Truncated Heegaard Floer homology and knot concordance invariants

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We construct a sequence of smooth concordance invariants $\nu_n(K)$ defined using truncated Heegaard Floer homology. The invariants generalize the concordance invariants ν of Ozsváth and Szabó and ν^+ of Hom and Wu. We exhibit an example in which the gap between two consecutive elements in the sequence ν_n can be arbitrarily large. We also prove that the sequence ν_n contains more concordance information than τ, ν, ν', ν^+ and $\nu^{+'}$.

1 Introduction

Two knots, K_0 and K_1, in S^3 are *smoothly concordant* if there is a smooth proper embedding of a cylinder $S^1 \times [0,1]$ into $S^3 \times [0,1]$ with boundary $S^1 \times \{0\} = K_0$ and $S^1 \times \{1\} = -K_1$. The Heegaard Floer homology package of Ozsváth and Szabó has led to a wealth of smooth concordance invariants. A survey on this topic can be found in Hom [6].

Two concordance invariants motivate this article:

(1) $\nu(K)$, defined by Ozsváth and Szabó [14] using maps on Heegaard Floer homology $\widehat{HF}$ induced by the two-handle cobordism corresponding to integral surgery along K.

(2) $\nu^+(K)$, defined by Hom and Wu [7] using maps induced by surgery on HF^+. Hom and Wu showed that $\nu^+(K)$ produces arbitrarily better four-ball genus bounds than $\nu(K)$.

We construct a sequence of concordance invariants $\nu_n(K)$ for $n \in \mathbb{Z}$ which are defined using maps induced by surgery on the truncated Heegaard Floer homology HF^n. The invariants $\nu_n(K)$ generalize $\nu(K)$ and $\nu^+(K)$, as $\nu_1(K) = \nu(K)$ and $\nu_n(K) = \nu^+(K)$ for n sufficiently large. The properties of $\nu_n(K)$ are stated below.

Theorem 1.1 *The knot invariants $v_n(K)$, where $n \in \mathbb{Z}$, satisfy the following properties:*

- $v_n(K)$ *is a concordance invariant.*

- $v_1(K) = v(K)$.

- $v_n(K) \leq v_{n+1}(K)$.

- *For sufficiently large n, $v_n(K) = v^+(K)$.*

- $v_n(-K) = -v_{-n}(K)$, *where $-K$ is the mirror of K.*

- $v_n(K) \leq g_4(K)$.

By an extension of the large integer surgery formulas to truncated Floer homology (see Propositions 3.1 and 3.2) the invariants $v_n(K)$ can be computed from the $\mathbb{Z} \oplus \mathbb{Z}$–filtered knot Floer chain complex $\mathrm{CFK}^\infty(K)$.

Homologically thin knots are a special class of knots whose knot Floer homology lies in a single $\delta = A - M$ grading, where A is the Alexander grading and M is the Maslov grading. We prove that $v_n(K)$ of thin knots only depends on $\tau(K)$.

Proposition 4.1 *Let K be a homologically thin knot with $\tau(K) = \tau$.*

(i) *If $\tau = 0$, $v_n(K) = 0$ for all n.*

(ii) *If $\tau > 0$,*
$$v_n(K) = \begin{cases} 0 & \text{for } n \leq -\tfrac{1}{2}(\tau + 1), \\ \tau + 2n + 1 & \text{for } -\tfrac{1}{2}\tau \leq n \leq -1, \\ \tau & \text{for } n \geq 0. \end{cases}$$

(iii) *If $\tau < 0$,*
$$v_n(K) = \begin{cases} \tau & \text{for } n \leq 0, \\ \tau + 2n - 1 & \text{for } 1 \leq n \leq -\tfrac{1}{2}\tau, \\ 0 & \text{for } n \geq \tfrac{1}{2}(-\tau + 1). \end{cases}$$

The computation of $v_n(K)$ for thin knots illustrates that the gap between $v_n(K)$ and $v_{n+1}(K)$ can be more than one. In fact, the gap between $v_n(K)$ and $v_{n+1}(K)$ can be arbitrarily big.

Theorem 1.2 *Let $T_{p,p+1}$ denote the $(p, p+1)$–torus knot. For $p > 3$,*
$$v_{-1}(T_{p,p+1}) - v_{-2}(T_{p,p+1}) = p.$$

Finally, we show that the sequence ν_n contains more concordance information than τ, ν, ν', ν^+ and $\nu^{+'}$.

Proposition 1.3 *There exist knots K and K' with equal τ, ν, ν', ν^+ and $\nu^{+'}$ invariants, but $\nu_n(K) \neq \nu_n(K')$ for some $n \in \mathbb{Z}$.*

Organization of the paper In Section 2 we review the constructions of the concordance invariants $\nu(K)$ and $\nu^+(K)$. In Section 3 we define the invariants $\nu_n(K)$ and prove their properties: monotonicity, stabilization and behavior under mirroring. In Section 4 we compute $\nu_n(K)$ for special families of knots and compare them to $\nu(K)$ and $\nu^+(K)$. In Section 5 we pose some questions about the concordance invariants $\nu_n(K)$.

2 Background on the invariants $\nu(K)$ and $\nu^+(K)$

A four-dimensional cobordism equipped with a Spin^c structure between two three-manifolds induces a map on the Heegaard Floer homology groups [12]. In particular, for a knot K in S^3, the 2–handle attachment cobordism from $S_N^3(K)$ or $S_{-N}^3(K)$ to S^3 induces maps

(1) $\hat{v}_{s,*}\colon \widehat{\mathrm{HF}}(S_N^3(K), \mathfrak{s}_s) \to \widehat{\mathrm{HF}}(S^3), \qquad \hat{v}_{s,*}'\colon \widehat{\mathrm{HF}}(S^3) \to \widehat{\mathrm{HF}}(S_{-N}^3(K), \mathfrak{s}_s),$

(2) $v_{s,*}^+\colon \mathrm{HF}^+(S_N^3(K), \mathfrak{s}_s) \to \mathrm{HF}^+(S^3), \qquad v_{s,*}^{+'}\colon \mathrm{HF}^+(S^3) \to \mathrm{HF}^+(S_{-N}^3(K), \mathfrak{s}_s),$

(3) $v_{s,*}^-\colon \mathrm{HF}^-(S_N^3(K), \mathfrak{s}_s) \to \mathrm{HF}^-(S^3), \qquad v_{s,*}^{-'}\colon \mathrm{HF}^-(S^3) \to \mathrm{HF}^-(S_{-N}^3(K), \mathfrak{s}_s),$

where $\mathfrak{s}_s$ denotes the restriction to $S_N^3(K)$ or $S_N^3(K)$ of a Spin^c structure $\mathfrak{t}$ on the corresponding 2–handle cobordism such that

$$\langle c_1(\mathfrak{t}), [\hat{F}]\rangle + N = 2s, \quad \langle c_1(\mathfrak{t}), [\hat{F}]\rangle - N = 2s,$$

where $\hat{F}$ is a capped-off Seifert surface for K. These cobordism maps on $\widehat{\mathrm{HF}}$ and HF^+ play a key role in defining the concordance invariants ν and ν^+.

Definition 2.1 [14, Section 9] The concordance invariant $\nu(K)$ is defined as

$$\nu(K) = \min\{s \in \mathbb{Z} \mid \widehat{v}_{s,*} \text{ is surjective}\}.$$

Definition 2.2 The concordance invariant $\nu'(K)$ is defined as

$$\nu'(K) = \max\{s \in \mathbb{Z} \mid \widehat{v}'_{s,*} \text{ is injective}\}.$$

For a rational homology 3–sphere Y with a Spin^c structure $\mathfrak{s}$, $\mathrm{HF}^+(Y, \mathfrak{s})$ can be decomposed as the direct sum of two groups: the first group is the image of $\mathrm{HF}^\infty(Y, \mathfrak{s}) \cong \mathbb{F}[U, U^{-1}]$ in $\mathrm{HF}^+(Y, \mathfrak{s})$, which is isomorphic to $\mathcal{T}^+ = \mathbb{F}[U, U^{-1}]/U\mathbb{F}[U]$; the second group is $\mathrm{HF}_{\mathrm{red}}(Y, \mathfrak{s}) = \mathrm{HF}^+(Y, \mathfrak{s})/\mathcal{T}^+$. That is,

$$\mathrm{HF}^+(Y, \mathfrak{s}) = \mathcal{T}^+ \oplus \mathrm{HF}_{\mathrm{red}}(Y, \mathfrak{s}).$$

Definition 2.3 [7] The concordance invariant ν^+ is defined as

$$\nu^+(K) = \min\{s \mid v_{s,*}^+ \colon \mathrm{HF}^+(S_N^3(K), \mathfrak{s}_s) \to \mathrm{HF}^+(S^3) \text{ sends } 1 \text{ to } 1\},$$

where 1 denotes the lowest-graded generator in the subgroup $\mathcal{T}^+$ of the homology, and N is sufficiently large that the integer surgery formula holds.

Equivalently, Hom [6] defines the invariant $\nu^-(K)$ in terms of the map

$$v_{s,*}^- \colon \mathrm{HF}^-(S_N^3 K, \mathfrak{s}_s) \to \mathrm{HF}^-(S^3).$$

Definition 2.4 [6] The concordance invariant

$$\nu^-(K) = \min\{s \in \mathbb{Z} \mid v_{s,*}^- \text{ is surjective}\}$$

is equal to $\nu^+(K)$.

Hom and Wu show that

$$\tau(K) \le \nu(K) \le \nu^+(K)$$

and $\nu^+(K) \ge 0$. In addition, ν^+ gives a lower bound on the four-ball genus, $\nu^+(K) \le g_4(K)$. Furthermore, Hom and Wu provide a family of knots where $\nu^+(K)$ is an arbitrarily better bound on $g_4(K)$ than $\tau(K)$.

The concordance invariants ν and ν^+ are easily computable from $\mathrm{CFK}^\infty(K)$ via the large integer surgery formulas. Let CX denote the subgroup of $\mathrm{CFK}^\infty(K)$ generated

by elements $[x, i, j]$ that lie in filtration level $(i, j) \in X \subset \mathbb{Z} \oplus \mathbb{Z}$. Consider the chain maps

$$\hat{v}_s \colon C\{\max(i, j - s) = 0\} \to C\{i = 0\},$$
$$v_s^+ \colon C\{\max(i, j - s) \geq 0\} \to C\{i \geq 0\},$$

defined by taking the quotient by $C\{i < 0, \, j = s\}$ or $C\{i < 0, \, j \geq s\}$, respectively, followed by the inclusions. The large integer surgery formula of Ozsváth and Szabó [11] asserts that the maps $\hat{v}_s$ and v_s^+ induce the maps from (1) and (2). Similarly, consider the chain maps

$$\hat{v}_s' \colon C\{i = 0\} \to C\{\min(i, j - s) = 0\},$$
$$v_s^{+'} \colon C\{i \geq 0\} \to C\{\min(i, j - s) \geq 0\},$$

consisting of quotienting by $C\{i = 0, \, j \leq s\}$ followed by the inclusion. Ozsváth and Szabó [11] show that these maps induce the maps from (1) and (2).

We introduce a concordance invariant $v^{+'}$, so that the pair v^+ and $v^{+'}$ is the HF$^+$ analogue to the pair v and v'.

Definition 2.5 The concordance invariant $v^{+'}$ is defined as

$$v^{+'}(K) = \max\{s \in \mathbb{Z} \mid v_{s,*}^{+'} \colon \mathrm{HF}^+(S^3) \to \mathrm{HF}^+(S^3_{-N}(K), \mathfrak{s}_s) \text{ is injective}\},$$

where $-N$ is sufficiently negative that the (negative) large integer surgery formula holds.

We prove a mirroring property which relates $v^{+'}(K)$ to the invariant $v^+(-K)$ of the mirror of K:

Lemma 2.6
$$v^{+'}(K) = -v^+(-K).$$

Proof Recall the symmetry of CFK$^\infty$ under mirroring [11, Section 3.5],

$$\mathrm{CFK}^\infty(-K) \simeq \mathrm{CFK}^\infty(K)^*,$$

where CFK$^\infty(K)^*$ is the dual complex $\mathrm{Hom}_{\mathbb{F}[U,U^{-1}]}(\mathrm{CFK}^\infty(K), \mathbb{F}[U, U^{-1}])$. Therefore,

$v_{-s,*}^{+'} \colon \mathrm{HF}^+(S^3) \to \mathrm{HF}^+(S^3_{-N}(K), \mathfrak{s}_{-s})$ is injective

$$\iff v_{s,*}^- \colon \mathrm{HF}^-(S^3_N(-K), \mathfrak{s}_s) \to \mathrm{HF}^-(S^3) \text{ is surjective,}$$

which implies the result. $\qquad\square$

It follows from the above lemma that the invariant $\nu^{+'}$ exhibits properties similar to ν^+,

$$\nu^{+'}(K) \leq \nu'(K) \leq \tau(K) \leq \nu(K) \leq \nu^+(K)$$

and $\nu^{+'}(K) \leq 0$. In addition, the absolute value of $\nu^{+'}(K)$ gives a lower bound on the four-ball genus:

Theorem 2.7 $$|\nu^{+'}(K)| \leq g_4(K).$$

Proof This follows from the fact that $\nu^+(K) \leq g_4(K)$ and Lemma 2.6. $\square$

3 The concordance invariants $\nu_n(K)$

The construction of the concordance invariants $\nu_n(K)$ uses truncated Heegaard Floer homology $\mathrm{HF}^n(Y, \mathfrak{s})$, described in [9; 13]. $\mathrm{HF}^n(Y, \mathfrak{s})$ is the homology of the kernel $\mathrm{CF}^n(Y, \mathfrak{s})$ of the multiplication map

$$U^n \colon \mathrm{CF}^+(Y, \mathfrak{s}) \to \mathrm{CF}^+(Y, \mathfrak{s}),$$

where $n \in \mathbb{Z}_+$. The two-handle cobordism from $S_N^3 K$ or $S_{-N}^3 K$, respectively, to S^3 induces a map on the truncated Floer chain complex,

$$v_s^n \colon \mathrm{CF}^n(S_N^3 K, \mathfrak{s}_s) \to \mathrm{CF}^n(S^3), \quad v_s^{-n} \colon \mathrm{CF}^n(S^3) \to \mathrm{CF}^n(S_{-N}^3 K, \mathfrak{s}_s),$$

and on the truncated Floer homology,

$$v_{s,*}^n \colon \mathrm{HF}^n(S_N^3 K, \mathfrak{s}_s) \to \mathrm{HF}^n(S^3), \quad v_{s,*}^{-n} \colon \mathrm{HF}^n(S^3) \to \mathrm{HF}^n(S_{-N}^3 K, \mathfrak{s}_s),$$

where $\mathfrak{s}_s$ denotes the restriction to $S_N^3(K)$ or $S_{-N}^3(K)$, respectively, of a Spin^c structure $\mathfrak{t}$ on the corresponding 2–handle cobordism such that

$$\langle c_1(\mathfrak{t}), [\widehat{F}] \rangle + N = 2s, \quad \langle c_1(\mathfrak{t}), [\widehat{F}] \rangle - N = 2s,$$

where $\widehat{F}$ is a capped-off Seifert surface for K. These cobordism maps on $\widehat{\mathrm{HF}}$ and HF^+ play a key role in defining the concordance invariants ν and ν^+.

We extend the large integer surgery formula of Ozsváth and Szabó to truncated Heegaard Floer homology:

Proposition 3.1 (large negative integer surgery formula for HF^n) *Consider the subquotient complex*

$$\mathrm{CFK}^{-n}(S^3, K, m) = C\{0 \leq \min(i, j - m) \leq n - 1\}$$

of $\mathrm{CFK}^+(S^3, K, m)$ generated by $[\boldsymbol{x}, i, j]$ with $0 \leq \min(i, j - m) \leq n - 1$. For each $m \in \mathbb{Z}$, there is an integer $N(m)$ such that for all $p \geq N(m)$, the map Φ of Ozsváth and Szabó induces isomorphisms in the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{CFK}^{-n}(S^3, K, m) & \longrightarrow & \mathrm{CFK}^+(S^3, K, m) & \xrightarrow{U^n} & \mathrm{CFK}^+(S^3, K, m) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Phi(n)} & & \downarrow{\scriptstyle \Phi^+} & & \downarrow{\scriptstyle \Phi^+} & & \\
0 & \longrightarrow & \mathrm{CF}^n(S^3_{-p}K, [m]) & \longrightarrow & \mathrm{CF}^+(S^3_{-p}K, [m]) & \xrightarrow{U^n} & \mathrm{CF}^+(S^3_{-p}K, [m]) & \longrightarrow & 0
\end{array}
$$

Proof The map Φ^+ is an isomorphism of chain complexes by Theorem 4.1 of [11]. By the five lemma, so is $\Phi(n)$. □

Proposition 3.2 (large positive integer surgery formula for HF^n) *Consider the subquotient complex*

$$\mathrm{CFK}^n(S^3, K, m) = C\{0 \leq \max(i, j - m) \leq n - 1\}$$

of $\mathrm{CFK}^+(S^3, K, m)$ generated by $[\boldsymbol{x}, i, j]$ with $0 \leq \max(i, j - s) \leq n - 1$. For each $m \in \mathbb{Z}$, there is an integer $N(m)$ such that for all $p \geq N(m)$, the map Ψ of Ozsváth and Szabó induces isomorphisms in the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{CF}^n(S^3_p K, [m]) & \longrightarrow & \mathrm{CF}^+(S^3_p K, [m]) & \xrightarrow{U^n} & \mathrm{CF}^+(S^3_p K, [m]) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Psi(n)} & & \downarrow{\scriptstyle \Psi^+} & & \downarrow{\scriptstyle \Psi^+} & & \\
0 & \longrightarrow & \mathrm{CFK}^n(S^3, K, m) & \longrightarrow & {}^b\mathrm{CFK}^+(S^3, K, m) & \xrightarrow{U^n} & {}^b\mathrm{CFK}^+(S^3, K, m) & \longrightarrow & 0
\end{array}
$$

Proof The map Ψ^+ is an isomorphism of chain complexes by Theorem 4.4 of [11]. By the five lemma, so is $\Psi(n)$. □

Notation We sometimes use the notation, for $n > 0$,

$$A^n_m = C\{0 \leq \max(i, j - m) \leq n - 1\}, \quad A^+_m = C\{0 \leq \max(i, j - m)\},$$
$$A^{-n}_m = C\{0 \leq \min(i, j - m) \leq n - 1\}, \quad A^{+\prime}_m = C\{0 \leq \min(i, j - m)\}$$

and

$$B^n = C\{0 \leq i \leq n - 1\},$$
$$B^+ = C\{0 \leq i\}.$$

The cobordism maps on truncated Heegaard Floer groups lead us to define concordance invariants v_n.

Definition 3.3 For $n > 0$, define

$$v_n(K) = \min\{s \in \mathbb{Z} \mid v_s^n \colon \mathrm{CF}^n(S_N^3(K), s) \to \mathrm{CF}^n(S^3)$$
$$\text{induces a surjection on homology}\},$$

and for $n < 0$, define

$$v_n(K) = \max\{s \in \mathbb{Z} \mid v_s^n \colon \mathrm{CF}^{-n}(S^3) \to \mathrm{CF}^{-n}(S_{-N}^3(K), s)$$
$$\text{induces an injection on homology}\},$$

where N is sufficiently large that the Ozsváth–Szabó large integer surgery formula of [11] holds. For $n = 0$, we define $v_0(K) = \tau(K)$.

Remark 3.4 For $n = \pm 1$, these invariants are already known as $v_1(K) = v(K)$ and $v_{-1}(K) = v'(K)$.

Proposition 3.5 $v_n(K)$ is a concordance invariant.

Proof Suppose K_1 is concordant to K_2. Then $S_N^3(K_1)$ is homology cobordant to $S_N^3(K_2)$. This implies that there exists a (smooth, connected, oriented) cobordism W from $S_N^3(K_1)$ to $S_N^3(K_2)$ with $H_i(W, \mathbb{Q}) = 0$ for $i = 1, 2$.

The map

$$\mathrm{HF}^n(S_N^3(K_1), s) \to \mathrm{HF}^n(S^3)$$

induced by the cobordism obtained by adding a two-handle along K_1 factors through $\mathrm{HF}^n(S_N^3(K_2), s)$. So, if it is surjective, then the map

$$\mathrm{HF}^n(S_N^3(K_2), s) \to \mathrm{HF}^n(S^3)$$

is also surjective. So we get that $v_n(K_1) \geq v_n(K_2)$. The same argument with K_1 and K_2 switched shows the inequality $v_n(K_2) \geq v_n(K_1)$. Therefore, $v_n(K_1) = v_n(K_2)$.

For negative n, that $v_n(K)$ is a concordance invariant follows from a similar argument to the above. □

Proposition 3.6 (mirroring property) $v_n(-K) = -v_{-n}(K)$.

Proof Recall the symmetry of CFK^∞ under mirroring [11, Section 3.5],

$$\mathrm{CFK}^\infty(-K) \simeq \mathrm{CFK}^\infty(K)^*,$$

where $\mathrm{CFK}^\infty(K)^*$ is the dual complex $\mathrm{Hom}_{\mathbb{F}[U,U^{-1}]}(\mathrm{CFK}^\infty(K), \mathbb{F}[U,U^{-1}])$. Letting $C = \mathrm{CFK}^\infty(S^3, K)$ and $n > 0$, the following conditions are equivalent:

- $v_{-s,*}^{-n}(K)$: $\mathrm{HF}^n(S^3) \to \mathrm{HF}^n(S_{-N}^3(K), \mathfrak{s}_{-s})$ is injective.
- $v_{-s}^{-n}(K)$: $C\{0 \le i \le n-1\} \to C\{0 \le \min(i, j+s) \le n-1\}$ is injective on H_*.
- $U^{n-1}v_s^n(-K)$: $C\{-(n-1) \le \max(i, j-s) \le 0\} \to C\{-(n-1) \le i \le 0\}$ is surjective on H_*.
- $v_s^n(-K)$: $C\{0 \le \max(i, j-s) \le n-1\} \to C\{0 \le i \le n-1\}$ is surjective on H_*.
- $v_{s,*}^n(-K)$: $\mathrm{HF}^n(S_N^3(-K), \mathfrak{s}_s) \to \mathrm{HF}^n(S^3)$ is surjective.

Here U^{n-1} is a degree-shifting isomorphism on $\mathrm{CFK}^\infty(K)$. Therefore,

$$v_n(-K) = \min(s \in \mathbb{Z} \mid v_{s,*}^n(-K) \text{ is surjective})$$
$$= -\max(-s \in \mathbb{Z} \mid v_{-s,*}^{-n}(K) \text{ is injective}) = -v_{-n}(K). \qquad \square$$

Proposition 3.7 (monotonicity) $v_n(K) \le v_{n+1}(K)$.

Proof It is known that $v_{-1}(K) \le \tau(K) \le v_1(K)$, so we focus on the two separate cases where $n > 0$ and $n < 0$.

For $n > 0$, consider the commutative diagram

$$
\begin{array}{ccc}
\mathrm{HF}^{n+1}(S_N^3 K, s) & \xrightarrow{v_{s,*}^{n+1}} & \mathrm{HF}^{n+1}(S^3) \\
\downarrow{\cdot U} & & \downarrow{\cdot U} \\
\mathrm{HF}^n(S_N^3 K, s) & \xrightarrow{v_{s,*}^n} & \mathrm{HF}^n(S^3)
\end{array}
$$

where the vertical maps are given by multiplication by U. The vertical map on the right is surjective. Thus, if $v_{s,*}^{n+1}$ is surjective, then so is $v_{s,*}^n$.

For $n < 0$, consider the commutative diagram

$$
\begin{array}{ccc}
\mathrm{HF}^{-n}(S^3) & \xrightarrow{v_{s,*}^n} & \mathrm{HF}^{-n}(S_{-N}^3 K, s) \\
\downarrow{i_a'} & & \downarrow{i_b'} \\
\mathrm{HF}^{-(n-1)}(S^3) & \xrightarrow{v_{s,*}^{n-1}} & \mathrm{HF}^{-(n-1)}(S_{-N}^3 K, s)
\end{array}
$$

where the vertical maps are induced by inclusion of chain groups. In particular, the left map i'_a is injective on homology. Therefore, if $v^{n-1}_{s,*}(K)$ is injective, then so is $v^n_{s,*}(K)$. We conclude that $v_{n-1}(K) \leq v_n(K)$. $\square$

Proposition 3.8 (boundedness) $v^{+'}(K) \leq v_n(K) \leq v^+(K)$ for all n.

Proof It is known that $v(K) \leq v^+(K)$ from [7]. For $n \geq 1$, consider the commutative diagram

$$
\begin{array}{ccc}
H_*(A^-_k) & \xrightarrow{\;j_A\;} & H_*(A^n_k) \\
\Big\downarrow{\scriptstyle v^-_{k,*}} & & \Big\downarrow{\scriptstyle v^n_{k,*}} \\
H_*(B^-) & \xrightarrow{\;j_B\;} & H_*(B^n)
\end{array}
$$

The map j_B is surjective, so if $v^-_{k,*}$ is surjective, then so is $v^n_{k,*}$.

For $n \leq -1$, consider the commutative diagram

$$
\begin{array}{ccc}
H_*(B^n) & \xrightarrow{\;i_B\;} & H_*(B^+) \\
\Big\downarrow{\scriptstyle v^n_{k,*}} & & \Big\downarrow{\scriptstyle v^{+'}_{k,*}} \\
H_*(A^n_k) & \xrightarrow{\;i_A\;} & H_*(A^{+'}_k)
\end{array}
$$

The map i_B is injective, so if $v^{+'}_{k,*}$ is injective, then so is $v^n_{k,*}$. $\square$

Proposition 3.9 (stabilization) *For sufficiently large positive n, $v_n(K) = v^+(K)$ and $v_{-n}(K) = v^{+'}(K)$.*

Proof Let $C_1 = \mathrm{CF}^-(S^3_N K, s)$ and $C_2 = \mathrm{CF}^-(S^3)$. There is a canonical degree-shifting isomorphism

$$
\mathrm{CF}^n(Y, \mathfrak{s}) \cong \mathrm{CF}^-(Y, \mathfrak{s}) \otimes_{\mathbb{F}[U]} \frac{\mathbb{F}[U]}{U^n}.
$$

Moreover, the map

$$
v^-_{n,s} \colon C_1 \otimes_{\mathbb{F}[U]} \frac{\mathbb{F}[U]}{U^n} \to C_2 \otimes_{\mathbb{F}[U]} \frac{\mathbb{F}[U]}{U^n}
$$

is the same as the map $v^n_s \colon \mathrm{CF}^n(S^3_N K, s) \to \mathrm{CF}^n(S^3)$. We show that if $v^-_{s,*}$ is not surjective, then neither is $v^n_{s,*}$ for sufficiently large n. By the universal coefficient

theorem,

$$0 \to H_*(C_1) \otimes \frac{\mathbb{F}[U]}{U^n} \xrightarrow{i_1} H_*\left(C_1 \otimes \frac{\mathbb{F}[U]}{U^n}\right) \longrightarrow \mathrm{Tor}\left(H_* C_1, \frac{\mathbb{F}[U]}{U^n}\right) \to 0$$

$$\downarrow{v^-_{s,*} \otimes \mathrm{id}} \qquad\qquad \downarrow{v^n_{s,*}} \qquad\qquad\qquad \downarrow{\mathrm{Tor}(v^-_s)}$$

$$0 \to H_*(C_2) \otimes \frac{\mathbb{F}[U]}{U^n} \xrightarrow{i_2} H_*\left(C_2 \otimes \frac{\mathbb{F}[U]}{U^n}\right) \longrightarrow \mathrm{Tor}\left(H_* C_2, \frac{\mathbb{F}[U]}{U^n}\right) \to 0$$

where all tensor products are taken over $\mathbb{F}[U]$.

We note the following facts:

- For a rational homology 3–sphere Y, $\mathrm{HF}^-(Y, \mathfrak{s})/\{U\text{–torsion}\} = \mathfrak{T}^- = \mathbb{F}[X]$. So $H_*(C_1) = \mathfrak{T}^- \oplus \left(\bigoplus \mathbb{F}[U]/U^{m_i}\right)$.
- $\mathfrak{T}^- \otimes \mathbb{F}[U]/U^n = \mathbb{F}[U]/U^n$ and $\mathbb{F}[U]/U^{m_i} \otimes \mathbb{F}[U]/U^n = \mathbb{F}[U]/U^{m_i}$. So $H_*(C_1) \otimes \mathbb{F}[U]/U^n = \mathbb{F}[U]/U^n \oplus \left(\bigoplus \mathbb{F}[U]/U^{m_i}\right)$.
- $\mathrm{Tor}(\mathbb{F}[U]/U^m, \mathbb{F}[U]/U^n) = \mathbb{F}[U]/U^m$ if $m < n$.
- $H_*(C_2 \otimes \mathbb{F}[U]/U^n) = \mathbb{F}[U]/U^n$.

Assume n is sufficiently large that $m_i < n$ for all m_i. So the above Tor groups are $n-1$ torsion.

If v^-_s is not surjective, we can further choose n sufficiently large that the image of $v^-_{s,*} \otimes \mathrm{id}$ is $n-1$ U–torsion. By commutativity of the diagram, the image of $v^n_{s,*} \circ i_1$ is $n-1$ U–torsion.

Suppose $\xi \in H_*(C_1 \otimes \mathbb{F}[U]/U^n)$ is such that $v^n_{s,*}(\xi)$ is an element of order n. Then, since the short exact sequence in the universal coefficient theorem splits, $\xi = \alpha + \beta$, where $\alpha \in H_*(C_1) \otimes \mathbb{F}[U]/U^n$ and $\beta \in \mathrm{Tor}(H_* C_1, \mathbb{F}[U]/U^n)$. But

$$U^{n-1} \cdot v^n_{s,*}(\alpha + \beta) = v^n_{s,*}(U^{n-1}\alpha) + v^n_{s,*}(U^{n-1}\beta) = 0.$$

Since $H_*(C_2 \otimes \mathbb{F}[U]/U^n) = \mathbb{F}[U]/U^n$, the invariant $v^n_{s,*}$ is not surjective. Therefore, for sufficiently large n, $v_n(K) = v^+(K)$.

Finally, by the mirroring property, $v_n(K) = v^{+'}(K)$ for sufficiently large negative integers n. $\qquad\square$

The fact that $v_n(K)$ are not concordance homomorphisms from $\mathcal{C}$ to $\mathbb{Z}$ can easily be seen. Note $v_n(K)$ is not additive under connected sum of knots. For $n = 1$, just

consider two knots with $\varepsilon(K) = \varepsilon(K') = -1$. Then

$$\nu(K) = \tau(K) + 1 \quad \text{and} \quad \nu(K') = \tau(K') + 1,$$

but

$$\nu(K \# K') = \tau(K \# K') + 1 = \tau(K) + \tau(K') + 1 < \nu(K) + \nu(K').$$

4 Computations

Knot Floer homology groups can be easily computed for certain special families of knots. For example, homologically thin knots are knots with $\widehat{\mathrm{HFK}}(K)$ supported in a single δ–grading, where $\delta = A - M$. If the homology is supported on the diagonal $\delta = -\frac{1}{2}\sigma(K)$, where $\sigma(K)$ denotes the knot signature, then we say the knot is σ–thin. The class of σ–thin knots contains as a proper subset all quasialternating knots, and in particular all alternating knots. The following theorem shows that $\nu_n(K)$ of thin knots only depends on $\tau(K)$:

Proposition 4.1 *Let K be a homologically thin knot with $\tau(K) = \tau$.*

(i) *If $\tau = 0$, $\nu_n(K) = 0$ for all n.*

(ii) *If $\tau > 0$,*

$$\nu_n(K) = \begin{cases} 0 & \text{for } n \leq -\frac{1}{2}(\tau + 1), \\ \tau + 2n + 1 & \text{for } -\frac{1}{2}\tau \leq n \leq -1, \\ \tau & \text{for } n \geq 0. \end{cases}$$

(iii) *If $\tau < 0$,*

$$\nu_n(K) = \begin{cases} \tau & \text{for } n \leq 0, \\ \tau + 2n - 1 & \text{for } 1 \leq n \leq -\frac{1}{2}\tau, \\ 0 & \text{for } n \geq \frac{1}{2}(-\tau + 1). \end{cases}$$

Proof In [15, Theorem 4] Petkova constructs model complexes for $\mathrm{CFK}^\infty(K)$ of homologically thin knots. She shows the model chain complex contains a direct summand (called the "staircase") isomorphic to

$$\begin{cases} \mathrm{CFK}^\infty(T_{2,2\tau+1}) & \text{if } \tau(K) > 0, \\ \mathrm{CFK}^\infty(T_{2,2\tau-1}) & \text{if } \tau(K) \leq 0. \end{cases}$$

The "staircase" summand supports $H_*(\mathrm{CFK}^\infty(K))$; that is,

$$H_*(\mathrm{CFK}^\infty(K)) = H_*(\mathrm{CFK}^\infty(T_{2,2\tau-1})).$$

The maps induced on homology by v_s^n (or $v_{n,s}^-$) will thus only depend on the "staircase" summand and not the acyclic summands. Thus, $v_n(T_{2,2\tau\pm1}) = v_n(K)$.

Without loss of generality, assume $\tau(K) > 0$. The chain complex $\mathrm{CFK}^-(T_{2,2\tau+1})$ is generated over $\mathbb{F}[U]$ by generators $\{z_p\}_{p=1}^{2\tau+1}$ with U–filtration levels i and Alexander filtration levels j specified (for all $1 \le p \le 2\tau + 1$) by

$$j(z_p) = \begin{cases} \tau - \frac{1}{2}(p-1) & \text{if } p \text{ odd,} \\ \tau - \frac{1}{2}(p-2) & \text{if } p \text{ even,} \end{cases} \qquad i(z_p) = \begin{cases} \frac{1}{2}(p-1) & \text{if } p \text{ odd,} \\ \frac{1}{2}p & \text{if } p \text{ even,} \end{cases}$$

and differential

$$\partial z_p = \begin{cases} 0 & \text{if } p \text{ odd,} \\ z_{p-1} + z_{p+1} & \text{if } p \text{ even.} \end{cases}$$

The above complex with generators $\{z_p\}_{p=1}^{2\tau+1}$ and given differential maps forms the generating "staircase" complex C_τ, and $\mathrm{CFK}^\infty(T_{2,2\tau+1}) = C_\tau \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]$. The U–action lowers i and j by one.

Computation of $v^+(T_{2,2\tau+1})$ and $v_n(T_{2,2\tau+1})$ for $n > 0$ Since

$$C\{i < 0, \, j \ge \tau\} = 0,$$

the map v_τ^+ is the same as the inclusion

$$C\{0 \le i \le n-1, \, j \le \tau + n - 1\} \to C\{0 \le i \le n-1\} = B^n.$$

Moreover, the generator with the highest Alexander grading in $C\{0 \le i \le n-1\}$ is $U^{n-1}z_1$, with

$$j(U^{n-1}z_1) = \tau + n - 1.$$

Thus, $C\{0 \le i \le n-1, \, j > \tau+n-1\} = 0$. That is, the inclusion v_τ^+ is an isomorphism of chain complexes, so $v^+(K) = \tau(K)$. Therefore, $v_n(K) = \tau(K)$ for all $n \ge 0$.

Computation of $v^{+'}(T_{2,2\tau+1})$ The homology of B^+ is generated by $\{[U^{-i}z_1]\}$ for all $i \ge 0$. The subquotient complex $A_0^{+'}$ contains $U^{-i}C_\tau$ for all $i \ge 0$, and the homology of $U^{-i}C_\tau$ is generated by the class $[U^{-i}z_1]$. Therefore, $v_{0,*}^{+'}[U^{-i}z_1] \ne 0$ in $H_*(A_0^{+'})$, and $v_{0,*}^{+'}$ is injective. So $v^{+'}(T_{2,2\tau+1}) \ge 0$. But since $v^{+'}(K) \le 0$ for any knot K, we conclude $v^{+'}(T_{2,2\tau+1}) = 0$.

Computation of $v_n(T_{2,2\tau+1})$ for $-\frac{1}{2}\tau \le n \le -1$ Consider the subquotient complex A_k^n where $k = \tau + 2n + 1$. For each $1 \le p \le 2\tau + 1$,

$$\min(i(z_p), j(z_p) - k) = \begin{cases} \min\left(\frac{1}{2}(p-1), -\frac{1}{2}(p-1) - 2n - 1\right) & \text{if } p \text{ is odd,} \\ \min\left(\frac{1}{2}p, -\frac{1}{2}(p-2) - 2n - 1\right) & \text{if } p \text{ is even} \end{cases}$$

$$
= \begin{cases}
\frac{1}{2}(p-1) & \text{if } p \text{ is odd and } p \le -2n-1, \\
-\frac{1}{2}(p-1)-2n-1 & \text{if } p \text{ is odd and } p > -2n, \\
\frac{1}{2}p & \text{if } p \text{ is even and } p \le -2n, \\
-\frac{1}{2}(p-2)-2n-1 & \text{if } p \text{ is even and } p > -2n.
\end{cases}
$$

Using these formulas, it is straightforward to check that A_k^n contains z_p for $1 \le p \le -2n-1$ but $z_{-2n} \notin A_k^n$. Therefore, $[z_1] \ne 0$ in A_k^n. Similarly, for $1 \le i \le -n-1$, A_k^n contains $U^{-i}z_p$ for $1 \le p \le -2(n+i)-1$ but $z_{-2(n+i)} \notin A_k^n$. Therefore, $[U^{-i}z_1] \ne 0$ in $H_*(A_k^n)$. Since $H_*(B^n)$ is generated by $[U^{-i}z_1]$ for $0 \le i \le -n-1$, v_k^n is injective on homology.

To check that $v_n(T_{2,2\tau+1}) = \tau + 2n + 1$, consider the subquotient complex $A_{\tau+2n+2}^n$. For each $1 \le p \le 2\tau+1$,

$$
\min(i(z_p), j(z_p)-k) = \begin{cases}
\min\left(\frac{1}{2}(p-1), -\frac{1}{2}(p-1)-2n-2\right) & \text{if } p \text{ is odd}, \\
\min\left(\frac{1}{2}p, -\frac{1}{2}(p-2)-2n-2\right) & \text{if } p \text{ is even}
\end{cases}
$$

$$
= \begin{cases}
\frac{1}{2}(p-1) & \text{if } p \text{ is odd and } p \le -2n-1, \\
-\frac{1}{2}(p-1)-2n-2 & \text{if } p \text{ is odd and } p > -2n, \\
\frac{1}{2}p & \text{if } p \text{ is even and } p \le -2n-2, \\
-\frac{1}{2}(p-2)-2n-2 & \text{if } p \text{ is even and } p \ge -2n.
\end{cases}
$$

Using the above, it is straightforward to check that $A_{\tau+2n+2}^n$ contains z_p for $1 \le p \le -4n-2$ but $z_{-4n-1} \notin A_{\tau+2n+2}^n$. Therefore, $[z_1] = 0$ in $H_*(A_{\tau+2n+2}^n)$. Thus, $v_n(T_{2,2\tau+1}) = \tau + 2n + 1$.

Computation of $v_n(T_{2,2\tau+1})$ for $n \le -\frac{1}{2}(\tau+1)$ Consider $A_0^{n'}$, where

$$
n' = \begin{cases}
-\frac{1}{2}(\tau+1) & \text{if } \tau \text{ is odd}, \\
-\frac{1}{2}\tau - 1 & \text{if } \tau \text{ is even}.
\end{cases}
$$

For each $1 \le p \le 2\tau+1$,

$$
\min(i(z_p), j(z_p)-0) = \begin{cases}
\min\left(\frac{1}{2}(p-1), \tau-\frac{1}{2}(p-1)\right) & \text{if } p \text{ is odd}, \\
\min\left(\frac{1}{2}p, \tau-\frac{1}{2}(p-2)\right) & \text{if } p \text{ is even}
\end{cases}
$$

$$
= \begin{cases}
\frac{1}{2}(p-1) & \text{if } p \text{ is odd and } p \le \tau+1, \\
\tau-\frac{1}{2}(p-1) & \text{if } p \text{ is odd and } p > \tau+1, \\
\frac{1}{2}p & \text{if } p \text{ is even and } p \le \tau+1, \\
\tau-\frac{1}{2}(p-2) & \text{if } p \text{ is even and } p > \tau+1.
\end{cases}
$$

These computations show that $v_0^{n'}$ is injective on homology:

- If τ is odd, $A_0^{n'}$ contains z_p for $1 \le p \le -2n-1$ but $z_{-2n} \notin A_0^{n'}$. Similarly, for $1 \le i \le -n-1$, $A_0^{n'}$ contains $U^{-i}z_p$ for $1 \le p \le -2(n+i)-1$ but

$z_{-2(n+i)} \notin A_0^{n'}$. Therefore, $[U^{-i}z_1] \neq 0$ in $H_*(A_0^{n'})$ for $0 \leq i \leq -n-1$. So $v_0^{n'}$ is injective on homology.

- If τ is even, $A_0^{n'}$ contains all z_p for $1 \leq p \leq 2\tau + 1$. Furthermore, for $1 \leq i \leq -n-1$, $A_0^{n'}$ contains $U^{-i}z_p$ for $1 \leq p \leq -2(n+i)-1$ but $z_{-2(n+i)} \notin A_0^{n'}$. Therefore, $[U^{-i}z_1] \neq 0$ in $H_*(A_0^{n'})$ for $0 \leq i \leq -n-1$. So $v_0^{n'}$ is injective on homology.

Since $v^{+'}(T_{2,2\tau+1}) = 0$ is a lower bound on $v_n(T_{2,2\tau+1})$, we conclude that

$$v_n(T_{2,2\tau+1}) = 0$$

for all $n \leq -\frac{1}{2}(\tau + 1)$. □

We have the following result for strongly quasipositive knots (see [4] for background on strongly quasipositive knots):

Proposition 4.2 If K is strongly quasipositive, then $v_n(K) = \tau(K) = g_4(K) = g(K)$ for all positive n.

Proof Theorem 1.2 of [4] states that K is strongly quasipositive if and only if $\tau(K) = g_4(K) = g(K)$. The result immediately follows since $\tau(K) \leq v_n(K) \leq v^+(K) \leq g_4(K)$ for positive n. See also [7, Proposition 3]. □

Example 4.3 Figure 1 (top-left) shows the knot Floer chain complex CFK^∞ of the $(2, 9)$–torus knot. The computation of $v_{-2}(T_{2,9})$ is shown in Figure 1 (top-right and bottom-left). We have

$$v_n(T_{2,9}) = \begin{cases} 4 & \text{for all } n \geq 0, \\ 3 & \text{for } n = -1, \\ 1 & \text{for } n = -2, \\ 0 & \text{for all } n \leq -3. \end{cases}$$

The computation of $v_n(K)$ for thin knots shows that the sequence v_n can increase by more than one at a time, in contrast to the local h–invariants defined by Rasmussen, which jump by at most one [16, Proposition 7.6].

In fact, the gap between $v_n(K)$ and $v_{n+1}(K)$ can be arbitrarily big. For example, a straightforward (partial) computation of $v_n(T_{p,p+1})$ using $\mathrm{CFK}^\infty(T_{p,p+1})$ shows that for $p > 3$,
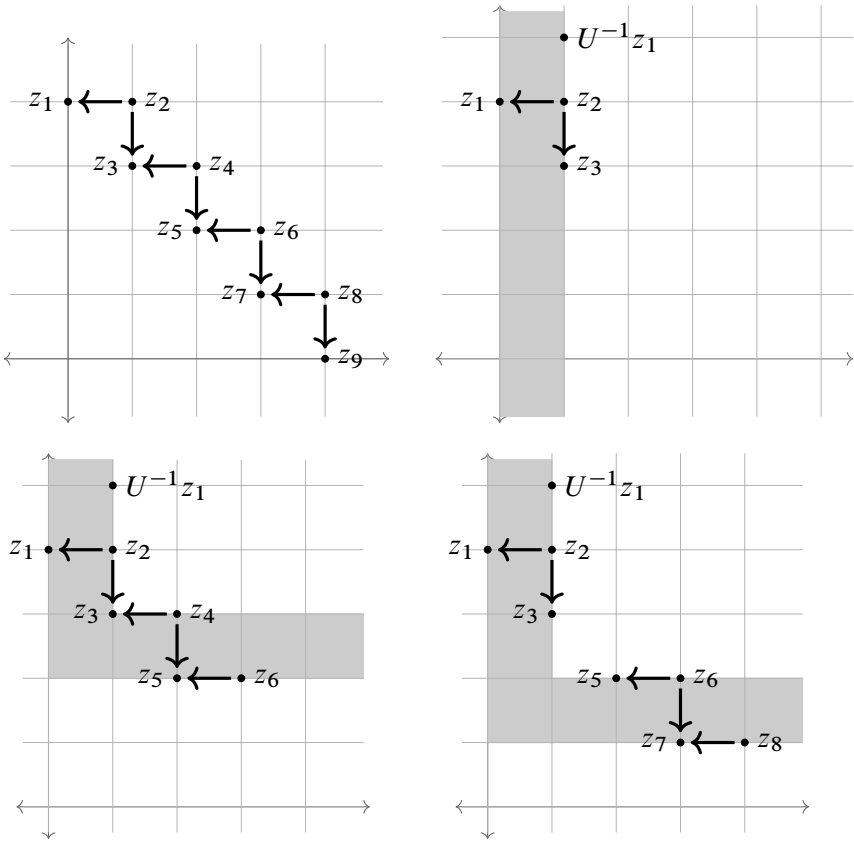
$$v_{-1}(T_{p,p+1}) - v_{-2}(T_{p,p+1}) = p.$$

Figure 1: Top-left: Generating complex for CFK^∞ of the $(2, 9)$-torus knot $T_{2,9}$. Top-right: The classes $[U^{-1}z_1]$ and $[z_1] = [z_3]$ generate $\mathrm{HF}^2(S^3)$. Bottom-left: The classes $[z_1]$ and $[z_3]$ vanish in $\mathrm{HF}^2(S^3_{-N}K, [2])$. Bottom-right: The classes $[z_1] = [z_3]$ and $[U^{-1}z_1]$ survive in $\mathrm{HF}^2(S^3_{-N}K, [1])$.

Theorem 4.4 Let $T_{p,p+1}$ denote the $(p, p+1)$–torus knot for $p > 3$. Let $\tau = \tau(T_{p,p+1}) = \frac{1}{2}(p-1)p$. Then

$$\nu_n(T_{p,p+1}) = \begin{cases} \tau & \text{for } n \geq 0, \\ \tau - 1 & \text{for } n = -1, \\ \tau - 1 - p & \text{for } n = -2. \end{cases}$$

Thus, $\nu_{-1}(T_{p,p+1}) - \nu_{-2}(T_{p,p+1}) = p$.

Proof In [1], Allen shows that the staircase model chain complex for $\mathrm{CFK}^\infty(T_{p,p+1})$ takes the form

$$[1, \ p-1, \ 2, \ p-2, \ \ldots, \ j, \ p-j, \ \ldots, \ p-1, \ 1],$$

where the indices alternate between the widths of the horizontal and vertical steps. From this staircase description, there exists a (i, j)–filtered basis for $\text{CFK}^\infty(T_{p,p+1})$ consisting of generators $\{b_l\}_{l=0}^{2(p-1)}$ lying in (i, j)–filtrations

$$b_{2m}: \quad \left(\sum_{k=1}^m k, \tfrac{1}{2}(p-1)p - \sum_{k=1}^m (p-k) \right),$$

$$b_{2m+1}: \quad \left(\sum_{k=1}^{m+1} k, \tfrac{1}{2}(p-1)p - \sum_{k=1}^m (p-k) \right),$$

and differential

$$\partial b_{2m} = 0, \quad \partial b_{2m+1} = b_{2m} + b_{2m+2}.$$

The same argument for showing that $\nu^+(T_{2,2\tau+1}) = \tau(T_{2,2\tau+1})$ in [Proposition 4.1](#) holds for the knots $T_{p,p+1}$. Moreover, in the terminology of [5], the basis $\{b_l\}_{l=0}^{2(p-1)}$ satisfies:

- b_0 is the vertically distinguished element of a vertically simplified basis.
- b_0 has a unique incoming horizontal arrow (from b_1) (and no outgoing horizontal arrows).

We immediately conclude that $\varepsilon(T_{p,p+1}) = 1$ and $\nu_{-1}(T_{p,p+1}) = \tau - 1$.

To show $\nu_{-2}(T_{p,p+1}) = \tau - p - 1$, we observe:

- $A_{\tau-p-1}^{-2}$ contains the generators b_0, b_1 and b_2, but $b_3 \notin A_{\tau-p-1}^{-2}$. Therefore, $[b_0] \neq 0$ in $H_*(A_{\tau-p-1}^{-2})$. Moreover, $[U^{-1}b_0] \neq 0$ in $H_*(A_{\tau-p-1}^{-2})$. Thus, $\nu_{\tau-p-1}^{-2}$ is injective on homology.
- $A_{\tau-p}^{-2}$ contains the generators b_0, b_1, b_2 and b_3, but $b_4 \notin A_{\tau-p}^{-2}$. Therefore, $[b_0] = 0$ in $H_*(A_{\tau-p}^{-2})$. $\qquad \square$

We show the concordance invariants $\{\nu_n(K)\}$ contain more concordance information than the collection $\{\tau, \nu, \nu', \nu^+, \nu^{+'}\}$:

Proposition 4.5 *There exist knots K and K' with equal τ, ν, ν', ν^+ and $\nu^{+'}$ invariants, but $\nu_n(K) \neq \nu_n(K')$ for some $n \in \mathbb{Z}$.*

Proof The torus knot $T_{4,5}$ and the torus knot $T_{2,13}$ share the following invariants in common:

$$\nu^{+'}(T_{4,5}) = 0 = \nu^{+'}(T_{2,13}),$$

$$\nu'(T_{4,5}) = 5 = \nu'(T_{2,13}),$$

$$\tau(T_{4,5}) = \nu(T_{4,5}) = \nu^+(T_{4,5}) = 6 = \tau(T_{2,13}) = \nu(T_{2,13}) = \nu^+(T_{2,13}).$$

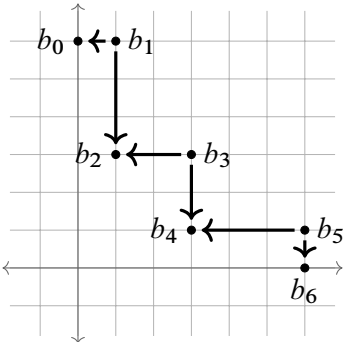Figure 2: Generating complex for CFK$^\infty$ of the left-handed $(4,5)$–torus knot $T_{4,5}$. CFK$^\infty(T_{4,5})$ is generated over $\mathbb{F}[U, U^{-1}]$ by the above chain complex. The arrows, representing terms in the differential, are drawn to scale, with lengths of arrows ranging between one and three.

However, the invariants $v_n(T_{4,5})$ are different from $v_n(T_{2,13})$:

$$
v_n(T_{4,5}) = \begin{cases} 6 & \text{for } n \geq 0, \\ 5 & \text{for } n = -1, \\ 1 & \text{for } n = -2, \\ 0 & \text{for } n \leq -3, \end{cases}
\qquad
v_n(T_{2,13}) = \begin{cases} 6 & \text{for } n \geq 0, \\ 5 & \text{for } n = -1, \\ 3 & \text{for } n = -2, \\ 1 & \text{for } n = -3, \\ 0 & \text{for } n \leq -4, \end{cases}
$$

where $v_n(T_{4,5})$ is computed from the knot Floer chain complex CFK$^\infty(T_{4,5})$ as shown in Figure 2.   □

5 Further directions

One question is the effectiveness of $v_n(K)$ when compared to other concordance invariants such as $\Upsilon_K(t)$, coming from t–modified knot Floer homology [10], or V_k, coming from surgery formulas of Ozsváth and Szabó [13].

The invariants $v_n(K)$ do not define concordance homomorphisms $\mathcal{C} \to \mathbb{Z}$, where $\mathcal{C}$ is the concordance group of knots. This implies that they do not necessarily vanish on knots of finite concordance order. The existence of p–torsion, with $p \neq 2$, in the concordance group $\mathcal{C}$ is an open question. A related conjecture, based on a question of Gordon [3], as phrased in [8] is:

Conjecture 5.1 (Gordon) A knot has order two in $\mathcal{C}$ if and only if $K = -K$ is negative amphichiral.

Recently, Hendricks and Manolescu defined involutive Heegaard Floer concordance invariants $\overline{V}_0$ and $\underline{V}_0$, which detects the nonsliceness of the figure eight knot. The nonsliceness of 4_1 was previously known through classical methods, but this is the first method of detection coming from the Heegaard Floer package. By additivity of τ and the behavior of ε under connected sums, $\tau(K)$ and $\nu(K)$ vanish for all knots K of finite concordance order. This leaves open the cases $\nu_n(K)$ for $n > 1$ and $n < -1$. We pose the question:

Question 5.2 Does there exist a knot K of finite concordance order such that $\nu_n(K) \neq 0$ for some n?

Another question is how the invariants $\nu_n(K)$ behave under connected sum. It is known that $\nu^+(K)$ is subadditive by [2]. That is,

$$\nu^+(K \# L) \leq \nu^+(K) + \nu^+(L).$$

Using mirroring relations and subadditivity of $\nu^+(K)$ shows that $\nu^{+'}(K)$ is superadditive:

Lemma 5.3 *For any two knots K and L,*

$$\nu^{+'}(K \# L) \geq \nu^{+'}(K) + \nu^{+'}(L).$$

Proof By subadditivity of ν^+ and the mirroring relations,

$$\nu^+(-K \# -L) \leq \nu^+(-K) + \nu^+(-L),$$
$$-\nu^{+'}(K \# L) \leq -\nu^{+'}(K) + -\nu^{+'}(L),$$
$$\nu^{+'}(K \# L) \geq \nu^{+'}(K) + \nu^{+'}(L). \qquad \square$$

As pointed out to the author by Jen Hom, it can also be seen by additivity of τ and the behavior of ε under connected sum that $\nu(K)$ is subadditive. A similar argument shows that $\nu'(K)$ is superadditive. This leads us to ask the following two questions:

Question 5.4 Is $\nu_n(K \# K') \leq \nu_n(K) + \nu_n(K')$ for all positive integers $n \in \mathbb{Z}_+$?

Question 5.5 Is $\nu_n(K \# K') \geq \nu_n(K) + \nu_n(K')$ for all negative integers $n \in \mathbb{Z}_-$?

The next question was posed by Zhongtao Wu:

Question 5.6 (Wu) If $\nu_n(K) = \nu_n(K')$ for all $n \in \mathbb{Z}$, then is $\nu^+(K \,\#\, {-}K') = \nu^+({-}K \,\#\, K') = 0$?

The condition that $\nu^+(K \,\#\, {-}K') = \nu^+({-}K \,\#\, K') = 0$ implies that

$$\mathrm{CFK}^\infty(K \,\#\, {-}K') \simeq \mathrm{CFK}^\infty(U) \oplus A,$$

where A is an acyclic complex [6].

References

[1] **S Allen**, *Using secondary Upsilon invariants to rule out stable equivalence of knot complexes*, preprint (2017) arXiv To appear in Algebr. Geom. Topol.

[2] **J Bodnár**, **D Celoria**, **M Golla**, *A note on cobordisms of algebraic knots*, Algebr. Geom. Topol. 17 (2017) 2543–2564 MR

[3] **C M Gordon**, *Some aspects of classical knot theory*, from "Knot theory" (J-C Hausmann, editor), Lecture Notes in Math. 685, Springer (1978) 1–60 MR

[4] **M Hedden**, *Notions of positivity and the Ozsváth–Szabó concordance invariant*, J. Knot Theory Ramifications 19 (2010) 617–629 MR

[5] **J Hom**, *Bordered Heegaard Floer homology and the tau-invariant of cable knots*, J. Topol. 7 (2014) 287–326 MR

[6] **J Hom**, *A survey on Heegaard Floer homology and concordance*, J. Knot Theory Ramifications 26 (2017) art. id. 1740015 MR

[7] **J Hom**, **Z Wu**, *Four-ball genus bounds and a refinement of the Ozsváth–Szabó tau invariant*, J. Symplectic Geom. 14 (2016) 305–323 MR

[8] **C Livingston**, *A survey of classical knot concordance*, from "Handbook of knot theory" (W Menasco, M Thistlethwaite, editors), Elsevier, Amsterdam (2005) 319–347 MR

[9] **C Manolescu**, **P Ozsváth**, *Heegaard Floer homology and integer surgeries on links*, preprint (2010) arXiv

[10] **P S Ozsváth**, **A I Stipsicz**, **Z Szabó**, *Concordance homomorphisms from knot Floer homology*, Adv. Math. 315 (2017) 366–426 MR

[11] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and knot invariants*, Adv. Math. 186 (2004) 58–116 MR

[12] **P Ozsváth**, **Z Szabó**, *Holomorphic triangles and invariants for smooth four-manifolds*, Adv. Math. 202 (2006) 326–400 MR

[13] **P S Ozsváth**, **Z Szabó**, *Knot Floer homology and integer surgeries*, Algebr. Geom. Topol. 8 (2008) 101–153 MR

[14]   **P S Ozsváth**, **Z Szabó**, *Knot Floer homology and rational surgeries*, Algebr. Geom. Topol. 11 (2011) 1–68 MR

[15]   **I Petkova**, *Cables of thin knots and bordered Heegaard Floer homology*, Quantum Topol. 4 (2013) 377–409 MR

[16]   **J A Rasmussen**, *Floer homology and knot complements*, PhD thesis, Harvard University (2003) MR Available at https://search.proquest.com/docview/305332635

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