

# Generalized B-Operas

Indranil BISWAS <sup>†</sup>, Laura P. SCHAPOSNIK <sup>‡</sup> and Mengxue YANG <sup>‡</sup>

<sup>†</sup> Tata Institute of Fundamental Research, India

E-mail: [indranil@math.tifr.res.in](mailto:indranil@math.tifr.res.in)

<sup>‡</sup> University of Illinois at Chicago, USA

E-mail: [schapos@uic.edu](mailto:schapos@uic.edu), [myang59@uic.edu](mailto:myang59@uic.edu)

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**Abstract.** Operas were introduced by Beilinson–Drinfeld [arXiv:math.AG/0501398]. In [*J. Math. Pures Appl.* **82** (2003), 1–42] a higher rank analog was considered, where the successive quotients of the oper filtration are allowed to have higher rank. We dedicate this paper to introducing and studying generalized *B*-opers (where “*B*” stands for “bilinear”), obtained by endowing the underlying vector bundle with a bilinear form which is compatible with both the filtration and the connection. In particular, we study the structure of these *B*-opers, by considering the relationship of these structures with jet bundles and with geometric structures on a Riemann surface.

**Key words:** operas; connection; projective structure; Higgs bundles; differential operator; Lagrangians

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*In celebration of Motohico Mulase’s 65th birthday.*

## 1 Introduction

The study of operas within geometry and mathematical physics has received much attention in the last years, and in particular in connection with Higgs bundles in the recent years. In fact, certain operas arise naturally as limits of Higgs bundles in the Hitchin components. Recall that the Higgs bundles, corresponding to a complex Lie group  $G_{\mathbb{C}}$ , on a compact Riemann surface  $\Sigma$  are given by the solutions of Hitchin’s equations:

$$F_A + [\Phi, \Phi^*] = 0, \tag{1.1}$$

$$\bar{\partial}_A \Phi = 0, \tag{1.2}$$

where  $F_A$  is the curvature of a unitary connection  $\nabla_A = \partial_A + \bar{\partial}_A$  associated to the Dolbeault operator  $\bar{\partial}_A$  on a principal  $G_{\mathbb{C}}$ -bundle  $P$  on  $\Sigma$  and  $\Phi$  is a  $(1, 0)$ -form on  $\Sigma$  with values in the adjoint bundle  $\text{ad}(P)$  [26]. Given any solution  $(A, \Phi)$  of (1.1) and (1.2), there is a 1-parameter family of flat connections

$$\nabla_{\xi} = \xi^{-1} \Phi + \bar{\partial}_A + \partial_A + \xi \Phi^*, \tag{1.3}$$

parametrized by  $\xi \in \mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$ . Then, given a solution of (1.1)–(1.2) in the  $\text{SL}(n, \mathbb{C})$ -Hitchin section, we can add a real parameter  $R > 0$  to (1.3) to obtain a natural family of connections with  $\text{SL}(n, \mathbb{R})$  monodromy

$$\nabla(\xi, R) := \xi^{-1} R \Phi + \bar{\partial}_A + \partial_A + \xi R \Phi^*.$$

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In [23] Gaiotto conjectured that the space of opers should be obtained as the  $\hbar$ -conformal limit of the Hitchin section: taking the limits  $R \rightarrow 0$  and  $\xi \rightarrow 0$  simultaneously while holding the ratio  $\hbar = \xi/R$  fixed. The conjecture was recently established for general simple Lie groups by Dumitrescu, Fredrickson, Kydonakis, Mazzeo, Mulase and Neitzke in [17], who also conjectured that this oper is the *quantum curve* in the sense of Dumitrescu and Mulase [18], a quantization of the spectral curve  $S$  of the corresponding Higgs bundle by *topological recursion* [19] – see also references and details in [16]. Moreover, very recently Collier and Wentworth showed in [12] that the above conformal limit exists for spaces other than the Hitchin components.

## opers

With views towards understanding other conformal limits of Higgs bundles and their appearance through quantum curves, in this note we introduce *generalized B-opers* and begin a program to study their geometry and topology, leaving for future work the extension of the above results to this new setting. Opers were introduced by Beilinson–Drinfeld [3, 4]; they were motivated by Drinfeld–Sokolov [14, 15]. Given a semisimple complex Lie group  $G$ , a  $G$ -oper on a compact Riemann surface  $\Sigma$  is

- a holomorphic principal  $G$ -bundle  $P$  on  $\Sigma$  with a holomorphic connection  $\nabla$ , and
- a holomorphic reduction of structure group of  $P$  to a Borel subgroup of  $G$ ,

such that reduction satisfies the Griffiths transversality with respect to the connection with the corresponding second fundamental forms being isomorphisms. They admit reformulation in terms of holomorphic differential operators on a Riemann surface. For example,  $\mathrm{SL}(n, \mathbb{C})$ -opers on  $\Sigma$  are precisely holomorphic differential operators on  $\Sigma$  of order  $n$  and symbol 1 with vanishing sub-principal term.

opers also provide a coordinate-free description of the connections in the base space of a generalized KdV hierarchy<sup>1</sup>. These connections can be translated to the usual  $n$ 'th order differential operators according to Drinfeld and Sokolov [14, 15]. The study of opers has been carried out from many different perspectives: as an example, Ben-Zvi and Frenkel studied how opers arise from homogeneous spaces for loop groups and found a natural morphism between the moduli spaces of spectral data and opers [5]. Moreover, Frenkel and Gaitsgory studied opers on the formal punctured disc in [22] and showed that there is a natural forgetful map from such opers to local systems on the punctured disc, which appears to be of much importance for the geometric Langlands correspondence. Opers are also related to the representations of affine Kac–Moody algebras at the critical level. In particular, from [21], the algebra of functions on the space of  ${}^L G$ -opers, where  ${}^L G$  is the Langlands dual of  $G$ , on the formal disc is isomorphic to the classical  $\mathcal{W}$ -algebra associated to the Langlands dual Lie algebra  ${}^L \mathfrak{g}$ .

In recent years, new types of opers were introduced, such as  $\mathfrak{g}$ -opers<sup>2</sup> and Miura opers [21]. We dedicate this paper to introducing and studying *generalized B-opers* (Definition 2.11; where “ $B$ ” stands for “bilinear” and not “Borel”) which we show are closely related to geometric structures on the base Riemann surface  $\Sigma$  (Theorem 4.6), and which encode certain classical  $G$ -opers. These are a generalization of [8], a higher rank analog of the work in Beilinson–Drinfeld [3, 4], where the successive quotients of the oper filtration are allowed to have higher rank. In our new setting, when the bilinear form is skew-symmetric and the rank of the bundle is 4, it allows us to recover some of the  $\mathrm{Sp}(4, \mathbb{C})$ -structure studied in [11].

The relationship between opers and geometric structures has been of much interests in the last half century, with the first results on this going back to work of R.C. Gunning in [25], where

<sup>1</sup>We may think of a generalized KdV hierarchy to be an infinite family of commuting flows on the space of connections on a principal  $G$ -bundle on a noncompact curve.

<sup>2</sup>A  $\mathfrak{g}$ -oper is a  $G$ -oper where  $G$  is the group of automorphisms of  $\mathfrak{g}$  [4].

it was shown that there is a one-to-one correspondence between  $\mathrm{SL}(n, \mathbb{C})$ -opers on a compact Riemann surface  $\Sigma$  and complex projective structures on  $\Sigma$ . One should note that the term oper did not exist at the time, and hence Gunning calls his family of flat bundles associated to the complex projective structures *indigenous bundles* on the Riemann surface. Much work has been done since then on geometric structures arising through opers, and the reader may want to refer to [30] and references therein for further details.

In [8], a more general class of opers was studied, where  $\mathrm{rank}(E) = nr$  and the rank of each successive quotient  $E_i/E_{i-1}$  is  $r$  (the above two conditions remain unchanged). In the present paper, we incorporate a non-degenerate bilinear form  $B$  and require the (not necessarily full) filtration and the connection appearing in a  $G$ -oper to be compatible with it, thus leading to the natural objects which we call *generalized B-opers*, and for which we consider geometric structures arising through them in our main Theorem 4.6.

The paper is organized as follows. We first review some of the basic definitions and properties of classical opers, following [4] and [5] (done in Section 2.1), and then introduce in Section 2.2 what we call *generalized B-opers* and their main properties. These are triples  $(E, \mathcal{F}, D)$ , where  $\mathcal{F}$  is a  $B$ -filtration of a holomorphic vector bundle  $E$  associated to a non-degenerate bilinear form  $B$  (as in Definition 2.6) and  $D$  is a  $B$ -connection on  $E$  (as in Definition 2.7), such that for every  $1 \leq i \leq n-1$  the condition  $D(E_i) \subset E_{i+1} \otimes K_\Sigma$  holds, and the resulting homomorphism

$$E_i/E_{i-1} \longrightarrow (E_{i+1}/E_i) \otimes K_\Sigma$$

(constructed in (2.7)) is an isomorphism. Generalized  $B$ -opers are closely related to jets, and we investigate this correspondence in Section 3, where in particular we show in Theorem 3.17 and Proposition 4.5 that there is a correspondence between the generalized  $B$ -opers  $(E, \mathcal{F}, D)$  on  $\Sigma$  and triples consisting of the following:

- (1) a fiberwise non-degenerate symmetric bilinear form  $\mathbb{B}_n$  on  $\mathcal{Q}^* := (E/E_{n-1} \otimes K_\Sigma^{(n-1)/2})^*$ ,
- (2) a holomorphic connection on  $\mathcal{Q}^*$  that preserves this bilinear form  $\mathbb{B}_n$ , and
- (3) a classical  $\mathrm{Sp}(n, \mathbb{C})$ -oper for  $n$  even and a classical  $\mathrm{SO}(n, \mathbb{C})$ -oper for  $n$  odd.

The above correspondence of spaces can be taken further, which is done in Section 4. More precisely, the known relation between classical opers and projective structures can be extended to this generalized setting, leading to one of our main results:

**Theorem 4.6.** *For integers  $n \geq 2$ ,  $n \neq 3$  and  $r \geq 1$ , the space of all generalized  $B$ -opers of filtration length  $n$  and  $\mathrm{rank}(E_i/E_{i-1}) = r$  is in correspondence with*

$$\mathcal{C}_\Sigma \times \mathfrak{P}(\Sigma) \times \left( \bigoplus_{i=2}^{[n/2]} H^0(\Sigma, K_\Sigma^{\otimes 2i}) \right),$$

where  $\mathcal{C}_\Sigma$  denotes the space of all flat orthogonal bundles of rank  $r$  on  $\Sigma$ , which is independent of  $i$ , and  $\mathfrak{P}(\Sigma)$  is the space of all projective structures on  $\Sigma$ .

We conclude the paper describing certain naturally defined Higgs bundles appearing through generalized  $B$ -opers which carry certain Lagrangian structure. In Section 5 we initiate the study of those Higgs bundles, leaving to future work their further study.

## 2 Generalized $B$ -opers

In what follows we shall first give a brief overview of classical  $G$ -opers for  $G$  a connected reductive complex Lie group (this is done in Section 2.1), and then introduce our generalization, what we call “Generalized  $B$ -opers”, in Section 2.2.

Let  $\Sigma$  be a compact connected Riemann surface of genus  $g$ ; its holomorphic cotangent bundle will be denoted by  $K_\Sigma$ .

## 2.1 $G$ -opers

We shall recall here the following definitions from Beilinson and Drinfeld's paper [4]. Fix a Borel subgroup  $B \subset G$  together with a maximal torus  $H \subset B$ . Let  $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$  be the Lie algebras of  $H \subset B \subset G$  respectively. Fix a set of positive simple roots  $\Gamma \subset \mathfrak{h}^*$  with respect to  $\mathfrak{b}$ . For each  $\alpha$  in the dual space  $\mathfrak{h}^*$ , set

$$\mathfrak{g}^\alpha := \{x \in \mathfrak{g} \mid [a, x] = \alpha(a)x \ \forall a \in \mathfrak{h}\}.$$

Then, there is a unique Lie algebra grading  $\mathfrak{g} = \bigoplus_k \mathfrak{g}_k$  such that the following holds

$$\mathfrak{g}_0 = \mathfrak{h}, \quad \mathfrak{g}_1 = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}^\alpha, \quad \mathfrak{g}_{-1} = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}^{-\alpha}.$$

Let  $P$  be a holomorphic principal  $B$ -bundle on  $\Sigma$ , and let  $\mathcal{E}_P$  be the Lie algebroid over  $\Sigma$  of infinitesimal symmetries of  $P$ . This  $\mathcal{E}_P$  is same as the Atiyah bundle for  $P$  [1]. Let  $\mathcal{E}_P^\mathfrak{g}$  be the Lie algebroid over  $\Sigma$  of infinitesimal symmetries of the holomorphic principal  $G$ -bundle  $Q = P \times_B G$  obtained by extending the structure group of  $P$  to  $G$ . Let  $\mathfrak{b}_P := P(\mathfrak{b})$  (respectively,  $\mathfrak{g}_P := P(\mathfrak{g})$ ) be the holomorphic vector bundle over  $\Sigma$  associated to  $P$  for the  $B$ -module  $\mathfrak{b}$  (respectively,  $\mathfrak{g}$ ); so  $\mathfrak{b}_P$  is the adjoint bundle  $\text{ad}(P)$ . Then, there is a filtration  $\mathfrak{g}_P^k \subset \mathfrak{g}_P$ , and we shall let  $\mathcal{E}_P^k$  be the preimage of  $\mathfrak{g}_P^k/\mathfrak{b}_P \subset \mathfrak{g}_P/\mathfrak{b}_P = \mathcal{E}_P^\mathfrak{g}/\mathcal{E}_P$ . One can then define another filtration  $\mathcal{E}_P^k \subset \mathcal{E}_P^\mathfrak{g}$ ,  $k \leq 0$ . Considering  $\mathfrak{g}_P^{-\alpha}$  the  $P$ -twist of the  $B$ -module  $\mathfrak{g}^{-\alpha}$ , one has that  $\mathcal{E}_P^{-1}/\mathcal{E}_P = \mathfrak{g}_P^{-1}/\mathfrak{b}_P = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_P^{-\alpha}$ .

**Definition 2.1** ( $G$ -oper). A  $G$ -oper on  $\Sigma$  is a holomorphic principal  $B$ -bundle  $P$  on  $\Sigma$  with a holomorphic connection  $\omega: T\Sigma \rightarrow \mathcal{E}_P^\mathfrak{g}$  such that  $\omega(T\Sigma) \subset \mathcal{E}_P^{-1}$ , and for every  $\alpha \in \Gamma$ , the following composition of homomorphisms is an isomorphism

$$T\Sigma \xrightarrow{\omega} \mathcal{E}_P^{-1} \rightarrow \mathcal{E}_P^{-1}/\mathcal{E}_P \rightarrow \mathfrak{g}_P^{-\alpha}.$$

In order to draw a parallel between  $G$ -opers and our generalization, we shall focus now on  $\text{SL}(n, \mathbb{C})$ -opers, which can be described as follows from [4]. Indeed, opers are some vector bundles satisfying certain conditions that give them an oper structure.

**Definition 2.2.** A  $\text{GL}(n, \mathbb{C})$ -oper on  $\Sigma$  is a holomorphic vector bundle  $E$  of rank  $n$  with a complete filtration of holomorphic subbundles

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_n = E$$

and a holomorphic connection  $D: E \rightarrow E \otimes K_\Sigma$  such that

- 1) (the filtration is a complete flag) the quotient  $E_i/E_{i-1}$  is a line bundle for all  $1 \leq i \leq n$ ;
- 2) (Griffiths' transversality)  $D(E_i) \subset E_{i+1} \otimes K_\Sigma$  for  $1 \leq i \leq n-1$ ;
- 3) (non-degeneracy) and  $D$  induces an isomorphism

$$E_i/E_{i-1} \xrightarrow{\sim} (E_{i+1}/E_i) \otimes K_\Sigma.$$

We denote this  $\text{GL}(n, \mathbb{C})$ -oper by  $(E, \{E_i\}, D)$ .

**Remark 2.3.** An  $\text{SL}(n, \mathbb{C})$ -oper is a  $\text{GL}(n, \mathbb{C})$ -oper  $(E, \{E_i\}, D)$  on  $\Sigma$  such that the determinant line bundle  $\bigwedge^n E$  is holomorphically trivial and the connection on  $\bigwedge^n E$  induced by  $D$  coincides with the trivial connection on  $\mathcal{O}_\Sigma$ . An  $\text{Sp}(2n, \mathbb{C})$ -oper is an  $\text{SL}(2n, \mathbb{C})$ -oper  $(E, \{E_i\}, D)$  with a horizontal symplectic form on  $E$  such that  $E_i^\perp = E_{n-i}$ . An  $\text{SO}(n, \mathbb{C})$ -oper is an  $\text{SL}(n)$ -oper  $(E, \{E_i\}, D)$  with a horizontal non-degenerate symmetric bilinear form  $B$  on  $E$  such that  $E_i^\perp = E_{n-i}$ .

## 2.2 Second fundamental form and generalized $B$ -opers

In order to generalize the above definitions to account for the structure appearing through bilinear forms and their preserved filtrations, consider a pair  $(E, B)$ , where  $E$  is a holomorphic vector bundle over  $\Sigma$ , and

$$B: E \otimes E \longrightarrow \mathcal{O}_\Sigma \quad (2.1)$$

is a holomorphic homomorphism such that for every point  $x \in \Sigma$ , the bilinear form

$$B(x): E_x \otimes E_x \longrightarrow \mathbb{C}$$

is non-degenerate, meaning for every point  $x \in \Sigma$  and each nonzero vector  $v \in E_x$  there is some  $w \in E_x$  such that  $B(x)(v, w) \neq 0$ .

**Definition 2.4.** The form  $B$  in (2.1) is called *symplectic* (respectively, *orthogonal*) if it satisfies  $B(x)(v, w) = -B(x)(w, v)$  (respectively,  $B(x)(v, w) = B(x)(w, v)$ ) for all  $x \in \Sigma$  and for all  $v, w \in E_x$ .

The bilinear forms we shall consider in the present paper will be either symplectic or orthogonal. Let

$$p_0: E \longrightarrow \Sigma$$

be the natural projection. For any holomorphic subbundle  $F \subset E$ , define the subbundle

$$F^\perp := \{w \in E \mid B(p_0(w))(w, v) = 0 \ \forall v \in F_{p_0(w)}\} \subset E.$$

Note that since  $B(p_0(w))(w, v) = 0$  if and only if  $B(p_0(w))(v, w) = 0$ , the subbundle  $F^\perp$  does not change if  $B(p_0(w))(w, v)$  in the definition of  $F^\perp$  is replaced by  $B(p_0(w))(v, w)$ .

**Lemma 2.5.** *The  $C^\infty$  subbundle  $F^\perp \subset E$  is a holomorphic subbundle. Moreover,  $F^\perp$  is canonically isomorphic to the dual bundle  $(E/F)^*$ . Also,  $(F^\perp)^\perp = F$ .*

**Proof.** Let  $\bar{\partial}_E: E \longrightarrow E \otimes \Omega_\Sigma^{0,1} = E \otimes \bar{K}_\Sigma$  be the Dolbeault operator defining the holomorphic structure of the holomorphic vector bundle  $E$ . The given condition that the bilinear form  $B$  is holomorphic implies that

$$\bar{\partial}B(s_1, s_2) = B(\bar{\partial}_E(s_1), s_2) + B(s_1, \bar{\partial}_E(s_2)) \quad (2.2)$$

for all locally defined  $C^\infty$  sections  $s_1$  and  $s_2$  of  $E$ . Since  $F$  is a holomorphic subbundle of  $E$ , it follows that  $\bar{\partial}(s)$  is a  $C^\infty$  (local) section of  $F \otimes \Omega_\Sigma^{0,1}$  for every (local)  $C^\infty$  section  $s$  of  $F$ . Therefore if  $s_1$  is a  $C^\infty$  locally defined section of  $F^\perp$  and  $s_2$  is a  $C^\infty$  locally defined section of  $F$ , then from (2.2) it follows immediately that

$$B(\bar{\partial}_E(s_1), s_2) = 0.$$

Now, this implies that  $\bar{\partial}_E(s_1)$  is a locally defined section of  $F^\perp \otimes \Omega_\Sigma^{0,1}$ . Consequently,  $F^\perp$  is a holomorphic subbundle on  $E$ .

The bilinear form  $B$  identifies  $E$  with its dual  $E^*$ . Consider the following composition of homomorphisms

$$F^\perp \hookrightarrow E \xrightarrow{\sim} E^*,$$

which is injective. The image of  $F^\perp$  in  $E^*$  evidently coincides with the image of the natural injective homomorphism

$$(E/F)^* \longrightarrow E^*.$$

Consequently,  $F^\perp$  is identified with  $(E/F)^*$ .

Finally, from the definition of  $F^\perp$  it follows immediately that  $(F^\perp)^\perp = F$ . ■

In the case of Hermitian holomorphic vector bundles, Kobayashi showed in [29] that  $F^\perp$  is not a holomorphic subbundle if  $F$  is not preserved by the Chern connection on  $E$  (see Definition 2.9).

**Definition 2.6.** A *B-filtration* of a holomorphic vector bundle  $E$  is an increasing filtration  $\mathcal{F}$  of holomorphic subbundles

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{n-1} \subsetneq E_n = E \quad (2.3)$$

for which the following two conditions hold:

- 1) the length  $n \in \mathbb{N}$  of the filtration is even when  $B$  is symplectic, and it is odd when  $B$  is orthogonal;
- 2)  $E_i^\perp = E_{n-i}$  for all  $1 \leq i \leq n-1$ .

Holomorphic connections were introduced by Atiyah in [1] (see also [2]). We recall that a holomorphic connection on a holomorphic vector bundle  $W$  on  $\Sigma$  is a holomorphic differential operator

$$D: W \longrightarrow W \otimes K_\Sigma$$

such that

$$D(fs) = fD(s) + s \otimes df \quad (2.4)$$

for all locally defined holomorphic sections  $s$  of  $W$  and all locally defined holomorphic functions  $f$  on  $\Sigma$ .

Note that the Leibniz condition in (2.4) implies that the order of the differential operator  $D$  is one. Moreover, a holomorphic connection on  $\Sigma$  is automatically flat because the sheaf of holomorphic two-forms on  $\Sigma$  is the zero sheaf. Through this, we can impose the following compatibility condition with respect to the bilinear form  $B$ .

**Definition 2.7.** Let  $E$  be a holomorphic vector bundle on  $\Sigma$  equipped with a bilinear form  $B$ . A *B-connection* on  $E$  is a holomorphic connection  $D$  on  $E$  such that

$$\partial(B(s, t)) = B(D(s), t) + B(s, D(t)) \quad (2.5)$$

for all locally defined holomorphic sections  $s$  and  $t$  of  $E$ . (Note that  $\partial(B(s, t)) = d(B(s, t))$  because  $B(s, t)$  is a holomorphic function.)

Given a holomorphic subbundle  $F \subset E$  and the corresponding quotient map

$$q_F: E \longrightarrow E/F,$$

it follows from (2.4) that the composition of homomorphisms

$$F \hookrightarrow E \xrightarrow{D} E \otimes K_\Sigma \xrightarrow{q_F \otimes \text{Id}_{K_\Sigma}} (E/F) \otimes K_\Sigma, \quad (2.6)$$

is in fact  $\mathcal{O}_\Sigma$ -linear.

**Definition 2.8.** The holomorphic section

$$S(D, F) \in H^0(\Sigma, \text{Hom}(F, E/F) \otimes K_\Sigma)$$

given by the composition of homomorphism in (2.6) is called the *second fundamental form* of  $F$  for the connection  $D$ .

The above construction of the second fundamental form in Definition 2.8 can be generalized from subbundles of  $E$  to filtration of subbundles of  $E$  as follows.

**Definition 2.9.** Let  $D$  be a holomorphic connection on  $E$ , and let

$$F_1 \subset F_2 \subset E \quad \text{and} \quad F_3 \subset F_4 \subset E$$

be holomorphic subbundles such that

$$D(F_1) \subset F_3 \otimes K_\Sigma \quad \text{and} \quad D(F_2) \subset F_4 \otimes K_\Sigma.$$

Then, the second fundamental form of  $(F_1, F_2, F_3, F_4)$  for the connection  $D$  is the map

$$\begin{aligned} S(D; F_1, F_2, F_3, F_4): F_2/F_1 &\longrightarrow (F_4/F_3) \otimes K_\Sigma, \\ s &\longmapsto D(\tilde{s}), \end{aligned} \tag{2.7}$$

that sends any locally defined holomorphic section  $s$  of  $F_2/F_1$  to the image of  $D(\tilde{s})$  in  $(F_4/F_3) \otimes K_\Sigma$ , where  $\tilde{s}$  is any locally defined holomorphic section of the subbundle  $F_2$  that projects to  $s$ .

In view of (2.4), note that the above conditions in Definition 2.9 imply that  $F_1 \subset F_3$  and  $F_2 \subset F_4$ .

**Lemma 2.10.** *The section  $S(D; F_1, F_2, F_3, F_4)(s)$  in (2.7) is independent of the lift  $\tilde{s}$  of  $s$  to  $F_2$ .*

**Proof.** This follows from the condition  $D(F_1) \subset F_3 \otimes K_\Sigma$ . Indeed, for  $s$  a local section of  $F_2/F_1$ , let  $\tilde{s}_1, \tilde{s}_2$  be two lifts of  $s$  to  $F_2$ . This means  $\tilde{s}_1 - \tilde{s}_2$  is a local section of  $F_1$ . Therefore, from the given condition that  $D(F_1) \subset F_3 \otimes K_\Sigma$  we know that

$$D(\tilde{s}_1) - D(\tilde{s}_2) = D(\tilde{s}_1 - \tilde{s}_2)$$

is a local section of  $F_3 \otimes K_\Sigma$ . Consequently, the section  $S(D; F_1, F_2, F_3, F_4)(s)$  is independent of the lift  $\tilde{s}$  of  $s$  to  $F_2$ .  $\blacksquare$

From (2.4) it follows immediately that for  $f$  a locally defined holomorphic function on  $\Sigma$  and  $s$  a locally defined holomorphic section of  $F_2/F_1$  to  $F_2$ , then the equality

$$S(D; F_1, F_2, F_3, F_4)(fs) = f \cdot S(D; F_1, F_2, F_3, F_4)(s)$$

holds. In other words,  $S(D; F_1, F_2, F_3, F_4)$  is an  $\mathcal{O}_\Sigma$ -linear homomorphism of coherent analytic sheaves.

Recall from Section 2.1 that a  $\mathrm{GL}(n, \mathbb{C})$ -oper on  $\Sigma$  is a  $\mathrm{GL}(n, \mathbb{C})$ -local system (that is a holomorphic vector bundle  $E$  with a holomorphic connection) equipped with a complete flag satisfying some extra conditions (2) and (3) in Definition 2.2. The following definition generalizes these two conditions.

**Definition 2.11.** A generalized  $B$ -oper is a triple  $(E, \mathcal{F}, D)$ , where  $\mathcal{F}$  is a  $B$ -filtration

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{n-2} \subsetneq E_{n-1} \subsetneq E_n = E$$

as in (2.3) and  $D$  is a  $B$ -connection on  $E$ , such that

- 1)  $D(E_i) \subset E_{i+1} \otimes K_\Sigma$  for all  $1 \leq i \leq n-1$ , and
- 2) for every  $1 \leq i \leq n-1$ , the homomorphism

$$S(D; E_{i-1}, E_i, E_i, E_{i+1}): E_i/E_{i-1} \longrightarrow (E_{i+1}/E_i) \otimes K_\Sigma \tag{2.8}$$

(constructed in (2.7)) is an isomorphism. For simplicity of notation, we shall sometimes write  $S_i(D)$  for the above map  $S(D; E_{i-1}, E_i, E_i, E_{i+1})$ .

**Remark 2.12.** Note that properties (1) and (2) from Definition 2.11 are natural counterparts to (2) and (3) in Definition 2.2. Moreover, the above definition can be seen as a generalization of the opers studied in [8]. Here we have the additional structure of a bilinear form  $B$ , and it fits within the broader picture of  $(G, P)$ -opers which is introduced in the work of Collier and Sanders in [11].

### 2.3 The underlying bundle of a generalized $B$ -oper

Let  $(E, \mathcal{F}, D)$  be a generalized  $B$ -oper as in Definition 2.11 above. In what follows we shall construct a natural  $K_{\Sigma}^{\otimes(n-1)}$ -valued symmetric form  $\mathbb{S}'$  on the subbundle  $E_1$  in Definition 2.11. For this, first note that for any  $1 \leq i \leq n-1$ , the isomorphism

$$S(D; E_{i-1}, E_i, E_i, E_{i+1}): E_i/E_{i-1} \longrightarrow (E_{i+1}/E_i) \otimes K_{\Sigma}$$

in (2.8) tensored with the identity map of  $K_{\Sigma}^{\otimes(i-1)}$  produces isomorphisms  $S_i$  defined by

$$\begin{aligned} S_i &:= S(D; E_{i-1}, E_i, E_i, E_{i+1}) \otimes \text{Id}_{K_{\Sigma}^{\otimes(i-1)}}: \\ (E_i/E_{i-1}) \otimes K_{\Sigma}^{\otimes(i-1)} &\longrightarrow (E_{i+1}/E_i) \otimes K_{\Sigma} \otimes K_{\Sigma}^{\otimes(i-1)} = (E_{i+1}/E_i) \otimes K_{\Sigma}^{\otimes i}. \end{aligned} \quad (2.9)$$

By composing the above maps from (2.9) we obtain an isomorphism

$$\mathbb{S}' := S_{n-1} \circ \cdots \circ S_1: E_1 \longrightarrow (E_n/E_{n-1}) \otimes K_{\Sigma}^{\otimes(n-1)}. \quad (2.10)$$

**Lemma 2.13.** *The bilinear form  $B$  produces the following isomorphism on a  $B$ -filtration as in Definition 2.6*

$$E_{n-i}^* \xrightarrow{\sim} E_n/E_i$$

for every  $1 \leq i \leq n-1$ .

**Proof.** Since  $E_i^{\perp} = E_{n-i}$  (see Definition 2.6), from Lemma 2.5 we conclude that  $(E/E_i)^* = E_i^{\perp} = E_{n-i}$ . The lemma follows from this.  $\blacksquare$

Using the identification between  $E_1^*$  and  $E_n/E_{n-1}$  in Lemma 2.13, the isomorphism  $\mathbb{S}'$  in (2.10) can be seen as a holomorphic section

$$\mathbb{S}' \in H^0(\Sigma, (E_1^*)^{\otimes 2} \otimes K_{\Sigma}^{\otimes(n-1)}) = H^0(\Sigma, (E_n/E_{n-1})^{\otimes 2} \otimes K_{\Sigma}^{\otimes(n-1)}). \quad (2.11)$$

**Lemma 2.14.** *Take a holomorphic vector field  $v \in H^0(U, T\Sigma)$  on an open subset  $U \subset \Sigma$ , and define the map*

$$\begin{aligned} V: H^0(U, E) &\longrightarrow H^0(U, E), \\ s &\longmapsto D(s)(v), \end{aligned} \quad (2.12)$$

where as usual,  $H^0(U, E)$  denotes the space of all holomorphic sections of  $E$  over  $U$ . The map  $V$  from (2.12) satisfies the identity

$$B(V(s), t) + B(s, V(t)) = v(B(s, t)).$$

**Proof.** This can be proved using (2.5). Indeed, since  $D$  is a  $B$ -connection, the following equality of 1-forms on  $\Sigma$  holds

$$d(B(s, t)) = B(D(s), t) + B(s, D(t)) \quad (2.13)$$

(see (2.5)), and thus contracting both sides of (2.13) by the vector field  $v$  the identity in the lemma is obtained.  $\blacksquare$

**Lemma 2.15.** *Let  $(E, \mathcal{F}, D)$  be a generalized  $B$ -oper as in Definition 2.11. Take  $s, t \in H^0(U, E_1)$ . Then,*

$$B(V^{n-1}(s), t) + (-1)^{n-2}B(s, V^{n-1}(t)) = 0 \quad (2.14)$$

for the map  $V$  in (2.12).

**Proof.** Through Lemma 2.14, for  $s, t \in H^0(U, E_1) \subset H^0(U, E)$  we have that

$$B(V^{n-1}(s), t) + B(V^{n-2}(s), V(t)) = v(B(V^{n-2}(s), t)) = 0, \quad (2.15)$$

because  $V^{n-2}(s) \in H^0(U, E_{n-1})$  by Definition 2.11(2), and  $E_1^\perp = E_{n-1}$  (see Definition 2.6(2)). Similarly, we have

$$B(V^{n-2}(s), V(t)) + B(V^{n-3}(s), V^2(t)) = v(B(V^{n-3}(s), V(t))) = 0, \quad (2.16)$$

because

- $V^{n-3}(s) \in H^0(U, E_{n-2})$  (by Definition 2.11(1)),
- $V(t) \in H^0(U, E_2)$  (by Definition 2.11(1)), and
- $E_2^\perp = E_{n-2}$  (Definition 2.6(2)).

Iterating the argument for (2.15) and (2.16), more generally we get that

$$\mathcal{T}_i := B(V^{n-i}(s), V^{i-1}(t)) + B(V^{n-i-1}(s), V^i(t)) = 0 \quad (2.17)$$

for all  $n-1 \leq i \geq 1$ . Taking alternating sum, from (2.17) we have

$$(-1)^{i-1} \mathcal{T}_i = B(V^{n-1}(s), t) + (-1)^{n-2} B(s, V^{n-1}(t)) = 0.$$

This completes the proof. ■

**Proposition 2.16.** *The section  $\mathbb{S}'$  in (2.11) lies in the subspace*

$$H^0(\Sigma, \text{Sym}^2(E_1^*) \otimes K_\Sigma^{\otimes(n-1)}) \subset H^0(\Sigma, (E_1^*)^{\otimes 2} \otimes K_\Sigma^{\otimes(n-1)}).$$

*This  $K_\Sigma^{\otimes(n-1)}$ -valued symmetric form  $\mathbb{S}'$  on  $E_1$  is fiberwise non-degenerate.*

**Proof.** We shall first assume that  $B$  is symmetric, hence an orthogonal form. In this case, from Definition 2.6(1) we know that the integer  $n$  in Definition 2.11 is odd. Since  $B$  is symmetric, and  $n$  is odd, from (2.14) in Lemma 2.15 it follows immediately that  $\mathbb{S}'$  in (2.11) is a section of

$$\text{Sym}^2(E_1^*) \otimes K_\Sigma^{\otimes(n-1)} \subset (E_1^*)^{\otimes 2} \otimes K_\Sigma^{\otimes(n-1)}.$$

Moreover, the  $K_\Sigma^{\otimes(n-1)}$ -valued bilinear form  $\mathbb{S}'$  on  $E_1$  is fiberwise non-degenerate, because the homomorphism  $\mathbb{S}'$  in (2.10) is an isomorphism.

Now assume that  $B$  is a symplectic form. In this case, from Definition 2.6(1) we know that the integer  $n$  in Definition 2.11 is even. Since  $B$  is anti-symmetric, and  $n$  is even, from (2.14) it follows that  $\mathbb{S}'$  in (2.11) is a section of

$$\text{Sym}^2(E_1^*) \otimes K_\Sigma^{\otimes(n-1)} \subset (E_1^*)^{\otimes 2} \otimes K_\Sigma^{\otimes(n-1)}.$$

Moreover, as in the previous case, the  $K_\Sigma^{\otimes(n-1)}$ -valued bilinear form  $\mathbb{S}'$  on  $E_1$  is fiberwise non-degenerate, because the homomorphism  $\mathbb{S}'$  in (2.10) is an isomorphism. ■

### 3 Generalized $B$ -opers and jet bundles

Generalized  $B$ -opers as introduced in Definition 2.11 are closely related to *jet bundles*, natural objects which have had different applications in geometry. In what follows, we shall first recall the basic definitions and properties of jet bundles, and then study their relation to generalized  $B$ -opers. The reader should refer, for instance, to [8] for a gentle introduction to the main ideas surrounding jet bundles which will come useful in the present paper.

### 3.1 Jet bundles

We briefly recall the definition of a jet bundle of a holomorphic vector bundle on the Riemann surface  $\Sigma$ . Consider the product  $\Sigma \times \Sigma$ , and let

$$p_i: \Sigma \times \Sigma \longrightarrow \Sigma, \quad i = 1, 2$$

be the projection to the  $i$ -th factor. Let

$$\Delta := \{(x, x) \in \Sigma \mid x \in \Sigma\} \subset \Sigma \times \Sigma \quad (3.1)$$

be the (reduced) diagonal divisor.

**Definition 3.1.** Given a holomorphic vector bundle  $W$  of rank  $r_0$  on  $\Sigma$  and any nonnegative integer  $m \geq 0$ , the  $m$ th order *jet bundle*  $J^m(W)$  of  $W$  is the holomorphic vector bundle of rank  $(m+1)r_0$  given by the direct image

$$J^m(W) := p_{1*} \left( \frac{p_2^* W}{p_2^* W \otimes \mathcal{O}_{\Sigma \times \Sigma}(-(m+1)\Delta)} \right) \longrightarrow \Sigma.$$

The natural inclusion of  $\mathcal{O}_{\Sigma \times \Sigma}(-(j+1)\Delta)$  in  $\mathcal{O}_{\Sigma \times \Sigma}(-j\Delta)$  produces a surjective homomorphism  $J^j(W) \longrightarrow J^{j-1}(W)$ .

**Lemma 3.2.** For  $j \geq 1$ , the kernel of the above natural projection

$$J^j(W) \longrightarrow J^{j-1}(W)$$

is the vector bundle  $K_{\Sigma}^{\otimes j} \otimes W$ .

**Proof.** Whilst this is something well known, we shall include here a short proof in order to be somewhat self-contained. We will show this algebraically, following the proof given by Eisenbud and Harris for bundles of principal parts [20, p. 247]. Since  $\Sigma$  can be viewed as a smooth projective curve, it suffices to prove the claim on  $\text{Spec}(R)$ , where  $R = \mathbb{C}[z]$ . The vector bundle  $W$  on  $R = \Sigma$  corresponds to a  $R$ -module, which we also denote by  $W$ . The pullback bundle  $p_2^* W = S \otimes_R W = R \otimes_{\mathbb{C}} W$  is just an  $S$ -module from changing base ring. Let  $I$  be the ideal sheaf of the diagonal  $\Delta$ . Then  $I$  is generated by  $z \otimes 1 - 1 \otimes z$ , where  $z$  is the generator of  $R$ . In this set-up, we see the  $R$ -module associated to the jet bundle is

$$J^m(W) = p_{1*} \left( \frac{R \otimes_{\mathbb{C}} W}{R \otimes_{\mathbb{C}} W \otimes_{\mathbb{C}} I^{m+1}} \right) = W/I^{m+1}W$$

viewed as an  $R$ -module with multiplication given by  $r \mapsto r \otimes_{\mathbb{C}} 1$ . Then the kernel of the natural projection  $J^j(W) \longrightarrow J^{j-1}(W)$  is given by  $(I^m/I^{m+1}) \otimes_{\mathbb{C}} W$ . It is easy to see that  $I^j/I^{j+1}$  is generated by  $z^m \otimes 1 - 1 \otimes z^m$  and is isomorphic to  $\text{Sym}^m(\Omega_{R/\mathbb{C}}) = \Omega_{R/\mathbb{C}}^{\otimes m}$  as  $R$ -modules, where the latter is generated by  $dz^{\otimes m}$ . Finally, the canonical bundle coincides with the sheaf of differentials for a curve. Hence the short exact sequence

$$0 \longrightarrow K_{\Sigma}^{\otimes m} \otimes W \longrightarrow J^m(W) \longrightarrow J^{m-1}(W) \longrightarrow 0$$

is obtained. ■

The following two definitions will be of much use here, and the reader should refer for instance to [8] and references therein for further details on them.

**Definition 3.3.** Given holomorphic vector bundles  $W, W'$  on  $\Sigma$ , we let  $\text{Diff}_{\Sigma}^m(W, W')$  be the holomorphic vector bundle on  $\Sigma$  given by the *sheaf of holomorphic differential operators* of order  $m$  from  $W$  to  $W'$ , that is,

$$\text{Diff}_{\Sigma}^m(W, W') := \text{Hom}(J^m(W), W') \longrightarrow \Sigma. \quad (3.2)$$

**Definition 3.4.** The *symbol map*  $\sigma: \text{Diff}_{\Sigma}^m(W, W') \longrightarrow (T\Sigma)^{\otimes m} \otimes \text{Hom}(W, W')$  is the composition of homomorphisms

$$\text{Diff}_{\Sigma}^m(W, W') = J^m(W)^* \otimes W' \longrightarrow (K_{\Sigma}^{\otimes m} \otimes W)^* \otimes W' = (T\Sigma)^{\otimes m} \otimes \text{Hom}(W, W'),$$

where the homomorphism is  $J^m(W)^* \otimes W' \longrightarrow (K_{\Sigma}^{\otimes m} \otimes W)^* \otimes W'$  is obtained from the inclusion  $K_{\Sigma}^{\otimes m} \otimes W \hookrightarrow J^m(W)$  in Lemma 3.2.

In what follows we shall construct an  $\mathcal{O}_{\Sigma}$ -linear homomorphism from the holomorphic vector bundle  $E$  of a generalized  $B$ -oper  $(E, \mathcal{F}, D)$  to the jet bundle  $J^k(Q)$ , where  $k$  is any nonnegative integer and  $Q = E/E_{n-1}$ ; this will be done using the connection  $D$ .

Take a generalized  $B$ -oper  $(E, \mathcal{F}, D)$ . Consider a point  $x \in \Sigma$ , and take a point in the corresponding fiber  $v \in E_x$ . We shall denote by  $\tilde{v}$  the unique locally defined flat section of  $E$  for the flat connection  $D$ , for which  $\tilde{v}(x) = v$  (note that  $\tilde{v}$  is defined on any simply connected open neighborhood of  $x$ ). Consider the holomorphic section

$$q(\tilde{v}): \Sigma \longrightarrow Q := E/E_{n-1}$$

of  $Q$  defined around  $x$ , where  $q: E \longrightarrow E/E_{n-1} = Q$  is the natural quotient map (see (2.3)). Now restricting  $q(\tilde{v})$  to the  $k$ -th order infinitesimal neighborhood of  $x$  we get an element  $q(\tilde{v})_k \in J^k(Q)_x$ , and we shall denote by

$$\begin{aligned} f_k: E &\longrightarrow J^k(Q), \\ v &\longmapsto q(\tilde{v})_k, \end{aligned} \quad (3.3)$$

the  $\mathcal{O}_{\Sigma}$ -linear homomorphism constructed this way.

From the construction of the homomorphisms  $f_k$  in (3.3) it follows immediately that the following diagram of homomorphisms is commutative:

$$\begin{array}{ccc} E & \xrightarrow{f_k} & J^k(Q) \\ \parallel & & \downarrow \\ E & \xrightarrow{f_{k-j}} & J^{k-j}(Q) \end{array}$$

for all  $0 < j \leq k$ , where the projection  $J^k(Q) \longrightarrow J^{k-j}(Q)$  is the composition of iterations of the projection in Lemma 3.2.

The holomorphic vector bundle  $J^k(Q)$  has an increasing filtration of holomorphic subbundles of length  $k+2$

$$0 = \mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \cdots \subset \mathcal{Q}_k \subset \mathcal{Q}_{k+1} := J^k(Q), \quad (3.4)$$

where  $\mathcal{Q}_i$ , for every  $0 \leq i \leq k$ , is the kernel of the natural projection  $J^k(Q) \longrightarrow J^{k-i}(Q)$  obtained by iterating the projection in Lemma 3.2.

The following result from [8] will be of much use.

**Theorem 3.5** ([8, Theorem 4.2]). *The homomorphism  $f_{n-1}$  in (3.3) is an isomorphism. This isomorphism  $f_{n-1}$  takes the filtration  $\mathcal{F}$  in (2.3) to the filtration of  $J^{n-1}(Q)$  constructed in (3.4).*

**Remark 3.6.** It should be clarified that the isomorphism  $f_{n-1}$  in (3.3) depends on the connection  $D$ . More precisely, given  $(E, \mathcal{F})$ , if  $D'$  and  $D''$  are two  $B$ -connections such that  $(E, \mathcal{F}, D')$  and  $(E, \mathcal{F}, D'')$  are generalized  $B$ -opers, then the two isomorphisms

$$f'_{n-1}: E \longrightarrow J^k(Q) \quad \text{and} \quad f''_{n-1}: E \longrightarrow J^k(Q)$$

corresponding to  $D'$  and  $D''$  respectively do not in general coincide.

The construction of the homomorphism  $f_k$  in (3.3) gives the following general result.

**Proposition 3.7.** *Let  $V$  and  $W$  be holomorphic vector bundles on  $\Sigma$ , and let  $D_W$  be a holomorphic connection on  $W$ . Then for any integer  $n \geq 1$ , there is a natural holomorphic isomorphism*

$$\varphi: J^n(V) \otimes W \longrightarrow J^n(V \otimes W).$$

**Proof.** Take any point  $x \in \Sigma$  and a vector  $w \in W_x$  in the fiber of  $W$  over  $x$ . Let  $s_w$  denote the unique holomorphic section of  $W$ , defined on a simply connected open neighborhood of  $x \in \Sigma$ , satisfying the following two conditions:

- $s_w(x) = w$ , and
- $s_w$  is flat with respect to the integrable connection  $D_W$  on  $W$ .

Let  $\tilde{s}_w \in J^n(W)_x$  be the element obtained by restricting  $s_w$  to the  $n$ -th order infinitesimal neighborhood of  $x$ . Consequently, we get a holomorphic homomorphism

$$\eta: W \longrightarrow J^n(W) \tag{3.5}$$

that for any  $x \in \Sigma$ , sends any  $w \in W_x$  to  $\tilde{s}_w \in J^n(W)_x$  constructed as above.

Given any two holomorphic vector bundles  $V_1$  and  $V_2$ , there is a natural holomorphic homomorphism

$$J^n(V_1) \otimes J^n(V_2) \longrightarrow J^n(V_1 \otimes V_2).$$

Now consider the composition of homomorphisms

$$J^n(V) \otimes W \xrightarrow{\text{Id} \otimes \eta} J^n(V) \otimes J^n(W) \longrightarrow J^n(V \otimes W),$$

where  $\eta$  is the homomorphism in (3.5). It is straightforward to check that this composition of homomorphisms is fiberwise injective. This implies that it is an isomorphism, because  $\text{rank}(J^n(V) \otimes W) = \text{rank}(J^n(V \otimes W))$ .  $\blacksquare$

The following is a special case of Proposition 3.7.

**Corollary 3.8.** *Let  $V$  be a holomorphic vector bundle on  $\Sigma$ , and  $\mathcal{L}$  be a holomorphic line bundle on  $\Sigma$  of order  $r$ . Then there is a canonical holomorphic isomorphism*

$$J^n(V) \otimes \mathcal{L} \xrightarrow{\sim} J^n(V \otimes \mathcal{L})$$

for every  $j \geq 1$ .

**Proof.** Take a holomorphic isomorphism

$$\xi: \mathcal{O}_\Sigma \longrightarrow \mathcal{L}^{\otimes r}.$$

There is a unique holomorphic connection  $D_{\mathcal{L}}$  on  $\mathcal{L}$  such that the isomorphism  $\xi$  takes the trivial connection on  $\mathcal{O}_\Sigma$ , given by the de Rham differential  $d$ , to the connection on  $\mathcal{L}^{\otimes r}$  induced by  $D_{\mathcal{L}}$ . This connection  $D_{\mathcal{L}}$  does not depend on the choice of the isomorphism  $\xi$ , because any two choices of  $\xi$  differ by an automorphism of  $\mathcal{O}_\Sigma$  given by a nonzero scalar multiplication. Note that a nonzero scalar multiplication preserves the trivial connection on  $\mathcal{O}_\Sigma$ .

Since  $\mathcal{L}$  is equipped with a canonical connection  $D_{\mathcal{L}}$ , the required isomorphism

$$J^n(V) \otimes \mathcal{L} \xrightarrow{\sim} J^n(V \otimes \mathcal{L})$$

is given by Proposition 3.7.  $\blacksquare$

### 3.2 A natural symmetric form

Let  $g$  be the genus of  $\Sigma$ . Recall that a theta characteristic on  $\Sigma$  (or spin structure) is a holomorphic line bundle  $\xi$  on  $\Sigma$  of degree  $g - 1$  equipped with a holomorphic isomorphism of  $\xi^{\otimes 2}$  with  $K_\Sigma$ . Fix a theta characteristic on  $\Sigma$ , and denote it by  $K_\Sigma^{1/2}$ . For any integer  $m$ , the holomorphic line bundle  $(K_\Sigma^{1/2})^{\otimes m}$  will be denoted by  $K_\Sigma^{m/2}$ .

Take a generalized  $B$ -oper  $(E, \mathcal{F}, D)$  on  $\Sigma$ . For ease of notation we define the holomorphic vector bundle

$$\mathcal{Q} := Q \otimes K_\Sigma^{(n-1)/2} = (E/E_{n-1}) \otimes K_\Sigma^{(n-1)/2}. \quad (3.6)$$

Having studied the natural  $K_\Sigma^{\otimes(n-1)}$ -valued symmetric form  $\mathbb{S}'$  on  $E_1$  in Section 2.3, in what follows we shall construct an equivalent symmetric form on  $\mathcal{Q}$ .

Using the isomorphism  $E_1^* = Q$  from Lemma 2.13, the section  $\mathbb{S}'$  in (2.11) produces a section

$$\mathbb{S} \in H^0(\Sigma, Q^{\otimes 2} \otimes K_\Sigma^{\otimes(n-1)}) = H^0(\Sigma, \mathcal{Q}^{\otimes 2}). \quad (3.7)$$

In view of Proposition 2.16, the section  $\mathbb{S}$  satisfies the following:

**Proposition 3.9.** *The section  $\mathbb{S}$  in (3.7) lies in the subspace*

$$H^0(\Sigma, \text{Sym}^2(\mathcal{Q})) \subset H^0(\Sigma, \mathcal{Q}^{\otimes 2}).$$

Moreover, the symmetric bilinear form on  $\mathcal{Q}^*$  defined by  $\mathbb{S}$  is fiberwise non-degenerate.

**Proof.** This follows immediately from Proposition 2.16 and Lemma 2.13. ■

Since the symmetric bilinear form  $\mathbb{S}$  in (3.7) is fiberwise non-degenerate, it produces a symmetric bilinear form

$$\mathbb{S}^\vee \in H^0(\Sigma, (\mathcal{Q}^*)^{\otimes 2})$$

on  $\mathcal{Q}$ . In view of (3.6), this  $\mathbb{S}^\vee$  defines a homomorphism

$$\mathbb{S}_0^\vee: Q \otimes Q \otimes K_\Sigma^{\otimes(n-1)} \longrightarrow \mathcal{O}_\Sigma. \quad (3.8)$$

**Proposition 3.10.** *A generalized  $B$ -oper  $(E, \mathcal{F}, D)$  produces a holomorphic connection on the holomorphic vector bundle  $\mathcal{Q}$ . For the holomorphic connection on  $\text{Sym}^2(\mathcal{Q})$  induced by this holomorphic connection on  $\mathcal{Q}$ , the section  $\mathbb{S}$  in Proposition 3.9 is flat.*

**Proof.** Let  $(E, \mathcal{F}, D)$  be a generalized  $B$ -oper on  $\Sigma$ . The triple  $(E, \mathcal{F}, D)$  produces a holomorphic differential operator  $\mathcal{D}$  on  $\Sigma$  of order  $n$

$$\mathcal{D} \in H^0(\Sigma, \text{Diff}_\Sigma^n(Q, Q \otimes K_\Sigma^n)) \quad (3.9)$$

(see [8, p. 18, equation (4.6)]). As before,  $\Delta \subset \Sigma \times \Sigma$  is the divisor in (3.1). Using  $\mathcal{D}$  one can construct a holomorphic section of  $p_1^* \mathcal{Q} \otimes p_2^* \mathcal{Q}^*$  over the non-reduced divisor  $2\Delta$  as in [8, p. 27] (note that in [8] this section is called  $\kappa|_{2\Delta} \otimes s$ ). Moreover, as shown in [8, p. 27], the restriction of this section to  $\Delta \subset 2\Delta$  coincides with the identity map of  $\mathcal{Q}$ . Therefore, this section of  $p_1^* \mathcal{Q} \otimes p_2^* \mathcal{Q}^*$  over  $2\Delta$  defines a holomorphic connection on the holomorphic vector bundle  $\mathcal{Q}$  [8, p. 7], [13]; this connection on  $\mathcal{Q}$  will be denoted by  $\nabla$ .

Let  $\widehat{\nabla}$  denote the holomorphic connection on  $\text{Sym}^2(\mathcal{Q})$  induced by the connection  $\nabla$  on  $\mathcal{Q}$ . To complete the proof of the proposition, we need to show that the section  $\mathbb{S}$  of  $\text{Sym}^2(\mathcal{Q})$  in Proposition 3.9 is flat (covariant constant) with respect to this induced connection  $\widehat{\nabla}$ . For that

we need to recall the construction, as well as some properties, of the differential operator  $\mathcal{D}$  in (3.9).

As before,  $p_1$  and  $p_2$  are the projections of  $\Sigma \times \Sigma$  to the first and second factor respectively. The holomorphic differential operator  $\mathcal{D}$  is given by a holomorphic section

$$\kappa \in H^0((n+1)\Delta, p_1^*(K_\Sigma^{\otimes n} \otimes Q) \otimes p_2^*(K_\Sigma \otimes Q^*) \otimes \mathcal{O}_{\Sigma \times \Sigma}((n+1)\Delta))$$

over the nonreduced divisor  $(n+1)\Delta$  [8, p. 25, equation (5.1)]. From (3.6) it follows immediately that

$$p_1^*(K_\Sigma^{\otimes n} \otimes Q) \otimes p_2^*(K_\Sigma \otimes Q^*) = p_1^*(K_\Sigma^{(n+1)/2} \otimes \mathcal{Q}) \otimes p_2^*(K_\Sigma^{(n+1)/2} \otimes \mathcal{Q}^*),$$

and we conclude that

$$\kappa \in H^0((n+1)\Delta, p_1^*(K_\Sigma^{(n+1)/2} \otimes \mathcal{Q}) \otimes p_2^*(K_\Sigma^{(n+1)/2} \otimes \mathcal{Q}^*) \otimes \mathcal{O}_{\Sigma \times \Sigma}((n+1)\Delta)). \quad (3.10)$$

We note that the holomorphic line bundle  $p_1^*(K_\Sigma^{(n+1)/2}) \otimes (p_2^*K_\Sigma^{(n+1)/2}) \otimes \mathcal{O}_{\Sigma \times \Sigma}((n+1)\Delta)$  has a canonical trivialization over  $2\Delta$  [9, p. 688, Theorem 2]. Since

$$\begin{aligned} & p_1^*(K_\Sigma^{(n+1)/2} \otimes \mathcal{Q}) \otimes p_2^*(K_\Sigma^{(n+1)/2} \otimes \mathcal{Q}^*) \otimes \mathcal{O}_{\Sigma \times \Sigma}((n+1)\Delta) \\ &= (p_1^*\mathcal{Q}) \otimes (p_2^*\mathcal{Q}^*) \otimes p_1^*(K_\Sigma^{(n+1)/2}) \otimes (p_2^*K_\Sigma^{(n+1)/2}) \otimes \mathcal{O}_{\Sigma \times \Sigma}((n+1)\Delta), \end{aligned}$$

using this trivialization of  $p_1^*(K_\Sigma^{(n+1)/2}) \otimes (p_2^*K_\Sigma^{(n+1)/2}) \otimes \mathcal{O}_{\Sigma \times \Sigma}((n+1)\Delta)$  over  $2\Delta$ , the section  $\kappa$  in (3.10) produces a section of  $(p_1^*\mathcal{Q}) \otimes (p_2^*\mathcal{Q}^*)$ . This section of  $(p_1^*\mathcal{Q}) \otimes (p_2^*\mathcal{Q}^*)$  over  $2\Delta$  gives the holomorphic connection on  $\nabla$  on  $\mathcal{Q}$ .

On the other hand,  $\mathcal{Q}$  is identified with its dual  $\mathcal{Q}^*$  using the pairing  $\mathbb{S}$  in Proposition 3.9. Consequently, we conclude that

$$\kappa \in H^0((n+1)\Delta, p_1^*(K_\Sigma^{(n+1)/2} \otimes \mathcal{Q}) \otimes p_2^*(K_\Sigma^{(n+1)/2} \otimes \mathcal{Q}) \otimes \mathcal{O}_{\Sigma \times \Sigma}((n+1)\Delta)). \quad (3.11)$$

Now from the construction of  $\kappa$  it follows that the section  $\kappa$  is symmetric, meaning

$$\eta^* \kappa = \kappa, \quad (3.12)$$

where  $\eta: \Sigma \times \Sigma \longrightarrow \Sigma \times \Sigma$  is the involution defined by  $(x, y) \mapsto (y, x)$ ; in (3.12), the section  $\kappa$  is considered as the section in (3.11).

In view of  $\eta$ , the following lemma implies that the section  $\mathbb{S}$  of  $\text{Sym}^2(\mathcal{Q})$  is covariantly constant with respect to the connection  $\tilde{\nabla}$ . ■

**Lemma 3.11.** *Let  $F$  be a holomorphic vector bundle over  $\Sigma$  equipped with a holomorphic connection  $\nabla_F$ . Let  $\theta^F$  be the section of  $(p_1^*F) \otimes (p_2^*F^*)$  over  $2\Delta$  corresponding to the connection  $\nabla_F$ . Let*

$$B \in H^0(\Sigma, \text{Sym}^2(F))$$

*be a nondegenerate symmetric bilinear form on  $F^*$ . Then the following statements hold:*

- *Using  $B$ , the section  $\theta^F$  produces a section  $\tilde{\theta}^F$  of  $(p_1^*F) \otimes (p_2^*F)$  over  $2\Delta$ .*
- *For the connection  $\tilde{\nabla}_F$  on  $F \otimes F$  induced by  $\nabla_F$ ,*

$$\tilde{\nabla}_F(B) = \tilde{\theta}^F - \eta^* \tilde{\theta}^F \in H^0(\Sigma, \text{Sym}^2(F) \otimes K_\Sigma),$$

*where  $\eta$  is the involution of  $\Sigma \times \Sigma$  in (3.12).*

*In particular,  $B$  is covariant constant with respect to the connection  $\tilde{\nabla}_F$  if and only if  $\eta^* \tilde{\theta}^F = \tilde{\theta}^F$ .*

**Proof.** Since  $B$  identifies  $F$  with  $F^*$ , we have  $(p_1^*F) \otimes (p_2^*F^*) = (p_1^*F) \otimes (p_2^*F)$ . So  $\theta^F$  produces a section  $\tilde{\theta}^F$  of  $(p_1^*F) \otimes (p_2^*F)$  over  $2\Delta$ .

The restriction of  $\theta^F$  to  $\Delta \subset 2\Delta$  is  $\text{Id}_F$ . Hence we have

$$\tilde{\theta}^F|_{\Delta} = B = (\eta^* \tilde{\theta}^F)|_{\Delta}.$$

This, and the fact that  $\mathcal{O}_{\Sigma \times \Sigma}(-\Delta)|_{\Delta} = K_{\Delta}$  with  $K_{\Delta}$  being the holomorphic cotangent bundle of  $\Delta$ , together imply that

$$\tilde{\theta}^F - \eta^* \tilde{\theta}^F \in H^0(\Sigma, \text{Sym}^2(F) \otimes K_{\Sigma})$$

after we identify  $\Delta$  with  $\Sigma$  using the natural map  $x \mapsto (x, x)$ . Now it is straightforward to check that  $\tilde{\theta}^F - \eta^* \tilde{\theta}^F = \tilde{\nabla}_F(B)$ .  $\blacksquare$

Let

$$\mathbb{H}: Q \otimes (Q \otimes K_{\Sigma}^n) \longrightarrow K_{\Sigma}$$

be the homomorphism defined by

$$Q \otimes (Q \otimes K_{\Sigma}^n) = (Q \otimes (Q \otimes K_{\Sigma}^{\otimes(n-1)})) \otimes K_{\Sigma} \xrightarrow{\mathbb{S}_0^{\vee} \otimes \text{Id}} K_{\Sigma},$$

where  $\mathbb{S}_0^{\vee}$  is the homomorphism in (3.8). Consider  $\mathbb{H}$  as a paring  $\langle -, - \rangle$  between  $Q$  and  $Q \otimes K_{\Sigma}^n$  with values in  $K_{\Sigma}$ . Then it can be shown that the differential operator  $\mathcal{D}$  in (3.9) satisfies the equation

$$\langle \mathcal{D}(s), t \rangle = \langle s, \mathcal{D}(t) \rangle \tag{3.13}$$

for any locally defined holomorphic sections  $s$  and  $t$  of  $Q$ . Indeed, (3.13) is an exact reformulation of (3.12). In other words, the differential operator  $\mathcal{D}$  is “self-adjoint”.

**Remark 3.12.** Given the above defined pairing  $\langle -, - \rangle$  between  $Q$  and  $Q \otimes K_{\Sigma}^n$  with values in  $K_{\Sigma}$ , for any differential operator

$$D_n \in H^0(\Sigma, \text{Diff}_{\Sigma}^n(Q, Q \otimes K_{\Sigma}^n)),$$

the adjoint differential operator  $D_n^*$  defined by the equation

$$\langle D_n(s), t \rangle = \langle s, D_n^*(t) \rangle,$$

where  $s$  and  $t$  are any locally defined holomorphic sections of  $Q$ , is similar to the classical Lagrange adjoint [28]. More precisely, when the rank of  $Q$  is one, the above defined adjoint map coincides with the Lagrange adjoint.

### 3.3 Symplectic and orthogonal opers

In what follows, using the connection  $\nabla$  on  $Q$  from Proposition 3.10, we shall construct a holomorphic isomorphism between the bundles  $J^{n-1}(Q)$  and  $Q \otimes J^{n-1}(K_{\Sigma}^{(1-n)/2})$ , which in turn would enable us to express generalized  $B$ -opers in terms of classical opers together with a fiberwise non-degenerate symmetric bilinear form on  $Q^*$ , and a holomorphic connection on  $Q^*$  that preserves this form.

As before,  $(E, \mathcal{F}, D)$  is a generalized  $B$ -oper on  $\Sigma$ .

Given any point  $x \in \Sigma$  and  $v \in \mathcal{Q}_x$ , let  $\tilde{v}$  be the unique flat section of  $\mathcal{Q}$ , defined around  $x$ , such that  $\tilde{v}(x) = v$ . Then, for each  $k$ , just as the map  $f_k$  in (3.3) is constructed, we have a homomorphism

$$\gamma_{\nabla}^k: \mathcal{Q} \longrightarrow J^k(\mathcal{Q})$$

that sends any  $v \in \mathcal{Q}_x$ ,  $x \in \Sigma$ , to the element of  $J^k(\mathcal{Q})_x$  obtained by restricting the flat section  $\tilde{v}$  to the  $k$ -th order infinitesimal neighborhood of  $x$ . Moreover, we shall denote by

$$\Gamma_{\nabla}^k: \mathcal{Q} \otimes J^k(K_{\Sigma}^{(1-n)/2}) \longrightarrow J^k(Q) \quad (3.14)$$

the homomorphism given by the composition of homomorphisms

$$\mathcal{Q} \otimes J^k(K_{\Sigma}^{(1-n)/2}) \xrightarrow{\gamma_{\nabla}^k \otimes \text{Id}} J^k(\mathcal{Q}) \otimes J^k(K_{\Sigma}^{(1-n)/2}) \longrightarrow J^k(\mathcal{Q} \otimes K_{\Sigma}^{(1-n)/2}) = J^k(Q),$$

where the homomorphism  $J^k(\mathcal{Q}) \otimes J^k(K_{\Sigma}^{(1-n)/2}) \longrightarrow J^k(\mathcal{Q} \otimes K_{\Sigma}^{(1-n)/2})$  is the natural homomorphism  $J^k(A) \otimes J^k(B) \longrightarrow J^k(A \otimes B)$  for any vector bundles  $A, B$  on  $\Sigma$ . For ease of notation, in what follows we denote by  $\mathcal{J}_k$  the jet bundle

$$\mathcal{J}_k := J^{k-1}(K_{\Sigma}^{(1-k)/2}). \quad (3.15)$$

Then, the following lemma is established.

**Lemma 3.13.** *The homomorphism*

$$\Gamma_{\nabla}^k: \mathcal{Q} \otimes J^k(K_{\Sigma}^{(1-n)/2}) \longrightarrow J^k(Q)$$

in (3.14) is an isomorphism for each  $k$ . Hence the isomorphism  $f_{n-1}$  in Theorem 3.5 produces a holomorphic isomorphism of  $E = J^{n-1}(Q)$  with  $\mathcal{Q} \otimes \mathcal{J}_n$ , where  $\mathcal{J}_n$  is defined in (3.15).

**Proof.** We will show that the homomorphism  $\Gamma_{\nabla}^k$  is fiberwise injective. For that, take any point  $x \in \Sigma$ , and consider the restriction of  $\mathcal{Q}$  to a simply connected open neighborhood  $U_x$  of  $x$ . Recall from Proposition 3.10 that  $\mathcal{Q}$  is equipped with a holomorphic connection  $\nabla$ . As mentioned earlier, any holomorphic connection on a Riemann surface is automatically flat. Using this flat connection  $\nabla$  on  $\mathcal{Q}$ , we trivialize the restriction  $\mathcal{Q}|_{U_x}$  of  $\mathcal{Q}$  to  $U_x$  (this is possible because  $U_x$  is simply connected). Since  $\mathcal{Q} \otimes K_{\Sigma}^{(1-n)/2} = Q$  (see (3.6)), using this trivialization of  $\mathcal{Q}|_{U_x}$ , the restriction  $Q|_{U_x}$  of  $Q$  to  $U_x$  gets identified with  $(K_{\Sigma}^{(1-n)/2})^{\oplus r}|_{U_x}$ , where  $r = \text{rank}(Q)$ . Using this identification of  $Q|_{U_x}$  with  $(K_{\Sigma}^{(1-n)/2})^{\oplus r}|_{U_x}$ , the restriction of the homomorphism  $\Gamma_{\nabla}^k$  in (3.14) to  $U_x$  gets identified with the natural homomorphism

$$J^k(K_{\Sigma}^{(1-n)/2})^{\oplus r} \longrightarrow J^k((K_{\Sigma}^{(1-n)/2})^{\oplus r}). \quad (3.16)$$

The homomorphism in (3.16) is evidently fiberwise injective. Consequently,  $\Gamma_{\nabla}^k$  is fiberwise injective. The homomorphism in (3.16) is also fiberwise surjective, using which it can be deduced that  $\Gamma_{\nabla}^k$  is an isomorphism. Alternatively, since

$$\text{rank}(\mathcal{Q} \otimes J^k(K_{\Sigma}^{(1-n)/2})) = (k+1)r = \text{rank}(J^k(Q)),$$

$\Gamma_{\nabla}^k$  is fiberwise surjective, because it is fiberwise injective. ■

**Corollary 3.14.** *For a generalized B-oper  $(E, \mathcal{F}, D)$  on  $\Sigma$ , the holomorphic vector bundle  $E$  is canonically identified with  $\mathcal{Q} \otimes J^{n-1}(K_{\Sigma}^{(1-n)/2})$ .*

**Proof.** This follows from the combination of Theorem 3.5 and Lemma 3.13.  $\blacksquare$

**Lemma 3.15.** *Associated to  $(E, \mathcal{F}, D)$ , there is a natural homomorphism*

$$\tau: \text{Diff}_{\Sigma}^n(Q, Q \otimes K_{\Sigma}^n) \longrightarrow \text{Diff}_{\Sigma}^n(K_{\Sigma}^{(1-n)/2}, K_{\Sigma}^{(n+1)/2}).$$

**Proof.** Combining the isomorphism in Lemma 3.13 for  $k=n$  with the definition of  $\text{Diff}_{\Sigma}^m(W, W')$  in (3.2) for  $m=n$ ,  $W=Q$  and  $W'=Q \otimes K_{\Sigma}^n$ , we have

$$\begin{aligned} \text{Diff}_{\Sigma}^n(Q, Q \otimes K_{\Sigma}^n) &= (Q \otimes K_{\Sigma}^n) \otimes J^n(Q)^* = Q \otimes K_{\Sigma}^n \otimes (\mathcal{Q} \otimes J^n(K_{\Sigma}^{(1-n)/2}))^* \\ &= Q \otimes K_{\Sigma}^n \otimes (Q \otimes K_{\Sigma}^{(n-1)/2} \otimes J^n(K_{\Sigma}^{(1-n)/2}))^* \\ &= \text{Diff}_{\Sigma}^n(K_{\Sigma}^{(1-n)/2}, K_{\Sigma}^{(n+1)/2}) \otimes \text{End}(Q). \end{aligned} \quad (3.17)$$

Using the trace homomorphism

$$\begin{aligned} \text{End}(Q) &\longrightarrow \mathcal{O}_{\Sigma}, \\ s &\longmapsto \frac{1}{\text{rank}(Q)} \text{trace}(s), \end{aligned}$$

the isomorphism in (3.17) gives the homomorphism

$$\tau: \text{Diff}_{\Sigma}^n(Q, Q \otimes K_{\Sigma}^n) \longrightarrow \text{Diff}_{\Sigma}^n(K_{\Sigma}^{(1-n)/2}, K_{\Sigma}^{(n+1)/2})$$

as required.  $\blacksquare$

Although the holomorphic connection  $\nabla$  on  $\mathcal{Q}$  does not explicitly arise in the statement or proof of Lemma 3.15, the homomorphism  $\tau$  in Lemma 3.15 crucially uses  $\nabla$ . Indeed, the isomorphism  $\Gamma_{\nabla}^k$  in Lemma 3.13, whose construction uses  $\nabla$ , is the key ingredient in the construction of  $\tau$  in Lemma 3.15.

Through the map  $\tau$  from Lemma 3.15 we shall see how generalized  $B$ -opers relate to the classical  $\text{Sp}(n, \mathbb{C})$  and  $\text{SO}(n, \mathbb{C})$ -opers as introduced in Section 2.1.

**Proposition 3.16.** *Consider the differential operator  $\mathcal{D}$  in (3.9). For the map  $\tau$  in Lemma 3.15, The differential operator*

$$\tau(\mathcal{D}) \in H^0(\Sigma, \text{Diff}_{\Sigma}^n(K_{\Sigma}^{(1-n)/2}, K_{\Sigma}^{(n+1)/2}))$$

*is a classical  $\text{Sp}(n, \mathbb{C})$ -oper for  $n$  even, and a classical  $\text{SO}(n, \mathbb{C})$ -oper for  $n$  odd.*

**Proof.** Holomorphic differential operators on  $\Sigma$  are given by holomorphic section of suitable bundles on the neighborhood of the diagonal (see [8, p. 25, equation (5.1)] for the precise statement). First we shall describe the homomorphism  $\tau$  in Lemma 3.15 in terms of such sections.

Recall from (3.11) that the differential operator  $\mathcal{D}$  in (3.9) is given by the section

$$\begin{aligned} \kappa &\in H^0((n+1)\Delta, p_1^*(K_{\Sigma}^{(n+1)/2} \otimes \mathcal{Q}) \otimes p_2^*(K_{\Sigma}^{(n+1)/2} \otimes \mathcal{Q}) \otimes \mathcal{O}_{\Sigma \times \Sigma}((n+1)\Delta)) \\ &= H^0((n+1)\Delta, (p_1^*K_{\Sigma}^{(n+1)/2}) \otimes (p_2^*K_{\Sigma}^{(n+1)/2}) \otimes (p_1^*\mathcal{Q}) \otimes (p_2^*\mathcal{Q}) \otimes \mathcal{O}_{\Sigma \times \Sigma}((n+1)\Delta)). \end{aligned}$$

It can be shown that using the flat connection  $\nabla$  on  $\mathcal{Q}$ , the two vector bundles  $p_1^*\mathcal{Q}$  and  $p_2^*\mathcal{Q}$  are identified on some open neighborhood of the diagonal  $\Delta \subset \Sigma \times \Sigma$ . Indeed, take an open subset  $U_{\Delta} \subset \Sigma \times \Sigma$  containing  $\Delta$  such that  $\Delta$  is a deformation retraction of  $U_{\Delta}$ . For  $i=1, 2$ , let  $q_i: U_{\Delta} \longrightarrow \Sigma$  be the projection to the  $i$ -th factor. Consider the two flat bundles

$(q_1^* \mathcal{Q}, q_1^* \nabla)$  and  $(q_2^* \mathcal{Q}, q_2^* \nabla)$  on  $U_\Delta$ . On the diagonal  $\Delta \subset U_\Delta$ , both are evidently identified with the flat bundle  $(\mathcal{Q}, \nabla)$  (once we identify  $\Delta$  with  $\Sigma$  using the map  $x \mapsto (x, x)$ ). Since  $\Delta$  is a deformation retraction of  $U_\Delta$ , this isomorphism between  $(q_1^* \mathcal{Q}, q_1^* \nabla)|_\Delta$  and  $(q_2^* \mathcal{Q}, q_2^* \nabla)|_\Delta$  extends to an isomorphism between  $(q_1^* \mathcal{Q}, q_1^* \nabla)$  and  $(q_2^* \mathcal{Q}, q_2^* \nabla)$  over the entire  $U_\Delta$ . Using this isomorphism between  $q_1^* \mathcal{Q}$  and  $q_2^* \mathcal{Q}$ , the above section  $\kappa$  becomes a section

$$\kappa \in H^0((n+1)\Delta, (p_1^* K_\Sigma^{(n+1)/2}) \otimes (p_2^* K_\Sigma^{(n+1)/2}) \otimes p_1^*(\mathcal{Q} \otimes \mathcal{Q}) \otimes \mathcal{O}_{\Sigma \times \Sigma}((n+1)\Delta)).$$

Now using the symmetric bilinear form  $\mathbb{S}^*$  on  $\mathcal{Q}$  (see (3.7)), the above section  $\kappa$  produces a section

$$\tilde{\kappa} \in H^0((n+1)\Delta, (p_1^* K_\Sigma^{(n+1)/2}) \otimes (p_2^* K_\Sigma^{(n+1)/2}) \otimes \mathcal{O}_{\Sigma \times \Sigma}((n+1)\Delta)).$$

Using the isomorphism in [8, p. 25, equation (5.1)], this section  $\tilde{\kappa}$  produces a holomorphic differential operator

$$\tilde{\mathcal{D}} \in H^0(\Sigma, \text{Diff}_\Sigma^n(K_\Sigma^{(1-n)/2}, K_\Sigma^{(n+1)/2})). \quad (3.18)$$

The differential operator  $\tilde{\mathcal{D}}$  in (3.18) coincides with the differential operator  $\tau(\mathcal{D})$  in the statement of Proposition 3.16.

Let

$$K_\Sigma^{(1-n)/2} \otimes K_\Sigma^{(n+1)/2} \longrightarrow K_\Sigma$$

be the natural homomorphism; we shall denote this  $K_\Sigma$ -valued pairing between  $K_\Sigma^{(1-n)/2}$  and  $K_\Sigma^{(n+1)/2}$  by  $\langle -, - \rangle$ . Now from (3.13) it follows that

$$\langle \tilde{\mathcal{D}}(s), t \rangle = \langle s, \tilde{\mathcal{D}}(t) \rangle \quad (3.19)$$

for locally defined holomorphic sections  $s$  and  $t$  of  $K_\Sigma^{(1-n)/2}$ .

The symbol of the differential operator  $\mathcal{D}$  is  $\text{Id}_\mathcal{Q}$  [8, p. 18] (see the paragraph following (4.6)). From this it follows that the symbol of  $\tilde{\mathcal{D}}$  is the constant function 1. Such a differential operator produces a holomorphic connection on the jet bundle  $J^{n-1}(K_\Sigma^{(1-n)/2})$  (see [7, p. 15, equation (4.3)]; the spaces  $\mathcal{A}$  and  $\mathcal{B}$  in [7, equation (4.3)] are defined in [7, p. 13]). The holomorphic connection on  $J^{n-1}(K_\Sigma^{(1-n)/2})$  given by the differential operator  $\tilde{\mathcal{D}}$  will be denoted by  $\hat{\mathcal{D}}$ .

Consequently, all the three vector bundles in Corollary 3.14, namely  $E$ ,  $\mathcal{Q}$  and  $J^{n-1}(K_\Sigma^{(1-n)/2})$ , are equipped with holomorphic connections. The isomorphism in Corollary 3.14 between  $E$  and  $\mathcal{Q} \otimes J^{n-1}(K_\Sigma^{(1-n)/2})$  is connection preserving. In particular, the above connection  $\hat{\mathcal{D}}$  is an  $\text{SL}(n, \mathbb{C})$ -connection; recall that  $D$  on  $E$  is an  $\text{SL}(nr, \mathbb{C})$ -connection and  $\nabla$  on  $\mathcal{Q}$  is an  $\text{SL}(r, \mathbb{C})$ -connection, where  $r = \text{rank}(\mathcal{Q})$ .

Therefore,  $\tilde{\mathcal{D}}$  defines an  $\text{SL}(n, \mathbb{C})$ -oper on  $\Sigma$ . We recall that the  $\text{Sp}(n, \mathbb{C})$  opers ( $n$  is an even integer) and  $\text{SO}(n, \mathbb{C})$  opers ( $n \geq 5$  is an odd integer) have the following property: Let  $(F, \nabla^F)$  be a holomorphic principal  $\text{Sp}(n, \mathbb{C})$ -bundle, or a holomorphic principal  $\text{SO}(n, \mathbb{C})$ -bundle,  $F$  equipped with a holomorphic connection  $\nabla^F$ , giving such an oper. Consider the holomorphic principal  $\text{SL}(n, \mathbb{C})$ -bundle equipped with a holomorphic connection obtained by extending the structure group of  $(F, \nabla^F)$  using the natural inclusions of  $\text{Sp}(n, \mathbb{C})$  and  $\text{SO}(n, \mathbb{C})$  in  $\text{SL}(n, \mathbb{C})$ . Then this holomorphic principal  $\text{SL}(n, \mathbb{C})$ -bundle equipped with a holomorphic connection is an  $\text{SL}(n, \mathbb{C})$  oper. It should be mentioned that this is not true for  $\text{SO}(2n, \mathbb{C})$ -opers. This is one of the reasons why our method does not apply to the even orthogonal cases, which shall be treated separately in future work [10].

Since  $\tilde{\mathcal{D}}$  satisfies (3.19), the  $\text{SL}(n, \mathbb{C})$ -oper given by it is actually an  $\text{Sp}(n, \mathbb{C})$ -oper when  $n$  is even, and it is an  $\text{SO}(n, \mathbb{C})$ -oper when  $n$  is odd. In other words, the holomorphic connection  $\hat{\mathcal{D}}$

on  $J^{n-1}(K_{\Sigma}^{(1-n)/2})$  defines a  $\mathrm{Sp}(n, \mathbb{C})$ -oper when  $n$  is even, and it defines a  $\mathrm{SO}(n, \mathbb{C})$ -oper when  $n$  is odd. To construct the bilinear form on  $J^{n-1}(K_{\Sigma}^{(1-n)/2})$  for this  $\mathrm{Sp}(n, \mathbb{C})$  or  $\mathrm{SO}(n, \mathbb{C})$  oper structure, consider the isomorphism

$$\mathcal{Q} \otimes J^{n-1}(K_{\Sigma}^{(1-n)/2}) \xrightarrow{\sim} E$$

in Corollary 3.14. This produces an injective homomorphism

$$J^{n-1}(K_{\Sigma}^{(1-n)/2}) \hookrightarrow E \otimes \mathcal{Q}^*.$$

The bilinear form  $B$  on  $E$  and the bilinear form  $\mathbb{S}$  on  $\mathcal{Q}^*$  (see (3.7) and Proposition 3.9) together produce a bilinear form on  $E \otimes \mathcal{Q}^*$ . The bilinear form on  $J^{n-1}(K_{\Sigma}^{(1-n)/2})$  is the restriction of this bilinear form on  $E \otimes \mathcal{Q}^*$ . (See also Remark 4.3.)

Let  $\nabla$  denote the holomorphic connection on  $\mathcal{Q}^*$  induced by the holomorphic connection  $\nabla$  on  $\mathcal{Q}$ . Since the holomorphic connection  $D$  (respectively,  $\nabla^*$ ) on  $E$  (respectively,  $\mathcal{Q}^*$ ) preserves the bilinear form  $B$  (respectively,  $\mathbb{S}$ ) on  $E$  (respectively,  $\mathcal{Q}^*$ ), the holomorphic connection  $\widehat{D}$  on  $J^{n-1}(K_{\Sigma}^{(1-n)/2})$  preserves the bilinear form on  $J^{n-1}(K_{\Sigma}^{(1-n)/2})$  constructed above. ■

From the above results, one has the following:

**Theorem 3.17.** *Let  $(E, \mathcal{F}, D)$  be a generalized B-oper over  $\Sigma$  of filtration length  $n > 0$ . Then, the following three objects are canonically associated to  $(E, \mathcal{F}, D)$ :*

- 1) a fiberwise non-degenerate symmetric bilinear form on  $\mathcal{Q} = E/E_{n-1} \otimes K^{\frac{n-1}{2}}$ ,
- 2) a holomorphic connection on  $\mathcal{Q}$  that preserves this bilinear form, and
- 3) a classical  $\mathrm{Sp}(n, \mathbb{C})$ -oper for  $n$  even and a classical  $\mathrm{SO}(n, \mathbb{C})$ -oper for  $n$  odd.

**Proof.** Consider the symmetric bilinear form  $\mathbb{S}$  on  $\mathcal{Q}^*$  in (3.7). Since  $\mathbb{S}$  is nondegenerate (Proposition 3.9), it produces a nondegenerate symmetric bilinear form on the dual vector bundle  $\mathcal{Q}$ .

Then, from Proposition 3.10, there is a holomorphic connection  $\nabla$  on  $\mathcal{Q}$  which preserves the symmetric bilinear form on  $\mathcal{Q}$  given by  $\mathbb{S}$ .

Finally, the classical  $\mathrm{Sp}(n, \mathbb{C})$ -oper for  $n$  even, and classical  $\mathrm{SO}(n, \mathbb{C})$ -oper for  $n$  odd, are given by the differential operator  $\tau(\mathcal{D})$  in the statement of Proposition 3.16. ■

The correspondence in Theorem 3.17 can be shown to be a one-to-one correspondence, and therefore in the next section we shall prove a converse of Theorem 3.17.

## 4 Generalized $B$ -opers and projective structures

### 4.1 A projective structure from a $B$ -oper

Given a projective structure  $P$  on  $\Sigma$ , from [7, p. 13, equation (3.6)] we have a ( $P$  dependent) decomposition of the space of differential operators

$$H^0(\Sigma, \mathrm{Diff}_{\Sigma}^n(K_{\Sigma}^{(1-n)/2}, K_{\Sigma}^{(n+1)/2})) = \bigoplus_{j=0}^n H^0(\Sigma, K_{\Sigma}^{\otimes j}). \quad (4.1)$$

The component in  $H^0(\Sigma, \mathcal{O}_{\Sigma})$  corresponding to a differential operator

$$D' \in H^0(\Sigma, \mathrm{Diff}_{\Sigma}^n(K_{\Sigma}^{(1-n)/2}, K_{\Sigma}^{(n+1)/2}))$$

is the symbol  $\sigma(D')$  of  $D'$ . Moreover, given a differential operator

$$\mathbf{D} \in H^0(\Sigma, \text{Diff}_\Sigma^n(K_\Sigma^{(1-n)/2}, K_\Sigma^{(n+1)/2})),$$

whose symbol is a nonzero (constant) function, there is a unique projective structure  $P_{\mathbf{D}}$  on  $\Sigma$  such that the component of  $\mathbf{D}$  in  $H^0(\Sigma, K_\Sigma^2)$  vanishes identically for the decomposition of  $H^0(\Sigma, \text{Diff}_\Sigma^n(K_\Sigma^{(1-n)/2}, K_\Sigma^{(n+1)/2}))$  (as in (4.1)) associated to  $P_{\mathbf{D}}$  (see [7, p. 14, equation (3.7)]).

**Definition 4.1.** Consider the differential operator  $\mathcal{D} \in H^0(\Sigma, \text{Diff}_\Sigma^n(Q, Q \otimes K_\Sigma^n))$  in (3.9), and the map  $\tau$  in Lemma 3.15. We shall denote by  $\mathbf{P}$  the unique projective structure corresponding to the differential operator

$$\tau(\mathcal{D}) \in H^0(\Sigma, \text{Diff}_\Sigma^n(K_\Sigma^{(1-n)/2}, K_\Sigma^{(n+1)/2})),$$

for which the component of  $\tau(\mathcal{D})$  in  $H^0(\Sigma, K_\Sigma^2)$  vanishes identically for the decomposition of  $H^0(\Sigma, \text{Diff}_\Sigma^n(K_\Sigma^{(1-n)/2}, K_\Sigma^{(n+1)/2}))$  as in (4.1) associated to  $\mathbf{P}$ .

**Lemma 4.2.** *The projective structure  $\mathbf{P}$  induces a canonical bilinear form*

$$\mathbf{B}_n: \mathcal{J}_n \otimes \mathcal{J}_n \longrightarrow \mathcal{O}_\Sigma$$

on  $\mathcal{J}_n$  (defined in (3.15)) which is orthogonal when  $n$  is odd and symplectic when  $n$  is even.

**Proof.** Any two choices of theta characteristic on  $\Sigma$  differ by tensoring with a holomorphic line bundle of order two on  $\Sigma$ . If  $K^{1/2}$  and  $\mathbb{K}^{1/2} = K^{1/2} \otimes \mathcal{L}$  are two theta characteristics on  $\Sigma$ , where  $\mathcal{L}$  is a holomorphic line bundle of order two, then  $\mathbb{K}_\Sigma^{(1-k)/2} = K_\Sigma^{(1-k)/2} \otimes (\mathcal{L}^*)^{\otimes(k-1)}$ . Therefore, from Corollary 3.8 we conclude that

$$J^{k-1}(\mathbb{K}_\Sigma^{(1-k)/2}) = J^{k-1}(K_\Sigma^{(1-k)/2}) \otimes (\mathcal{L}^*)^{\otimes(k-1)} = \mathcal{J}_k \otimes (\mathcal{L}^*)^{\otimes(k-1)}, \quad (4.2)$$

where  $\mathcal{J}_k$  is defined in (3.15).

In view of (4.2), using [6, p. 10, Theorem 3.7], the projective structure  $\mathbf{P}$  from Definition 4.1 produces a holomorphic isomorphism

$$\beta: \text{Sym}^{n-1}(\mathcal{J}_2) \longrightarrow \mathcal{J}_n. \quad (4.3)$$

We note that

$$\bigwedge^2 \mathcal{J}_2 = K_\Sigma^{-1/2} \otimes K_\Sigma^{-1/2} \otimes K_\Sigma = \mathcal{O}_\Sigma,$$

and hence the fibers of the vector bundle  $\mathcal{J}_2$  are equipped with a symplectic structure. For any  $x \in \Sigma$ , and for any  $v, w$  in the fiber  $(\mathcal{J}_2)_x$  of  $\mathcal{J}_2$  over  $x$ , let

$$\langle v, w \rangle_1 \in \mathbb{C} \quad (4.4)$$

be this symplectic pairing. Note that we have

$$\langle v, w \rangle_1 = -\langle w, v \rangle_1 \quad (4.5)$$

as the pairing  $\langle -, - \rangle_1$  is symplectic.

We will show that the above symplectic structure  $\langle -, - \rangle_1$  in (4.4) on the fibers of  $\mathcal{J}_2$  produces a bilinear form on the fibers of the symmetric product  $\text{Sym}^d(\mathcal{J}_2)$  for every  $d \geq 1$ . For this, take a point  $x \in \Sigma$ , and take

$$v_1, \dots, v_d \in \text{Sym}^d(\mathcal{J}_2)_x \quad \text{and} \quad w_1, \dots, w_d \in \text{Sym}^d(\mathcal{J}_2)_x.$$

Then we have

$$\underline{v} := v_1 \otimes \cdots \otimes v_d \in \text{Sym}^d(\mathcal{J}_2)_x \quad \text{and} \quad \underline{w} := w_1 \otimes \cdots \otimes w_d \in \text{Sym}^d(\mathcal{J}_2)_x.$$

Now define the pairing

$$\langle \underline{v}, \underline{w} \rangle_d := \prod_{i=1}^d \langle v_i, w_i \rangle_1 \in \mathbb{C}, \quad (4.6)$$

where  $\langle -, - \rangle_1$  is the pairing in (4.4). Note that the pairing  $\langle -, - \rangle_d$  in (4.6) coincides with  $\langle -, - \rangle_1$  in (4.4) when  $d = 1$ . It is straightforward to check that  $\langle -, - \rangle_d$  in (4.6) produces a nodegenerate bilinear form on  $\text{Sym}^d(\mathcal{J}_2)_x$ .

Next note that from (4.5) and (4.6) we have

$$\langle \underline{w}, \underline{v} \rangle_d = \prod_{i=1}^d \langle w_i, v_i \rangle_1 = (-1)^d \prod_{i=1}^d \langle v_i, w_i \rangle_1 = (-1)^d \langle \underline{v}, \underline{w} \rangle_d.$$

Therefore, the nodegenerate bilinear form  $\langle -, - \rangle_d$  on  $\text{Sym}^d(\mathcal{J}_2)_x$  is symmetric if  $d$  is even and  $\langle -, - \rangle_d$  is anti-symmetric if  $d$  is odd.

In particular,  $\text{Sym}^{n-1}(\mathcal{J}_2)$  is equipped with the orthogonal (respectively, symplectic) form  $\langle -, - \rangle_{n-1}$  if  $n$  is odd (respectively, even).

For  $n$  odd (respectively, even), using the isomorphism  $\beta$  in (4.3), the orthogonal (respectively, symplectic) form  $\langle -, - \rangle_{n-1}$  on  $\text{Sym}^{n-1}(\mathcal{J}_2)$  produces an orthogonal (respectively, symplectic) structure on the fibers of  $\mathcal{J}_n$ .  $\blacksquare$

**Remark 4.3.** The bilinear form on  $\mathcal{J}_n$  in Lemma 4.2 coincides with the bilinear form on  $\mathcal{J}_n$  constructed in the proof of Proposition 3.16.

The filtration (see Lemma 3.2)

$$0 \longrightarrow K_\Sigma^{1/2} \longrightarrow \mathcal{J}_2 = J^1(K_\Sigma^{-1/2}) \longrightarrow J^0(K_\Sigma^{-1/2}) = K_\Sigma^{-1/2} \longrightarrow 0$$

of  $\mathcal{J}_2$  produces a filtration of  $\text{Sym}^{n-1}(\mathcal{J}_2)$  such that all the successive quotients are of the form  $K_\Sigma^{j/2}$ , for  $1 - n \leq j \leq n - 1$ .

**Lemma 4.4.** *The holomorphic isomorphism  $\beta$  in (4.3) takes the above filtration of  $\text{Sym}^{n-1}(\mathcal{J}_2)$  to the filtration of  $\mathcal{J}_n$  constructed in (3.4).*

**Proof.** This is straightforward. When  $\text{genus}(\Sigma) \geq 2$ , both these filtrations coincide with the Harder–Narasimhan filtration of  $\text{Sym}^{n-1}(\mathcal{J}_2) = \mathcal{J}_n$ .  $\blacksquare$

## 4.2 Construction of $B$ -opers

In order to build a converse statement to that of Theorem 3.17, let  $W$  be a holomorphic vector bundle on  $\Sigma$  equipped with a symmetric fiberwise non-degenerate bilinear form

$$\mathbb{S}_W: \text{Sym}^2(W) \longrightarrow \mathcal{O}_\Sigma. \quad (4.7)$$

Moreover, consider a holomorphic connection  $\nabla^W$  on  $W$  that preserves the bilinear form  $\mathbb{S}_W$ . For an integer  $n \geq 2$ , we shall let  $\omega$  be an  $\text{Sp}(n, \mathbb{C})$ -oper (respectively,  $\text{SO}(n, \mathbb{C})$ -oper) on  $\Sigma$  if  $n$  is even (respectively, odd), which defines a holomorphic connection  $\nabla^\omega$  on the holomorphic vector bundle  $\mathcal{J}_n$  as defined in (3.15). Then, one can show the following.

**Proposition 4.5.** *The holomorphic connection*

$$(\nabla^\omega \otimes \text{Id}_W) \oplus (\text{Id}_{\mathcal{J}_n} \otimes \nabla^W)$$

on  $\mathcal{J}_n \otimes W$ , induced by  $\nabla^\omega$  and  $\nabla^W$ , produces a generalized  $B$ -oper on  $\Sigma$ .

**Proof.** Consider the filtration of holomorphic subbundles on  $\mathcal{J}_n$  given by

$$0 = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_{n-1} \subset A_n = \mathcal{J}_n,$$

where  $A_i$  is the kernel of the natural projection  $\mathcal{J}_n \longrightarrow J^{n-i-1}(K_\Sigma^{(1-n)/2})$  (similar to the filtration in (3.4)). By tensoring with  $W$ , the above filtration produces a filtration of holomorphic subbundles of  $\mathcal{J}_n \otimes W$  given by

$$0 = A_0 \otimes W \subset A_1 \otimes W \subset A_2 \otimes W \subset \cdots \subset A_{n-1} \otimes W \subset A_n \otimes W = \mathcal{J}_n \otimes W. \quad (4.8)$$

From [7, p. 13, equation (3.4)], the oper  $\omega$  produces a holomorphic differential operator of order  $n$  from  $K_\Sigma^{(1-n)/2}$  to  $K_\Sigma^{(n+1)/2}$ , and from [7, p. 14, equation (3.7)] this differential operator produces a projective structure on  $\Sigma$ . Then, as in (4.3), from [6, p. 10, Theorem 3.7] this projective structure produces a holomorphic isomorphic between  $\text{Sym}^{n-1}(\mathcal{J}_2)$  and  $\mathcal{J}_n$ . Moreover, as observed in the proof of Lemma 4.2, the vector bundle  $\text{Sym}^{n-1}(\mathcal{J}_2)$  is equipped with an orthogonal (respectively, symplectic) form if the integer  $n$  is odd (respectively, even) and thus we get a non-degenerate bilinear form  $\mathbb{S}_\omega$  on  $\mathcal{J}_n$  via the above isomorphism.

The form  $\mathbb{S}_\omega$  and the form  $\mathbb{S}_W$  in (4.7) together define a bilinear form on  $\mathcal{J}_n \otimes W$ , which we shall denote by  $\mathbb{S}_0$ . Moreover, one can see that the holomorphic connection  $(\nabla^\omega \otimes \text{Id}_W) \oplus (\text{Id}_{\mathcal{J}_n} \otimes \nabla^W)$  on  $\mathcal{J}_n \otimes W$  preserves  $\mathbb{S}_0$ , because  $\nabla^W$  preserves  $\mathbb{S}_W$  and  $\nabla^\omega$  preserves  $\mathbb{S}_\omega$ . Then, it is straightforward to check that the triple

$$(\mathcal{J}_n \otimes W, \mathbb{S}_0, (\nabla^\omega \otimes \text{Id}_W) \oplus (\text{Id}_{\mathcal{J}_n} \otimes \nabla^W))$$

and the filtration in (4.8) together define a generalized  $B$ -oper on  $\Sigma$ , completing the proof. ■

The constructions in Theorem 3.17 and Proposition 4.5 are evidently inverses of each other, and can be further understood in terms of projective structures on the Riemann surface  $\Sigma$ . For this, let  $\mathfrak{P}(\Sigma)$  denote the space of all projective structures on the Riemann surface  $\Sigma$ . We recall that  $\mathfrak{P}(\Sigma)$  is an affine space for  $H^0(\Sigma, K_\Sigma^{\otimes 2})$  (see [24]).

In the case of classical opers, from [7, p. 17, Theorem 4.9] and [7, p. 19, equation (5.4)], the space of  $\text{SL}(n, \mathbb{C})$ -opers on  $\Sigma$  is in bijection with

$$\mathfrak{P}(\Sigma) \times \left( \bigoplus_{i=3}^n H^0(\Sigma, K_\Sigma^{\otimes i}) \right).$$

Note that the above description of  $\text{SL}(n, \mathbb{C})$ -opers allows one to relate them with the Hitchin base of the  $\text{SL}(n, \mathbb{C})$ -Hitchin fibration introduced in [27].

When the integer  $n$  is even (respectively, odd) the subclass of  $\text{Sp}(n, \mathbb{C})$ -opers (respectively,  $\text{SO}(n, \mathbb{C})$ -opers) corresponds to the subspace

$$\mathfrak{P}(\Sigma) \times \left( \bigoplus_{i=2}^{[n/2]} H^0(\Sigma, K_\Sigma^{\otimes 2i}) \right) \subset \mathfrak{P}(\Sigma) \times \left( \bigoplus_{i=3}^n H^0(\Sigma, K_\Sigma^{\otimes i}) \right).$$

In the case of generalized  $B$ -opers, combining Theorem 3.17 and Proposition 4.5, we have the following equivalent result.

**Theorem 4.6.** *For integers  $n \geq 2$ ,  $n \neq 3$  and  $r \geq 1$ , the space of all generalized B-oper of filtration length  $n$  and  $\text{rank}(E_i/E_{i-1}) = r$  is in correspondence with*

$$\mathcal{C}_\Sigma \times \mathfrak{P}(\Sigma) \times \left( \bigoplus_{i=2}^{[n/2]} H^0(\Sigma, K_\Sigma^{\otimes 2i}) \right),$$

where  $\mathcal{C}_\Sigma$  denotes the space all flat orthogonal bundles of rank  $r$  on  $\Sigma$ , which is independent of  $i$ , and  $\mathfrak{P}(\Sigma)$  is the space of all projective structures on  $\Sigma$ .

In Theorem 4.6, the case of  $n = 3$  is excluded because  $\text{SO}(3, \mathbb{C}) = \text{Sp}(2, \mathbb{C})/(\mathbb{Z}/2\mathbb{Z})$ .

The summation in Theorem 4.6 should be clarified – the summation is from 2 to  $[n/2]$  in increasing order. So  $(\bigoplus_{i=2}^{[n/2]} H^0(\Sigma, K_\Sigma^{\otimes 2i})) = 0$  when  $n = 2$ .

### 4.3 Generalized B-oper on jet bundles

Consider now a holomorphic vector bundle  $W$  on  $\Sigma$  of rank  $n$  equipped with a fiberwise non-degenerate symmetric pairing

$$\nu: W \otimes W \longrightarrow K_\Sigma^{1-n}.$$

As before, using  $\nu$  we have a non-degenerate pairing  $\langle -, - \rangle$  between  $W$  and  $W \otimes K_\Sigma^n$  with values in  $K_\Sigma$ . The jet bundle  $J^{n-1}(W)$  has a filtration

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_{n-1} \subset \mathcal{S}_n := J^{n-1}(W),$$

which is constructed as done in (3.4) using the natural projections  $J^{n-1}(W) \longrightarrow J^{n-1-i}(W)$ . More precisely,  $\mathcal{S}_i$  is the kernel of the projection  $J^{n-1}(W) \longrightarrow J^{n-1-i}(W)$ .

Given a holomorphic differential operator

$$\mathbb{D} \in H^0(\Sigma, \text{Diff}_\Sigma^n(W, W \otimes K_\Sigma^n)),$$

its symbol  $\sigma(\mathbb{D})$  as in Definition 3.4 is a holomorphic section of

$$(T\Sigma)^{\otimes n} \otimes \text{Hom}(W, W \otimes K_\Sigma^n) = \text{End}(W).$$

**Definition 4.7.** We define the differential operator

$$\mathbb{D}^* \in H^0(\Sigma, \text{Diff}_\Sigma^n(W, W \otimes K_\Sigma^n))$$

by the equation  $\langle \mathbb{D}(s), t \rangle = \langle s, \mathbb{D}^*(t) \rangle$ , where  $s$  and  $t$  are any locally defined holomorphic sections of the vector bundle  $W$ , and  $\langle -, - \rangle$  is the  $K_\Sigma$  valued pairing between  $W$  and  $W \otimes K_\Sigma^n$  mentioned earlier.

From this definition one can obtain a correspondence between differential operators  $\mathbb{D}$  as above and generalized B-oper structures on the jet bundle  $J^{n-1}(W)$ .

**Proposition 4.8.** *The differential operator  $\mathbb{D}$  defines a generalized B-oper structure on the vector bundle  $J^{n-1}(W)$  equipped with the filtration constructed as in (3.4) if*

- $\sigma(\mathbb{D}) = \text{Id}_W$ , and
- $\mathbb{D}^* = \mathbb{D}$ .

**Proof.** In (3.13) we saw that the holomorphic differential operator associated to a generalized  $B$ -oper satisfies the above condition  $\mathbb{D}^* = \mathbb{D}$ . Also, it was noted in the proof of Proposition 3.16 that symbol of the operator is  $\text{Id}$  (see [8, p. 18]).

Consider the subset of  $H^0(\Sigma, \text{Diff}_\Sigma^n(W, W \otimes K_\Sigma^n))$  defined by all differential operators whose symbol is  $\text{Id}_W$ . From [7, p. 17, Theorem 4.9] and [7, p. 19, equation (5.4)], this space is in bijection with

$$\mathfrak{P}(\Sigma) \times \left( \bigoplus_{i=3}^n H^0(\Sigma, K_\Sigma^{\otimes i}) \right).$$

Furthermore, a holomorphic differential  $D$  lying in this subset satisfies the equation  $D^* = D$  if the component of  $D$  in  $H^0(\Sigma, K_\Sigma^{\otimes(2j+1)})$  vanishes for all  $1 \leq j \leq \lfloor (n-1)/2 \rfloor$ , from which the proposition follows.  $\blacksquare$

## 5 Concluding remarks: generalized $B$ -opers and Higgs bundles

For those having worked with Higgs bundles, introduced in [26] as solutions of the so-called Hitchin equations on  $\Sigma$ , there is proximity between some aspects of  $B$ -opers and Higgs bundles on  $\Sigma$ , which are pairs  $(E, \Phi)$  where

- $E$  is a holomorphic vector bundle on  $\Sigma$ ,
- the Higgs field  $\Phi: E \rightarrow E \otimes K$ , is a holomorphic  $K$ -valued endomorphism.

To illustrate this, we shall focus on  $B$ -opers of even rank where  $B$  is anti-symmetric, which shall naturally lead to symplectic Higgs bundles. For further details on Higgs bundles, the reader may refer to standard references such as Hitchin [26, 27] and Simpson [32, 33, 34]. Finally, we should mention that generalized  $B$ -opers carry particularly interesting properties when the length of the filtration is 2, a case which leads to an intermediate generalization of opers with short filtrations, whose study is carried on in more detail in the third author's PhD thesis [36]. In what follows, we shall describe some aspects initiating the program. For this, some properties of determinant bundles shall be of much use, and thus we will give a brief description of them first.

We want to find a formula for the determinant bundle for each subbundle in the filtration of a generalized  $B$ -oper. The proof to the next proposition follows the idea given by Wentworth in [35, Lemma 4.9].

**Proposition 5.1.** *Suppose  $(E, \{E_i\}, D)$  is a generalized  $B$ -oper of filtration length  $n$  and each associated graded piece has rank  $r = \text{rank}((E_i/E_{i-1})$ . Then there are smooth isomorphisms*

$$\det(E_i) \simeq \det(Q)^i \otimes K_\Sigma^{r(ni-i(i+1)/2)}, \quad (5.1)$$

where  $Q = E/E_{n-1}$ . In particular, when the  $B$ -oper is a complete flag, we have

$$\det(E_i) \simeq Q^i \otimes K_\Sigma^{ni-i(i+1)/2}.$$

**Proof.** The composition of second fundamental forms  $S_j(D) \circ \dots \circ S_{n-1}(D)$  gives an isomorphism for each associated graded piece:

$$E_j/E_{j-1} \simeq E_n/E_{n-1} \otimes K_\Sigma^{n-j} = Q \otimes K_\Sigma^{n-j}, \quad \forall j = 1, 2, \dots, n.$$

Combining these isomorphisms with the fact that there is a smooth decomposition of  $E$  into associated graded pieces one gets the following isomorphism:

$$E_i \simeq (Q \otimes K_\Sigma^{n-1}) \oplus \dots \oplus (Q \otimes K_\Sigma^{n-i}).$$

Since  $E_i$  has rank  $ri$ , the determinant bundle of  $E_i$  is given by

$$\bigwedge^{ri} E_i \simeq \bigotimes_{j=1}^i \bigwedge^r (Q \otimes K_\Sigma^{n-j}) \simeq \det(Q)^i \otimes K_\Sigma^{r(ni-i(i+1)/2)},$$

where we have used the exterior product isomorphisms  $\wedge^k(V \oplus W) \simeq \wedge^k V \otimes \wedge^k W$  and  $\wedge^k(V \otimes W_1) \simeq \wedge^k V \otimes W_1^{\otimes k}$ , for  $W_1$  a one dimensional vector space.  $\blacksquare$

### 5.1 Generalized $B$ -opers and filtrations of length $n = 2$

From their construction, given an anti-symmetric form  $B$  and a rank  $2r$  holomorphic bundle  $E$ , from Definition 2.11 a generalized  $B$ -oper  $(E, \mathcal{F}, D)$  naturally defines a Lagrangian subspace  $E_1|_x \subset E|_x$  for each  $x \in \Sigma$  and thus a Lagrangian subbundle  $E_1 \subset E$ .

**Proposition 5.2.** *Let  $(E, E_1, D)$  be a generalized  $B$ -oper of rank  $2r$  and filtration length 2. Then there is a rank  $2r$  Higgs bundle  $(E, \Phi)$ , where the Higgs field is induced by*

$$S_1(D): E_1 \longrightarrow E/E_1 \otimes K_\Sigma.$$

In particular, when  $r = 1$ , the Higgs bundle is  $(K^{1/2} \oplus K^{-1/2}, \Phi)$ .

**Proof.** Note that  $S_1(D)$  induces an isomorphism  $E_1 \rightarrow (E/E_1) \otimes K_\Sigma = E_1^* \otimes K_\Sigma$ , because the definition of a  $B$ -filtration gives  $E/E_1 \simeq E_1^*$ . Then,  $E_1 \oplus (E/E_1) \simeq E_1 \oplus E_1^*$  has the induced Higgs field

$$\Phi = \begin{bmatrix} 0 & 0 \\ S_1(D) & 0 \end{bmatrix}: E_1 \oplus (E/E_1) \longrightarrow (E_1 \oplus (E/E_1)) \otimes K_\Sigma.$$

When  $r = 1$  we have  $E_1$  is just a line bundle over  $\Sigma$ . Then the isomorphism given by  $S_1(D)$  implies  $E_1^2 \simeq K_\Sigma$ . Hence  $E_1 \simeq K_\Sigma^{1/2}$  is a choice of theta characteristic, and  $E \simeq K_\Sigma^{1/2} \oplus K_\Sigma^{-1/2}$ .  $\blacksquare$

### 5.2 Other induced Higgs bundles

We shall finally consider the appearance of Higgs bundles through generalized  $B$ -opers with flags of length bigger than 2. In this case, one has the following Higgs bundles constructed in a natural way from generalized  $B$ -opers.

**Proposition 5.3.** *On a compact connected Riemann surface of genus  $g \geq 2$ , a generalized  $B$ -oper  $(E, \{E_i\}, D)$  of filtration length  $n$  and associated graded rank  $r = \text{rank}((E_i/E_{i-1})$  induces a naturally defined stable Higgs bundle  $(E, \Phi)$  given by*

$$E = \bigoplus_{i=1}^n E_i/E_{i-1} \quad \text{and} \quad \Phi = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ S_1(D) & 0 & 0 & 0 & 0 \\ 0 & S_2(D) & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & S_{n-1}(D) & 0 \end{bmatrix}: E \longrightarrow E \otimes K_\Sigma,$$

where  $S_i(D): E_i/E_{i-1} \longrightarrow E_{i+1}/E_i \otimes K_\Sigma$  are the second fundamental forms with respect to  $D$ .

**Proof.** We need to check that  $(E, \Phi)$  is a stable Higgs bundle. From the decomposition of  $E$ , we see that  $E/E_{n-1}$  is the only  $\Phi$ -invariant subbundle, because  $\Phi(E/E_{n-1}) = 0 \subset E/E_{n-1} \otimes K$ .

So according to the slope stability definition, we know  $(E, \Phi)$  is stable if  $\mu(E) - \mu(E/E_{n-1}) > 0$ . Write  $Q = E/E_{n-1}$  and apply equation (5.1) to compute the degree of  $E = E_n$ :

$$\begin{aligned}\mu(E) - \mu(E/E_{n-1}) &= \frac{\deg(E_n)}{\text{rank}(E_n)} - \frac{\deg(Q)}{\text{rank}(Q)} = \frac{n \deg(Q) + rn(n-1)(g-1)}{rn} - \frac{\deg(Q)}{r} \\ &= (n-1)(g-1).\end{aligned}$$

Thus for  $n > 1$  and  $g > 1$ , the Higgs bundle  $(E, \Phi)$  is always stable.  $\blacksquare$

**Remark 5.4.** One should note that all of the Higgs bundles induced by  $B$ -opers we constructed in this section lie in the nilpotent cone in the Hitchin fibration, because the characteristic polynomial of  $\Phi$  is  $\det(xI - \Phi) = x^n$ .

### 5.2.1 The rank 4 case

We shall conclude here with some further comments on the lowest possible rank of generalized  $B$ -opers which have a symmetric bilinear form  $B$ . For this, consider now a  $B$ -filtration of a holomorphic vector bundle  $E$  of rank 4 on  $\Sigma$ , which is given by an increasing filtration of holomorphic subbundles

$$0 = E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq E_3 \subsetneq E_4 = E \tag{5.2}$$

for which  $E_i^\perp = E_{4-i}$  for all  $1 \leq i \leq 3$ . Then, from Definition 2.11, we have that a generalized  $B$ -oper is a triple  $(E, \mathcal{F}, D)$ , where  $\mathcal{F}$  is a  $B$ -filtration as in (5.2) and  $D$  is a  $B$ -connection on  $E$ , such that  $D(E_i) \subset E_{i+1} \otimes K_\Sigma$  for all  $1 \leq i \leq 3$ , and the homomorphisms

$$\begin{aligned}S_1(D): E_1 &\longrightarrow (E_2/E_1) \otimes K_\Sigma, \\ S_2(D): E_2/E_1 &\longrightarrow (E_3/E_2) \otimes K_\Sigma, \\ S_3(D): E_3/E_2 &\longrightarrow (E/E_3) \otimes K_\Sigma\end{aligned}$$

are isomorphisms. Through the bilinear form  $B$  there is an isomorphism  $E_1^* \xrightarrow{\sim} E/E_3$ , and in this case one has that  $E_2 = E_2^\perp$  and thus we obtain again a Lagrangian sub-fibration. Hence, a rank 4 generalized  $B$ -oper  $(E, E_1, D)$  induces the following naturally defined Higgs bundle  $(P, \Phi)$ , where  $P = E_3$  and  $\Phi: E_3 \rightarrow E_3 \otimes K$  is induced by

$$S_1 \oplus S_2 \oplus S_3: E_1 \oplus E_2/E_1 \oplus E_3/E_2 \longrightarrow (E_2/E_1 \otimes K) \oplus (E_3/E_2 \otimes K) \oplus (E/E_3 \otimes K),$$

since  $E_1 \oplus E_2/E_1 \oplus E_3/E_2 \simeq E_3$  and  $(E_2/E_1 \otimes K) \oplus (E_3/E_2 \otimes K) \oplus (E/E_3 \otimes K) \simeq E_3 \otimes K$ .

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