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A REFINEMENT OF THE OZSVÁTH-SZABÓ LARGE INTEGER SURGERY FORMULA AND KNOT CONCORDANCE

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Abstract. We compute the knot Floer filtration induced by a cable of the meridian of a knot in the manifold obtained by large integer surgery along the knot. We give a formula in terms of the original knot Floer complex of the knot in the three-sphere. As an application, we show that a knot concordance invariant of Hom can equivalently be defined in terms of filtered maps on the Heegaard Floer homology groups induced by the two-handle attachment cobordism of surgery along a knot.

1. Introduction

We provide a description of the knot Floer homology of certain knots lying in three-manifolds obtained by Dehn surgeries. This description generalizes the large surgery formulae of Ozvsváth-Szabó to a knot Floer context. Our results expand the set of knot Floer calculations that are practical outside of the three-sphere, where available computational tools are more limited than for knots in the three-sphere.

Let K denote a null-homologous oriented knot in an oriented, closed three-manifold Y. Let $Y_t(K)$ denote the manifold constructed as Dehn surgery along $K \subset Y$ with surgery coefficient t. In [OS04] Ozsváth and Szabó construct a chain homotopy equivalence between certain subquotient complexes of the full knot Floer chain complex $\mathrm{CFK}^\infty(Y,K)$ and Heegaard Floer chain complexes $\widehat{\mathrm{CF}}(Y_t(K),\mathfrak{s}_m)$ for sufficiently large integers t for each spin^c structure $\mathfrak{s}_m \in \mathrm{Spin}^c(Y_t(K))$ (see Remark 2.2). This equivalence is known as the *large integer surgery formula*.

The meridian μ of K naturally lies inside of the knot complement $Y \setminus K$ and the surgered manifold $Y_t(K)$. The meridian μ induces a filtration on $\widehat{\mathrm{CF}}(Y_t(K),\mathfrak{s}_m)$ for each spin^c structure $\mathfrak{s}_m$. In [Hed07] Hedden gives a formula for the filtered complex $\widehat{\mathrm{CFK}}(S^3_t(K),\mu,\mathfrak{s}_m)$ in terms of $\mathrm{CFK}^\infty(S^3,K)$ for sufficiently large t. As an application of this formula, Hedden computes the knot Floer homology of Whitehead doubles and the Ozsváth-Szabó concordance invariant τ of Whitehead doubles. In [HKL16] Hedden, Kim, and Livingston generalize Hedden's formula by computing the full knot Floer complex $\mathrm{CFK}^\infty(Y_t(K),\mu,\mathfrak{s}_m)$ in terms of $\mathrm{CFK}^\infty(Y,K)$ for sufficiently large t. As an application to knot concordance, they show that the subgroup of topologically slice knots of the concordance group contains a $\mathbb{Z}_2^\infty$ subgroup.

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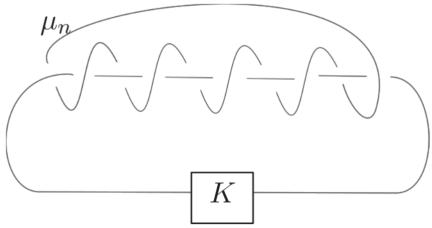


FIGURE 1. The two-component link μ_n and K for $n = 5$

We refine the theorems of Ozsváth-Szabó, Hedden, and Hedden-Kim-Livingston to determine the filtered chain homotopy type of $\mathrm{CFK}^\infty(Y_t(K), \mu_n)$, where μ_n denotes the $(n,1)$–cable of the meridian of K, viewed as a knot in $Y_t(K)$. See Figure 1. For each spin^c structure $\mathfrak{s}_m$, we show that the complex $\mathrm{CFK}^\infty(Y_t(K), \mu_n, \mathfrak{s}_m)$ is isomorphic to $\mathrm{CFK}^\infty(Y, K)$, but endowed with a different $\mathbb{Z} \oplus \mathbb{Z}$ filtration and an overall shift in the homological grading.

Theorem 1.1. *Let K be a null-homologous knot in Y and fix $m, n \in \mathbb{Z}$. Then there exists $T = T(m,n) > 0$ such that for all $t > T$, the complex $\mathrm{CFK}^\infty(Y_t(K), \mu_n, \mathfrak{s}_m)$ is isomorphic to $\mathrm{CFK}^\infty(Y, K)[\epsilon]$ as an unfiltered complex, where $[\epsilon]$ denotes a grading shift that depends only on m and t. Given a generator $[x, i, j]$ for $\mathrm{CFK}^\infty(Y, K)$, the $\mathbb{Z} \oplus \mathbb{Z}$ filtration level of the same generator, viewed as a chain in $\mathrm{CFK}^\infty(Y_t(K), \mu_n, \mathfrak{s}_m)$, is given by:*

$$\mathcal{F}([x,i,j]) = \begin{cases} [i,i] & \text{if } j \leq m+i, \\ [j-m, j-m-k] & \text{if } j = m+i+k, \text{ where } 1 \leq k < n, \\ [j-m, j-m-n] & \text{if } j \geq m+i+n. \end{cases}$$

Similarly, the complex $\mathrm{CFK}^\infty(Y_{-t}(K), \mu_n, \mathfrak{s}_m)$ is isomorphic to $\mathrm{CFK}^\infty(Y, K)[\epsilon']$ as an unfiltered complex, where $[\epsilon']$ denotes a grading shift that depends only on m and t. Given a generator $[x, i, j]$ for $\mathrm{CFK}^\infty(Y, K)$, the $\mathbb{Z} \oplus \mathbb{Z}$ filtration level of the same generator, viewed as a chain in $\mathrm{CFK}^\infty(Y_{-t}(K), \mu_n, \mathfrak{s}_m)$, is given by:

$$\mathcal{F}([x,i,j]) = \begin{cases} [i,i] & \text{if } j \geq m+i, \\ [j-m, j-m+k] & \text{if } j = m+i-k, \text{ where } 1 \leq k < n, \\ [j-m, j-m+n] & \text{if } j \leq m+i-n. \end{cases}$$

As a corollary, the $\mathbb{Z}$–filtered complex $\widehat{\mathrm{CFK}}(Y_t(K), \mu_n, \mathfrak{s}_m)$ is isomorphic to a subquotient complex of $\mathrm{CFK}^\infty(Y, K)$, endowed with an $(n+1)$ step filtration $\mathcal{F}$:

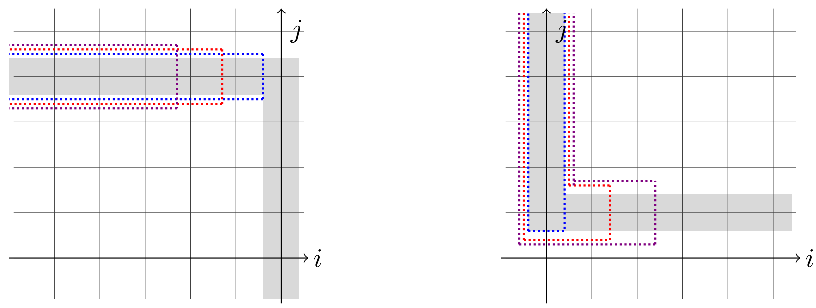
$$0 \subseteq C_{\{i<-n+1, j=m\}} \subseteq \cdots \subseteq C_{\{i<0, j=m\}} \subseteq C_{\{\max(i, j-m)=0\}}.$$

This filtration is illustrated in Figure 2(A) in the case $n = 3$. Similarly, the $\mathbb{Z}$–filtered complex $\widehat{\mathrm{CFK}}(Y_{-t}(K), \mu_n, \mathfrak{s}_m)$ is isomorphic to a subquotient complex of $\mathrm{CFK}^\infty(Y, K)$, endowed with an $(n+1)$ step filtration $\mathcal{F}'$:

$$0 \subseteq C_{\{\min(i,j-m)=0\} \cap \{i<1\}} \subseteq \cdots \subseteq C_{\{\min(i,j-m)=0\} \cap \{i<n\}} \subseteq C_{\{\min(i,j-m)=0\}}.$$

See Figure 2(B) for an illustration in the case $n = 3$.

Corollary 1.2. *Let $K \subset Y$ be a null-homologous knot, and fix $m, n \in \mathbb{Z}$. Then there exists $T = T(m,n) > 0$ such that for all $t > T$, the $\mathbb{Z}$–filtration on $\widehat{CF}(Y_t(K), \mathfrak{s}_m)$ induced by $\mu_n \subset Y_t(K)$ is isomorphic to the filtered chain homotopy type of the $(n+1)$ step filtration on $C\{\max(i, j-m) = 0\}$ described above. Similarly, the $\mathbb{Z}$–filtration on $\widehat{CF}(Y_{-t}(K), \mathfrak{s}_m)$ induced by $\mu_n \subset Y_{-t}(K)$ is isomorphic to the filtered chain homotopy type of the $(n+1)$ step filtration on $C\{\min(i, j-m) = 0\}$ described above.*



(A) $C\{\max(i, j-m) = 0\}$ is the shaded region. The subregions bounded by the colored dots represent subcomplexes of the filtration $\mathcal{F}$ in the case $n = 3$ and $m = 4$.

(B) $C\{\min(i, j-m) = 0\}$ is the shaded region. The subregions bounded by the colored dots represent subcomplexes of the filtration $\mathcal{F}'$ in the case $n = 3$ and $m = 1$.

FIGURE 2. Filtrations $\mathcal{F}$ and $\mathcal{F}'$.

As an application, we show that the concordance invariant $a_1(K)$ of Hom [Hom14b] can equivalently be defined in terms of filtered maps on the Heegaard Floer homology groups induced by the two-handle attachment cobordism of surgery along a knot K in S^3. We will be particularly interested in the spin^c structure $\mathfrak{s}_\tau$ corresponding to the $\tau = \tau(K)$ concordance invariant of Ozsváth-Szabó [OS03]. The rationally null-homologous knot $\mu_n \subset S^3_t(K)$ induces a $\mathbb{Z}$-filtration of $\widehat{\mathrm{CF}}(S^3_t(K), \mathfrak{s}_\tau)$ and $\widehat{\mathrm{CF}}(S^3_{-t}(K), \mathfrak{s}_\tau)$, that is, a sequence of subcomplexes:

$$0 \subset \mathcal{F}_{bottom} \subset \mathcal{F}_{bottom+1} \subset \cdots \subset \mathcal{F}_{top-1} \subset \mathcal{F}_{top} = \widehat{\mathrm{CF}}(S^3_t(K), \mathfrak{s}_\tau),$$
$$0 \subset \mathcal{F}'_{bottom} \subset \mathcal{F}'_{bottom+1} \subset \cdots \subset \mathcal{F}'_{top-1} \subset \mathcal{F}'_{top} = \widehat{\mathrm{CF}}(S^3_{-t}(K), \mathfrak{s}_\tau).$$

Using the knot filtrations, an equivalent definition of $a_1(K)$ can be formulated in terms of the filtration $\mathcal{F}$ and $\mathcal{F}'$ induced by μ_n as a knot inside $S^3_t(K)$ and $S^3_{-t}(K)$.

Theorem 1.3. *Let $n > 2g(K)$. For sufficiently large surgery coefficient t, the concordance invariant $a_1(K)$ is equal to:*

$$a_1(K) = \begin{cases} \max\left\{ m \;\middle|\; \begin{array}{c} \widehat{\mathrm{CF}}(S^3_t K,\ \mathfrak{s}_\tau)/\mathcal{F}_{top-1-m} \to \widehat{\mathrm{CF}}(S^3) \\ \textit{induces a trivial map on homology} \end{array} \right\} & \textit{if } \varepsilon(K) = -1, \\ 0 & \textit{if } \varepsilon(K) = 0, \\ \min\left\{ m \;\middle|\; \begin{array}{c} \widehat{\mathrm{CF}}(S^3) \to \mathcal{F}'_{bottom+m} \subset \widehat{\mathrm{CF}}(S^3_{-t} K,\ \mathfrak{s}_\tau) \\ \textit{induces a trivial map on homology} \end{array} \right\} & \textit{if } \varepsilon(K) = 1. \end{cases}$$

This interpretation of the invariant $a_1(K)$ offers a topological perspective that complements the original algebraic definition of $a_1(K)$. We will also include properties of the invariant $a_1(K)$ as well as computations of $a_1(K)$ for homologically thin knots and L–space knots.

2. The knot Floer filtration of cables of the meridian in Dehn surgery along a knot

In this section we will refine the theorem of Ozsváth-Szabó to determine the filtered chain homotopy type of the knot Floer complex of $(Y_t(K), \mu_n)$.

We begin by recalling the large integer surgery formula from Ozsváth and Szabó [OS04]. Let $(\Sigma_g,\ \alpha_1, \dots, \alpha_g,\ \gamma_1, \dots, \gamma_g,\ w,\ z)$ be a doubly-pointed Heegaard diagram for (Y, K), where

- the curve $\gamma_g = \mu$ is a meridian of the knot K,
- the curve α_g is a longitude for K,
- there is a single intersection point in $\alpha_g \cap \gamma_g = x_0$,
- the basepoints w and z lie on either side of γ_g.

Let $\beta = \{\gamma_1, \dots, \gamma_{g-1}, \lambda_t\}$ be the set of curves in γ, with γ_g replaced by a longitude $\beta_g = \lambda_t$ winding t times around μ. Label the unique intersection point $\gamma_g \cap \beta_g = \theta$. The Heegaard triple diagram $(\Sigma, \alpha, \beta, \gamma, w, z)$ represents a cobordism between Y and $Y_t(K)$. See Figure 3.

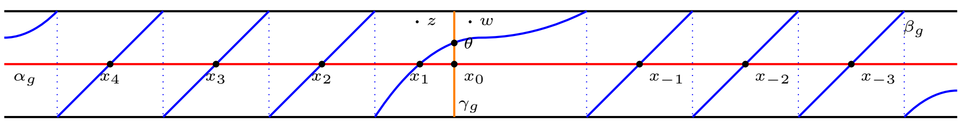


Figure 3. Local picture of the winding region of the Heegaard triple diagram $(\Sigma, \alpha, \beta, \gamma, w, z)$ for the cobordism between Y_tK and Y

Let $C\{\max(i, j - m) = 0\}$ denote the subquotient complex of $\mathrm{CFK}^\infty(Y, K)$ generated by triples $[x, i, j]$ with the i and j filtration levels satisfying the specified constraints.

Theorem 2.1 ([OS04])**.** *Let $K \subset Y$ be a knot, and fix $m \in \mathbb{Z}$. Then there exists $T = T(m) > 0$ such that for all $t > T$, the chain map*

$$\Phi_m : \widehat{\mathrm{CF}}(Y_t(K), \mathfrak{s}_m) \to C\{\max(i, j - m) = 0\}$$

defined by

$$\Phi_m([x]) = \sum_{y \in T_\alpha \cap T_\gamma} \ \sum_{\{\psi \in \pi_2(x, \theta, y) \ | \ n_z(\psi) - n_w(\psi) = m - \mathcal{F}(y),\ \mu(\psi) = 0\}} [y, -n_w(\psi), m - n_z(\psi)]$$

induces an isomorphism of chain complexes.

Remark 2.2. Here, as usual, the labeling of the spin^c structures is determined by the condition that $\mathfrak{s}_m$ can be extended over the cobordism $-W_t$ from $-Y_t(K)$ to $-Y$ associated to the two-handle addition along K with framing t, yielding a spin^c structure $\mathfrak{r}_m$ satisfying

$$\langle c_1(\mathfrak{r}_m, [S]) \rangle + \ t = 2m.$$

Above, S denotes a surface in W_t obtained from closing off a Seifert surface for K in Y to produce a surface S of square t.

We refine the theorem of Ozsváth-Szabó to determine the filtered chain homotopy type of the knot Floer complex of $(Y_t(K), \mu_n)$. Consider the meridian $\mu = \mu_K$ of a knot K. The meridian μ naturally lies inside of the knot complement $Y \setminus K$ and the surgered manifold $Y_t(K)$. For $n \in \mathbb{N}$, μ_n denotes the $(n, 1)$–cable of μ_K, and also lies inside $Y \setminus K$ and the surgered manifold $Y_t(K)$. The knot μ_n is homologically equivalent to $n \cdot [\mu]$ in $H_1(Y_t(K))$. When $n = 1$, $\mu_1 = \mu$. See Figure 1 for a picture of the two-component link $K \cup \mu_n$.

For all $n \geq 1$ there is a natural $(n+1)$ step algebraic filtration $\mathcal{F}$ on the subquotient complex $C_{\{\max(i,j-m)=0\}}$ of $\mathrm{CFK}^\infty(Y, K)$:

$$0 \subseteq C_{\{i<-n+1,j=m\}} \subseteq \cdots \subseteq C_{\{i<0,j=m\}} \subseteq C_{\{\max(i,j-m)=0\}}.$$

This filtration is illustrated in the case $n = 3$ in Figure 2(A).

Theorem 2.3 says that this algebraic filtration $\mathcal{F}$ corresponds to a relative $\mathbb{Z}$-filtration on $\widehat{\mathrm{CF}}(Y_t(K), \mathfrak{s}_m)$ induced by $\mu_n \in Y_t(K)$. This generalizes work of Hedden [Hed07] who studied the $n = 1$ case of the filtered complex $\widehat{\mathrm{CFK}}(S^3_t(K), \mu, \mathfrak{s}_m)$.

Theorem 2.3. *Let $K \subset Y$ be a null-homologous knot, and fix $m, n \in \mathbb{Z}$. Then there exists $T = T(m, n) > 0$ such that for all $t > T$, the following holds: The filtered chain homotopy type of the $(n + 1)$ step filtration $\mathcal{F}$ on $C\{\max(i, j - m) = 0\}$ described above is filtered chain homotopy equivalent to that of the filtration on $\widehat{CF}(Y_t(K), \mathfrak{s}_m)$ induced by $\mu_n \subset Y_t(K)$.*

Proof. The key observation will be that the triple diagram $(\Sigma, \alpha, \beta, \gamma, w, z)$ used to define Φ_m not only specifies a Heegaard diagram for the knot (Y, K), but also a Heegaard diagram for the knot $(Y_t(K), \mu_n)$ with the addition of a basepoint z'. Place an extra basepoint $z' = z_n$ so that it is n regions away from the basepoint w in the Heegaard triple diagram representing the cobordism between Y and $Y_t(K)$ as in Figure 4. (This can be accomplished if t is sufficiently large, e.g., if $t > 2n$). The knot represented by the doubly-pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z_n)$ is μ_n in $Y_t(K)$.

An intersection point $x' \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is said to be supported in the winding region if the component of x' in α_g lies in the local picture of Figure 4. Intersection points in the winding region are in t to 1 correspondence with intersection points x in $\mathbb{T}_\alpha \cap \mathbb{T}_\gamma$.

Fix a Spin^c structure $\mathfrak{s}_m$ where $m \in \mathbb{Z}$. For t (the surgery coefficient) sufficiently large, any generator $x' \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ representing Spin^c structure $\mathfrak{s}_m$ is supported in the winding region. In this case, there is a uniquely determined $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$ and a canonical small triangle $\psi \in \pi_2(x, \theta, x')$.

Suppose $\psi \in \pi_2(x, \theta, x')$ is the canonical small triangle and $x' \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is a generator representing Spin^c structure $\mathfrak{s}_m$. If $k = n_z(\psi) \geq 0$ (so $n_w(\psi) = 0$), then the α_g component of x' is x_k (and lies k units to the left of x_0) in Figure 4. In this case, Φ_m maps x' to $C\{i = 0, j \leq m\}$. On the other hand, if x' is a generator with $n_z(\psi) = 0$ and $l = n_w(\psi) > 0$, then the α_g component of x' is x_{-l} (and lies l steps to the right of x_0) in Figure 4. In this case, Φ_m maps x' to the subcomplex $C\{i \leq -l, j = m\} \subset C\{i < 0, j = m\}$. The following lemma (which generalizes Lemma 4.2 of [Hed07]) will be used to finish the proof.

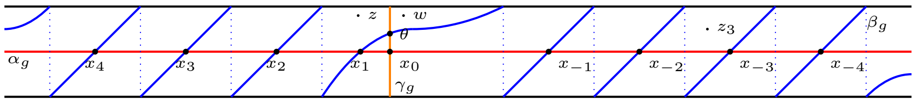


FIGURE 4. Local picture of the winding region of the Heegaard triple diagram $(\Sigma, \alpha, \beta, \gamma, w, z_n)$ for the cobordism between $Y_t K$ and Y. The basepoint z_n is located n regions away from the basepoint w in the Heegaard diagram $(\Sigma, \alpha, \beta, w, z_n)$. Here we depict the basepoint z_n for $n = 3$.

Lemma 2.4. *Let $p \in \widehat{\mathrm{CFK}}(Y_t(K), \mu_n, \mathfrak{s}_m)$ be a generator supported in the winding region, and let x_i denote the α_g component of the corresponding intersection point in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, where the x_i are labeled as in Figure 4. Then*

$$\mathcal{F}(p) = \begin{cases} \mathcal{F}_{top} & i > 0; \\ \mathcal{F}_{top+i} & -n < i < 0; \\ \mathcal{F}_{bottom} & i \leq -n. \end{cases}$$

Here, $\mathcal{F}_{top}$ (respectively, $\mathcal{F}_{bottom}$) denotes the top (respectively, bottom) filtration level of $\widehat{\mathrm{CFK}}(Y_t(K), \mu_n, \mathfrak{s}_m)$. $\mathcal{F}_{top-i}$ denotes the filtration level that is i lower than $\mathcal{F}_{top}$. In addition $\mathcal{F}_{bottom} = \mathcal{F}_{top-n}$, so this is an $(n+1)$ step filtration.

Proof. The $\mathbb{Z}$-filtration $\mathcal{F}$ is defined by the relative *Alexander* grading A'_n induced by μ_n on $\mathrm{CF}^\infty(Y_t K, \mathfrak{s}_m)$. That is,

$$\mathcal{F}(p) - \mathcal{F}(q) = n_{z_n}(\phi) - n_w(\phi),$$

where $\phi \in \pi_2(p, q)$ is a Whitney disk connecting $p, q \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

Let $p, q \in \widehat{\mathrm{CFK}}(Y_t K, \mu_n, \mathfrak{s}_m)$ be generators supported in the winding region, and let x_i, x_j denote the α_g components of the corresponding intersection points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Assume without loss of generality that $i < j$ (so that x_i lies to the right of x_j).

We will define a set of n arcs $\delta_1, \ldots, \delta_n$ on β as follows. Let δ_1 denote the arc on β connecting x_1 to x_{-1}. Let δ_k denote on the arc on β connecting $x_{-(k-1)}$ to x_{-k} for $k \in \{2, \ldots, n\}$.

We will construct a Whitney disk $\phi_{p,q} \in \pi_2(p, q)$ with the following properties:

- If $i > 0$ and $j > 0$, (that is, x_i, x_j both lie on the left of x_0), then $\partial\phi_{p,q}$ doesn't contain any arc δ_k. Therefore,

$$\mathcal{F}(p) - \mathcal{F}(q) = 0.$$

- If $i \leq -n$ and $j \leq -n$, (that is, x_i, x_j both lie $\geq n$ steps to the right of x_0), then $\partial\phi_{p,q}$ doesn't contain any arc δ_k. Therefore,

$$\mathcal{F}(p) - \mathcal{F}(q) = 0.$$

- If $i < -n$ and $j > 0$, (that is, x_j lies to the left of x_0 and x_i lies i steps to the right of x_0), then $\partial\phi_{p,q}$ contains the n arcs $\delta_1, \ldots, \delta_n$, each with multiplicity one. Therefore,

$$\mathcal{F}(p) - \mathcal{F}(q) = -n.$$

- If $-n \leq i < 0$ and $j > 0$, (that is, x_j lies to the left of x_0 and x_i lies i steps to the right of x_0), then $\partial\phi_{p,q}$ contains the i arcs $\delta_1, \dots, \delta_i$, each with multiplicity one. Moreover, $\partial\phi_{p,q}$ doesn't contain the arcs δ_k for $k > i$. Therefore,
$$\mathcal{F}(p) - \mathcal{F}(q) = -i.$$
- If $-n < j < 0$ and $i \leq -n$, (that is, x_i lies $\geq n$ steps to the right of x_0 and x_j lies j steps to the right of x_0), then $\partial\phi_{p,q}$ contains the $n+j$ arcs $\delta_{|j|+1}, \dots, \delta_n$, each with multiplicity one. Moreover, $\partial\phi_{p,q}$ doesn't contain the arcs δ_k for $k \leq |j|$. Therefore,
$$\mathcal{F}(p) - \mathcal{F}(q) = -n - j.$$
- If $-n < i < 0$ and $-n < j < 0$, (that is, x_j lies j steps to the right of x_0 and x_i lies i steps to the right of x_0), then $\partial\phi_{p,q}$ contains the $j-i$ arcs $\delta_{|j|+1}, \dots, \delta_{|i|}$, each with multiplicity one. Therefore,
$$\mathcal{F}(p) - \mathcal{F}(q) = i - j.$$

Assuming the existence of such $\phi_{p,q}$, the lemma follows immediately.

In [Hed07, Lemma 4.2] Hedden constructs a Whitney disk $\phi_{p,q} \in \pi_2(p,q)$. The above enumerated properties of $\partial\phi_{p,q}$ will be immediate from the construction. We restate his construction here. Note first since p, q lie in the winding region, they correspond uniquely to intersection points $\tilde{p}$, $\tilde{q} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma$. These intersection points $\tilde{p}$, $\tilde{q}$ can be connected by a Whitney disk $\phi \in \pi_2(\tilde{p}, \tilde{q})$ with $n_w(\phi) = 0$ and $n_z(\phi) = k$ for some $k \in \mathbb{Z}_{\geq 0}$. This means that $\partial\phi$ contains γ_g with multiplicity k, which further implies that the distance between x_i and x_j is k, that is, $i - j = k$. The domain of $\phi_{p,q}$ can then be obtained from the domain of ϕ by a simple modification in the winding region as described in [Hed07]. This modification is shown in Figure 5. It replaces the boundary component $k \cdot \gamma_g$ by a simple closed curve from an arc connecting x_i and x_j along α_g followed by an arc connecting x_j to x_i along β_g, and which wraps k times around the neck of the winding region. □

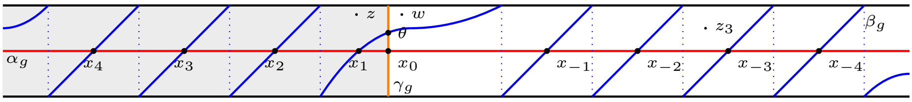
This completes the description of the knot Floer complex $\widehat{\mathrm{CFK}}(Y_t(K), \mu_n)$ in terms of the complex $\mathrm{CFK}^\infty(Y, K)$. □

Theorem 2.3 described the $\mathbb{Z}$-filtered chain homotopy type of knot Floer chain complex $\widehat{\mathrm{CFK}}(Y_t(K), \mu_n, \mathfrak{s}_m)$ for t large with respect to m and n. In Theorem 1.1, we describe the $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain homotopy type of $\mathrm{CFK}^\infty(Y_t K, \mu_n, \mathfrak{s}_m)$. This generalizes Theorem 4.2 of Hedden-Kim-Livingston [HKL16] which studies the $n = 1$ case.

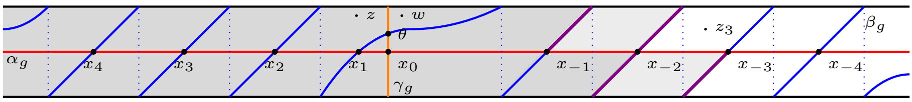
Proof of Theorem 1.1. The isomorphism of chain complexes induced by the map (defined in [OS04])
$$\Phi_m : \mathrm{CF}^\infty(Y_t(K), \mathfrak{s}_m) \to \mathrm{CFK}^\infty(Y, K)$$
respects the $\mathbb{F}[U, U^{-1}]$-module structure of both complexes, and hence determines one of the $\mathbb{Z}$-filtrations (called the U-filtration) of $\mathrm{CFK}^\infty(Y_t(K), \mu_n, \mathfrak{s}_m)$.

The knot $\mu_n \subset Y_t(K)$ induces an additional $\mathbb{Z}$-filtration (the Alexander filtration) on $\mathrm{CF}^\infty(Y_t(K), \mathfrak{s}_m)$. The additional $\mathbb{Z}$-filtration on $\mathrm{CF}^\infty(Y_t(K), \mathfrak{s}_m)$ induced by μ_n can be determined in exactly the same way as it was determined for the case of $\widehat{\mathrm{CF}}(Y_t(K), \mathfrak{s}_m)$. Lemma 2.4 identifies the $\mathbb{Z}$-filtration induced on any given

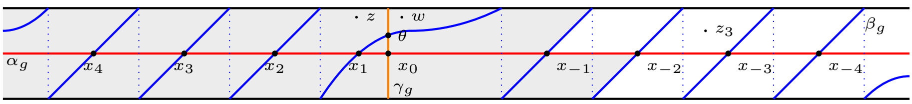
(A) The domain of a disk $\phi \in \pi_2(\tilde{p}, \tilde{q})$.



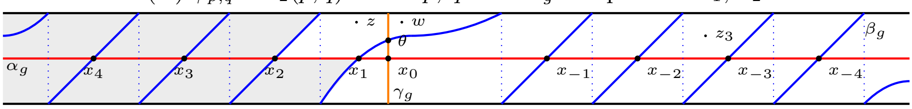
(B) $\phi_{p,q} \in \pi_2(p, q)$ where p, q have α_g components x_{-3}, x_{-1}. $\partial\phi_{p,q}$ contains arcs δ_2 and δ_3 on β drawn in violet.



(C) $\phi_{p,q} \in \pi_2(p, q)$ where p, q have α_g components x_{-2}, x_{-1}.



(D) $\phi_{p,q} \in \pi_2(p, q)$ where p, q have α_g components x_1, x_2.



(E) $\phi_{p,q} \in \pi_2(p, q)$ where p, q have α_g components x_{-2}, x_1.

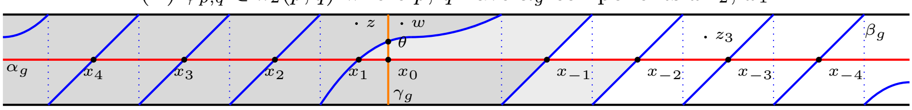


FIGURE 5. The domain of a disk $\phi_{p,q} \in \pi_2(p, q)$ for $p, q \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ in the winding region can be identified with the domain of a disk $\phi \in \pi_2(\tilde{p}, \tilde{q})$.

$i = constant$ slice in $\mathrm{CF}^\infty(Y_t(K), \mathfrak{s}_m)$ with an $(n+1)$ step filtration as above. This yields the statement of the theorem $t > 0$. The case $t < 0$ follows similarly.

Alternatively, the additional (Alexander) $\mathbb{Z}$-filtration on $\mathrm{CFK}^\infty(Y_t(K), \mu_n, \mathfrak{s}_m)$ can be obtained from the Alexander filtration on $\widehat{\mathrm{CFK}}(Y_t(K), \mu_n, \mathfrak{s}_m)$ by the fact that the U variable decreases Alexander grading by one, i.e., we have the relation $A(U \cdot x) = A(x) - 1$. $\square$

Corollary 2.5. *Let K be a knot in Y and fix $m, n \in \mathbb{Z}$. Then there exists $T = T(m, n) > 0$ such that for all $t > T$ the following holds: Up to a grading shift, the pth filtration level of $\mathrm{CFK}^\infty(Y_t(K), \mu_n, \mathfrak{s}_m)$ is described in terms of the original $\mathbb{Z} \oplus \mathbb{Z}-$filtered knot Floer homology $\mathrm{CFK}^\infty(Y, K)$ as*

$$\max(i, j - m - n) = p.$$

That is, each Alexander filtration level p of $\mathrm{CFK}^\infty(Y_t(K), \mu_n, \mathfrak{s}_m)$ is a "hook" shaped region in $\mathrm{CFK}^\infty(Y, K)$.

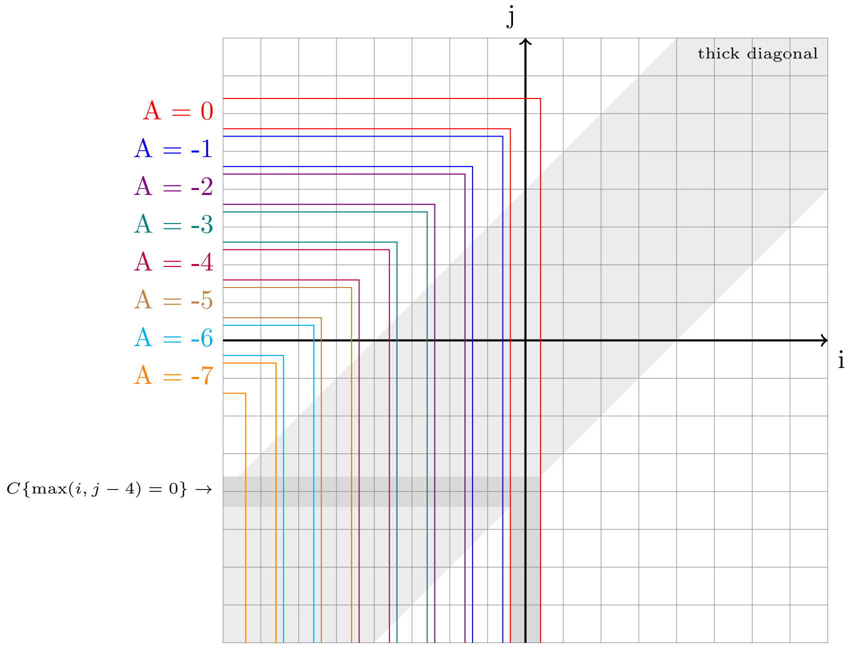


FIGURE 6. $\mathrm{CFK}^\infty(S^3, K)$ is supported along a thick diagonal of width $2g(K)+1$. The regions labeled $A = 0, \dots, A = -7$ have constant Alexander grading A'_n induced by μ_n on $\mathrm{CF}^\infty(S^3_t(K), \mathfrak{s}_m)$. For spin^c structures $\mathfrak{s}_m$ where $|m| \leq g(K)$, sufficiently large surgery coefficient t, the algebraic filtration i on $C\{\max(i, j-m) = 0\}$ corresponds to the $\mathbb{Z}$-filtration induced by μ_n on $\mathrm{CF}^\infty(S^3_t(K), \mathfrak{s}_m)$ where $n > 2g(K)$.

Proof. This follows from Theorem 1.1. □

Proposition 2.6. *Let $m \in \mathbb{Z}$ with $|m| \leq g(K)$ and let $n > 2g(K)$. For sufficiently large surgery coefficient t, the Alexander filtration induced by μ_n on $\mathrm{CF}^\infty(Y_t(K), \mathfrak{s}_m)$ coincides with the algebraic i-filtration on $\mathrm{CFK}^\infty(Y, K)$ under the correspondence given by Φ_m.*

Proof. Since $\widehat{\mathrm{CFK}}(Y, K)$ has degree equal to the Seifert genus of the knot, $\mathrm{CFK}^\infty(Y, K)$ is supported along a thick diagonal of width $2g(K) + 1$. By the hypothesis, we have

$$m + n > g(K).$$

Therefore the corner $(p, m+n+p)$ of the hook region $C\{\max(i, j-m-n) = p\}$ of each constant Alexander filtration level p of $\mathrm{CFK}^\infty(Y_t(K), \mu_n, \mathfrak{s}_m)$ lies above the thick diagonal along which $\mathrm{CFK}^\infty(Y, K)$ is supported. See Figure 6. For spin^c structures $\mathfrak{s}_m$ where $|m| \leq g(K)$, this means that the Alexander filtration induced by μ_n on $\mathrm{CFK}^\infty(Y_t(K), \mu_n, \mathfrak{s}_m)$ coincides with the algebraic i-filtration on $\mathrm{CFK}^\infty(Y, K)$ under the correspondence given by Φ_m. □

Because the algebraic i-filtration is used to define concordance invariants (such as $a_1(K)$, which can be interpreted as an integer lift of the Hom ε invariant [Hom14a]), the filtration induced by μ_n on $\mathrm{CF}^\infty(S^3_t(K), \mathfrak{s}_m)$ can be used to study the concordance class of a knot K. We will see that we can extract concordance invariants of K from $\mathrm{CFK}^\infty(S^3_t(K), \mu_n, \mathfrak{s}_m)$.

3. A knot concordance invariant

As an application for the results in the previous section on the $\mathbb{Z}$–filtration induced on $\widehat{\mathrm{CF}}(S^3_N(K), \mathfrak{s}_m)$ by the $(n,1)$–cable of the meridian μ_n, our main result in this section (Theorem 1.3) shows that the concordance invariant $a_1(K)$ of Hom [Hom14b], which has an algebraic definition in terms of maps on subquotient complexes of $\mathrm{CFK}^\infty(K)$, can be equivalently defined by studying filtered maps on the (hat version of the) Heegaard Floer homology groups induced by the two-handle attachment cobordism of large integer surgery along a knot K in S^3 and the filtration induced by the knot μ_n inside of the surgered manifold.

Our result is analogous to the statement that the concordance invariants $\nu(K)$ of Ozsváth-Szabó [OS11] and $\varepsilon(K)$ of Hom [Hom14a] can be defined algebraically or in terms of maps on the (hat version of the) Heegaard Floer homology groups induced by the two-handle attachment cobordism of large integer surgery along a knot K in S^3. Definition 3.1 gives an algebraic definition of $\varepsilon(K)$ in terms of certain chain maps on the subquotient complexes of the knot Floer chain complex $\mathrm{CFK}^\infty(K)$. Due to the Ozsváth-Szabó large integer surgery formula [OS04], $\varepsilon(K)$ can equivalently be defined in terms of maps on the Heegaard Floer chain complexes induced by the two-handle attachment cobordism of (large integer) surgery.

We begin by recalling the definition of the concordance invariants $\varepsilon(K)$. Let N be a sufficiently large integer relative to the genus of a knot K. Consider the map

$$F_s : \widehat{\mathrm{HF}}(S^3) \to \widehat{\mathrm{HF}}(S^3_{-N}(K), [s]),$$

induced by the two-handle cobordism W^4_{-N}. Here, $[s]$ denotes the restriction to $S^3_{-N}(K)$ of the Spin^c structure $\mathfrak{s}_s$ over W^4_{-N} with the property that

$$\langle c_1(\mathfrak{s}_s), [\widehat{F}]\rangle - N = 2s,$$

where $|s| \leq \frac{N}{2}$ and $\widehat{F}$ denotes the capped off Seifert surface in the four-manifold. We also consider the map

$$G_s : \widehat{\mathrm{HF}}(S^3_N(K), [s]) \to \widehat{\mathrm{HF}}(S^3),$$

induced by the two-handle cobordism $-W^4_N$.

The maps F_s and G_s can be defined algebraically by studying certain natural maps on subquotient complexes of $\mathrm{CFK}^\infty(K)$, as in [OS04]. The map F_s is induced by the chain map

$$C\{i = 0\} \to C\{\min(i, j - s) = 0\}$$

consisting of quotienting by $C\{i = 0, j < s\}$ followed by the inclusion. Similarly, the map G_s is induced by the chain map

$$C\{\max(i, j - s) = 0\} \to C\{i = 0\}$$

consisting of quotienting by $C\{i < 0, j = s\}$ followed by the inclusion.

Definition 3.1 ([Hom14a], [Hom14b])**.** Let $\tau = \tau(K)$ be the Ozsváth-Szabó concordance invariant. The invariant $\varepsilon(K)$ is defined as follows:

- $\varepsilon(K) = 1$ if F_τ is trivial (in which case G_τ is necessarily non-trivial).
- $\varepsilon(K) = -1$ if G_τ is trivial (in which case F_τ is necessarily non-trivial).

- $\varepsilon(K) = 0$ if F_τ and G_τ are both non-trivial.

In [Hom14b], Hom defines a concordance invariant $a_1(K)$ for knots with $\varepsilon(K) = 1$ that is a refinement of $\varepsilon(K)$.

Definition 3.2 ([Hom14b])**.** If $\varepsilon(K) = 1$ (F_τ is trivial), define

$$a_1(K) = \min\{s \mid H_s : H_*(C\{i = 0\}) \to H_*(C\{\min(i, j - \tau) = 0, i \leq s\}) \text{ is trivial}\}.$$

We extend this definition of $a_1(K)$ to all knots (to include knots with $\varepsilon(K) \neq 1$). Consider the maps

$$\begin{aligned} G_{-s,\tau} &: C\{\max(i, j - \tau) = 0, i \geq -s\} \to C\{i = 0\} \\ F_{s,\tau} &: C\{i = 0\} \to C\{\min(i, j - \tau) = 0, i \leq s\}. \end{aligned}$$

Definition 3.3. Given a knot K inside S^3, define:

$$a_1(K) = \begin{cases} \max\{-s \mid G_{-s,\tau} \text{ is trivial on homology}\} & \text{if } \varepsilon(K) = -1; \\ 0 & \text{if } \varepsilon(K) = 0; \\ \min\{s \mid F_{s,\tau} \text{ is trivial on homology}\} & \text{if } \varepsilon(K) = 1. \end{cases}$$

Note that $a_1(K)$ only depends on the doubly-filtered chain homotopy type of the knot Floer chain complex $CFK^\infty(K)$, so it is a knot invariant.

Remark 3.4. When $\varepsilon(K) = 1$, the definition of $a_1(K)$ agrees with the invariant $a_1(K)$ defined in Lemma 6.1 in [Hom14b]. As remarked in [Hom14b], $a_1(K)$ measures the "length" of the horizontal differential hitting the special class generating the vertical homology of $\widehat{\mathrm{CF}}(S^3)$. Similarly, when $\varepsilon(K) = -1$, $a_1(K)$ measures the "length" of the horizontal differential coming out of the special class generating the vertical homology of $\widehat{\mathrm{CF}}(S^3)$.

Recall that the rationally null-homologous knot $\mu_n \subset S^3_t(K)$ induces a $\mathbb{Z}$-filtration of $\widehat{\mathrm{CF}}(S^3_t(K), \mathfrak{s}_\tau)$ and $\widehat{\mathrm{CF}}(S^3_{-t}(K), \mathfrak{s}_\tau)$, that is, a sequence of subcomplexes:

$$0 \subset \mathcal{F}_{bottom} \subset \mathcal{F}_{bottom+1} \subset \cdots \subset \mathcal{F}_{top-1} \subset \mathcal{F}_{top} = \widehat{\mathrm{CF}}(S^3_t(K), \mathfrak{s}_\tau),$$

$$0 \subset \mathcal{F}'_{bottom} \subset \mathcal{F}'_{bottom+1} \subset \cdots \subset \mathcal{F}'_{top-1} \subset \mathcal{F}'_{top} = \widehat{\mathrm{CF}}(S^3_{-t}(K), \mathfrak{s}_\tau).$$

Using Theorem 2.3 and Proposition 2.6, an equivalent definition of $a_1(K)$ can be formulated in terms of the filtration $\mathcal{F}$ and $\mathcal{F}'$ induced by μ_n as a knot inside $S^3_t(K)$ and $S^3_{-t}(K)$. This interpretation of the invariant $a_1(K)$ offers a topological perspective that complements the original algebraic definition of $a_1(K)$.

Theorem 3.5. *Let $n > 2g(K)$. For sufficiently large surgery coefficient t, the concordance invariant $a_1(K)$ is equal to*

$$a_1(K) = \begin{cases} \max\left\{ m \,\middle|\, \begin{array}{c} \widehat{\mathrm{CF}}(S^3_t K,\ s_\tau)/\mathcal{F}_{top-1-m} \to \widehat{\mathrm{CF}}(S^3) \\ \textit{induces a trivial map on homology} \end{array} \right\} & \textit{if } \varepsilon(K) = -1, \\ 0 & \textit{if } \varepsilon(K) = 0, \\ \min\left\{ m \,\middle|\, \begin{array}{c} \widehat{\mathrm{CF}}(S^3) \to \mathcal{F}'_{bottom+m} \subset \widehat{\mathrm{CF}}(S^3_{-t} K,\ \mathfrak{s}_\tau) \\ \textit{induces a trivial map on homology} \end{array} \right\} & \textit{if } \varepsilon(K) = 1. \end{cases}$$

Proof. Since $|\tau| \leq g_4(K) \leq g(K)$, we can apply Proposition 2.6 which states that in the spin^c structure $\mathfrak{s}_\tau$, the algebraic i-filtration on $\mathrm{CFK}^\infty(S^3, K)$ coincides with the filtration induced by μ_n on $\widehat{\mathrm{CF}}(S^3_N(K), \mathfrak{s}_\tau)$ under the identification of the two filtered chain complexes in Theorem 2.3. □

Remark 3.6. Recall that $a_1(K)$ is a concordance invariant (see Proposition 3.7) that fits into a family of concordance invariants studied by Dai, Hom, Stoffregen, and the author in [DHST19]. It would be interesting to see if an analogue of Theorem 3.5 exists for this entire family of algebraically defined invariants corresponding to the standard local representative (over $\mathbb{F}[U,V]/(UV)$) of the knot.

Proposition 3.7 ([Hom14b])**.** *The invariant $a_1(K)$ is a concordance invariant.*

Proof. Suppose K_1 and K_2 are concordant knots, i.e., $K_1 \# \overline{K_2}$ is slice. Then $\varepsilon(K_1 \# \overline{K_2}) = 0$. By Proposition 3.11 in [Hom15], we may find a basis for $\mathrm{CFK}^\infty(K_1 \# \overline{K_2})$ with a distinguished element x that generates the homology $\mathrm{HFK}^\infty(K_1 \# \overline{K_2})$ and splits off as a direct summand of $\mathrm{CFK}^\infty(K_1 \# \overline{K_2})$. Similarly, we can find a basis for $\mathrm{CFK}^\infty(K_2 \# \overline{K_2})$ with a distinguished element y with the same properties. Then to compute $a_1(K_2 \# K_1 \# \overline{K_2})$, by the Kunneth principle [OS04] we can consider either chain complex:

$$\mathrm{CFK}^\infty(K_1 \# \overline{K_2}) \otimes_{\mathbb{Z}[U,U^{-1}]} \mathrm{CFK}^\infty(K_2) \quad \text{or} \quad \mathrm{CFK}^\infty(K_1) \otimes_{\mathbb{Z}[U,U^{-1}]} \mathrm{CFK}^\infty(K_2 \# \overline{K_2}).$$

Using the special bases from above, the relevant summands to a_1 are

$$\{x\} \otimes \mathrm{CFK}^\infty(K_2) \quad \text{or} \quad \mathrm{CFK}^\infty(K_1) \otimes \{y\}.$$

Thus, $a_1(K_2) = a_1(K_2 \# K_1 \# \overline{K_2}) = a_1(K_1)$. □

Example 3.8 (Homologically thin knots)**.** Model complexes for CFK^∞ of homologically thin knots are studied in [Pet13]. Petkova shows that if $\tau(K) = n$, the model complex contains a direct summand isomorphic to

$$\mathrm{CFK}^\infty(T_{2,2n+1}) \text{ if } n > 0 \quad \text{and} \quad \mathrm{CFK}^\infty(T_{2,2n-1}) \text{ if } n < 0.$$

This summand supports $H_*(\mathrm{CFK}^\infty(K))$ and thus determines the value of $a_1(K)$. It is easy to see from the complex that $a_1(K) = \mathrm{sgn}(\tau(K))$.

Proposition 3.9. *The following are properties of $a_1(K)$:*

(1) *If K is smoothly slice, then $a_1(K) = 0$.*
(2) $\mathrm{sgn}(a_1(K)) = \varepsilon(K)$.
(3) $a_1(K) = -a_1(\overline{K})$.
(4) *If $a_1(K) = 0$, then $a_1(K \# K') = a_1(K')$.*

Proof of (1). If K is smoothly slice, then $\varepsilon(K) = 0$; therefore, $a_1(K) = 0$. □

Proof of (2). By construction, if $a_1(K) > 0$, then $\varepsilon(K) = 1$; if $a_1(K) < 0$, then $\varepsilon(K) = -1$.

If $a_1(K) = 0$, we show that $\varepsilon(K) = 0$. Suppose $\varepsilon(K) = -1$. Then the vanishing of

$$a_1(K) = \max\{n \mid G_{n,\tau} \text{ is trivial on homology}\}$$

implies that the map $G_{0,\tau} : C\{i = 0, j \leq \tau\} \to C\{i = 0\}$ is trivial on homology, which contradicts the definition of τ. Similarly, $\varepsilon(K) \neq 1$ if $a_1(K) = 0$.

Finally, according to [Hom14a], $\varepsilon(K) = 0$ implies that $\tau(K) = 0$. □

Proof of (3). The symmetry properties of CFK^∞ of Section 3.5 in [OS04] imply that $a_1(K) = -a_1(\overline{K})$. □

Proof of (4). If $a_1(K) = 0$, $\varepsilon(K) = 0$. By Lemma 3.3 from [Hom14a], we may find a basis for $\mathrm{CFK}^\infty(K)$ with a distinguished element x which is the generator of both vertical and horizontal homology. Then $a_1(K\#K')$ can be computed from $\{x\} \otimes \mathrm{CFK}^\infty(K')$. □

In fact, we can extend Proposition 3.9(4) to describe the behavior of a_1 under connect sum in many (but not all) cases.

Proposition 3.10.

(1) *If* $a_1(K_1) > 0$ *and* $a_1(K_2) < 0$ *and* $a_1(K_1) + a_1(K_2) < 0$, *then*
$$a_1(K_1\#K_2) = a_1(K_1).$$
(2) *If* $a_1(K_1) > 0$ *and* $a_1(K_2) < 0$ *and* $a_1(K_1) + a_1(K_2) > 0$, *then*
$$a_1(K_1\#K_2) = a_1(K_2).$$
(3) *If* $a_1(K_1) > 0$ *and* $a_1(K_2) > 0$, *then* $a_1(K_1\#K_2) = \min(a_1(K_1), a_1(K_2))$.
(4) *If* $a_1(K_1) < 0$ *and* $a_1(K_2) < 0$, *then* $a_1(K_1\#K_2) = \max(a_1(K_1), a_1(K_2))$.

Proof. Note that we use $-K$ to denote the mirror of a knot K.

(1) See the proof of Lemma 6.3 of [Hom14b].
(2) The mirrors $-K_1$ and $-K_2$ satisfy the hypothesis of (1), so
$$a_1(-K_1\# - K_2) = a_1(-K_2).$$
Apply the symmetry property of a_1 under mirroring (3.9):
$$-a_1(K_1\#K_2) = -a_1(K_2).$$
(3) By Lemma 6.2 of [Hom14b], there exists a basis $\{x_i\}$ over $\mathbb{F}[U, U^{-1}]$ for $\mathrm{CFK}^\infty(K_1)$ with basis elements x_0 and x_1 with the property that
   (a) There is a horizontal arrow of length a_1 from x_1 to x_0.
   (b) There are no other horizontal arrows or vertical arrows to or from x_0.
   (c) There are no other horizontal arrows to or from x_1.

Similarly, we may find a basis $\{y_i\}$ over $\mathbb{F}[U, U^{-1}]$ for $\mathrm{CFK}^\infty(K_2)$ with basis elements y_0 and y_1 with the above properties. Without loss of generality, assume that $a_1(K_1) \leq a_1(K_2)$.

Notice x_0y_0 generates the vertical homology $H_*(C(\{i = 0\}))$ of $\mathrm{CFK}^\infty(K_1\#K_2)$. Let $\tau = \tau(K_1\#K_2)$. Consider the subquotient complex
$$A = C\{\min(i, j - \tau) = 0\}.$$
There is a direct summand of A consisting of the generators x_0y_0, x_0y_1, x_1y_0, and x_1y_1, and four horizontal arrows as shown in Figure 7. The arrow x_1y_0 to x_0y_0 has length $a_1(K_1)$. Clearly, $\varepsilon(K_1\#K_2) = 1$ and $a_1(K_1\#K_2) = a_1(K_1)$.

(4) The mirrors $-K_1$ and $-K_2$ satisfy the hypothesis of (3). So
$$\begin{aligned} -a_1(K_1\#K_2) &= a_1(-K_1\# - K_2) = \min(a_1(-K_1), a_1(-K_2)) \\ &= \min(-a_1(K_1), -a_1(K_2)) = -\max(a_1(K_1), a_1(K_2)). \end{aligned}$$
□

Proposition 3.10 can be rewritten as the following.

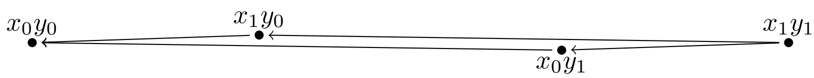


FIGURE 7. A direct summand of $A = C\{\min(i, j - \tau) = 0\}$ in Proposition 3.10(3).
This is the summand that is relevant for computing a_1, as it contains the generator x_0y_0 of vertical homology $H_*(C\{i = 0\})$.

Proposition 3.11. *If $a_1(K_1) \neq 0$ and $a_1(K_2) \neq 0$:*

(1) *If $a_1(K_1) + a_1(K_2) < 0$, then $a_1(K_1\#K_2) = \max(a_1(K_1), a_1(K_2))$.*
(2) *If $a_1(K_1) + a_1(K_2) > 0$, then $a_1(K_1\#K_2) = \min(a_1(K_1), a_1(K_2))$.*

Remark 3.12. If $a_1(K) \neq 0$ and $a_1(K') \neq 0$, and $a_1(K) + a_1(K') = 0$, then $a_1(K\#K')$ is indeterminate. The next two examples illustrate this case.

Example 3.13. The connect sum of any knot K with the reverse of its mirror $-K$, i.e., the inverse of K in the concordance group $\mathcal{C}$, has vanishing $a_1(K\# - K) = 0$.

We conclude with some computations of the a_1–invariant.

Example 3.14. The full knot Floer chain complexes CFK^∞ of the mirror $-T_{2,3;2,5}$ of the $(2, 5)$-cable of the torus knot $T_{2,3}$, the torus knot $T_{2,9}$, and the connect sum $-T_{2,3;2,5}\#T_{2,9}$ are described in [HW14]. It is easy to see that $a_1(-T_{2,3;2,5}) = -1$, $a_1(T_{2,9}) = 1$, and $a_1(-T_{2,3;2,5}\#T_{2,9}) = -1$.

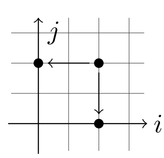
Example 3.15. In [Hom16, Figure 1(b)] Hom produces the relevant summand of $\mathrm{CFK}^\infty(T_{4,5}\# - T_{2,3;2,5})$ for computing $\varepsilon(T_{4,5}\# - T_{2,3;2,5})$ and $a_1(T_{4,5}\# - T_{2,3;2,5})$. Note that $\tau(T_{4,5}\# - T_{2,3;2,5}) = 2$. Using Figure 8, the map

$$F_{2,2} \colon C\{i = 0\} \to C\{\min(i, j - 2) = 0, i \leq 2\}$$

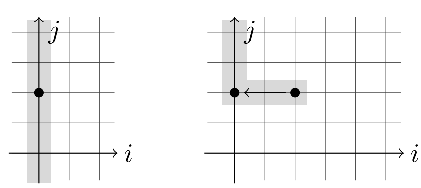
is trivial on homology, whereas the map

$$F_{1,2} \colon C\{i = 0\} \to C\{\min(i, j - 2) = 0, i \leq 1\}$$

is not trivial on homology. This gives $a_1(T_{4,5}\# - T_{2,3;2,5}) = 2$.



(A) The relevant summand of $\mathrm{CFK}^\infty(T_{4,5}\# - T_{2,3;2,5})$ for computing a_1.



(B) The map $C\{i = 0\} \to C\{\min(i, j - 2) = 0, i \leq 2\}$ is trivial on homology.

FIGURE 8. Computing a_1 for the knot $T_{4,5}\# - T_{2,3;2,5}$.

Example 3.16. The Conway knot $C_{2,1}$ has $a_1(C_{2,1}) = 0$. According to [Pet10], the knot Floer chain complex $\mathrm{CFK}^\infty(C_{2,1})$ is generated as an $\mathbb{F}[U, U^{-1}]$–module by a single isolated $\mathbb{F}$ at the origin plus a collection of null-homologous "boxes".

Example 3.17. The knot Floer chain complex of an L-space knot is a given by Theorem 2.1 in [OSS14]. If K is an L-space knot, with Alexander polynomial $\Delta_K(t) = \sum_{i=0}^{k}(-1)^i t^{n_i}$, where $n_0 > n_1 > \cdots > n_k$, then $a_1(K) = n_0 - n_1$ by Lemma 6.5 in [Hom14b].

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References

[DHST19] Irving Dai, Jennifer Hom, Matthew Stoffregen, and Linh Truong, *More concordance homomorphisms from knot Floer homology*, 2019, preprint, arXiv:1902.03333.

[Hed07] Matthew Hedden, *Knot Floer homology of Whitehead doubles*, Geom. Topol. **11** (2007), 2277–2338, DOI 10.2140/gt.2007.11.2277. MR2372849

[HKL16] Matthew Hedden, Se-Goo Kim, and Charles Livingston, *Topologically slice knots of smooth concordance order two*, J. Differential Geom. **102** (2016), no. 3, 353–393. MR3466802

[Hom14a] Jennifer Hom, *Bordered Heegaard Floer homology and the tau-invariant of cable knots*, J. Topol. **7** (2014), no. 2, 287–326, DOI 10.1112/jtopol/jtt030. MR3217622

[Hom14b] Jennifer Hom, *The knot Floer complex and the smooth concordance group*, Comment. Math. Helv. **89** (2014), no. 3, 537–570, DOI 10.4171/CMH/326. MR3260841

[Hom15] Jennifer Hom, *A survey on Heegaard Floer homology and concordance*, J. Knot Theory Ramifications **26** (2017), no. 2, 1740015, 24, DOI 10.1142/S0218216517400156. MR3604497

[Hom16] Jennifer Hom, *A note on the concordance invariants epsilon and upsilon*, Proc. Amer. Math. Soc. **144** (2016), no. 2, 897–902, DOI 10.1090/proc/12706. MR3430863

[HW14] Jennifer Hom and Zhongtao Wu, *Four-ball genus bounds and a refinement of the Ozváth-Szabó tau invariant*, J. Symplectic Geom. **14** (2016), no. 1, 305–323, DOI 10.4310/JSG.2016.v14.n1.a12. MR3523259

[OS03] Peter Ozsváth and Zoltán Szabó, *Knot Floer homology and the four-ball genus*, Geom. Topol. **7** (2003), 615–639, DOI 10.2140/gt.2003.7.615. MR2026543

[OS04] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and knot invariants*, Adv. Math. **186** (2004), no. 1, 58–116, DOI 10.1016/j.aim.2003.05.001. MR2065507

[OS11] Peter S. Ozsváth and Zoltán Szabó, *Knot Floer homology and rational surgeries*, Algebr. Geom. Topol. **11** (2011), no. 1, 1–68, DOI 10.2140/agt.2011.11.1. MR2764036

[OSS14] Peter Ozsváth, András Stipsicz, and Zoltán Szabó, *Concordance homomorphisms from knot Floer homology*, arXiv:1407.1795 (2014).

[Pet10] Thomas D. Peters, *A concordance invariant from the Floer homology of +/- 1 surgeries*, arXiv:1003.3038 (2010).

[Pet13] Ina Petkova, *Cables of thin knots and bordered Heegaard Floer homology*, Quantum Topol. **4** (2013), no. 4, 377–409, DOI 10.4171/QT/43. MR3134023

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