THE ISOPERIMETRIC INEQUALITY FOR A MINIMAL SUBMANIFOLD IN EUCLIDEAN SPACE

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ABSTRACT. We prove a Sobolev inequality which holds on submanifolds in Euclidean space of arbitrary dimension and codimension. This inequality is sharp if the codimension is at most 2. As a special case, we obtain a sharp isoperimetric inequality for minimal submanifolds in Euclidean space of codimension at most 2.

1. INTRODUCTION

The isoperimetric inequality for a domain in $\mathbb{R}^n$ is one of the most beautiful results in geometry. It has long been conjectured that the isoperimetric inequality still holds if we replace the domain in $\mathbb{R}^n$ by a minimal hypersurface in $\mathbb{R}^{n+1}$. In this paper, we prove this conjecture, as well as a more general inequality which holds for submanifolds of arbitrary dimension and codimension.

Theorem 1. *Let Σ be a compact n-dimensional submanifold of $\mathbb{R}^{n+m}$ (possibly with boundary $\partial\Sigma$), where $m \geq 2$. Let f be a positive smooth function on Σ. Then*

$$\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2} + \int_{\partial\Sigma} f \geq n\,\Big(\frac{(n+m)\,|B^{n+m}|}{m\,|B^m|}\Big)^{\frac{1}{n}}\,\Big(\int_\Sigma f^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}}.$$

Here, H denotes the mean curvature vector of Σ, and B^n denotes the open unit ball in $\mathbb{R}^n$.

Let us consider the special case $m = 2$. The standard recursion formula for the volume of the unit ball in Euclidean space gives $(n+2)\,|B^{n+2}| = 2\pi\,|B^n| = 2\,|B^2|\,|B^n|$. Thus, Theorem 1 implies a sharp Sobolev inequality for submanifolds of codimension 2:

Corollary 2. *Let Σ be a compact n-dimensional submanifold of $\mathbb{R}^{n+2}$ (possibly with boundary $\partial\Sigma$), and let f be a positive smooth function on Σ. Then*

$$\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2} + \int_{\partial\Sigma} f \geq n\,|B^n|^{\frac{1}{n}}\,\Big(\int_\Sigma f^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}},$$

where H denotes the mean curvature vector of Σ.

Finally, we characterize the case of equality in Corollary 2:

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Theorem 3. *Let Σ be a compact n-dimensional submanifold of $\mathbb{R}^{n+2}$ (possibly with boundary $\partial\Sigma$), and let f be a positive smooth function on Σ. If*

$$\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2} + \int_{\partial\Sigma} f = n\,|B^n|^{\frac{1}{n}} \Big(\int_\Sigma f^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}},$$

then f is constant and Σ is a flat round ball.

In particular, if Σ is a compact n-dimensional minimal submanifold of $\mathbb{R}^{n+2}$, then Σ satisfies the sharp isoperimetric inequality

$$|\partial\Sigma| \geq n\,|B^n|^{\frac{1}{n}}\,|\Sigma|^{\frac{n-1}{n}},$$

and equality holds if and only if Σ is a flat round ball.

Every n-dimensional submanifold of $\mathbb{R}^{n+1}$ can be viewed as a submanifold of $\mathbb{R}^{n+2}$. Hence, Corollary 2 and Theorem 3 imply a sharp isoperimetric inequality in codimension 1.

The isoperimetric inequality on a minimal surface has a long history. In 1921, Torsten Carleman [4] proved that every two-dimensional minimal surface Σ which is diffeomorphic to a disk satisfies the sharp isoperimetric inequality $|\partial\Sigma|^2 \geq 4\pi\,|\Sigma|$. Various authors have weakened the topological assumption in Carleman's theorem. In particular, the sharp isoperimetric inequality has been verified for two-dimensional minimal surfaces with connected boundary (see [11], [15]); for two-dimensional minimal surfaces diffeomorphic to annuli (cf. [9], [14]); and for two-dimensional minimal surfaces with two boundary components (cf. [6], [12]). On the other hand, using different techniques, Leon Simon showed that every two-dimensional minimal surface satisfies the non-sharp isoperimetric inequality $|\partial\Sigma|^2 \geq 2\pi\,|\Sigma|$ (see [17], Section 4). Stone [16] subsequently improved the constant in this inequality: he proved that $|\partial\Sigma|^2 \geq 2\sqrt{2}\,\pi\,|\Sigma|$ for every two-dimensional minimal surface Σ. We refer to [7] for a survey of these developments.

In higher dimensions, the famous Michael-Simon Sobolev inequality (cf. [1], Section 7, and [13]) implies an isoperimetric inequality for minimal submanifolds, albeit with a non-sharp constant. Castillon [5] gave an alternative proof of the Michael-Simon Sobolev inequality using methods from optimal transport. Finally, Almgren [2] proved a sharp version of the filling inequality of Federer and Fleming [8]. In particular, this gives a sharp isoperimetric inequality for area-minimizing submanifolds in all dimensions.

Our method of proof is inspired in part by the Alexandrov-Bakelman-Pucci maximum principle (cf. [3], [18]). An alternative way to prove Theorem 1 would be to use optimal transport; in that case, we would consider the transport map from a thin annulus in $\mathbb{R}^{n+m}$ to the submanifold Σ equipped with the measure $f^{\frac{n}{n-1}}\,d\mathrm{vol}$.

2. Proof of Theorem 1

Let Σ be a compact n-dimensional submanifold of $\mathbb{R}^{n+m}$ (possibly with boundary $\partial\Sigma$), where $m \geq 2$. For each point $x \in \Sigma$, we denote by $T_x\Sigma$

and $T_x^\perp \Sigma$ the tangent and normal space to Σ at x, respectively. Moreover, we denote by $I\!I$ the second fundamental form of Σ. Recall that $I\!I$ is a symmetric bilinear form on $T_x\Sigma$ which takes values in $T_x^\perp \Sigma$. If X and Y are tangent vector fields on Σ and V is a normal vector field along Σ, then $\langle I\!I(X,Y), V\rangle = \langle \bar{D}_X Y, V\rangle = -\langle \bar{D}_X V, Y\rangle$, where $\bar{D}$ denotes the standard connection on $\mathbb{R}^{n+m}$. The trace of the second fundamental form gives the mean curvature vector, which we denote by H. Finally, we denote by η the co-normal to $\partial\Sigma$.

We now turn to the proof of Theorem 1. We first consider the special case that Σ is connected. By scaling, we may assume that

$$\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2} + \int_{\partial\Sigma} f = n \int_\Sigma f^{\frac{n}{n-1}}.$$

Since Σ is connected, we can find a function $u : \Sigma \to \mathbb{R}$ with the property that

$$\operatorname{div}_\Sigma(f\,\nabla^\Sigma u) = n\, f^{\frac{n}{n-1}} - \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2}$$

on Σ and $\langle \nabla^\Sigma u, \eta\rangle = 1$ at each point on $\partial\Sigma$. Since the function $\sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2}$ is Lipschitz continuous, it follows from standard elliptic regularity theory that the function u is of class $C^{2,\gamma}$ for each $0 < \gamma < 1$ (see [10], Theorem 6.30).

We define

$$\begin{aligned} \Omega &:= \{x \in \Sigma \setminus \partial\Sigma : |\nabla^\Sigma u(x)| < 1\}, \\ U &:= \{(x,y) : x \in \Sigma \setminus \partial\Sigma,\ y \in T_x^\perp\Sigma,\ |\nabla^\Sigma u(x)|^2 + |y|^2 < 1\}, \\ A &:= \{(x,y) \in U : D_\Sigma^2 u(x) - \langle I\!I(x), y\rangle \geq 0\}. \end{aligned}$$

Moreover, we define a map $\Phi : U \to \mathbb{R}^{n+m}$ by

$$\Phi(x,y) = \nabla^\Sigma u(x) + y$$

for all $(x,y) \in U$. Note that Φ is of class $C^{1,\gamma}$ for each $0 < \gamma < 1$. Since $\nabla^\Sigma u(x) \in T_x\Sigma$ and $y \in T_x^\perp\Sigma$ are orthogonal, we obtain $|\Phi(x,y)|^2 = |\nabla^\Sigma u(x)|^2 + |y|^2 < 1$ for all $(x,y) \in U$.

Lemma 4. *The image $\Phi(A)$ is the open unit ball B^{n+m}.*

Proof. Clearly, $\Phi(A) \subset \Phi(U) \subset B^{n+m}$. To prove the reverse inclusion, we consider an arbitrary vector $\xi \in \mathbb{R}^{n+m}$ such that $|\xi| < 1$. We define a function $w : \Sigma \to \mathbb{R}$ by $w(x) := u(x) - \langle x, \xi\rangle$. Using the Cauchy-Schwarz inequality, we obtain

$$\langle \nabla^\Sigma w(x), \eta(x)\rangle = \langle \nabla^\Sigma u(x), \eta(x)\rangle - \langle \eta(x), \xi\rangle = 1 - \langle \eta(x), \xi\rangle > 0$$

for each point $x \in \partial\Sigma$. Consequently, the function w must attain its minimum in the interior of Σ. Let $\bar{x} \in \Sigma \setminus \partial\Sigma$ be a point in the interior of Σ such that $w(\bar{x}) = \inf_{x\in\Sigma} w(x)$. Clearly, $\nabla^\Sigma w(\bar{x}) = 0$. This implies $\xi = \nabla^\Sigma u(\bar{x}) + \bar{y}$ for some $\bar{y} \in T_{\bar{x}}^\perp\Sigma$. Consequently, $|\nabla^\Sigma u(\bar{x})|^2 + |\bar{y}|^2 = |\xi|^2 < 1$. Moreover, we have $D_\Sigma^2 w(\bar{x}) \geq 0$. From this, we deduce that $D_\Sigma^2 u(\bar{x}) - \langle I\!I(\bar{x}), \xi\rangle \geq$

0. Since $\langle I\!I(\bar{x}), \xi\rangle = \langle I\!I(\bar{x}), \nabla^\Sigma u(\bar{x}) + \bar{y}\rangle = \langle I\!I(\bar{x}), \bar{y}\rangle$, we conclude that $D^2_\Sigma u(\bar{x}) - \langle I\!I(\bar{x}), \bar{y}\rangle \geq 0$. Therefore, $(\bar{x}, \bar{y}) \in A$ and $\Phi(\bar{x}, \bar{y}) = \xi$. Thus, $B^{n+m} \subset \Phi(A)$.

Lemma 5. *The Jacobian determinant of Φ is given by*

$$\det D\Phi(x, y) = \det(D^2_\Sigma u(x) - \langle I\!I(x), y\rangle)$$

for all $(x, y) \in U$.

Proof. Fix a point $(\bar{x}, \bar{y}) \in U$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space $T_{\bar{x}}\Sigma$, and let $(x_1, \ldots, x_n)$ be a local coordinate system on Σ such that $\frac{\partial}{\partial x_i} = e_i$ at the point $\bar{x}$. Moreover, let $\{\nu_1, \ldots, \nu_m\}$ denote a local orthonormal frame for the normal bundle $T^\perp\Sigma$. Every normal vector y can be written in the form $y = \sum_{\alpha=1}^m y_\alpha \nu_\alpha$. With this understood, $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ is a local coordinate system on the total space of the normal bundle $T^\perp\Sigma$. We compute

$$\begin{aligned}\Big\langle \frac{\partial\Phi}{\partial x_i}(\bar{x}, \bar{y}), e_j\Big\rangle &= \langle \bar{D}_{e_i}(\nabla^\Sigma u), e_j\rangle + \sum_{\alpha=1}^m \bar{y}_\alpha \, \langle \bar{D}_{e_i}\nu_\alpha, e_j\rangle \\ &= (D^2_\Sigma u)(e_i, e_j) - \langle I\!I(e_i, e_j), \bar{y}\rangle.\end{aligned}$$

In the last step, we have used the identity $\langle I\!I(e_i, e_j), \nu_\alpha\rangle = -\langle \bar{D}_{e_i}\nu_\alpha, e_j\rangle$. Moreover,

$$\Big\langle \frac{\partial\Phi}{\partial y_\alpha}(\bar{x}, \bar{y}), e_j\Big\rangle = \langle \nu_\alpha, e_j\rangle = 0$$

and

$$\Big\langle \frac{\partial\Phi}{\partial y_\alpha}(\bar{x}, \bar{y}), \nu_\beta\Big\rangle = \langle \nu_\alpha, \nu_\beta\rangle = \delta_{\alpha\beta}.$$

Thus, we conclude that

$$\det D\Phi(\bar{x}, \bar{y}) = \det \begin{bmatrix} D^2_\Sigma u(\bar{x}) - \langle I\!I(\bar{x}), \bar{y}\rangle & 0 \\ * & \mathrm{id}\end{bmatrix} = \det(D^2_\Sigma u(\bar{x}) - \langle I\!I(\bar{x}), \bar{y}\rangle).$$

This proves the assertion.

Lemma 6. *The Jacobian determinant of Φ satisfies*

$$0 \leq \det D\Phi(x, y) \leq f(x)^{\frac{n}{n-1}}$$

for all $(x, y) \in A$.

Proof. Consider a point $(x, y) \in A$. Using the inequality $|\nabla^\Sigma u(x)|^2 + |y|^2 < 1$ and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&- \langle \nabla^\Sigma f(x), \nabla^\Sigma u(x)\rangle - f(x)\, \langle H(x), y\rangle \\ &\leq \sqrt{|\nabla^\Sigma f(x)|^2 + f(x)^2 \, |H(x)|^2} \, \sqrt{|\nabla^\Sigma u(x)|^2 + |y|^2} \\ &\leq \sqrt{|\nabla^\Sigma f(x)|^2 + f(x)^2 \, |H(x)|^2}.\end{aligned}$$

Using the identity $\text{div}_\Sigma(f\,\nabla^\Sigma u) = n\,f^{\frac{n}{n-1}} - \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2}$, we deduce that

$$\begin{aligned}
&\Delta_\Sigma u(x) - \langle H(x), y\rangle \\
&= n\,f(x)^{\frac{1}{n-1}} - f(x)^{-1}\,\sqrt{|\nabla^\Sigma f(x)|^2 + f(x)^2\,|H(x)|^2} \\
&- f(x)^{-1}\,\langle \nabla^\Sigma f(x), \nabla^\Sigma u(x)\rangle - \langle H(x), y\rangle \\
&\leq n\,f(x)^{\frac{1}{n-1}}.
\end{aligned}$$

Moreover, $D^2_\Sigma u(x) - \langle I\!I(x), y\rangle \geq 0$ since $(x, y) \in A$. Hence, the arithmetic-geometric mean inequality implies

$$0 \leq \det(D^2_\Sigma u(x) - \langle I\!I(x), y\rangle) \leq \Big(\frac{\text{tr}(D^2_\Sigma u(x) - \langle I\!I(x), y\rangle)}{n}\Big)^n \leq f(x)^{\frac{n}{n-1}}.$$

Using Lemma 5, we conclude that $0 \leq \det D\Phi(x, y) \leq f(x)^{\frac{n}{n-1}}$. This completes the proof of Lemma 6.

We now continue with the proof of Theorem 1. Using Lemma 4 and Lemma 6, we obtain

$$\begin{aligned}
&|B^{n+m}|\,(1 - \sigma^{n+m}) \\
&= \int_{\{\xi \in \mathbb{R}^{n+m} : \sigma^2 < |\xi|^2 < 1\}} 1\,d\xi \\
&\leq \int_\Omega \Big(\int_{\{y \in T_x^\perp \Sigma : \sigma^2 < |\Phi(x,y)|^2 < 1\}} |\det D\Phi(x, y)|\,1_A(x, y)\,dy\Big)\,d\text{vol}(x) \\
&\leq \int_\Omega \Big(\int_{\{y \in T_x^\perp \Sigma : \sigma^2 < |\nabla^\Sigma u(x)|^2 + |y|^2 < 1\}} f(x)^{\frac{n}{n-1}}\,dy\Big)\,d\text{vol}(x) \\
&= |B^m| \int_\Omega \Big[(1 - |\nabla^\Sigma u(x)|^2)^{\frac{m}{2}} - (\sigma^2 - |\nabla^\Sigma u(x)|^2)_+^{\frac{m}{2}}\Big]\,f(x)^{\frac{n}{n-1}}\,d\text{vol}(x)
\end{aligned}$$

for all $0 \leq \sigma < 1$. Since $m \geq 2$, the mean value theorem gives $b^{\frac{m}{2}} - a^{\frac{m}{2}} \leq \frac{m}{2}\,(b - a)$ for $0 \leq a \leq b \leq 1$. Consequently,

$$\begin{aligned}
&(1 - |\nabla^\Sigma u(x)|^2)^{\frac{m}{2}} - (\sigma^2 - |\nabla^\Sigma u(x)|^2)_+^{\frac{m}{2}} \\
&\leq \frac{m}{2}\,\Big[(1 - |\nabla^\Sigma u(x)|^2) - (\sigma^2 - |\nabla^\Sigma u(x)|^2)_+\Big] \leq \frac{m}{2}\,(1 - \sigma^2)
\end{aligned}$$

for all $x \in \Omega$ and all $0 \leq \sigma < 1$. Putting these facts, together, we obtain

$$|B^{n+m}|\,(1 - \sigma^{n+m}) \leq \frac{m}{2}\,|B^m|\,(1 - \sigma^2) \int_\Omega f^{\frac{n}{n-1}}$$

for all $0 \leq \sigma < 1$. In the next step, we divide by $1 - \sigma$ and take the limit as $\sigma \to 1$. This gives

$$(n + m)\,|B^{n+m}| \leq m\,|B^m| \int_\Omega f^{\frac{n}{n-1}} \leq m\,|B^m| \int_\Sigma f^{\frac{n}{n-1}}.$$

On the other hand, $\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2} + \int_{\partial\Sigma} f = n \int_\Sigma f^{\frac{n}{n-1}}$ in view of our normalization. Thus, we conclude that

$$\begin{aligned}&\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2} + \int_{\partial\Sigma} f \\ &= n \int_\Sigma f^{\frac{n}{n-1}} \geq n \Big(\frac{(n+m)\,|B^{n+m}|}{m\,|B^m|}\Big)^{\frac{1}{n}} \Big(\int_\Sigma f^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}}.\end{aligned}$$

This proves Theorem 1 in the special case when Σ is connected.

It remains to consider the case when Σ is disconnected. In that case, we apply the inequality to each individual connected component of Σ, and take the sum over all connected components. Since

$$a^{\frac{n-1}{n}} + b^{\frac{n-1}{n}} > a\,(a+b)^{-\frac{1}{n}} + b\,(a+b)^{-\frac{1}{n}} = (a+b)^{\frac{n-1}{n}}$$

for $a, b > 0$, we conclude that

$$\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2} + \int_{\partial\Sigma} f > n \Big(\frac{(n+m)\,|B^{n+m}|}{m\,|B^m|}\Big)^{\frac{1}{n}} \Big(\int_\Sigma f^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}}.$$

if Σ is disconnected. This completes the proof of Theorem 1.

3. Proof of Theorem 3

Suppose that Σ is a compact n-dimensional submanifold in $\mathbb{R}^{n+2}$ (possibly with boundary $\partial\Sigma$), and f is a positive smooth function on Σ satisfying

$$\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2} + \int_{\partial\Sigma} f = n\,|B^n|^{\frac{1}{n}} \Big(\int_\Sigma f^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}}.$$

Clearly, Σ must be connected.

By scaling, we may arrange that $\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2} + \int_{\partial\Sigma} f = n\,|B^n|$ and $\int_\Sigma f^{\frac{n}{n-1}} = |B^n|$. In particular,

$$\int_\Sigma \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2} + \int_{\partial\Sigma} f = n \int_\Sigma f^{\frac{n}{n-1}}.$$

Since Σ is connected, we can find a function $u : \Sigma \to \mathbb{R}$ such that

$$\operatorname{div}_\Sigma(f\,\nabla^\Sigma u) = n\,f^{\frac{n}{n-1}} - \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2}$$

on Σ and $\langle \nabla^\Sigma u, \eta\rangle = 1$ on $\partial\Sigma$. Moreover, u is of class $C^{2,\gamma}$ for each $0 < \gamma < 1$.

Let Ω, U, A, and $\Phi : U \to \mathbb{R}^{n+2}$ be defined as in Section 2.

Lemma 7. *Suppose that $\bar{x} \in \Omega$, $\bar{y} \in T_{\bar{x}}^\perp \Sigma$, $|\nabla^\Sigma u(\bar{x})|^2 + |\bar{y}|^2 = 1$, and $D_\Sigma^2 u(\bar{x}) - \langle I\!I(\bar{x}), \bar{y}\rangle \neq f(\bar{x})^{\frac{1}{n-1}}\,g$. Then there exists a real number $\varepsilon \in (0,1)$ and an open neighborhood V of the point $(\bar{x}, \bar{y})$ such that $\det D\Phi(x,y) \leq (1-\varepsilon)\,f(x)^{\frac{n}{n-1}}$ for all $(x,y) \in A \cap V$.*

Proof. We distinguish two cases:

Case 1: Suppose that $D_\Sigma^2 u(\bar{x}) - \langle I\!I(\bar{x}), \bar{y}\rangle \geq 0$. Since $|\nabla^\Sigma u(\bar{x})|^2 + |\bar{y}|^2 = 1$, the Cauchy-Schwarz inequality implies

$$-\langle \nabla^\Sigma f(\bar{x}), \nabla^\Sigma u(\bar{x})\rangle - f(\bar{x})\,\langle H(\bar{x}), \bar{y}\rangle \leq \sqrt{|\nabla^\Sigma f(\bar{x})|^2 + f(\bar{x})^2\,|H(\bar{x})|^2}.$$

Using the identity $\text{div}_\Sigma(f\,\nabla^\Sigma u) = n\,f^{\frac{n}{n-1}} - \sqrt{|\nabla^\Sigma f|^2 + f^2\,|H|^2}$, we obtain

$$\Delta_\Sigma u(\bar{x}) - \langle H(\bar{x}), \bar{y}\rangle \leq n\,f(\bar{x})^{\frac{1}{n-1}}.$$

Since $D_\Sigma^2 u(\bar{x}) - \langle I\!I(\bar{x}), \bar{y}\rangle \geq 0$ and $D_\Sigma^2 u(\bar{x}) - \langle I\!I(\bar{x}), \bar{y}\rangle \neq f(\bar{x})^{\frac{1}{n-1}}\,g$, the arithmetic-geometric mean inequality gives

$$\det(D_\Sigma^2 u(\bar{x}) - \langle I\!I(\bar{x}), \bar{y}\rangle) < f(\bar{x})^{\frac{n}{n-1}}.$$

Let us choose a real number $\varepsilon \in (0,1)$ such that $\det(D_\Sigma^2 u(\bar{x}) - \langle I\!I(\bar{x}), \bar{y}\rangle) < (1-\varepsilon)\,f(\bar{x})^{\frac{n}{n-1}}$. Since u is of class $C^{2,\gamma}$, we can find an open neighborhood V of $(\bar{x}, \bar{y})$ such that $\det(D_\Sigma^2 u(x) - \langle I\!I(x), y\rangle) \leq (1-\varepsilon)\,f(x)^{\frac{n}{n-1}}$ for all $(x,y) \in V$. Using Lemma 5, we obtain $\det D\Phi(x,y) \leq (1-\varepsilon)\,f(x)^{\frac{n}{n-1}}$ for all $(x,y) \in U \cap V$.

Case 2: Suppose that the smallest eigenvalue of $D_\Sigma^2 u(\bar{x}) - \langle I\!I(\bar{x}), \bar{y}\rangle$ is strictly negative. Since u is of class $C^{2,\gamma}$, we can find an open neighborhood V of $(\bar{x}, \bar{y})$ with the property that the smallest eigenvalue of $D_\Sigma^2 u(x) - \langle I\!I(x), y\rangle$ is strictly negative for all $(x,y) \in V$. Consequently, $A \cap V = \emptyset$. This completes the proof of Lemma 7.

Lemma 8. *We have* $D_\Sigma^2 u(x) - \langle I\!I(x), y\rangle = f(x)^{\frac{1}{n-1}}\,g$ *for all* $x \in \Omega$ *and all* $y \in T_x^\perp \Sigma$ *satisfying* $|\nabla^\Sigma u(x)|^2 + |y|^2 = 1$.

Proof. We argue by contradiction. Suppose that there exists a point $\bar{x} \in \Omega$ and a vector $\bar{y} \in T_{\bar{x}}^\perp \Sigma$ such that $|\nabla^\Sigma u(\bar{x})|^2 + |\bar{y}|^2 = 1$ and $D_\Sigma^2 u(\bar{x}) - \langle I\!I(\bar{x}), \bar{y}\rangle \neq f(\bar{x})^{\frac{1}{n-1}}\,g$. By Lemma 7, we can find a real number $\varepsilon \in (0,1)$ and an open neighborhood V of the point $(\bar{x}, \bar{y})$ such that $\det D\Phi(x,y) \leq (1-\varepsilon)\,f(x)^{\frac{n}{n-1}}$ for all $(x,y) \in A \cap V$. Using Lemma 6, we deduce that

$$0 \leq \det D\Phi(x,y) \leq (1 - \varepsilon \cdot 1_V(x,y))\,f(x)^{\frac{n}{n-1}}$$

for all $(x,y) \in A$. Arguing as in Section 2, we obtain

$$\begin{aligned}
&|B^{n+2}|\,(1-\sigma^{n+2}) \\
&= \int_{\{\xi \in \mathbb{R}^{n+2}:\sigma^2<|\xi|^2<1\}} 1\,d\xi \\
&\leq \int_\Omega \bigg(\int_{\{y \in T_x^\perp \Sigma:\sigma^2<|\Phi(x,y)|^2<1\}} |\det D\Phi(x,y)|\,1_A(x,y)\,dy\bigg)\,d\text{vol}(x) \\
&\leq \int_\Omega \bigg(\int_{\{y \in T_x^\perp \Sigma:\sigma^2<|\nabla^\Sigma u(x)|^2+|y|^2<1\}} (1-\varepsilon \cdot 1_V(x,y))\,f(x)^{\frac{n}{n-1}}\,dy\bigg)\,d\text{vol}(x) \\
&= |B^2| \int_\Omega \Big[(1-|\nabla^\Sigma u(x)|^2) - (\sigma^2-|\nabla^\Sigma u(x)|^2)_+\Big]\,f(x)^{\frac{n}{n-1}}\,d\text{vol}(x) \\
&- \varepsilon \int_\Omega \bigg(\int_{\{y \in T_x^\perp \Sigma:\sigma^2<|\nabla^\Sigma u(x)|^2+|y|^2<1\}} 1_V(x,y)\,f(x)^{\frac{n}{n-1}}\,dy\bigg)\,d\text{vol}(x) \\
&\leq |B^2|\,(1-\sigma^2) \int_\Omega f(x)^{\frac{n}{n-1}}\,d\text{vol}(x) \\
&- \varepsilon \int_\Omega \bigg(\int_{\{y \in T_x^\perp \Sigma:\sigma^2<|\nabla^\Sigma u(x)|^2+|y|^2<1\}} 1_V(x,y)\,f(x)^{\frac{n}{n-1}}\,dy\bigg)\,d\text{vol}(x)
\end{aligned}$$

for all $0 \leq \sigma < 1$. Dividing by $1-\sigma$ and taking the limit as $\sigma \to 1$ gives

$$(n+2)\,|B^{n+2}| < 2\,|B^2| \int_\Omega f^{\frac{n}{n-1}} \leq 2\,|B^2| \int_\Sigma f^{\frac{n}{n-1}} = 2\,|B^2|\,|B^n|.$$

This contradicts the fact that $(n+2)\,|B^{n+2}| = 2\,|B^2|\,|B^n|$.

Lemma 9. *We have $D^2_\Sigma u(x) = f(x)^{\frac{1}{n-1}}\,g$ and $I\!I(x) = 0$ for all $x \in \Omega$.*

Proof. Lemma 8 implies $D^2_\Sigma u(x) - \langle I\!I(x), y\rangle = f(x)^{\frac{1}{n-1}}\,g$ for all $x \in \Omega$ and all $y \in T_x^\perp \Sigma$ satisfying $|\nabla^\Sigma u(x)|^2 + |y|^2 = 1$. Replacing y by $-y$ gives $D^2_\Sigma u(x) + \langle I\!I(x), y\rangle = f(x)^{\frac{1}{n-1}}\,g$ for all $x \in \Omega$ and all $y \in T_x^\perp \Sigma$ satisfying $|\nabla^\Sigma u(x)|^2 + |y|^2 = 1$. Consequently, $D^2_\Sigma u(x) = f(x)^{\frac{1}{n-1}}\,g$ and $\langle I\!I(x), y\rangle = 0$ for all $x \in \Omega$ and all $y \in T_x^\perp \Sigma$ satisfying $|\nabla^\Sigma u(x)|^2 + |y|^2 = 1$. From this, the assertion follows.

Lemma 10. *We have $\nabla^\Sigma f(x) = 0$ for all $x \in \Omega$.*

Proof. Using Lemma 9, we obtain $\Delta_\Sigma u = n\,f^{\frac{1}{n-1}}$ at each point in Ω. This implies $\text{div}_\Sigma(f\,\nabla^\Sigma u) = n\,f^{\frac{n}{n-1}} + \langle \nabla^\Sigma f, \nabla^\Sigma u\rangle$ at each point in Ω. On the other hand, by definition of u, we have $\text{div}_\Sigma(f\,\nabla^\Sigma u) = n\,f^{\frac{n}{n-1}} - |\nabla^\Sigma f|$ at each point in Ω. Consequently, $\langle \nabla^\Sigma f, \nabla^\Sigma u\rangle = -|\nabla^\Sigma f|$ at each point in Ω. Since $|\nabla^\Sigma u| < 1$ at each point in Ω, we conclude that $\nabla^\Sigma f = 0$ at each point in Ω.

Lemma 11. *The set* Ω *is dense in* Σ.

Proof. We argue by contradiction. Suppose that Ω is not dense in Σ. Then $\int_\Omega f^{\frac{n}{n-1}} < \int_\Sigma f^{\frac{n}{n-1}}$. Hence, the arguments in Section 2 imply

$$(n+2)\,|B^{n+2}| \leq 2\,|B^2| \int_\Omega f^{\frac{n}{n-1}} < 2\,|B^2| \int_\Sigma f^{\frac{n}{n-1}} = 2\,|B^2|\,|B^n|.$$

This contradicts the fact that $(n+2)\,|B^{n+2}| = 2\,|B^2|\,|B^n|$.

Using Lemma 9, Lemma 10, and Lemma 11, we conclude that $D^2_\Sigma u = f^{\frac{1}{n-1}}\,g$, $I\!I = 0$, and $\nabla^\Sigma f = 0$ at each point on Σ. Since Σ is connected and $\nabla^\Sigma f = 0$ at each point on Σ, it follows that $f = \lambda^{n-1}$ for some positive constant λ. Since Σ is connected and $I\!I = 0$ at each point on Σ, Σ is contained in an n-dimensional plane P. Since $D^2_\Sigma u = f^{\frac{1}{n-1}}\,g = \lambda\,g$ at each point on Σ, the function u must be of the form $u(x) = \frac{1}{2}\,\lambda\,|x-p|^2 + c$ for some point $p \in P$ and some constant c. On the other hand, we know that $|\nabla^\Sigma u| < 1$ at each point on Ω. Using Lemma 11, it follows that $|\nabla^\Sigma u| \leq 1$ at each point on Σ. This implies $\Sigma \subset \{x \in P : \lambda\,|x-p| \leq 1\}$. Since $\lambda^n\,|\Sigma| = \int_\Sigma f^{\frac{n}{n-1}} = |B^n|$, we conclude that $\Sigma = \{x \in P : \lambda\,|x-p| \leq 1\}$. This completes the proof of Theorem 3.

References

[1] W. Allard, *On the first variation of a varifold,* Ann. of Math. 95, 417–491 (1972)

[2] F.J. Almgren, Jr., *Optimal isoperimetric inequalities,* Indiana Univ. Math. J. 35, 451–547 (1986)

[3] X. Cabré, *Elliptic PDEs in probability and geometry. Symmetry and regularity of solutions,* Discrete Cont. Dyn. Systems A 20, 425–457 (2008)

[4] T. Carleman, *Zur Theorie der Minimalflächen,* Math. Z. 9, 154–160 (1921)

[5] P. Castillon, *Submanifolds, isoperimetric inequalities and optimal transportation,* J. Funct. Anal. 259, 79–103 (2010)

[6] J. Choe, *The isoperimetric inequality for a minimal surface with radially connected boundary,* Ann. Scuola Norm. Sup. Pisa 17, 583–593 (1990)

[7] J. Choe, *Isoperimetric inequalities of minimal submanifolds,* Global Theory of Minimal Surfaces, Clay Math. Proc. vol. 2, pp. 325–369, Amer. Math. Soc., Providence RI, 2005

[8] H. Federer and W.H. Fleming, *Normal and integral currents,* Ann. of Math. 72, 458–520 (1960)

[9] J. Feinberg, *The isoperimetric inequality for doubly-connected minimal surfaces in* $\mathbb{R}^n$*,* J. d'Anal. Math. 32, 249–278 (1977)

[10] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Springer-Verlag, 2001

[11] C.C. Hsiung, *Isoperimetric inequalities for two-dimensional Riemannian manifolds with boundary,* Ann. of. Math. 73, 213–220 (1961)

[12] P. Li, R. Schoen, and S.T. Yau, *On the isoperimetric inequality for minimal surfaces,* Ann. Scuola Norm. Sup. Pisa 11, 237–244 (1984)

[13] J.H. Michael and L.M. Simon, *Sobolev and mean value inequalities on generalized submanifolds of* $\mathbb{R}^n$*,* Comm. Pure Appl. Math. 26, 361–379 (1973)

[14] R. Osserman and M. Schiffer, *Doubly-connected minimal surfaces,* Arch. Rational Mech. Anal. 58, 285–307 (1975)

[15] W.T. Reid, *The isoperimetric inequality and associated boundary problems,* J. Math. Mech. 8, 897–906 (1959)

[16] A. Stone, *On the isoperimetric inequality on a minimal surface,* Calc. Var. PDE 17, 369–391 (2003)

[17] P. Topping, *Relating diameter and mean curvature for submanifolds in Euclidean space,* Comment. Math. Helv. 83, 539–546 (2008)

[18] N. Trudinger, *Isoperimetric inequalities for quermassintegrals,* Ann. Inst. H. Poincaré Anal. Non Linéaire 11, 411–425 (1994)

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