A CONTINUATION PRINCIPLE FOR PERIODIC BV-CONTINUOUS STATE-DEPENDENT SWEEPING PROCESSES*

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Abstract. We consider a Caratheodory differential equation with a state-dependent convex constraint that changes BV-continuously in time (a perturbed BV-continuous state-dependent sweeping processes). By setting up an appropriate catching-up algorithm we prove solvability of the initial value problem. Then, for sweeping processes with T-periodic right-hand-sides, we prove the existence of at least one T-periodic solution. Finally, we investigate a T-periodic sweeping process which is close to an autonomous sweeping process with a constant constraint and prove the existence of a T-periodic solution specifically located near the boundary switched equilibrium of the autonomous sweeping process.

Key words. Sweeping process, perturbation theory, continuation principle, periodic solution, BV-continuous state-dependent convex constraint

AMS subject classifications. 47H11; 70H45; 26B30; 34A60

1. Introduction. A variety of applications in elastoplasticity, economics, electrical circuits (see Adly et al [1, 2] and references therein) lead to a constrained differential equation

$$-\dot{x}(t) \in N_{A(t)}(x(t)) + f(t, x(t)), \qquad x \in E, \tag{1.1}$$

with a convex moving set $t \mapsto A(t)$ of just bounded variation (with respect to the Hausdorff metric). Here E is a finite-dimensional vector space and $N_A(x)$ is a so-called normal cone defined for closed convex $A \subset E$ as

$$N_A(x) = \begin{cases} \{\xi \in E : \langle \xi, c - x\rangle \le 0, \text{ for any } c \in A\}, & \text{if } x \in A, \\ \emptyset, & \text{if } x \notin A. \end{cases} \tag{1.2}$$

Whereas the case of Lipschitz $t \mapsto A(t)$ always leads (Edmond-Thibault [15]) to the existence and uniqueness of an absolutely continuous solution $x(t)$ for any initial condition (under natural assumptions on f), the case where $t \mapsto A(t)$ is a convex-valued function of bounded variation doesn't ensure solvability of (1.1) in the class of absolutely continuous functions. That is why an extended concept of the derivative (called Radon-Nikodym concept) is required in (1.1) when the map $t \mapsto A(t)$ is a function of bounded variation, in which case equation (1.1) is usually formulated in terms of differential measure dx of BV-continuous function x and Lebesgue measure dt as

$$-dx \in N_{A(t)}(x) + f(t, x)dt, \quad x \in E. \tag{1.3}$$

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The formulation (1.3) follows Edmond and Thibault [15]. A formulation of a slightly different form $-Dx \in N_{A(t)}(x) + f(t,x)$ is used in Castaing and Monteiro Marques [10]. Existence and uniqueness of solutions to (1.3) as well as the existence of periodic solutions has been established by Castaing and Monteiro Marques in [10]. We also note that [10] is probably the only paper that addresses BV periodic solutions in sweeping processes for measures (compared to a relatively intensively studied case of absolutely continuous periodic solutions, see [3, 23, 22, 4, 19, 20, 30, 37] or [8], that can be used to design tracking to a periodic solution). The problem of existence and uniqueness of solutions in the unperturbed case ($f \equiv 0$) was addressed in Moreau [34], Monteiro Marques [32], Valadier [41]. Further state-independent extensions of (1.3) were considered in Adly et al [2], Edmond-Thibault [15], Colombo and Monteiro Marques [11].

Motivated by applications to elastoplastic models with hardening and softening (see e.g. Francfort-Stefanelli [17], Kunze [23]), the goal of the present paper is to investigate the existence of T-periodic solutions to the following state-dependent version of (1.3)

$$-dx \in N_{A(t,x)}(x) + f(t,x)dt, \quad x \in E, \tag{1.4}$$

whose right-hand-side is T-periodic in t. To achieve this goal, we use topological degree methods. Our main idea is in establishing a nondegenerate deformation between sweeping process (1.4) and a simpler sweeping process, for which certain topological characteristics (topological degree) are readily available. Such a strategy is known as *continuation principle* in the literature, see e.g. Capietto et al [9] in references therein. To implement the proposed strategy, the continuity of solutions of (1.4) with respect to both initial conditions and small deformations is required. Important results on the existence of solutions to (1.4) have been achieved in Kunze and Monteiro Marques [25], Nacry [36] (that also contain interesting ideas towards continuity), but continuity results are not explicitly stated. Furthermore, the analysis in [25, 36] assumes $A(t,x)$ Lipschitz-continuous in both the variables, so the solutions of (1.4) are absolutely continuous and don't require any Radon-Nikodym derivative concepts.

That is why a significant part of the paper is devoted to developing a solution approximation scheme for (1.4) (called *catching-up algorithm*), whose individual approximations satisfy the requirement of continuous dependence on initial conditions and parameter perturbation. As explained in Remark 6.5, we can establish such a catching-up algorithm only for sweeping process (1.4) of the following particular form

$$-dx \in N_{A+a(t)+c(x)}(x) + f(t,x)dt, \quad x \in E, \tag{1.5}$$

where a is a BV-continuous function and $c : E \to E$ is a Lipschitz function. The existence of T-periodic solutions to (1.5) is then proved by showing that

$$d(I - P^n, Q) \neq 0 \tag{1.6}$$

for the Poincaré maps $P^n : Q \to E$ of the n-th approximation of the catching-up scheme and suitable $Q \subset E$. Here $d(I - P^n, Q)$ is the topological degree of the map P^n with respect to an open bounded set Q (see Krasnoselskii-Zabreiko [29]). After we get the existence of a fixed point for P^n we pass to the limit as $n \to \infty$ on the respective T-periodic solutions of sweeping process (1.5) and get the existence of a T-periodic solution to (1.5) even though we don't know whether $\lim_{n\to\infty} P^n(x)$ exists or not.

The paper offers both global and local sufficient conditions to ensure (1.6). The global sufficient condition is based on construction of such a convex set Q which contains all possible values of the set $A + a(t) + c(x(t))$ for all possible solutions of (1.5). In this way, we can show that $P^n(Q) \subset \overline{Q}$ for sufficiently large $n \in \mathbb{N}$, which ensures (1.6) (here $P^n(Q)$ stays for the image of Q under the action of the map P^n and $\overline{Q}$ stays for the closure of Q). Global sufficient conditions don't need constructing any nondegenerate transformation of sweeping process (1.5) (i.e. doesn't require any continuation principle).

A continuation principle is required to discover local sufficient conditions ensuring (1.6). To design sufficient conditions that ensure the validity of (1.6) in a desired region Q (called *local sufficient conditions*), we are no longer allowed to enlarge Q as much as we want, so we have to seek for alternative deformations of (1.6) that stick to the given region Q. We go here by a continuation approach and replace (1.5) by a parameter dependent sweeping process

$$-dx \in N_{A+a(t,\lambda)+c(x,\lambda)}(x) + f(t,x,\lambda)dt, \quad x \in E, \ \lambda \in \mathbb{R}. \tag{1.7}$$

Accordingly, we don't deal with the relation (1.6) but rather replace it by

$$d(I - P^{\lambda,n}, Q) \neq 0, \tag{1.8}$$

where $P^{\lambda,n} : Q \to E$ is the Poincaré map of the n-th approximation of the catching-up scheme for sweeping process (1.7) (the formal definition of $P^{\lambda,n}$ is given in Section 7.1).

We, therefore, assume that (1.6) corresponds to (1.8) with some $\lambda = \lambda_1$ and prove the validity of (1.8) for $\lambda = \lambda_1$ building upon some good properties of $P^{\lambda,n}$ for $\lambda = 0$ that guarantee

$$d(I - P^{0,n}, Q) \neq 0 \tag{1.9}$$

combined with nondegenerate homotopy between $P^{\lambda_1,n}$ and $P^{0,n}$.

To ensure (1.9), we offer both topological (Theorem 5.1) and algebraic (Theorem 8.6) conditions. The topological condition simply assumes that (1.8) holds for $n = \infty$, that leads to an analogue of standard continuation principles available for ordinary differential equations, see e.g. Capietto et al [9] and Kamenskii et al [21].

To obtain easily verifiable algebraic conditions ensuring (1.9), this paper takes a straightforward route and offers sufficient conditions for asymptotic stability of a point x_0 of the target set Q. Such an approach is based on the fact that the topological degree of a Poincaré map in the neighborhood of an asymptotically stable fixed point equals 1. However, considering x_0 to be an asymptotically stable equilibrium of (1.7) with $\lambda = 0$ that doesn't interact with the boundary of the constraint is not of interest. Periodic solutions generated by such an equilibrium when λ changes from $\lambda = 0$ to $\lambda > 0$ will simply be solutions of the differential equation

$$-dx = f(t,x,\lambda)dt, \quad x \in E, \ \lambda \in \mathbb{R}. \tag{1.10}$$

Considering x_0 to be an equilibrium of (1.10) that belongs to the boundary of $A+c(x_0)$ is structurally unstable. Small deviation of λ from $\lambda = 0$ to $\lambda > 0$ may include a perturbation that moves the equilibrium towards the interior of the constraint in which case the periodic solutions obtained will again be solutions of (1.10) rather than solutions of (1.7).

That is why a non-equilibrium concept of an asymptotically stable point x_0 is required to design periodic solutions of (1.7) which are intrinsically sweeping (i.e. interact with the boundary of the constraint of (1.7)).

The required concept of asymptotically stable point x_0 has been recently developed in Kamenskii-Makarenkov [20] based on the notion of switched boundary equilibrium well known in control theory (see e.g. Bolzern-Spinelli [6]). To introduce the concept of switched boundary equilibrium for sweeping process (1.7) at $\lambda = 0$, we will assume that, at $\lambda = 0$, sweeping process (1.7) takes the form

$$-\dot{x} \in N_A(x) + f_0(x), \qquad x \in E, \tag{1.11}$$

where A is just a constant convex closed bounded set and f_0 is Lipschitz continuous. Following Kamenskii-Makarenkov [20], x_0 is a switched boundary equilibrium of (1.11), if $f_0(x_0)$ is normal to the boundary ∂A of A and if $f_0(x_0)$ points inwards A. To prove the validity of (1.8) for $\lambda = 0$, we use a result by Krasnoselskii-Zabreiko [29] which allows to deduce the value of the topological degree of a Poincaré map from asymptotic stability of a fixed point of this map. To this end, we offer sufficient conditions for asymptotic stability of switched boundary equilibrium x_0 of (1.11). Such a result (Theorem 8.5) is established in the present paper for the first time ever.

The paper is organized as follows. In the next section of the paper we introduce a formal definition of sweeping process (1.7) following Castaing and Monteiro Marques [10]. The fundamental result of the paper (Theorem 3.1) on solvability of the initial-value problem for (1.5) (and, therefore, for (1.7) too) is formulated in Section 3. In the same section we introduce our concept of *generalized initial condition* that we repeatedly use in the paper later and which allows us to consider solutions of (1.5) with initial conditions outside of $A+a(t)+c(x)$. We simply say that $x(t)$ is a solution of (1.5) with a generalized initial condition $q \in E$, if $x(0)$ is the solution of the equation $x(0) = \mathrm{proj}(q, A + a(0) + c(x(0)))$, which has a unique solution $x(0)$ according to Lemma 6.3 (this unique solution is denoted by $V^0(q)$ in the sequel).

Sections 4 and 5 contain formulations of our results on the existence of T-periodic solutions to (1.5). Section 4 offers a theorem (Theorem 4.1) saying that any T-periodic state-dependent sweeping process (1.5) always admits at least one T-periodic solution, if the right-hand-sides of (1.5) are T-periodic. Remarkably, the theorem doesn't assume uniqueness or continuous dependence of solutions of (1.5) on initial conditions.

Abstract results on continuation of T-periodic solutions to (1.7) are presented in Section 5. We assume that for $\lambda = 0$ the sweeping process (1.7) admits a Poincaré map P^0 (over time T) and formulate (Theorem 5.1) a standard continuation principle: *if the topological degree $d(I - P^0 \circ V^0, Q) \neq 0$ for some open bounded set $Q \subset E$ and if none points of the boundary of Q are initial conditions of T-periodic solutions of sweeping process (1.7) for any $\lambda \in [0, \lambda_1]$ (non-degenerate deformation), then, for any $\lambda \in [0, \lambda_1]$, sweeping process (1.7) admits a T-periodic solution x.* A result on the existence of $\lambda_1 > 0$ such that the non-degenerate deformation assumption of Theorem 5.1 holds is also presented (Theorem 5.2) in Section 5.

Section 6 contains proofs of Theorems 3.1-5.2. The proof of the existence of solutions is based on introducing (section 6.2) an implicit catching-up scheme (6.4)-(6.7), which in turn relies on the following two ideas: **(i)** Castaing and Monteiro Marques change of the variables [10, Theorem 4.1] that converts (section 6.1) the perturbed sweeping process (1.7) with differential measure dx into a non-perturbed sweeping process (6.2) for the derivative $\frac{du}{|du|}$ with respect to the variation measure $|du|$ of du; **(ii)** Kunze and Monteiro Marques lemma ([23, Lemma 7]) to resolve (Lemma 6.3) the implicit catching-up scheme (6.4)-(6.7) with respect to the implicit variable. Furthermore, our Lemma 6.3 extends [23, Lemma 7] by proving continuous dependence of scheme (6.4)-(6.7) on initial condition, that gave us continuity of Poincaré maps $P^{\lambda,n}$ (section 7.1). The convergence of the scheme (6.4)-(6.7) is established in section 6.3 where we prove (Lemma 6.7) convergence of the approximations $\{u_n\}_{n\in\mathbb{N}}$ of solution u of (6.2) and then prove (Lemma 6.9) convergence of the respective approximations $\{x_n\}_{n\in\mathbb{N}}$ of solution x of sweeping process (1.7). In other words, Lemma 6.9 states that the change of the variables of Castaing and Monteiro Marques [10, Theorem 4.1] is continuous with respect to time-discretization. Finally, a result by Monteiro Marques [32, p. 15-16] (which is also Proposition 6 in Valadier [41]) is used to prove (Theorem 6.11 of section 6.4) that the limit of catching-up scheme (6.4)-(6.7) is a solution of (1.7).

Section 8 is devoted to establishing conditions for the validity of (1.8) at $\lambda = 0$ in a neighborhood Q of a switched boundary equilibrium x_0. Specifically, as mentioned earlier, we assume that, for $\lambda = 0$ sweeping process (1.7) takes the form (1.11) and discover conditions for asymptotic stability of $x_0 \in \partial A$. In particular, in section 8 we extend the two-dimensional approach of Makarenkov and Niwanthi Wadippuli [30] and derive a differential equation of sliding motion along ∂A, for which x_0 is a regular equilibrium whose stability can be investigated (Theorem 8.5) over the eigenvalues of the respective linearization. Assuming that the real parts of these eigenvalues are negative we conclude that $d(I - P^0 \circ V^0, Q) = 1$ and establish (Theorem 8.6) the existence of T-periodic solutions near x_0 for all BV-continuous state-dependent sweeping processes (1.7) that approaches (1.11) when $\lambda \to 0$.

Conclusions and Acknowledgments sections follows Section 8.

The very end of the paper (Appendix A) includes a flowchart of the results of the paper that the reader can use to navigate through the proofs easier.

2. Definition of solution. In what follows, $\mathcal{B}([0,T])$ is the family of Borel subsets of $[0,T]$. A *Borel vector measure* on $[0,T]$ is a map $\mu : \mathcal{B}([0,T]) \to E$ such that $\mu\left(\cup_{n=1}^{\infty} B_n\right) = \Sigma_{n=1}^{\infty}\mu(B_n)$ for any sequence $\{B_n\}_{n=1}^{\infty}$ of mutually disjoint elements of $\mathcal{B}([0,T])$, see Recupero [38, §2.4] or Dinculeanu [14, Definition 1, §III.14.4, p. 297].

According to Dinculeanu [14, Theorem 1, § III.17.2, p. 358] (see also Recupero [38]), any BV-continuous function $x : [0,T] \to E$ admits a unique vector measure of bounded variation $dx : \mathcal{B}([0,T]) \to E$ (called *Stieltjes measure* in [14]) such that for every $0 < t_1 < t_2 < T$ we have

$$\begin{aligned} dx((t_1,t_2)) &= x(t_2) - x(t_1), \quad & dx([t_1,t_2]) &= x(t_2) - x(t_1), \\ dx([t_1,t_2)) &= x(t_2) - x(t_1), \quad & dx((t_1,t_2]) &= x(t_2) - x(t_1). \end{aligned}$$

A vector Borel measure $d\mu$ is called continuous with respect to a scalar Borel measure $d\nu$ (or simply $d\nu$-continuous), if $\lim_{\nu(D)\to 0} \mu(D) = 0$, see Diestel-Uhl [13, p. 11]. If a vector measure $d\mu$ is $d\nu$-continuous then, according to Radon-Nikodym Theorem [13, p. 59] there is a $d\nu$-integrable function $g : [0,T] \to E$ such that

$$d\mu(D) = \int_D g\, d\nu, \quad \text{for all } D \in \mathcal{B}([0,T]).$$

In this case, the function g is called Radon-Nikodym derivative of $d\mu$ with respect to $d\nu$ (or density) and is denoted by $\dfrac{d\mu}{d\nu}$. Furthermore, according to Moreau-Valadier [35, Proposition 1] (see also Valadier [41, Theorem 3]), the Radon-Nikodym derivative $\dfrac{d\mu}{d\nu}$ can be computed as

$$\frac{d\mu}{d\nu}(t) = \lim_{\varepsilon \to 0,\, \varepsilon > 0} \frac{d\mu([t, t+\varepsilon])}{d\nu([t, t+\varepsilon])}, \quad d\nu - a.e.\ on\ [0, T].$$

We will use the following definition of the solution of (1.5) (Castaing and Monteiro Marques [10, §1]).

Definition 2.1. *A BV-continuous function x is called a solution of (1.5), if there exists a finite measure $d\nu$ for which both differential measure dx and Lebesgue measure dt are $d\nu$-continuous, such that*

$$-\frac{dx}{d\nu}(t) \in N_{A+a(t)+c(x(t))}(x(t)) + f(t, x(t))\frac{dt}{d\nu}(t), \quad d\nu - a.e.\ on\ [0, T],$$

and $x(t) \in A + a(t) + c(x(t))$, for all $t \in [0, T]$.

3. Existence of solutions. It is customary (see [24, Theorem 6]) to assume that the initial condition q of sweeping process (1.5) satisfies

$$q \in A + a(0) + c(q). \tag{3.1}$$

However, it will be convenient for our analysis to define solutions of (1.5) for any initial condition $q \in E$, that we will term a *generalized initial condition.* We take advantage of the fact, that for a contracting map c, the equation

$$v = \mathrm{proj}(q, A + a(0) + c(v))$$

always has a solution $v = V(q)$ (see Lemma 6.3) and $V \in C^0(E, E)$. In other words, we say that x is a solution of (1.5) with a generalized initial condition $q \in E$, if x is a solution of (1.5) with the initial condition $x(0) = V(q)$.

As itself, the theorem won't loose anything by dropping the generalized initial condition concept. However, considering the generalized initial conditions will be convenient for applications of Theorem 3.1 to the problem of the occurrence of periodic solutions from a boundary equilibrium, that we consider in this paper later (Theorem 8.6).

Following Filippov [16, Ch. 1, §1], we say that a function $f : [0, T] \times E \to E$ satisfies a Carathéodory condition, if

(i) the function $f(t, x)$ is continuous in x for a.a. $t \in [0, T]$;
(ii) the function $f(t, x)$ is Lebesgue measurable in t for each x;
(iii) for each bounded $D \subset E$, there exists a function $m(t)$ that is summable on $[0, T]$ and such that $\|f(t, x)\| \le m(t)$, for all $t \in [0, T]$ and all $x \in D$.

The Carathéodory conditions (i)-(ii) ensure that the composition $t \mapsto f(t, x(t))$ is Lebesgue measurable on $[0, T]$ for any Lebesgue measurable $x(t)$, see e.g. Krasnoselskii et al [28, §17.1]. Condition (iii) further ensures that $t \mapsto f(t, x(t))$ is summable on $[0, T]$. In what follows, we will always assume that $f(t, x)$ is Lipschitz in x, so that condition (i) will always hold automatically. Therefore, assuming that $(t, x) \mapsto f(t, x)$ is Carathéodory we effectively impose conditions (ii) and (iii).

THEOREM 3.1. *Assume that $A \subset E$ is a nonempty closed convex bounded set, $a : [0,T] \to E$ is BV-continuous on $[0,T]$, $x \mapsto c(x)$ is globally Lipschitz with Lipschitz constant $L_2 \in [0,1)$, and $(t,x) \mapsto f(t,x)$ is Carathéodory in (t,x) and globally Lipschitz in x. Then, for any generalized initial condition $q \in E$, the sweeping process (1.5) admits a solution, defined on $[0,T]$, with the initial condition $x(0) = V(q)$. In particular, sweeping process (1.5) admits a solution on $[0,T]$, for any initial condition $x(0) = q$, where q satisfies (3.1).*

Theorem 3.1 can be potentially derived using the ideas of Nacry [36] (and even for a general form $A(t,x)$ of the constraint $A + a(t) + c(x)$), but in the present paper Theorem 3.1 comes as a corollary of a more general Theorem 6.11, that we were unable to extend to the case of a general constraint $A(t,x)$.

4. Global existence of periodic solutions. In this section we offer a result saying that, under the conditions of Theorem 3.1, sweeping process (1.5) always has a periodic solution, if the right-hand-sides are T-periodic.

We remind the reader that a solution $t \mapsto x(t)$ of sweeping process (1.5) that is defined on $\mathbb{R}$ is called T-periodic, if $x(t+T) = x(t)$, for a fixed positive constant T and for all $t \in \mathbb{R}$.

THEOREM 4.1. *Assume that conditions of theorem 3.1 hold and let $L_2 \in [0,1)$ be the Lipschitz constant of c as introduced in theorem 3.1. Denoting by $\xi \in E$ the unique solution of $c(\xi) = \xi$, consider the set*

$$\Omega = \bigcup_{t\in[0,T]} \Omega_t, \quad \Omega_t = \bigcup_{b\in A+a(t)} \left\{ x : \|x - \xi\| < \frac{\|b\|}{1-L_2} \right\}.$$

Then sweeping process (1.5) admits a solution $t \mapsto x(t)$ such that

$$x(T) = x(0) \in \overline{\Omega}. \tag{4.1}$$

In particular, $t \mapsto x(t)$ is a T-periodic solution of (1.5) (i.e. verifies $x(t+T) = x(t)$, $t \in \mathbb{R}$), if both $t \mapsto a(t)$ and $t \mapsto f(t,x)$ are T-periodic on $\mathbb{R}$.

A constant solution $x(t) \equiv const$, $t \in \mathbb{R}$, is a T-periodic solution for any $T > 0$. For example, when $A = [-1,1]$ and $a(t) \equiv c(x) \equiv 0$, we have $\Omega = (-1,1)$. If, further, $f(t,x) = x$, then $x(t) \equiv 0$ is a T-periodic solution of (1.5) for any $T > 0$. If, however, $f(t,x) = x - 2$, then a T-periodic solution of (1.5) is given by $x(t) \equiv 1$ for any $T > 0$.

REMARK 4.2. *Throughout the paper we prefer to work with functions defined on $[0,T]$ only. When saying $t \mapsto x(t)$ is a T-periodic solution of (1.5), we mean that $t \mapsto x(t)$ becomes a T-periodic solution after both functions $t \mapsto a(t)$ and $t \mapsto f(t,x)$ are extended to $\mathbb{R}$ by T-periodicity. In particular, a solution $t \mapsto x(t)$ with a generalized initial condition q and defined on $[0,T]$ is called T-periodic, if $x(T) = q$. If we happen to establish the existence of a T-periodic solution x with a generalized initial condition q, then q must necessary equal $x(0)$. Indeed, every solution $t \mapsto x(t)$ of (1.5) satisfies $x(T) \in A + a(T) + c(x(T))$ which implies $q \in A + a(0) + c(q)$. Therefore, $v = q$ is one of the solutions of $v = \mathrm{proj}(q, A + a(0) + c(v))$. Therefore, $x(0) = q$.*

5. The continuation principle. This section introduces the main abstract results of the paper. These abstract results are mainly motivated by an application to a specific practical situation considered in section 8.

We consider a λ-dependent sweeping process (1.7) for measures dx and dt, and discover how the existence of periodic solutions for $\lambda = \lambda_1 > 0$ (where even continuous dependence of solutions on initial conditions is not given) can be concluded from an appropriate knowledge about sweeping process (1.7) for $\lambda = 0$ (which possesses fairly good properties). In this way, we obtain conditions for the existence of periodic solutions to sweeping process (1.7) for $\lambda = \lambda_1$, which denotes sweeping process (1.5) of our initial interest. We recall that such an approach is referred to as "continuation principle" in the literature.

We will assume that BV-continuity of a of Theorem 3.1 holds uniformly with respect to λ, i.e.

$$\begin{aligned}&\operatorname{var}(a(\cdot,\lambda),[s,t]) \le \operatorname{var}(\bar{a},[s,t]), \quad \lambda \in [0,1],\\ &\text{where } \bar{a} : [0,T] \to \mathbb{R} \text{ is a BV-continuous function.}\end{aligned} \tag{5.1}$$

The map V^λ for (1.7) now depends on the parameter λ and is defined as the unique solution (according to Lemma 6.3) of the equation

$$v = \operatorname{proj}(q, A + a(0,\lambda) + c(v,\lambda)).$$

Accordingly, we say that x is a solution of (1.7) with a generalized initial condition $q \in E$, if x is a solution of (1.7) with the initial condition $x(0) = V^\lambda(q)$.

We will call sweeping process (1.7) T-periodic, if

$$a(t+T,\lambda) \equiv a(t,\lambda), \quad f(t+T,x,\lambda) \equiv f(t,x,\lambda). \tag{5.2}$$

In what follows, $d(I-\bar{P},Q)$ is the topological degree of the vector field $I-\bar{P}$ on an open bounded set $Q \subset E$, see e.g. Krasnoselskii-Zabreiko [29]. To give a brief intuition to the reader unfamiliar with the notion of topological degree, we can mention that, in $\mathbb{R}^2$, the quantity $d(I-\bar{P},Q)$ is the number of complete turns that the vector $x-\bar{P}(x)$ makes (clockwise and counter-clockwise turns are counted with opposite signs) when the point x makes one complete turn along the boundary of Q (provided that the boundary is smooth). Computing $d(I-\bar{P},Q)$ for an arbitrary vector field $I-\bar{P}$ in $\mathbb{R}^n$ requires the knowledge of $I-\bar{P}$ on the entire boundary of Q too and is discussed in Remark 5.3 for completeness. However, important classes of vector fields $I-\bar{P}$ lead to $d(I-\bar{P},Q) = 1$ just because of certain easily verifiable qualitative criteria, such as e.g. $(I-\bar{P})\left(\overline{Q}\right) \subset Q$ with Q being convex (Brouwer-Bohl Theorem, see [29, Theorem 6.2]). The present paper takes advantage of these qualitative criteria (Theorem 8.1), rather than computes $d(I-\bar{P},Q) = 1$ from first principles of Remark 5.3.

We remind the reader that the Poincaré map $P^0(q)$ of T-periodic sweeping sweeping process (1.7) with $\lambda = 0$ and $q \in E$ is the value of the solution $x(t)$ of (1.7) with the initial condition $x(0) = q$ at time $t = T$.

THEOREM 5.1. *Assume that T-periodic sweeping process (1.7) possesses the following regularity:*

- I) *The set $A \subset E$ is nonempty, convex, closed, and bounded. The function a satisfies (5.1). The function $x \mapsto c(x,\lambda)$ is globally Lipschitz with Lipschitz constant $0 < L_2 < 1$. The function $(t,x) \mapsto f(t,x,\lambda)$ is Carathéodory in (t,x) and globally Lipschitz in x, and both the Lipschitz constants are independent of $\lambda \in [0,1]$. Furthermore, a, c, and f are continuous in $\lambda \in [0,1]$ uniformly with respect to $t \in [0,T]$ and $x \in E$.*

Assume, that the existence of a T-periodic solution for $\lambda = 0$ is given in the following extended way:

II) *There exists an open bounded $Q \subset E$ such that, when $\lambda = 0$, the solution of (1.7) is unique for any initial condition $x(0) \in V^0(\overline{Q})$, none of the elements of ∂Q are generalized initial conditions of T-periodic solutions of (1.7) with $\lambda = 0$, and for the Poincaré map P^0 of (1.7) with $\lambda = 0$ one has*

$$d(I - P^0 \circ V^0, Q) \neq 0.$$

Finally, assume the following homotopy through $\lambda \in [0, \lambda_1]$:

III) *There exists $\lambda_1 \in (0, 1]$ such that sweeping process (1.7) doesn't have periodic solutions x with generalized initial conditions $x(0) \in \partial Q$, $\lambda \in [0, \lambda_1]$.*

Then, for any $\lambda \in [0, \lambda_1]$, sweeping process (1.7) admits a T-periodic solution x with the initial condition $x(0) \in V^\lambda(Q)$.

To illustrate conditions of Theorem 5.1, we slightly extend the example that we considered right after the formulation of Theorem 4.1. Let $E = \mathbb{R}$, $A = [-1, 1]$, $a(t, \lambda) = e^{\lambda \sin t} - 1$, $f(t, x, \lambda) = x + \lambda \sin t - 2$, $c(x) \equiv 0$, $Q = (0, 2)$, which implies that $T = 2\pi$. Condition I) holds trivially and we first focus on verifying condition II). Observe that $V^0(x)$ computes as

$$V^0(x) = \begin{cases} 1, & x > 1, \\ x, & x \in [-1, 1], \\ -1, & x < -1. \end{cases} \tag{*}$$

Ignoring the moving constraint $A + a(t, \lambda)$, the solution of (1.7) with the initial condition $x(0) = x_0$ computes for $\lambda = 0$ as

$$x(t) = (x_0 - 2)e^{-t} + 2. \tag{**}$$

In particular, $x(2\pi) > 1$ for all $x_0 \in A$. Therefore, accounting for A gives

$$P^0(V^0(x)) = 1, \quad x \in \partial Q = \{0, 2\},$$

which implies that the map $x \mapsto x - P^0(V^0(x))$ points outwards of the interval Q at its boundary point and so (see e.g. Krasnoselskii-Zabreiko [29, §3.2])

$$d(I - P^0 \circ V^0, Q) = 1.$$

We now claim that Condition III) holds with e.g. $\lambda_1 = \dfrac{1}{2}$. In other words, we claim that solutions of sweeping process (1.7) with generalized initial conditions in $\{0, 2\}$ never get T-periodic when one varies λ in $[0, \lambda_1]$.

Step 1. Consider the solution x of (1.7) with a generalized initial condition 0, i.e. consider the solution x of (1.7) with the initial condition $x(0) = V^\lambda(0)$. Since the moving constraint $A + a(t, \lambda)$ computes as $A + a(t, \lambda) = \left[-2 + e^{\lambda \sin t}, e^{\lambda \sin t}\right]$, we have $A + a(0, \lambda) = [-1, 1]$ for any $\lambda \in [0, 1/2]$. Therefore, $V^\lambda(x) = V^0(x)$ for all $x \in \mathbb{R}$, and based on (*) we conclude $x(0) = 0$. It now follows from (**) that, in the absence of the moving constraint, the solution $t \mapsto x(t)$ would be strictly increasing for all $\lambda \in [0, 1/2]$. But the right end of the constraint is always greater than $1/2$ when $\lambda \in [0, 1/2]$ and $t \in [0, 2\pi]$. Therefore, even though the increasing solution $x(t)$ will interact with the right end of $A + a(t, \lambda)$ when t grows from 0 to 2π, the right end of $A + a(t, \lambda)$ is not capable to pull (sweep) the solution $x(t)$ to $x = 0$. In other words, $t \mapsto x(t)$ cannot be T-periodic.

Step 2. Consider the solution x of (1.7) with a generalized initial condition 2, i.e. consider the solution x of (1.7) with the initial condition $x(0) = V^{\lambda}(2)$. Since $V^{\lambda}(2) = V^{0}(2)$ for any $\lambda \in \mathbb{R}$, we conclude from (*) that $x(0) = 1$. Since the right end of $A + a(t, \lambda)$ never exceeds $3/2$ for $t \in [0, 2\pi]$, $\lambda \in [0, 1/2]$, the value $x(T)$ can never get larger than $3/2$. In particular, the solution x of (1.7) with a generalized initial condition 2 can never be T-periodic (see also Remark 4.2).

We showed that all conditions of Theorem 5.1 hold and, therefore, for any $\lambda \in [0, 1/2]$, sweeping process (1.7) admits a 2π-periodic solution x with the initial condition $x(0) \in [0, 1]$.

Note, for $\lambda > 0$, we don't know whether or not the solutions of sweeping process (1.7) are uniquely defined by the initial condition or depend continuously on λ. That is why the statement of the theorem is not a direct consequence of II) as it usually happens in topological degree based existence results. In particular, we cannot establish any type of continuity of solutions as $\lambda \to 0$. That is why the next theorem is not a direct consequence of Theorem 5.1.

Theorem 5.2. *Assume that sweeping process (1.7) is T-periodic. Assume that conditions I) and II) of Theorem 5.1 hold. Then, there exists $\lambda_1 > 0$ such that condition III) of Theorem 5.1 holds, and, therefore, for any $\lambda \in [0, \lambda_1]$, sweeping process (1.7) admits a T-periodic solution x with the initial condition $x(0) \in V^{\lambda}(Q)$.*

We conclude this section with a remark on the computation of the topological degree.

Remark 5.3. Here we follow Krasnoselskii-Zabreiko [29, §3.4] and the interested reader is referred to this reference for further details and for alternative formulas for $d(I - \bar{P}, Q)$. Assuming that the boundary ∂Q is smooth (one passes to a suitable smooth approximation of the boundary otherwise), we consider an auxiliary vector field $\Psi(x) = (x - \bar{P}(x))/\left\|x - \bar{P}(x)\right\|$ which is assumed to be smooth as well (a suitable smooth approximation is introduced otherwise). A point y_0 with $\|y_0\| = 1$ is called *regular* with respect to the differentiable mapping Ψ, if the following two conditions are satisfied: (i) the pre-image of y_0 under Ψ must be a finite set; (ii) for each point that is mapped into y_0 by Ψ the Jacobian (with respect to a certain local coordinate system) must be different from zero. Let t and s be the numbers of points which are mapped to y_0 with positive and negative Jacobian respectively. The difference $t - s$ is independent of the choice of y_0 and $d(I - \bar{P}, U) = t - s$.

6. The catching-up algorithm and proofs of the abstract existence results.

6.1. An equivalent non-perturbed formulation of the initial perturbed sweeping process.

Recall, that for a BV-continuous function $u : [0, T] \to E$, the *variation measure* $|du|$ (also called *modulus measure*) is defined, for any $D \in \mathcal{B}([0, T])$, as (see Diestel-Uhl [13, Definition 4, p. 2], Recupero [38, §2.4])

$$|du|(D) = \\ = \sup \left\{ \sum_{n=1}^{\infty} \|u(D_n)\| : D = \bigcup_{n=1}^{\infty} D_n,\ D_n \in \mathcal{B}([0, T]),\ D_i \cap D_j = \emptyset \text{ if } i \neq j \right\}.$$

For a BV-continuous function $u : [0, T] \to \mathbb{R}$, the differential measure du is always $|du|$-continuous (it follows e.g. from Diestel-Uhl [13, Theorem 1, p. 10]), i.e. a $|du|$-integrable density $\dfrac{du}{|du|}$ is well defined. Moreover, according to Castaing and Monteiro

Marques [10, Theorem 4.1], if x is a BV-continuous solution of the perturbed sweeping process (1.7), then the BV-continuous function u defined by

$$u(t) = x(t) + \int_0^t f(\tau, x(\tau))d\tau \tag{6.1}$$

is a solution to the non-perturbed sweeping process

$$-\frac{du}{|du|}(t) \in N_{A+a(t,\lambda)+c(x(t),\lambda)+\int_0^t f(\tau,x(\tau),\lambda)d\tau}(u(t)), \quad |du| - a.e.\ on\ [0,T]. \tag{6.2}$$

LEMMA 6.1. *Assume that $(t, x, \lambda) \mapsto f(t, x, \lambda)$ is Carathéodory in (t, x) and is globally Lipschitz in x with Lipschitz constant independent of $t \in [0, T]$ and $\lambda \in [0, 1]$. Then, for any BV-continuous $u : [0, T] \to E$, the integral equation (6.1) admits a unique BV-continuous solution $x : [0, T] \to E$.*

Lemma 6.1 is a direct consequence of Lemma 6.9 that we prove below.

Combining [10, Theorem 4.1] and Lemma 6.1, we can formulate the following equivalent definition of the solution of (1.7).

DEFINITION 6.2. *A BV-continuous function x is called a solution of perturbed sweeping process (1.7), if the function u given by (6.1) is a solution of the non-perturbed sweeping process (6.2).*

6.2. The catching-up algorithm. For each fixed $n \in \mathbb{N}$, we partition $[0, T]$ into smaller intervals by the points $\{t_0, t_1, ..., t_n\} \subset [0, T]$ defined by

$$t_0 = 0,\ t_n = T,\ t_{i+1} - t_i = \frac{T}{n}, \quad i \in \overline{1, n}.$$

In what follows, we fix some initial condition

$$x(0) = u(0) = q,$$

where q satisfies

$$q \in A + a(0, \lambda) + c(q, \lambda), \tag{6.3}$$

and use the ideas of Definition 6.2 in order to construct pieceiwise-linear functions u_n and x_n (linear on each $[t_i, t_{i+1}]$) that serve as approximations of the solutions u and x of Definition 6.2. The construction will be implemented iteratively through the intervals $[t_i, t_{i+1}]$ starting from $i = 0$, and moving towards $i = n - 1$.

Denoting

$$u_n(0) = q,\ x_n(0) = q,\ u_i^n = u_n(t_i),\ x_i^n = x_n(t_i), \quad i \in \overline{0, n},$$

we apply the implicit iterative scheme

$$u^n_{i+1} = \mathrm{proj}\left[u^n_i, A + a(t_{i+1},\lambda) + c\left(u^n_{i+1} - \int\limits_0^{t_i} f(\tau, x_n(\tau),\lambda)d\tau, \lambda\right)\right.$$

$$\left. + \int\limits_0^{t_i} f(\tau, x_n(\tau),\lambda)d\tau\right], \tag{6.4}$$

$$x^n_{i+1} = u^n_{i+1} - \int\limits_0^{t_i} f(\tau, x_n(\tau),\lambda)d\tau, \tag{6.5}$$

$$u_n(t) = u^n_i + \frac{t - t_i}{t_{i+1} - t_i}(u^n_{i+1} - u^n_i), \quad t \in [t_i, t_{i+1}], \tag{6.6}$$

$$x_n(t) = x^n_i + \frac{t - t_i}{t_{i+1} - t_i}(x^n_{i+1} - x^n_i), \quad t \in [t_i, t_{i+1}], \tag{6.7}$$

successively from $i = 0$ to $i = n - 1$. Next lemma uses the idea of the implicit scheme of Kunze and Monteiro Marques ([24, Lemma 7]) and it proves that for each $i \in \overline{0, n-1}$ we can extend the definition of u_n and x_n from $[0, t_i]$ to $[0, t_{i+1}]$ according to (6.4)-(6.7).

Lemma 6.3. *Consider a set-valued function*

$$C(s_1, s_2, u, \xi) = A + \tilde{a}(s_1, \xi) + \tilde{c}(s_2, u, \xi), \quad s_1, s_2 \in [0, T],\ u \in E,\ \xi \in W,$$

where $A \subset E$ is a nonempty closed convex bounded set, $\tilde{a} : \mathbb{R} \times W \to E$, $\tilde{c} : \mathbb{R} \times E \times W \to E$, and W is a finite dimensional Euclidean space. Assume that

$$\begin{gathered}\mathrm{var}(\tilde{a}(\cdot,\xi), [s, t]) \le \mathrm{var}(\bar{a}, [s, t]), \quad \xi \in W,\\ \textit{where } \bar{a} : [0, T] \to \mathbb{R} \textit{ is a BV-continuous function,}\end{gathered}$$

and $(s, \xi) \to \tilde{a}(s, \xi)$ is continuous in $\xi \in W$ uniformly in $s \in [0, T]$. Assume that $(s, u, \xi) \mapsto \tilde{c}(s, u, \xi)$ is continuous in $\xi \in W$ uniformly in $(s, u) \in [0, T] \times E$ and satisfies the Lipschitz condition

$$\begin{gathered}\|\tilde{c}(s, u, \xi) - \tilde{c}(t, v, \xi)\| \le L_1|s - t| + L_2\|u - v\|,\\ \textit{for any } s, t \in [0, T],\ u, v \in E,\ \xi \in W,\end{gathered}$$

with $L_1 > 0$ and $L_2 \in (0, 1)$. Then, for any $\tau_1, \tau_2, s_1, s_2 \in [0, T]$ and any $u \in E$ there exists an unique $v = v(\tau_1, \tau_2, s_1, s_2, u, \xi)$ such that

$$v \in C(\tau_1, \tau_2, v, \xi) \quad \text{and} \quad v = \mathrm{proj}(u, C(\tau_1, \tau_2, v, \xi)). \tag{6.8}$$

Moreover, $v \in C^0([0, T] \times [0, T] \times [0, T] \times [0, T] \times E \times W, E)$. If, in addition,

$$u \in C(s_1, s_2, u, \xi),$$

then

$$\|v - u\| \le \frac{\mathrm{var}(\bar{a}, [s_1, \tau_1]) + L_1|\tau_2 - s_2|}{1 - L_2}. \tag{6.9}$$

The following key estimate is required for the proof of Lemma 6.3.

Lemma 6.4. *Let C be a convex set of E. Then, for any vectors $u, c \in E$,*

$$\|\mathrm{proj}(u, C) - \mathrm{proj}(u, C + c)\| \le \|c\|.$$

Proof. From the definition of projections $v_1 = \mathrm{proj}(u, C)$ and $v_2 = \mathrm{proj}(u, C + c)$ we have (see e.g. Kunze and Monteiro Marques [24, §2])

$$u - v_1 \in N_C(v_1) \quad \text{and} \quad u - v_2 \in N_{C+c}(v_2). \tag{6.10}$$

Since $v_2 - c \in C$ and $v_1 + c \in C + c$, we conclude from (6.10) that

$$\langle u - v_1, v_2 - c - v_1\rangle \le 0 \quad \text{and} \quad \langle u - v_2, v_1 + c - v_2\rangle \le 0,$$

or, rearranging the terms,

$$\langle v_1 - u, v_1 - v_2\rangle \le \langle u - v_1, c\rangle \quad \text{and} \quad \langle u - v_2, v_1 - v_2\rangle \le \langle v_2 - u, c\rangle\,.$$

Finally, we add both inequalities together and get

$$\langle v_1 - v_2, v_1 - v_2\rangle \le \langle v_2 - v_1, c\rangle \le \|v_1 - v_2\| \cdot \|c\|,$$

which implies the statement. ∎

Proof of Lemma 6.3. **Step 1.** *The existence of* $v(\tau_1, \tau_2, s_1, s_2, u, \xi)$. Define $F \in C^0(E, E)$ as $F(v) = \mathrm{proj}(u, C(\tau_1, \tau_2, v, \xi))$. Using Lemma 6.4, we have

$$\begin{aligned} &\|F(v_1) - F(v_2)\| = \\ &\qquad = \|\mathrm{proj}(u, A + \tilde a(\tau_1, \xi) + \tilde c(\tau_2, v_1, \xi)) - \mathrm{proj}(u, A + \tilde a(\tau_1, \xi) + \tilde c(\tau_2, v_2, \xi))\| \le \\ &\qquad \le \|\tilde c(\tau_2, v_1, \xi) - \tilde c(\tau_2, v_2, \xi)\| \le \\ &\qquad \le L_2\|v_1 - v_2\|, \end{aligned} \tag{6.11}$$

so the existence of $v = v(\tau_1, \tau_2, s_1, s_2, u, \xi)$ with the required property (6.8) follows by applying the contraction mapping theorem (see e.g. Rudin [39, Theorem 9.23]).

Step 2. *Continuity of* $v(\tau_1, \tau_2, s_1, s_2, u, \xi)$. To prove the continuity of v, let $v = v(\tau_1, \tau_2, s_1, s_2, u, \xi)$ and $\bar v = v(\bar\tau_1, \bar\tau_2, \bar s_1, \bar s_2, \bar u, \bar\xi)$ where $s_1, s_2, \bar s_1, \bar s_2 \in [0, T]$, $\tau_1, \tau_2, \bar\tau_1, \bar\tau_2 \in [0, T]$, $\xi, \bar\xi \in W$ and $u, \bar u \in E$.

First observe that

$$\begin{aligned} &\|\bar v - v\| = \\ &\quad = \|\mathrm{proj}(\bar u, A + \tilde a(\bar\tau_1, \bar\xi) + \tilde c(\bar\tau_2, \bar v, \bar\xi)) - \mathrm{proj}(u, A + \tilde a(\tau_1, \xi) + \tilde c(\tau_2, v, \xi))\| \\ &\quad \le \|\mathrm{proj}(\bar u, A + \tilde a(\bar\tau_1, \bar\xi) + \tilde c(\bar\tau_2, \bar v, \bar\xi)) - \mathrm{proj}(u, A + \tilde a(\bar\tau_1, \bar\xi) + \tilde c(\bar\tau_2, \bar v, \bar\xi))\| \\ &\quad + \|\mathrm{proj}(u, A + \tilde a(\bar\tau_1, \bar\xi) + \tilde c(\bar\tau_2, \bar v, \bar\xi)) - \mathrm{proj}(u, A + \tilde a(\tau_1, \xi) + \tilde c(\tau_2, v, \xi))\|. \end{aligned}$$

Since for any nonempty, closed, convex set $C \subset E$ and any vectors $\bar u, u \in E$, we have (see e.g. Mordukhovich-Nam [33, Proposition 1.79])

$$\|\mathrm{proj}(\bar u, C) - \mathrm{proj}(u, C)\| \le \|\bar u - u\|, \tag{6.12}$$

then, using also Lemma 6.4, we conclude that

$$\begin{aligned} \|\bar v - v\| &\le \|\bar u - u\| + \left\|\tilde a(\bar\tau_1, \bar\xi) + \tilde c(\bar\tau_2, \bar v, \bar\xi) - \tilde a(\tau_1, \xi) - \tilde c(\tau_2, v, \xi)\right\| \le \\ &\le \|\bar u - u\| + \left\|\tilde a(\bar\tau_1, \bar\xi) - \tilde a(\bar\tau_1, \xi)\right\| + \mathrm{var}(\bar a, [\tau_1, \overline{\tau}_1]) + \\ &\quad + \|\tilde c(\bar\tau_2, v, \bar\xi) - \tilde c(\tau_2, v, \xi)\| + L_1|\bar\tau_2 - \tau_2| + L_2\|\bar v - v\|, \end{aligned} \tag{6.13}$$

so that the required continuity of $v(\tau_1, \tau_2, s_1, s_2, u, \xi)$ follows from $0 \le L_2 < 1$.

Step 3. *Proof of the estimate (6.9).* Assuming that $u \in C(s_1, s_2, u, \xi)$, we follow the lines of (6.13) to get

$$\|v - u\| = \|\text{proj}(u, C(\tau_1, \tau_2, v, \xi)) - u\| = \min_{\bar{v} \in C(\tau_1, \tau_2, v, \xi)} \|u - \bar{v}\|.$$

But $C(s_1, s_2, u, \xi) = A + \tilde{a}(s_1, \xi) + \tilde{c}(s_2, u, \xi)$ and $C(\tau_1, \tau_2, v, \xi) = A + \tilde{a}(\tau_1, \xi) + \tilde{c}(\tau_2, v, \xi)$. Therefore,

$$\min_{\bar{v} \in C(\tau_1, \tau_2, v, \xi)} \|u - \bar{v}\| \le \|\tilde{a}(s_1, \xi) + \tilde{c}(s_2, u, \xi) - \tilde{a}(\tau_1, \xi) - \tilde{c}(\tau_2, v, \xi)\| \le$$
$$\le \text{var}(\bar{a}, [s_1, \tau_1]) + L_1|\tau_2 - s_2| + L_2\|u - v\|, \tag{6.14}$$

which implies (6.9).

The proof of the lemma is complete. ∎

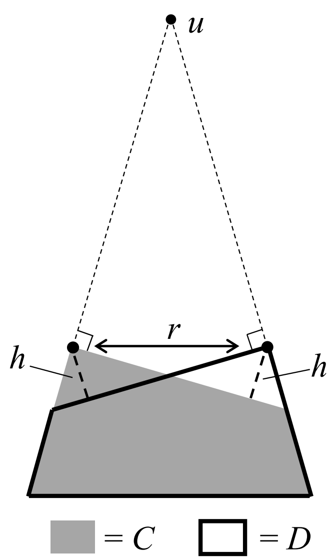

Fig. 1. *Illustration of the incorrectness of formula (6.16). Here $h = d_H(C, D)$, and $r = \|\text{proj}(u, C) - \text{proj}(u, D)\|$.*

Remark 6.5. *On the validity of Lemma 6.3 when $A + c(t, \xi)$ is replaced by a more general term $A(t, \xi)$.*

One can observe that estimate (6.11) holds also in the case where $A + \tilde{a}(t, \xi)$ takes a more general form $A(t, \xi)$. Furthermore, if $d_H(A_1, A_2)$ is the Hausdorff distance between nonempty closed sets $A_1, A_2 \subset E$ and $A(t, \xi)$ satisfies

$$d_H(A(s), A(t)) \le \text{var}(\bar{a}, [s, t]), \tag{6.15}$$

then (6.14) holds as well since

$$\min_{\bar{v} \in C(\tau_1, \tau_2, v, \xi)} \|u - \bar{v}\| \le d_H(A(s_1, \xi) +$$
$$+ \tilde{c}(s_2, u, \xi), A(\tau_1, \xi) + \tilde{c}(\tau_2, v, \xi)).$$

To summarize, the existence of $v(\tau_1, \tau_2, s_1, s_2, u, \xi)$ (Step 1) and the estimate (6.9) (Step 3) still hold, if $A + \tilde{a}(t, \xi)$ is replaced by $A(t, \xi)$ satisfying (6.15).

At the same time, the continuity of the function $v(\tau_1, \tau_2, s_1, s_2, u, \xi)$ can no longer be established when $A + \tilde{a}(t, \xi)$ is replaced by $A(t, \xi)$. Indeed, the core of estimate (6.13) is Lemma 6.4 which doesn't allow a generalization when $C + c$ is replaced by an arbitrary set D. One might be tempted to believe that the conclusion of Lemma 6.4 can be replaced by

$$\|\mathrm{proj}(u, C) - \mathrm{proj}(u, D)\| \le d_H(C, D), \tag{6.16}$$

when $C + c$ is replaced by just D, but formula (6.16) appears to be wrong as our Fig. 1 illustrates.

On the other hand Monteiro Marques [31, Proposition 4.7, p. 26] implies that

$$\|\mathrm{proj}(u, C) - \mathrm{proj}(u, D)\| \le \sqrt{2(\mathrm{dist}(u, C) + \mathrm{dist}(u, D))} \cdot \sqrt{d_H(C, D)}, \tag{6.17}$$

which could potentially help to obtain other versions of Lemma 6.3, that we don't pursue in this paper.

Corollary 6.6. *Assume that condition I) of Theorem 5.1 holds. Then, for any (q, λ) satisfying (6.3) the implicit scheme (6.4)-(6.7) is solvable iteratively from $i = 0$ to $i = n - 1$ and the respective iterations $x_i^n = x_i^n(q, \lambda)$ and $u_i^n = u_i^n(q, \lambda)$ are continuous in (q, λ) on $E \times [0, 1]$. Moreover,*

$$\|u_{i+1}^n(q, \lambda) - u_i^n(q, \lambda)\| \le \frac{\mathrm{var}(\bar{a}, [t_i, t_{i+1}]) + L_1 T/n}{1 - L_2}, \quad i \in \overline{0, n-1},$$

where $L_1 > 0$ and $L_2 \in (0, 1)$.

Proof. Let $\xi = ((\xi_1, \xi_2, \cdots, \xi_{n+1}), \xi_{n+2}) \in E^{n+1} \times \mathbb{R}$ be defined as

$$\xi_i = x_{i-1}^n, \quad i \in \overline{1, n+1} \quad , \quad \xi_{n+2} = \lambda.$$

Therefore, the rule (6.7) defines a function $\Psi : E^{n+1} \times \mathbb{R} \to C^0([0, T], E)$ that relates $\xi \in E^{n+1} \times \mathbb{R}$ to a piecewise linear function $x_n(t)$ defined on $[0, T]$. The statement of the Corollary 6.6 now follows by applying Lemma 6.3 with

$$\begin{aligned} \tilde{c}(s, u, \xi) &= \left(u - \int_0^s f(\tau, \Psi(\xi)(\tau), \xi_{n+2}) d\tau, \xi_{n+2}\right) + \int_0^s f(\tau, \Psi(\xi)(\tau), \xi_{n+2}) d\tau, \\ \tilde{a}(s, \xi) &= a(s, \xi_{n+2}). \end{aligned}$$

The proof of the corollary is complete. ∎

6.3. Convergence of the catching-up algorithm. Let $(u_n(t, q, \lambda), x_n(t, q, \lambda))$ be the solution $(u_n(t), x_n(t))$ of the catching-up algorithm (6.4)-(6.7) with the parameter $\lambda \in [0, 1]$ and the initial condition $u_n(0) = x_n(0) = q$.

Lemma 6.7. *Assume that condition I) of Theorem 5.1 holds. Consider a sequence $(\lambda_n, q_n) \to (\lambda_0, q_0)$ as $n \to \infty$ of $[0, 1] \times E$ satisfying (6.3) for each $n \in \mathbb{N}$. Then, there exists a subsequence $\{n_k\}_{k \in \mathbb{N}}$ such that $\{u_{n_k}(t, q_{n_k}, \lambda_{n_k})\}_{k \in \mathbb{N}}$ converges as $k \to \infty$ uniformly in $t \in [0, T]$.*

Proof. **Step 1.** *Boundedness of* $\{u_n(t, q_n, \lambda_n)\}_{n\in\mathbb{N}}$. Let u_i^n, $i \in \overline{0, n}$, be the approximations given by (6.4)-(6.7) with $q = q_n$ and $\lambda = \lambda_n$. By Corollary 6.6,

$$\|u_n(t, q_n, \lambda_n)\| \le \|q_n\| + \frac{1}{(1 - L_2)} (\mathrm{var}(\bar{a}, [0, T]) + L_1 T),$$

so the sequence $\{u_n(t, q_n, \lambda_n)\}_{n\in\mathbb{N}}$ is bounded uniformly on $[0, T]$.

Step 2. *Equicontinuity of* $\{u_n(t, q_n, \lambda_n)\}_{n\in\mathbb{N}}$. Fix $\varepsilon > 0$. Since $\mathrm{var}(\bar{a}, [s, t]) \to 0$ as $|s - t| \to 0$ (see e.g. Lojasiewicz [27, Theorem 1.3.4, p. 16]), we can choose $\delta_1 > 0$ such that

$$\frac{\mathrm{var}(\bar{a}, [s, t]) + L_1(t - s)}{1 - L_2} < \frac{\varepsilon}{3}, \quad \text{for all } 0 \le s \le t \le T \text{ with } t - s < \delta_1. \tag{6.18}$$

Fix some $0 \le s \le t \le T$ satisfying $t - s < \delta_1$ and denote by $i_s, i_t \in \overline{0, n-1}$ such indexes that

$$s \in [t_{i_s}, t_{i_s+1}], \quad t \in [t_{i_t}, t_{i_t+1}].$$

Then we can estimate $\|u_n(t) - u_n(s)\|$ as follows:

$$\begin{aligned}
&\|u_n(t) - u_n(s)\| \le \\
&\quad \le \|u_n(s) - u_n(t_{i_s+1})\| + \|u_n(t_{i_s+1}) - u_n(t_{i_t})\| + \|u_n(t_{i_t}) - u_n(t)\| \le \\
&\quad \le \mathrm{var}(u_n, [t_{i_s}, t_{i_s+1}]) + \mathrm{var}(u_n, [t_{i_s+1}, t_{i_t}]) + \mathrm{var}(u_n, [t_{i_t}, t_{i_t+1}]).
\end{aligned}$$

The second term is smaller than $\varepsilon/3$ by (6.18) right away. Assuming that $n \ge T/\delta_1$, the property (6.18) ensures that first and third terms are each smaller than $\varepsilon/3$ as well. So we proved that

$$\|u_n(t) - u_n(s)\| < \varepsilon, \quad \text{for all } 0 \le s \le t \le T \text{ with } t - s < \delta_1, \text{ and } n \ge T/\delta_1.$$

Since there is only a finite number of $n \in \mathbb{N}$ with $n < T/\delta_1$, we can find $\delta_2 > 0$ such that

$$\|u_n(t) - u_n(s)\| < \varepsilon, \quad \text{for all } 0 \le s \le t \le T \text{ with } t - s < \delta_2, \text{ and } n < T/\delta_1.$$

Letting $\delta = \min\{\delta_1, \delta_2\}$, we finally obtain

$$\|u_n(t) - u_n(s)\| < \varepsilon, \quad \text{for all } 0 \le s \le t \le T \text{ with } t - s < \delta, \text{ and } n \in \mathbb{N}.$$

The conclusion of the Lemma now follows by applying the Arzela-Ascoli theorem (see e.g. Rudin [39, Theorem 7.25]). ∎

Remark 6.8. *Establishing the existence of a converging subsequence* $\{x_{n_k}(t, q_{n_k}, \lambda_{n_k})\}_{k\in\mathbb{N}}$ *needs more work compared to what we did in the proof of Lemma 6.7 because the direct corollary of (6.5)*

$$x_{i+1}^n - x_i^n = u_{i+1}^n - u_i^n + \int_{t_{i-1}}^{t_i} f(\tau, x_n(\tau), \lambda) d\tau$$

doesn't imply uniform boundedness of $x_n(t, q_n, \lambda_n)$, $n \in \mathbb{N}$, *directly.*

To prove the convergence of $\{x_{n_k}(t, q_{n_k}, \lambda_{n_k})\}_{k\in\mathbb{N}}$ we will now extend the discrete map (6.6) to such an operator $F_n : C([0, T], E) \to C([0, T], E)$ whose fixed point is exactly $t \mapsto x_n(t, q_n, \lambda_n)$. The convergence of x_{n_k} will then follow from the continuity of F_n in n at $n = \infty$.

Let us define $P_n : C([0,T],E) \to E^{n+1}$, $l^- : E^{n+1} \to E^{n+1}$ and $Q_n : E^{n+1} \to C([0,T],E^{n+1})$ as

$$P_n(x) = \left(x(0), x\left(\frac{T}{n}\right), ..., x\left((n-1)\frac{T}{n}\right), x(T)\right), \quad x \in C([0,T],E),$$

$$[l^-(y)]_1 = 0 \ , \qquad [l^-(y)]_i \ = \ y_{i-1} \ , \quad i \in \overline{2, n+1} \ , \ y \in E^{n+1} \ ,$$

$$Q_n(y)(t) = \frac{t - t_{i-1}}{1/n} y_{i+1} + \frac{t_i - t}{1/n} y_i \ , \quad y \in E^{n+1} \ , \ t \in [t_{i-1}, t_i), \ i \in \overline{1, n},$$

$$Q_n(y)(t_n) = y_{n+1} \ , \quad y \in E^{n+1}, \text{ since } t_n = T.$$

For a fixed $\lambda \in [0,1]$ and a continuous function $u : [0,T] \to E$, we introduce a continuous extension of (6.6) as

$$\begin{aligned} (F_n x)(t) \quad &= \quad (Q_n P_n u)(t) - \left(Q_n l^- P_n J\right)(t), \quad t \in [0,T], \\ &\quad\ \ \text{where } J(t) = \int\limits_0^t f(\tau, x(\tau), \lambda) d\tau. \end{aligned} \tag{6.19}$$

Then, for $x \in C([0,T],E)$ satisfying $x = F_n x$, one has

$$\begin{aligned} x(0) &= (Q_n P_n u)(0) - \left(Q_n l^- P_n J\right)(0) = [P_n u]_1 - [l^- P_n J]_1 = u(0) - 0, \\ x(t_1) &= [P_n u]_2 - [l^- P_n J]_2 = u(t_1) - [P_n J]_1 = u(t_1) - J(0) = u(t_1), \\ x(t_2) &= u(t_2) - J(t_1), \\ &\ \ \ldots \\ x(t_n) &= u(t_n) - J(t_{n-1}). \end{aligned}$$

Therefore, if u_n and x_n are given by (6.4)-(6.7), then, letting $u = u_n$ in (6.19), the fixed point x of F_n verifies $x(t_i) = x_n(t_i)$, $i \in \overline{0, n}$. And, since the function $t \mapsto (F_n x)(t)$ is linear on $[t_i, t_{i+1}]$, $i \in \overline{0, n-1}$, we conclude $x_n = x$. In other words, if u in (6.19) is given by $u = u_n$, then x_n is the unique fixed point of F_n.

Lemma 6.9. *Assume that the conditions of Lemma 6.1 hold. Then, there exists $\alpha > 0$ and $L \in [0,1)$ such that*

$$\|F_n(x_1) - F_n(x_2)\|^* \le L\|x_1 - x_2\|^*, \quad n \in \mathbb{N},$$

for any $x_1, x_2, u \in C([0,T],E)$, $\lambda \in [0,1]$, and

$$\|x\|^* = \max_{t \in [0,T]} e^{-\alpha t}\|x(t)\|.$$

Moreover, for each $x, u \in C([0,T],E)$, and $\lambda \in [0,1]$, one has

$$\lim_{n \to \infty} \|F_n(x) - F(x)\| = 0, \quad \text{with } F(x)(t) = u(t) - \int_0^t f(\tau, x(\tau), \lambda) d\tau,$$

where $\|\cdot\|$ is the max-norm on $[0,T]$. Moreover, F is a contraction in the norm $\|\cdot\|^*$.

Proof. **Step 1.** Using the definition of Q_n, l^-, and P_n, we have

$$(Q_n l^- P_n J)(t_{i-1}) = [l^- P^n J]_i = [P_n J]_{i-1} = J(t_{i-2}), \quad i \in \overline{2, n+1}.$$

So that

$$(F_n x)(t_i) = u(t_i) - J(t_{i-1}).$$

Fix $i \in \overline{1, n-1}$ and choose any $t \in [t_i, t_{i+1}]$. Then,

$$\begin{aligned}
&\|F_n(x_1)(t) - F_n(x_2)(t)\| \leq \\
&\quad \leq \max\left\{\|F_n(x_1)(t_i) - F_n(x_2)(t_i)\|, \|F_n(x_1)(t_{i+1}) - F_n(x_2)(t_{i+1})\|\right\} = \\
&\quad = \max\left\{\left\|\int_0^{t_{i-1}} f(\tau, x_1(\tau), \lambda)d\tau - \int_0^{t_{i-1}} f(\tau, x_2(\tau), \lambda)d\tau\right\|,\right. \\
&\qquad\qquad \left.\left\|\int_0^{t_i} f(\tau, x_1(\tau), \lambda)d\tau - \int_0^{t_i} f(\tau, x_2(\tau), \lambda)d\tau\right\|\right\} \leq \\
&\quad \leq \bar{L}\int_0^{t_i} \|x_1(\tau) - x_2(\tau)\|d\tau \leq \bar{L}\int_0^{t_i} e^{\alpha\tau}\|x_1 - x_2\|^* d\tau,
\end{aligned}$$

where $\bar{L} > 0$ is the global Lipschitz constant of $x \mapsto f(t, x, \lambda)$ and $\alpha > 0$ is an arbitrary constant. Therefore,

$$e^{-\alpha t}\|F_n(x_1)(t) - F_n(x_2)(t)\| \leq \frac{\bar{L}}{\alpha}\left(e^{\alpha(t_i - t)} - e^{-\alpha t}\right)\|x_1 - x_2\|^* \leq \frac{\bar{L}}{\alpha}\|x_1 - x_2\|^*,$$

which holds for any $t \in [0, T]$. The case $t \in [0, t_1]$ can be considered along the same lines. This proves the contraction part of the lemma.

Step 2. To prove the convergence part, fix $i \in \overline{1, n-1}$ again and consider $t \in [t_i, t_{i+1}]$. Since $(Q_n P_n u)(t_i) = u(t_i)$, we have

$$\begin{aligned}
\|(Q_n P_n u)(t) - u(t)\| &\leq \|(Q_n P_n u)(t) - (Q_n P_n u)(t_i)\| + \|u(t) - u(t_i)\| \leq \\
&\leq \|u(t_{i+1}) - u(t_i)\| + \|u(t) - u(t_i)\|,
\end{aligned}$$

so that the convergence of $(Q_n P_n u)(t)$ to $u(t)$ as $n \to \infty$ follows from continuity of u. The convergence of $(Q_n l^- P_n J)(t)$ follows same lines. Indeed, since $(Q_n l^- P_n J)(t_{i+1}) = J(t_i)$, one has

$$\begin{aligned}
&\|(Q_n l^- P_n J)(t) - J(t)\| \leq \\
&\quad \leq \|(Q_n l^- P_n J)(t) - (Q_n l^- P_n J)(t_{i+1})\| + \|J(t) - J(t_i)\| \leq \\
&\quad \leq \|J(t_{i-1}) - J(t_i)\| + \|J(t) - J(t_i)\|
\end{aligned}$$

and the convergence of $(Q_n l^- P_n J)(t)$ to $J(t)$ as $n \to \infty$ follows from continuity of $J(t)$.

The proof of the lemma is complete.

Corollary 6.10. *Assume that condition I) of Theorem 5.1 holds. Let $\{n_k\}_{k\in\mathbb{N}}$ be the subsequence given by Lemma 6.7 (which ensures the convergence of $\{u_{n_k}(t, q_{n_k}, \lambda_{n_k})\}_{k\in\mathbb{N}}$). Consider the limit*

$$u(t) = \lim_{k\to\infty} u_{n_k}(t, q_{n_k}, \lambda_{n_k}).$$

Let $x(t)$ be the solution of the respective equation (6.1) (which exists according to Lemma 6.1). Then $\{x_{n_k}(t, q_{n_k}, \lambda_{n_k})\}_{k\in\mathbb{N}}$ converges uniformly in $t \in [0, T]$, and

$$x(t) = \lim_{k\to\infty} x_{n_k}(t, q_{n_k}, \lambda_{n_k}). \tag{6.20}$$

Proof. The conclusion follows from the inequality

$$\begin{aligned}\|x - x_n\|^* = &\|F(x) - F_n(x_n)\|^* \le \|F(x) - F_n(x)\|^* + \|F_n(x) - F_n(x_n)\|^* \le \\ &\le \|F(x) - F_n(x)\|^* + L\|x - x_n\|^*,\end{aligned}$$

where $L \in (0, 1)$ is given by Lemma 6.9. □

6.4. Verifying that the limit of the catching-up algorithm is indeed a solution.

THEOREM 6.11. *Let the conditions of Corollary 6.10 hold and let $u(t)$ and $x(t)$ be as given by this corollary. Then, $u(t)$ is a solution of sweeping process (6.2) with the parameters $x(t)$, $\lambda = \lim\limits_{k\to\infty} \lambda_{n_k}$, and the initial condition $u(0) = \lim\limits_{k\to\infty} q_{n_k}$. Accordingly, by Definition 6.2, $x(t)$ is a solution of perturbed sweeping process (1.7).*

Proof. Let us first observe that both $u(t)$ and $x(t)$ are BV-continuous. Indeed, according to Corollary 6.6,

$$\mathrm{var}(u_n, [s, t]) \le \frac{\mathrm{var}(\bar{a}, [s, t])}{1 - L_2} + \frac{L_1|t - s|}{1 - L_2}, \tag{6.21}$$

which implies BV-continuity of $u(t)$. The BV-continuity of $x(t)$ follows from (6.1) because the difference of a BV-continuous function and an absolutely continuous function is a BV-continuous function.

Let us now $\phi(t)$, $t \in [0, T]$, be an arbitrary continuous selector of the moving set of (6.2), i.e.

$$\phi(t) \in A + a(t, \lambda) + c(x(t), \lambda) + \int_0^t f(\tau, x(\tau), \lambda)d\tau, \quad t \in [0, T].$$

According to Monteiro Marques [32, p. 15-16] (see also Valadier [41, Proposition 6]) it is sufficient to prove that

$$\int_s^t \langle \phi(\tau), du(\tau)\rangle \ge \frac{1}{2}\left(\|u(t)\|^2 - \|u(s)\|^2\right), \quad 0 \le s \le t \le T, \tag{6.22}$$

which we now establish using the ideas of Kunze and Monteiro Marques [24].

Without loss of generality we will assume that $\{n_k\}_{k\in\mathbb{N}} = \mathbb{N}$, and replace n_k, $k \in \mathbb{N}$, by n, $n \in \mathbb{N}$ in the formulation of the theorem. Fix $t > 0$ and select $i \in \overline{0, n-1}$ such that $t \in [t_i, t_{i+1}]$. Introduce $\hat{c}_n(t)$ as

$$\hat{c}_n(t) = \mathrm{proj}\left(\phi(t), A + a(t_{i+1}, \lambda_n) + c(x^n_{i+1}, \lambda_n) + \int_0^{t_i} f(\tau, x_n(\tau), \lambda_n)d\tau\right).$$

Then, by (6.4) and by convexity of A, we have (see e.g. Kunze and Monteiro Marques [24, formula (4)])

$$\langle u_n(t_{i+1}) - u_n(t_i), u_n(t_{i+1}) - \hat{c}_n(t)\rangle \le 0, \quad t \in [t_i, t_{i+1}],$$

from where

$$\begin{aligned}&\langle u_n(t_{i+1})-u_n(t_i),u_n(t)-\hat{c}_n(t)\rangle\le\\&\qquad\le\langle u_n(t_{i+1})-u_n(t_i),u_n(t)-u_n(t_{i+1})\rangle\le\|u_n(t_{i+1})-u_n(t)\|^2,\end{aligned}$$

or

$$\langle u_n(t_{i+1})-u_n(t_i),\hat{c}_n(t)\rangle\ge-\|u_n(t_{i+1})-u_n(t_i)\|^2+\langle u_n(t_{i+1})-u_n(t_i),u_n(t)\rangle\,,$$

for any $t\in[t_i,t_{i+1}]$. Using the linearity of u_n on $[t_i,t_{i+1}]$, we conclude

$$\begin{aligned}&\langle\hat{c}_n(t),u_n(\bar{t}_{i+1})-u_n(\bar{t}_i)\rangle\ge\\&\qquad\ge\langle u_n(t),u_n(\bar{t}_{i+1})-u_n(\bar{t}_i)\rangle-\langle u_n(\bar{t}_{i+1})-u_n(\bar{t}_i)),(u_n(t_{i+1})-u_n(t_i))\rangle\,,\end{aligned}$$

for any $t_i\le\bar{t}_i\le t\le\bar{t}_{i+1}\le t_{i+1}$. Therefore, denoting $\tau_{j,k}=\bar{t}_i+\left(j+\frac{1}{2}\right)\frac{\bar{t}_{i+1}-\bar{t}_i}{k}$ for $j\in\{0,1,\cdots,k-1\}$, one has (same approach is used e.g. in part (ii) of the proof of Monteiro Marques [31, Theorem 2.1, p. 12, second formula from below])

$$\begin{aligned}&\int_{\bar{t}_i}^{\bar{t}_{i+1}}\langle\hat{c}_n(\tau),du_n(d\tau)\rangle=\\&=\lim_{k\to\infty}\sum_{j=0}^{k-1}\left\langle\hat{c}_n\left(\tau_{j,k}\right),u_n\left(\bar{t}_i+(j+1)\frac{\bar{t}_{i+1}-\bar{t}_i}{k}\right)-u_n\left(\bar{t}_i+j\frac{\bar{t}_{i+1}-\bar{t}_i}{k}\right)\right\rangle\ge\\&\ge\lim_{k\to\infty}\sum_{j=0}^{k-1}\left\langle u_n\left(\tau_{j,k}\right),u_n\left(\bar{t}_i+(j+1)\frac{\bar{t}_{i+1}-\bar{t}_i}{k}\right)-u_n\left(\bar{t}_i+j\frac{\bar{t}_{i+1}-\bar{t}_i}{k}\right)\right\rangle+R_n,\end{aligned}$$

where the reminder R_n is given by

$$\begin{aligned}R_n=-\lim_{k\to\infty}\sum_{j=0}^{k-1}\Bigg\langle u_n&\left(\bar{t}_i+(j+1)\frac{\bar{t}_{i+1}-\bar{t}_i}{k}\right)-\\&-u_n\left(\bar{t}_i+j\frac{\bar{t}_{i+1}-\bar{t}_i}{k}\right),u_n(t_{i+1})-u_n(t_i)\Bigg\rangle=\\=-\langle u_n(\bar{t}_{i+1})&-u_n(\bar{t}_i),u_n(t_{i+1})-u_n(t_i)\rangle\,.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{\bar{t}_i}^{\bar{t}_{i+1}}\langle\hat{c}_n(\tau),du_n(d\tau)\rangle=\\&=\lim_{k\to\infty}\sum_{j=0}^{k-1}\left\langle u_n(\tau_{j,k}),u_n'(\tau_{j,k})\frac{\bar{t}_{i+1}-\bar{t}_i}{k}\right\rangle+R_n=\\&=\frac{1}{2}\lim_{k\to\infty}\sum_{j=0}^{k-1}\left(\frac{d}{d\tau}\|u_n(\tau)\|^2\right)\Big|_{\tau=\tau_{j,k}}\cdot\frac{\bar{t}_{i+1}-\bar{t}_i}{k}+R_n=\\&=\frac{1}{2}\int_{\bar{t}_i}^{\bar{t}_{i+1}}\frac{d}{d\tau}\|u_n(\tau)\|^2d\tau+R_n=\frac{1}{2}\left(\|u_n(\bar{t}_{i+1})\|^2-\|u_n(\bar{t}_i)\|^2\right)+R_n.\end{aligned}$$

This result can now be used to estimate the required integral (6.22) as follows

$$\int_s^t\langle\hat{c}_n(\tau),du_n(d\tau)\rangle\ge\frac{1}{2}\left(\|u_n(t)\|^2-\|u_n(s)\|^2\right)+R,\tag{6.23}$$

where

$$|R| \le \mathrm{var}(u_n, [s,t]) \cdot \max_{i \in \overline{0,n-1}} \|u_n(t_{i+1}) - u_n(t_i)\|.$$

Using (6.21) we now conclude that the desired statement (6.22) follows from (6.23) by passing to the limit as $n \to \infty$ (the passage to the limit is valid e.g. by Monteiro Marques [31, Theorem 2.1(ii)-(iii)] combined with formula (26) of p. 7 of the same book).

The proof of the theorem is complete. ◻

6.5. Proof of Theorem 3.1 (sweeping process without a parameter). Theorem 3.1 is a direct consequence of Theorem 6.11. One just views sweeping process (1.5) as sweeping process (1.7) with $\lambda = 0$.

REMARK 6.12. *Using Remark 6.5, Theorem 3.1 can be directly extended to sweeping processes of the form*

$$-\,dx \in N_{A(t,\lambda)+c(x,\lambda)}(x) + f(t,x,\lambda)dt, \quad x \in E, \;\; \lambda \in \mathbb{R}, \tag{6.24}$$

where A is a set-valued function with nonempty closed convex bounded values that satisfies the property

$$\begin{array}{l} d_H(A(s,\lambda), A(t,\lambda)) \le \mathrm{var}(\bar{a}, [s,t]), \quad \lambda \in [0,1], \\ \textit{where } \bar{a} : [0,T] \to \mathbb{R} \textit{ is a } BV\textit{-continuous function.} \end{array} \tag{6.25}$$

7. Proofs of the theorems on the existence of periodic solutions.

7.1. The Poincaré map associated to the catching-up algorithm. Even though we cannot ensure the existence of a Poincaré map for sweeping process (1.7), we can associate the following Poincaré map

$$P^{\lambda,n}(q) = x_n(T)$$

to the approximations x_n of the catching-up algorithm (6.4)-(6.7). Corollary 6.6 allows to formulate the following property of the map $P^{\lambda,n}$.

COROLLARY 7.1. *Assume that condition I) of Theorem 5.1 holds. Consider an open bounded set $Q \subset E$. Then, for each fixed $\lambda \in [0,1]$ and $n \in \mathbb{N}$, the Poincaré map $q \mapsto P^{\lambda,n}(q)$ is continuous on $\overline{Q}$.*

7.2. Proof of Theorem 4.1 (sweeping process without a parameter). To prove Theorem 4.1 we will use the following well-known result (see e.g. Krasnoselskii-Zabreiko [29, Theorem 6.2]):

THEOREM 7.2. *Let $\bar{P} : E \to E$ be a continuous map and let $Q \subset E$ be an open bounded convex set. If $\bar{P}(Q) \subset \overline{Q}$ and if $\bar{P}$ doesn't have fixed points on ∂Q, then*

$$d(I - \bar{P}, Q) = 1.$$

Proof of Theorem 4.1. Let Ω_1 be the neighborhood of Ω of radius 1. Since Ω is convex, then Ω_1 is convex as well. We will view sweeping process (1.5) as sweeping process (1.7) with $\lambda = 0$, i.e. we identify $a(t), c(x), f(t,x)$ with $a(t,0), c(x,0), f(t,x,0)$. So we consider the map

$$\overline{P}^{0,n}(x) = P^{0,n}(V(x)),$$

where $P^{0,n}$ is as introduced in Section 7.1 and V is as introduced in Section 3. We claim that

$$\overline{P}^{0,n}(\Omega_1) \subset \Omega, \quad \text{for all } n \in \mathbb{N}. \tag{7.1}$$

We have $V(x) \in \Omega$ by the definition of the map V. Then, according to the catching-up scheme (6.4)-(6.7), we have that

$$x_{i+1}^n \in A + a(t_{i+1}, 0) + c(x_{i+1}^n, 0), \quad \text{i.e. } x_{i+1}^n \in \Omega_{t_{i+1}}, \ i \in \overline{0, n-1}.$$

Therefore, $x_{i+1}^n = b + c(x_{i+1}^n, 0)$, where $b \in A + a(t_{i+1}, 0)$. And using $\xi = c(\xi, 0)$, we conclude $\|x_{i+1}^n - \xi\| \leq \|b\| + L_2\|x_{i+1}^n - \xi\|$, which means that $x_n(T) \in \Omega_T \subset \Omega$, i.e. (7.1) holds.

Using the continuity of $P^{0,n}$ (Corollary 7.1) and V (Lemma 6.3) along with Theorem 7.2, we get the existence of $q_n \in \Omega$ such that $\overline{P}^{0,n}(q_n) = q_n$, which implies

$$P^{0,n}(q_n) = q_n, \quad n \in \mathbb{N},$$

because $V(q_n) \in \Omega$. In other words, we have $x_n(T, q_n, 0) = x_n(0, q_n, 0)$ for all $n \in \mathbb{N}$. Now, Theorem 6.11 applied with $\lambda_n = 0$, implies the existence of a convergent subsequence $\{x_{n_k}(t, q_{n_k}, 0)\}$ whose limit $x(t)$ is solution of (1.5) with the required T-periodicity property (4.1). The proof is complete. ∎

7.3. Proofs of Theorems 5.1 and 5.2 (sweeping process with a parameter).

Proof of Theorem 5.1. **Step 1.** First we prove that there exists $N > 0$ such that $d(I - P^{\lambda,n} \circ V^{\lambda,n}, Q)$ is defined for $n \geq N$ and $\lambda \in [0, \lambda_1]$. Assuming the contrary, we get a sequence $n_k \to \infty$, $\lambda_k \to \lambda_0 \in [0, \lambda_1]$, and a converging sequence $\{q_k\}_{k\in\mathbb{N}} \subset \partial Q$ such that

$$P^{\lambda_k, n_k} \circ V^{\lambda_k, n_k}(q_k) = q_k, \quad k \in \mathbb{N}. \tag{7.2}$$

Applying Lemma 6.7, Corollary 6.10, and Theorem 6.11 we conclude that $q_0 = \lim_{k\to\infty} q_k \in \partial Q$ is a generalized initial condition of a T-periodic solution (6.20) of sweeping process (1.7) with $\lambda = \lambda_0$, which contradicts conditions III) of Theorem 5.1.

The conclusion of Step 1, in particular, implies that

$$d(I - P^{\lambda,n} \circ V^{\lambda,n}, Q) = d(I - P^{0,n} \circ V^{0,n}, Q), \quad n \geq N, \ \lambda \in [0, \lambda_1].$$

Step 2. Here we use assumption II (uniqueness) of Theorem 5.1 to conclude that

$$P^{0,n} \circ V^{0,n}(q) \to P^0 \circ V^0(q), \quad \text{as } n \to \infty,$$

uniformly with respect to $q \in \overline{Q}$. Thus, we can diminish $N > 0$ in such a way that $d(I - P^{0,n} \circ V^{0,n}, Q) = d(I - P^0 \circ V^0, Q)$, $n \geq N$, which gives

$$d(I - P^{\lambda,n} \circ V^{\lambda,n}, Q) \neq 0, \quad n \geq N, \ \lambda \in [0, \lambda_1].$$

Therefore, for each $\lambda \in [0, \lambda_1]$ there exists $q_n \in Q$ such that the approximations $\{x_n(\cdot, q_n, \lambda)\}_{n\geq N}$ are T-periodic, so this sequence has a convergent subsequence which converges to a T-periodic solution $x(\cdot)$ of (1.7) with initial condition $q = \lim_{n_k\to\infty} q_{n_k}$ as $n \to \infty$ according to Corollary 6.10.

We can see that $q \in V^\lambda(Q)$ directly from the catching-up algorithm (6.4)-(6.7). Indeed, as in the proof of Theorem 4.1, we observe from (6.4)-(6.7) that $x_{i+1}^n \in A + a(t_{i+1},\lambda) + c(x_{i+1}^n,\lambda)$, for all $i \in \overline{0,n-1}$. Plugging $i = n-1$ and passing to the limit as $n \to \infty$, we get $x(T) \in A + a(T,\lambda) + c(x(T),\lambda)$. Therefore, $x(T) = V^\lambda(x(T))$. But $q = x(T)$ and so $q = V^\lambda(q) \in V^\lambda(Q)$.

The proof of the theorem is complete. ∎

The proof of Theorem 5.2 follows the lines of the proof of Theorem 5.1. The only difference is in the beginning of Step 1, which now proves the existence of both $N > 0$ and $\lambda_1 \in (0,1]$ such that $d(I - P^{\lambda,n} \circ V^{\lambda,n}, Q)$ is defined for $n \geq N$ and $\lambda \in [0,\lambda_1]$. Assuming the contrary, we get a sequence $n_k \to \infty$, $\lambda_k \to 0 \in [0,1]$, and a converging sequence $\{q_k\}_{k\in\mathbb{N}} \subset \partial Q$ such that (7.2) holds, that leads to the existence of a T-periodic solution to sweeping process (1.7) with $\lambda = 0$, contradicting condition II) of Theorem 5.1. The rest of the proof of Theorem 5.2 follows the proof of Theorem 5.1 just literally.

8. Existence of periodic solutions in the neighborhood of a boundary equilibrium (the theorem and its proof). This section uses the following extension of Theorem 7.2 (see e.g. Krasnoselskii-Zabreiko [29, Theorem 31.1]):

THEOREM 8.1. *Let $\bar{P} : E \to E$ be a continuous map and let $Q \subset E$ be an open bounded set. If $(\bar{P})^m$ maps Q strictly into itself for all $m \in \mathbb{N}$ sufficiently large, then*

$$d(I - \bar{P}, Q) = 1.$$

The main assumption of this section is that sweeping processes (1.7) reduces to

$$-\dot{x} \in N_A(x) + f_0(x), \qquad x \in E, \tag{8.1}$$

when $\lambda = 0$ and that (8.1) posses a switched equilibrium on the boundary ∂A (as was earlier introduced in Kamenskii-Makarenkov [20] in 2d). To introduce the definition of a switched boundary equilibrium $x_0 \in \partial A$, we assume that in some neighborhood $Q \subset \mathbb{R}^n$ of x_0 the boundary ∂A is smooth and can be described as

$$\partial A \cap Q = \{x \in Q : H(x) = 0\}, \quad \text{where } H \in C^1(\mathbb{R}^n, \mathbb{R}).$$

DEFINITION 8.2. *A point $x_0 \in \partial A$ is a switched boundary equilibrium of sweeping process (8.1), if*

$$H(x) > 0, \quad \text{for all } x \in Q \backslash A,$$

and

$$H'(x_0) = \alpha f_0(x_0) \quad \text{for some } \alpha < 0.$$

As the definition says, x_0 is not an equilibrium of f_0, however the next two lemmas imply that the solution of (8.1) with the initial condition at x_0 don't leave x_0.

If x_0 is a switched equilibrium, then Q can be considered so small that

$$\langle f_0(x), H'(x)\rangle < 0, \quad \text{for all } x \in \partial A \cap Q. \tag{8.2}$$

The next lemma claims that $\partial A \cap Q$ is a sliding region for sweeping process (8.1).

LEMMA 8.3. *Let $x_0 \in \partial A$ be a switched equilibrium of (8.1) and let $Q \subset E$ be such a neighborhood of x_0 that (8.2) holds. Consider a solution x of (8.1) with an initial condition $x_0 \in \partial A \cap Q$. Let $t_1 > 0$ be such that $x(t) \in Q$ for all $t \in [0,t_1]$. Then $x(t) \in \partial A$ for all $t \in [0,t_1]$.*

Proof. Let us assume, by contradiction, that there exists $t_{escape} \in [0, t_1]$ where $x(t)$ escapes from ∂A, i.e.

$$t_{escape} = \max\{t_0 \geq 0 : x(t) \in Q, H(x(t)) = 0,\ t \in [0, t_0]\} < t_1.$$

By the definition of t_{escape}, for any $\delta > 0$ there exist $t_\delta \in [t_{escape}, t_{escape} + \delta]$ such that $H(x(t)) < 0$ for each $t \in (t_{escape}, t_\delta]$. Since, the solution $x(t)$ satisfies $\dot{x}(t) = -f_0(x(t))$ on $(t_{escape}, t_\delta]$, by the Mean-Value Theorem

$$H(x(t_\delta)) - H(x(t_{escape}) = -H'(x(t_\delta^*))f_0(x(t_\delta^*))(t_\delta - t_{escape}),$$

for some $t_\delta^* \in (t_{escape}, t_\delta)$. This yields

$$H'(x(t_{escape}))f_0(x(t_{escape})) \geq 0,$$

as $\delta \to 0$, contradicting (8.2).

The proof of the lemma is complete. ◻

As it happens in the theory of Filippov systems (see [16]), the dynamics of (8.1) in the sliding region is described by a smooth differential equation. Indeed, let us introduce the differential equation

$$\begin{aligned} &-\dot{x} = \bar{f}(x), \\ &\text{where } \bar{f}(x) = f_0(x) - \pi_{H'(x)}(f_0(x)) \text{ and } \pi_L(\xi) = \frac{1}{\|L\|^2}\langle \xi, L\rangle L. \end{aligned} \tag{8.3}$$

Next lemma says that (8.3) is the equation of sliding motion for sweeping process (8.1) in the neighborhood of switched equilibrium $x_0 \in \partial A$.

Lemma 8.4. *Let the conditions of Lemma 8.3 hold and let $x(t)$ be the sliding solution $x(t)$, $t \in [0, t_1]$, of sweeping process (8.1) as introduced in Lemma 8.3. Then $x(t)$ is a solution of (8.3) on $[0, t_1]$.*

Proof. Fix $t \in [0, t_1]$ such that $\dot{x}(t)$ exists. Then, from (8.1),

$$-\dot{x}(t) = \alpha H'(x(t)) + f_0(x(t)), \quad \text{with some } \alpha > 0,$$

or

$$\alpha H'(x(t)) = -\pi_{H'(x(t))}(f_0(x(t))) + \left[-f_0(x(t)) + \pi_{H'(x(t))}(f_0(x(t)))\right] - \dot{x}(t). \tag{8.4}$$

From the definition of $\pi_L(\xi)$ we have

$$\left\langle -f_0(x(t)) + \pi_{H'(x(t))}(f_0(x(t))), H'(x(t))\right\rangle = 0.$$

On the other hand, from Lemma 8.3,

$$\langle \dot{x}(t), H'(x(t))\rangle = 0.$$

Therefore, taking the scalar product of (8.4) with $H'(x(t))$, we get

$$\alpha = -\frac{1}{\|H'(x(t))\|^2}\langle f_0(x(t)), H'(x(t))\rangle,$$

which completes the proof. ◻

Lemma 8.4 implies that the boundary ∂A is an invariant manifold for the differential equation (8.3). The definition (8.3) reduces the dimension of the image of f_0 by 1. Therefore, the image of the map $\bar{f}$ acts to a space of dimension $dimE - 1$, which implies that one eigenvalue of the Jacobian $\bar{f}'(x_0)$ is always zero.

We now offer an asymptotic stability result which can be of independent interest in applications of perturbed sweeping processes.

THEOREM 8.5. *Let $x_0 \in \partial A$ be a switched equilibrium of (8.1). If real parts of $dimE - 1$ eigenvalues of the Jacobian $\bar{f}'(x_0)$ are negative, then x_0 is a uniformly asymptotically stable point of sweeping process (8.1).*

Proof. **Step 1.** *Convergence to ∂A.* Let $B_r(x_0)$ be a ball of radius r centered at x_0. Let us show that there exists $r > 0$ such that for any $\xi \in B_r(x_0) \cap A$, the solution $t \mapsto X(t, \xi)$ of

$$\dot{x} = -f_0(x) \tag{8.5}$$

with the initial condition $X(0, \xi) = \xi$ reaches ∂A at time some time $\tau(\xi) > 0$. The proof will be through the Implicit Function Theorem applied to

$$F(t, x) = H(X(t, x)).$$

We have $F(0, x_0) = 0$ and $F_t(0, x_0) = -H'(x_0)f_0(x_0) \neq 0$ by the definition of switched equilibrium. Therefore, Implicit Function Theorem (see e.g. Rudin [39, Theorem 9.28]) ensures the existence of $\xi \to \tau(\xi)$ defined and continuous on a sufficiently small ball $B_r(x_0)$ and such that $\tau(x_0) = 0$.

It remains to show that $\tau(\xi) > 0$ for all $\xi \in B_r(x_0) \cap A$. Since, according to the definition of switched equilibrium, $H'(x_0)^T$ is a normal to A pointing outwards to A, it is sufficient to prove that $\tau(\xi) > 0$ for $\xi = x_0 - \lambda H'(x_0)^T$ with all $\lambda > 0$ sufficiently small. So we introduce a scalar function

$$G(\lambda) = \tau(x_0 - \lambda H'(x_0)^T)$$

and want to prove that $G'(0) > 0$. Using the formula for the derivative of the implicit function (see [39, Theorem 9.28])

$$\tau'(x_0) = -(H'(x_0)f_0(x_0))^{-1}H'(x_0)$$

and so

$$G'(0) = -(H'(x_0)f_0(x_0))^{-1}H'(x_0)(-H'(x_0)^T) = H'(x_0)f_0(x_0)\|H'(x_0)\|^2,$$

which is indeed positive according to Definition 8.2.

Finally, let us fix $\xi \in B_r(x_0) \cap A$ and let $x(t)$ be the solution of (8.1) with the initial condition $x(0) = \xi$. Since the conclusion of the Implicit Function Theorem comes with uniqueness, we have that $X(t, \xi) \notin \partial A$, $t \in [0, \tau(\xi))$. Therefore, $X(t, \xi) = x(t)$, for any $t \in [0, \tau(\xi))$, which implies that $\lim_{t \to \tau(\xi)} X(t, \xi) = \lim_{t \to \tau(\xi)} x(t)$ and so $x(\tau(\xi)) \in \partial A$.

Step 2. *Convergence along ∂A.* Lemmas 8.3 and 8.4 combined with the negativeness of real parts of $dimE - 1$ eigenvalues of $\bar{f}'(x_0)$ imply that there exists an neighborhood $x_0 \in Q \subset E$ such that any solution of (8.1) with the initial condition $x(0) \in Q \cap \partial A$

converges to x_0 along ∂A as $t \to \infty$ and the convergence is uniform with respect to the initial condition.

Making now $r > 0$ in Step 1 so small that $\cup_{\xi \in B_r(x_0)} X(\tau(\xi), \xi) \in Q$ (which is possible by continuity of $\xi \to \tau(\xi)$), we combine Step 1 and Step 2 to conclude that any solution of (8.1) with $x(0) \in B_r(x_0)$ approaches x_0 as $t \to \infty$.

The proof of the theorem is complete. ∎

We are now in the position to combine theorems 5.2, 8.1, and 8.5 when the following condition holds for (1.7) at $\lambda = 0$:

$$a(t, 0) \equiv 0, \quad c(x, 0) \equiv 0, \quad f(t, x, 0) \equiv f_0(x) \text{ with } f_0 \in C^1(E, E), \tag{8.6}$$

and the T-periodicity condition (5.2) holds for $\lambda > 0$.

THEOREM 8.6. *Assume that condition I) of Theorem 5.1 holds. Assume, that for $\lambda = 0$ sweeping process (1.7) is smooth autonomous, i.e. satisfies (8.6). If real parts of $n - 1$ eigenvalues of $\bar{f}'(x_0)$ are negative for some switched equilibrium $x_0 \in \partial A$, then there exists $\lambda_1 > 0$ such that for all $\lambda \in (0, \lambda_1]$ the T-periodic sweeping process (1.7) admits a T-periodic solution $x_\lambda(t) \to x_0$ as $\lambda \to 0$.*

Proof. Let $\bar{P}(x) = P^0(V^0(x))$. By Theorem 8.5, there exists an open bounded set $x_0 \in Q \subset E$ such that $(\bar{P})^m$ maps Q strictly into itself for all $m \in \mathbb{N}$ sufficiently large. Therefore, Theorem 8.1 ensures that condition II) of Theorem 5.1 holds, so Theorem 5.2 applies. ∎

Similar to Theorem 8.6 results about periodic perturbations of autonomous ordinary differential equations have been obtained by Berstein-Halanai [5] and Cronin [12].

9. Conclusions. By extending the implicit catching-up scheme of Kunze and Monteiro Marques [24] to perturbed sweeping processes, we proved solvability of BV-continuous state-dependent sweeping processes with Lipschitz dependence on the state. We further used topological degree arguments to establish the existence of periodic solutions to sweeping processes of this type. The analysis is carried out for the simplest possible moving set $C(t) = A + a(t) + c(x)$ throughout the entire paper, that allowed us to focus on the development of core mathematical ideas rather than on its possible generalizations. We explain in Remarks 6.5 and 6.12 how the existence result (Theorem 3.1) immediately extends to the moving set of the form $C(t) = A(t) + c(x)$. At the same time, Remark 6.5 shows that our method of proof of continuity of approximations $x_n(t)$ on the initial condition fails for moving sets of the form $C(t) = A(t) + c(x)$. Since continuous dependence of $x_n(t)$ on the initial condition is the main ingredient in our proof of the existence of periodic solutions, the respective main theorems (Theorem 4.1 and Theorem 8.6) do not readily extend even to the moving set of the form $C(t) = A(t) + c(x)$. We don't know whether or not an alternative approach (e.g. formula (6.17) quoted from [31, Proposition 4.7, p. 26]) can deal with any more general state-dependent moving constraints. Perhaps the ideas of Nacry [36] can be helpful here.

The existence of T-periodic solutions to a sweeping process with T-periodic right-hand-sides and convex moving set would be an immediate result when uniqueness and continuous dependence of solutions on the initial conditions holds. The difficulty we overcame when proving the existence of periodic solutions comes from the fact that uniqueness and continuous dependence on the initial conditions of solutions of

BV-continuous state-dependent sweeping processes is still an open question even when the dependence on the state is Lipschitz continuous (for state-independent sweeping processes uniqueness and continuous dependence is established e.g in Castaing and Monteiro Marques [10] and Adly et al [2]). The way how we overcame the possible lack of uniqueness and continuous dependence is by considering an approximation of the sweeping process by a discretization coming from a suitable catching-up algorithm. We then used the uniqueness and continuous dependence of the solutions of the discretization obtained in order to establish the existence of periodic solutions to the respective discretized equations. After this step was complete, the existence of periodic solutions in the initial sweeping process came by passing to the limit upon the increasing accuracy of the discretization.

The second part of the paper concerns sweeping processes with a parameter λ, for which we developed a topological degree based continuation principle. As an application of the continuation principle, we proved the occurrence of periodic solutions at a specific location being a neighborhood of a switched boundary equilibrium. Specifically, we assumed that for $\lambda = 0$, the sweeping process is autonomous and admits an asymptotically stable switched boundary equilibrium x_0. We then proved the occurrence of T-periodic solutions from x_0 when the parameter λ increases and the sweeping process becomes nonautonomous (and T-periodic). The condition for asymptotic stability of x_0 can be replaced by assuming that the topological index of x_0 is different from 0. Such a condition can be also expressed in terms of the eigenvalues of the linearization $\bar{f}(x_0)$ of sliding differential equation (8.3), see e.g. Krasnoselskii-Zabreiko [29, Theorem 6.1] and [29, Theorem 7.4] (which will be required to account for the vector field outside of the boundary of the constraint).

We stress that the question of asymptotic stability, Lyapunov stability, and invariance of absolutely continuous (or more regular) solutions of perturbed sweeping processes has been intensively addressed in the recent literature, see e.g. Brogliato-Tanwani [7], Tanwani et al [40], Leine-van de Wouw [26], Kunze and Monteiro Marques [25], Adly [1], Kamenskii et al [19], Makarenkov-Niwanthi [30], Niwanthi et al [37], Hantoute-Vilches [18]. Combined with Theorem 8.1, these results can be eventually used for verifying condition II (the topological degree condition) of Theorem 5.1 on a suitable region $Q \subset E$.

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Appendix A. Flowchart of the results. To ease navigation through the proofs of the paper a flowchart of the results is given in Fig. 2. An arrow from one statement to another indicates that the target statement uses the source statement for the proof.

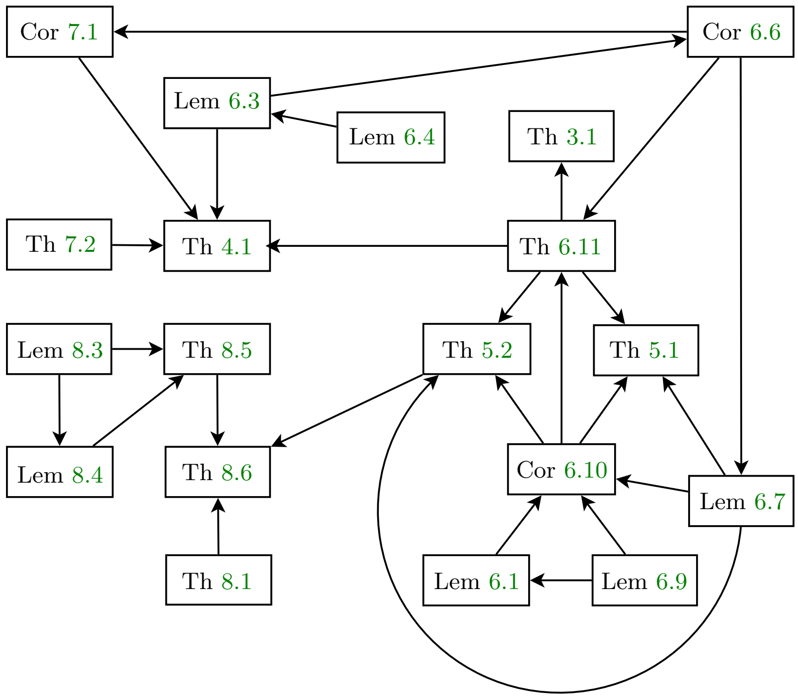

FIG. 2. *The flowchart of the results of the paper.*

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