POTENTIALS OF A FAMILY OF ARRANGEMENTS OF HYPERPLANES AND ELEMENTARY SUBARRANGEMENTS

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Abstract. We consider the Frobenius algebra of functions on the critical set of the master function of a weighted arrangement of hyperplanes in C^k with normal crossings. We construct two *potential* functions (of first and second kind) of variables labeled by hyperplanes of the arrangement and prove that the matrix coefficients of the Grothendieck residue bilinear form on the algebra are given by the $2k$-th derivatives of the potential function of first kind and the matrix coefficients of the multiplication operators on the algebra are given by the $(2k + 1)$-st derivatives of the potential function of second kind. Thus the two potentials completely determine the Frobenius algebra. The presence of these potentials is a manifestation of a Frobenius like structure similar to the Frobenius manifold structure.

We introduce the notion of an elementary subarrangement of an arrangement with normal crossings. It turns out that our potential functions are local in the sense that the potential functions are sums of contributions from elementary subarrangements of the given arrangement. This is a new phenomenon of locality of the Grothendieck residue bilinear form and multiplication on the algebra.

It is known that this Frobenius algebra of functions on the critical set is isomorphic to the Bethe algebra of this arrangement. (That Bethe algebra is an analog of the Bethe algebras in the theory of quantum integrable models.) Thus our potential functions describe that Bethe algebra too.

1. Introduction

It is well known that the algebra of functions on the set of solutions of the Bethe ansatz equations plays an important role in the study of quantum integrable systems since in many cases the algebra of functions is isomorphic to the Bethe algebra of Hamiltonians of the system, see for example [NS, MTV1, GRTV, R]. An interesting problem is to describe the algebra. In this paper we consider the model case of the algebra of functions on the critical set of the master function associated with a family of arrangements with normal crossings. Such algebras appear in the KZ-Gaudin type integrable systems, see for example [SV, RV]. We describe the algebra of functions on the critical set together with the Grothendieck residue bilinear form in terms of derivatives of two potential functions in the spirit of Frobenius structures.

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1.1. **Statement of results.** Denote $J = \{1,...,n\}$. Consider $\mathbb{C}^n \times \mathbb{C}^k$ with coordinates $(z,t) = (z_1,...,z_n,t_1,...,t_k)$ and the projection $\tau : \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^n$. Fix n nonzero linear functions on $\mathbb{C}^k$, $g_j = b^1{}_j t_1 + \cdots + b^k{}_j t_k$, $j \in J$. Assume that $\{g_j\}_{j\in J}$ span the dual space $(\mathbb{C}^k)^*$. Define the functions $f_j = g_j + z_j$ on $\mathbb{C}^n \times \mathbb{C}^k$. We obtain on $\mathbb{C}^n \times \mathbb{C}^k$ an arrangement $\mathcal{C} = \{H_j\}_{j=1}^n$, where H_j is the zero set of f_j. Let $U(\mathcal{C}) := \mathbb{C}^n \times \mathbb{C}^k - \cup_{j\in J} H_j$ be the complement. For every $x \in \mathbb{C}^n$, the arrangement $\mathcal{C}$ restricts to an arrangement $\mathcal{C}(x)$ on $\tau^{-1}(x) \tilde{=} \mathbb{C}^k$ with the complement $U(\mathcal{C}(x)) := \tau^{-1}(x) \cap U(\mathcal{C})$. For almost all $x \in \mathbb{C}^k$ the arrangement $\mathcal{C}(x)$ is with normal crossings. The subset $\Delta \subset \mathbb{C}^n$, where this does not hold, is a hypersurface and is called the discriminant.

A set $I = \{i_1,...,i_k\} \subset J$ is called independent if $g_{i_1},...,g_{i_k}$ are linearly independent. Denote J^{ind} the set of all independent k-element subsets of J.

Let $a = (a_1,...,a_n) \in (\mathbb{C}^*)^n$ be a system of weights such that for any $x \in \mathbb{C}^n - \Delta$ the weighted arrangement $(\mathcal{C}(x),a)$ is unbalanced, see Section 2.10, e.g. $a \in \mathbb{R}^n_{>0}$ is unbalanced, also a generic system of weights is unbalanced. The *master function* of the weighted arrangement $(\mathcal{C},a)$ is

$$\Phi_{\mathcal{C},a}(z,t) := \sum_{i\in J} a_i \log f_i. \tag{1.1}$$

For $x \in \mathbb{C}^n - \Delta$ all critical points of $\Phi_{\mathcal{C},a}|_{z=x}$ with respect to the variables t, are isolated, and the sum μ of their Milnor numbers is independent of the unbalanced weight a and the parameter $x \in \mathbb{C}^n - \Delta$. The main object of this paper is the μ-dimensional algebra

$$\mathcal{O}(C_{\mathcal{C}(x),a}) := \mathcal{O}(U(\mathcal{C}(x))) \Big/ \Big(\frac{\partial \Phi_{\mathcal{C},a}}{\partial t_j} \,\Big|\, j = 1,...,k \Big) \tag{1.2}$$

of functions on the critical set of the master function $\Phi_{\mathcal{C},a}|_{z=x}$, see Section 3.4. Define (1.3)

$$p_j := \Big[\frac{a_j}{f_j}\Big] = \Big[\frac{\partial \Phi_{\mathcal{C},a}}{\partial z_j}\Big] \in \mathcal{O}(C_{\mathcal{C}(x),a}), \qquad j \in J.$$

The elements $\{p_j\}_{j\in J}$ generate $\mathrm{O}(C_{\mathcal{C}(x),a})$ as an algebra. The elements $\{p_{i_1}\cdots p_{i_k}\}_{\{i_1,...,i_k\}\in J^{\text{ind}}}$ generate $\mathrm{O}(C_{\mathcal{C}(x),a})$ as a vector space. The Grothendieck residue defines a nondegenerate bilinear form $(\ ,\ )_{C_{\mathcal{C}(x),a}}$ on $\mathrm{O}(C_{\mathcal{C}(x),a})$. The algebra $(\mathrm{O}(C_{\mathcal{C}(x),a}),(\ ,\ )_{C_{\mathcal{C}(x),a}})$ is a Frobenius algebra.

The main result of this paper is a construction of two functions P, Q on $\mathbb{C}^n - \Delta$ called the *potentials of first and second kind*, respectively. The potentials have the following properties.

Theorem 1.1. *Let $x \in \mathbb{C}^n - \Delta$. Then for any two independent subsets $\{i_1,...,i_k\},\{l_1,...,l_k\} \subset J$ and any $i_0 \in J$, we have*

$$(p_{i_1}\cdots p_{i_k}, p_{l_1}\cdots p_{l_k})_{C_{\mathcal{C}(x),a}} = (-1)^k \frac{\partial^{2k} P}{\partial z_{i_1}\ldots\partial z_{i_k}\partial z_{l_1}\ldots\partial z_{l_k}}(x), \tag{1.4}$$

$$(p_{i_0}p_{i_1}\cdots p_{i_k}, p_{l_1}\cdots p_{l_k})_{C_{\mathcal{C}(x),a}} = (-1)^k \frac{\partial^{2k+1} Q}{\partial z_{i_0}\partial z_{i_1}\ldots\partial z_{i_k}\partial z_{l_1}\ldots\partial z_{l_k}}(x). \tag{1.5}$$

Formula (1.4) determines the Grothendieck residue bilinear form $(,)_{C_{\mathcal{C}(x),a}}$ in terms of the potential of first kind. Formula (1.5) determines the operators of multiplication by generators $\{p_j\}_{j\in J}$ in terms of the potential of second kind.

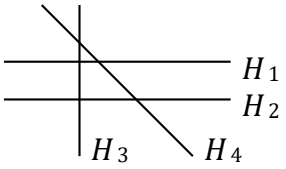

Figure 1

Example. For the arrangement of four lines shown in Figure 1 and given by equations $t_2 + z_1 = 0$, $t_2 + z_2 = 0$, $t_1 + z_3 = 0$, $t_1 + t_2 + z_4 = 0$ we have

$$P = \frac{1}{a_1+a_2+a_3+a_4}\Big(a_1a_3a_4\frac{(z_1+z_3-z_4)^4}{4!} + a_2a_3a_4\frac{(z_2+z_3-z_4)^4}{4!} + \frac{a_1a_2a_3a_4}{a_3+a_4}\frac{(z_1-z_2)^2}{2!}\frac{(z_1+z_3-z_4)^2}{2!}\Big),$$

$$Q = a_1a_3a_4\ln(z_1+z_3-z_4)\frac{(z_1+z_3-z_4)^4}{4!} + a_2a_3a_4\ln(z_2+z_3-z_4)\frac{(z_2+z_3-z_4)^4}{4!} + \frac{a_1a_2a_3a_4}{a_3+a_4}\ln(z_1-z_2)\frac{(z_1-z_2)^2}{2!}\frac{(z_1+z_3-z_4)^2}{2!}.$$

Theorem 1.1 in particular says that $(p_1p_3, p_2p_4)_{C_{\mathcal{C}(x),a}} = \frac{a_1a_2a_3a_4}{(a_1+a_2+a_3+a_4)(a_3+a_4)}$ and it does not depend on $x \in \mathbb{C}^n - \Delta$, and $(p_4p_1p_3, p_3p_4)_{C_{\mathcal{C}(x),a}} = \frac{a_1a_3a_4}{z_1(x)+z_3(x)-z_4(x)}$.

In this example the potentials are sums of terms corresponding to subarrangements consisting of three or four lines. It turns out that this is the general case. In Section 4.1 we introduce the notion of an elementary arrangement in $\mathbb{C}^k$ of type $\lambda = (\lambda_1,\ldots,\lambda_m)$, $\lambda_h \in \mathbb{Z}_{>0}$, $\lambda_1 + \cdots + \lambda_m = k$. In particular, such an elementary arrangement consists of $k + m$ hyperplanes, and an elementary arrangement in $\mathbb{C}^k$ has at most $2k$ hyperplanes. We show that the potentials are sums, over all elementary subarrangements, of the prepotentials of the subarrangements taken with suitable weights, see Corollary 6.4 and Theorem 7.1. The fact that the potentials are sums of contributions from elementary subarrangements indicates a new phenomenon of *locality* of the Grothendieck residue bilinear form and multiplication on $\mathcal{O}(C_{\mathcal{C}(x),a})$.

The existence of the potentials of first and second kind locally on $\mathbb{C}^n - \Delta$ was established in [HV].

1.2. **Frobenius like structure of order** (n,k,m)**.** The potential of the second kind is an analog of the potential in the theory of Frobenius manifolds. A Frobenius manifold is a manifold with a flat metric and a Frobenius algebra structure on tangent spaces at points of the manifold such that the structure constants of multiplication are given by third derivatives of a potential function on the manifold with respect to flat coordinates, see [D, M]. As an analogy of that, for our family of arrangements the structure constants of multiplication are given by $2k + 1$-st derivatives of the potential of second kind, see Theorem 1.1.

The notion of potentials of a family of arrangements was introduced and studied in [V5, V7, HV]. In [V5] the potentials were constructed for the families of *generic arrangements*, that is, such that the linear functions $g_{i_1},...,g_{i_k}$ are linearly independent for any distinct $i_1,...,i_k \in J$. In [V5, V7, HV] different axiomatizations of the structure leading to the existence of the potentials were given. In particular in [HV] Frobenius like structures of order (n,k,m) were introduced. Our case of a family of arrangements corresponds to the case of order $(n,k,2)$. Under the axioms of [HV] the existence of the potential of second kind was deduced in [HV] from a surprising elementary study of finite sets of vectors in a finite-dimensional vector space W. Given a natural number m and a finite set $\{w_i\}$ of vectors, a necessary and sufficient condition was given to find in the set $\{w_i\}$ m bases of W. If m bases in the set $\{w_i\}$ are selected, then some elementary transformations of such a selection are defined. It was shown in [HV] that any two selections are connected by a sequence of elementary transformations. These structures are fundamental and one may expect a matroid version of them.

1.3. **Bethe algebra.** Given a family of weighted arrangements in $\mathbb{C}^k$ as in Section 1.1, one considers the Gauss-Manin differential equations for associated k-dimensional hypergeometric integrals, $\kappa\frac{\partial I}{\partial z_j}(z) = K_j(z)I(z),\ j \in J,\ z \in \mathbb{C}^n - \Delta$, where $K_i(z)$ are suitable linear operators on the space of singular vectors $\mathrm{Sing}_a V$, see Section 3.3. For every $x \in \mathbb{C}^n - \Delta$, the operators $K_j(x), j \in J$, commmute and are symmetric with respect to the contravariant bilinear form $S^{(a)}$ on $\mathrm{Sing}_a V$. The unital subalgebra of $\mathrm{End}(\mathrm{Sing}_a V)$ generated by the operators $K_j(x), j \in J$, is called the Bethe algebra of the weighted arrangement $(\mathbb{C}(x),a)$. This algebra is the analog of the Bethe algebra in the theory of quantum integrable systems, see [V4]. It is known that the Bethe algebra together with the bilinear form $S^{(a)}$ is isomorphic to the pair consisting of the algebra of multiplication operators on $\mathrm{O}(C_{\mathbb{C}(x),a})$ and the Grothendieck residue bilinear form $(,)_{C_{\mathbb{C}(x),a}}$. Thus Theorem 1.1 gives us a description of the Bethe algebra in terms of the derivatives of the potential functions, see Theorem 6.1 and Corollary 7.4.

Our construction of potential functions is based on the isomorphism of the Bethe algebra and the algebra of functions on the critical set.

The Bethe algebra of our family of arrangements is an example of the Bethe algebra of a quantum integrable system. One may expect to determine glimpses of Frobenius like structures in the Bethe algebras of standard quantum integrable systems.

1.4. **Exposition of material.** In Section 2 we remind general facts about arrangements. In Section 3 we consider families of arrangements. In Section 4 we introduce elementary arrangements and define potential functions. In Section 5 we prove an important formula for the orthogonal projection $\pi : V \to \mathrm{Sing}_a V$ with respect to the bilinear form $S^{(a)}$. Based on that formula we prove the first part of Theorem 1.1 in Section 6 and the second part of Theorem 1.1 in Section 7.

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2. Arrangements

2.1. **Affine arrangement.** Let k,n be positive integers, $k < n$. Denote $J = \{1,...,n\}$.

Consider the complex affine space $\mathbb{C}^k$ with coordinates $t_1,...,t_k$. Let $\mathrm{C} = (H_j)_{j\in J}$, be an arrangement of n affine hyperplanes in $\mathbb{C}^k$. Denote $U(\mathrm{C}) = \mathbb{C}^k - \cup_{j\in J} H_j$, the complement.

An *edge* $X_\alpha \subset \mathbb{C}^k$ of C is a nonempty intersection of some hyperplanes of C. Denote by $J_\alpha \subset J$ the subset of indices of all hyperplanes containing X_α. Denote $l_\alpha = \mathrm{codim}_{\mathbb{C}^k} X_\alpha$. We assume that C is *essential*, that is, C has a vertex, an edge which is a point.

An edge is called *dense* if the subarrangement of all hyperplanes containing it is irreducible: the hyperplanes cannot be partitioned into nonempty sets so that, after a change of coordinates, hyperplanes in different sets are in different coordinates. In particular, each hyperplane of C is a dense edge.

2.2. **Orlik-Solomon algebra.** Define complex vector spaces $A^p(\mathrm{C})$, $p = 0,...,k$. For $p = 0$, we set $A^p(\mathrm{C}) = \mathbb{C}$. For $p > 1$, $A^p(\mathrm{C})$ is generated by symbols $(H_{j_1},...,H_{j_p})$ with $j_i \in J$, such that (i) $(H_{j_1},...,H_{j_p}) = 0$ if $H_{j_1},...,H_{j_p}$ are not in general position, that is, if the intersection $H_{j_1} \cap ... \cap H_{j_p}$ is empty or has codimension less than p; (ii) $(H_{j_{\sigma(1)}},...,H_{j_{\sigma(p)}}) = (-1)^{|\sigma|}(H_{j_1},...,H_{j_p})$ for any element σ of the symmetric group Σ_p; (iii) $\sum_{i=1}^{p+1}(-1)^i(H_{j_1},...,\widehat{H_{j_i}},...,H_{j_{p+1}}) = 0$ for any $(p+1)$-tuple $H_{j_1},...,H_{j_{p+1}}$ of hyperplanes in C which are not in general position and such that $H_{j_1} \cap ... \cap H_{j_{p+1}} \neq \emptyset$.

The direct sum $\mathcal{A}(\mathcal{C}) = \oplus_{i=1}^{N} \mathcal{A}^p(\mathcal{C})$ is the *Orlik-Solomon* algebra with respect to multiplication $(H_{j_1},...,H_{j_p}) \cdot (H_{j_{p+1}},...,H_{j_{p+q}}) = (H_{j_1},...,H_{j_p},H_{j_{p+1}},...,H_{j_{p+q}})$.

2.3. **Aomoto complex.** Fix a point $a = (a_1,...,a_n) \in (\mathbb{C}^\times)^n$ called the *weight*. Then the arrangement C is *weighted*: for $j \in J$, we assign weight a_j to hyperplane H_j. For an edge X_α,

define its weight $a_\alpha = \sum_{j\in J_\alpha} a_j$. We denote $a_J = \sum_{j\in J} a_j$ and $\omega^{(a)} = \sum_{j\in J} a_j \cdot (H_j) \in A^1(C)$. Multiplication by $\omega^{(a)}$ defines the differential $d^{(a)} : A^p(C) \to A^{p+1}(C)$, $x \mapsto \omega^{(a)} \cdot x$, on $A(C)$.

2.4. **Flag complex, see** [SV]**.** For an edge X_α, $l_\alpha = p$, a *flag* starting at X_α is a sequence $X_{\alpha 0} \supset X_{\alpha 1} \supset \cdots \supset X_{\alpha p} = X_\alpha$ of edges such that $l_{\alpha j} = j$ for $j = 0,\dots,p$. For an edge X_α, we define $(\overline{F_\alpha})_Z$ as the free Z-module generated by the elements $F_{\alpha 0,\dots,\alpha p=\alpha}$ labeled by the elements of the set of all flags starting at X_α. We define $(F_\alpha)_Z$ as the quotient of $(\overline{F_\alpha})_Z$ by the submodule generated by all the elements of the form

$$\sum_{X_\beta,\ X_{\alpha_{j-1}} \supset X_\beta \supset X_{\alpha_{j+1}}} \overline{F}_{\alpha_0,\dots,\alpha_{j-1},\beta,\alpha_{j+1},\dots,\alpha_p=\alpha}. \tag{2.1}$$

Such an element is determined by $j \in \{1,\dots,p-1\}$ and an incomplete flag $X_{\alpha 0} \supset \dots \supset X_{\alpha j-1} \supset X_{\alpha j+1} \supset \dots \supset X_{\alpha p} = X_\alpha$ with $l_{\alpha i} = i$.

We denote by $F_{\alpha 0,\dots,\alpha p}$ the image in $(F_\alpha)_Z$ of the element $\overline{F}_{\alpha 0,\dots,\alpha p}$. For $p = 0,\dots,k$, we set $\mathcal{F}^p(\mathcal{C}))_{\mathbb{Z}} = \oplus_{X_\alpha, l_\alpha = p} (\mathcal{F}_\alpha)_{\mathbb{Z}}$, $\mathcal{F}^p(\mathcal{C}) = (\mathcal{F}^p(\mathcal{C}))_{\mathbb{Z}} \otimes \mathbb{C}$, $\mathcal{F}(\mathcal{C}) = \oplus_{p=1}^N \mathcal{F}^p(\mathcal{C}$). We define the differential $d_Z : (F^p(C))_Z \to (F^{p+1}(C))_Z$ by

$$d_{\mathbb{Z}} \ : \ F_{\alpha_0,\dots,\alpha_p} \ \mapsto \ \sum_{X_\beta,\ X_{\alpha_p} \supset X_\beta} F_{\alpha_0,\dots,\alpha_p,\beta}, \tag{2.2}$$

$d^2_Z = 0$. Tensoring d_Z with C, we obtain the differential $d : F^p(C) \to F^{p+1}(C)$. In particular, we have $H^p(F(C),d) = H^p((F(C))_Z, d_Z) \otimes C$.

We have $H^p(F(C),d) = 0$ for $p \neq k$ and $\dim H^k(F(C),d) = |\chi(U(C))|$, where $\chi(U(C))$ is the Euler characteristic of the complement $U(C)$, see [SV, Corollary 2.8]

2.5. **Duality.** The vector spaces $A^p(C)$ and $F^p(C)$ are dual, see [SV]. The pairing $A^p(C) \otimes F^p(C) \to C$ is defined as follows. For $H_{j1},\dots,H_{jp}$ in general position, set $F(H_{j1},\dots,H_{jp}) = F_{\alpha 0,\dots,\alpha p}$, where $X_{\alpha 0} = C^k$, $X_{\alpha 1} = H_{j1}$, ..., $X_{\alpha p} = H_{j1} \cap \cdots \cap H_{jp}$. Then we define $\langle H_{j1},\dots,H_{jp}), F_{\alpha 0,\dots,\alpha p}\rangle = (-1)^{|\sigma|}$, if $F_{\alpha 0,\dots,\alpha p} = F(H_{j\sigma(1)},\dots,H_{j\sigma(p)})$ for some $\sigma \in S_p$, and $\langle H_{j1},\dots,H_{jp}), F_{\alpha 0,\dots,\alpha p}\rangle = 0$ otherwise.

An element $F \in F^k(C)$ is called *singular* if F annihilates the image of the map $d^{(a)} : A^{k-1}(C) \to A^k(C)$, see [V3]. Denote by $\mathrm{Sing}_a F^k(C) \subset F^k(C)$ the subspace of all singular vectors.

2.6. **Contravariant map and form, see** [SV]**.** The weights a determines the *contravariant map*

$$S_{(a)} : F_p(C) \to A_p(C), \qquad F_{\alpha 0,\dots,\alpha p} \mapsto 7 \qquad X a_{j1} \cdots a_{jp} (H_{j1},\dots,H_{jp}), \tag{2.3}$$

where the sum is taken over all p-tuples $(H_{j_1},...,H_{j_p})$ such that $H_{j_1} \supset X_{\alpha_1},..., H_{j_p} \supset X_{\alpha_p}$. Identifying $A^p(C)$ with $F^p(C)^*$, we consider the map as a bilinear form, $S^{(a)} : F^p(C) \otimes F^p(C) \to C$. The bilinear form is called the *contravariant form*. The contravariant form is symmetric. The contravariant map (2.3) defines a homomorphism of complexes $S^{(a)} : (F(C),d) \to (A(C),d^{(a)})$, see [SV, Lemma 3.2.5].

2.7. **Generic weights.**

Theorem 2.1 ([SV, Theorem 3.7])**.** *If the weight a is such that none of the dense edges has weight zero, then the contravariant form is nondegenerate. In particular, we have an isomorphism of complexes* $S : (F(C),d) \to (A(C),d^{(a)})$.

Notice that none of the dense edges has weight zero if all weights are positive.

If the weight a is such that none of the dense edges has weight zero, then the isomorphism of Theorem 2.1 and the graded algebra structure on A(C) induce a graded algebra structure on F(C).

2.8. **Differential forms.** For $j \in J$, fix defining equations $f_j = 0$ for the hyperplanes H_j, where $f_j = b^1_j t_1 + \cdots + b^k_j t_k + z_j$ with $b^i_j, z_j \in C$. Consider the logarithmic differential 1-forms $\omega_j = df_j/f_j$ on C^k. Let $A^-(C)$ be the exterior C-algebra of differential forms generated by 1 and ω_j, $j \in J$. The map $A(C) \to A^-(C)$, $(H_j) \to 7\ \omega_j$, is an isomorphism. We identify A(C) and $A^-(C)$.

For $I = \{i_1,...,i_k\} \subset J$, denote $d_I = d_{i_1,...,i_k} = \det_{i,l=1}^k (b^i_{i_l})$. Then $\omega_{i_1} \wedge \cdots \wedge \omega_{i_k} = \frac{d_{i_1,...,i_k}}{f_{i_1} \cdots f_{i_k}}\, dt_1 \wedge \cdots \wedge dt_k$.

2.9. **Master function.** The *master function* of the weighted arrangement (C,a) is

$$\Phi_{C,a} = \sum_{j \in J} a_j \log f_j, \tag{2.4}$$

a multivalued function on $U(C)$. Let $C_{\mathcal{C},a} = \{u \in U(\mathcal{C}) \mid \frac{\partial \Phi_{\mathcal{C},a}}{\partial t_i}(u) = 0 \text{ for } i = 1,...,k\}$ be the critical set of $\Phi_{C,a}$.

2.10. **Isolated critical points.** For generic weight $a \in (C^\times)^n$, all critical points of $\Phi_{C,a}$ are nondegenerate and the number of critical points equals $|\chi(U(C))|$, see [V2, OT1, Si].

Consider the projective space P^k compactifying C^k. Assign the weight $a_\infty = -\sum_{j \in J} a_j$ to the hyperplane $H_\infty = P^k - C^k$. Denote by $\bar{C}$ the arrangement $(H_j)_{j \in J \cup \infty}$ in P^k. The weighted arrangement (C,a) is called *unbalanced* if the weight of any dense edge of $\bar{C}$ is nonzero, see [V4]. For example, (C,a) is unbalanced if all weights $(a_j)_{j \in J}$ are positive.

If (C,a) is unbalanced, then all critical points of $\Phi_{\mathsf{C},a}$ are isolated and the sum of their Milnor numbers equals $|\chi(U(\mathsf{C}))|$, see [V4, Section 4].

2.11. **Residue.** Let $\mathrm{O}(U(\mathsf{C}))$ be the algebra of regular functions on $U(\mathsf{C})$ and $I_{\mathcal{C},a} = \langle \frac{\partial \Phi_{\mathcal{C},a}}{\partial t_i} \mid i = 1, \dots, k \rangle \subset \mathcal{O}(U(\mathcal{C}))$ the ideal generated by first derivatives of $\Phi_{\mathsf{C},a}$. Let $\mathrm{O}(C_{\mathsf{C},a}) = \mathrm{O}(U(\mathsf{C}))/I_{\mathsf{C},a}$ be the algebra of functions on the critical set and $[\,] : \mathrm{O}(U(\mathsf{C})) \to \mathrm{O}(C_{\mathsf{C},a})$, $f \to 7\,[f]$, the projection. We assume that all critical points are isolated. In that case the algebra $\mathrm{O}(C_{\mathsf{C},a})$ is finite-dimensional and the elements $[1/f_j]$, $j \in J$, generate $\mathrm{O}(C_{\mathsf{C},a})$ as an algebra, see [V4, Lemma 2.5].

Let $\mathrm{R} : \mathrm{O}(C_{\mathsf{C},a}) \to \mathsf{C}$ be the Grothendieck residue,

$$[f] \;\mapsto\; \frac{1}{(2\pi i)^k}\mathrm{Res}\frac{f}{\prod_{j=1}^k \frac{\partial \Phi_{\mathcal{C},a}}{\partial t_j}} = \frac{1}{(2\pi i)^k}\int_\Gamma \frac{f\; dt_1 \wedge \cdots \wedge dt_k}{\prod_{j=1}^k \frac{\partial \Phi_{\mathcal{C},a}}{\partial t_j}}\,.$$

Here Γ is the real k-cycle defined by the equations $\left|\frac{\partial \Phi_{\mathcal{C},a}}{\partial t_j}\right| = \epsilon_j,\;\; j = 1, \dots, k$, where ϵ_j are small positive numbers, see [GH]. Define the *residue bilinear form* $(\,,\,)_{C_{\mathsf{C},a}}$ on $\mathrm{O}(C_{\mathsf{C},a})$ by $([f],[g])_{C_{\mathsf{C},a}} = \mathrm{R}([f][g])$. This form is nondegenerate, see [AGV], and $([f][g],[h])_{C_{\mathsf{C},a}} = ([f],[g][h])_{C_{\mathsf{C},a}}$ for all $[f],[g],[h] \in \mathrm{O}(C_{\mathsf{C},a})$, thus $(\mathrm{O}(C_{\mathsf{C},a}),(\,,\,)_{C_{\mathsf{C},a}})$ is a *Frobenius algebra*.

2.12. **Orthogonal projection.** Let $\pi^\perp : \mathrm{F}^k(\mathsf{C}) \to \mathrm{Sing}_a\,\mathrm{F}^k(\mathsf{C})$ be the orthogonal projection with respect to $S^{(a)}$.

If the weight $a \in (\mathsf{C}^\times)^n$ is unbalanced, then $d\mathrm{F}^{k-1}(\mathsf{C}) = \mathrm{Sing}_a\,\mathrm{F}^k(\mathsf{C})^\perp$, where $d\mathrm{F}^{k-1}(\mathsf{C}) \subset \mathrm{F}^k(\mathsf{C})$ is the image of the differential defined by (2.2) and $\mathrm{Sing}_a\,\mathrm{F}^k(\mathsf{C})^\perp \subset \mathrm{F}^k(\mathsf{C})$ is the orthogonal complement to $\mathrm{Sing}_a\,\mathrm{F}^k(\mathsf{C})$ with respect to $S^{(a)}$, see [V6, Lemma 2.14]. Define the map

$$\nu_{\mathsf{C}} : \mathrm{F}^k(\mathsf{C}) \to \mathrm{O}(C_{\mathsf{C},a}), \qquad F\;7\to\;[f], \tag{2.5}$$

where f is defined by the formula $\mathrm{S}^{(a)}(F) = f dt_1 \wedge \cdots \wedge dt_k$. Clearly, $\nu_{\mathsf{C}}(\mathrm{Sing}_a\,\mathrm{F}^k(\mathsf{C})^\perp) = \nu_{\mathsf{C}}(d\mathrm{F}^{k-1}(\mathsf{C})) = 0$, since $\omega^{(a)} = 0$ on $C_{\mathsf{C},a}$.

Theorem 2.2 ([V6]). *If the weight $a \in (\mathsf{C}^\times)^n$ is unbalanced, then the map* $\nu_{\mathcal{C}}\big|_{\mathrm{Sing}_a\,\mathcal{F}^k(\mathcal{C})}$: $\mathrm{Sing}_a\,\mathrm{F}^k(\mathsf{C}) \to \mathrm{O}(C_{\mathsf{C},a})$ *is an isomorphism of vector spaces. The isomorphism ν_{C} identifies the residue form on* $\mathrm{O}(C_{\mathsf{C},a})$ *and the contravariant form on* $\mathrm{Sing}\mathrm{F}^k(\mathsf{C})$ *multiplied by* $(-1)^k$, $S^{(a)}(f,g) = (-1)^k(\nu_{\mathsf{C}}(f),\nu_{\mathsf{C}}(g))_{C_{\mathsf{C},a}}$ *for* $f,g \in \mathrm{Sing}_a\,\mathrm{F}^k(\mathsf{C})$.

Remark. If the weight $a \in (\mathsf{C}^\times)^n$ is unbalanced, then the isomorphism μ_{C} induces a commutative associative algebra structure on $\mathrm{Sing}_a\mathrm{F}^k(\mathsf{C})$. Together with the contravariant form $S^{(a)}|_{\mathrm{Sing}_a\,\mathrm{F}^k}$ it is a Frobenius algebra. The algebra of multiplication operators on $\mathrm{Sing}_a\mathrm{F}^k(\mathsf{C})$ is called the *Bethe algebra* of the weighted arrangement (C,a). This Bethe algebra is an analog of the Bethe algebra in the theory of quantum integrable models, see, for example, [MTV1, MTV2, V3, V4].

2.13. **Integral structure on** $O(C_{C,a})$ **and** $\mathrm{Sing}_a F^k(C)$**.** If the weight a is unbalanced, the formula $H^p(F(C),d) = H^p((F(C))_Z, d_Z)\otimes C$ and the isomorphism $\nu_C|_{\mathrm{Sing}_a \mathcal{F}^k(C)} : H^k(\mathcal{F}(C), d)$ $\tilde{}= \mathrm{Sing}_a F(C) \to O(C_{C,a})$ define an integral structure on $O(C_{C,a})$. More precisely, for a k-flag of edges $X_{\alpha 0} \supset X_{\alpha 1} \supset \cdots \supset X_{\alpha k}$, let $S_{(a)}(F_{\alpha 0,\dots,\alpha k}) = f_{\alpha 0,\dots,\alpha k} dt_1 \wedge \cdots \wedge dt_k$. Denote by $w_{\alpha 0,\dots,\alpha k}$ the element $[f_{\alpha 0,\dots,\alpha k}] \in O(C_{C,a})$.

Corollary 2.3 ([V6])**.** *If the weight a is unbalanced, then the set of all elements* $\{w_{\alpha 0,\dots,\alpha k}\}$*, labeled by all k-flag of edges of* C*, spans the vector space* $O(C_{C,a})$*. All linear relations between the elements of the set are corollaries of the relations*

$$\sum_{X_\beta, X_{\alpha j-1} \supset X_\beta \supset X_{\alpha j+1}} w_{\alpha 0,\dots,\alpha j-1,\beta,\alpha j+1,\dots,\alpha p=\alpha} = 0, \tag{2.6}$$

$$\sum w_{\alpha 0,\dots,\alpha p,\beta} = 0, \; X_\beta, X_{\alpha p} \supset X_\beta$$

cf. formulas (2.1)*,* (2.2)*.*

Similarly, for a k-flag of edges $X_{\alpha 0} \supset X_{\alpha 1} \supset \cdots \supset X_{\alpha k}$, let $v_{\alpha 0,\dots,\alpha k}$ be the orthogonal projection of $F_{\alpha 0,\dots,\alpha k}$ to $\mathrm{Sing}_a F^k(C)$.

Corollary 2.4 ([V6])**.** *If the weight a is unbalanced, then the set of all elements* $\{v_{\alpha 0,\dots,\alpha k}\}$*, labeled by all k-flag of edges of* C*, spans the vector space* $\mathrm{Sing}_a F^k(C)$*. All linear relations between the elements of the set are corollaries of the relations*

$$\sum_{X_\beta, X_{\alpha j-1} \supset X_\beta \supset X_{\alpha j+1}} v_{\alpha 0,\dots,\alpha j-1,\beta,\alpha j+1,\dots,\alpha p=\alpha} = 0, \tag{2.7}$$

$$\sum v_{\alpha 0,\dots,\alpha p,\beta} = 0, \; X_\beta, X_{\alpha p} \supset X_\beta$$

cf. formulas (2.1)*,* (2.2)*.*

We have $\nu_C : v_{\alpha 0,\dots,\alpha k} \mapsto w_{\alpha 0,\dots,\alpha k}$. The elements $\{w_{\alpha 0,\dots,\alpha k}\} \subset O(C_{C,a})$ and $\{v_{\alpha 0,\dots,\alpha k}\} \subset \mathrm{Sing}_a F^k(C)$ are called the *marked elements*. The relations (2.6), (2.7) are called the *marked relations*.

2.14. **Combinatorial connection, I.** Consider a deformation $C(s)$ of the arrangement C, which preserves the combinatorics of C. Assume that the edges of $C(s)$ can be identified with the edges of C so that the elements in formula (2.1) and the differential in formula (2.2) do not depend on s. Then for every s, the elements $\{w_{\alpha 0,\dots,\alpha k}(s)\}$ span $O(C_{C(s),a})$ as a vector space with linear relations (2.6) not depending on s. This allows us to identify all the vector spaces $O(C_{C(s),a})$. In particular, if an element $w(s) \in O(C_{C(s),a})$ is given, then the derivative $\frac{dw}{ds}$ is well-

defined. This construction is called the *combinatorial connection* on the family of algebras $O(C_{C(s),a})$, see [V5]. All the elements $\{w_{\alpha_0,\dots,\alpha_k}(s)\}$ are flat sections of the combinatorial connection.

Similarly we can define the combinatorial connection on the family of vector spaces Sing ${}_aF^k(C(s))$.

2.15. **Arrangement with normal crossings.** An essential arrangement C is *with normal crossings*, if exactly k hyperplanes meet at every vertex of C. Assume that C is an essential arrangement with normal crossings.

A basis of $A^p(C)$ is formed by $(H_{j_1},\dots,H_{j_p})$, where $\{j_1 < \dots < j_p\}$ are independent ordered p-element subsets of J. The dual basis of $F^p(C)$ is formed by the corresponding vectors $F(H_{j_1},\dots,H_{j_p})$. These bases of $A^p(C)$ and $F^p(C)$ are called *standard*. We have

$$F(H_{j_1},\dots,H_{j_p}) = (-1)^{|\sigma|}F(H_{j_{\sigma(1)}},\dots,H_{j_{\sigma(p)}}), \qquad \text{for } \sigma \in \Sigma_p. \tag{2.8}$$

For an independent subset $\{j_1,\dots,j_p\}$, we have $S^{(a)}(F(H_{j_1},\dots,H_{j_p}),F(H_{j_1},\dots,H_{j_p})) = a_{j_1}\cdots a_{j_p}$ and $S^{(a)}(F(H_{j_1},\dots,H_{j_p}),F(H_{i_1},\dots,H_{i_k})) = 0$ for distinct elements of the standard basis. If a is unbalanced, then the marked elements in $O(C_{C,a})$ are

$$w_{i_1,\dots,i_k} = d_{i_1,\dots,i_k}\frac{a_{i_1}}{[f_{i_1}]}\cdots\frac{a_{i_k}}{[f_{i_k}]}, \tag{2.9}$$

where $\{i_1,\dots,i_k\}$ runs through the set of all independent k-element subsets of J. We have $w_{i_{\sigma(1)},\dots,i_{\sigma(k)}} = (-1)^\sigma w_{i_1,\dots,i_k}$ for $\sigma \in \Sigma_k$. We put $w_{i_1,\dots,i_k} = 0$ if the set $\{i_1,\dots,i_k\}$ is dependent. The marked relations are labeled by independent subsets $\{i_2,\dots,i_k\}$ and have the form $\sum_{j\in J} w_{j,i_2,\dots,i_k} = 0$. The marked elements $v_{i_1,\dots,i_k}$ in Sing ${}_aF^k(C)$ are orthogonal projections to Sing ${}_aF^k(C)$ of the elements $F(H_{i_1},\dots,H_{i_k})$ with the skew-symmetry property $v_{i_{\sigma(1)},\dots,i_{\sigma(k)}} = (-1)^\sigma v_{i_1,\dots,i_k}$ for $\sigma \in \Sigma_k$ and the marked relations $\sum_{j\in J} v_{j,i_2,\dots,i_k} = 0$ labeled by independent subsets $\{i_2,\dots,i_k\}$.

For any independent ordered subset $j_1,\dots,j_p \in J$ we denote $F_{j_1,\dots,j_p} = F(H_{j_1},\dots,H_{j_p}) \in F^p(C)$ and set $F_{j_1,\dots,j_p} = 0$ if $j_1,\dots,j_p$ is a dependent subset.

Corollary 2.5. *The orthogonal complement* Sing ${}_aF^k(C)$ *is generated by the elements* $\sum_{j\in J} F_{j,i_1,\dots,i_{k-1}}$ *labeled by independent subsets* $\{i_1,\dots,i_{k-1}\} \in J$, *and an element of* $F^k(C)$ *lies in* Sing ${}_aF^k(C)$ *if and only if it is orthogonal to all the elements* $\sum_{j\in J} F_{j,i_1,\dots,i_{k-1}}$.

3. Family of parallelly transported hyperplanes

3.1. **Arrangement in $C^n\times C^k$.** Consider C^k with coordinates $t_1,\dots,t_k$, C^n with coordinates $z_1,\dots,z_n$, the projection $\tau : C^n \times C^k \to C^n$. Fix n nonzero linear functions on C^k, $g_j = b_j^1 t_1 + \dots + b_j^k t_k$, $j \in J$, where $b_j^i \in \mathbb{C}$. Assume that the functions $\{g_j\}_{j\in J}$, span the dual

space $(\mathbb{C}^k)^*$.

Define n linear functions on $\mathbb{C}^n \times \mathbb{C}^k$, $f_j = g_j + z_j$, $j \in J$. Consider the arrangement of hyperplanes $\mathrm{C} = \{H_j\}_{j\in J}$ in $\mathbb{C}^n \times \mathbb{C}^k$, where H_j is the zero set of f_j, and denote by $U(\mathrm{C}) = \mathbb{C}^n \times \mathbb{C}^k - \cup_{j\in J} H_j$ the complement. For every $x = \in \mathbb{C}^n$, the arrangement C induces an arrangement $\mathrm{C}(x)$ in the fiber $\tau^{-1}(x) \tilde{=} \mathbb{C}^k$. Then $\mathrm{C}(x)$ consists of hyperplanes $\{H_j(x)\}_{j\in J}$, defined in $\mathbb{C}^k$ by the equations $g_j + x_j = 0$. Thus $\{\mathrm{C}(x)\}_{x\in\mathbb{C}^n}$ is a family of arrangements in $\mathbb{C}^k$, whose hyperplanes are transported parallelly to themselves as x changes. Denote by $U(\mathrm{C}(x)) = \mathbb{C}^k - \cup_{j\in J} H_j(x)$ the complement. For almost all $x \in \mathbb{C}^k$ the arrangement $\mathrm{C}(x)$ is with normal crossings. The subset $\Delta \subset \mathbb{C}^n$ where this does not hold, is a hypersurface called the discriminant. On the discriminant see, for example, [BB, V4].

3.2. **Combinatorial connection, II.** For any $x^1, x^2 \in \mathbb{C}^n - \Delta$, the spaces $\mathrm{F}^p(\mathrm{C}(x^1))$, $\mathrm{F}^p(\mathrm{C}(x^2))$ are canonically identified if a vector $F(H_{j_1}(x^1),\dots,H_{j_p}(x^1))$ of the first space is identified with the vector $F(H_{j_1}(x^2),\dots,H_{j_p}(x^2))$ of the second, in other words, if we identify the standard bases of these spaces.

Assume that a weight $a \in (\mathbb{C}^\times)^n$ is given. Then each arrangement $\mathrm{C}(x)$ is weighted. The identification of spaces $\mathrm{F}^p(\mathrm{C}(x^1))$, $\mathrm{F}^p(\mathrm{C}(x^2))$ for $x^1, x^2 \in \mathbb{C}^n - \Delta$ identifies the corresponding subspaces $\mathrm{Sing}_a \mathrm{F}^k(\mathrm{C}(x^1))$, $\mathrm{Sing}_a \mathrm{F}^k(\mathrm{C}(x^2))$ and contravariant forms.

Assume that the weighted arrangement $(\mathrm{C}(x), a)$ is unbalanced for some $x \in \mathbb{C}^n - \Delta$, then $(\mathrm{C}(x), a)$ is unbalanced for all $x \in \mathbb{C}^n - \Delta$. The identification of $\mathrm{Sing}_a \mathrm{F}^k(\mathrm{C}(x^1))$ and $\mathrm{Sing}_a \mathrm{F}^k(\mathrm{C}(x^2))$ also identifies the marked elements $v_{j_1,\dots,j_k}(x^1)$ and $v_{j_1,\dots,j_k}(x^2)$, see Section 2.15. For $x \in \mathbb{C}^n - \Delta$, denote $V = \mathrm{F}^k(\mathrm{C}(x))$, $\mathrm{Sing}_a V = \mathrm{Sing}_a \mathrm{F}^k(\mathrm{C}(x))$, $v_{j_1,\dots,j_k} = v_{j_1,\dots,j_k}(x)$. The triple $(V, \mathrm{Sing}_a V, S^{(a)})$, with marked elements $v_{j_1,\dots,j_k}$, does not depend on x under the identification.

As a result of this reasoning we obtain the canonically trivialized vector bundle

$$\mathrm{t}_{x\in\mathbb{C}^n-\Delta} \mathrm{F}^k(\mathrm{C}(x)) \to \mathbb{C}^n - \Delta, \tag{3.1}$$

with the canonically trivialized subbundle $\mathrm{t}_{x\in\mathbb{C}^n-\Delta} \mathrm{Sing}_a \mathrm{F}^k(\mathrm{C}(x)) \to \mathbb{C}^n - \Delta$ and the constant contravariant form on the fibers. This trivialization identifies the bundle in (3.1) with the bundle $(\mathbb{C}^n - \Delta) \times V \to \mathbb{C}^n - \Delta$ and identifies the subbundle $\mathrm{t}_{x\in\mathbb{C}^n-\Delta} \mathrm{Sing}_a \mathrm{F}^k(\mathrm{C}(x)) \to \mathbb{C}^n - \Delta$ with the subbbundle

$$(\mathbb{C}^n - \Delta) \times (\mathrm{Sing}_a V) \to \mathbb{C}^n - \Delta. \tag{3.2}$$

The bundle in (3.2) is called the *combinatorial bundle*, the flat connection on it is called *combinatorial*, see Section 2.14 and [V4, V5].

3.3. **Gauss-Manin connection on** $(\mathbb{C}^n - \Delta) \times (\mathrm{Sing}_a V) \to \mathbb{C}^n - \Delta$**.** The *master function* is $\Phi_{\mathrm{C},a} = \mathrm{P}_{j\in J} a_j \log f_j$, a multivalued function on $U(\mathrm{C})$. Let $\kappa \in \mathbb{C}^\times$. The function $e^{\Phi_{\mathrm{C},a}/\kappa}$ defines a rank one

local system L_κ on $U(\mathrm{C})$ whose horizontal sections over open subsets of $U(\mathrm{C})$ are univalued branches of $e^{\Phi_{\mathrm{C},a}/\kappa}$ multiplied by complex numbers, see, for example,
[SV, V2]. The vector bundle

$$\mathrm{t}_{x\in\mathrm{C}^n-\Delta} H_k(U(\mathrm{C}(x)),\mathrm{L}_\kappa|_{U(\mathrm{C}(x))}) \to \mathrm{C}^n - \Delta$$

is called the *homology bundle*. The homology bundle has a canonical flat Gauss-Manin connection.

For a fixed $x \in \mathrm{C}^n - \Delta$, choose $\gamma \in H_k(U(\mathrm{C}(x)),\mathrm{L}_\kappa|_{U(\mathrm{C}(x))})$. The linear map $\{\gamma\} : \mathrm{A}^k(\mathrm{C}(x)) \to \mathrm{C}$, $\omega \mapsto \int_\gamma e^{\Phi_{\mathrm{C},a}/\kappa}\omega$, is an element of $\mathrm{Sing}_a\mathrm{F}^k(\mathrm{C}(x))$ by Stokes' theorem. It is known that for generic κ any element of $\mathrm{Sing}_a\mathrm{F}^k(\mathrm{C}(x))$ corresponds to a certain γ and in that case this construction gives the *integration isomorphism*

$$H_k(U(\mathrm{C}(x)),\mathrm{L}_\kappa|_{U(\mathrm{C}(x))}) \to \mathrm{Sing}_a\,\mathrm{F}^k(\mathrm{C}(x)), \tag{3.3}$$

see [SV]. The precise values of κ, such that (3.3) is an isomorphism, can be deduced from the determinant formula in [V1].

For generic κ the fiber isomorphisms (3.3) define an isomorphism of the homology bundle and the combinatorial bundle (3.2). The Gauss-Manin connection induces a connection on the combinatorial bundle. That connection on the combinatorial bundle is also called the *Gauss-Manin connection*.

Thus, there are two connections on the combinatorial bundle: the combinatorial connection and the Gauss-Manin connection depending on κ. In this situation we consider the differential equations for flat sections of the Gauss-Manin connection with respect to the combinatorially flat standard basis. Namely, let $\gamma(x) \in H_k(U(\mathrm{C}(x)),\mathrm{L}_\kappa|_{U(\mathrm{C}(x))})$ be a flat section of the Gauss-Manin connection. Let us write the corresponding section $I_\gamma(x)$ of the bundle $(\mathrm{C}^n -$

$$I_\gamma(x) = \sum_{\substack{\text{independent} \\ \{j_1<\cdots<j_k\}\subset J}} I_\gamma^{j_1,\dots,j_k}(x) F(H_{j_1},\dots,H_{j_k}),\ I_\gamma^{j_1,\dots,j_k}(x) = \int_{\gamma(x)} e^{\Phi_{\mathcal{C},a}/\kappa}\omega_{j_1}\wedge\cdots\wedge\omega_{j_k}$$

$\Delta) \times \mathrm{Sing}_a V \to \mathrm{C}^n - \Delta$ in the combinatorially flat standard basis,.

We may rewrite it as $I_\gamma(x) = \mathrm{P}_{\text{independent}} I_{\gamma j_1,\dots,j_k}(x) v_{j_1,\dots,j_k}$ since $I_\gamma(x) \in \mathrm{Sing}_a V$. For $I = \mathrm{P} I_{j_1,\dots,j_k} v_{j_1,\dots,j_k}$ and $j \in J$, we denote $\frac{\partial I}{\partial z_j} = \sum \frac{\partial I^{j_1,\dots,j_k}}{\partial z_j} v_{j_1,\dots,j_k}$. This formula defines the combinatorial connection on the combinatorial bundle.

The section I_γ satisfies the Gauss-Manin differential equations

$$\kappa\frac{\partial I}{\partial z_j}(x) = K_j(x) I(x), \qquad j \in J, \tag{3.4}$$

where $K_j(x) \in \mathrm{End}(\mathrm{Sing}_a V)$. See a description of the operators $K_j(x)$, for example, in [OT2, V2, V4].

3.4. **Critical set.** Denote by $C_{\mathrm{C},a}$ the critical set of $\Phi_{\mathrm{C},a}$ in the C^k-direction,

(3.5)) $C_{\mathcal{C},a} = \left\{(x,u) \in U(\mathcal{C}) \subset \mathbb{C}^n \times \mathbb{C}^k \mid \frac{\partial \Phi_{\mathcal{C},a}}{\partial t_i}(x,u \right.$ o $= 0$ for $i = 1,\dots,k$.

Let $O(C_{C(x),a})$ be the algebra of regular functions on $C_{C(x),a} = C_{C,a} \cap \tau^{-1}(x)$. Namely, for $x \in C^n$, let $I_{C(x),a}$ be the ideal in $O(U(C(x)))$ generated by $\frac{\partial \Phi_{\mathcal{C},a}}{\partial t_i}, i = 1,\dots,k$. We set $O(C_{C(x),a}) = O(U(C(x)))/I_{C(x),a}$. Assume that the weight a is such that the pair $(C(x),a)$ is unbalanced for some $x \in C^n - \Delta$. Then we obtain the vector *bundle of algebras* $t_{x \in C^n - \Delta} O(C_{C(x),a}) \to C^n - \Delta$. For $x \in C^n - \Delta$, recall the isomorphism

(3.6) $$\nu(x) := \nu_{\mathcal{C}(x)}\Big|_{\mathrm{Sing}_a \mathcal{F}^k(\mathcal{C}(x)} : \mathrm{Sing}_a \mathcal{F}^k(\mathcal{C}(x) \to \mathcal{O}(C_{\mathcal{C}(x),a})$$

of Theorem 2.2. This 'fiber' isomorphism establishes an isomorphism of the bundle $t_{x \in C^n - \Delta} O(C_{C(x),a}) \to C^n - \Delta$ and the bundle $(C^n - \Delta) \times (\mathrm{Sing}_a V) \to C^n - \Delta$. This isomorphism together with the combinatorial and Gauss-Manin connections on the bundle $(C^n - \Delta) \times (\mathrm{Sing}_a V) \to C^n - \Delta$ induces two connections on the bundle of algebras $t_{x \in C^n - \Delta} O(C_{C(x),a}) \to C^n - \Delta$, which also are called the *combinatorial and Gauss-Manin connections*, respectively.

Theorem 3.1 ([V4]). *If the pair* $(C(x),a)$ *is unbalanced for* $x \in C^n - \Delta$*, then for all* $j \in J$*, we have*

(3.7) $$\left(\left[\frac{a_j}{f_j}\right]*_x\right) \circ \nu(x) = \nu(x) \circ K_j(x),$$

where $\left[\frac{a_j}{f_j}\right]*_x$ *is the operator of multiplication by* $\frac{a_j}{f_j}$ *on* $\mathcal{O}(C_{\mathcal{C}(x),a})$ *and* $K_j(x) \in \mathrm{End}(\mathrm{Sing}_a V)$ *is the operator defined in* (3.4).

Remark. Recall that $a_j/f_j = \partial \Phi_{C,a}/\partial z_j$ and the elements $[a_j/f_j]$, $j \in J$, generate the algebra $O(C_{C(x),a})$. Theorem 3.1 says that under the isomorphism $\nu(x)$ the operators of multiplication $[a_j/f_j]*_x$ on $O(C_{C(x),a})$ are identified with the operators $K_j(x)$ in the GaussManin differential equations (3.4). The correspondence of Theorem 3.1 defines a commutative algebra structure on $\mathrm{Sing}_a V$, the structure depending on x. The multiplication in this commutative algebra is generated by the operators $K_j(x), j \in J$. The correspondence of Theorem 3.1 also defines the Gauss-Manin differential equations on the bundle of algebras in terms of the multiplication in the fiber algebras, see these differential equations in [V6, Theorem 3.9].

3.5. **Formulas for multiplication.** Recall that for $j \in J$, we denote $p_j = \left[\frac{\partial \Phi}{\partial z_j}\right] = \left[\frac{a_j}{f_j}\right] \in O(C_{C(x),a})$. Then $w_{i_1,\dots,i_k} = d_{i_1,\dots,i_k} p_{i_1} \dots p_{i_k}$, and for any $i_1,\dots,i_{k-1} \in J$ we have

(3.8) $$\sum_{j \in J} d_{j,i_1,\dots,i_{k-1}} p_j = 0.$$

For a subset $I = \{i_1,\dots,i_{k+1}\} \subset J$ denote

(3.9) $$f_I(z) = \sum_{j=1}^{k+1} (-1)^{j+1} z_{i_j} d_{i_1,\dots,\widehat{i_j},\dots,i_{k+1}}.$$

Lemma 3.2. *We have*

(3.10) $$f_I(z) \prod_{j=1}^{k+1} \frac{a_{i_j}}{f_{i_j}} = \sum_{j=1}^{k+1} (-1)^{j+1} a_{i_j} d_{i_1,\dots,\widehat{i_j},\dots,i_{k+1}} \prod_{m=1,\ m\neq j}^{k+1} \frac{a_{i_m}}{f_{i_m}}.$$

Proof. We have $\sum_{j=1}^{k+1} (-1)^{j+1} (f_{i_j} - z_{i_j}) d_{i_1,\dots,\widehat{i_j},\dots,i_{k+1}} = 0$. That implies the lemma.

Corollary 3.3. *Assume that a subset* $\{i_1,\dots,i_{k+1}\} \subset J$ *consists of distinct elements and contains a* k*-element independent subset. Then we have an identity in* $\mathrm{O}(C_{C(x),a})$*:*

(3.11) $$f_{i_1,\dots,i_{k+1}}(z)\, p_{i_1} \dots p_{i_{k+1}} = \sum_{j=1}^{k+1} (-1)^{j+1} a_{i_j} w_{i_1,\dots,\widehat{i_j},\dots,i_{k+1}}.$$

Proof. The corollary follows from Lemma 3.2.

4. Elementary arrangements

4.1. **Definition.** Consider a k-dimensional vector space X with coordinates $t_1,t_2,\dots,t_k$. Let $\lambda = (\lambda_1,\lambda_2,\dots,\lambda_m)$ be a collection of positive integers such that $\sum_{i=1}^{m} \lambda_i = k$. Assume that we have a collection of non-intersecting index sets J_h, $h = 1,\dots,m$, each with $|J_h| = \lambda_h + 1$ elements. Denote $J_\lambda := \cup_{h=1}^{m} J_h,\ \lambda^h = \lambda_1 + \dots + \lambda_h,\ \lambda^0 = 0.$

Let $\mathsf{C}_{J_\lambda} = \{H_i\}_{i\in J_\lambda}$ be a weighted arrangement of affine hyperplanes in X with normal crossings. Each hyperplane H_i has weight a_i and is defined by an equation $g_i(t_1,\dots,t_k)+z_i = 0$, where $g_i = \sum_{l=1}^{k} b_i^l t_l$ is an element of the dual space X^* and $z_i \in \mathsf{C}$. Define a subspace $X_h^*(J_\lambda) = \operatorname{span}\{g_i\}_{i\in\cup_{l=1}^{h} J_l}$ of X^*. We have a filtration

(4.1) $$X_1^*(J_\lambda) \subset X_2^*(J_\lambda) \subset \dots \subset X_m^*(J_\lambda) \subset X^*.$$

The arrangement C_{J_λ} is called *elementary of type* λ if

(i) For any $h \in \{1,\dots,m\}$ we have $\dim X_h^*(J_\lambda) = \lambda^h$.

(ii) For any $h \in \{1,\dots,m\}$, any $j \in J_h$ and $A := \{\cup_{i=1}^{h} J_i\} - \{j\}$, we have $\dim(\operatorname{span}\{g_l\}_{l\in A}) = \lambda^h$.

In Figure 2 the first elementary arrangement is of type $\lambda = (2)$ and the second is of type $\lambda = (1,1)$ with $J_1 = \{1,2\}, J_2 = \{3,4\}$. In this section we always assume that C_{J_λ} is an elementary arrangement.

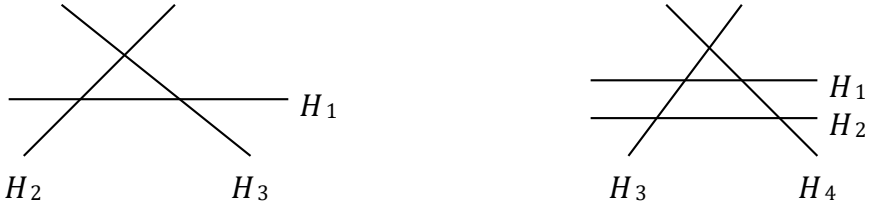

Figure 2. Elementary Arrangements in Dimension 2.

4.2. **Distinguished Elements.** For $h = 1,\dots,m$, let $K_h = \{j_1,\dots,j_{\lambda_h}\}$ be an ordered $\lambda_h$element subset of J_h. Recall the notation $F_{K_h} = F(H_{j_1},\dots,H_{j_{\lambda_h}}) \in \mathrm{F}^{\lambda_h}(\mathrm{C}_{J_\lambda})$. The elements of the flag space $\mathrm{F}^k(\mathrm{C}_{J_\lambda})$ of the form

$$F_{K_1,\dots,K_m} = F_{K_1} \wedge F_{K_2} \wedge \cdots \wedge F_{K_m} \in \mathrm{F}_k(\mathrm{C}_{J_\lambda}),$$

are called the *distinguished elements* of the elementary arrangement C_{J_λ}.

Lemma 4.1. *Counted up to permutation of indices potentially changing sign, there are exactly* $\prod_{h=1}^{m}(\lambda_h+1)$ *distinguished elements of* C_{J_λ}.

For example, in Figure 2 the distinguished elements of the first arrangement are $\pm F_{1,2}, \pm F_{2,3}, \pm F_{1,3}$ and distinguished elements of the second are $\pm F_{1,3}, \pm F_{1,4}, \pm F_{2,3}, \pm F_{2,4}$.

Let $J_h = \{j_1^h, j_2^h, \dots, j_{\lambda_h+1}^h\}$, and let $K_{\widehat{j_i^h}} = \{j_1^h, \dots, j_i^h, \dots, j_{\lambda_h+1}^h\}$ be an ordered $\lambda_h$element subset of J_h. The element

$$s(\mathcal{C}_{J_\lambda}) = \left(\sum\nolimits_{i=1}^{\lambda_1+1}(-1)^{i+1}a_{j_i^1}F_{K_{\widehat{j_i^1}}}\right) \wedge \left(\sum\nolimits_{i=1}^{\lambda_2+1}(-1)^{i+1}a_{j_i^2}F_{K_{\widehat{j_i^2}}}\right) \wedge \dots \cdots \wedge \left(\sum\nolimits_{i=1}^{\lambda_m+1}(-1)^{i+1}a_{j_i^m}F_{K_{\widehat{j_i^m}}}\right) \in \mathcal{F}^k(\mathcal{C}_{J_\lambda}) \tag{4.2}$$

is called the *singular element* of C_{J_λ}.

Lemma 4.2. *We have* $s(\mathrm{C}_{J_\lambda}) \in \mathrm{Sing}_A\mathrm{F}^k(\mathrm{C}_{J_\lambda})$.

Proof. By Corollary 2.5, the element $s(\mathrm{C}_{J_\lambda})$ lies in $\mathrm{Sing}_A\mathrm{F}^k(\mathrm{C}_{J_\lambda})$ if and only if it is orthogonal to each of the elements $\mathrm{P}_{j\in J}F_{j,i_1,\dots,i_{k-1}}$ labeled by independent subsets $I = \{i_1,\dots,i_{k-1}\} \subset J_\lambda$. If $\{i_1,\dots,i_{k-1}\} \subset J_\lambda$ is independent, then there is $l \in \{1,\dots,m\}$ such that $|I\cap J_l| = \lambda_l - 1$ and $|I \cap J_h| = \lambda_h$ for $h \neq l$. Then it is clear that $S^{(a)}(s(\mathrm{C}_{J_\lambda}), \mathrm{P}_{j\in J}F_{j,i_1,\dots,i_{k-1}}) = 0$ as the sum of two opposite terms coming from the l-th factor in formula (4.2).

The singular element $s(\mathrm{C}_{J_\lambda})$ has the following properties. It is defined uniquely up to multiplication by ±1, and this sign depends on the ordering put on each subset J_h. Each destinguished element of C_{J_λ}, considered up to sign, enters the singular element exactly once.

Example. In Figure 2 the singular element of the first elementary arrangement is $\pm(a_3F_{1,2} + a_2F_{3,1} + a_1F_{2,3})$ and the singular element of the second is $\pm(a_2F_1 - a_1F_2) \wedge (a_3F_4 - a_4F_3)$.

4.3. **Decomposing Determinants.** Recall the elements $g_j = \sum_{i=1}^{k} b_j^i t_i \in X^*$, used in defining the hyperplanes H_j, and the notation $d_{j_1,\dots,j_k} = \det\left(b_{j_\ell}^i\right)_{i,\ell=1}^{k}$.

Let $s_1,\dots,s_k$ be a basis of X^* adjacent to the filtration $\{X_h^*(J_\lambda)\}_{h=1}^m$ in (4.1), i.e., let $s_1,\dots,s_k$ be such that for any $h = 1,\dots,m$, the elements $s_1,\dots,s_{\lambda^h}$ form a basis of $X_h^*(J_\lambda)$. Additionally, we select $s_1,\dots,s_k$ such that the change of basis matrix from $t_1,\dots,t_k$ to $s_1,\dots,s_k$ has determinant one.

Let $g_j = \sum_{i=1}^k c_j^i s_i$ be the expansion of g_j with respect to the new basis, then for any $j_1,\dots,j_k \in J_\lambda$ we have $\det(c_{j_\ell}^i)_{i,\ell=1}^k = d_{j_1,\dots,j_k}$ and $c^i{}_j = 0$ for all $j \in J_h$, $i > \lambda^h$.

For $h = 1,\dots,m$, let $K_h = \{k_1^h, \dots, k_{\lambda_h}^h\}$ be an ordered λ_h-element subset of J_h. Consider $K = (K_1,K_2,\dots,K_m)$ as an ordered k-element subset of J_λ. The matrix $(c_j^i)_{j\in K}^{i=1,\dots,k}$ is lower block-triangular. Define the diagonal $\lambda_h \times \lambda_h$-blocks as the matrices $C_h = (c_{k_j^h}^i)_{j=1,\dots,\lambda_h}^{i=\lambda^{h-1}+1,\dots,\lambda^h}$.

Lemma 4.3. *Let K be constructed as above. Then, $d_K = \prod_{h=1}^m \det C_h$.*

4.4. **Auxiliary Arrangements.** For $h = 1,\dots,m$, let Y_h be a vector space of dimension λ_h with coordinates $s_{1,h},\dots,s_{\lambda_h,h}$. For $j \in J_h$, let $g_{j,h}$ be elements of Y_{h^*} given by the formula $g_{j,h} = \sum_{i=1}^{\lambda_h} c_{j,h}^i s_{i,h}$ where $c_{j,h}^i := c_j^{\lambda^{h-1}+i}$ and the $c_j^{\lambda^{h-1}+i}$ are the coefficients introduced in Section 4.3.

Define $C_{J_\lambda,h} = \{H_{j,h}\}_{j\in J_h}$ to be the following weighted arrangement of affine hyperplanes in Y_h. Each hyperplane $H_{j,h}$ has weight a_j and is defined by the equation $g_{j,h}(s_{1,h},\dots,s_{\lambda_h,h}) + z_j = 0$, where z_j and a_j are the same as in Section 4.1. We call $C_{J_\lambda,h}$ the *auxiliary arrangement of type h* associated with the elementary arrangement C_{J_λ}.

For an ordered λ_h-element subset $I = \{j_1,\dots,j_{\lambda_h}\} \subset J_h$ denote $d_{I;h} = \det(c_{j_l,h}^i)_{i,l=1}^{\lambda_h}$. Let $J_h = \{j_1^h, \dots, j_{\lambda_h+1}^h\}$. By the construction, the linear combination $\sum_{i=1}^{\lambda_h+1}(-1)^{i+1} d_{j_1,\dots,\widehat{j_i},\dots,j_{\lambda_h+1};h} g_{j_i^h}$ lies in X_{h^*-1}. Choose some numbers e_j, $j \in \cup_{i=1}^{h-1} J_i$, such that the linear combination

$$\sum_{i=1}^{\lambda_h+1}(-1)^{i+1} d_{j_1,\dots,\widehat{j_i},\dots,j_{\lambda_h+1};h} g_{j_i^h} + \sum_{j\in\cup_{i=1}^{h-1} J_i} e_j g_j = 0 \tag{4.3}$$

as an element of the space X^*. Such numbers exist since e_j, $j \in \cup_{i=1}^{h-1} J_i$, span X_{h-1}^*. Denote

$$f_{c_{J_\lambda,h}} = \sum_{i=1}^{\lambda_h+1}(-1)^{i+1} d_{j_1,\dots,\widehat{j_i},\dots,j_{\lambda_h+1};h} z_{j_i^h} + \sum_{j\in\cup_{i=1}^{h-1} J_i} e_j z_j. \tag{4.4}$$

We call the function

$$P_{c_{J_\lambda,h}} = \frac{\prod_{j\in J_h} a_j}{(2\lambda_h)!} \frac{(f_{c_{J_\lambda,h}})^{2\lambda_h}}{(\prod_{i=1}^{\lambda_h+1} d_{j_1^h,\dots,\widehat{j_i^h},\dots,j_{\lambda_h+1}^h;h})^2} \tag{4.5}$$

a *prepotential of first kind* of the auxiliary arrangement C_{J_h}. We call the function $P_{C_{J\lambda}} = \prod_{h=1}^m P_{c_{J_\lambda,h}}$ a *prepotential of first kind* of the elementary arrangement C_{J_λ}. We call the function

$$Q_{c_{J_\lambda}} = \ln(f_{c_{J_\lambda,1}}) \prod_{h=1}^m P_{c_{J_\lambda,h}} \tag{4.6}$$

a *prepotential of second kind* of the elementary arrangement $C_{J\lambda}$. The prepotentials are not unique due to the choice of the numbers e_j above.

4.5. **Elementary subarrangements.** Let us return to the situation of Section 3. For $x \in C^n - \Delta$ consider the weighted arrangement $C(x)$ with normal crossings.

Let $C_{J\lambda}(x) = \{H_i(x)\}_{i\in J\lambda}$ be an elementary subarrangement of the arrangement $C(x)$ of type $\lambda = (\lambda_1,...,\lambda_m)$. Recall that $J_\lambda = \cup_{h=1}^m J_h \subset J$ with subsets J_h satisfying properties described in Section 4.1. According to those properties if a subarrangement $C_{J\lambda}(x) = \{H_i(x)\}_{i\in J\lambda}$ is an elementary subarrangement of $C(x)$ for some $x \in C^n - \Delta$, then the subarrangement $C_{J\lambda}(x^0) = \{H_i(x^0)\}_{i\in J\lambda}$, associated with the same J_λ, is an elementary subarrangement of $C(x^0)$ for every $x^0 \in C^n - \Delta$.

Example. If $C(x)$ is a generic arrangement, then all elementary subarrangements are of type $\lambda = (k)$, they are given by $k + 1$-element subsets of J.

Since $C(x)$ is with normal crossings we have a natural embeddings of graded exterior algebras $F(C_{J\lambda}(x)) \subset F(C(x))$ and an embedding of spaces $\mathrm{Sing}\ {}_AF^k(C_{J\lambda}(x)) \subset \mathrm{Sing}\ {}_AF^k(C(x))$. In particular, the singular element $s(C_{J\lambda}(x))$ of $C_{J\lambda}(x)$ can be considered as an element of $\mathrm{Sing}\ {}_AF^k(C(x))$.

Recall that for $h = 1,...,m$, there are auxiliary arrangements $C_{J\lambda,h}(x)$ associated with $C_{J\lambda}(x)$. For $h = 1,...,m - 1$, we define the *weight of the auxiliary arrangement* $C_{J\lambda,h}(x)$ with respect to $C(x)$ as the sum $a(J_\lambda,J,h) = \sum_{i\in J \text{ suchthat } g_i 6\in X_{h*}(J\lambda)} a_i$ and the *weight of the elementary subarrangement* $C_{J\lambda}(x)$ with respect to $C(x)$ as the product $a(J_\lambda,J) = a_J \cdot \prod_{h=1}^{m-1} a(J_\lambda, J, h)$. We define the *potentials of first and second kind* of the family of arrangements $C(x)$, $x \in C^n - \Delta$, to be respectively the following functions on $C^n - \Delta$:

$$P(x_1,\dots,x_n) = \sum \frac{1}{a(J_\lambda,J)} P_{\mathcal{C}_{J_\lambda}}(x_1,\dots,x_n), \tag{4.7}$$

$$Q(x_1,\dots,x_n) = \sum \frac{a_J}{a(J_\lambda,J)} Q_{\mathcal{C}_{J_\lambda}}(x_1,\dots,x_n), \tag{4.8}$$

where the sums are over all elementary subarrangements $C_{J\lambda}(x)$ of $C(x)$ and $P_{J\lambda}(x),Q_{J\lambda}(x)$ are the prepotentials of first and second kind, respectively, of the elementary subarrangements $C_{J\lambda}(x)$ of the arrangement $C(x)$. The potentials are not uniques, since the prepotentials are not unique, see Section 4.4.

Example. The second arrangement in Figure 2 has three elementary subarrangements with J_λ being $\{1,3,4\}$ or $\{2,3,4\}$ or $\{1,2\} \cup \{3,4\}$ of types $\lambda = (2),(2),(1,1)$, respectively. The potential of second kind for that arrangement is

$$Q(z_1,z_2,z_3,z_4) = a_1a_3a_4\ln(f_{\{1,3,4\},1}(z_1,z_2,z_3))\frac{(f_{\{1,3,4\},1}(z_1,z_3,z_4))^4}{4!\,(d_{1,3}d_{3,4}d_{4,1})^2}$$
$$+ a_2a_3a_4\ln(f_{\{2,3,4\},1}(z_2,z_3,z_4))\frac{(f_{\{2,3,4\},1}(z_2,z_3,z_4))^4}{4!\,(d_{2,3}d_{3,4}d_{4,2})^2}$$
$$+ \frac{a_1a_2a_3a_4}{a_3+a_4}\ln(f_{\{1,2\}\cup\{3,4\},1}(z_1,z_2))\frac{(f_{\{1,2\}\cup\{3,4\},1}(z_1,z_2))^2}{2!(d_{1;1}d_{2;1})^2}\frac{(f_{\{1,2\}\cup\{3,4\},2}(z_1,z_2,z_3,z_4))^2}{2!(d_{3;2}d_{4;2})^2},$$

c.f. this formula with the formula in the example of Section 1.1.

The potentials *P,Q* for families of generic arrangements were constructed in [V5], c.f. [V7].

Lemma 4.4. *For any independent subset* $\{s_1,\dots,s_{k-1}\} \subset J$, *we have*

$$\sum_{j\in J} d_{j,s_1,\dots,s_{k-1}}\frac{\partial P}{\partial z_j} = \sum_{j\in J} d_{j,s_1,\dots,s_{k-1}}\frac{\partial Q}{\partial z_j} = 0 \tag{4.9}$$

Proof. It is enough to prove that $\sum_{j\in J} d_{j,s_1,\dots,s_{k-1}}\frac{\partial P_{\mathcal{C}_{J_\lambda}(z)}}{\partial z_j} = \sum_{j\in J} d_{j,s_1,\dots,s_{k-1}}\frac{\partial Q_{\mathcal{C}_{J_\lambda}(z)}}{\partial z_j} = 0$ for every elementary subarrangement $C_{J\lambda}(z)$ of $C(x)$. To prove that, it is enough to prove that $\sum_{j\in J} d_{j,s_1,\dots,s_{k-1}}\frac{\partial f_{\mathcal{C}_{J_\lambda},h}}{\partial z_j} = 0$ for any h, see formula (4.4), but that is clear.

5. Orthogonal projection

5.1. **Formula for orthogonal projection.** Recall the objects of Section 3. For $x \in C^n - \Delta$, we denoted $V = F^k(C(x))$, $\mathrm{Sing}_a V = \mathrm{Sing}_a F^k(C(x))$, $F_{j_1,\dots,j_k} = F_{j_1,\dots,j_k}(x)$. Let $\pi : V \to \mathrm{Sing}_a V$ be the orthogonal projection with respect to $S^{(a)}$.

For an ordered independent subset $I = \{i_1,\dots,i_k\} \subset J$, let E_I be the set of all elementary subarrangements $C_{J\lambda}(x)$ of $C(x)$ which have F_I as a distinguished element. Let $C_{J\lambda}(x) \in E_I$ be such a subarrangement. Let $s(C_{J\lambda}(x))$ be the singular element of $C_{J\lambda}(x)$ considered as an element of $\mathrm{Sing}_a V$. The singular element is defined up to multiplication by ±1. We fix the sign so that the distinguished element F_I enters $s(C_{J\lambda}(x))$ with coefficient 1.

Theorem 5.1. *For an independent ordered subset* $I = \{i_1,\dots,i_k\} \subset J$ *we have*

$$\pi(F_I) = \sum_{C_{J_\lambda}(x)\in E_I}\frac{1}{a(J_\lambda,J)}s(\mathcal{C}_{J_\lambda}(x)) \ \in \mathrm{Sing}_a V. \tag{5.1}$$

Corollary 5.2. *The space* $\mathrm{Sing}_a V$ *is generated by singular elements of elementary subarrangements.*

Notice that the singular element of an elementary subarrangement in C^k is a linear combination of at most 2^k basis vectors $F_{l_1,\dots,l_k} \in V$, while the dimension of V could be arbitrarily big and grow with n.

Example. For the second arrangement in Figure 2, we have

$$\pi(F_{3,4}) = \frac{1}{a_1+a_2+a_3+a_4}\Big((a_1F_{3,4}+a_3F_{4,1}+a_4F_{1,3}) + (a_2F_{3,4}+a_3F_{4,2}+a_4F_{2,3})\Big)$$

$$\pi(F_{2,3}) = \frac{1}{a_1+a_2+a_3+a_4}\Big((a_4F_{2,3}+a_2F_{3,4}+a_3F_{4,2}) + \frac{1}{a_3+a_4}(a_1F_2-a_2F_1)\wedge(a_4F_3-a_3F_4)\Big),$$

where $(a_1F_2 - a_2F_1) \wedge (a_4F_3 - a_3F_4) = a_1a_4F_{2,3} - a_1a_3F_{2,4} - a_2a_4F_{1,3} + a_1a_3F_{1,4}$.

5.2. **Proof of Theorem 5.1.** Recall that every element of the form $P_{j\in J}F_{j,l_1,\dots,l_{k-1}}$ is orthogonal to Sing $_aV$. In order to construct $\pi(F_I)$ from F_I we add to F_I a linear combination of elements of the form $P_{j\in J}F_{j,l_1,\dots,l_{k-1}}$ so that the result is a linear combination of the singular elements of elementary subarrangements $C_{J\lambda}(x)$. That means that the result lies in Sing $_aV$ by Lemma 4.2.

The transition from F_I to $\pi(F_I)$ is done in k steps and this reasoning is by induction on the number m appearing in the presentation $\lambda = (\lambda_1,\dots,\lambda_m)$. As the first step we add to F_I a linear combination of elements of the form $P_{j\in J}F_{j,l_1,\dots,l_{k-1}}$ and transform F_I to the sum $P_{C_{J\lambda}(x)\in E_I \text{ with } m=1}\frac{1}{a(J_\lambda,J)}s(\mathcal{C}_{J_\lambda}(x)) + R_1$, where R_1 is a remainder. Then we add a new linear combination of elements of the form $P_{j\in J}F_{j,l_1,\dots,l_{k-1}}$ and transform the result to the sum $P_{C_{J\lambda}(x)\in E_I \text{ with } m\leqslant 2}\frac{1}{a(J_\lambda,J)}s(\mathcal{C}_{J_\lambda}(x)) + R_2$, and so on. After m steps the result will be the right-hand side in (5.1) and there will be no remainder.

We illustrate that reasoning by considering the case $k = 3$. We construct the orthogonal projection of the element $F_{1,2,3}$, which could be an arbitrary basis vector of V after reordering hyperplanes. Formula (5.1) says

(5.2) $\pi(F_{1,2,3}) = \Sigma_{1,2,3} + \Sigma_{1,2;3} + \Sigma_{1,3;2} + \Sigma_{2,3;1} + \Sigma_{1;2,3} + \Sigma_{2;1,3} + \Sigma_{3;1,2} +$
$+ \Sigma_{1;2;3} + \Sigma_{1;3;2} + \Sigma_{2;1;3} + \Sigma_{2;3;1} + \Sigma_{3;1;2} + \Sigma_{3;2;1}$,

where

$$\Sigma_{1,2,3} := \frac{1}{a_J}\sum{}^{1,2,3}(a_jF_{1,2,3} - a_1F_{j,2,3} + a_2F_{1,j,3} - a_3F_{1,2,j}),$$

$$\Sigma_{1,2;3} := \frac{1}{a_Ja(1,2)}\sum{}^{1,2}(a_jF_{1,2} - a_1F_{j,2} + a_2F_{j,1})\wedge\sum{}^{1,2;3}(a_hF_3 - a_3F_h),$$

$$\Sigma_{1,3;2} := \frac{-1}{a_Ja(1,3)}\sum{}^{1,3}(a_jF_{1,3} - a_1F_{j,3} + a_2F_{j,1})\wedge\sum{}^{1,3;2}(a_hF_2 - a_2F_h),$$

$$\Sigma_{2,3;1} := \frac{1}{a_Ja(2,3)}\sum{}^{2,3}(a_jF_{2,3} - a_2F_{j,3} + a_3F_{j,2})\wedge\sum{}^{2,3;1}(a_hF_1 - a_1F_h),$$

$$\Sigma_{1;2,3} := \frac{1}{a_Ja(1)}\sum{}^{1}(a_jF_1 - a_1F_j)\wedge\sum{}^{1;2,3}(a_hF_{2,3} - a_2F_{h,3} + a_3F_{h,2}),$$

$$\Sigma_{2;1,3} := \frac{-1}{a_J a(2)} \sum{}^{2}(a_j F_2 - a_2 F_j) \wedge \sum{}^{2;1,3}(a_h F_{1,3} - a_1 F_{h,3} + a_3 F_{h,1}),$$

$$\Sigma_{3;1,2} := \frac{1}{a_J a(3)} \sum{}^{3}(a_j F_3 - a_3 F_j) \wedge \sum{}^{3;1,2}(a_h F_{1,2} - a_1 F_{h,2} + a_2 F_{h,1}),$$

$$\Sigma_{1;2,3} := \frac{1}{a_J a(1) a(1,2)} \sum{}^{1}(a_j F_1 - a_1 F_j) \wedge \sum{}^{1;2}(a_h F_2 - a_2 F_h) \wedge \sum{}^{1,2;3}(a_i F_3 - a_3 F_i),$$

$$\Sigma_{1;3,2} := \frac{-1}{a_J a(1) a(1,3)} \sum{}^{1}(a_j F_1 - a_1 F_j) \wedge \sum{}^{1;3}(a_h F_3 - a_3 F_h) \wedge \sum{}^{1,3;2}(a_i F_2 - a_2 F_i),$$

$$\Sigma_{2;3,1} := \frac{1}{a_J a(2) a(2,3)} \sum{}^{2}(a_j F_2 - a_2 F_j) \wedge \sum{}^{2;3}(a_h F_3 - a_3 F_h) \wedge \sum{}^{2,3;1}(a_i F_1 - a_1 F_i),$$

$$\Sigma_{2;1,3} := \frac{-1}{a_J a(2) a(2,1)} \sum{}^{2}(a_j F_2 - a_2 F_j) \wedge \sum{}^{2;1}(a_h F_1 - a_1 F_h) \wedge \sum{}^{2,1;3}(a_i F_3 - a_3 F_i),$$

$$\Sigma_{3;2,1} := \frac{-1}{a_J a(3) a(2,3)} \sum{}^{3}(a_j F_3 - a_3 F_j) \wedge \sum{}^{3;2}(a_h F_2 - a_2 F_h) \wedge \sum{}^{3,2;1}(a_i F_1 - a_1 F_i),$$

$$\Sigma_{3;1,2} := \frac{1}{a_J a(3) a(1,3)} \sum{}^{3}(a_j F_3 - a_3 F_j) \wedge \sum{}^{3;1}(a_h F_1 - a_1 F_h) \wedge \sum{}^{3,1;2}(a_i F_2 - a_2 F_i).$$

In these formulas we use the following notations.

We denote $a(h,l) = \sum^{P} a_j$, where the sum is over all $j \in J$ such that $g_j \in/ \operatorname{span}(g_h, g_l)$. We denote $a(h) = \sum^{P} a_j$, where the sum is over all $j \in J$ such that $g_j \in/ \operatorname{span}(g_h)$.

The sum $\sum^{P_{1,2,3}}$ is over all $j \in J$ such that the subset $\{j,1,2,3\}$ forms a circuit in J. The sum $\sum^{P_{h,l}}$ is over all $j \in J$ such that the subset $\{j,h,l\}$ forms a circuit in J. The sum $\sum^{P_h}$ is over all $j \in J$ such that the subset $\{j,h\}$ forms a circuit in J.

The sum $\sum^{P_{j,l,s}}$ is over all $h \in J$ such that $g_h \in/ \operatorname{span}(g_j, g_l)$. The sum $\sum^{P_{j;l,s}}$ is over all $h \in J$ such that $\operatorname{span}(g_j,g_l,g_h) = \operatorname{span}(g_j,g_s,g_h) = (\mathbb{C}^3)^*$. The sum $\sum^{P_{j;l}}$ is over all $h \in J$ such that $\operatorname{span}(g_j,g_l) = \operatorname{span}(g_j,g_h)$. The first transformation is

$$F_{1,2,3} \mapsto F_{1,2,3} - \frac{a_1}{a_J} \sum\nolimits_{j \in I} F_{j,2,3} - \frac{a_2}{a_J} \sum\nolimits_{j \in I} F_{1,j,3} - \frac{a_3}{a_J} \sum\nolimits_{j \in I} F_{1,2,j}, \tag{5.3}$$

the added terms are a linear combination of elements of the form $\sum^{P_{j \in J}} F_{j,l_1,l_2}$. We rearrange the right-hand side of (5.3) as follows:

$$= a_{-J}\, 1\, X_{1,2,3}(a_j F_{1,2,3} - a_1 F_{j,2,3} + a_2 F_{1,j,3} - a_3 F_{1,2,j}) \tag{5.4}$$

(5.5) $+a_{-J\,1}\,\mathrm{X}_{\,1,2}(a_jF_{1,2} - a_1F_{j,2} + a_2F_{j,1}) \wedge F_3$

(5.6) $-a_{-J\,1}\,\mathrm{X}_{\,1,3}(a_jF_{1,3} - a_1F_{j,3} + a_3F_{j,1}) \wedge F_2$

(5.7) $$+a_J^{-1}\sum{}^{2,3}(a_jF_{2,3} - a_2F_{j,3} + a_3F_{j,2}) \wedge F_1$$

(5.8) $+a_{-J\,1}\,\mathrm{X}_{\,1}(a_jF_1 - a_1F_j) \wedge F_{2,3} - a_{J-\,1}\,\mathrm{X}_{\,2}(a_jF_2 - a_2F_j) \wedge F_{1,3}$

(5.9) $+a_{-J\,1}\,\mathrm{X}_{\,3}(a_jF_3 - a_3F_j) \wedge F_{1,2}.$

The sum in (5.4) is exactly the sum $\mathrm{P}_{C_J\lambda(x)\in E\{1,2,3\}\text{ with }} \frac{1}{a\{J_\lambda,J\}}s(\mathcal{C}_{J_\lambda}(x$)) and the sums in (5.5)-(5.9) form the first remainder R_1.

Now we add to each of the sums in (5.5)-(5.9) a linear combination of elements of the form $\mathrm{P}_{j\in J}\,F_{j,l_1,l_2}$ as follows. Let

$$a_J^{-1}\sum{}^{1,2}(a_jF_{1,2} - a_1F_{j,2} + a_2F_{j,1}) \wedge F_3 \to$$
$$a_J^{-1}\sum{}^{1,2}(a_jF_{1,2} - a_1F_{j,2} + a_2F_{j,1}) \wedge \left(F_3 - \frac{a_3}{a(1,2)}\sum\nolimits_{i\in J}F_i\right)$$
$$= \frac{1}{a_Ja(1,2)}\sum{}^{1,2}(a_jF_{1,2} - a_1F_{j,2} + a_2F_{j,1}) \wedge \sum{}^{1,2;3}(a_iF_3 - a_3F_i).$$

Similarly,

$$-a_J^{-1}\sum{}^{1,3}(a_jF_{1,3} - a_1F_{j,3} + a_3F_{j,1}) \wedge F_2 \to$$
$$-\frac{1}{a_Ja(1,3)}\sum{}^{1,3}(a_jF_{1,3} - a_1F_{j,3} + a_2F_{j,1}) \wedge \sum{}^{1,3;2}(a_iF_2 - a_2F_i),$$

$$a_{-J\,1}\,\mathrm{X}_{\,2,3}(a_jF_{2,3} - a_2F_{j,3} + a_3F_{j,2}) \wedge F_1 \to$$
$$\frac{1}{a_Ja(2,3)}\sum{}^{2,3}(a_jF_{2,3} - a_2F_{j,3} + a_3F_{j,2}) \wedge \sum{}^{2,3;1}(a_iF_1 - a_1F_i).$$

Similarly

$$a_J^{-1}\sum{}^{1}(a_jF_1 - a_1F_j) \wedge F_{2,3} \to$$
$$a_J^{-1}\sum{}^{1}(a_jF_1 - a_1F_j) \wedge \left(F_{2,3} - \frac{a_2}{a(1)}\sum\nolimits_{i\in J}F_{i,3} - \frac{a_3}{a(1)}\sum\nolimits_{i\in J}F_{2,i}\right)$$
$$= \frac{1}{a_Ja(1)}\sum{}^{1}(a_jF_1 - a_1F_j) \wedge \sum{}^{1;2,3}(a_iF_{2,3} - a_2F_{i,3} + a_3F_{i,2})$$

$$+\frac{1}{a_Ja(1)}\sum{}^{1}(a_jF_1 - a_1F_j) \wedge \sum{}^{1;2}(a_iF_2 - a_2F_i) \wedge F_3$$

$$-\frac{1}{a_J a(1)} \sum{}^{1}(a_j F_1 - a_1 F_j) \wedge \sum{}^{1;3}(a_i F_3 - a_3 F_i) \wedge F_2$$

We transform similarly the remaining two sums in (5.8)-(5.9). This finishes step two of the procedure. After the two steps the result is

(5.10) $$\Sigma_{1,2,3} + \Sigma_{1,2;3} + \Sigma_{1,3;2} + \Sigma_{2,3;1} + \Sigma_{1;2,3} + \Sigma_{2;1,3} + \Sigma_{3;1,2}$$

(5.11) $$+\frac{1}{a_J a(1)} \sum{}^{1}(a_j F_1 - a_1 F_j) \wedge \sum{}^{1;2}(a_i F_2 - a_2 F_i) \wedge F_3$$

(5.12) $$-\frac{1}{a_J a(1)} \sum{}^{1}(a_j F_1 - a_1 F_j) \wedge \sum{}^{1;3}(a_i F_3 - a_3 F_i) \wedge F_2$$

(5.13) $$-\frac{1}{a_J a(2)} \sum{}^{2}(a_j F_2 - a_2 F_j) \wedge \sum{}^{2;1}(a_i F_1 - a_1 F_i) \wedge F_3$$

(5.14) $$+\frac{1}{a_J a(2)} \sum{}^{2}(a_j F_2 - a_2 F_j) \wedge \sum{}^{2;3}(a_i F_3 - a_3 F_i) \wedge F_1$$

(5.15) $$-\frac{1}{a_J a(3)} \sum{}^{3}(a_j F_3 - a_3 F_j) \wedge \sum{}^{3;2}(a_i F_2 - a_2 F_i) \wedge F_1$$

(5.16) $$+\frac{1}{a_J a(3)} \sum{}^{3}(a_j F_3 - a_3 F_j) \wedge \sum{}^{3;1}(a_i F_1 - a_1 F_i) \wedge F_2$$

The sum in (5.10) is the sum $\mathrm{P}_{C\lambda(x)\in E\{1,2,3\}}$ with $m\leqslant 2\ \frac{1}{a(J_\lambda,J)} s(\mathcal{C}_{J_\lambda}(x))$ and the sums in (5.11)-(5.16) form the second remainder R_2. As the third step we transform the expression in (5.11) to

$$\frac{1}{a_J a(1)} \sum{}^{1}(a_j F_1 - a_1 F_j) \wedge \sum{}^{1;2}(a_i F_2 - a_2 F_i) \wedge (F_3 - \frac{a_3}{a(1,2)} \sum\nolimits_{l\in J} F_l)$$
$$= \frac{1}{a_J a(1) a(1,2)} \sum{}^{1}(a_j F_1 - a_1 F_j) \wedge \sum{}^{1;2}(a_i F_2 - a_2 F_i) \wedge \sum{}^{1,2;3}(a_l F_3 - a_3 F_l),$$

and similarly we transform the expressions in (5.12)-(5.16). After these three steps we obtain formula (5.2). The case of arbitrary k is similar to this case of $k = 3$. Theorem 5.1 is proved.

6. Potential of first kind

Recall the objects of Section 3. Recall that $v_{j_1,\dots,j_k} = \pi(F_{j_1,\dots,j_k})$, where $\pi : V \to \mathrm{Sing}\ {}_aV$ is the orthogonal projection with respect to $S^{(a)}$. Let P be the potential of first kind of the family of arrangements $\mathrm{C}(x), x \in \mathrm{C}^n - \Delta$.

Theorem 6.1. *For any two ordered independent subsets $I = \{i_1,\dots,i_k\}, L = \{l_1,\dots,l_k\} \subset J$, we have*

$$S^{(a)}(v_{i_1,\dots,i_k}, v_{l_1,\dots,l_k}) = d_{i_1,\dots,i_k} d_{l_1,\dots,l_k} \frac{\partial^{2k} P}{\partial z_{i_1} \dots \partial z_{i_k} \partial z_{l_1} \dots \partial z_{l_k}}. \tag{6.1}$$

For families of generic arrangements this theorem was proved in [V5], c.f. [V7]. *Proof.*

Since $v_{l_1,\dots,l_k} = \pi(F_{l_1,\dots,l_k})$ we have

$$\begin{aligned} S^{(a)}(v_{i_1,\dots,i_k}, v_{l_1,\dots,l_k}) &= S^{(a)}(v_{i_1,\dots,i_k}, F_{l_1,\dots,l_k}) \\ &= \sum\nolimits_{C_{J_\lambda}(x) \in E_{i_1,\dots,i_k}} \frac{1}{a(J_\lambda, J)} S^{(a)}(s(\mathcal{C}_{J_\lambda}(x)), F_{l_1,\dots,l_k}) \end{aligned}.$$

Formula (4.2) for the singular element $s(\mathrm{C}_{J_\lambda}(x))$ shows that the number $S^{(a)}(s(\mathrm{C}_{J_\lambda}(x)), F_{l_1,\dots,l_k})$ is nonzero if and only if $F_{l_1,\dots,l_k}$ is a distinguished elements of the elementary arrangement $\mathrm{C}_{J_\lambda}(x)$. Let this condition be satisfied for an elementary arrangement $\mathrm{C}_{J_\lambda}(x) \in E_{i_1,\dots,i_k}$. Let $J_\lambda = \cup_{h=1}^m J_h,\ J_h = \{j_1^h, j_2^h, \dots, j_{\lambda_h+1}^h\}$, and let $K_{\widehat{j_i^h}} = \{j_1^h, \dots, j_i^h, \dots, j_{\lambda_h+1}^h\}$ be an ordered λ_h-element subset of J_h. Then

$$\begin{aligned} s(\mathcal{C}_{J_\lambda}(x)) &= \left(\sum\nolimits_{i=1}^{\lambda_1+1} (-1)^{i+1} a_{j_i^1} F_{K_{\widehat{j_i^1}}}\right) \wedge \left(\sum\nolimits_{i=1}^{\lambda_2+1} (-1)^{i+1} a_{j_i^2} F_{K_{\widehat{j_i^2}}}\right) \wedge \dots \\ &\quad \dots \wedge \left(\sum\nolimits_{i=1}^{\lambda_m+1} (-1)^{i+1} a_{j_i^m} F_{K_{\widehat{j_i^m}}}\right). \end{aligned} \tag{6.2}$$

Due to the choice of sign of $s(\mathrm{C}_{J_\lambda}(x))$ in Section 5.1 we may assume that

$$F_{i_1,\dots,i_k} = F_{K_{\widehat{j_1^1}}} \wedge F_{K_{\widehat{j_1^2}}} \wedge \dots F_{K_{\widehat{j_1^m}}}. \tag{6.3}$$

We may also assume that

$$F_{l_1,\dots,l_k} = F_{K_{\widehat{j_{s_1}^1}}} \wedge F_{K_{\widehat{j_{s_2}^2}}} \wedge \dots F_{K_{\widehat{j_{s_m}^m}}} \tag{6.4}$$

for some $s_h \in \{1,\dots,\lambda_h + 1\}, h = 1,\dots,m$. [1]Equations (6.2)-(6.4) imply

$$S^{(a)}(s(\mathcal{C}_{J_\lambda}(x)), F_{l_1,\dots,l_k}) = \prod\nolimits_{h=1}^m (-1)^{s_h+1} \prod\nolimits_{h=1}^m \prod\nolimits_{j \in J_h} a_j. \tag{6.5}$$

Recall formula (4.7) for the potential of first kind. It is a linear combination of prepotentials of first kind $P_{\mathrm{C}_{J_\lambda}}(x)$ of all elementary subarrangements $\mathrm{C}_{J_\lambda}(x)$ of $\mathrm{C}(x)$. To finish the proof of Theorem 6.1 we need to show that if $\mathrm{C}_{J_\lambda}(x)$ is as in formula (6.5), then

$$d_{i_1,\dots,i_k} d_{l_1,\dots,l_k} \frac{\partial^{2k} P_{\mathcal{C}_{J_\lambda}}}{\partial z_{i_1} \dots \partial z_{i_k} \partial z_{l_1} \dots \partial z_{i_k}} = \prod\nolimits_{h=1}^m (-1)^{s_h+1} \prod\nolimits_{h=1}^m \prod\nolimits_{i \in J_h} a_i \tag{6.6}$$

and

[1] Notice that if the indices of $F_{l_1,\dots,l_k}$ are permuted then $F_{l_1,\dots,l_k}$ is multiplied by ±1 and $d_{l_1,\dots,l_k}$ in the right-hand side of (6.1) is multiplied by the same ±1.

$$\frac{\partial^{2k} P_{\mathcal{C}_{J_\lambda}}}{\partial z_{i_1} \dots \partial z_{i_k} \partial z_{l_1} \dots \partial z_{i_k}} = 0 \tag{6.7}$$

if $C_{J\lambda}(x)$ 6∈ $E_{i1,\dots,ik}$ or if $C_{J\lambda}(x) \in E_{i1,\dots,ik}$ but $F_{l1,\dots,lk}$ is not a distinguished element of $C_{J\lambda}(x)$. Recall the formula

$$P_{\mathcal{C}_{J_\lambda}} = \prod_{h=1}^{m} \frac{\prod_{j \in J_h} a_j}{(2\lambda_h)!} \frac{(f_{\mathcal{C}_{J_\lambda,h}})^{2\lambda_h}}{(\prod_{i=1}^{\lambda_h+1} d_{j_1^h,\dots,\widehat{j_i^h},\dots,j_{\lambda_h+1}^h;h})^2}.$$

Lemma 6.2. *The* $2\lambda_h$*-th derivative of* $\frac{1}{(2\lambda_h)!} \frac{(f_{\mathcal{C}_{J_\lambda,h}})^{2\lambda_h}}{(\prod_{i=1}^{\lambda_h+1} d_{j_1^h,\dots,\widehat{j_i^h},\dots,i_{\lambda_h+1}^h;h})^2}$ *with respect to the variables* $z_{j_2^h}, \dots, z_{j_{\lambda_h+1}^h}, z_{j_1^h}, \dots, \widehat{z_{j_{s_h}^h}}, \dots, z_{j_{\lambda_h+1}^h}$ *equals* $\frac{(-1)^{s_h+1}}{d_{j_2^h,\dots,j_{\lambda_h+1}^h;h} d_{j_1^h,\dots,\widehat{j_{s_h}^h},\dots,j_{\lambda_h+1}^h;h}}$.

Now Lemmas 6.2 and 4.3 imply formula (6.6).

Lemma 6.3. *Formula* (6.7) *holds if* $C_{J\lambda}(x)$ 6∈ $E_{i1,\dots,ik}$ *or if* $C_{J\lambda}(x) \in E_{i1,\dots,ik}$ *but* $F_{l1,\dots,lk}$ *is not a distinguished element of* $C_{J\lambda}(x)$.

Proof. Let $C_{J\lambda}$ be any elementary subarrangement and $F_{i1,\dots,ik}$,$F_{l1,\dots,lk}$ two nonzero elements. Clearly formula (6.7) holds if $\{i_1, \dots, i_k, \ l_1, \dots, l_k\} \not\subset \cup_{h=1}^{m} J_h$. Assume that $\{i_1,\dots,i_k, l_1, \dots, l_k\} \subset \cup_{h=1}^{m} J_h$. For h = 1,...,m denote $i^h = |\{s \mid i_s \in J_h\}|$ and $l^h = |\{s \mid l_s \in J_h\}|$. We have $i^1 + \dots + i^h$ 6 $\lambda_1 + \dots + \lambda_h$, $l^1 + \dots + l^h$ 6 $\lambda_1 + \dots + \lambda_h$ for any h and $i_1 + \dots + i_m = l_1 + \dots + l_m = \lambda_1 + \dots + \lambda_m = k$. If $i_h = l_h = \lambda_h$ for all h, then $F_{i1,\dots,ik}$,$F_{l1,\dots,lk}$ are distinguished elements of $C_{J\lambda}$. So we assume that at least one of the numbers i^h,l^h differs from λ_h. Let h^{max} be the maximal h such that at least one of the numbers i^h,l^h differs from λ_h. Then: (a) each of $i^{h_{max}}$,$l^{h_{max}}$ is not less than $\lambda_{h_{max}}$; (b) at least one of them is greater than $\lambda_{h_{max}}$; (c) $h^{max} > 1$.

Then the derivative in (6.7) is zero due to the fact that the set $\{i_1,\dots,i_k, l_1,\dots,l_k\}$ has too many elements of $\cup_{h=h^{max}}^{m} J_h$, c.f. formulas (4.4), (4.5).

Theorem 6.1 is proved.

Recall the elements $p_j \in O(C_{C(x),a})$, $j \in J$, and the Grothiendick residue bilinear form $(,)_{C_{C(x),a}}$ on $O(C_{C(x),a})$.

Corollary 6.4. *For any two independent subsets* $I = \{i_1,\dots,i_k\}$,$L = \{l_1,\dots,l_k\} \subset J$, *we have*

$$(p_{i_1} \dots, p_{i_k}, p_{l_1} \dots p_{l_k})_{C_{C(x),a}} = (-1)^k \frac{\partial^{2k} P}{\partial z_{i_1} \dots \partial z_{i_k} \partial z_{l_1} \dots \partial z_{i_k}}. \tag{6.8}$$

Proof. Recall that the isomorphism of vector spaces

$$\nu(x) : \mathrm{Sing}_a \mathrm{F}^k(C(x)) \to O(C_{C(x),a}) \tag{6.9}$$

sends $\pi(F_{i_1,\dots,i_k})$ to $d_{i_1,\dots,i_k}p_{i_1}\dots p_{i_k}$ for all independent subsets $\{i_1,\dots,i_k\} \subset J$ and also identifies the form $S^{(a)}(\,,\,)$ on $\mathrm{Sing}_a \mathrm{F}^k(\mathrm{C}(x))$ and the the form $(-1)^k(\,,\,)_{C_{\mathrm{C}(x),a}}$ on $\mathrm{O}(C_{\mathrm{C}(x),a})$, see Theorem 2.2.

7. Potential of second kind

Recall the objects of Section 3. Let Q be the potential of second kind of the family of arrangements $\mathrm{C}(x), x \in \mathrm{C}^n - \Delta$.

Theorem 7.1. *Let $x \in \mathrm{C}^n - \Delta$. Then for any two independent subsets $I = \{i_1,\dots,i_k\}, L = \{l_1,\dots,l_k\} \subset J$ and any $i_0 \in J$, we have*

$$(p_{i_0}p_{i_1}\dots p_{i_k}, p_{l_1}\dots p_{l_k})_{C_{\mathrm{C}(x),a}} = (-1)^k \frac{\partial^{2k+1}Q}{\partial z_{i_0}\partial z_{i_1}\dots\partial z_{i_k}\partial z_{l_1}\dots\partial z_{l_k}}(x). \tag{7.1}$$

For families of generic arrangements this theorem was proved in [V5].

Proof. Due to relations (3.8) and (4.9) it is enough to prove (7.1) in the case when $i_0,i_1,\dots,i_k$ are distinct elements of J. Thus assume that $i_0,i_1,\dots,i_k$ are distinct. After reordering $i_0,i_1,\dots,i_k$, we may assume that $i_0,i_1,\dots,i_\mu$ form a circuit, where μ is some number $\leq k$. Recall the function

$$f_{i_0,i_1,\dots,i_k}(z) = \sum\nolimits_{j=0}^{k}(-1)^j z_{i_j} d_{i_0,\dots,\widehat{i_j},\dots,i_k} = \sum\nolimits_{j=0}^{\mu}(-1)^j z_{i_j} d_{i_0,\dots,\widehat{i_j},\dots,i_{\mu+1},\dots,i_k}$$

in (3.9) and relation (3.11):

$$p_{i_0}p_{i_1}\dots p_{i_k} = \frac{1}{f_{i_0,i_1,\dots,i_k}} \sum\nolimits_{j=0}^{\mu}(-1)^j a_{i_j} w_{i_0,\dots,\widehat{i_j},\dots,i_{\mu+1},\dots,i_k}.$$

First we analyze the left-hand side in (7.1). We have

$$\begin{aligned}
&(p_{i_0}p_{i_1}\dots p_{i_k}, p_{l_1}\dots p_{l_k})_{C_{\mathrm{C}(x),a}} \\
&= \frac{1}{f_{i_0,i_1,\dots,i_k}} \Big(\sum\nolimits_{j=0}^{\mu}(-1)^j a_{i_j} w_{i_0,\dots,\widehat{i_j},\dots,i_{\mu+1},\dots,i_k}, p_{l_1}\dots p_{l_k}\Big)_{C_{\mathrm{C}(x),a}} \\
&= \frac{1}{d_{l_1,\dots,l_k} f_{i_0,i_1,\dots,i_k}} \Big(\sum\nolimits_{j=0}^{\mu}(-1)^j a_{i_j} w_{i_0,\dots,\widehat{i_j},\dots,i_{\mu+1},\dots,i_k}, w_{l_1,\dots,l_k}\Big)_{C_{\mathrm{C}(x),a}} \\
&= \frac{(-1)^k}{d_{l_1,\dots,l_k} f_{i_0,i_1,\dots,i_k}} S^{(a)}\Big(\sum\nolimits_{j=0}^{\mu}(-1)^j a_{i_j} v_{i_0,\dots,\widehat{i_j},\dots,i_{\mu+1},\dots,i_k}, v_{l_1,\dots,l_k}\Big),
\end{aligned} \tag{7.2}$$

where the last equality holds by Theorem 2.2. We have

$$\begin{aligned}
&S^{(a)}\Big(\sum\nolimits_{j=0}^{\mu}(-1)^j a_{i_j} v_{i_0,\dots,\widehat{i_j},\dots,i_{\mu+1},\dots,i_k}, v_{l_1,\dots,l_k}\Big) \\
&= S^{(a)}\Big(\pi\Big(\sum\nolimits_{j=0}^{\mu}(-1)^j a_{i_j} F_{i_0,\dots,\widehat{i_j},\dots,i_{\mu+1},\dots,i_k}\Big), F_{l_1,\dots,l_k}\Big),
\end{aligned} \tag{7.3}$$

where π is the orthogonal projection, see Section 5.1.

Let $E(i_0,i_1,\dots,i_\mu)$ be the set of all elementary subarrangements $\mathrm{C}_{J_\lambda}(x)$ of $\mathrm{C}(x)$ with $J_\lambda = \cup_{h=1}^{m} J_h$ such that $J_1 = \{i_0,i_1,\dots,i_\mu\}$ and such that $F_{i_1,\dots,i_k}$ is a distinguished element of $\mathrm{C}_{J_\lambda}(x)$. Let $s(\mathrm{C}_{J_\lambda}(x))$ be the singular element of $\mathrm{C}_{J_\lambda}(x)$ considered as an element of $\mathrm{Sing}_a V$. The singular element is

defined up to multiplication by ±1. We fix the sign so that the distinguished element $F_{i_1,\dots,i_k}$ enters $s(\mathcal{C}_{J_\lambda}(x))$ with coefficeint 1.

Lemma 7.2. *We have*

$$\pi\left(\sum_{j=0}^{\mu}(-1)^j a_{i_j} F_{i_0,\dots,\widehat{i_j},\dots,i_{\mu+1},\dots,i_k}\right) = \sum_{\mathcal{C}_{J_\lambda}(x)\in E(i_0,i_1,\dots,i_\mu)} \frac{a_J}{a(J_\lambda, J)} s(\mathcal{C}_{J_\lambda}(x))$$

Proof. Indeed we have

$$\sum_{j=0}^{\mu}(-1)^j a_{i_j} F_{i_0,\dots,\widehat{i_j},\dots,i_{\mu+1},\dots,i_k} = \left(\sum_{j=0}^{\mu}(-1)^j a_{i_j} F_{i_0,\dots,\widehat{i_j},\dots,i_\mu}\right) \wedge F_{i_{\mu+1},\dots,i_k} \tag{7.4}$$

To construct the orthogonal projection of the right-hand side in (7.4), we need to apply the construction of the orthogonal projection, described in the proof of Theorem 5.1, but starting with step 2 since the result of the first step is already presented by the factor $\left(\sum_{j=0}^{\mu}(-1)^j a_{i_j} F_{i_0,\dots,\widehat{i_j},\dots,i_\mu}\right)$ in the right-hand side of (7.4), c.f. formulas (5.4)-(5.9).

By Lemma 7.2 the expression in (7.2) equals

$$\frac{(-1)^k}{d_{l_1,\dots,l_k} f_{i_0,i_1,\dots,i_k}} \sum_{\mathcal{C}_{J_\lambda}(x)\in E(i_0,i_1,\dots,i_\mu)} \frac{a_J}{a(J_\lambda, J)} S^{(a)}(s(\mathcal{C}_{J_\lambda}(x)), F_{l_1,\dots,l_k}) \tag{7.5}$$

We have

$$\begin{aligned} s(\mathcal{C}_{J_\lambda}(x)) \;=\; & \left(\sum_{j=0}^{\mu}(-1)^j a_{i_j} F_{i_0,\dots,\widehat{i_j},\dots,i_\mu}\right) \wedge \left(\sum_{i=1}^{\lambda_2+1}(-1)^{i+1} a_{j_i^2} F_{K_{\widehat{j_i^2}}}\right) \wedge \dots \\ & \dots \wedge \left(\sum_{i=1}^{\lambda_m+1}(-1)^{i+1} a_{j_i^m} F_{K_{\widehat{j_i^m}}}\right), \end{aligned}$$

where we use the notations of Section 4.2, namely, we have $J_h = \{j_1^h, j_2^h, \dots, j_{\lambda_h+1}^h\}$ for h = 2,...,m, and $K_{\widehat{j_i^h}} = \{j_1^h, \dots, \widehat{j_i^h}, \dots, j_{\lambda_h+1}^h\}$.

Due to our choice of sign of $s(\mathrm{C}_{J\lambda}(x))$ we may assume that we have the equality of ordered sets

$$\{i_1, \dots, i_k\} = \{i_1, \dots, i_\mu, K_{\widehat{j_1^2}}, K_{\widehat{j_1^3}}, \dots, K_{\widehat{j_1^m}}\} \tag{7.6}$$

The term $S^{(a)}(s(\mathrm{C}_{J\lambda}(x)),F_{l_1,\dots,l_k})$ is nonzero if and only if

$$\{l_1, \dots, l_k\} = \{i_0, \dots, \widehat{i_s}, \dots, i_\mu, K_{\widehat{j_{s_2}^2}}, K_{\widehat{j_{s_3}^3}}, \dots, K_{\widehat{j_{s_m}^m}}\} \tag{7.7}$$

for some 0 6 s 6 μ and some 1 6 s_h 6 λ_h + 1 for h = 2,...,m. In this case

$$\begin{aligned} & \frac{(-1)^k}{d_{l_1,\dots,l_k} f_{i_0,i_1,\dots,i_k}} \frac{a_J}{a(J_\lambda, J)} S^{(a)}(s(\mathcal{C}_{J_\lambda}(x)), F_{l_1,\dots,l_k}) \\ & = \frac{(-1)^k}{d_{l_1,\dots,l_k} f_{i_0,i_1,\dots,i_k}} \frac{a_J}{a(J_\lambda, J)} (-1)^{s+\sum_{h=2}^m s_h} \prod_{q=0}^{\mu} a_{i_q} \prod_{h=2}^{m} \prod_{q=1}^{\lambda_h+1} a_{j_q^h} \end{aligned} \tag{7.8}$$

Consider the right-hand side of (7.1). The potential Q of second kind is the sum $\sum \frac{a_J}{a(J_\lambda,J)} Q_{\mathcal{C}_{J_\lambda}}$ shown in (4.8), where the sum is over all elementary subarrangements $\mathsf{C}_{J\lambda}(x)$ of $\mathsf{C}(x)$.

Lemma 7.3. *If the derivative*

$$\frac{\partial^{2k+1} Q_{\mathcal{C}_{J_\lambda}}}{\partial z_{i_0}\partial z_{i_1}\dots\partial z_{i_k}\partial z_{l_1}\dots\partial z_{l_k}} \tag{7.9}$$

is nonzero, then $F_{i_1,\dots,i_k}$ and $F_{l_1,\dots,l_k}$ are distinguished elements of $\mathsf{C}_{J\lambda}$.

Proof. The proof is the same as the proof of Lemma 6.3.

Clearly the function $\frac{1}{f_{i_0,\dots,i_\mu}(z)}$ multiplied by a constant can be obtained by this differentiation only if $\mathsf{C}_{J\lambda}(x) \in E(i_0,i_1,\dots,i_\mu)$. In this case we have

$$\frac{a_J}{a(J_\lambda,J)} Q_{\mathcal{C}_{J_\lambda}} = \frac{a_J}{a(J_\lambda,J)} \frac{\prod_{q=0}^{\mu} a_{i_q}}{(2\mu)!} \ln(f_{\mathcal{C}_{J_\lambda,1}}) \frac{(f_{\mathcal{C}_{J_\lambda,1}})^{2\mu}}{(\prod_{j=0}^{\mu} d_{i_0,\dots,\widehat{i_j},\dots,i_\mu;1})^2} \prod_{h=2}^{m} P_{\mathcal{C}_{J_\lambda,h}}, \tag{7.10}$$

see formula (4.6). Derivatives of this summand do not depend on the ordering the elements of the sets J_h, $h = 1,\dots,m$, and we may assume that the equality (7.6) of ordered sets holds.

By Lemma 7.3 we may assume that the equality of ordered sets in (7.7) holds. In that case, the operator $(-1)^k \frac{\partial^{2k+1}}{\partial z_{i_0}\dots\widehat{\partial z_{i_s}}\dots\partial z_{i_\mu}\partial z_{i_{\mu+1}}\dots\partial z_{l_k}}$ applied to the expression in (7.10) gives

$$(-1)^k \frac{a_J}{a(J_\lambda,J)} \prod_{q=0}^{\mu} a_{i_q} \prod_{h=2}^{m} \prod_{q=1}^{\lambda_h+1} a_{j_q^h} \times \frac{(-1)^{s+\sum_{h=2}^m s_h}}{d_{i_0,\dots,\widehat{i_s},\dots,i_\mu;1} \prod_{h=2}^{m} d_{j_1^h,\dots,\widehat{j_{s_h}^h},\dots,j_{\lambda_h+1}^h;h}} \frac{1}{f_{\mathcal{C}_{J_\lambda,1}} \prod_{h=2}^{m} d_{j_2^h,\dots,j_{\lambda_h+1}^h;h}}. \tag{7.11}$$

Lemma 4.3 implies that

$$d_{i_0,\dots,\widehat{i_s},\dots,i_\mu;1} \prod_{h=2}^{m} d_{j_1^h,\dots,\widehat{j_{s_h}^h},\dots,j_{\lambda_h+1}^h;h} = d_{l_1,\dots,l_k}, \qquad f_{\mathcal{C}_{J_\lambda,1}} \prod_{h=2}^{m} d_{j_2^h,\dots,j_{\lambda_h+1}^h;h} = f_{i_0,\dots,i_k}. \tag{7.12}$$

Now (7.11) equals (7.8). This proves Theorem 7.1.

Corollary 7.4. *Let $x \in \mathsf{C}^n - \Delta$. Then for any two ordered independent subsets $I = \{i_1,\dots,i_k\}, L = \{l_1,\dots,l_k\} \subset J$ and any $i_0 \in J$, we have*

$$S^{(a)}(K_{i_0}(x) v_{i_1,\dots,i_k}, v_{l_1,\dots,l_k}) = d_{i_1,\dots,i_k} d_{l_1,\dots,l_k} \frac{\partial^{2k+1} Q}{\partial z_{i_0}\partial z_{i_1}\dots\partial z_{i_k}\partial z_{l_1}\dots\partial z_{l_k}}(x). \tag{7.13}$$

Proof. The corollary follows from formula (2.9) and Theorems 2.2, 3.1.

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