! 1.$$

We will see shortly the utility of this strange order. Note that for $a, b \neq 0$, we have $a <^! b$ if and only if $\frac{1}{a} < \frac{1}{b}$, where $<$ denotes the usual order on $\mathbb{Q}$. Since $a >^! 0$ if and only if $a > 0$, the sign of $a \in \mathbb{Z}^!$ coincides with the usual definition (that is, a is positive if $a > 0$ and negative if $a < 0$).

4.3 Ordering standard complexes

We consider $\mathbb{Z}^!$–valued sequences, with the lexicographic order induced by $\leq^!$. We take the convention that in order to compare two sequences of different lengths, we append sufficiently many trailing zeros to the shorter sequence so that the sequences have the same length.

Proposition 4.10 *Standard complexes are ordered lexicographically as* $\mathbb{Z}^!$*–valued sequences with respect to the total order on* $\mathfrak{K}$.

The proof of Proposition 4.10 consists of a number of straightforward but technical verifications regarding local maps between standard complexes. We have included the details so that the reader will become accustomed to routine manipulations involving these definitions.

Lemma 4.11 *Let* $(a_1,\dots,a_m) \leq^! (b_1,\dots,b_n)$ *in the lexicographic order on* $\mathbb{Z}^!$*–valued sequences. Then* $C(a_1,\dots,a_m) \leq C(b_1,\dots,b_n)$ *in* $\mathfrak{K}$.

Proof If $(a_1,\dots,a_m) = (b_1,\dots,b_n)$, then it is clear that the complexes in question are locally equivalent by taking the obvious identity map. Thus, assume that $(a_1,\dots,a_m) < (b_1,\dots,b_n)$. Suppose that the two sequences agree up to index k, so that $a_i = b_i$ for $1 \leq i < k$ and $a_k <^! b_k$.

Let $\{x_i\}$ and $\{y_i\}$ be the preferred bases for $C(a_1,\dots,a_m)$ and $C(b_1,\dots,b_n)$, respectively. Define

$$f : C(a_1,\dots,a_m) \to C(b_1,\dots,b_n)$$

by

$$f(x_i) = \begin{cases} y_i & \text{if } 0 \le i < k, \\ 0 & \text{if } i > k. \end{cases}$$

In order to define $f(x_k)$, we proceed with some elementary casework based on the value of k. First, suppose that $k \le \min\{m,n\}$, and consider the parity of k:

(1) If k is odd:

   (a) If $a_k <^! b_k < 0$, then let $f(x_k) = U^{a_k - b_k} y_k$. It is straightforward to verify that f is a chain map; the only nontrivial checks are that $\partial_U f(x_{k-1}) = f \partial_U(x_{k-1})$ and $\partial f(x_k) = f\partial(x_k)$. To verify the former, we see that

   $$\partial_U f(x_{k-1}) = \partial_U y_{k-1} = U^{-b_k} y_k,$$

   while

   $$f\partial_U(x_{k-1}) = f(U^{-a_k} x_k) = U^{-a_k} U^{a_k - b_k} y_k.$$

   To verify the latter, we see that

   $$\partial f(x_k) = \partial U^{a_k - b_k} y_k = U^{a_k - b_k} \partial_V y_k.$$

   This is zero, since either $\partial_V y_k = 0$ or $\partial_V y_k = V^{-b_{k+1}} y_{k+1}$ and $UV = 0$. Meanwhile, $f\partial(x_k) = 0$ since ∂x_k is either equal to zero or $V^{-a_{k+1}} x_{k+1}$.

   (b) If $a_k < 0 < b_k$, then let $f(x_k) = 0$. It is straightforward to verify f is a chain map; the only nontrivial check is that $\partial_U f(x_{k-1}) = f\partial_U(x_{k-1})$. This follows from the fact that $b_k > 0$ (ie $\partial_U y_{k-1} = 0$).

   (c) If $0 < a_k <^! b_k$, then let $f(x_k) = U^{a_k - b_k} y_k$. It is straightforward to verify that f is a chain map; the only nontrivial check is that $\partial f(x_k) = f\partial(x_k)$. This follows from the fact that

   $$\partial f(x_k) = \partial U^{a_k - b_k} y_k = U^{a_k - b_k}(\partial_U y_k + \partial_V y_k) = U^{a_k - b_k} U^{b_k} y_{k-1},$$

   while

   $$f\partial(x_k) = f U^{a_k} x_{k-1} = U^{a_k} y_{k-1}.$$

(2) The case when k even is similar, but with V playing the role of U.

Now assume that $k > \min\{m,n\}$. We consider the following two cases:

(1) Suppose that $n > m$. Then $k = m+1$, and

$$(b_i)_{i=1}^n = (a_1, \dots, a_m, b_{m+1}, \dots, b_n)$$

with $b_{m+1} > 0$. Let f be the obvious inclusion map. As above, it is easily checked that f commutes with ∂.

(2) Suppose that $m > n$. Then $k = n + 1$, and

$$(a_i)_{i=1}^m = (b_1, \dots, b_n, a_{n+1}, \dots, a_m)$$

with $a_{n+1} < 0$. Let f be the obvious projection map. As above, it is easily checked that f commutes with ∂.

It is clear that f is local, since $f(x_0) = y_0$. This completes the proof. □

Lemma 4.12 *Let $C_1 = C(a_1, \dots, a_m)$ and $C_2 = C(b_1, \dots, b_n)$ be standard complexes with preferred bases $\{x_i\}$ and $\{y_i\}$, respectively. Suppose that $a_i = b_i$ for all $1 \le i \le k$ and that $f : C_1 \to C_2$ is a local map. Then $f(x_i)$ is supported by y_i for all $0 \le i \le k$.*

Proof We proceed by induction on i. The base case $i = 0$ follows from the fact that f is local. Thus, let $i < k$, and assume that $f(x_i)$ is supported by y_i. We show that $f(x_{i+1})$ is supported by y_{i+1}. Suppose that i is even. We consider the following two cases:

(1) Suppose that $a_{i+1} = b_{i+1} < 0$. Then $\partial_U f(x_i) = f \partial_U (x_i) = U^{|a_{i+1}|} f(x_{i+1})$. By the induction hypothesis, $f(x_i)$ is supported by y_i. We have that $\partial_U y_i = U^{|b_{i+1}|} y_{i+1}$ and that y_i is the unique element in C_2 such that ∂_U of it is supported by a U–power of y_{i+1}. It follows that $f(x_{i+1})$ must be supported by y_{i+1}.

(2) Suppose that $a_{i+1} = b_{i+1} > 0$. Then $\partial_U f(x_{i+1}) = f \partial_U (x_{i+1}) = U^{|a_{i+1}|} f(x_i)$. By the induction hypothesis, $f(x_i)$ is supported by y_i. We have that $\partial_U y_{i+1} = U^{|b_{i+1}|} y_i$ and that y_{i+1} is the unique basis element in C_2 such that ∂_U of it is supported by a U–power of y_i. It follows that $f(x_{i+1})$ must be supported by y_{i+1}.

The case i odd is similar, but with V playing the role of U. □

Lemma 4.13 *Let $(a_1, \dots, a_m) >^! (b_1, \dots, b_n)$ in the lexicographic order on $\mathbb{Z}^!$–valued sequences. Then there is no local map from $C_1 = C(a_1, \dots, a_m)$ to $C_2 = C(b_1, \dots, b_n)$.*

Proof Suppose that $a_i = b_i$ for $i < k$ and that $a_k >^! b_k$. We proceed by contradiction. Assume there is a local map $f : C_1 \to C_2$. We begin by considering the case when $k \le \min\{m, n\}$:

(1) Suppose that k is odd. We have three further subcases:

(a) Suppose that $b_k <^! a_k < 0$. Then $\partial_U x_{k-1} = U^{|a_k|} x_k$ and $\partial_U y_{k-1} = U^{|b_k|} y_k$. Furthermore, y_{k-1} is the unique basis element of C_2 such that ∂_U of it is supported by a U–power of y_k. By Lemma 4.12, $f(x_{k-1})$ is supported by y_{k-1}. It follows that $f\partial_U(x_{k-1}) = \partial_U f(x_{k-1})$ is supported by $U^{|b_k|} y_k$. Hence $f(U^{|a_k|} x_k)$ must be supported by $U^{|b_k|} y_k$, which is a contradiction, since $b_k <^! a_k < 0$, ie $|b_k| < |a_k|$, where $<$ denotes the usual ordering on $\mathbb{Z}$.

(b) Suppose that $b_k < 0 < a_k$. Then $\partial_U y_{k-1} = U^{|b_k|} y_k$ and $\partial_U x_{k-1} = 0$. Furthermore, y_{k-1} is the unique basis element in C_2 such that ∂_U of it is supported by a U–power of y_k. By Lemma 4.12, $f(x_{k-1})$ is supported by y_{k-1}. But $0 = f\partial_U(x_{k-1}) = \partial_U f(x_{k-1})$, a contradiction, since the right-hand side is supported by $U^{|b_k|} y_k$.

(c) Suppose that $0 < b_k <^! a_k$. Then $\partial_U x_k = U^{a_k} x_{k-1}$ and $\partial_U y_k = U^{b_k} y_{k-1}$. Furthermore, y_k is the unique basis element in C_2 such that ∂_U of it is supported by a U–power of y_{k-1}. By Lemma 4.12, $f(x_{k-1})$ is supported by y_{k-1}. Then $\partial_U f(x_k) = f\partial_U(x_k) = f(U^{a_k} x_{k-1})$, where the right-hand side is supported by $U^{a_k} y_{k-1}$. Hence $f(x_k)$ must be supported by $U^{a_k - b_k} y_k$, a contradiction since $0 < b_k <^! a_k$, ie $b_k > a_k$, where $<$ denotes the usual ordering on $\mathbb{Z}$.

(2) The case when k is even is similar, but with V playing the role of U.

Now assume that $k > \min\{m, n\}$. We consider the following two cases:

(1) Suppose $n > m$. Then $k = m + 1$ and $(b_i)_{i=1}^n = (a_1, \ldots, a_m, b_{m+1}, \ldots, b_n)$. Then $b_{m+1} < 0$, that is, $\partial_U y_m = U^{|b_{m+1}|} y_{m+1}$ and y_m is the unique element in C_2 such that ∂_U of it is supported by a U–power of y_{m+1}. By Lemma 4.12, $f(x_m)$ is supported by y_m. But $0 = f\partial_U(x_m) = \partial_U f(x_m) \neq 0$ since $\partial_U f(x_m)$ is supported by $U^{|b_{m+1}|} y_{m+1}$.

(2) Suppose $m > n$. Then $k = n + 1$ and $(a_i)_{i=1}^m = (b_1, \ldots, b_n, a_{n+1}, \ldots, a_m)$. Then $a_{n+1} > 0$, that is, $\partial_U x_{n+1} = U^{|a_{n+1}|} x_n$. Furthermore, no U–power of y_n appears as ∂_U of any element in C_2. By Lemma 4.12, $f(x_n)$ is supported by y_n. But $\partial_U f(x_{n+1}) = f\partial_U(x_{n+1}) = f(U^{|a_{n+1}|} x_n)$ is supported by $U^{|a_{n+1}|} y_n$, a contradiction.

This completes the proof. □

Proof of Proposition 4.10 The proposition follows immediately from Lemmas 4.11 and 4.13. □

4.4 Semistandard complexes

In future sections, we will also find it useful to have the following generalization of standard complexes:

Definition 4.14 Let $n \in 2\mathbb{N}-1$, and let $(b_1, \dots, b_n)$ be a sequence of nonzero integers. The *semistandard complex* $C'(b_1, \dots, b_n)$ is the subcomplex of the standard complex $C(b_1, \dots, b_n, 1)$ generated by $x_0, x_1, \dots, x_n$. We call these the *preferred generators* of $C'(b_1, \dots, b_n)$. (The choice $b_{n+1} = 1$ here is unimportant; any $b_{n+1} > 0$ is allowed.)

We stress that a semistandard complex is *not* a knot-like complex; indeed, for C' a semistandard complex, $H_*(C'/U)/V$–torsion has two V–towers, which are generated by x_0 and x_n. Note that since n is odd, the gradings of x_0 and x_n have opposite parities.

We use the symbol $'$ to distinguish semistandard complexes from standard complexes; that is, $C'(b_1, \dots, b_n)$ denotes a semistandard complex (where n is odd) while $C(b_1, \dots, b_n)$ denotes a standard complex (where n is even).

Definition 4.15 A grading-preserving $\mathcal{R}$–equivariant chain map

$$f : C'(b_1, \dots, b_n) \to C$$

from a semistandard complex to a knot-like complex C is said to be *local* if the class of $f(x_0)$ generates $H_*(C/U)/V$–torsion.

Example 4.16 The semistandard complex $C'(1, -2, -1, 1, 2)$ is generated over $\mathcal{R}$ by

$$x_0, x_1, x_2, x_3, x_4, x_5$$

with nonzero differentials

$$\partial x_1 = Ux_0 + V^2 x_2, \quad \partial x_2 = Ux_3, \quad \partial x_4 = Vx_3, \quad \partial x_5 = U^2 x_4.$$

See Figure 6 for a visual depiction.

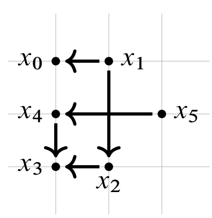

Figure 6: The semistandard complex $C'(1, -2, -1, 1, 2)$.

4.5 Short maps

It will often be useful for us to consider module maps from a standard complex $C(b_1,\dots,b_n)$ to a knot-like complex C that are chain maps except possibly at x_n. We make this notion precise with the following definition:

Definition 4.17 Let $C_1=C(b_1,\dots,b_n)$ be a standard complex and C_2 a knot-like complex. An absolutely U–graded, relatively V–graded module map $f\colon C_1\to C_2$ is called a *short map*, denoted by

$$f\colon C(b_1,\dots,b_n)\rightsquigarrow C_2,$$

if $f\partial(x_i)+\partial f(x_i)=0$ for $1\le i\le n-1$ and $f\partial_V(x_n)+\partial_V f(x_n)=0$. If f induces an isomorphism on $H_*(C_i/U)/V$–torsion, then we call f a *short local map*.

We similarly define short maps for semistandard complexes:

Definition 4.18 Let $C_1=C'(b_1,\dots,b_n)$ be a semistandard complex and C_2 a knot-like complex. An absolutely U–graded, relatively V–graded module map $f\colon C_1\to C_2$ is called a *short map*, denoted by

$$f\colon C'(b_1,\dots,b_n)\rightsquigarrow C,$$

if $f\partial(x_i)+\partial f(x_i)=0$ for $1\le i\le n-1$ and $f\partial_U(x_n)+\partial_U f(x_n)=0$. If the class of $f(x_0)$ generates $H_*(C_2/U)/V$–torsion, then we call f a *short local map*.

The following lemma states that given a short map, we can extend it to an actual chain map (from a different domain).

Lemma 4.19 (extension lemma) *Let*

$$f\colon C(b_1,\dots,b_n)\rightsquigarrow C$$

be a short map from a standard complex to C. Then there exists an $\mathcal{R}$–equivariant chain map

$$g\colon C(b_1,\dots,b_n,b_{n+1},\dots,b_m)\to C$$

for some b_i with $n+1\le i\le m$ such that f and g agree on the generators of $C(b_1,\dots,b_n)$ (viewed as generators of $C(b_1,\dots,b_n,b_{n+1},\dots,b_m)$ in the obvious way). Moreover, if f is local, then g is local.

Proof Consider $f(x_n)$. If $\partial_U f(x_n) = 0$, then f is already a chain map and we are done. Thus, suppose that $\partial_U f(x_n) = U^c z$ for some $z \in C$ and $c \geq 1$. Define a short map $f' \colon C'(b_1, \dots, b_n, -1) \rightsquigarrow C$ by setting $f'(x_i) = f(x_i)$ for $0 \leq i \leq n$ and $f'(x_{n+1}) = U^{c-1} z$. We now consider several cases:

(1) If $c > 1$, then extend the domain of f' to $C(b_1, \dots, b_n, -1, -1)$ by setting $f'(x_{n+2}) = 0$. It is easily checked that f' then provides the desired $\mathcal{R}$–equivariant chain map.

(2) If $c = 1$ and $\partial_V z = 0$, then we may again extend the domain of f' to $C'(b_1, \dots, b_n, -1, -1)$ by setting $f'(x_{n+2}) = 0$. It is easily checked that f' then provides the desired $\mathcal{R}$–equivariant chain map.

(3) If $c = 1$ and $\partial_V z = V^d w$ for some $w \in C$ and $d \geq 1$, then we proceed as in the beginning of the proof, except replacing the role of U with V. That is, extend the short map
$$f' \colon C'(b_1, \dots, b_n, -1) \rightsquigarrow C$$
to a short map
$$f'' \colon C(b_1, \dots, b_n, -1, -1) \rightsquigarrow C.$$
Iterate this procedure. Note that both the U– and V–gradings of the final preferred generator of $C(b_1, \dots, b_n, -1, -1, \dots, -1, -1)$ increase as the length of standard complex increases. Since C is finitely generated, the gradings of its generators are bounded above. Hence it is easily checked that at some point this process must terminate, yielding the desired extension.

Since $g(x_0) = f(x_0)$, it is clear that g is local if f is local. □

The analogous result holds for semistandard complexes:

Lemma 4.20 *Let*
$$f \colon C'(b_1, \dots, b_n) \rightsquigarrow C$$
be a short map from a semistandard complex to C. Then there exists an $\mathcal{R}$–equivariant chain map
$$g \colon C(b_1, \dots, b_n, b_{n+1}, \dots, b_m) \to C$$
for some b_i with $n + 1 \leq i \leq m$ such that f and g agree on the generators of $C'(b_1, \dots, b_n)$. Moreover, if f is local, then g is local.

Proof This is analogous to the proof of Lemma 4.19. □

5 Numerical invariants a_i

In this section, we define a sequence of numerical invariants (a_i) for any knot-like complex C, analogous to those constructed in [2, Section 6]. Up to sign, these are the same as the invariants defined in [9, Section 3], which are also denoted by (a_i). In the next section, we will see that the a_i compute successive parameters in the standard complex representative of C.

Let C be a knot-like complex. Define

$$a_1(C) = \sup^!\{b_1 \in \mathbb{Z}^! \mid C(b_1, \dots, b_n) \leq C\}.$$

Here, $\sup^!$ denotes the supremum taken with respect to the (unusual!) order on $\mathbb{Z}^!$. We define $a_k(C)$ for $k \geq 2$ inductively, as follows. Suppose that we have already defined $a_i = a_i(C)$ for $1 \leq i \leq k$. If $a_k = 0$, define $a_{k+1}(C) = 0$. Otherwise, define

$$a_{k+1}(C) = \sup^!\{b_{k+1} \in \mathbb{Z}^! \mid C(a_1, \dots, a_k, b_{k+1}, \dots, b_n) \leq C\}.$$

That is, we consider the set of standard complexes $\leq C$ whose first k symbols agree with the previously defined a_i. We then take the supremum over the family of $(k+1)^{\text{st}}$ symbols appearing in this set.[3]

It will be convenient for us to have the following terminology:

Definition 5.1 Let C be a knot-like complex, and let n be a positive integer. Let $(a_1, \dots, a_n)$ be the sequence given by the first n invariants, $a_i = a_i(C)$ for $1 \leq i \leq n$. We say that $(a_1, \dots, a_n)$ — and, similarly, the standard complex $C(a_1, \dots, a_n)$ — is *n–maximal with respect to C*. Here, we identify $C(a_1, \dots, a_n, 0, \dots, 0) = C(a_1, \dots, a_n)$.

The following proposition (combined with the extension lemma) shows that the supremum in the definition of a_i is always realized:

Proposition 5.2 *Let $a_i = a_i(C)$. For each $n \in \mathbb{N}$, there is a short local map*

$$f : C(a_1, \dots, a_n) \rightsquigarrow C.$$

Here, we identify $C(a_1, \dots, a_n, 0, \dots, 0) = C(a_1, \dots, a_n)$.

This is a consequence of the following lemmas:

[3] It will be implicit in the proof of Proposition 5.2 that this set of standard complexes is nonempty. More precisely, if $a_1, \dots, a_k$ are all defined and nonzero, then there exists a standard complex of the form $C(a_1, \dots, a_k, b_{k+1}, \dots, b_n)$ which is $\leq C$.

Lemma 5.3 *Let*

$$f: C'(b_1, \dots, b_n) \to C$$

be a local map from a semistandard complex to a knot-like complex C*. Then there is some* $b_{n+1} > 0$ *such that we have a short local map from the standard complex* $C(b_1, \dots, b_n, b_{n+1})$ *to* C*,*

$$g: C(b_1, \dots, b_n, b_{n+1}) \rightsquigarrow C.$$

Proof Let $C' = C'(b_1, \dots, b_n)$. Since x_n is a cycle in C'/U, we have that $f(x_n)$ is a cycle in C/U. Moreover, the class of $f(x_n)$ must be V–torsion in C/U, since x_n has odd grading and $H_*(C/U)/V$–torsion is supported in U–grading zero. It follows that there exists some $y \in C$ and $m > 0$ for which $\partial_V y = V^m f(x_n)$. Now define

$$g(x_i) = \begin{cases} f(x_i) & \text{if } i = 1, \dots, n, \\ y & \text{if } i = n+1. \end{cases}$$

Note that $b_{n+1} = m$. By construction, g is a short local map. □

Lemma 5.4 *Let* $\{t_i\}_{i \in \mathbb{N}}$ *be a sequence of integers with* $t_i \to \infty$*, and let*

$$f_i: C(b_1, \dots, b_{n-1}, -t_i) \rightsquigarrow C$$

be a sequence of short local maps from standard complexes to a knot-like complex C*. Then there exists a short local map*

$$f: C(b_1, \dots, b_{n-1}, b_n) \rightsquigarrow C$$

for some $b_n > 0$*.*

Proof As i increases, the V–grading of the final generator x_n of $C(b_1, \dots, b_{n-1}, -t_i)$ also increases. Since C is finitely generated, it follows that for sufficiently large i, we have $f_i(x_n) = 0$. Restriction to the first $n-1$ generators thus yields a local map from the semistandard complex $C'(b_1, \dots, b_{n-1})$ to C. Now apply Lemma 5.3 to obtain the desired result. □

Lemma 5.5 *Let* $\{t_i\}_{i \in \mathbb{N}}$ *be a sequence of integers with* $t_i \to \infty$*, and let*

$$f_i: C'(b_1, \dots, b_{n-1}, -t_i) \rightsquigarrow C$$

be a sequence of short local maps from semistandard complexes to a knot-like complex C*. Then there exists a short local map*

$$f: C(b_1, \dots, b_{n-1}) \rightsquigarrow C$$

from the standard complex $C(b_1, \dots, b_{n-1})$ *to* C*.*

Proof As i increases, the U–grading of the last generator x_n of $C'(b_1, \dots, b_{n-1}, -t_i)$ also increases. Since C is finitely generated, it follows that for sufficiently large i, we have $f_i(x_n) = 0$. Restriction to the first $n-1$ generators then yields a local map from the standard complex $C(b_1, \dots, b_{n-1})$ to C. □

We are now ready to prove Proposition 5.2:

Proof of Proposition 5.2 We prove that the supremum in the definition of a_i is always realized (modulo trailing zeros). We proceed by induction. Suppose that $a_1, \dots, a_k$ are defined and nonzero. Let $\mathcal{F}$ be the family of standard complexes appearing in the definition of a_{k+1}. By examining the order on $\mathbb{Z}^!$, we see that the only subsets of $\mathbb{Z}^!$ which fail to attain their supremum are those which are unbounded below (in the usual sense). Hence the only case we have to worry about is when the family of $(k+1)^{\text{st}}$ symbols appearing in $\mathcal{F}$ has $\sup^!$ equal to zero.

If k is odd, then truncating each element of $\mathcal{F}$ to its first $k+1$ generators provides a family of standard complexes and local maps as in the statement of Lemma 5.4. This is a contradiction, since Lemma 5.4 (combined with the extension lemma) then implies that the relevant $\sup^!$ is strictly greater than zero. Thus, we may assume that k is even. Then truncating each element of $\mathcal{F}$ to its first $k+1$ generators yields a family of semistandard complexes to which we may apply Lemma 5.5. In this situation, we see that a_{k+1} is realized as a trailing zero, completing the proof. □

6 Characterization of knot-like complexes up to local equivalence

We now prove that every knot-like complex is locally equivalent to a standard complex. In fact, we prove a slightly stronger statement in Corollary 6.2 below:

Theorem 6.1 *Every knot-like complex is locally equivalent to a standard complex.*

Corollary 6.2 *Let C be a knot-like complex, and assume C is locally equivalent to $C(a_1, \dots, a_n)$. Then C is homotopy equivalent to $C(a_1, \dots, a_n) \oplus A$ for some $\mathcal{R}$–complex A.*

Theorem 6.1 immediately implies Theorem 1.2:

Proof of Theorem 1.2 Following Section 2, to every knot in S^3, we can associate a knot-like complex. By Theorem 6.1, every knot-like complex is locally equivalent to a standard complex, and by Proposition 4.10, standard complexes are ordered lexicographically. This proves Theorem 1.2 modulo the claim that the standard complex associated to any knot is symmetric. We delay this until the end of the section; see Lemma 6.10. □

Roughly speaking, we will show that if C is a knot-like complex, then the numerical invariants $a_i(C)$ defined in the previous section compute successive parameters in the desired standard complex representative of C. Our main technical result will be to show that the a_i (as defined previously) eventually become equal to zero:

Proposition 6.3 *Let C be a knot-like complex. Then $a_i(C) = 0$ for all i sufficiently large.*

The proof of Proposition 6.3 will be given at the end of the section. First, we show how this implies Theorem 6.1:

Proof of Theorem 6.1 Let C be a knot-like complex with numerical invariants a_i. By Propositions 5.2 and 6.3, there exists some standard complex $C_1 \leq C$ which realizes the a_i. It is easily checked from the fact that standard complexes are lexicographically ordered that C_1 must be the maximal standard complex $\leq C$. Dualizing, let C_2 be the minimal standard complex with $C \leq C_2$. If $C_1 \neq C_2$, then (using the fact that standard complexes are lexicographically ordered) there exists a standard complex C_3 lying strictly between them. This complex contradicts either the maximality of C_1 or the minimality of C_2. Thus we must have the local equivalence $C_1 = C = C_2$. □

To prove the more refined Corollary 6.2, we use the following series of lemmas concerning self-maps of standard complexes:

Lemma 6.4 *Let*

$$f : C(b_1, \dots, b_n) \to C(b_1, \dots, b_n)$$

be a local map such that $f(x_i)$ is supported by x_j for some $i \neq j$. Then

$$(b_{i+1}, \dots, b_n) <^! (b_{j+1}, \dots, b_n).$$

Here, we mean that $(b_{k+1}, \dots, b_n) = (0)$ if $k = n$.

Proof First assume that i is even. By grading considerations, this implies that j is also even. We have the following casework:

(1) Suppose that $b_{i+1} < 0$. Then $\partial_U x_i = U^{|b_{i+1}|} x_{i+1}$. Hence $\partial_U f(x_i) = f \partial_U x_i \in \operatorname{im} U^{|b_{i+1}|}$. Since $f(x_i)$ is supported by x_j, it follows that $\partial_U x_j \in \operatorname{im} U^{|b_{i+1}|}$. This implies that $b_{j+1} \geq^! b_{i+1}$. (Here, we use the fact that no U–power of x_{j+1} appears in ∂_U of any standard basis element other than x_j.)

(2) Suppose that $b_{i+1} > 0$. Then $\partial_U x_{i+1} = U^{b_{i+1}} x_i$. Hence $\partial_U f(x_{i+1}) = U^{b_{i+1}} f(x_i)$ is supported by $U^{b_{i+1}} x_j$. In particular, $U^{b_{i+1}} x_j$ is in the image of ∂_U, which implies that $b_{j+1} \geq^! b_{i+1}$. (Here, we use the fact that x_{j+1} is the unique basis element whose image under ∂_U can be supported by a U–power of x_j.)

(3) Suppose that $i = n$, so that $b_{i+1} = 0$. Then $\partial_U x_i = 0$. Hence $\partial_U f(x_i) = 0$. Since $f(x_i)$ is supported by x_j, it follows that $\partial_U x_j = 0$. (Here, we use the fact that no U–power of x_{j+1} can appear in ∂_U of any standard basis element other than x_j.) This implies that $b_{j+1} > 0$.

If strict inequality holds in any of the above cases, then we are done. On the other hand, if $b_{i+1} = b_{j+1}$, then it is easily seen that $f(x_{i+1})$ is supported by x_{j+1}, and we proceed inductively. By the hypothesis that $i \neq j$, the sequences $(b_{i+1}, \dots, b_n)$ and $(b_{j+1}, \dots, b_n)$ are of different lengths, and hence cannot be equal. The case i odd is similar, with the role of U played by V. □

Lemma 6.5 *Any local map*

$$f : C(b_1, \dots, b_n) \to C(b_1, \dots, b_n)$$

must be injective.

Proof Suppose not. Then there exists some linear combination $\sum_i r_i x_i$ with $r_i \in \mathcal{R}$ such that $f\left(\sum_i r_i x_i\right) = 0$. Since f is graded, we may assume that $\sum_i r_i x_i$ is grading-homogenous, so that each r_i is a monomial (that is, $r_i \in \{0, 1, U, U^2, \dots, V, V^2, \dots\}$).

We impose a partial order on the set of monomials in $\mathcal{R}$ by defining $1 > U > U^2 > \cdots > 0$ and $1 > V > V^2 > \cdots > 0$. Among the nonzero coefficients r_i, choose a maximal element r_{i_0} with respect to this partial order. Let $I = \{j \mid r_j = r_{i_0}\}$. For each $j \in I$, consider $(b_{j+1}, \dots, b_n)$. Label the elements of $I = \{j_1, \dots, j_m\}$ so that

$$(b_{j_1+1}, \dots, b_n) <^! (b_{j_2+1}, \dots, b_n) <^! \cdots <^! (b_{j_m+1}, \dots, b_n).$$

Consider $f(x_{j_1})$. By Lemma 4.12, $f(x_{j_1})$ is supported by x_{j_1}. By Lemma 6.4, $f(x_{j_i})$ for $i=2,\dots,m$ cannot be supported by x_{j_1}. By the $\mathcal{R}$–equivariance of f and maximality of r_{i_0}, there is no other term in $f\big(\sum_{i\neq j_1} r_i x_i\big)$ that can cancel $r_{j_1}x_{j_1}$, contradicting the fact that $f\big(\sum_i r_i x_i\big)=0$. Hence f must be injective. □

We thus have:

Lemma 6.6 *Any local self-map of a standard complex to itself is an isomorphism.*

Proof Let f be a local self-map of a standard complex C. It is clear that f must be absolutely V–graded. Hence f restricted to each bigrading is a linear map from a finite-dimensional $\mathbb{F}$–vector space to itself, which is injective by Lemma 6.5. (Note that C is finitely generated.) It follows that f is surjective. □

Using Lemma 6.6, we now prove Corollary 6.2:

Proof of Corollary 6.2 By Theorem 6.1, for a knot-like complex C, we have local maps

$$f\colon C(a_1,\dots,a_n)\to C \quad \text{and} \quad g\colon C\to C(a_1,\dots,a_n).$$

Then $g\circ f$ is a local map from $C(a_1,\dots,a_n)$ to itself, which is an isomorphism by Lemma 6.6. It follows that the short exact sequence

$$0\to C(a_1,\dots,a_n)\xrightarrow{f} C\to C/\operatorname{im} f\to 0$$

splits. □

We now turn to the proof of Proposition 6.3. We begin with the following lemma:

Lemma 6.7 *Let C be a knot-like complex and let $a_i=a_i(C)$. Suppose we have a short local map*

$$f\colon C(a_1,\dots,a_n)\rightsquigarrow C.$$

Then $f(x_i)$ is not in $\operatorname{im}(U,V)$ for any $0\le i\le n$.[4] *In particular, $f(x_i)\neq 0$ for $0\le i\le n$.*

Proof We first show by contradiction that $f(x_i)\notin\operatorname{im} U$. Let $j=\min\{i\mid f(x_i)\in\operatorname{im} U\}$ be the minimal index for which $f(x_j)\in\operatorname{im} U$, and let $f(x_j)=U\eta_j$. (Note that η_j is allowed to be zero.) Since f is local, we have that $f(x_0)\neq 0\in H_*(C/U)$, so $j\neq 0$.

[4] Note that 0 is considered to be in $\operatorname{im}(U,V)$.

Suppose that j is odd. If $a_j \neq 1$, define a local map

$$g\colon C'(a_1, \dots, a_j - 1) \rightsquigarrow C$$

by setting $g(x_i) = f(x_i)$ for $1 \leq i < j$ and $g(x_j) = \eta_j$. By the extension lemma, g extends to a local map. This contradicts the maximality of a_j, since $a_j - 1 >^! a_j$. If $a_j = 1$, we have that $\partial_U x_j = U x_{j-1}$. Since $\partial_U f(x_j) = f \partial_U(x_j) = U f(x_{j-1})$, we have $\partial_U \eta_j = f(x_{j-1})$. Since C is reduced, it follows that $f(x_{j-1}) \in \operatorname{im} U$, contradicting the minimality of j.

Now suppose that j is even. Assume $a_j < 0$. Since $f(x_j) \equiv 0 \bmod U$, it is easily checked that the restriction of f gives a local map

$$g\colon C'(a_1, \dots, a_{j-1}) \to C.$$

Applying Lemma 5.3 and then the extension lemma shows that this contradicts the maximality of a_j. Thus, we may assume $a_j > 0$. Then

$$V^{a_j} f(x_{j-1}) = \partial_V f(x_j) = \partial_V U \eta_j = 0.$$

This implies that $f(x_{j-1}) \in \operatorname{im} U$, contradicting the minimality of j.

The case $f(x_i) \notin \operatorname{im} V$ is similar. Indeed, let $j = \min\{i \mid f(x_i) \in \operatorname{im} V\}$, and let $f(x_j) = V\eta_j$. (Note that η_j is allowed to be zero.) Since $H_*(C/U)$ does not have any V–nontorsion classes of positive grading, it follows that $j \neq 0$. The remainder of the proof follows by interchanging the roles of U and V in the argument above. □

Before proceeding, we will need the following technical result, which will allow us to rule out when certain complexes are n–maximal. The reader may wish to postpone reading the proof of Lemma 6.8 until after seeing its utilization in the proof of Proposition 6.3.

Lemma 6.8 *Let*

$$f\colon C(b_1, \dots, b_m) \rightsquigarrow C \quad \text{and} \quad g\colon C(c_1, \dots, c_n) \rightsquigarrow C$$

be short local maps from standard complexes to a knot-like complex C. Let $\{y_i\}_{i=1}^m$ and $\{x_i\}_{i=1}^n$ denote the standard bases for $C(b_1, \dots, b_m)$ and $C(c_1, \dots, c_n)$, respectively. Suppose that $f(y_m) = g(x_n)$, and we have the inequality of reversed sequences

$$(b_m, \dots, b_1) <^! (c_n, \dots, c_1)$$

with respect to the lexicographic order on $\mathbb{Z}^!$–valued sequences. Then $C(c_1, \dots, c_n)$ is not n–maximal (with respect to C).

Proof Assume that the sequences $(b_m, \dots, b_1)$ and $(c_n, \dots, c_1)$ first differ in their $(l+1)^{\text{st}}$ terms, so that $b_{m-i} = c_{n-i}$ for $0 \le i < l$ and $b_{m-l} <^! c_{n-l}$.[5] This means that the final $l+1$ generators of $C(b_1, \dots, b_m)$ (and the arrows going between them) are isomorphic to the final $l+1$ generators of $C(c_1, \dots, c_n)$. Our goal will be to define a new local map

$$h\colon C(c_1, \dots, c_n) \to C$$

which has the property that $h(x_{n-i}) = g(x_{n-i}) + f(y_{m-i})$ for all $0 \le i \le l$. Since f and g are chain maps, it is evident that h is a chain map, at least when restricted to the generators x_{n-i} for $0 \le i < l$. Below, we give the full verification and construction of h. In order to conclude the proof, we then note that $h(x_n) = g(x_n) + f(y_m) = 0$, and apply Lemma 6.7.

We define h on all generators except x_{n-l-1} as follows. Let

$$h(x_i) = g(x_i) \qquad \text{for } 0 \le i \le n-l-2, \tag{6-1}$$

$$h(x_{n-i}) = g(x_{n-i}) + f(y_{m-i}) \qquad \text{for } 0 \le i \le l. \tag{6-2}$$

It is clear that the chain map condition $\partial h = h \partial$ holds for all generators x_i with $i < n-l-2$, as well as all generators with $i > n-l$. The main subtlety will thus be to define $h(x_{n-l-1})$. We have the following casework:

$$h(x_{n-l-1}) = \begin{cases} g(x_{n-l-1}) + U^{b_{m-l} - c_{n-l}} f(y_{m-l-1}) & \text{if } c_{n-l} \text{ and } b_{m-l} \text{ same sign and } n-l \text{ odd}, \\ g(x_{n-l-1}) + V^{b_{m-l} - c_{n-l}} f(y_{m-l-1}) & \text{if } c_{n-l} \text{ and } b_{m-l} \text{ same sign and } n-l \text{ even}, \\ g(x_{n-l-1}) & \text{if } c_{n-l} \text{ and } b_{m-l} \text{ have different signs.} \end{cases} \tag{6-3}$$

Here, we consider $b_0 = c_0 = 0$ to be of a different sign than either positive or negative. For the sake of concreteness, we explicitly describe h in the two cases when $m < n$ and $n < m$. If $m < n$, then all three of (6-1), (6-2) and (6-3) are utilized when defining h. In particular, since m and n are both even and $l \le \min(m, n)$, we have $n - l - 2 \ge 0$, and thus $h(x_0) = g(x_0)$. However, if $n < m$, then the form of h may change slightly depending on the value of l. More precisely, if we are in the boundary case when $l = n$, then h is defined on all generators by (6-2):

$$h(x_{n-i}) = g(x_{n-i}) + f(y_{m-i}) \quad \text{for } 0 \le i \le n.$$

[5] Here, $l \le \min(m, n)$. Note that we allow $l = \min(m, n)$, with the convention that $b_0 = c_0 = 0$.

Similarly, if $l = n-1$, then only (6-2) and (6-3) are used:

$$h(x_{n-i}) = g(x_{n-i}) + f(y_{m-i}) \qquad \text{for } 0 \le i < n,$$
$$h(x_0) = g(x_0) + U^{b_{m-n+1}-c_1} f(y_{m-n}).$$

Note that in all other cases, we again have $h(x_0) = g(x_0)$.

We now check that h is a chain map. As in Section 4, this consists of a number of technical but straightforward verifications. For simplicity, assume for the moment that $l < n-1$. First consider the case when $n-l$ is odd. Note that this also implies $m-l$ is odd, so $m-l > 0$. It is clear that $\partial_U h(x_{n-l-2}) = h\partial_U(x_{n-l-2})$ and $\partial_V h(x_{n-l}) = h\partial_V(x_{n-l})$. For the remaining chain map conditions, we proceed with casework based on the signs of c_{n-l-1} and c_{n-l}. First, we consider the possible signs of c_{n-l-1} to verify that $h\partial_V(x_{n-l-2}) = \partial_V h(x_{n-l-2})$ and $h\partial_V(x_{n-l-1}) = \partial_V h(x_{n-l-1})$. We then consider the possible signs of c_{n-l} to verify that $h\partial_U(x_{n-l-1}) = \partial_U h(x_{n-l-1})$ and $h\partial_U(x_{n-l}) = \partial_U h(x_{n-l})$.

(1) Suppose $c_{n-l-1} < 0$. Then $\partial_V x_{n-l-2} = V^{|c_{n-l-1}|} x_{n-l-1}$ and $\partial_V x_{n-l-1} = 0$. Assume that c_{n-l} and b_{n-l} have the same sign. We compute

$$\begin{aligned} h\partial_V(x_{n-l-2}) &= V^{|c_{n-l-1}|} h(x_{n-l-1}) \\ &= V^{|c_{n-l-1}|}(g(x_{n-l-1}) + U^{b_{m-l}-c_{n-l}} f(y_{m-l-1})) \\ &= V^{|c_{n-l-1}|} g(x_{n-l-1}), \\ \partial_V h(x_{n-l-2}) = \partial_V g(x_{n-l-2}) &= g\partial_V(x_{n-l-2}) = V^{|c_{n-l-1}|} g(x_{n-l-1}). \end{aligned}$$

Similarly,

$$\begin{aligned} &h\partial_V(x_{n-l-1}) = 0, \\ &\partial_V h(x_{n-l-1}) = \partial_V(g(x_{n-l-1}) + U^{b_{m-l}-c_{n-l}} f(y_{m-l-1})) = g\partial_V(x_{n-l-1}) = 0, \end{aligned}$$

as desired. If c_{n-l} and b_{m-l} have different signs, then the same computation holds, except that the $U^{b_{m-l}-c_{n-l}} f(y_{m-l-1})$ terms vanish.

(2) Suppose $c_{n-l-1} > 0$. Then $\partial_V x_{n-l-2} = 0$ and $\partial_V x_{n-l-1} = V^{c_{n-l-1}} x_{n-l-2}$. Assume that c_{n-l} and b_{n-l} have the same sign. We compute

$$h\partial_V(x_{n-l-2}) = 0, \quad \partial_V h(x_{n-l-2}) = \partial_V g(x_{n-l-2}) = g\partial_V(x_{n-l-2}) = 0.$$

Similarly,

$$h\partial_V(x_{n-l-1}) = h(V^{c_{n-l-1}} x_{n-l-2}) = V^{c_{n-l-1}} g(x_{n-l-2}),$$

$$\begin{aligned}\partial_V h(x_{n-l-1}) &= \partial_V(g(x_{n-l-1}) + U^{b_{m-l}-c_{n-l}} f(y_{m-l-1})) = g\partial_V(x_{n-l-1}) \\ &= V^{c_{n-l-1}} g(x_{n-l-2}),\end{aligned}$$

as desired. If c_{n-l} and b_{m-l} have different signs, then the same computation holds, except that the $U^{b_{m-l}-c_{n-l}} f(y_{m-l-1})$ terms vanish.

(3) Suppose $c_{n-l} < 0$. Then $\partial_U x_{n-l-1} = U^{|c_{n-l}|} x_{n-l}$ and $\partial_U x_{n-l} = 0$. We compute

$$\begin{aligned}h\partial_U(x_{n-l-1}) &= h(U^{|c_{n-l}|} x_{n-l}) = U^{|c_{n-l}|}(g(x_{n-l}) + f(y_{m-l})), \\ \partial_U h(x_{n-l-1}) &= \partial_U(g(x_{n-l-1}) + U^{b_{m-l}-c_{n-l}} f(y_{m-l-1})) \\ &= g\partial_U(x_{n-l-1}) + U^{b_{m-l}-c_{n-l}} f\partial_U(y_{m-l-1}) \\ &= U^{|c_{n-l}|} g(x_{n-l}) + U^{b_{m-l}-c_{n-l}} f(U^{|b_{m-l}|} y_{m-l}) \\ &= U^{|c_{n-l}|}(g(x_{n-l}) + f(y_{m-l})).\end{aligned}$$

In the penultimate equality above, we are using the fact that $b_{m-l} <^! c_{n-l} < 0$ to conclude that $\partial_U(y_{m-l-1}) = U^{|b_{m-l}|} y_{m-l}$; we use this again in the final equality to write $|b_{m-l}| = -b_{m-l}$ and $|c_{n-l}| = -c_{n-l}$. Similarly,

$$h\partial_U(x_{n-l}) = 0, \quad \partial_U h(x_{n-l}) = \partial_U(g(x_{n-l}) + f(y_{m-l})) = 0,$$

where in the second equality above, we again use $b_{m-l} <^! c_{n-l} < 0$.

(4) Suppose $c_{n-l} > 0$. Then $\partial_U x_{n-l-1} = 0$ and $\partial_U x_{n-l} = U^{c_{n-l}} x_{n-l-1}$. We consider two further subcases, based on whether $b_{m-l} < 0$ or $b_{m-l} > 0$.

(a) Suppose $b_{m-l} < 0$, so that $\partial_U y_{m-l-1} = U^{|b_{m-l}|} y_{m-l}$ and $\partial_U y_{m-l} = 0$. Then

$$h\partial_U(x_{n-l-1}) = 0, \quad \partial_U h(x_{n-l-1}) = \partial_U g(x_{n-l-1}) = 0.$$

Similarly,

$$\begin{aligned}h\partial_U(x_{n-l}) &= h(U^{c_{n-l}} x_{n-l-1}) = U^{c_{n-l}} g(x_{n-l-1}), \\ \partial_U h(x_{n-l}) &= \partial_U(g(x_{n-l}) + f(y_{m-l})) = U^{c_{n-l}} g(x_{n-l-1}),\end{aligned}$$

as desired.

(b) Suppose $b_{m-l} > 0$, so that $\partial_U y_{m-l-1} = 0$ and $\partial_U y_{m-l} = U^{b_{m-l}} y_{m-l-1}$. Then

$$\begin{aligned}&h\partial_U(x_{n-l-1}) = 0, \\ &\partial_U h(x_{n-l-1}) = \partial_U(g(x_{n-l-1}) + U^{b_{m-l}-c_{n-l}} f(y_{m-l-1})) = 0.\end{aligned}$$

Similarly,

$$\begin{aligned} h\partial_U(x_{n-l}) &= h(U^{c_{n-l}} x_{n-l-1}) \\ &= U^{c_{n-l}}(g(x_{n-l-1}) + U^{b_{m-l}-c_{n-l}} f(y_{m-l-1})) \\ &= U^{c_{n-l}} g(x_{n-l-1}) + U^{b_{m-l}} f(y_{m-l-1}), \\ \partial_U h(x_{n-l}) &= \partial_U(g(x_{n-l}) + f(y_{m-l})) \\ &= U^{c_{n-l}} g(x_{n-l-1}) + U^{b_{m-l}} f(y_{m-l-1}), \end{aligned}$$

as desired.

This shows that h is a chain map, at least when $l < n-1$ and $n-l$ is odd. The proof when $n-l$ is even follows by interchanging the roles of U and V. (There is a slight reinterpretation of case (4) when $l = m$, which we leave to the reader.)

Finally, we consider the remaining cases when $l = n$ or $l = n-1$. If $l = n$, then the only nontrivial check is to show that $\partial h(x_0) = h\partial(x_0)$. In this situation, we have $c_{m-n} <^! b_0 = 0$. First, suppose that $c_{n-m+1} = b_1 < 0$. Then

$$\begin{aligned} \partial h(x_0) &= \partial f(x_0) + \partial g(y_{n-m}) = U^{|b_1|} f(x_1) + U^{|c_{n-m+1}|} g(y_{n-m+1}), \\ h\partial(x_0) &= h(U^{|b_1|} x_1) = U^{|b_1|} h(x_1) = U^{|b_1|}(f(x_1) + g(y_{n-m+1})). \end{aligned}$$

The case $b_1 > 0$ is analogous. The situation when $l = n-1$ is similar in flavor, and we leave it to the reader.

We now claim that h is a local map. If $l < n-1$, then $h(x_0) = g(x_0)$, and so clearly h is local. If $l = n-1$, then $h(x_0) = g(x_0) + U^{b_{m-n+1}-c_1} f(y_{m-n})$. Hence $h(x_0)$ and $g(x_0)$ are equal in C/U, and h is again local. Finally, if $l = n$, then $h(x_0) = g(x_0) + f(y_{m-n})$. Since $b_{m-n} <^! c_0 = 0$, we have that $\partial_V y_{m-n} = 0$ and $\partial_V y_{m-n-1} = V^{|b_{m-n}|} y_{m-n}$. Hence $f(y_{m-n})$ is a V–torsion cycle in $H_*(C/U)$. Since $g(x_0)$ generates $H_*(C/U)/V$–torsion, this shows that h is local, as desired.

By construction, $h(x_n) = f(y_m) + g(x_n) = 0$. Applying Lemma 6.7, we conclude that $C(c_1, \dots, c_n)$ is not n–maximal with respect to C. □

We are now ready to prove Proposition 6.3:

Proof of Proposition 6.3 We proceed by contradiction. Suppose that $a_i \neq 0$ for all indices i. Let n be very large. By Proposition 5.2, we have a short local map

$$g\colon C(a_1, \dots, a_n) \rightsquigarrow C.$$

Since C is finitely generated, it follows from Lemma 6.7 that for n sufficiently large, we must have $g(x_m) = g(x_n)$ for some $m < n$. Indeed, Lemma 6.7 implies that the gradings of the $g(x_i)$ must lie in a bounded interval, since otherwise some $g(x_i)$ would be in $\operatorname{im} U$ or $\operatorname{im} V$. Hence $g(x_m) = g(x_n)$ for some $m < n$.

Consider the short local map

$$f\colon C(a_1, \dots, a_m) \rightsquigarrow C$$

obtained by restricting g. On one hand, $C(a_1, \dots, a_m)$ and $C(a_1, \dots, a_n)$ are evidently m– and n–maximal with respect to C. However, since $m \neq n$, we have that either $(a_m, \dots, a_1) <^! (a_n, \dots, a_1)$ or $(a_n, \dots, a_1) <^! (a_m, \dots, a_1)$. Hence we may apply Lemma 6.8, either with the maps f and g, or vice versa. This gives a contradiction. □

We now justify Remark 3.5 and show that if C_1 and C_2 are locally equivalent via maps f and g, then f and g take U–tower classes to U–tower classes:

Lemma 6.9 *Let C_1 and C_2 be knot-like complexes. If C_1 and C_2 are locally equivalent via f and g, then f and g induce isomorphisms on $H_*(C_i/V)/U$–torsion.*

Proof By passing to the same local representative, we may assume that C_1 is a standard complex. Then $g \circ f$ is a local map from a standard complex to itself, which is an isomorphism by Lemma 6.6. In particular, $g \circ f$ induces an isomorphism from $H_*(C_1/V)/U$–torsion to itself, factoring through the composition

$$H_*(C_1/V)/U\text{–torsion} \xrightarrow{f} H_*(C_2/V)/U\text{–torsion} \xrightarrow{g} H_*(C_1/V)/U\text{–torsion}.$$

Since each of the above terms consists of a single U–tower, it is clear that the induced maps must individually be isomorphisms. □

Finally, we show that the standard complex associated to any knot is symmetric:

Lemma 6.10 *Let K be a knot in S^3, and let $C = C(a_1, \dots, a_n)$ be the standard complex representative of $\mathrm{CFK}_{\mathcal{R}}(K)$. Then C is symmetric.*

Proof Given Lemma 6.9, it is clear that the definition of local equivalence is in fact completely symmetric with respect to interchanging the roles of U and V. That is, we may require the maps f and g in Definition 3.3 to be absolutely U– and V–graded, and induce isomorphisms on both $H_*(C_i/U)/V$–torsion and $H_*(C_i/V)/U$–torsion. Now suppose that f and g are such local equivalences between C and $\mathrm{CFK}_{\mathcal{R}}(K)$. Then it is not hard to see that we have local equivalences between these two complexes with the roles of U and V reversed; ie

$$\overline{C} \sim \overline{\mathrm{CFK}}_{\mathcal{R}}(K).$$

However, we already know that $\mathrm{CFK}_{\mathcal{R}}(K)$ is homotopy equivalent to $\overline{\mathrm{CFK}}_{\mathcal{R}}(K)$, so $C \sim \overline{C}$. It is easily checked that passing from C to $\overline{C}$ reverses the order of the standard complex parameters, showing that C is symmetric, as desired. □

7 Homomorphisms

In this section, we construct an infinite family of linearly independent homomorphisms from $\mathfrak{K}$ to $\mathbb{Z}$.

7.1 Some $\mathbb{Z}$–valued homomorphisms

We begin with the following definition:

Definition 7.1 Let $C = C(a_1, \dots, a_n)$ be a standard complex. Define

$$\varphi_j(C) = \#\{a_i \mid a_i = j,\ i \text{ odd}\} - \#\{a_i \mid a_i = -j,\ i \text{ odd}\}.$$

That is, $\varphi_j(C)$ is the signed count of the number of times that j appears as an odd parameter a_{2k+1}. Equivalently, $\varphi_j(C)$ is the signed count of horizontal arrows of length j. If C is any knot-like complex, then we define $\varphi_j(C)$ by passing to the standard complex representative of C afforded by Theorem 6.1.

The goal of this section is to prove the following theorem:

Theorem 7.2 *For each $j \in \mathbb{N}$, the function*

$$\varphi_j : \mathfrak{K} \to \mathbb{Z}$$

is a homomorphism.

Note that the product of two standard complexes is not a standard complex. Thus, to compute $\varphi_j(C_1 \otimes C_2)$ directly, we would first have to determine the standard complex representative of $C_1 \otimes C_2$. However, it turns out that we do not currently have an explicit description of the group law on $\mathfrak{K}$ in terms of the standard complex parameters (see Section 11). Instead, we prove Theorem 7.2 by expressing each φ_j as a linear combination of other auxiliary homomorphisms. The construction of these (and the proof that they are additive) will occupy our attention for the next two subsections.

Before proceeding, we show that Theorem 1.1 follows readily from Theorem 7.2:

Proof of Theorem 1.1 By Theorem 2.5 and the behavior of $\mathrm{CFK}_{\mathcal{R}}(K)$ under connected sum, we have a homomorphism

$$\mathcal{C} \to \mathfrak{K}$$

sending $[K]$ to $[\mathrm{CFK}_{\mathcal{R}}(K)]$. Now compose with φ_j. (We henceforth abuse notation slightly and also refer to the composition $\mathcal{C} \to \mathfrak{K} \to \mathbb{Z}$ as φ_j.) Surjectivity of

$$\bigoplus_{j=1}^{\infty} \varphi_j : \mathcal{C} \to \bigoplus_{j=1}^{\infty} \mathbb{Z}$$

follows from the observation that $\varphi_j(T_{i+1,i+2} \# -T_{i,i+1}) = \delta_{ij}$ (see Example 1.4), or alternatively by considering the knots in Proposition 9.1. □

We now introduce the first of our auxiliary homomorphisms:

Definition 7.3 Let C be a knot-like complex and let $C(a_1, \dots, a_n)$ be the standard complex representative of C given by Theorem 6.1. Define

$$P(C) = -2 \sum_{j>0} j\varphi_j(C) + \sum_{i=1}^{n} \operatorname{sgn} a_i .$$

It is clear that φ_j is an invariant of the local equivalence class of C. To see that P is a homomorphism, we use the following alternative definition:

Lemma 7.4 *The integer $P(C)$ is equal to the U–grading of a U–tower generator.*

Proof By Corollary 6.2, C is homotopy equivalent to $C(a_1, \dots, a_n) \oplus A$, where $a_i = a_i(C)$ and A is some $\mathcal{R}$–complex. Since C is a knot-like complex, $U^{-1}H_*(C) \cong \mathbb{F}[U, U^{-1}]$, and so the U–nontorsion classes in C are supported by the standard summand $C(a_1, \dots, a_n)$. It is then clear that x_n is a U–tower generator in C. A straightforward computation shows that $\mathrm{gr}_U(x_n)$ is given by the expression in Definition 7.3. □

Given this, we immediately have:

Proposition 7.5 *The function $P : \mathfrak{K} \to 2\mathbb{Z}$ is a surjective homomorphism.*

Proof The fact that P is a homomorphism follows from the Künneth formula. To see that P is surjective, we observe that $P(C(1,-1)) = -2$. □

Before proceeding, we show Proposition 7.6 from the introduction:

Proposition 7.6 *Let K be a knot in S^3. Then we have the following equality relating the Ozsváth-Szabó τ–invariant with φ_j:*

$$\tau(K) = \sum_{j \in \mathbb{N}} j\varphi_j(K).$$

Proof It is sufficient to consider the local equivalence class of $\mathrm{CFK}_{\mathcal{R}}(K)$. Let $C = C(a_1, \dots, a_n)$ be the local equivalence class of $\mathrm{CFK}_{\mathcal{R}}(K)$. Then C is symmetric, so $\sum_{i=1}^{n} \operatorname{sgn} a_i = 0$ and $P(C) = \mathrm{gr}_U(x_n) = -2\tau(K)$. □

7.2 Shift homomorphisms

We now introduce an auxiliary family of endomorphisms $\mathrm{sh}_m : \mathfrak{K} \to \mathfrak{K}$ for $m \in \mathbb{N}$. Composing these with P gives an infinite sequence of homomorphisms $P \circ \mathrm{sh}_m : \mathfrak{K} \to 2\mathbb{Z}$. In the next subsection, we show that the φ_j are certain linear combinations of the $P \circ \mathrm{sh}_m$ (divided by 2). Our present goal will be to define the sh_m and show that they are additive. This will be the most technical part of the argument, and will require the introduction of several auxiliary definitions.

Definition 7.7 Let $C = C(a_1, \dots, a_n)$ be a standard complex. Let $\mathrm{sh}_m(C)$ be the standard complex given by

$$\mathrm{sh}_m(C) = C(a_1', \dots, a_n'),$$

where

$$a_i' = \begin{cases} a_i + 1 & \text{if } a_i \geq m, \\ a_i - 1 & \text{if } a_i \leq -m, \\ a_i & \text{if } |a_i| < m. \end{cases}$$

That is, sh_m fixes U^n– and V^n–arrows for $n < m$ and takes U^n– and V^n–arrows to U^{n+1}– and V^{n+1}–arrows, respectively, for $n \geq m$.

The majority of this subsection will be devoted to proving the following theorem:

Theorem 7.8 *For all $m \geq 1$, the function $\mathrm{sh}_m \colon \mathfrak{K} \to \mathfrak{K}$ is a homomorphism; that is, for knot-like complexes C_1 and C_2, we have the local equivalence*

$$\mathrm{sh}_m(C_1 \otimes C_2) \sim \mathrm{sh}_m(C_1) \otimes \mathrm{sh}_m(C_2).$$

It will also be helpful to decompose sh_m as a composition of a shift in U and a shift in V (denoted by $\mathrm{sh}_{U,m}$ and $\mathrm{sh}_{V,m}$, respectively):

Definition 7.9 Given a standard complex $C = C(a_1, \dots, a_n)$, let

$$\mathrm{sh}_{U,m}(C) = C(a_1', \dots, a_n'),$$

where, for i odd,

$$a_i' = \begin{cases} a_i + 1 & \text{if } a_i \geq m, \\ a_i - 1 & \text{if } a_i \leq -m, \\ a_i & \text{if } |a_i| < m, \end{cases}$$

and, for i even,

$$a_i' = a_i.$$

Similarly, let

$$\mathrm{sh}_{V,m}(C) = C(a_1', \dots, a_n')$$

where, for i even,

$$a_i' = \begin{cases} a_i + 1 & \text{if } a_i \geq m, \\ a_i - 1 & \text{if } a_i \leq -m, \\ a_i & \text{if } |a_i| < m, \end{cases}$$

and, for i odd,

$$a_i' = a_i.$$

It follows from the definitions that $\mathrm{sh}_m = \mathrm{sh}_{V,m} \circ \mathrm{sh}_{U,m}$.

Lemma 7.10 *Let $C = C(a_1, \dots, a_n)$ be a standard complex. Then*

$$\mathrm{sh}_{U,m}(C)^\vee = \mathrm{sh}_{U,m}(C^\vee) \quad \textit{and} \quad \mathrm{sh}_{V,m}(C)^\vee = \mathrm{sh}_{V,m}(C^\vee).$$

Proof The result follows from the definition of $\mathrm{sh}_{U,m}$ and $\mathrm{sh}_{V,m}$ combined with Lemma 4.9. □

We now introduce some convenient terminology:

Definition 7.11 Let C be a knot-like complex (not necessarily a standard complex) with an $\mathcal{R}$–basis $\{x_i\}$. We say that $\{x_i\}$ is *U–simplified* if for each x_i, exactly one of the following holds:

(1) $\partial_U x_i = U^k x_j$ for some j and k,

(2) $\partial_U x_j = U^k x_i$ for some j and k, or

(3) $\partial_U x_i = 0$ and $x_i \notin \operatorname{im} \partial_U$.

If $\partial x_i = U^k x_j$ (or vice versa), we say that x_i and x_j are *U–paired*. Since $H_*(C/V)$ has a single U–tower, it follows that at most one of the x_i satisfies (3). We define a *V–simplified basis* and *V–paired* basis elements analogously. (See for example the proof of Lemma 3.15.)

Example 7.12 Let $C = C(a_1, \dots, a_n)$ be a standard complex with preferred basis $\{x_i\}_{i=0}^n$; this basis is clearly both U– and V–simplified. We will find it convenient to relabel our basis elements slightly. We denote the *U–simplified basis* $\{w, y_i, z_i\}_{i=1}^{n/2}$ for C by

$$w = x_n$$

and, for each $1 \le i \le \frac{1}{2}n$,

$$y_i = \begin{cases} x_{2i-1} & \text{if } a_{2i-1} > 0, \\ x_{2i-2} & \text{if } a_{2i-1} < 0, \end{cases} \qquad z_i = \begin{cases} x_{2i-1} & \text{if } a_{2i-1} < 0, \\ x_{2i-2} & \text{if } a_{2i-1} > 0. \end{cases}$$

Setwise, the U–simplified basis is of course identical to the standard preferred basis, but we fix notation so that $\partial_U y_i = U^{|a_{2i-1}|} z_i$. (That is, y_i and z_i are U–paired.) We can likewise define the *V–simplified basis* in the obvious way.

Definition 7.13 For $C = C(a_1, \dots, a_n)$, let $\{w, y_i, z_i\}$ and $\{w', y_i', z_i'\}$ be the U–simplified bases for C and $\mathrm{sh}_{U,m}(C)$, respectively. Define an $\mathcal{R}$–module map

$$s_{U,m} \colon C \to \mathrm{sh}_{U,m}(C)$$

by setting

$$s_{U,m}(r) = r'$$

for each $r \in \{w, y_i, z_i\}$, and extending $\mathcal{R}$–linearly. That is, $s_{U,m}$ simply effects the correspondence between the unprimed generators of C and the primed generators of $\mathrm{sh}_{U,m}(C)$. Note that $s_{U,m}$ induces an isomorphism of ungraded $\mathcal{R}$–modules, although we stress that $s_{U,m}$ is *not* graded (even relatively). Furthermore, it is easily checked that

$s_{U,m}\partial_V = \partial_V s_{U,m}$. On the other hand, $s_{U,m}$ does not commute with ∂_U. Explicitly, we have

$$s_{U,m}(\partial_U y_i) = s_{U,m}(U^{|a_{2i-1}|}z_i) = U^{|a_{2i-1}|}z_i', \quad \partial_U(s_{U,m}y_i) = \partial_U(y_i') = U^{|a_{2i-1}'|}z_i'.$$

Note that the above expressions may differ by a power of U, depending on the value of $|a_{2i-1}|$.

Example 7.14 Let $C_1 = C(a_1, \dots, a_{n_1})$ and $C_2 = C(b_1, \dots, b_{n_2})$ be standard complexes. Abusing notation slightly, let w, y_i and z_i denote the U–simplified bases for both C_1 and C_2; it will be clear from context which generators lie in C_1 and C_2. Then the obvious tensor product basis for $C_1 \otimes C_2$ is not U–simplified. Instead, we define a U–simplified basis for $C_1 \otimes C_2$ as follows. For $1 \le i \le \frac{1}{2}n_2$, let

$$\alpha_i = w \otimes y_i, \quad \beta_i = w \otimes z_i,$$

and, for $1 \le i \le \frac{1}{2}n_1$, let

$$\gamma_i = y_i \otimes w, \quad \delta_i = z_i \otimes w.$$

For $1 \le i \le \frac{1}{2}n_1$ and $1 \le j \le \frac{1}{2}n_2$, define

$$\epsilon_{i,j} = y_i \otimes y_j, \qquad \zeta_{i,j} = \begin{cases} U^{|b_{2j-1}|-|a_{2i-1}|}y_i \otimes z_j + z_i \otimes y_j & \text{if } |a_{2i-1}| \le |b_{2j-1}|, \\ y_i \otimes z_j + U^{|a_{2i-1}|-|b_{2j-1}|}z_i \otimes y_j & \text{if } |a_{2i-1}| > |b_{2j-1}|, \end{cases}$$

and

$$\eta_{i,j} = \begin{cases} y_i \otimes z_j & \text{if } |a_{2i-1}| \le |b_{2j-1}|, \\ z_i \otimes y_j & \text{if } |a_{2i-1}| > |b_{2j-1}|, \end{cases} \qquad \theta_{i,j} = z_i \otimes z_j.$$

Finally, let

$$\omega = w \otimes w.$$

Note that the following basis elements are U–paired:

$$\{\alpha_i, \beta_i\}, \quad \{\gamma_i, \delta_i\}, \quad \{\epsilon_{i,j}, \zeta_{i,j}\}, \quad \{\eta_{i,j}, \theta_{i,j}\}.$$

For notational convenience, we relabel the basis elements

$$\{\kappa_l\} = \{\alpha_i\} \cup \{\gamma_i\} \cup \{\epsilon_{i,j}\} \cup \{\eta_{i,j}\}, \quad \{\lambda_l\} = \{\beta_i\} \cup \{\delta_i\} \cup \{\zeta_{i,j}\} \cup \{\theta_{i,j}\},$$

so that $\{\omega, \kappa_l, \lambda_l\}$ is a U–simplified basis and $\partial_U \kappa_l = U^{e_l}\lambda_l$ for some e_l. The reader should check that if κ_l is one of $\epsilon_{i,j}$ or $\eta_{i,j}$, then

$$e_l = \min(|a_{2i-1}|, |b_{2j-1}|).$$

If κ_l is an α_i, then $e_l = |b_{2i-1}|$, while if κ_l is a γ_i, then $e_l = |a_{2i-1}|$.

We analogously define a U–simplified basis $\{\alpha'_i, \beta'_i, \gamma'_i, \delta'_i, \epsilon'_{i,j}, \zeta'_{i,j}, \eta'_{i,j}, \theta'_{i,j}, \omega'\}$ for $\mathrm{sh}_{U,m}(C_1) \otimes \mathrm{sh}_{U,m}(C_2)$ by considering both factors as standard complexes in their own right. (That is, $\alpha'_i = w' \otimes y'_i$, and so on.) We relabel this basis $\{\omega', \kappa'_l, \lambda'_l\}$ as before, so that κ'_l and λ'_l are U–paired. As above, we have $\partial_U \kappa'_l = U^{e'_l} \lambda'_l$, where

$$e'_l = \min(|a'_{2i-1}|, |b'_{2j-1}|),$$

whenever κ'_l is one of $\epsilon'_{i,j}$ or $\eta'_{i,j}$ (similarly for the other cases). An examination of Definition 7.9 then shows that we may write

$$e'_l = e_l + \tau(l),$$

where

$$\tau(l) = \begin{cases} 0 & \text{if } e_l < m, \\ 1 & \text{if } e_l \geq m. \end{cases}$$

Definition 7.15 Let C_1 and C_2 be standard complexes. Define an $\mathcal{R}$–module map

$$\sigma_{U,m} \colon C_1 \otimes C_2 \to \mathrm{sh}_{U,m}(C_1) \otimes \mathrm{sh}_{U,m}(C_2)$$

by setting

$$\sigma_{U,m}(\xi) = \xi'$$

for $\xi \in \{\omega, \kappa_l, \lambda_l\}$, and extending $\mathcal{R}$–linearly. As in Definition 7.13, $\sigma_{U,m}$ induces an isomorphism of ungraded $\mathcal{R}$–modules. Furthermore, we claim that $\sigma_{U,m}\partial_V = \partial_V \sigma_{U,m}$. To see this, observe that

$$\sigma_{U,m} \equiv s_{U,m} \otimes s_{U,m} \mod U.$$

Indeed, this congruence is obviously an equality for all basis elements not of the form $\gamma_{i,j}$ or $\eta_{i,j}$. For $\eta_{i,j}$, we again have equality using the fact that $|a_{2i-1}| \leq |b_{2j-1}|$ if and only if $|a'_{2i-1}| \leq |b'_{2j-1}|$. For basis elements of the form $\gamma_{i,j}$, a straightforward casework check establishes the congruence. The fact that $s_{U,m}$ commutes with ∂_V then shows that $\sigma_{U,m}\partial_V = \partial_V \sigma_{U,m}$. Again, however, note that $\sigma_{U,m}$ does not commute with ∂_U.

We now introduce an auxiliary technical definition which we will need to prove Theorem 7.8:

Definition 7.16 An *almost chain map* $f \colon C(a_1, \dots, a_n) \to C$ from a standard complex with preferred basis $\{x_i\}_{i=0}^n$ to a knot-like complex is an ungraded $\mathcal{R}$–module map such that, for $1 \leq i \leq n$,

(1) for i odd,

(a) if $a_i < 0$, that is, $\partial_U x_{i-1} = U^{|a_i|} x_i$, we have

$$\partial_U f(x_{i-1}) \equiv U^{|a_i|} f(x_i) \mod U^{|a_i|+1};$$

(b) if $a_i > 0$, that is, $\partial_U x_i = U^{|a_i|} x_{i-1}$, we have

$$\partial_U f(x_i) \equiv U^{|a_i|} f(x_{i-1}) \mod U^{|a_i|+1};$$

(2) for i even,

(a) if $a_i < 0$, that is, $\partial_V x_{i-1} = V^{|a_i|} x_i$, we have

$$\partial_V f(x_{i-1}) \equiv V^{|a_i|} f(x_i) \mod V^{|a_i|+1};$$

(b) if $a_i > 0$, that is, $\partial_V x_i = V^{|a_i|} x_{i-1}$, we have

$$\partial_V f(x_i) \equiv V^{|a_i|} f(x_{i-1}) \mod V^{|a_i|+1}.$$

We stress that an almost chain map is *not* in general a chain map, and may not even be grading-homogeneous.

The main import of the (admittedly unmotivated) notion of an almost chain map will be the following lemma, which explains how to extract a genuine chain map from a given almost chain map. In our context, it will be easier to construct almost chain maps, which is why we have introduced Definition 7.16. In what follows, let $[x]_{p,q}$ denote the homogeneous part of x in bigrading (u, v).

Lemma 7.17 *Let $f: C(a_1, \dots, a_n) \to C$ be an almost chain map. Let (u_i, v_i) be the bigrading of the generator x_i in $C(a_1, \dots, a_n)$. Suppose that $[f(x_0)]_{u_0,v_0}$ represents a V–tower class in C and $\partial_U [f(x_n)]_{u_n,v_n} = 0$. Then there exists a genuine local map*

$$g: C(a_1, \dots, a_n) \to C$$

such that $g(x_i) \equiv [f(x_i)]_{u_i,v_i} \mod (U, V)$ for all $0 \leq i \leq n$.

Proof For each $0 \leq i \leq n$, consider the ansatz

$$g(x_i) = [f(x_i)]_{u_i,v_i} + Up_i + Vq_i,$$

where p_i and q_i are undetermined elements of $C(a_1, \dots, a_n)$ with bigrading (u_i, v_i). In order to determine p_i and q_i, we substitute our ansatz into the chain map condition for g. We begin by using the condition $\partial_U g = g \partial_U$ to help determine the p_i:

(1) Let i be odd, and suppose $a_i < 0$. Then $\partial_U x_{i-1} = U^{|a_i|} x_i$ and $\partial_U x_i = 0$. Using Definition 7.16, write

$$\partial_U f(x_{i-1}) = U^{|a_i|} f(x_i) + U^{|a_i|+1} \eta_i$$

for some (possibly nonhomogeneous) element $\eta_i \in C(a_1, \dots, a_n)$. Note that since $\partial_U^2 = 0$, we have $\partial_U f(x_i) + U \partial_U \eta_i = 0$. We now compute

$$\begin{aligned} g(\partial_U x_{i-1}) &= U^{|a_i|} g(x_i) \\ &= U^{|a_i|}([f(x_i)]_{u_i,v_i} + U p_i + V q_i) \\ &= U^{|a_i|}[f(x_i)]_{u_i,v_i} + U^{|a_i|+1} p_i, \\ \partial_U g(x_{i-1}) &= \partial_U([f(x_{i-1})]_{u_{i-1},v_{i-1}} + U p_{i-1} + V q_{i-1}) \\ &= U^{|a_i|}[f(x_i)]_{u_i,v_i} + U^{|a_i|+1}[\eta_i]_{u_i+2,v_i+2} + U \partial_U p_{i-1}, \end{aligned}$$

where in the last line, we have used the fact that $\partial_U f(x_{i-1}) = U^{|a_i|} f(x_i) + U^{|a_i|+1} \eta_i$. We likewise compute

$$\begin{aligned} g(\partial_U x_i) &= g(0) = 0, \\ \partial_U g(x_i) &= \partial_U([f(x_i)]_{u_i,v_i} + U p_i + V q_i) = \partial_U [f(x_i)]_{u_i,v_i} + U \partial_U p_i. \end{aligned}$$

Examining the first pair of equalities above, we see that it suffices to set $p_{i-1} = 0$ and $p_i = [\eta_i]_{u_i+2,v_i+2}$. The second pair of equalities then follows from the fact that $\partial_U f(x_i) + U \partial_U \eta_i = 0$.

(2) Let i be odd, and suppose $a_i > 0$. Then $\partial x_i = U^{a_i} x_{i-1}$ and $\partial_U x_{i-1} = 0$. A similar analysis as above (interchanging the roles of i and $i-1$ and replacing $|a_i|$ with a_i) shows that if we set $p_{i-1} = [\eta_{i-1}]_{u_{i-1}+2,v_{i-1}+2}$ and $p_i = 0$, then we have $(g\partial_U + \partial_U g)(x_{i-1}) = (g\partial_U + \partial_U g)(x_i) = 0$.

In this manner, by considering all odd indices $1 \le i \le n$, we see that we can choose the p_i for $0 \le i < n$ so that $(g\partial_U + \partial_U g)(x_i) = 0$ for all $0 \le i < n$. Define $p_n = 0$. Then $\partial_U g(x_n) = \partial_U [f(x_n)]_{u_i,v_i} = 0$ by hypothesis, while $g\partial_U(x_n) = 0$. This establishes the ∂_U–condition for all generators x_i.

Interchanging the roles of U and V, an analogous argument (where we consider the case when i is even) allows us to choose the q_i such that $(g\partial_V + \partial_V g)(x_i) = 0$ for all $0 \le i \le n$. (To establish the ∂_V–condition for x_0, we use the fact that $\partial_V [f(x_0)]_{u_0,v_0} = 0$, since $[f(x_0)]_{u_0,v_0}$ represents a V–tower class in C by hypothesis.) By construction, g is a graded, $\mathcal{R}$–equivariant chain map which is clearly local. This completes the proof. $\square$

Now let $C_3 = C(c_1, \dots, c_{n_3})$ be a standard complex, and let $f\colon C_3 \to C_1 \otimes C_2$ be a local map. Our goal will be to construct a shifted map $f_{U,m}$ from $\mathrm{sh}_m C_3$ to $\mathrm{sh}_m C_1 \otimes \mathrm{sh}_m C_2$. We do this by first constructing an almost chain map between the desired complexes, and then applying Lemma 7.17. The construction of $f_{U,m}$ (and the verification that it is an almost chain map) will be the most technical part of the argument and will occupy our attention for the next few pages.

Definition 7.18 Let $\{w, y_i, z_i\}$ and $\{w', y_i', z_i'\}$ be the U–simplified bases for C_3 and $\mathrm{sh}_{U,m}(C_3)$, respectively. Define

$$f_{U,m}\colon \mathrm{sh}_{U,m}(C_3) \to \mathrm{sh}_{U,m}(C_1) \otimes \mathrm{sh}_{U,m}(C_2)$$

by first setting

$$f_{U,m}(r') = \sigma_{U,m} f(r)$$

whenever $r' \in \{w', z_i'\}$. To define $f_{U,m}(y_i')$, we proceed with some casework. Write $f(y_i)$ in terms of the U–simplified basis for $C_1 \otimes C_2$, so that

$$f(y_i) = \sum_{j \in J_1} \kappa_j + \sum_{j \in J_2} U^{p_j} \kappa_j + \sum_{j \in J_3} V^{q_j} \kappa_j + \sum_j P_j(U,V)\lambda_j + Q(U,V)\omega$$

for some $p_j, q_j \in \mathbb{N}$, $P_j, Q \in \mathcal{R}$ and disjoint index sets J_1, J_2 and J_3. We define $f_{U,m}(y_i')$ based on the value of $|c_{2i-1}|$. If $|c_{2i-1}| < m$, let

$$\begin{aligned} f_{U,m}(y_i') &= \sigma_{U,m} f(y_i) \\ &= \sigma_{U,m}\Bigl(\sum_{j \in J_1} \kappa_j + \sum_{j \in J_2} U^{p_j} \kappa_j + \sum_{j \in J_3} V^{q_j} \kappa_j + \sum_j P_j(U,V)\lambda_j + Q(U,V)\omega\Bigr) \end{aligned}$$

as before. If $|c_{2i-1}| \geq m$, let

$$\begin{aligned} f_{U,m}(y_i') = \sigma_{U,m}\Bigl(\sum_{j \in J_1} U^{\overline{\tau}(j)} \kappa_j + \sum_{j \in J_2} U^{p_j + \overline{\tau}(j)} \kappa_j + \sum_{j \in J_3} V^{q_j} \kappa_j \\ + \sum_j P_j(U,V)\lambda_j + Q(U,V)\omega\Bigr), \end{aligned}$$

where

$$\overline{\tau}(j) = \begin{cases} 1 & \text{if } e_j < m, \\ 0 & \text{if } e_j \geq m. \end{cases}$$

Observe that $\tau(j) + \overline{\tau}(j) = 1$. In addition, note that if $f(y_i)$ is supported by κ_j, then $e_j \geq |c_{2i-1}|$. This follows from the fact that $\partial_U f(y_i) = f(\partial y_i)$ is in $\operatorname{im} U^{|c_{2i-1}|}$, while $\partial_U \kappa_j = U^{e_j} \lambda_j$. Hence, in particular, if $|c_{2i-1}| \geq m$, then for any $j \in J_1$, we must have $\overline{\tau}(j)=0$. (Thus we could have omitted the very first instance of $U^{\overline{\tau}(j)}$ in the above definition of $f_{U,m}(y_i')$, but we have left it in for future notational convenience.)

We also note that

$$f(U^{|c_{2i-1}|}z_i) = f\partial_U(y_i) = \partial_U f(y_i) = \sum_{j\in J_1} U^{e_j}\lambda_j + \sum_{j\in J_2} U^{p_j+e_j}\lambda_j,$$

hence

$$U^{|c_{2i-1}|}\sigma_{U,m} f(z_i) = \sum_{j\in J_1} U^{e_j}\sigma_{U,m}(\lambda_j) + \sum_{j\in J_2} U^{p_j+e_j}\sigma_{U,m}(\lambda_j). \tag{7-1}$$

Finally, note that

$$\sigma_{U,m} f(r) \equiv f_{U,m} s_{U,m}(r) \mod U \tag{7-2}$$

for all $r\in\{w, y_i, z_i\}$. Indeed, if $r=w$ or z_i, this congruence is an equality by definition, whereas if $r=y_i$, then the claim follows from the fact that (in the $|c_{2i-1}|\geq m$ case) $\overline{\tau}(j)=0$ for all $j\in J_1$.

Lemma 7.19 *Let $f\colon C_3\to C_1\otimes C_2$ be a local map. Then $f_{U,m}$ is an almost chain map.*

Proof Let $\{w, y_i, z_i\}$ be the U–simplified basis for $C_3 = C(c_1,\dots,c_{n_3})$. It suffices to show

$$\partial_U f_{U,m}(y_i') \equiv \begin{cases} U^{|c_{2i-1}|}\sigma_{U,m} f(z_i) & \mod U^{|c_{2i-1}|+1} \quad \text{if } |c_{2i-1}|<m,\\ U^{|c_{2i-1}|+1}\sigma_{U,m} f(z_i) & \mod U^{|c_{2i-1}|+2} \quad \text{if } |c_{2i-1}|\geq m,\end{cases} \tag{7-3}$$

and that

$$\partial_V f_{U,m}(r') \equiv f_{U,m}\partial_V(r') \mod U \tag{7-4}$$

for all $r'\in\{w', y_i', z_i'\}$. (The mod U in the above equation is not necessary, since our complexes are reduced by assumption, but is included for emphasis.)

We first consider (7-3). Suppose $|c_{2i-1}|<m$. Then

$$\begin{aligned}\partial_U f_{U,m}(y_i') &= \partial_U\Bigg(\sum_{j\in J_1}\sigma_{U,m}(\kappa_j) + \sum_{j\in J_2} U^{p_j}\sigma_{U,m}(\kappa_j)\Bigg)\\ &= \sum_{j\in J_1} U^{e_j+\tau(j)}\lambda_j' + \sum_{j\in J_2} U^{p_j}U^{e_j+\tau(j)}\lambda_j'\\ &= \sum_{j\in J_1} U^{e_j+\tau(j)}\sigma_{U,m}(\lambda_j) + \sum_{j\in J_2} U^{p_j}U^{e_j+\tau(j)}\sigma_{U,m}(\lambda_j)\\ &\equiv U^{|c_{2i-1}|}\sigma_{U,m} f(z_i) \mod U^{|c_{2i-1}|+1}.\end{aligned}$$

Here, to obtain the last line, we compare the third line with (7-1), and use the fact that if $\tau(j)=1$, then $e_j \geq m > |c_{2i-1}|$.

Now suppose $|c_{2i-1}| \geq m$. We have

$$\begin{aligned}
\partial_U f_{U,m}(y_i') &= \partial_U\Bigl(\sum_{j\in J_1} U^{\bar\tau(j)}\sigma_{U,m}(\kappa_j) + \sum_{j\in J_2} U^{p_j+\bar\tau(j)}\sigma_{U,m}(\kappa_j)\Bigr)\\
&= \sum_{j\in J_1} U^{e_j+1}\lambda_j' + \sum_{j\in J_2} U^{p_j+\bar\tau(j)}U^{e_j+\tau(j)}\lambda_j'\\
&= \sum_{j\in J_1} U^{e_j+1}\sigma_{U,m}(\lambda_j) + \sum_{j\in J_2} U^{p_j+e_j+1}\sigma_{U,m}(\lambda_j)\\
&= U^{|c_{2i-1}|+1}\sigma_{U,m} f(z_i),
\end{aligned}$$

where in the last line we have used (7-1).

We now consider (7-4). We have

$$\begin{aligned}
\partial_V f_{U,m}(r') &\equiv \partial_V \sigma_{U,m} f(r) && \bmod U\\
&\equiv \sigma_{U,m} f\, \partial_V(r) && \bmod U\\
&\equiv f_{U,m} s_{U,m} \partial_V(r) && \bmod U\\
&\equiv f_{U,m} \partial_V(s_{U,m}(r)) && \bmod U\\
&\equiv f_{U,m} \partial_V(r') && \bmod U
\end{aligned}$$

for any $r' \in \{w', y_i', z_i'\}$ (in fact, for any $r \in C_3$), where the first equivalence is by definition, the second since ∂_V commutes with $\sigma_{U,m}$ and f, the third by (7-2) and the fourth since ∂_V and $s_{U,m}$ commute. □

We now verify the remaining hypotheses of Lemma 7.17. In the proofs of the following lemmas, we denote the standard preferred basis for $\mathrm{sh}_m(C_3)$ by $\{x_i'\}$, and the U–simplified basis by $\{w', y_j', z_j'\}$ as usual.

Lemma 7.20 *With the notation as above, $[f_{U,m}(x_0')]_{u_0',v_0'}$ represents a V–tower class.*

Proof Note that x_0' is one of w', y_j' or z_j' for some j. If $x_0' = w'$ or z_j', then $f_{U,m}(x_0') = \sigma_{U,m} f(x_0)$. The result now follows from the fact that f is local and $\sigma_{U,m}$ induces an ungraded isomorphism between $(C_1 \otimes C_2)/U$ and $(\mathrm{sh}_m(C_1)\otimes \mathrm{sh}_m(C_2))/U$. If $x_0' = y_j'$, then $f_{U,m}(x_0') \equiv \sigma_{U,m} f(x_0) \bmod U$, and the result follows as before. □

Lemma 7.21 *With the notation as above, $\partial_U [f_{U,m}(x_n')]_{u_n',v_n'} = 0$.*

Proof Recall that $x'_n = w'$. Therefore, we have $f_{U,m}(x'_n) = \sigma_{U,m} f(x_n)$. Since f is an $\mathcal{R}$–equivariant chain map and x_n is a U–cycle, it follows that $f(x_n)$ is also a U–cycle. An examination of the definition shows that $\sigma_{U,m}$ takes U–cycles to U–cycles, so $\partial_U f_{U,m}(x'_n) = 0$. □

Putting everything together, we have:

Lemma 7.22 *Let $f: C_3 \to C_1 \otimes C_2$ be a local map. Then there exists a local map*

$$g: \mathrm{sh}_{U,m}(C_3) \to \mathrm{sh}_{U,m}(C_1) \otimes \mathrm{sh}_{U,m}(C_2).$$

Proof By Lemma 7.19, $f_{U,m}$ is an almost chain map; by Lemma 7.20, $[f_{U,m}(x'_0)]_{u'_0,v'_0}$ represents a V–tower class; and by Lemma 7.21, $\partial_U [f_{U,m}(x'_n)]_{u'_n,v'_n} = 0$. Thus Lemma 7.17 gives us the desired local map. □

By reversing the roles of U and V, we may similarly define $f_{V,m}$. We record the analogous set of lemmas below:

Lemma 7.23 *Let $f: C_3 \to C_1 \otimes C_2$ be a local map. With the notation as above, $f_{V,m}$ is an almost chain map.*

Proof The proof is identical to the proof of Lemma 7.19 after reversing the roles of U and V. □

Lemma 7.24 *With the notation as above, $[f_{V,m}(x'_0)]_{u'_0,v'_0}$ represents a V–tower class.*

Proof By definition, $f_{V,m}(x'_0) = \sigma_{V,m} f(x_0)$. Since f is local, $f(x_0)$ represents a V–tower class, and it is easy to check that $\sigma_{V,m}$ takes V–tower classes to V–tower classes. □

Lemma 7.25 *With the notation as above, $\partial_U [f_{V,m}(x'_n)]_{u'_n,v'_n} = 0$.*

Proof We have

$$\partial_U f_{V,m}(x'_n) = f_{V,m} \partial_U (x'_n) = 0,$$

where the first equality follows by the analogue of (7-4). □

Lemma 7.26 *Let $f: C_3 \to C_1 \otimes C_2$ be a local map. Then there exists a local map*

$$g: \mathrm{sh}_{V,m}(C_3) \to \mathrm{sh}_{V,m}(C_1) \otimes \mathrm{sh}_{V,m}(C_2).$$

Proof By Lemma 7.23, $f_{V,m}$ is an almost chain map; by Lemma 7.24, $[f_{V,m}(x'_0)]_{u'_0,v'_0}$ represents a V–tower class; and by Lemma 7.25, $\partial_U[f_{V,m}(x'_n)]_{u'_n,v'_n} = 0$. Thus Lemma 7.17 gives the desired local map. □

We now finally turn to the proof of Theorem 7.8:

Proof of Theorem 7.8 Suppose that $C_3 \sim C_1 \otimes C_2$. Let $f : C_3 \to C_1 \otimes C_2$ be a local map. By Lemma 7.22, we have a local map

$$g : \mathrm{sh}_{U,m}(C_3) \to \mathrm{sh}_{U,m}(C_1) \otimes \mathrm{sh}_{U,m}(C_2),$$

that is,

$$\mathrm{sh}_{U,m}(C_3) \le \mathrm{sh}_{U,m}(C_1) \otimes \mathrm{sh}_{U,m}(C_2). \tag{7-5}$$

Dually, we have $C_3^\vee \sim C_1^\vee \otimes C_2^\vee$, and, by the same argument,

$$\mathrm{sh}_{U,m}(C_3^\vee) \le \mathrm{sh}_{U,m}(C_1^\vee) \otimes \mathrm{sh}_{U,m}(C_2^\vee). \tag{7-6}$$

Dualizing (7-6), applying Lemma 7.10, and combining with (7-5), we obtain

$$\mathrm{sh}_{U,m}(C_1) \otimes \mathrm{sh}_{U,m}(C_2) \le \mathrm{sh}_{U,m}(C_3) \le \mathrm{sh}_{U,m}(C_1) \otimes \mathrm{sh}_{U,m}(C_2).$$

Thus we have

$$\mathrm{sh}_{U,m}(C_3) \sim \mathrm{sh}_{U,m}(C_1) \otimes \mathrm{sh}_{U,m}(C_2).$$

The analogous argument replacing U with V (using Lemma 7.26 instead of Lemma 7.22) shows that

$$\mathrm{sh}_{V,m}(\mathrm{sh}_{U,m}(C_3)) \sim \mathrm{sh}_{V,m}(\mathrm{sh}_{U,m}(C_1)) \otimes \mathrm{sh}_{V,m}(\mathrm{sh}_{U,m}(C_2)).$$

Since $\mathrm{sh}_{V,m} \circ \mathrm{sh}_{U,m} = \mathrm{sh}_m$, it follows that

$$\mathrm{sh}_m(C_1 \otimes C_2) \sim \mathrm{sh}_m(C_1) \otimes \mathrm{sh}_m(C_2),$$

as desired. □

7.3 Proof of Theorem 7.2

We now turn to the proof that the φ_j are additive. By considering the composition

$$P \circ \mathrm{sh}_m : \mathfrak{K} \to 2\mathbb{Z} \quad \text{for } m \in \mathbb{N},$$

we obtain infinitely many homomorphisms from $\mathfrak{K}$ to $2\mathbb{Z}$. The proof of Theorem 7.2 relies on considering certain linear combinations of these homomorphisms.

Proof of Theorem 7.2 Let $C \in \mathfrak{K}$. Since all of our maps are local equivalence invariants, we may assume that $C = C(a_1, \dots, a_n)$ is a standard complex. For any $m \in \mathbb{N}$, write

$$P(\mathrm{sh}_m(C)) = -2 \sum_{1 \le j < m} j\varphi_j(C) - 2 \sum_{j \ge m} (j+1)\varphi_j(C) + \sum_{i=0}^{n} \operatorname{sgn} a_i.$$

Here, we have simply used the definition of sh_m, together with the definition of φ_j as a count of standard complex parameters. This implies that

$$P(\mathrm{sh}_m(C)) - P(C) = -2 \sum_{j \ge m} \varphi_j(C). \tag{7-7}$$

We now use (strong, downward) induction to show that φ_j is a homomorphism for all $j \in \mathbb{N}$. Fix $C_1, C_2 \in \mathfrak{K}$, where $C_1 = C(a_1, \dots, a_{n_1})$ and $C_2 = C(b_1, \dots, b_{n_2})$. For

$$N > \max\{a_i, b_j\},$$

we have $\varphi_N(C_1) = \varphi_N(C_2) = \varphi_N(C_1 \otimes C_2) = 0$. This establishes the base case. Thus, assume that φ_j is a homomorphism for all $j \ge M+1$. We will show that φ_M is also a homomorphism. Indeed,

$$\begin{aligned} -2 \sum_{j \ge M} (\varphi_j(C_1) + \varphi_j(C_2)) &= P(\mathrm{sh}_M(C_1)) + P(\mathrm{sh}_M(C_2)) - P(C_1) - P(C_2) \\ &= P(\mathrm{sh}_M(C_1 \otimes C_2)) - P(C_1 \otimes C_2) \\ &= -2 \sum_{j \ge M} \varphi_j(C_1 \otimes C_2), \end{aligned}$$

where the first and third equalities follow from (7-7), and the second equality follows from the fact that P and sh_M are homomorphisms. By the inductive hypothesis, we have that φ_j is a homomorphism for all $j \ge M+1$. It follows that φ_M is a homomorphism as well. This completes the proof. □

7.4 HFK$^-$ and φ_j

We are now ready to prove Proposition 7.27. Recall that

$$N(K) = \begin{cases} 0 & \text{if } \varphi_j(K) = 0 \text{ for all } j, \\ \max\{j \mid \varphi_j(K) \ne 0\} & \text{otherwise.} \end{cases}$$

Proposition 7.27 *If $U^M \cdot \mathrm{Tors}_U \mathrm{HFK}^-(K) = 0$, then $\varphi_j(K) = 0$ for all $j > M$. In particular, $N(K) \le M$.*

Proof Let $C = C(a_1, \dots, a_n)$ be the standard complex representative of $\mathrm{HFK}^-(K)$ given by Theorem 6.1 and Corollary 6.2. Recall that $H_*(\mathrm{CFK}_{\mathcal{R}}(K)/V) \cong \mathrm{HFK}^-(K)$. Then $U^M \cdot \mathrm{Tors}_U\, \mathrm{HFK}^-(K) = 0$ implies $U^M \cdot \mathrm{Tors}_U\, H_*(C/V) = 0$, which in turn implies that $a_i \le M$ for i odd. The result now follows from the definition of φ_j. □

8 Thin knots and L–space knots

In this section, we prove Propositions 8.1 and 8.2.

Proposition 8.1 *If K is homologically thin, then*

$$\varphi_j(K) = \begin{cases} \tau(K) & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof By [25, Theorem 4], it follows that if K is a thin knot, then $\mathrm{CFK}_{\mathcal{R}}(K)$ is locally equivalent to the standard complex $C(a_1, \dots, a_n)$ where $n = 2|\tau(K)|$ and $a_i = \operatorname{sgn} \tau(K)$ for i odd and $a_i = -\operatorname{sgn} \tau(K)$ for i even. That is, the a_i are an alternating sequence of ± 1, starting with $+1$ if $\tau(K) > 0$ and -1 if $\tau(K) < 0$. The result follows. □

Proposition 8.2 *Let K be an L–space knot with Alexander polynomial*

$$\Delta_K(t) = \sum_{i=0}^{n} (-1)^i t^{b_i},$$

where $(b_i)_{i=0}^n$ is a decreasing sequence of integers and n is even. Define

$$c_i = b_{2i-2} - b_{2i-1} \quad \text{for } 1 \le i \le \tfrac{1}{2}n.$$

Then

$$\varphi_j(C) = \#\{c_i \mid c_i = j\}.$$

Proof By [22, Theorem 1.2] (cf [18, Theorem 2.10]), we have that if K is an L–space knot, then $\mathrm{CFK}_{\mathcal{R}}(K)$ is the standard complex

$$C(c_1, -c_n, c_2, -c_{n-1}, c_3, -c_{n-2}, \dots, c_n, -c_1).$$

The result now follows from the definition of φ_j. □

9 An infinite-rank summand of topologically slice knots

The goal of this section is to prove Theorem 1.9. Let D be the (untwisted, positively clasped) Whitehead double of the right-handed trefoil. Let $K_n = D_{n,n+1} \# -T_{n,n+1}$. The knots K_n are topologically slice and will generate a $\mathbb{Z}^\infty$–summand of $\mathcal{C}_{\mathrm{TS}}$. Indeed, the knot D has Alexander polynomial one, and hence is topologically slice. Thus, the cable $D_{n,n+1}$ is topologically concordant to the underlying pattern torus knot $T_{n,n+1}$, and so $D_{n,n+1} \# -T_{n,n+1}$ is topologically slice.

Proposition 9.1 *Let $D_{n,n+1}$ denote the $(n, n+1)$ cable of the (untwisted, positively clasped) Whitehead double of the right-handed trefoil. Then*

$$\varphi_j(D_{n,n+1}) = \begin{cases} n & \text{if } j = 1, \\ 1 & \text{if } 1 < j < n-1 \text{ or } j = n, \\ 0 & \text{if } j = n-1 > 1 \text{ or } j > n. \end{cases}$$

Proof By [8, Lemma 6.12], the knot D is ε–equivalent to $T_{2,3}$. Thus, by Proposition 4 of [8], we may consider $\mathrm{CFK}_{\mathcal{R}}(T_{2,3;n,n+1})$, where $T_{2,3;n,n+1}$ denotes the $(n,n+1)$–cable of $T_{2,3}$, instead of the locally equivalent $\mathrm{CFK}_{\mathcal{R}}(D_{n,n+1})$. The advantage of this approach is that $T_{2,3;n,n+1}$ is an L–space knot [4, Theorem 1.10] (cf [7]), and so $\mathrm{CFK}_{\mathcal{R}}(T_{2,3;n,n+1})$ is a standard complex and completely determined by its Alexander polynomial [22, Theorem 1.2].

It follows from [8, Lemma 6.7] (also see the proof of [5, Proposition 6.1]) that

$$\Delta_{T_{n,n+1}}(t) = \sum_{i=0}^{n-1} t^{in} - t \sum_{i=0}^{n-2} t^{i(n+1)}.$$

Recall that the Alexander polynomial of a cable knot is determined by

$$\Delta_{K_{p,q}}(t) = \Delta_K(t^p) \cdot \Delta_{T_{p,q}}(t).$$

This gives

$$\Delta_{T_{2,3;n,n+1}}(t) = \Delta_{T_{2,3}}(t^n) \cdot \Delta_{T_{n,n+1}}(t) = (t^{2n} - t^n + 1) \cdot \left(\sum_{i=0}^{n-1} t^{in} - t \sum_{i=0}^{n-2} t^{i(n+1)} \right).$$

For small values of n, we have

$$\Delta_{T_{2,3;2,3}}(t) = t^6 - t^5 + t^3 - t + 1, \quad \Delta_{T_{2,3;3,4}}(t) = t^{12} - t^{11} + t^8 - t^7 + t^6 - t^5 + t^4 - t + 1.$$

For $n \geq 4$, we rearrange and simplify as follows. We first observe the telescoping sum

$$(-t^n+1)\cdot\left(\sum_{i=0}^{n-1} t^{in}\right) = 1 - t^{n^2}.$$

We also have

$$(-t^n+1)\cdot\left(-t\sum_{i=0}^{n-2} t^{i(n+1)}\right) = -t + \sum_{j=1}^{n-2}(t^{j(n+1)} - t^{j(n+1)+1}) + t^{(n-1)(n+1)}$$

and

$$(t^{2n})\cdot\left(\sum_{i=0}^{n-1} t^{in} - t\sum_{i=0}^{n-2} t^{i(n+1)}\right) = \sum_{i=2}^{n+1} t^{in} - \sum_{j=2}^{n} t^{j(n+1)-1}.$$

Putting the two simplifications together, we get

$$\begin{aligned}\Delta_{T_{2,3;n,n+1}}(t) &= 1 - t + t^{n+1} - t^{n+2} + \sum_{j=2}^{n-2}(-t^{j(n+1)-1} + t^{j(n+1)} - t^{j(n+1)+1}) \\ &\qquad + \sum_{i=2}^{n+1} t^{in} - t^{n^2} - t^{(n-1)(n+1)-1} - t^{n(n+1)-1} + t^{(n-1)(n+1)} \\ &= 1 - t + t^{n+1} - t^{n+2} + \sum_{j=2}^{n-2}(t^{jn} - t^{jn+j-1} + t^{jn+j} - t^{jn+j+1}) \\ &\qquad + t^{n^2-n} - t^{n^2-2} + t^{n^2-1} - t^{n^2+n-1} + t^{n^2+n}.\end{aligned}$$

In particular, the number of terms in the Alexander polynomial is $4\cdot(n-1)+1$.

Thus, we have

$$\Delta_{T_{2,3;n,n+1}}(t) = \sum_{i=0}^{4(n-1)} (-1)^i t^{b_i},$$

where $(b_i)_{i=0}^{4(n-1)}$ is the decreasing sequence of integers found above. Defining

$$c_i = b_{2i-2} - b_{2i-1} \quad \text{for } 1 \leq i \leq 2(n-1),$$

one readily checks that, for $1 \leq i \leq 2(n-1)$,

$$c_i(T_{2,3;n,n+1}) = \begin{cases} \frac{1}{2}(i-1) & \text{if } i \text{ is odd, } i > 1, \\ n & \text{if } i = 2(n-1). \\ 1 & \text{otherwise.} \end{cases}$$

Since $T_{2,3;n,n+1}$ is an L–space knot, by Proposition 8.2 we have $\varphi_j(T_{2,3;n,n+1}) = \#\{c_i \mid c_i = j\}$, and the calculation of $\varphi_j(D_{n,n+1})$ (which equals $\varphi_j(T_{2,3;n,n+1})$) follows immediately. □

We now prove Theorem 1.9 to produce an infinite-rank summand of $\mathcal{C}_{\mathrm{TS}}$.

Proof of Theorem 1.9 Recall Example 1.4, which states that the torus knot $T_{n,n+1}$ has

$$\varphi_j(T_{n,n+1}) = \begin{cases} 1 & \text{if } j = 1, 2, \dots, n-1, \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 9.1 and the fact that φ_j is a homomorphism (Theorem 7.2), we have that

$$\varphi_n(K_n) = \varphi_n(D_{n,n+1}) - \varphi_n(T_{n,n+1}) = 1$$

and $\varphi_j(K_n) = 0$ if $j > n$. The theorem now follows from a straightforward linear algebra argument; see for example [18, Lemma 6.4]. □

10 Concordance genus and concordance unknotting number

In this section, we discuss applications of our homomorphisms to concordance genus and concordance unknotting number.

10.1 Concordance genus

Recall that knot Floer homology detects genus [20]. Using the conventions and notation from Section 2, we have that

$$g(K) = \tfrac{1}{2}\max\{A([x]) - A([y]) \mid [x], [y] \neq 0 \in H_*(\mathrm{CFK}_{\mathcal{R}}(K)/(U, V))\}.$$

Proof of Theorem 1.11(1) Suppose that K' is concordant to K. Let $N = N(K) = N(K') = \max\{j \mid \varphi_j(K) \neq 0\}$. By Theorem 6.1 and Corollary 6.2, we have that there exist $[x], [y] \neq 0 \in H_*(\mathrm{CFK}_{\mathcal{R}}(K')/(U, V))$ with $\mathrm{gr}(x) - \mathrm{gr}(y) = (1 - 2N, 1)$. Then

$$|A(x) - A(y)| = N,$$

implying that $g(K') \geq \frac{1}{2}N$. Thus, $g_c(K) \geq \frac{1}{2}N$, as desired. □

10.2 Concordance unknotting number

We recall the following definitions and results from [1]. (The results are originally stated over the ring $\mathbb{F}[U, V]$; quotienting by UV yields the results as stated here.)

Let $u'(K)$ be the least integer m such that there exist grading-homogenous $\mathcal{R}$–equivariant chain maps

$$f\colon \mathrm{CFK}_{\mathcal{R}}(K) \to \mathcal{R} \quad \text{and} \quad g\colon \mathcal{R} \to \mathrm{CFK}_{\mathcal{R}}(K)$$

such that $g \circ f$ is homotopic to multiplication by U^m and $f \circ g$ is multiplication by U^m.

Theorem 10.1 [1, Theorem 1.1] *The integer $u'(K)$ is a lower bound for the unknotting number $u(K)$.*

Proof of Theorem 1.11(2) Suppose that K' is concordant to K. Let $N = N(K) = N(K') = \max\{j \mid \varphi_j(K') \neq 0\}$. This implies

$$U^{N-1}\, \mathrm{Tors}_U\, H_*(\mathrm{CFK}_{\mathcal{R}}(K')/V) \neq 0,$$

where $\mathrm{Tors}_U\, M$ denotes the U–torsion submodule of an $\mathbb{F}[U]$–module M.

Let $u' = u'(K')$. Then there exist grading-homogenous $\mathcal{R}$–equivariant chain maps

$$f\colon \mathrm{CFK}_{\mathcal{R}}(K') \to \mathcal{R} \quad \text{and} \quad g\colon \mathcal{R} \to \mathrm{CFK}_{\mathcal{R}}(K')$$

such that $g \circ f$ is homotopic to multiplication by $U^{u'}$ and $f \circ g$ is multiplication by $U^{u'}$. Now quotient by V. Since $g \circ f$ factors through $\mathcal{R}$, it follows that $U^{u'}$ must annihilate $\mathrm{Tors}_U\, H_*(\mathrm{CFK}_{\mathcal{R}}(K')/V)$, ie $u' \geq N$. This implies that $u_c(K) \geq N$, as desired. □

Proof of Theorem 1.12 Let K_n denote $D_{n,1} \# -D_{n-1,1}$ for $n \in \mathbb{N}$, where, as above, D denotes the positively clasped, untwisted Whitehead double of the right-handed trefoil. The knots K_n are topologically slice, since $D_{m,1}$ is. These knots are used in [10, Theorem 3]. In particular, by [10, Lemma 3.1], we have that $g_4(K_n) = 1$ for all n. By [10, Lemma 3.3], we have that $a_1(D_{n,1}) = 1$ and $a_2(D_{n,1}) = -n$. (There is a difference in sign conventions between a_2 in [10] and the present paper.) By [10, Lemma 3.2], we have that $|a_{2i}(D_{n,1})| \leq n$ for all i, with equality if and only if $i = 1$ by [10, Lemma 3.3]. It follows that $\varphi_n(D_{n,1}) = 1$ and $\varphi_i(D_{n,1}) = 0$ for all $i > n$. Hence $N(K_n) = n$, and, by Theorem 1.11(2), we have that $u_c(K_n) \geq n$. □

11 Further remarks

We conclude with some remarks on knot-like complexes.

11.1 Realizability

The question of which knot-like complexes can be realized by knots in S^3 is difficult. See [6; 14] for some restrictions. Note that their restrictions apply to the homotopy type, rather than local equivalence type, of knot-like complexes. For example, the standard complex $C(2,-2)$ is not realizable [6, Theorem 7] up to homotopy, but is realizable up to local equivalence [11, Lemma 2.1].

Instead, we turn to the following purely algebraic question:

Question 11.1 Which knot-like complexes are the mod UV reduction of chain complexes over $\mathbb{F}[U,V]$?

Indeed, in Section 2, we defined the complex $\mathrm{CFK}_{\mathcal{R}}(K)$ over the ring $\mathcal{R}$, but the definition works equally well over $\mathbb{F}[U,V]$. Thus, in order for a knot-like complex C to be realizable as coming from a knot $K\subset S^3$ up to homotopy (resp. local) equivalence, it is necessary for C to be homotopy (resp. locally) equivalent to a complex that is the mod UV reduction of a complex over $\mathbb{F}[U,V]$.

Naively, one may hope to "undo" modding out by UV. That is, given a standard complex $C(a_1,\dots,a_n)=(\mathcal{R}\langle x_i\rangle,\partial)$, one may hope to define a chain complex over $\mathbb{F}[U,V]$ by $C'=(\mathbb{F}[U,V]\langle x_i\rangle,\partial')$, where ∂' is obtained by extending ∂ linearly with respect to $\mathbb{F}[U,V]$. However, in general, ∂'^2 will not be zero. As the following examples show, in some cases, the failure of $\partial'^2=0$ can be remedied, while in other cases, it is fatal.

Example 11.2 We apply the above procedure to the standard complex

$$C(1,-2,-1,1,2,-1)$$

from Example 4.8. Let C' be generated over $\mathbb{F}[U,V]$ by

$$x_0,x_1,x_2,x_3,x_4,x_5,x_6$$

with nonzero differentials

$$\partial' x_1=Ux_0+V^2x_2,\quad \partial' x_2=Ux_3,\quad \partial' x_4=Vx_3,\quad \partial' x_5=U^2x_4+Vx_6.$$

Then $\partial'^2x_1=UV^2x_3\neq 0$ and $\partial'^2x_5=U^2Vx_3\neq 0$. However, if we instead endow C' with the differentials

$$\begin{aligned}\partial' x_1&=Ux_0+V^2x_2+UVx_4, & \partial' x_2&=Ux_3,\\ \partial' x_4&=Vx_3, & \partial' x_5&=UVx_2+U^2x_4+Vx_6,\end{aligned}$$

then C' becomes a chain complex, as desired. Note that this change to the differential is equivalent to adding diagonals arrow from x_1 to x_4 and from x_5 to x_2 in Figure 5.

Example 11.3 We attempt to apply the above procedure to the standard complex $C(1,1)$, generated by x_0, x_1 and x_2 with

$$\partial x_0 = 0, \quad \partial x_1 = U x_0, \quad \partial x_2 = V x_1.$$

Then $\partial' x_2 = UVx_0 \neq 0$ and there is no way to modify ∂' so that is squares to zero and reduces mod UV to ∂.

More generally, one can show that any standard complex beginning with the parameters $a_1 = 1$ and $a_2 > 0$ cannot be realized as the mod UV reduction of a chain complex over $\mathbb{F}[U,V]$, even up to local equivalence.

11.2 Group structure of $\mathfrak{K}$

Theorem 6.1 gives us a complete description of $\mathfrak{K}$ as a set; namely, the elements of $\mathfrak{K}$ are in bijection with finite sequences of nonzero integers. A natural question is the following:

Question 11.4 Is there is an explicit description of the group structure on $\mathfrak{K}$?

In many simple cases, the group operation in $\mathfrak{K}$ simply concatenates or merges the sequences associated to the standard representatives.

Example 11.5 It follows from [25, Theorem 4] that

$$C(1,-1) \otimes C(1,-1) \sim C(1,-1,1,-1).$$

More generally,

$$C(1,-1,1,-1,\dots,1,-1) \otimes C(1,-1) \sim C(1,-1,1,-1,\dots,1,-1),$$

where the length of the right-hand side is the sum of the lengths of the factors on the left-hand side.

Example 11.6 By [11, Lemma 2.1], we have that

$$C(1,-3,3,-1) \otimes C(2,-2) \sim C(1,-3,2,-2,3,-1).$$

However, in general, the group operation in $\mathfrak{K}$ is more complicated:

Example 11.7 One can show that

$$C(2,-2) \otimes C(1,-1) \sim C(1,-1,2,1,-1,-2,1,-1).$$

Despite the seemingly complicated product structure exhibited in Example 11.7, the standard complex representative of a product of two standard complexes is highly constrained by the fact that φ_j is a homomorphism for each $j \in \mathbb{N}$.

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