THE K'–THEORY OF MONOID SETS

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C. A. Weibel dedicates this paper to his father, C. G. Weibel.

There are two flavors of K-theory for a pointed monoid A in the literature; see [D], [CLS], [M] and [Sz] for example. One is $K(A)$, the K-theory of the category of finitely generated projective A–sets; the other is $G(A)$, the K-theory of the category of all finitely generated A–sets. While $K(A)$ is quite simple, $G(A)$ turns out to be complicated, even when A is $\mathbb{N} = \{*, 1, t, t^2, ...\}$; see Example 2.1.2.

One important class of monoids is the class of *partially cancellative* monoids (or pc monoids); see Definition 1.2. This class includes cancellative monoids (such as $\mathbb{N}$ and the polyhedral cones underlying affine toric varieties) as well as the quotients A/I of cancellative monoids by an ideal, which are useful in studying blow-ups of toric varieties; see [CHWW].

In this paper, we study $K'(A)$, the K-theory of the category of finitely generated *partially cancellative* A–sets over a pc monoid (see Definition 1.2). Such A–sets are well-behaved, include free A–sets, and quotients A/I of A by an ideal, such as $\mathbb{N}/t^N\mathbb{N} = \{1, t, ..., t^{N-1}, t^N = *\}$.

Our main result is an analogue of the "Fundamental Theorem" in algebraic K-theory (Theorem 4.6): if A is an abelian partially cancellative monoid then

$$K'(A) \simeq K'(A \wedge \mathbb{N}) \quad \text{and} \quad K'(A \wedge \mathbb{Z}_+) \simeq K'(A) \vee \Omega^{-1}K'(A).$$

In particular, $K'(\mathbb{N}) \simeq \mathbb{S}$, the sphere spectrum. The corresponding result fails dramatically for G-theory; see Example 2.1.2. We establish similar results for the K'-theory of pc monoid schemes in Section 5.

Our key tools are the additivity, devissage and localization theorems for "CGW"-categories, developed by Campbell and Zakharevich [CZ].

We define partially cancellative monoids in Section 1. In Section 2 we describe the K-theory of various categories of A–sets, using the constructions and techniques of [CZ]. In Section 3 we establish a number of structural theorems. In Section 4 we prove the FundamentalTheorem alluded to above, as well as a monoid version of the Farrell–Hsiang computation of the K-theory of twisted polynomial rings [FH]. Finally, in Section 5 we discuss the K'-theory of monoid schemes and in particular compute it for projective spaces.

We shall always assume all categories are skeletally small. We shall write $\mathbb{S}$ for the sphere spectrum and $S^\infty X$ for the suspension spectrum of a space X.

1. Partially cancellative monoids

Definition 1.1. By a *monoid A* we mean a pointed set with an associative product and distinguished elements $*, 1$ such that $a\cdot 1 = a = 1\cdot a$ and $a \cdot * = * = * \cdot a$ for all $a \in A$; these are sometimes called "pointed monoids." It is *left noetherian* if it has the ascending chain condition on left ideals; right noetherian monoids are defined similarly.

The units (i.e., invertible elements) of A form a group. We note that the initial monoid $\{*, 1\}$ is called $\mathbb{F}_1$ in [CLS].

A (left) A–set is a pointed set X with an action of A; in particular, $1 \cdot x = x$ and $* \cdot x = *$ for all $x \in X$. For example, a free A–set is just a wedge of copies of A.

Definition 1.2. A (left) noetherian monoid A is *partially cancellative*, or *pc* for short, if $ac = bc \neq *$ (or $ca = cb \neq *$) implies $a = b$ for all a, b, c in A. A prototype finite pc monoid is $\{1, t, ..., t^N, t^{N+1} = *\}$.

If A is a pc monoid, we say that a pointed (left) A–set X is *partially cancellative* if for every $x \in X$ and a, b in A, if $ax = bx \neq *$ then $a = b$. (We allow $ax = ay \neq *$ for $x \neq y$ in X.) Note that pc A–sets form a subcategory of A–sets which is closed under subobjects and quotients, and contains A.

Remark 1.2.1. If A is a pc monoid, then the subset $\mathfrak{m}$ of non-units in A is a two-sided ideal, and is the unique maximal (two-sided) ideal of A. Indeed, if $xy = 1$, then $xyx = x$ and therefore $yx = 1$.

Example 1.3. Let $\mathbb{N}$ denote the pointed monoid $\{*, 1, t, t^2, ...\}$. A (pointed) $\mathbb{N}$–set is just a pointed set X with a successor function $x \mapsto tx$. Thus we may identify a finite $\mathbb{N}$–set with a (pointed) directed graph such that every vertex has outdegree 1. Every finite rooted tree is a pc $\mathbb{N}$–set; the successor of a vertex x is the adjacent vertex closer to

the root vertex $*$. In fact, a finite $\mathbb{N}$–set is partially cancellative if and only if it is a rooted tree, because for every $x \in X$, the sequence $\{x, tx, t^2x, ...\}$ terminates at the basepoint.

An $\mathbb{N}$–set is not pc if and only if it contains a loop, i.e., there is an element $x \neq *$ and an integer d such that $t^d x = x$. A typical non-pc $\mathbb{N}$–set is $\{*, 1, t, ..., t^d, ..., t^{N-1} \mid t^N = t^d\}$ (a loop with a tail).

Example 1.4. If G is a group, we write G_+ for the pointed monoid $G \coprod \{*\}$ (with the evident product). Then every G_+–set is a wedge of copies of cosets $(G/H)_+$. If H is a proper subgroup of G, then $(G/H)_+$ is not pc because $h \cdot H = 1 \cdot H$ for $h \in H$. Thus a G_+–set is pc if and only if it is free.

2. Quasi-exact categories

Definition 2.1. ([D], [WK, Ex. IV.6.14]) A *quasi-exact category* is a category $\mathcal{C}$ with a distinguished zero object, and a coproduct $\vee$, equipped with a family $\mathcal{S}$ of sequences of the form

$$0 \to Y \xrightarrow{i} X \xrightarrow{j} Z \to 0,$$

called "admissible," such that: (i) any sequence isomorphic to an admissible sequence is admissible;
(ii) for any admissible sequence, j is a cokernel for i and i is a kernel for j;
(iii) $\mathcal{S}$ contains all split sequences (those with $X \cong Y \vee Z$); and
(iv) the class of admissible epimorphisms (resp., admissible monics) is closed under composition and pullback along admissible monics (resp., pullback along admissible epimorphisms).
We will often write X/Y for the cokernel of $Y \rightarrowtail X$.

Quillen's construction in [Q] yields a category $Q\mathcal{C}$, and $K(\mathcal{C})$ is the connective spectrum with initial space $\Omega BQ\mathcal{C}$; we write $K_n(\mathcal{C})$ for $\pi_n K(\mathcal{C})$. The group $K_0(\mathcal{C})$ is generated by the objects of $\mathcal{C}$, modulo the relations that $[X] = [Y] + [Z]$ for every admissible sequence.

Remark. Dyckerhoff and Kapranov [DK] have a similar construction for the weaker notion of "proto-exact categories."

Example 2.1.1. The category $\mathbf{Sets}_f$ of finite pointed sets is quasi-exact; every admissible sequence is split exact. It is well known that the Barratt–Priddy–Quillen theorem implies that $K(\mathbf{Sets}_f) \simeq \mathbb{S}$ (see [D], [CLS], [WK, Ex. IV.6.15].)

More generally, the category of finitely generated projective[1] A–sets is quasi-exact for any monoid A; every admissible exact sequence is

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[1]i.e., A-sets with the projective lifting property

split. The K-theory of the category of finitely generated projective A–sets is written as $K(A)$. Following [D, 3.1] and [CLS, 2.27], we see that if A has no idempotents or units then $K(A) \simeq \mathbb{S}$.

Example 2.1.2. If A is a left noetherian monoid, the category of finitely generated pointed left A–sets is quasi-exact; a sequence (2.1) is admissible if $Y \rightarrowtail X$ is an injection, and Z is isomorphic to the quotient A–set X/Y. (See [WK, Ex. IV.6.16].)

Following [CLS], we define $G(A)$ to be the K-theory of this category. For example, the group $G_0(\mathbb{N}) = \pi_0 G(\mathbb{N})$ is the free abelian group on $[\mathbb{N}]$ and the infinite set of loops of varying lengths.

If $A = G_+$ for a group G, then $G_0(A)$ is the Burnside ring of G; for example, $G_0(\mathbb{Z}_+)$ is free abelian on the classes of the (pointed) cyclic groups $\mathbb{Z}_+$ and the loops $(\mathbb{Z}/n\mathbb{Z})_+$. By [CDD, 5.2], the spectrum $G(A)$ is the G–fixed points of the equivariant sphere spectrum, at least if G is an abelian group.

Example 2.1.3. If A is a pc monoid, the category $A-\mathbf{Sets}_{\mathrm{pc}}$ of finitely generated pc A–sets is quasi-exact. We write $K'(A)$ for $K(A-\mathbf{Sets}_{\mathrm{pc}})$ and set $K'_n(A) = \pi_n K'(A)$.

Example 2.1.4. When $A = G_+$ for a group G, we saw in Example 1.4 that every pc A–set is a free A–set. Therefore $K(G_+) = K'(G_+)$. Note that $K_0(G_+) = \mathbb{Z}$ differs from $G_0(G_+)$ when G is not simple.

If $X = (G_+)^{\vee r}$, then $\mathrm{Aut}(X)$ is the wreath product $G \wr \Sigma_r$. By the Barratt–Priddy–Quillen theorem, $K(A)$ is the group completion of $\coprod B(G \wr \Sigma_r)$, which is $\Omega^\infty S^\infty(BG_+)$. In particular when $A = \mathbb{Z}_+$ (i.e., $G = \mathbb{Z}$), we have

$$K'(\mathbb{Z}_+) \simeq K(\mathbb{Z}_+) \simeq \mathbb{S} \vee \Omega^{-1}\mathbb{S}, \quad K'_n(\mathbb{Z}_+) \cong \pi_n^s \oplus \pi_{n-1}^s$$

This calculation of $K(G_+)$ is well known; see [CLS, 5.9].[2]

If $\mathcal{C}$ is a quasi-exact category, we can form a double category $(\mathcal{M}, \mathcal{E})$ with the same objects as $\mathcal{C}$; the horizontal and vertical maps are the admissible monics and epis (composed backwards), respectively, and the 2–cells are commutative diagrams of the form

(2.2)
$$
\begin{array}{ccc}
Y & \rightarrowtail & X \\
\upuparrows & & \upuparrows \\
Y' & \rightarrowtail & X'.
\end{array}
$$

We say that a square (2.2) is *distinguished* if the natural map of cokernels $X'/Y' \to X/Y$ is an isomorphism. Thus distinguished squares are

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[2]The formula in [D, p. 146] is incorrect, as $\mathrm{Aut}(G_+^{\vee n})$ is the wreath product $G \wr \Sigma_n$.

both pushout squares and pullback squares. Note that $\mathcal{C}$ is an "ambient category" for $(\mathcal{M}, \mathcal{E})$ in the sense of [CZ, 2.2]. We define $k(X \twoheadrightarrow X/Y)$ to be $(Y \rightarrowtail X)$ and $c(Y \rightarrowtail X) = (X \twoheadrightarrow X/Y)$.

Lemma 2.3. *If $\mathcal{C}$ is a quasi-exact category, $(\mathcal{M}, \mathcal{E})$ is a CGW-category in the sense of* [CZ, 2.4].

Proof. We need to verify the axioms in *loc. cit.* The isomorphism $\mathrm{iso}(\mathcal{E}) \cong \mathrm{iso}(\mathcal{M})$ is [CZ, 2.2]. Axiom (Z) holds because $\mathcal{C}$ has a zero object; Axiom (I) follows from (i) and (iii); and axiom (A) is (iii). Axiom (K) is immediate from the definitions of c and k, and Axiom (M) is (ii). □

Remark 2.3.1. In [CZ], Campbell and Zakharevich define the K-theory of a CGW-category using an appropriate version of the Q-construction. If $\mathcal{C}$ is a quasi-exact category with associated CGW-category $(\mathcal{M}, \mathcal{E})$, then $Q\mathcal{C}$ and $Q(\mathcal{M}, \mathcal{E})$ are isomorphic categories, and thus the two possible definitions of the K-theory coincide.

With this in mind, we will abuse notation and consider a quasi-exact category as a CGW-category.

Lemma 2.4. *If X is a pointed A–set and Y, Z are pointed A–subsets, then $Y/(Y \cap Z) \rightarrowtail X/Z \twoheadrightarrow X/(Y \cup Z)$ is an admissible sequence. That is, $Y/(Y \cap Z) \overset{\cong}{\rightarrow} (Y \cup Z)/Z$.*

The proof of Lemma 2.4 is standard, and left to the reader.

Theorem 2.5. *Let A be a pointed monoid, and $\mathcal{C}$ a quasi-exact subcategory of A–$\mathbf{Sets}$ closed under pushouts and pullbacks. Then the associated double category $(\mathcal{M}, \mathcal{E})$ is an ACGW-category in the sense of* [CZ, 5.2–5.3].

Proof. We let "commutative square" mean a 2–cell of the form (2.2). For this, we need to check the axioms (P), (U), (S) and (PP). Axiom (P) is the evident assertion that $\mathcal{M}$ is closed under pullbacks, and $\mathcal{E}$ is closed under pushouts.

Given Lemma 2.4, Axiom (U) is equivalent to the following assertion: given a pointed A–set X and pointed A–subsets Y, Z (i.e., a diagram $Y \rightarrowtail X \twoheadrightarrow X/Z$), the pushout $(X/Z) \cup_X (X/Y)$ and pullback $Y \times_X Z =$

$Y \cap Z$ fit into a commutative diagram:

$$\begin{array}{ccccc} \square & \rightarrowtail & X/Z & \twoheadrightarrow & (X/Z) \cup_X (X/Y) \\ \uparrow & & \uparrow & & \uparrow \\ Y & \rightarrowtail & X & \twoheadrightarrow & X/Y \\ \uparrow & & \uparrow & & \uparrow \\ Y \cap Z & \rightarrowtail & Z & \twoheadrightarrow & D \end{array}$$

(The object $\square$ is $Y/(Y \cap Z) \cong (Y \cup Z)/Z$.)

Axiom (S) says that given a pullback square in $\mathcal{M}$ of the form

$$\begin{array}{ccc} Y \cap Z & \rightarrowtail & Z \\ \downarrow & & \downarrow \\ Y & \rightarrowtail & X, \end{array}$$

the pushout $P = Y \cup_{Y \cap Z} Z$ exists, and X/P is the pushout of X/Y and X/Z along X/K. Dually, given a pushout square in $\mathcal{E}$, of $X \twoheadrightarrow X/Y$ and $X \twoheadrightarrow X/Z$, the pullback $L = X/(Y \cup Z)$ of X/Y and X/Z along $(X/Y) \cup_X (X/Z)$ exists, and the kernel of $X \to L$ is $Y \cup Z$. Both of these assertions are elementary, in the spirit of Lemma 2.4.

Axiom (PP) says that if V is an A–subset of both X and Y, then V is the intersection of X and Y in $X \cup_V Y$. (The pushout $X \cup_V Y$ of X and Y along V is the quotient of the wedge $X \vee Y$, modulo the equivalence relation identifying the two copies of V; it is the object $X \star_V Y$ of [CZ, 5.3].) Dually, given epis $X \twoheadrightarrow Z$ and $Y \twoheadrightarrow Z$, the same argument shows that the kernel of $(X \times_Z Y) \twoheadrightarrow X$ is isomorphic to the kernel of $Y \twoheadrightarrow Z$. $\qquad \square$

Remark 2.5.1. It would suffice to prove Theorem 2.5 for pointed sets, because the forgetful functor from $A-$**Sets** to pointed sets creates colimits and limits; its left adjoint is the free functor, and its right adjoint sends a pointed set X to its co-induced A–set $\mathrm{Hom}(A, X)$. However, the proof would be no shorter.

Corollary 2.6. *If A is a left noetherian monoid, the associated double category $(\mathcal{M}, \mathcal{E})$ associated to $A-$**Sets**$_{fg}$ (fin. gen. A–sets) is an ACGW–category. The same is true for any subcategory closed under pushouts and pullbacks, such as the categories of pc A–sets (if A is pc) and finite A–sets.*

Proof. It is straightforward that finitely generated (or pc, if A is pc, or finite) A–sets are closed under pushouts and pullbacks in the category $A-$**Sets** when A is left noetherian. Thus Theorem 2.5 applies. $\qquad \square$

3. K'-THEORY OF A MONOID

We recall the following definition from Example 2.1.3.

Definition 3.1. If A is a pc monoid, $K'(A)$ denotes the K-theory of the category of finitely generated, partially cancellative A–sets.

Remark 3.1.1. K'-theory is contravariantly functorial for normal monoid maps betweeen pc monoids, as these maps preserve the condition $ax = bx \neq *$. (Recall from [CLS] that a monoid map $f : A \to B$ is *normal* if it is the composition of a quotient by an ideal and an injection.) Since B is pc, f is normal iff B is a pc A-set.

K'-theory is covariantly functorial for flat monoid maps (a monoid map $A \to B$ is *flat* if the base extension functor $X \mapsto B \wedge_A X$ is exact), as flat base extensions preserve distinguished squares.

We will need the following theorem, taken from [CZ, 6.1] and based on [Q, Thm. 5.4].

Theorem 3.2 (Devissage)**.** *Let $\mathcal{A}$ be a full pre-ACGW subcategory of a pre-ACGW category $\mathcal{B}$, closed under subobjects and quotient objects, and such that $\mathcal{E}_\mathcal{A} \to \mathcal{E}_\mathcal{B}$ creates pushouts. Suppose that every object of $\mathcal{B}$ has a finite filtration*

$$1 \rightarrowtail F_1 B \rightarrowtail \cdots \rightarrowtail F_n B = B$$

such that each $F_i B / F_{i-1} B$ is in $\mathcal{A}$. Then $K(\mathcal{A}) \cong K(\mathcal{B})$.

Remark 3.2.1. The hypothesis that $\mathcal{E}_\mathcal{A} \to \mathcal{E}_\mathcal{B}$ creates pushouts will be satisfied in our applications.

Lemma 3.3. *If A is a pointed pc monoid of finite length[3] with units G, then*

$$K'(A) \simeq K'(A/\mathfrak{m}) \simeq S^\infty(BG_+).$$

In particular, if A has no nontrivial units then $K'(A) \simeq \mathbb{S}$. In particular, $K'(\mathbb{N}/t^n\mathbb{N}) \simeq \mathbb{S}$ for all $n \geq 1$.

Proof. The unique maximal ideal $\mathfrak{m}$ of A (see Remark 1.2.1) defines a finite filtration of any finitely generated A–set X: if $\mathfrak{m}^n = *$ then

$$X \supset \mathfrak{m} X \supset \mathfrak{m}^2 X \supset \cdots \mathfrak{m}^i X \supset \cdots \supset \mathfrak{m}^n X = *.$$

The lemma follows by Devissage (3.2), and the fact that $A/\mathfrak{m} = G_+$. $\square$

Remark 3.3.1. If A is any pointed monoid of finite length, the same proof shows that $G(A) \simeq G(A/\mathfrak{m})$. However, if A has nontrivial units then $G(A/\mathfrak{m}) \not\simeq S^\infty(BG_+)$; see Example 2.1.4.

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[3]the length of A is the length of the longest chain of ideals

The following Additivity Theorem is a special case of [CZ, 7.14]. Recall from Theorem 2.5 that the double category $(\mathcal{M}, \mathcal{E})$ associated to a quasi-exact category is an ACGW-category in many cases of interest.

Theorem 3.4 (Additivity)**.** *If $\mathcal{B}$ is a quasi-exact category such that $(\mathcal{M}, \mathcal{E})$ is an ACGW-category, and $s \rightarrowtail t \twoheadrightarrow q$ is an admissible sequence of exact functors $\mathcal{B} \to \mathcal{A}$, then*

$$t_* = s_* + q_* : K(\mathcal{B}) \to K(\mathcal{A}).$$

Proof. By [CZ, 7.5, 7.11], there is a CGW-category $S_2\,\mathcal{B}$ of admissible sequences $B_0 \rightarrowtail B_1 \twoheadrightarrow B_2$. The given admissible sequence defines exact functors $s, q, t : S_2\,\mathcal{B} \to \mathcal{A}$, and if $\amalg : \mathcal{B} \times \mathcal{B} \to S_2\,\mathcal{B}$ is the coproduct functor then the following compositions agree:

$$\mathcal{B} \times \mathcal{B} \xrightarrow{\amalg} S_2\,\mathcal{B} \underset{s \vee q}{\overset{t}{\rightrightarrows}} \mathcal{A}.$$

Hence they give the same map on K-theory. By [CZ, 7.14], the source and target functors yield an equivalence

$$K(\mathcal{B}) \vee K(\mathcal{B}) \xrightarrow{\simeq} K(S_2\mathcal{B}).$$

Hence $t_* = (s \vee q)_* = s_* + q_*$, as required. $\square$

If $i : A \to A/I$ is a surjection of pointed monoids, there is an exact functor $(A/I)-\mathbf{Sets} \to A-\mathbf{Sets}$. If A is a pc monoid, we have a map $i_* : K'(A/I) \to K'(A)$.

Theorem 3.5. *Let A be a pc monoid. Suppose that $s \in A$ is such that the set $\{s^n\}_{n \geq 0}$ is a 2–sided denominator set (that is, s-left fractions and s-right fractions coincide, see [WK, II.A]). Then we have a fibration sequence*

$$K'(A/sA) \xrightarrow{i_*} K'(A) \longrightarrow K'(A[s^{-1}]).$$

Proof. Consider the category $\mathcal{T}$ of (pc) s-torsion A–sets; by Corollary 2.6, $\mathcal{T}$ is an ACGW-category. In fact, it is a sub ACGW-category of $A-\mathbf{Set}_{\mathrm{pc}}$. Since every s-torsion A–set X is finitely generated, X has a finite filtration by the subsets $s^n X$. By Devissage, $K'(A/sA) \simeq K(\mathcal{T})$.

We will apply the localization theorem [CZ, 8.5] to $\mathcal{T} \subset A-\mathbf{Sets}_{\mathrm{pc}}$. The argument of [CZ, 8.6] shows that the category $A-\mathbf{Sets}_{\mathrm{pc}}\backslash\mathcal{T}$ is equivalent to $A[1/s]-\mathbf{Sets}_{\mathrm{pc}}$, and hence is a CGW category, so axiom (CGW) holds. Axiom (W) holds: $\mathcal{T}$ is "m-negligible" in $A-\mathbf{Sets}_{\mathrm{pc}}$ because every pc A-set N contains $s^n N$ (which is s-torsion-free for sufficiently large n), and if the kernel of $M \twoheadrightarrow N$ is in $\mathcal{T}$, then $s^n M \xrightarrow{\sim} s^n N$

for n large enough. Moreover, $\mathcal{T}$ is "e-well represented" because, given pc A–sets N_1, N_2 and V, and A–set maps

$$N_1 \to N_1[s^{-1}] \xrightarrow{\sim} V[s^{-1}] \text{ and } N_2 \to N_2[s^{-1}] \xrightarrow{\sim} V[s^{-1}]$$

the pullback $N = N_1 \times_{V[s^{-1}]} N_2$ is a pc A–set and $N[s^{-1}] \to V[s^{-1}]$ is an isomorphism. Finally, axiom (E) holds because every morphism $A \to B$ in $A-\mathbf{Sets}_{\mathrm{pc}} \backslash \mathcal{T}$ is represented as $A \twoheadrightarrow A/(s-\text{torsion}) \rightarrowtail s^n B$ for some n. $\qquad\square$

Remark 3.5.1. The same proof works to show that if A is left noetherian there is a fibration sequence

$$G(A/sA) \xrightarrow{i_*} G(A) \longrightarrow G(A[s^{-1}]).$$

The following corollary is an analogue of [WK, Ex. V.6.4], [Q, Exercise in §6] and [FH].

Corollary 3.6. *Let A be a pc monoid and $\phi\colon A \to A$ an automorphism. Write $A \rtimes \mathbb{N}$ and $A \rtimes \mathbb{Z}$, respectively, for the corresponding semi-direct product monoids (in which $ta = \phi(a)t$). Write $i\colon A \to A \rtimes \mathbb{N}$ for the inclusion. Then there is a fibration sequence*

$$K'(A) \xrightarrow{i_* - i_* \phi_*} K'(A \rtimes \mathbb{N}) \longrightarrow K'(A \rtimes \mathbb{Z}).$$

In particular (taking ϕ to be the identity), we get a fibration sequence

$$K'(A) \xrightarrow{0} K'(A \wedge \mathbb{N}) \to K'(A \wedge \mathbb{Z}).$$

Thus $K'(A \wedge \mathbb{Z}) \simeq K'(A \wedge \mathbb{N}) \vee \Omega^{-1} K'(A)$.

Proof. Apply Theorem 3.5 to $A \rtimes \mathbb{N}$ and $s = t$. The fact that $K'(A) \to K'(A \rtimes \mathbb{N})$ is $i_* - i_* \phi_*$ follows from additivity applied to the characteristic exact sequence of an A–set: $\phi_*(X) \wedge \mathbb{N} \xrightarrow{t} X \wedge \mathbb{N} \xrightarrow{i} X$, where the action of $A \rtimes \mathbb{N}$ on $\phi_*(X) \wedge \mathbb{N}$ is twisted by ϕ. $\qquad\square$

Remark 3.6.1. Again, the same proof works if A is (left) noetherian to yield a fibration sequence $G(A) \xrightarrow{0} G(A \wedge \mathbb{N}) \to G(A \wedge \mathbb{Z})$, and hence that

$$G(A \wedge \mathbb{Z}) \simeq G(A \wedge \mathbb{N}) \vee \Omega^{-1} G(A).$$

4. Fundamental Theorem

We now apply the results in the previous section to prove Theorems 4.6 and 4.7.

Lemma 4.1. *Let R, S and T be spectra, with S and T of finite type. Given an equivalence $f : S \vee T \to R \vee S \vee T$, which is an equivalence on S, then f induces an equivalence $T \simeq T$ and $R \simeq 0$.*

Proof. Localizing at $\mathbb{Q}$, a dimension count shows that the homotopy groups of R are finite abelian groups. Taking coefficients modulo a prime, a cardinality argument shows that all homotopy groups of R vanish. Hence $R \simeq 0$, and the result follows easily. $\qquad\square$

Lemma 4.2. *The composition $K'(A) \xrightarrow{f^*} K'(A \wedge \mathbb{N}) \xrightarrow{j^*} K'(A \wedge \mathbb{Z}_+)$ splits, where f^* and j^* are induced from the base change $X \mapsto X \wedge_A \mathbb{N}$ and localization functor $X \mapsto X[1/t]$.*

Proof. Since both f^* and j^* are exact, the composition is defined. Consider the orbit functor γ from $A \wedge \mathbb{Z}_+ - \mathbf{Sets}_{\mathrm{pc}}$ to $A - \mathbf{Sets}_{\mathrm{pc}}$, sending X to $X/\sim$, where $x \sim y$ if $y = t^n x$ for some integer n. It is easy to see that γ is an exact functor, and that $\gamma j^* f^*$ is the identity. $\qquad\square$

Theorem 4.3. $K'(\mathbb{N}) \simeq \mathbb{S}$.

The conclusion of Theorem 4.3 fails for $G(\mathbb{N})$; see Example 2.1.2.

Proof. Let $\mathcal{C}$ denote the category of finite pc $\mathbb{N}$–sets; it is an *ACGW* category by Theorem 2.5. Since each finite pc $\mathbb{N}$–set X has a finite filtration by $t^n X$, Devissage 3.2 and Example 2.1.1 imply that $K(\mathcal{C}) \simeq K(\mathbf{Sets}_{\mathrm{f}}) \simeq \mathbb{S}$. Then the localization sequence 3.6 yields a fibration sequence

$$K(\mathbf{Sets}_{\mathrm{f}}) \xrightarrow{0} K'(\mathbb{N}) \longrightarrow K'(\mathbb{Z}_+).$$

Hence $K'(\mathbb{Z}_+) \simeq K'(\mathbb{N}) \vee \Omega^{-1} K(\mathbf{Sets}_{\mathrm{f}})$. By Example 2.1.4, we see that $K'(\mathbb{Z}_+) \simeq \mathbb{S} \vee \Omega^{-1} K(\mathbf{Sets}_{\mathrm{f}})$. Since the inclusion (given by base extension) $\mathbb{S} \simeq K(\mathbf{Sets}_{\mathrm{f}}) \to K'(\mathbb{Z}_+)$ factors through $K'(\mathbb{N})$, Lemma 4.2 implies that there is a spectrum R such that $K'(\mathbb{N}) \simeq R \vee \mathbb{S}$. By Lemma 4.1, $K'(\mathbf{Sets}_{\mathrm{f}}) \simeq K'(\mathbb{N})$. $\qquad\square$

Porism 4.4. If G is a group, then the proof of Theorem 4.3 applies, using Lemma 3.3, to show that $K'(G_+ \wedge \mathbb{N}) \simeq K'(G_+) \simeq S^\infty(BG_+)$.

Theorem 4.5. *If A is a pc monoid of finite length with units G, then*

$$K'(A \wedge \mathbb{Z}_+) \cong K'(G \times \mathbb{Z}_+) \simeq S^\infty(BG_+) \vee \Omega^{-1} S^\infty(BG_+).$$

In particular, if $G = \{1\}$ then $K'(A \wedge \mathbb{Z}_+) \cong K'(\mathbb{Z}_+) \simeq \mathbb{S} \vee \Omega^{-1}\mathbb{S}$.

Proof. $A \wedge \mathbb{Z}_+$ has finite length, and its units are $G \times \mathbb{Z}$. By Lemma 3.3, we conclude that $K'(A \wedge \mathbb{Z}_+) \cong K'(G \times \mathbb{Z}_+)$. By Example 2.1.4, this is $S^\infty B(G \times \mathbb{Z})_+$, and

$$S^\infty B(G \times \mathbb{Z})_+ \simeq S^\infty(BG_+) \vee \Omega^{-1} S^\infty(BG_+). \qquad\square$$

Theorem 4.6. *If A is a pc abelian monoid, then*

$$K'(A) \simeq K'(A \wedge \mathbb{N}) \quad and \quad K'(A \wedge \mathbb{Z}_+) \simeq K'(A) \vee \Omega^{-1} K'(A).$$

Proof. It suffices to prove the first equivalence, as the second equivalence follows from it, using Theorem 3.5. If $A = G_+$ then $K'(G_+) \simeq K'(G_+ \wedge \mathbb{N})$ by Porism 4.4. Inductively, suppose that the result holds for pc monoids with n generators over its group of units, and that A is generated by $s = s_0, ..., s_n$, where s is in $\mathfrak{m}_A$. Then the result holds for the monoids A/sA and $A[1/s]$ (which are both pc by [CHWW, Prop. 9.1]), so by naturality and Theorem 3.5 (which applies as A is abelian) we have a map of fibration sequences whose outside maps are equivalences:

$$\begin{array}{ccccc} K'(A/sA) & \longrightarrow & K'(A) & \longrightarrow & K'(A[1/s]) \\ \downarrow{\scriptstyle\simeq} & & \downarrow & & \downarrow{\scriptstyle\simeq} \\ K'(A/sA \wedge \mathbb{N}) & \longrightarrow & K'(A \wedge \mathbb{N}) & \longrightarrow & K'(A[1/s] \wedge \mathbb{N}). \end{array}$$

By the 5–lemma, the middle map $K'(A) \to K'(A \wedge \mathbb{N})$ is an equivalence. $\square$

Theorem 4.7. *Let G be a group, $\phi : G \to G$ an automorphism, and $G_+ \rtimes \mathbb{N}$ and $G_+ \rtimes \mathbb{Z}$ the corresponding semidirect product monoids. Then the base extension map $K'(G_+) \to K'(G_+ \rtimes \mathbb{N})$ is an equivalence and we have a fibration sequence $K'(G_+) \xrightarrow{1-\phi_*} K'(G_+) \to K'(G_+ \rtimes \mathbb{Z})$.*

Proof. We consider the diagram

$$\begin{array}{ccccc} K'(G_+) & \xrightarrow{1-\phi_*} & K'(G_+) & \longrightarrow & K'(G_+ \rtimes \mathbb{Z}) \\ \downarrow{\scriptstyle=} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle=} \\ K'(G_+) & \xrightarrow{i_*-i_*\phi_*} & K'(G_+ \rtimes \mathbb{N}) & \longrightarrow & K'(G_+ \rtimes \mathbb{Z}), \end{array}$$

where i_* is the base extension map along the inclusion $G_+ \to G_+ \rtimes \mathbb{N}$. The bottom sequence is the localization sequence of Corollary 3.6, and the whole diagram commutes by inspection. It therefore suffices to show that the top row is a cofibration sequence; it is canonically equivalent to the sequence of suspension spectra of classifying spaces $S^\infty BG_+ \xrightarrow{1-\phi_*} S^\infty BG_+ \to S^\infty B(G \rtimes \mathbb{Z})_+$. This last sequence is a cofibration sequence by the mapping torus construction: unstably, the homotopy coequalizer of 1 and ϕ_* is computed by the mapping torus $T = BG \times I/\{(x,0) \sim (\phi_*(x),1)\}$. This mapping torus has $BG \times \mathbb{R}$ as a covering space with deck transformation group $\mathbb{Z}$. It follows that $\pi_n(T) = 0$ for $n \geq 2$, and it is easy to verify that $\pi_1(T) \cong G \rtimes \mathbb{Z}$. Thus $T \simeq B(G \rtimes \mathbb{Z})$, and stabilisation yields the desired conclusion. $\square$

5. Monoid schemes

The constructions of Section 3 can be generalized to define the K'-theory of noetherian partially cancellative monoid schemes. For relevant definitions see [CHWW]. In particular, given an abelian pc monoid A, we have an affine pc monoid scheme $\mathrm{MSpec}(A)$; a general pc monoid scheme is a noetherian ringed space locally isomorphic to such an affine pc monoid scheme. Since any A–set F defines a sheaf $\widetilde{F}$ on $\mathrm{MSpec}(A)$, we can replace A-sets by sheaves on $\mathrm{MSpec}(A)$. Recall as well from [CHWW, Defn. 2.5] that an equivariant closed subscheme of a monoid scheme X is one that is defined by a sheaf of ideals.

Definition 5.1. Let X be a pc monoid scheme. A pc set on X is a sheaf that is locally of the form $\widetilde{F}$ for a finitely generated pc A–set F; compare [CLS, 3.3]. We write $X-\mathbf{Sets}_{\mathrm{pc}}$ for the (quasi-exact) category of pc sets on X; its admissible sequences are those that are locally admissible in the sense of Example 2.1.2. We define $K'(X)$ to be the K-theory of $X-\mathbf{Sets}_{\mathrm{pc}}$.

Remark 5.1.1. If $X = \mathrm{MSpec}(A)$, then there is a natural equivalence of quasi-exact categories $X-\mathbf{Sets}_{\mathrm{pc}} \cong A-\mathbf{Sets}_{\mathrm{pc}}$ (see [CLS, Cor. 3.13]), and therefore $K'(X) \simeq K'(A)$.

Remark 5.1.2. The K'-theory of pc monoid schemes is contravariantly functorial for flat morphisms, and covariantly functorial for equivariant closed immersions. This is immediate from Remark 3.1.1.

As pushouts and pullbacks of sheaves, and admissible sequences, are all detected locally, the following result is an immediate consequence of Corollary 2.6.

Theorem 5.2. *Let X be a pc monoid scheme. Then any quasi-exact subcategory of $X-\mathbf{Sets}_{\mathrm{pc}}$ that is closed under pushouts and pullbacks is an ACGW-category.*

Here is the analogue of Quillen's localisation theorem [Q, 7.3.2].

Theorem 5.3. *Let X be a pc monoid scheme and $Z \xrightarrow{i} X$ an equivariant closed subscheme with open complement $U \xrightarrow{j} X$. Then there is a fibration sequence of spectra*

$$K'(Z) \xrightarrow{i_*} K'(X) \xrightarrow{j^*} K'(U).$$

Proof. Let J be the ideal sheaf defining Z. The proof of Theorem 3.5 applies *mutatis mutandis*, with $\mathcal{T}$ the subcategory of $X-\mathbf{Sets}_{\mathrm{pc}}$ consisting of those sheaves supported on Z, or equivalently, the subcategory of J-torsion sheaves. By Devissage, $K(\mathcal{T})$ is equivalent to $K'(Z)$.

The localized (bi-)category $X-\mathbf{Sets}_{\mathrm{pc}}\backslash\mathcal{T}$ is equivalent to $U-\mathbf{Sets}_{\mathrm{pc}}$; in particular, axiom (CGW) of [CZ, Theorem 8.5] holds. The remaining axioms are checked as in the proof of Theorem 3.5. $\qquad\square$

Remark 5.3.1. The localisation sequence of the theorem is natural for flat pullbacks $X' \to X$. In particular, if $V \subset X$ is an open subscheme containing Z, then

$$
\begin{array}{ccccc}
K'(Z) & \longrightarrow & K'(X) & \longrightarrow & K'(U) \\
\downarrow{\scriptstyle =} & & \downarrow & & \downarrow \\
K'(Z) & \longrightarrow & K'(V) & \longrightarrow & K'(U \cup V).
\end{array}
$$

is a map of fibration sequences of spectra.

The global version of the Fundamental Theorem is now easily derived. As in [CHWW], we writee $\mathbb{A}^1$ for $\mathrm{MSpec}(\mathbb{N})$.

Theorem 5.4. *Let X be a pc monoid scheme. Then the pullback map $K'(X) \to K'(X \times \mathbb{A}^1)$ is an equivalence, and*

$$K'(X \wedge \mathbb{Z}_+) \simeq K'(X) \vee \Omega^{-1}K'(X).$$

Proof. Note that $\mathrm{MSpec}(A) \times \mathbb{A}^1 = \mathrm{MSpec}(A \wedge \mathbb{N})$. By Theorem 4.6 the assertion holds for affine pc monoid schemes, and in particular for pc monoid schemes of dimension 0. The general case is now proved using induction on the dimension of X using Theorem 5.3. $\qquad\square$

Corollary 5.5. *Let $\xi : E \to X$ be a vector bundle over a pc monoid scheme. Then $\xi^* : K'(X) \to K'(E)$ is an equivalence.*

Proof. For a trivial vector bundle, this follows from the theorem by induction on the rank. Now the general case follows from localization 5.3, applied to an open cover of X trivializing E. $\qquad\square$

Theorem 5.6. *Let X be a pc monoid scheme. Then there is a natural equivalence $K'(X \times \mathbb{P}^1) \simeq K'(X) \vee K'(X)$.*

Proof. We have an open covering $X \times \mathbb{P}^1 = U_1 \cup U_2$, with each $U_i \cong X \times \mathbb{A}^1$ and $U_1 \cap U_2 \cong X \wedge \mathbb{Z}_+$. The complement of each U_i is isomorphic to X. Applying Remark 5.3.1 and the Fundamental Theorem 5.4, we obtain a homotopy cocartesian square of spectra

$$
\begin{array}{ccc}
K'(X \times \mathbb{P}^1) & \longrightarrow & K'(X) \\
\downarrow & & \downarrow \\
K'(X) & \longrightarrow & K'(X) \vee \Omega^{-1}K'(X).
\end{array}
$$

The right vertical and bottom map are isomorphisms onto the first summand; the assertion follows.   □

Theorem 5.7. *Let X be a pc monoid scheme, and n a non-negative integer. Then there is a natural equivalence*

$$K'(X \times \mathbb{P}^n) \simeq K'(X)^{\vee n+1}.$$

In particular, $K'(\mathbb{P}^n) \simeq \mathbb{S} \vee \mathbb{S} \vee \cdots \vee \mathbb{S}$ ($n + 1$ copies of $\mathbb{S}$).

Proof. We proceed by induction on n, the case $n = 0$ being tautological. Suppose $n > 0$ and we have proved the assertion for $n - 1$. Applying Theorem 5.3 to the equivariant closed immersion $X \times \mathbb{P}^{n-1} \to X \times \mathbb{P}^n$ with open complement $U \cong X \times \mathbb{A}^n$ we obtain a fibration sequence of spectra

$$K'(X \times \mathbb{P}^{n-1}) \to K'(X \times \mathbb{P}^n) \to K'(X \times \mathbb{A}^n) \simeq K'(X).$$

Because $K'(X) \to K'(X \times \mathbb{P}^n) \to K'(X \times \mathbb{A}^n)$ is an equivalence by Corollary 5.5, this sequence is split; the assertion for n follows.   □

Remark 5.7.1. In particular, $K'_0(\mathbb{P}^1) \cong \mathbb{Z} \oplus \mathbb{Z} \cong \mathbb{Z} \times \mathrm{Pic}(\mathbb{P}^1)$, and the isomorphism is given by the rank and determinant. See [FW] for the computation of the Picard group of monoid schemes.

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