q-HYPERGEOMETRIC SOLUTIONS OF QUANTUM DIFFERENTIAL EQUATIONS, QUANTUM PIERI RULES, AND GAMMA THEOREM

VITALY TARASOV[1] AND ALEXANDER VARCHENKO[2]

[?]*Department of Mathematics, University of North Carolina at Chapel Hill Chapel Hill, NC 27599-3250, USA*

[?]*Faculty of Mathematics and Mechanics, Lomonosov Moscow State University Leninskiye Gory 1, 119991 Moscow GSP-1, Russia*

[°] *Department of Mathematical Sciences, Indiana University–Purdue University Indianapolis 402 North Blackford St, Indianapolis, IN 46202-3216, USA*

[°] *St.Petersburg Branch of Steklov Mathematical Institute Fontanka 27, St.Petersburg, 191023, Russia*

Key words: Flag varieties, quantum differential equation, dynamical connection, q-hypergeometric solutions

2010 Mathematics Subject Classification: 82B23, 17B80, 14N15, 14N35

Abstract. We describe q-hypergeometric solutions of the equivariant quantum differential equations and the associated qKZ difference equations for the cotangent bundle T^*F_λ of a partial flag variety F_λ . These q-hypergeometric solutions manifest a Landau-Ginzburg mirror symmetry for the cotangent bundle. We formulate and prove Pieri rules for quantum equivariant cohomology of the cotangent bundle. Our Gamma theorem for T^*F_λ says that the leading term of the asymptotics of the q-hypergeometric solutions can be written as the equivariant Gamma class of the tangent bundle of T^*F_λ multiplied by the exponentials of the equivariant first Chern classes of the associated vector bundles. That statement is analogous to the statement of the gamma conjecture by B.Dubrovin and by S.Galkin, V.Golyshev, and H.Iritani, see also the Gamma theorem for F_λ in Appendix B.

In memory of Victor Lomonosov (1946–2018)

Contents

---

[1] *E-mail*: vtarasov@iupui.edu, vt@pdmi.ras.ru, supported in part by Simons Foundation grant 430235

[2] *E-mail*: anv@email.unc.edu, supported in part by NSF grants DMS-1362924, DMS-1665239

1. Introduction

In [MO], D.Maulik and A.Okounkov develop a general theory connecting quantum groups and equivariant quantum cohomology of Nakajima quiver varieties, see [N1, N2]. In particular, in [MO] the operators of quantum multiplication by divisors are described. As it is well-known, these operators determine the equivariant quantum differential equations of a quiver variety. In this paper we apply this description to the cotangent bundles T^*F_λ of the $\mathfrak{gl}_n$ N-step partial flag varieties and construct q-hypergeometric solutions of the associated equivariant quantum differential equations and *qKZ* difference equations. The q-hypergeometric solutions are constructed in the form of Jackson integrals.

Studying solutions of the equivariant quantum differential equations may lead to better understanding Gromov-Witten invariants of the cotangent bundle, cf. Givental's study of the J-function in [Gi1, Gi2, Gi3].

The presentation of solutions of the equivariant quantum differential equations as $q$hypergeometric integrals manifests a version of the Landau-Ginzburg mirror symmetry for the cotangent bundle.

In [MO] the equivariant quantum differential equations come together with a compatible system of difference equations called the *qKZ* equations. In [GRTV, RTV1] the equivariant quantum differential equations and *qKZ* difference equations were identified with the dynamical differential equations and *qKZ* difference equations with values in the tensor product $(\mathbb{C}^N)^{\otimes n}$ of vector representations of $\mathfrak{gl}_N$. The q-hypergeometric solutions of the $(\mathbb{C}^N)^{\otimes n}$-valued *qKZ* difference equations were constructed long time ago in [TV1], see also [TV2]-[TV4]. It was expected that those q-hypergeometric solutions are also solutions of the compatible dynamical differential equations. That fact is proved in this paper and is the first main result of the paper. The proof is based on some new rather nontrivial identities for the integrand of the Jackson integral. The integrand is the product of the scalar master function and a vector-valued function, whose coordinates are called weight functions. In [RTV1] it was shown that the weight functions are nothing else but the stable envelopes of [MO] for the cotangent bundle of the partial flag varieties. Our new identities can be interpreted as new identities for stable envelopes. We interpret these new identities as Pieri rules in quantum equivariant cohomology of the cotangent bundle of the partial flag variety. That is our second main result.

Our Gamma theorem for $T^*\mathcal{F}_\lambda$ (Theorem B.1) says that the leading term of the asymptotics of the q-hypergeometric solutions for $T^*\mathcal{F}_\lambda$ is the product of the equivariant gamma class of the tangent bundle of $T^*\mathcal{F}_\lambda$ and the exponentials of the equivariant first Chern classes of the associated vector bundles. That statement is analogous to the statement of the gamma conjecture by B.Dubrovin and by S.Galkin, V.Golyshev, and H.Iritani, see Appendix B. See also the Gamma theorem for $\mathcal{F}_\lambda$ (Theorem B.2).

The paper is organized as follows. In Section 2 we introduce the $(\mathbb{C}^N)^{\otimes n}$-valued dynamical and *qKZ* equations. In Section 3 we define the weight functions and list their basic properties. In Section 4 we introduce the master function and describe the discrete differentials — the quantities with zero Jackson integrals. We also formulate there two key identities for the weight functions — Theorems 4.3 and 4.4. We prove Theorem 4.3 in Section 5 and Theorem 4.4 in Section 6. In Section 7, we summarize Theorems 4.3 and 4.4 as a statement about the integrand of the main Jackson integral. In Section 8 we construct integral representations for solutions of the $(\mathbb{C}^N)^{\otimes n}$-valued dynamical equations. In Section 9 we introduce the equivariant quantum differential equations and explain how their q-hypergeometric solutions are obtained from solutions of the $(\mathbb{C}^N)^{\otimes n}$-valued dynamical equations. In Section 10 we formulate and prove Pieri rules. In Section 11 we show that the space of solutions of the quantum differential equation can be identified with the vector space of the equivariant Ktheory algebra. We also discuss two limiting cases of the quantum differential equation. In Appendix A we discuss the basic properties of Schubert polynomials, and in Appendix B we formulate our Gamma theorems.

The authors thank G.Cotti, V.Golyshev, and R.Rimanyi for useful discussions. The second author thanks the Hausdorff Institute for Mathematics in Bonn for hospitality in March 2018, when the Gamma theorem was discovered. The second author also thanks the Max Planck Institute for Mathematics in Bonn for hospitality in May–June 2018.

2. Dynamical and *qKZ* equations

2.1. **Notations.** Fix $N,n \in \mathbb{Z}_{>0}$ and $h,\kappa \in \mathbb{C}^\times$. Let $\boldsymbol{\lambda} \in \mathbb{Z}^N_{\geqslant 0}$, $|\boldsymbol{\lambda}| = \lambda_1 + \ldots + \lambda_N = n$. Let $I = (I_1,\ldots,I_N)$ be a partition of $\{1,\ldots,n\}$ into disjoint subsets $I_1,\ldots,I_N$. Denote $\mathcal{I}_\lambda$ the set of all partitions I with $|I_j| = \lambda_j$, $j = 1,\ldots,N$.

Consider $\mathbb{C}^N$ with basis $v_i = (0,\ldots,0,1_i,0,\ldots,0)$, $i = 1,\ldots,N$, and the tensor product $(\mathbb{C}^N)^{\otimes n}$ with basis

$$v_I = v_{i_1} \otimes \cdots \otimes v_{i_n},$$

where the index I is a partition $(I_1,\ldots,I_N)$ of $\{1,\ldots,n\}$ into disjoint subsets $I_1,\ldots,I_N$ and $i_j = m$ if $j \in I_m$.

The space $(\mathbb{C}^N)^{\otimes n}$ is a module over the Lie algebra $\mathfrak{gl}_N$ with basis $e_{i,j}$, $i,j = 1,\ldots,N$. The $\mathfrak{gl}_N$-module $(\mathbb{C}^N)^{\otimes n}$ has weight decomposition $(\mathbb{C}^N)^{\otimes n} = \sum_{|\boldsymbol{\lambda}|=n} (\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$, where $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$ is the subspace with basis $(v_I)_{I\in\mathcal{I}_\lambda}$.

2.2. **Dynamical differential equations.** Define the linear operators $X_1,\ldots,X_n$ acting on $(\mathbb{C}^N)^{\otimes n}$-valued

$$X_i(\boldsymbol{z};h;\boldsymbol{q}) = \sum_{a=1}^n z_a e^{(a)}_{i,i} - h\Big(\frac{\tilde e_{i,i}(1-\tilde e_{i,i})}{2} + \sum_{1\leqslant a<b\leqslant n}\sum_{k=1}^N e^{(a)}_{i,k}e^{(b)}_{k,i} + \sum_{\substack{j=1\\ j\neq i}}^N \frac{q_j}{q_i-q_j}\,(\tilde e_{i,j}\tilde e_{j,i} - \tilde e_{i,i})\Big)$$

functions of $z = (z_1,\ldots,z_n)$, $q = (q_1,\ldots,q_N)$ and called the dynamical Hamiltonians:

(2.1)

,

where $\tilde e_{s,t} = \sum_{a=1}^n e^{(a)}_{s,t}$ and a superscript means that the corresponding operator acts on the corresponding tensor factor. The differential operators

$$\nabla_{q,\kappa,i} = \kappa q_i \frac{\partial}{\partial q_i} - X_i(z;h;q), \qquad i = 1,\dots,N, \tag{2.2}$$

preserve the weight decomposition of $(\mathbb{C}^N)^{\otimes n}$ and pairwise commute, see [TV2], also [GRTV, Section 3.4], [RTV1, Section 7.1], [MTV1]. The operators $\nabla_{q,\kappa,i}$ define the $(\mathbb{C}^N)^{\otimes n}$-valued *dynamical connection*. The system of differential equations

$$\kappa q_i \frac{\partial f}{\partial q_i} = X_i(z;h;q) f, \qquad i = 1,\dots,N, \tag{2.3}$$

on a $(\mathbb{C}^N)^{\otimes n}$-valued function $f(z;h;q)$ is called the *dynamical equations*.

2.3. **Difference** *qKZ* **equations.** Define the R-matrices acting on $(\mathbb{C}^N)^{\otimes n}$,

$$R^{(i,j)}(u) = \frac{u - hP^{(i,j)}}{u-h}, \qquad i,j = 1,\dots,n, \quad i \neq j.$$

Define the *qKZ* operators $K_1,\dots,K_n$ acting on $(\mathbb{C}^N)^{\otimes n}$:

$$K_i(z;h;q;\kappa) = R^{(i,i-1)}(z_i - z_{i-1} + \kappa) \dots R^{(i,1)}(z_i - z_1 + \kappa) \times$$
$$\times\, q_1^{e_{1,1}^{(i)}} \dots q_N^{e_{N,N}^{(i)}} R^{(i,n)}(z_i - z_n) \dots R^{(i,i+1)}(z_i - z_{i+1}).$$

The *qKZ* operators preserve the weight decomposition of $(\mathbb{C}^N)^{\otimes n}$ and form a discrete flat connection,

$K_i(z_1,\dots,z_j+\kappa,\dots,z_n;q;\kappa)K_j(z;h;q;\kappa) = K_j(z_1,\dots,z_i+\kappa,\dots,z_n;q;\kappa)K_i(z;h;q;\kappa)$ for all i,j, see [FR]. The system of difference equations with step κ,

$$f(z_1,\dots,z_i+\kappa,\dots,z_n;q) = K_i(z;h;q;\kappa)f(z_1,\dots,z_n;q), \qquad i = 1,\dots,N, \tag{2.4}$$

on a $(\mathbb{C}^N)^{\otimes n}$-valued function $f(z,q)$ is called the *qKZ* equations.

Theorem 2.1 ([TV2])**.** *The systems of dynamical and qKZ equations are compatible.*

3. Weight functions

3.1. **Weight functions** $\check W_I$**.** For $I \in \mathcal{I}_\lambda$, we define the weight functions $\check W_I(t;z)$, cf. [TV1, TV4, RTV1]. The functions $\check W_I(t;z)$ here coincide with the functions $W_I(t;z;h)$ defined in [RTV1, Section 3.1].

Recall $\lambda = (\lambda_1,\dots,\lambda_N)$. Denote $\lambda^{(i)} = \lambda_1 + \dots + \lambda_i$, $i = 1,\dots,N-1$, $\lambda^{(N)} = n$, and $\lambda^{\{1\}} = \sum_{i=1}^{N-1} \lambda^{(i)} = \sum_{i=1}^{N-1} (N-i)\lambda_i$. Recall $I = (I_1,\dots,I_N)$. Set $\bigcup_{k=1}^{j} I_k = \{ i_1^{(j)} < \dots <$

$i^{(j)}_{\lambda^{(j)}}\}$. Consider the variables h and $t^{(j)}_a,\ j=1,\dots,N-1,\ a=1,\dots,\lambda^{(j)}$. Set $t^{(N)}_a = z_a$, a = $1,\dots,n$. Denote $t^{(j)} = (t^{(j)}_k)_{k\leqslant\lambda^{(j)}}$ and $t = (t^{(1)},\dots,t^{(N-1)})$.

The weight functions are

$$\check W_I(t;z) = (-h)^{\lambda^{\{1\}}}\,\mathrm{Sym}_{t^{(1)}_1,\dots,t^{(1)}_{\lambda^{(1)}}}\cdots\,\mathrm{Sym}_{t^{(N-1)}_1,\dots,t^{(N-1)}_{\lambda^{(N-1)}}}\,\check U_I(\boldsymbol{t};\boldsymbol{z}), \tag{3.1}$$

$$\check U_I(\boldsymbol{t};\boldsymbol{z}) \,=\, \prod_{j=1}^{N-1}\prod_{a=1}^{\lambda^{(j)}}\Bigg(\prod_{\substack{c=1\\ i^{(j+1)}_c<i^{(j)}_a}}^{\lambda^{(j+1)}}(t^{(j)}_a-t^{(j+1)}_c-h)\prod_{\substack{d=1\\ i^{(j+1)}_d>i^{(j)}_a}}^{\lambda^{(j+1)}}(t^{(j)}_a-t^{(j+1)}_d)\prod_{b=a+1}^{\lambda^{(j)}}\frac{t^{(j)}_a-t^{(j)}_b-h}{t^{(j)}_a-t^{(j)}_b}\Bigg).$$

In these formulas for a function $f(t_1,\dots,t_k)$ of some variables, we denote

$$\mathrm{Sym}_{t_1,\dots,t_k} f(t_1,\dots,t_k) = \sum_{\sigma\in S_k} f(t_{\sigma_1},\dots,t_{\sigma_k}).$$

Example. Let $N=2$, $n=2$, $\lambda=(1,1)$, $I=(\{1\},\{2\})$, $J=(\{2\},\{1\})$. Then

$$\check W_I(\boldsymbol{t};\boldsymbol{z}) = -h\,(t^{(1)}_1-z_2), \qquad \check W_J(\boldsymbol{t};\boldsymbol{z}) = -h\,(t^{(1)}_1-z_1-h).$$

Example. Let $N=2$, $n=3$, $\lambda=(1,2)$, $I=(\{2\},\{1,3\})$. Then

$$\check W_I(\boldsymbol{t};\boldsymbol{z}) = -h\,(t^{(1)}_1-z_1-h)\,(t^{(1)}_1-z_3).$$

Example. Let $N=2$, $n=3$, $\lambda=(2,1)$, $I=(\{1,3\},\{2\})$. Then

$$\begin{aligned}\check W_I(\boldsymbol{t};\boldsymbol{z}) \,=\, (-h)^2\Bigg(&(t^{(1)}_1-z_2)\,(t^{(1)}_1-z_3)\,(t^{(1)}_2-z_1-h)(t^{(1)}_2-z_2-h)\,\frac{t^{(1)}_1-t^{(1)}_2-h}{t^{(1)}_1-t^{(1)}_2}\,+\\ &+\,(t^{(1)}_2-z_2)\,(t^{(1)}_2-z_3)\,(t^{(1)}_1-z_1-h)(t^{(1)}_1-z_2-h)\,\frac{t^{(1)}_2-t^{(1)}_1-h}{t^{(1)}_2-t^{(1)}_1}\Bigg).\end{aligned}$$

For a subset $A=\{a_1,\dots,a_j\}\subset\{1,\dots,n\}$, denote $z_A=(z_{a_1},\dots,z_{a_j})$. For $I\in\mathcal{I}_\lambda$, denote $z_I=(z_{I_1},\dots,z_{I_N})$. For $f(t_{(1)},\dots,t_{(N)})\in\mathbb{C}[t_{(1)},\dots,t_{(N)}]^{S_{\lambda(1)}\times\dots\times S_{\lambda(N)}}$, we define $f(z_I)$ by substituting $t^{(j)}=(z_{I_1},\dots,z_{I_j})$, $j=1,\dots,N$.

3.2. **Weight functions $\check W_{\sigma,I}$.** For $\sigma\in S_n$ and $I\in\mathcal{I}_\lambda$, we define

$$\check W_{\sigma,I}(t;z) = \check W_{\sigma^{-1}(I)}(t;z_{\sigma(1)},\dots,z_{\sigma(n)}),\qquad \check U_{\sigma,I}(t;z) = \check U_{\sigma^{-1}(I)}(t;z_{\sigma(1)},\dots,z_{\sigma(n)}), \tag{3.2}$$

where $\sigma^{-1}(I) = (\sigma^{-1}(I_1),\dots,\sigma^{-1}(I_N))$.

Example. Let $N = 2$, $n = 2$, $\lambda = (1,1)$, $I = (\{1\},\{2\})$, $J = (\{2\},\{1\})$. Then

$$\check W_{\mathrm{id},I}(\boldsymbol t;\boldsymbol z) = -h\,(t^{(1)}_1 - z_2), \qquad \check W_{\mathrm{id},J}(\boldsymbol t;\boldsymbol z) = -h\,(t^{(1)}_1 - z_1 - h),$$
$$\check W_{s,I}(\boldsymbol t;\boldsymbol z) = -h\,(t^{(1)}_1 - z_2 - h), \qquad \check W_{s,J}(\boldsymbol t;\boldsymbol z) = -h\,(t^{(1)}_1 - z_1),$$

where s is the transposition.

3.3. **Three-term relation.**

Lemma 3.1 ([RTV1, Lemma 3.6])**.** *For any $\sigma \in S_n$, $I \in \mathcal{I}_\lambda$, $i = 1,\dots,n-1$, we have*

$$\check W_{\sigma s_{i,i+1},I} = \frac{z_{\sigma(i)} - z_{\sigma(i+1)}}{z_{\sigma(i)} - z_{\sigma(i+1)} + h}\check W_{\sigma,I} + \frac{h}{z_{\sigma(i)} - z_{\sigma(i+1)} + h}\check W_{\sigma,s_{\sigma(i),\sigma(i+1)}(I)}, \tag{3.3}$$

where $s_{i,j} \in S_n$ is the transposition of i and j.

3.4. **Weight functions $W_I(t;z)$.** Let $\sigma_0 \in S_n$ be the longest permutation, $\sigma_0(i) = n+1-i$, $i = 1,\dots,n$. For $I \in \mathcal{I}_\lambda$, denote

$$W_I(t;z) = (-h)^{-\lambda^{\{1\}}}\check W_{\sigma_0,I}(t;z), \qquad U_I(t;z) = \check U_{\sigma_0,I}(t;z). \tag{3.4}$$

In other words, we have

$$W_I(t;z) = \mathrm{Sym}_{t^{(1)}_1,\dots,t^{(1)}_{\lambda^{(1)}}} \cdots \mathrm{Sym}_{t^{(N-1)}_1,\dots,t^{(N-1)}_{\lambda^{(N-1)}}} U_I(\boldsymbol t;\boldsymbol z) \tag{3.5}$$

$$U_I(t;z) = \tag{3.6}$$
$$= \prod_{j=1}^{N-1}\prod_{a=1}^{\lambda^{(j)}}\Bigg(\prod_{\substack{c=1\\ i^{(j+1)}_c < i^{(j)}_a}}^{\lambda^{(j+1)}} (t^{(j)}_a - t^{(j+1)}_c) \prod_{\substack{d=1\\ i^{(j+1)}_d > i^{(j)}_a}}^{\lambda^{(j+1)}} (t^{(j)}_a - t^{(j+1)}_d - h) \prod_{b=a+1}^{\lambda^{(j)}} \frac{t^{(j)}_b - t^{(j)}_a - h}{t^{(j)}_b - t^{(j)}_a}\Bigg).$$

Example. Let $N = 2$, $n = 2$, $\lambda = (1,1)$, $I = (\{1\},\{2\})$, $J = (\{2\},\{1\})$. Then

$$W_I(\boldsymbol t;\boldsymbol z) = t^{(1)}_1 - z_2 - h\,, \qquad W_J(\boldsymbol t;\boldsymbol z) = t^{(1)}_1 - z_1.$$

3.5. **Modification of the three-term relation.** For a function $f(z_1,\dots,z_n)$ and $i = 1,\dots,n-1$, define the operator $S_{i,i+1}$ by the formula

$$S_{i,i+1} f(z_1,\dots,z_n) = \frac{z_i - z_{i+1} - h}{z_i - z_{i+1}} f(z_1,\dots,z_{i+1},z_i,\dots,z_n) + \frac{h}{z_i - z_{i+1}} f(z_1,\dots,z_n). \tag{3.7}$$

Lemma 3.1 can be reformulated as follows.

Lemma 3.2. *For any $I \in \mathcal{I}_\lambda$, $i = 1,\dots,n-1$, we have*

$$
\begin{aligned}
(3.8)\qquad W_{s_{i,i+1}(I)}(t;z) &= (S_{i,i+1}W_I)(t;z)\\
&= \frac{z_i - z_{i+1} - h}{z_i - z_{i+1}}\,W_I(\boldsymbol{t}; z_1,\dots,z_{i+1},z_i,\dots,z_n) + \frac{h}{z_i - z_{i+1}}\,W_I(\boldsymbol{t}; z_1,\dots,z_n)\,.
\end{aligned}
$$

3.6. **Shuffle properties.** Let $n,n_1,n_2 \in \mathbb{Z}_{>0}$, $n = n_1 + n_2$. Let $\boldsymbol{\lambda}^1,\boldsymbol{\lambda}^2,\boldsymbol{\lambda} \in \mathbb{Z}^N_{\geqslant 0}$, $|\boldsymbol{\lambda}^1| = n_1$, $|\lambda^2| = n_2$, $\lambda = \lambda^1 + \lambda^2$. Let $I^1 = (I^1_1,\dots,I^1_N)$ be a decomposition of the set $\{1,\dots,n_1\}$ into subsets such that $|I^1_j| = \lambda^1_j$. Let $I^2 = (I^2_1,\dots,I^2_N)$ be a decomposition of the set $\{n_1+1,\dots,n\}$ into subsets such that $|I^2_j| = \lambda^2_j$. Define the decomposition $I = (I_1,\dots,I_N)$ of the set $\{1,\dots,n\}$ by the rule: $I_j = I^1_j \cup I^2_j$.

Consider the weight function W_I of variables $(t^{(1)},\dots,t^{(N)})$, where $t^{(j)} = (t^{(j)}_1,\dots,t^{(j)}_{\lambda^{(j)}})$, $\lambda^{(j)} = \lambda_1 + \dots + \lambda_j$, $j = 1,\dots,N-1$, and $t^{(N)} = (z_1,\dots,z_n)$.

Consider the weight function W_{I^1} of variables $(\tilde t^{(1)},\dots,\tilde t^{(N)})$, where $\tilde t^{(j)} = (t^{(j)}_1,\dots,t^{(j)}_{(\boldsymbol{\lambda}^1)^{(j)}})$, $(\boldsymbol{\lambda}^1)^{(j)} = \lambda^1_1 + \dots + \lambda^1_j$, $j = 1,\dots,N-1$, and $\tilde t^{(N)} = (z_1,\dots,z_{n_1})$. Consider the weight function W_{I^2} of variables $\check t = (\check t^{(1)},\dots,\check t^{(N)})$, where $\check t^{(j)} = (t^{(j)}_{(\boldsymbol{\lambda}^1)^{(j)}+1},\dots,t^{(j)}_{\lambda^{(j)}})$ for $j = 1,\dots,N-1$, and $\check t^{(N)} = (z_{n_1+1},\dots,z_n)$. Denote $(\boldsymbol{\lambda}^2)^{(j)} = \lambda^2_1 + \dots + \lambda^2_j$, $j = 1,\dots,N-1$. Define the connection coefficient

$$
\begin{aligned}
(3.9)\qquad C_{\lambda^1,\lambda^2}(t;z) = \prod_{j=1}^{N-1}\Bigg[&\Big(\prod_{a=1}^{(\boldsymbol{\lambda}^1)^{(j)}}\ \prod_{b=(\boldsymbol{\lambda}^1)^{(j)}+1}^{\lambda^{(j)}} \frac{t^{(j)}_b - t^{(j)}_a - h}{t^{(j)}_b - t^{(j)}_a}\Big)\times\\
&\times\Big(\prod_{a=1}^{(\boldsymbol{\lambda}^1)^{(j)}}\ \prod_{c=(\boldsymbol{\lambda}^1)^{(j+1)}+1}^{\lambda^{(j+1)}} (t^{(j)}_a - t^{(j)}_c - h)\Big)\Big(\prod_{a=(\boldsymbol{\lambda}^1)^{(j)}+1}^{\lambda^{(j)}}\ \prod_{c=1}^{(\boldsymbol{\lambda}^1)^{(j+1)}} (t^{(j)}_a - t^{(j)}_c)\Big)\Bigg]\,.
\end{aligned}
$$

Lemma 3.3. *We have*

$$
(3.10)\qquad W_I(t;z) = \frac{\mathrm{Sym}_{t^{(1)}_1,\dots,t^{(1)}_{\lambda^{(1)}}}\dots\,\mathrm{Sym}_{t^{(N-1)}_1,\dots,t^{(N-1)}_{\lambda^{(N-1)}}}\,\widetilde W_{I^1,I^2}(t;z)}{\prod_{j=1}^{N-1}((\boldsymbol{\lambda}^1)^{(j)})!\,((\boldsymbol{\lambda}^2)^{(j)})!}\,,
$$

where

$$
\begin{aligned}
\widetilde W_{I^1,I^2}(t;z) = C_{\lambda^1,\lambda^2}(t;z)\, &W_{I^1}(\tilde t^{(1)},\dots,\tilde t^{(N-1)}; z_1,\dots,z_{n_1})\\
&\times W_{I^2}(\check t^{(1)},\dots,\check t^{(N-1)}; z_{n_1+1},\dots,z_n).
\end{aligned}
$$

3.7. **Factorization.** Consider the $\mathfrak{gl}_N$ weight function $W_{\{1,\dots,n\},\emptyset,\dots,\emptyset}$ of variables $(t^{(1)},\dots,t^{(N)})$, where $t^{(j)} = (t_1^{(j)},\dots,t_n^{(j)})$ for $j = 1,\dots,N-1$, and $t^{(N)} = (z_1,\dots,z_n)$.

Lemma 3.4. *We have*

$$W_{\{1,\dots,n\},\emptyset,\dots,\emptyset}(t^{(1)},\dots,t^{(N-1)};t^{(N)}) = \prod_{j=1}^{N-1} W^{\mathfrak{gl}_2}_{\{1,\dots,n\},\emptyset}(t^{(j)};t^{(j+1)}), \tag{3.11}$$

where $W^{\mathfrak{gl}_2}_{\{1,\dots,n\},\emptyset}(\boldsymbol{t}^{(j)};\boldsymbol{t}^{(j+1)})$ *is the* $\mathfrak{gl}_2$ *weight function assigned to the partition of the set* $\{1,\dots,n\}$ *into two subsets* $\{1,\dots,n\}$ *and* $\emptyset$.

Proof. The function $W^{\mathfrak{gl}_2}_{\{1,\dots,n\},\emptyset}(\boldsymbol{t}^{(j)};\boldsymbol{t}^{(j+1)})$ is symmetric in variables $(t_1^{(j+1)},\dots,t_n^{(j+1)})$ due to the $\mathfrak{gl}_2$ three-term relations of Lemma 3.1. That symmetry and formula (3.1) imply formula (3.11).

3.8. **Useful identities.**

Theorem 3.5. *Given* $k \in \mathbb{Z}_{>0}$, *consider variables* $t_1^{(0)}, t_1^{(k+1)}$ *and* $t_1^{(i)}, t_2^{(i)}$ *for* $i = 1,\dots,k$. *Set*

$$F = (t_1^{(0)} - t_1^{(1)})(t_1^{(k)} - t_1^{(k+1)} - h) - (t_1^{(0)} - t_2^{(1)} - h)(t_2^{(k)} - t_1^{(k+1)}),$$

$$G = (t_1^{(0)} - t_2^{(1)} - h)(t_1^{(k)} - t_1^{(k+1)} - h) - (t_1^{(0)} - t_1^{(1)})(t_2^{(k)} - t_1^{(k+1)}),$$

and

$$H = \prod_{i=1}^{k-1}\bigl((t_1^{(i)} - t_2^{(i+1)} - h)(t_2^{(i)} - t_1^{(i+1)})\bigr)\prod_{i=1}^{k}\frac{t_2^{(i)} - t_1^{(i)} - h}{t_2^{(i)} - t_1^{(i)}}.$$

Then

$$\mathrm{Sym}_{t_1^{(1)},t_2^{(1)}}\cdots\mathrm{Sym}_{t_1^{(k)},t_2^{(k)}}(FH) = 0 \tag{3.12}$$

and

$$\mathrm{Sym}_{t_1^{(1)},t_2^{(1)}}\cdots\mathrm{Sym}_{t_1^{(k)},t_2^{(k)}}(GH) = 0. \tag{3.13}$$

Proof. Formulae (3.12), (3.13) are equivalent to

$$\mathrm{Sym}_{t_1^{(1)},t_2^{(1)}}\cdots\mathrm{Sym}_{t_1^{(k)},t_2^{(k)}}\bigl((F \pm G)H\bigr) = 0.$$

Observe that

$$F + G = (2t_1^{(0)} - t_1^{(1)} - t_2^{(1)} - h)(t_1^{(k)} - t_2^{(k)} - h),$$

$$F - G = (t_1^{(1)} - t_2^{(1)} - h)(t_1^{(k)} + t_2^{(k)} - 2t_1^{(k+1)} - h),$$

and

$$\mathrm{Sym}^{s_1,s_2}\Big((s_1-u_2-h)\,(s_2-u_1)\,\frac{s_2-s_1-h}{s_2-s_1}\Big) = \mathrm{Sym}^{u_1,u_2}\Big((s_1-u_2-h)\,(s_2-u_1)\,\frac{u_2-u_1-h}{s_2-s_1}\Big).$$

Hence the function

$$W^{\mathfrak{gl}_2}_{\{1,2\},\emptyset}(s_1,s_2,u_1,u_2) = \mathrm{Sym}^{s_1,s_2}\Big((s_1-u_2-h)\,(s_2-u_1)\,\frac{s_2-s_1-h}{s_2-s_1}\Big)$$

is symmetric both in s_1,s_2 and u_1,u_2. Therefore,

$$\begin{aligned}\mathrm{Sym}_{t_1^{(1)},t_2^{(1)}}\cdots\mathrm{Sym}_{t_1^{(k)},t_2^{(k)}}\big((F+G)\,H\big) &= (2t_1^{(0)}-t_1^{(1)}-t_2^{(1)}-h)\,\times\\ &\times \mathrm{Sym}_{t_1^{(k)},t_2^{(k)}}\Big((t_1^{(k)}-t_2^{(k)}-h)\,\frac{t_2^{(k)}-t_1^{(k)}-h}{t_2^{(k)}-t_1^{(k)}}\Big)\prod_{i=1}^{k-1} W^{\mathfrak{gl}_2}_{\{1,2\},\emptyset}(t_1^{(i)},t_2^{(i)},t_1^{(i+1)},t_2^{(i+1)}) = 0.\end{aligned}$$

and

$$\begin{aligned}\mathrm{Sym}_{t_1^{(1)},t_2^{(1)}}\cdots\mathrm{Sym}_{t_1^{(k)},t_2^{(k)}}\big((F-G)\,H\big) &= (t_1^{(k)}+t_2^{(k)}-2t_1^{(k+1)}-h)\,\times\\ &\times \mathrm{Sym}_{t_1^{(1)},t_2^{(1)}}\Big((t_1^{(1)}-t_2^{(1)}-h)\,\frac{t_2^{(1)}-t_1^{(1)}-h}{t_2^{(1)}-t_1^{(1)}}\Big)\prod_{i=1}^{k-1} W^{\mathfrak{gl}_2}_{\{1,2\},\emptyset}(t_1^{(i)},t_2^{(i)},t_1^{(i+1)},t_2^{(i+1)}) = 0.\end{aligned}$$

Theorem 3.5 is proved.

4. Master function and discrete differentials

4.1. **Master function.** Let $\phi(x) = \Gamma(x/\kappa)\,\Gamma\big((h-x)/\kappa\big)$. Define the master function:

$$\Phi_\lambda(t;z;h;\boldsymbol{q}) = \big(e^{\pi\sqrt{-1}\,(n-\lambda_N)}q_N\big)^{\sum_{a=1}^n z_a/\kappa}\prod_{i=1}^{N-1}\Big(e^{\pi\sqrt{-1}\,(\lambda_{i+1}-\lambda_i)}\,\frac{q_i}{q_{i+1}}\Big)^{\sum_{j=1}^{\lambda^{(i)}} t_j^{(i)}/\kappa}\times$$
$$\times\prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}}\Big(\prod_{\substack{b=1\\ b\neq a}}^{\lambda^{(i)}}\frac{1}{(t_a^{(i)}-t_b^{(i)}-h)\,\phi(t_a^{(i)}-t_b^{(i)})}\prod_{c=1}^{\lambda^{(i+1)}}\phi(t_a^{(i)}-t_c^{(i+1)})\Big) \tag{4.1}$$

It is a symmetric function of variables in each of the groups $t^{(i)}$, $i = 1,\dots,N-1$.

4.2. **Definition of discrete differentials.** Consider the space S of functions of the form $\Phi_\lambda(t;z;h;q)f(t;z;h;q)$ where $f(t;z;h;q)$ is a rational function. Consider the lattice $\kappa\mathbb{Z}^{\lambda^{\{1\}}}$ whose coordinates are labeled by variables $t_j^{(i)} \in t$. The shifts $t_j^{(i)} \mapsto t_j^{(i)} + \kappa$ of any of the t-variables preserve the space S and extend to an action of the lattice $\kappa\mathbb{Z}^{\lambda^{\{1\}}}$ on S. A *discrete differential* is a finite sum of rational functions of the form

$$\frac{\Phi_\lambda(t+w;z;h;q)}{\Phi_\lambda(t;z;h;q)}\, f(t+w;z;h;q) - f(t;z;h;q), \tag{4.2}$$

where $w \in \kappa\mathbb{Z}^{\lambda^{\{1\}}}$.

4.3. **Special discrete differentials.** For integers $1 \leqslant \alpha < \beta \leqslant N$, split the variables $t = (t^{(1)},\dots,t^{(N-1)})$, $t^{(i)} = (t^{(i)}_1,\dots,t^{(i)}_{\lambda^{(i)}})$, into two groups $t^{\{\alpha,\beta\}}$ and $t_{\{\alpha,\beta\}}$ as follows:

$$t^{\{\alpha,\beta\}} = (t^{(\alpha)}_{\lambda^{(\alpha)}},\, t^{(\alpha+1)}_{\lambda^{(\alpha+1)}},\, \dots,\, t^{(\beta-1)}_{\lambda^{(\beta-1)}})$$

and $t_{\{\alpha,\beta\}} = (t^{(1)}_{\{\alpha,\beta\}},\dots,t^{(N-1)}_{\{\alpha,\beta\}})$, where $t^{(i)}_{\{\alpha,\beta\}} = (t^{(i)}_1,\dots,t^{(i)}_{\lambda^{(i)}-1})$ if $\alpha \leqslant i < \beta$ and $t^{(i)}_{\{\alpha,\beta\}} = t^{(i)}$, otherwise.

For a rational function g of $t_{\{\alpha,\beta\}}, z, q$, denote

$$\begin{aligned} d_{t^{\{\alpha,\beta\}}}\, g \; := \; \frac{g(t_{\{\alpha,\beta\}};z;h;q)}{q_\alpha - q_\beta} \Big(& q_\beta\, (t^{(\alpha-1)}_{\lambda^{(\alpha-1)}} - t^{(\alpha)}_{\lambda^{(\alpha)}}) \prod_{i=\alpha}^{\beta-1} \prod_{a=1}^{\lambda^{(i-1)}-1} (t^{(i-1)}_a - t^{(i)}_{\lambda^{(i)}}) \;\times \\ & \times\, (t^{(\beta-1)}_{\lambda^{(\beta-1)}} - t^{(\beta)}_{\lambda^{(\beta)}} - h) \prod_{i=\alpha}^{\beta-1} \prod_{a=1}^{\lambda^{(i+1)}-1} (t^{(i)}_{\lambda^{(i)}} - t^{(i+1)}_a - h) \prod_{i=\alpha}^{\beta-1} \prod_{a=1}^{\lambda^{(i)}-1} \frac{t^{(i)}_a - t^{(i)}_{\lambda^{(i)}} - h}{t^{(i)}_a - t^{(i)}_{\lambda^{(i)}}} \;- \\ & -\, q_\alpha\, (t^{(\alpha-1)}_{\lambda^{(\alpha-1)}} - t^{(\alpha)}_{\lambda^{(\alpha)}} - h) \prod_{i=\alpha}^{\beta-1} \prod_{a=1}^{\lambda^{(i-1)}-1} (t^{(i-1)}_a - t^{(i)}_{\lambda^{(i)}} - h) \;\times \\ & \times\, (t^{(\beta-1)}_{\lambda^{(\beta-1)}} - t^{(\beta)}_{\lambda^{(\beta)}}) \prod_{i=\alpha}^{\beta-1} \prod_{a=1}^{\lambda^{(i+1)}-1} (t^{(i)}_{\lambda^{(i)}} - t^{(i+1)}_a) \prod_{i=\alpha}^{\beta-1} \prod_{a=1}^{\lambda^{(i)}-1} \frac{t^{(i)}_{\lambda^{(i)}} - t^{(i)}_a - h}{t^{(i)}_{\lambda^{(i)}} - t^{(i)}_a} \Big). \end{aligned} \tag{4.3}$$

Lemma 4.1. *The function $d_{t^{\{\alpha,\beta\}}} g$ is a discrete differential.*

Proof. Formula (4.3) is an example of formula (4.2), where

$$\begin{aligned} f(t;z;h;q) = g(t_{\{\alpha,\beta\}};z;h;q)\,(t^{(\alpha-1)}_{\lambda^{(\alpha-1)}} - t^{(\alpha)}_{\lambda^{(\alpha)}}) & \prod_{i=\alpha}^{\beta-1} \prod_{a=1}^{\lambda^{(i-1)}-1} (t^{(i-1)}_a - t^{(i)}_{\lambda^{(i)}}) \;\times \\ \times\, (t^{(\beta-1)}_{\lambda^{(\beta-1)}} - t^{(\beta)}_{\lambda^{(\beta)}} - h) \prod_{i=\alpha}^{\beta-1} \prod_{a=1}^{\lambda^{(i+1)}-1} (t^{(i)}_{\lambda^{(i)}} - t^{(i+1)}_a - h) & \prod_{i=\alpha}^{\beta-1} \prod_{a=1}^{\lambda^{(i)}-1} \frac{t^{(i)}_a - t^{(i)}_{\lambda^{(i)}} - h}{t^{(i)}_a - t^{(i)}_{\lambda^{(i)}}} \end{aligned}$$

and w has coordinates $t^{(\alpha)}_{\lambda^{(\alpha)}},\, t^{(\alpha+1)}_{\lambda^{(\alpha+1)}},\, \dots,\, t^{(\beta-1)}_{\lambda^{(\beta-1)}}$ equal to κ and other coordinates equal to zero.

We rewrite $d_{t\{\alpha,\beta\}}g$ as

$$d_{t\{\alpha,\beta\}}g = \frac{q_\beta}{q_\alpha - q_\beta}\, \tilde d_{t\{\alpha,\beta\}}g - \check d_{t\{\alpha,\beta\}}g,$$

where

$$\begin{aligned}
(4.4)\qquad \tilde d_{\boldsymbol t\{\alpha,\beta\}}\, g \;:=\; & g(\boldsymbol t_{\{\alpha,\beta\}};\boldsymbol z;h;\boldsymbol q)\Big((t^{(\alpha-1)}_{\lambda^{(\alpha-1)}} - t^{(\alpha)}_{\lambda^{(\alpha)}}) \prod_{i=\alpha}^{\beta-1}\prod_{a=1}^{\lambda^{(i-1)}-1} (t^{(i-1)}_a - t^{(i)}_{\lambda^{(i)}}) \times \\
& \times (t^{(\beta-1)}_{\lambda^{(\beta-1)}} - t^{(\beta)}_{\lambda^{(\beta)}} - h) \prod_{i=\alpha}^{\beta-1}\prod_{a=1}^{\lambda^{(i+1)}-1} (t^{(i)}_{\lambda^{(i)}} - t^{(i+1)}_a - h) \prod_{i=\alpha}^{\beta-1}\prod_{a=1}^{\lambda^{(i)}-1} \frac{t^{(i)}_a - t^{(i)}_{\lambda^{(i)}} - h}{t^{(i)}_a - t^{(i)}_{\lambda^{(i)}}} - \\
& - (t^{(\alpha-1)}_{\lambda^{(\alpha-1)}} - t^{(\alpha)}_{\lambda^{(\alpha)}} - h) \prod_{i=\alpha}^{\beta-1}\prod_{a=1}^{\lambda^{(i-1)}-1} (t^{(i-1)}_a - t^{(i)}_{\lambda^{(i)}} - h) \times \\
& \times (t^{(\beta-1)}_{\lambda^{(\beta-1)}} - t^{(\beta)}_{\lambda^{(\beta)}}) \prod_{i=\alpha}^{\beta-1}\prod_{a=1}^{\lambda^{(i+1)}-1} (t^{(i)}_{\lambda^{(i)}} - t^{(i+1)}_a) \prod_{i=\alpha}^{\beta-1}\prod_{a=1}^{\lambda^{(i)}-1} \frac{t^{(i)}_{\lambda^{(i)}} - t^{(i)}_a - h}{t^{(i)}_{\lambda^{(i)}} - t^{(i)}_a} \Big)
\end{aligned}$$

and

$$\begin{aligned}
\check d_{\boldsymbol t\{\alpha,\beta\}}\, g \;:=\; & g(\boldsymbol t_{\{\alpha,\beta\}};\boldsymbol z;h;\boldsymbol q)\, (t^{(\alpha-1)}_{\lambda^{(\alpha-1)}} - t^{(\alpha)}_{\lambda^{(\alpha)}} - h) \prod_{i=\alpha}^{\beta-1}\prod_{a=1}^{\lambda^{(i-1)}-1} (t^{(i-1)}_a - t^{(i)}_{\lambda^{(i)}} - h) \times \\
& \times (t^{(\beta-1)}_{\lambda^{(\beta-1)}} - t^{(\beta)}_{\lambda^{(\beta)}}) \prod_{i=\alpha}^{\beta-1}\prod_{a=1}^{\lambda^{(i+1)}-1} (t^{(i)}_{\lambda^{(i)}} - t^{(i+1)}_a) \prod_{i=\alpha}^{\beta-1}\prod_{a=1}^{\lambda^{(i)}-1} \frac{t^{(i)}_{\lambda^{(i)}} - t^{(i)}_a - h}{t^{(i)}_{\lambda^{(i)}} - t^{(i)}_a}\,.
\end{aligned}\qquad (4.5)$$

Denote

$$(4.6)\qquad d_{\{\alpha,\beta\}}g := \mathrm{Sym}_{t^{(1)}_1,\dots,t^{(1)}_{\lambda^{(1)}}} \cdots \mathrm{Sym}_{t^{(N-1)}_1,\dots,t^{(N-1)}_{\lambda^{(N-1)}}}\, d_{\boldsymbol t\{\alpha,\beta\}}\, g,$$

$$(4.7)\qquad \tilde d_{\{\alpha,\beta\}}g := \mathrm{Sym}_{t^{(1)}_1,\dots,t^{(1)}_{\lambda^{(1)}}} \cdots \mathrm{Sym}_{t^{(N-1)}_1,\dots,t^{(N-1)}_{\lambda^{(N-1)}}}\, \tilde d_{\boldsymbol t\{\alpha,\beta\}}\, g,$$

$$(4.8)\qquad \check d_{\{\alpha,\beta\}}g := \mathrm{Sym}_{t^{(1)}_1,\dots,t^{(1)}_{\lambda^{(1)}}} \cdots \mathrm{Sym}_{t^{(N-1)}_1,\dots,t^{(N-1)}_{\lambda^{(N-1)}}}\, \check d_{\boldsymbol t\{\alpha,\beta\}}\, g\,.$$

Then

$$d_{\{\alpha,\beta\}}g = \frac{q_\beta}{q_\alpha - q_\beta}\, \tilde d_{\{\alpha,\beta\}}g - \check d_{\{\alpha,\beta\}}g. \qquad (4.9)$$

Corollary 4.2. *The function $d_{\{\alpha,\beta\}}g$ is a discrete differential.*

4.4. **First key formula.** Let $\boldsymbol\lambda \in \mathbb Z^N_{\geqslant 0}$, $|\boldsymbol\lambda| = n$. For $\alpha,\beta = 1,\dots,N$, $\alpha \neq \beta$, denote

$$\lambda_{\alpha,\beta} = (\lambda_1,\dots,\lambda_\alpha - 1,\dots,\lambda_\beta + 1,\dots,\lambda_N). \tag{4.10}$$

Notice that $|\lambda_{\alpha,\beta}| = |\lambda|$.

Let $I = (I_1,\dots,I_N) \in \mathcal{I}_\lambda$, $I_k = (\ell_{k,1},\dots,\ell_{k,\lambda_k})$, $k = 1,\dots,N$. For $\alpha \neq \beta$, $a = 1,\dots,\lambda_\alpha$, $b = 1,\dots,\lambda_\beta$, denote

$$(I)_{\beta,\alpha}^{b} = (I_1,\dots,I_\alpha \cup \{\ell_{\beta,b}\},\dots,I_\beta - \{\ell_{\beta,b}\},\dots,I_N) \in \mathcal{I}_{\lambda_{\alpha,\beta}}. \tag{4.11}$$

For $J \in \mathcal{I}_{\lambda_{\alpha,\beta}}$ and $b = 1,\dots,\lambda_\beta + 1$, we have $(J)_{\beta,\alpha}^{b} \in \mathcal{I}_\lambda$. The function U_J defined by formula (3.5) is a function of variables $t_{\alpha,\beta}, z$.

Theorem 4.3. *We have*

$$(\tilde d_{\{\alpha,\beta\}} U_J)(t;z) = -h \sum_{b=1}^{\lambda_\beta+1} W_{(J)_{\beta,\alpha}^{b}}(t;z). \tag{4.12}$$

Theorem 4.3 is proved in Section 5.

4.5. **Second key formula.** Let $I = (I_1,\dots,I_N) \in \mathcal{I}_\lambda$, $I_k = (\ell_{k,1},\dots,\ell_{k,\lambda_k})$. For $k_1,k_2 = 1,\dots,N$, $k_1 \neq k_2$, and $m_1 = 1,\dots,\lambda_{k_1}$, $m_2 = 1,\dots,\lambda_{k_2}$, define the element $I_{k_1,k_2;m_1,m_2} = (\tilde I_1,\dots,\tilde I_N) \in \mathcal{I}_\lambda$ such that $\tilde I_k = I_k$ if $k \neq k_1,k_2$, and

$$\tilde I_{k_1} = I_{k_1} \cup \{\ell_{k_2,m_2}\} - \{\ell_{k_1,m_1}\}, \qquad \tilde I_{k_2} = I_{k_2} \cup \{\ell_{k_1,m_1}\} - \{\ell_{k_2,m_2}\}. \tag{4.13}$$

Theorem 4.4. *For $I \in \mathcal{I}_\lambda$ and $i = 1,\dots,N-1$, we have*

$$\begin{aligned}
&\Big(\sum_{j=1}^{\lambda^{(i)}} t_j^{(i)} - \sum_{j=1}^{\lambda^{(i-1)}} t_j^{(i-1)} - \sum_{a\in I_i} z_a\Big) W_I \\
&\quad = h \sum_{j=1}^{i-1} \sum_{m_1=1}^{\lambda_i} \sum_{\substack{m_2=1\\ \ell_{i,m_1} > \ell_{j,m_2}}}^{\lambda_j} W_{I_{i,j;m_1,m_2}} - h \sum_{j=i+1}^{N} \sum_{m_1=1}^{\lambda_i} \sum_{\substack{m_2=1\\ \ell_{i,m_1} < \ell_{j,m_2}}}^{\lambda_j} W_{I_{i,j;m_1,m_2}} + \\
&\qquad + \sum_{j=i+1}^{N} \sum_{a=1}^{\lambda_i} \check d_{\{i,j\}} U_{(I)^a_{i,j}} - \sum_{j=1}^{i-1} \sum_{a=1}^{\lambda_j} \check d_{\{j,i\}} U_{(I)^a_{j,i}} .
\end{aligned} \tag{4.14}$$

Theorem 4.4 is proved in Section 6.

5. Proof of Theorem 4.3

For $n = 1$, Theorem 4.3 is the following statement.

Lemma 5.1. *Let $n = 1$. For $1 \le \gamma \le n$, let $J^\gamma = (J_1,\dots,J_N)$ be the decomposition of the one-element set $\{1\}$, such that $J_\gamma = \{1\}$ and $J_j = \emptyset$ for $j \neq \gamma$. Let $1 \le \alpha < \beta \le N$.*

Then

(5.1) $\tilde d_{\{\alpha,\beta\}}\, U_{J^\gamma} = -hW_{J^\alpha}, \qquad \beta = \gamma\,,$

(5.2) $\tilde d_{\{\alpha,\beta\}}\, U_{J^\gamma} = 0, \qquad \beta \neq \gamma\,.$

Proof. For any γ, we have $W_{J^\gamma} = U_{J^\gamma}$, and U_{J^γ} is the function of $t_1^{(\gamma)},\dots,t_1^{(N-1)},t_1^{(N)}=z_1$, which is identically equal to 1, see (3.5).

If $\beta = \gamma$, then

$$\tilde d_{\{\alpha,\beta\}}U_{J^\gamma} \,=\, \tilde d_{\{\alpha,\gamma\}}U_{J^\gamma} \,=\, (t_1^{(\gamma-1)}-t_1^{(\gamma)}-h)-(t_1^{(\gamma-1)}-t_1^{(\gamma)}) \,=\, -h \,=\, -hW_{J^\alpha},$$

which proves (5.1).

The proof of (5.2) is by cases. If $\beta < \gamma$, then $\tilde d_{\{\alpha,\beta\}}U_{J^\gamma} = (1-1)\cdot 1 = 0$. If $\gamma < \alpha < \beta$, then $\tilde d_{\{\alpha,\beta\}}U_{J^\gamma} = 0$ by identity (3.12). If $\alpha < \gamma < \beta$, then $\tilde d_{\{\alpha,\beta\}}U_{J^\gamma} = 0$ by identity (3.13).

If $\alpha = \gamma < \beta$, then $\tilde d_{\{\alpha,\beta\}}U_{J^\gamma} = 0$ by the degeneration of identity (3.12) as $t_1^{(0)}\to\infty$.

For arbitrary n, Theorem 4.3 follows by induction on n from the shuffle properties of weight functions in Lemma 3.3. To avoid writing numerous indices we illustrate the reasoning by an example.

Let $N = 3$, $n = 2$, $J = (\emptyset,\{1,2\},\emptyset)$, $\alpha = 1$, $\beta = 2$. Then formula (4.12) reads

(5.3) $\tilde d_{\{1,2\}}U_J = -hW_{(\{1\},\{2\},\emptyset)} - hW_{(\{2\},\{1\},\emptyset)}\,.$

Indeed, we have

$$\tilde d_{\{1,2\}}U_J = \mathrm{Sym}_{t_1^{(2)},t_2^{(2)}}\Big(\big((t_1^{(1)}-t_1^{(2)}-h)\,(t_1^{(1)}-t_2^{(2)}-h)-(t_1^{(1)}-t_1^{(2)})\,(t_1^{(1)}-t_2^{(2)})\big)\ \times$$
$$\times\ (t_1^{(2)}-z_2-h)\,(t_2^{(2)}-z_1)\,\frac{t_2^{(2)}-t_1^{(2)}-h}{t_2^{(2)}-t_1^{(2)}}\Big)\ =$$
$$= \mathrm{Sym}_{t_1^{(2)},t_2^{(2)}}\Big(\big((t_1^{(1)}-t_1^{(2)}-h)\,(t_1^{(1)}-t_2^{(2)}-h)-(t_1^{(1)}-t_1^{(2)})\,(t_1^{(1)}-t_2^{(2)}-h)\ +$$
$$+\ (t_1^{(1)}-t_1^{(2)})\,(t_1^{(1)}-t_2^{(2)}-h)-(t_1^{(1)}-t_1^{(2)})(t_1^{(1)}-t_2^{(2)})\big)\ \times$$
$$\times\ (t_1^{(2)}-z_2-h)\,(t_2^{(2)}-z_1)\,\frac{t_2^{(2)}-t_1^{(2)}-h}{t_2^{(2)}-t_1^{(2)}}\Big).$$

This is the sum of four terms. The first two are

$$\mathrm{Sym}_{t_1^{(2)},t_2^{(2)}}\Big(\big((t_1^{(1)}-t_1^{(2)}-h)\,(t_1^{(1)}-t_2^{(2)}-h)-(t_1^{(1)}-t_1^{(2)})\,(t_1^{(1)}-t_2^{(2)}-h)\big)\ \times$$
$$\times\ (t_1^{(2)}-z_2-h)\,(t_2^{(2)}-z_1)\,\frac{t_2^{(2)}-t_1^{(2)}-h}{t_2^{(2)}-t_1^{(2)}}\Big)\ =$$

$= -h\ \mathrm{Sym},$

$$ {}_{t_1^{(2)},t_2^{(2)}}\Big(\big(t_1^{(1)}-t_2^{(2)}-h\big)\big(t_1^{(2)}-z_2-h\big)\big(t_2^{(2)}-z_1\big)\,\frac{t_2^{(2)}-t_1^{(2)}-h}{t_2^{(2)}-t_1^{(2)}}\Big) \;=\; -h\,W_{\{1\},\{2\},\emptyset} $$

the last two are

$$ \mathrm{Sym}_{t_1^{(2)},t_2^{(2)}}\Big(\big((t_1^{(1)}-t_1^{(2)})\,(t_1^{(1)}-t_2^{(2)}-h)-(t_1^{(1)}-t_1^{(2)})\,(t_1^{(1)}-t_2^{(2)})\big)\;\times $$
$$ \times\;(t_1^{(2)}-z_2-h)\,(t_2^{(2)}-z_1)\,\frac{t_2^{(2)}-t_1^{(2)}-h}{t_2^{(2)}-t_1^{(2)}}\Big)\;= $$
$$ =-h\,\mathrm{Sym}_{t_1^{(2)},t_2^{(2)}}\Big(\big(t_1^{(1)}-t_2^{(2)}\big)\big(t_1^{(2)}-z_2-h\big)\big(t_2^{(2)}-z_1\big)\,\frac{t_2^{(2)}-t_1^{(2)}-h}{t_2^{(2)}-t_1^{(2)}}\Big) \;=\; -h\,W_{\{2\},\{1\},\emptyset} $$

and we get (5.3).

The treatment of these four terms is an inductive step from $n = 1$ to $n = 2$. The analysis of the first two terms is the application of Theorem 4.3 for $n = 1$ at the first point z_1. Namely, the factor $(t_1^{(1)}-t_1^{(2)}-h)-(t_1^{(1)}-t_2^{(2)}))$ corresponds to $\tilde d_{\{1,2\}}$ at z_1 and the product

$$ (t_1^{(1)}-t_2^{(2)}-h)\,(t_1^{(2)}-z_2-h)\,(t_2^{(2)}-z_1)\,\frac{t_2^{(2)}-t_1^{(2)}-h}{t_2^{(2)}-t_1^{(2)}} $$

is the connection coefficient between $W_{\{1\},\emptyset,\emptyset}$ sitting at z_1 and $W_{\emptyset,\{2\},\emptyset}$ sitting at z_2, see Lemma 3.3. And the analysis of the last two terms is the application of Theorem 4.3 for $n = 1$ at the second point z_2. Namely, the factor $(t_1^{(1)}-t_1^{(2)}-h)-(t_1^{(1)}-t_2^{(2)}))$ corresponds to $\tilde d_{\{1,2\}}$ at z_2 and the product

$$ (t_1^{(1)}-t_2^{(2)})\,(t_1^{(2)}-z_2-h)\,(t_2^{(2)}-z_1)\,\frac{t_2^{(2)}-t_1^{(2)}-h}{t_2^{(2)}-t_1^{(2)}} $$

is the connection coefficient between $W_{\{2\},\emptyset,\emptyset}$ sitting at z_1 and $W_{\emptyset,\{1\},\emptyset}$ sitting at z_2, see Lemma 3.3.

6. Proof of Theorem 4.4

6.1. **Proof of Theorem 4.4 for $N = 2$, $\lambda = (n,0)$, $I = (\{1,\dots,n\},\emptyset)$.** **Lemma 6.1.** *We have*

$$ \sum_{l=1}^{n}\,(t_l^{(1)}-z_l)\,W^{\mathfrak{gl}_2}_{\{1,\dots,n\},\emptyset} \;=\; \sum_{a=1}^{n}\,\check d_{\{1,2\}}\,U^{\mathfrak{gl}_2}_{\{1,\dots,a-1,a+1,\dots,n\},\{a\}}\,. \tag{6.1} $$

Proof. We will prove formula (6.1) by induction on n. Denote

$$ t' = (t_1^{(1)},\dots,t_{n-1}^{(1)}), \qquad t'' = (t_1^{(1)},\dots,t_{n-2}^{(1)}), \qquad z^0 = (z_1,\dots,z_{n-1}), $$

$$A_n(\boldsymbol{t},\boldsymbol{z}') \,=\, \prod_{b=1}^{n-1}\Big((t_n^{(1)}-z_b)\,\frac{t_n^{(1)}-t_b^{(1)}-h}{t_n^{(1)}-t_b^{(1)}}\Big)\,, \qquad B(\boldsymbol{t}'',z) \,=\, \prod_{a=1}^{n-2}(t_a^{(1)}-z-h)\,.$$

By formulae (4.5), (4.8), equality (6.1) reads as follows:

$$\text{Sym}_{t_1^{(1)},\dots,t_n^{(1)}} \sum_{a=1}^{n}\Big((t_a^{(1)}-z_a)\,U^{\mathfrak{gl}_2}_{\{1,\dots,n\},\emptyset}(\boldsymbol{t},\boldsymbol{z}) \,- \tag{6.2}$$
$$-\,(t_n^{(1)}-z_n)\,A_n(\boldsymbol{t},\boldsymbol{z}')\,U^{\mathfrak{gl}_2}_{\{1,\dots,a-1,a+1,\dots,n\},\{a\}}(\boldsymbol{t}',\boldsymbol{z})\Big) \,=\, 0\,.$$

For $n=1$, formula (6.2) is clearly true. For the induction step, we explore formula (3.6). It implies that the summation term with $a=n$ in formula (6.2) vanishes,

$$U^{\mathfrak{gl}_2}_{\{1,\dots,n\},\emptyset}(\boldsymbol{t},\boldsymbol{z}) \,=\, A_n(\boldsymbol{t},\boldsymbol{z})\,U^{\mathfrak{gl}_2}_{\{1,\dots,n-1\},\emptyset}(\boldsymbol{t}',\boldsymbol{z}')\,B(\boldsymbol{t}'',z_n)\,(t_{n-1}^{(1)}-z_n-h)\,, \tag{6.3}$$

and for $a<n$,

$$U^{\mathfrak{gl}_2}_{\{1,\dots,a-1,a+1,\dots,n\},\{a\}}(\boldsymbol{t}',\boldsymbol{z}) = \tag{6.4}$$
$$=\,(t_{n-1}^{(1)}-z_{n-1})\,A_{n-1}(\boldsymbol{t}',\boldsymbol{z}')\,U^{\mathfrak{gl}_2}_{\{1,\dots,a-1,a+1,\dots,n-1\},\{a\}}(\boldsymbol{t}'',\boldsymbol{z}')\,B(\boldsymbol{t}'',z_n)\,.$$

The last formula and the identity

$$\text{Sym}_{t_{n-1}^{(n)},t_n^{(n)}}\,(t_n^{(n)}-z_n)\,\frac{t_n^{(n)}-t_{n-1}^{(n)}-h}{t_n^{(n)}-t_{n-1}^{(n)}} \,=\, \text{Sym}_{t_{n-1}^{(n)},t_n^{(n)}}\,(t_{n-1}^{(n)}-z_n-h)\,\frac{t_n^{(n)}-t_{n-1}^{(n)}-h}{t_n^{(n)}-t_{n-1}^{(n)}}\,,$$

yield

$$\text{Sym}_{t_1^{(1)},\dots,t_n^{(1)}}\,(t_n^{(1)}-z_n)\,A_n(\boldsymbol{t},\boldsymbol{z}')\,U^{\mathfrak{gl}_2}_{\{1,\dots,a-1,a+1,\dots,n\},\{a\}}(\boldsymbol{t}',\boldsymbol{z}) = \tag{6.5}$$
$$=\text{Sym}_{t_1^{(1)},\dots,t_n^{(1)}}\,(t_{n-1}^{(1)}-z_{n-1})\,A_n(\boldsymbol{t},\boldsymbol{z}')\,A_{n-1}(\boldsymbol{t}',\boldsymbol{z}')\,\times$$
$$\times\,U^{\mathfrak{gl}_2}_{\{1,\dots,a-1,a+1,\dots,n-1\},\{a\}}(\boldsymbol{t}'',\boldsymbol{z}')\,B(\boldsymbol{t}'',z_n)\,(t_{n-1}^{(1)}-z_n-h)\,.$$

Summarizing all observations, we see that formula (6.2) follows from the equality

$$\text{Sym}_{t_1^{(1)},\dots,t_{n-1}^{(1)}}\,A_n(\boldsymbol{t},\boldsymbol{z}')\,B(\boldsymbol{t}'',z_n)\,(t_{n-1}^{(1)}-z_n-h)\,\times \tag{6.6}$$
$$\times\sum_{a=1}^{n-1}\Big((t_a^{(1)}-z_a)\,U^{\mathfrak{gl}_2}_{\{1,\dots,n-1\},\emptyset}(\boldsymbol{t}',\boldsymbol{z}') \,-$$
$$-\,(t_{n-1}^{(1)}-z_{n-1})\,A_{n-1}(\boldsymbol{t}',\boldsymbol{z}')\,U^{\mathfrak{gl}_2}_{\{1,\dots,a-1,a+1,\dots,n-1\},\{a\}}(\boldsymbol{t}'',\boldsymbol{z}')\Big) \,=\, 0\,,$$

with $t_n^{(1)}$ not involved in the symmetrization. Since the product

$$A_n(\boldsymbol{t},\boldsymbol{z}')\,B(\boldsymbol{t}'',z_n)\,(t_{n-1}^{(1)}-z_n-h)$$

is symmetric in $t_1^{(1)},\dots,t_{n-1}^{(1)}$, formula (6.6) follows from the induction assumption

$$\mathrm{Sym}_{t_1^{(1)},\dots,t_{n-1}^{(1)}} \sum_{a=1}^{n-1} \Big((t_a^{(1)} - z_a)\, U^{\mathfrak{gl}_2}_{\{1,\dots,n-1\},\emptyset}(\boldsymbol{t}',\boldsymbol{z}') - \qquad (6.7)$$

$$- (t_{n-1}^{(1)} - z_{n-1})\, A_{n-1}(\boldsymbol{t}',\boldsymbol{z}')\, U^{\mathfrak{gl}_2}_{\{1,\dots,a-1,a+1,\dots,n-1\},\{a\}}(\boldsymbol{t}'',\boldsymbol{z}')\Big) = 0 .$$

Lemma 6.1 is proved.

6.2. **Proof of Theorem 4.4 for $N = 2$ and $I = I^{\max}$.** For $N = 2$, $\lambda = (k,n-k)$, we denote $I^{\max} = (\{n-k+1,\dots,n\},\{1,\dots,n-k\})$. Then formula (4.14) becomes formula (6.8) below.

Lemma 6.2. *We have*

$$\sum_{l=1}^{k} (t_l^{(1)} - z_{n-k+l})\, W^{\mathfrak{gl}_2}_{\{n-k+1,\dots,n\},\{1,\dots,n-k\}} = \sum_{a=n-k+1}^{n} \check{d}_{\{1,2\}}\, U^{\mathfrak{gl}_2}_{\{n-k+1,\dots,a-1,a+1,\dots,n\},\{1,\dots,n-k,a\}} \qquad (6.8)$$

Proof. Dividing both sides of the equation by $\prod_{i=1}^{k}\prod_{a=1}^{n-k}(t_i^{(1)} - z_a)$, turns formula (6.8) into formula (6.1).

6.3. **Proof of Theorem 4.4 for $i = 1$, arbitrary N, and $I = (\{1,\dots,n\},\emptyset,\dots,\emptyset)$.**

Proposition 6.3. *For $I = (\{1,\dots,n\},\emptyset,\dots,\emptyset)$, we have*

$$\Big(\sum_{l=1}^{n} t_l^{(1)} - \sum_{a=1}^{n} z_a\Big) W_I = \sum_{j=2}^{n}\sum_{a=1}^{n} \check{d}_{\{1,j\}}\, U_{(I)'^{a}_{1,j}} . \qquad (6.9)$$

Proof. Formula (6.9) is equivalent to the formula

$$\sum_{j=2}^{n}\sum_{l=1}^{n} (t_l^{(j-1)} - t_l^{(j)})\, W_I = \sum_{j=2}^{n}\sum_{a=1}^{n} \check{d}_{\{1,j\}}\, U_{(I)'^{a}_{1,j}}$$

which follows from the next lemma. **Lemma 6.4.** *For $j = 2,\dots,N$ we have*

$$\sum_{l=1}^{n} (t_l^{(j-1)} - t_l^{(j)})\, W_I = \sum_{a=1}^{n} \check{d}_{\{1,j\}}\, U_{(I)'^{a}_{1,j}} \qquad (6.10)$$

Proof. By Lemma 3.4, the left-hand side of (6.10) equals

$$\Big(\sum_{l=1}^{n}(t_l^{(j-1)}-t_l^{(j)})\,W^{\mathfrak{gl}_2}_{\{1,\dots,n\},\emptyset}(\boldsymbol{t}^{(j-1)};\boldsymbol{t}^{(j)})\Big)\prod_{\substack{i=2\\ i\neq j}}^{N}W^{\mathfrak{gl}_2}_{\{1,\dots,n\},\emptyset}(\boldsymbol{t}^{(i-1)};\boldsymbol{t}^{(i)}) \tag{6.11}$$

It is easy to see that the right hand side equals

$$\Big(\sum_{a=1}^{n}\check{d}_{\{j-1,j\}}\,U_{(\{1,\dots,a-1,a+1,\dots,n\},\{a\})}(\boldsymbol{t}^{(j-1)};\boldsymbol{t}^{(j)})\Big)\prod_{\substack{i=2\\ i\neq j}}^{N}W^{\mathfrak{gl}_2}_{\{1,\dots,n\},\emptyset}(\boldsymbol{t}^{(i-1)};\boldsymbol{t}^{(i)}) \tag{6.12}$$

Hence, Lemma 6.4 follows from formula (6.1).

Proposition 6.3 is proved.

6.4. **Proof of Theorem 4.4 for** $i = 1$**, arbitrary** N**, and** $I = I^{\max}$**.** For $\lambda = (\lambda_1,\dots,\lambda_N)$, we denote $I^{\max} = (\{n-\lambda_1+1,\dots,n\},\dots,\{1,\dots,\lambda_N\})$. Then formula (4.14) takes the form

$$\sum_{j=2}^{N}\sum_{l=1}^{\lambda_1}\big(t^{(j-1)}_{l+\lambda^{(j-1)}-\lambda_1}-t^{(j)}_{l+\lambda^{(j)}-\lambda_1}\big)W_{I^{\max}}=\sum_{j=2}^{N}\sum_{a=1}^{\lambda_1}\check{d}_{\{1,j\}}W_{(I^{\max})'_{1a,j}}. \tag{6.13}$$

The following lemma implies formula (6.13).

Lemma 6.5. *For* $j = 2,\dots,N$*, we have*

$$\sum_{l=1}^{\lambda_1}\big(t^{(j-1)}_{l+\lambda^{(j-1)}-\lambda_1}-t^{(j)}_{l+\lambda^{(j)}-\lambda_1}\big)W_{I^{\max}}=\sum_{a=1}^{\lambda_1}\check{d}_{\{1,j\}}\,W_{(I^{\max})'_{1a,j}}. \tag{6.14}$$

Proof. The left-hand side of formula (6.14) equals

$$\operatorname{Sym}_{t^{(1)}_1,\dots,t^{(1)}_{\lambda^{(1)}}}\cdots\operatorname{Sym}_{t^{(N-1)}_1,\dots,t^{(N-1)}_{\lambda^{(N-1)}}}\Big(\sum_{l=1}^{\lambda_1}\big(t^{(j-1)}_{l+\lambda^{(j-1)}-\lambda_1}-t^{(j)}_{l+\lambda^{(j)}-\lambda_1}\big)\times\prod_{m=2}^{N}U^{\mathfrak{gl}_2}_{\{\lambda_m+1,\dots,\lambda^{(m)}\},\{1,\dots,\lambda_m\}}(\boldsymbol{t}^{(m-1)};\boldsymbol{t}^{(m)})\Big) \tag{6.15}$$

while the right-hand side of (6.14) equals by definition

$$t_1^{(N-1)},\dots,t_{\lambda^{(N-1)}}^{(N-1)} \Big(\prod (t_{\lambda^{(j-1)}}^{(j-1)} - t_l^{(j)}) \prod \frac{t_{\lambda^{(j-1)}} - t_l}{t^{(j-1)} - t^{(j-1)}} - \Big.$$

(6.16) $\quad t_1^{(1)},\dots,t_{(1)}^{(1)} \cdots \quad$ SymSym $\quad \lambda \quad \lambda^{(j)} \quad \lambda^{(j-1)}-1 \quad (j-1) \quad (j-1)\ h \quad l=1 \quad l=1 \quad \lambda^{(j-1)} \quad l \quad \lambda^{(j)}$

$$\times \sum U^{\mathfrak{gl}_2}_{\{\dots(j)-\lambda_1,\dots,b-1,b+1,\dots,\lambda^{(j)}\},\{1,\dots,\lambda_j,b\}}\big(t^{(j-1)} \setminus \{t^{(j-1)}_{\lambda^{(j-1)}_1}\},\, t^{(j)}\big) \quad \{\lambda+1,\dots,\lambda \quad b=1+\lambda^{(j)}-\lambda_1$$

$$\times \prod_{\substack{m=2\\ m\neq j}}^{N} U^{\mathfrak{gl}_2}_{\{\lambda_m+1,\dots,\lambda^{(m)}\},\{1,\dots,\lambda_m\}}(\boldsymbol{t}^{(m-1)},\boldsymbol{t}^{(m)})\Big)$$

.

The equality of (6.15) and (6.16) follows from the following case of formula (6.1):

$$\sum_{l=1}^{\lambda_1} \big(t^{(j-1)}_{(l+j-\lambda_1)^{(j-1)}-\lambda_1} - t^{(j)}_{(l+j)\lambda^{(j)}-\lambda_1}\big) W^{\mathfrak{gl}_2}_{\{1,\dots,\lambda_1\},\emptyset}\big(t^{(j-1)}_{1+\lambda^{(j-1)}-\lambda_1},\dots,t^{(j-1)}_{\lambda^{(j-1)}};t^{(j)}_{1+\lambda^{(j)}-\lambda_1},\dots,t^{(j)}_{\lambda^{(j)}}\big)$$

$$= \sum_{a=1}^{\lambda_1} \check{d}_{\{1,2\}}\, U^{\mathfrak{gl}_2}_{\{1,\dots,a-1,a+1,\dots,\lambda_1\},\{a\}}\big(t^{(j-1)}_{1+\lambda^{(j-1)}-\lambda_1},\dots,t^{(j-1)}_{\lambda^{(j-1)}};t^{(j)}_{1+\lambda^{(j)}-\lambda_1},\dots,t^{(j)}_{\lambda^{(j)}}\big).$$

6.5. **Proof of Theorem 4.4 for $i>1$, arbitrary N, and $I=(\{1,\dots,n\},\emptyset,\dots,\emptyset)$.** For $i=2,\dots,N-1$, and $I=(\{1,\dots,n\},\emptyset,\dots,\emptyset)$, Theorem 4.4 says that

$$\sum_{l=1}^{n} (t_l^{(i)} - t_l^{(i-1)})\, W_I \;=\; -\sum_{a=1}^{n} \check{d}_{\{1,i\}}\, U_{(I)'^{a}_{1,j}},$$

which is formula (6.10).

6.6. **Proof of Theorem 4.4 for $i>1$, arbitrary λ, and $I=I^{\max}$.** To prove this case of Theorem 4.4, we introduce a partition $I^{\max,j} = (I_1^{\max,j},\dots,I_j^{\max,j})$ of the set $(1,\dots,\lambda^{(j)})$ by the rule

(6.17)
$$I_a^{\max,j} = \{\, i \mid \lambda^{(j)} - \lambda^{(a)} < i \leqslant \lambda^{(j)} - \lambda^{(a-1)} \,\},$$

so that $|I_a^{\max,j}| = \lambda_a$. For example, $I^{\max,N} = I^{\max}$.

Formula (4.14) for $I = I^{\max}$ can be written as

$$\sum_{l=1}^{j-1}\Big(\Big(\sum_{i\in I_l^{\max,j}} t_i^{(j)} - \sum_{i\in I_l^{\max,j-1}} t_i^{(j-1)}\Big) W_{I^{\max}} + \sum_{a=1}^{\lambda_l} \check d_{\{l,j\}}\, U_{(I^{\max})'^{a}_{l,j}}\Big)$$
$$+ \sum_{l=j+1}^{N}\Big(\Big(\sum_{i\in I_j^{\max,l-1}} t_j^{(l-1)} - \sum_{j\in I_i^{\max,l}} t_i^{(l)}\Big) W_{I^{\max}} - \sum_{a=1}^{\lambda_l} \check d_{\{j,l\}}\, U_{(I^{\max})'^{a}_{j,l}}\Big) = 0 \tag{6.18}$$

Formula (6.18) follows from the next Proposition.

Proposition 6.6. *For $i = 1,2,...,N-1$, and $j = i,i+1,...,N-1$, we have*

$$\Big(\sum_{l\in I_i^{\max,j}} t_l^{(j)} - \sum_{l\in I_i^{\max,j}} t_l^{(j+1)}\Big) W_{I^{\max}} = \sum_{a=1}^{\lambda_i} \check d_{\{i,j+1\}}\, U_{(I^{\max})'^{a}_{i,j+1}} \tag{6.19}$$

Proof. For $i = 1$, formula (6.19) follows from Lemma 6.5. For $i > 1$, we prove formula (6.19) by induction on $\lambda^{(i-1)}$, see Lemmas 6.7 and 6.8 below. If $\lambda^{(i-1)} = 0$, that is, $\lambda_j = 0$ for all $j = 1,...,i-1$, formula (6.19) follows from Lemma 6.5 by renaming variables.

We will indicate explicitly the dependence of the partitions $I^{\max}$, $I^{\max,l}$ on λ:

$$I^{\max}_{\boldsymbol\lambda} = (I^{\max}_{\boldsymbol\lambda,1},\dots,I^{\max}_{\boldsymbol\lambda,N}), \qquad I^{\max,l}_{\boldsymbol\lambda} = (I^{\max,l}_{\boldsymbol\lambda,1},\dots,I^{\max,l}_{\boldsymbol\lambda,l}).$$

We fix i,j until the end of the proof of Proposition 6.6, and omit the condition $|\lambda| = n$.

Lemma 6.7. *Assume that formula (6.19) holds for $\lambda = (0,...,0,\lambda_k,...,\lambda_N)$ with $k \leqslant i$. Then formula (6.19) holds for $\tilde\lambda = (0,...,0,1,\lambda_k,...,\lambda_N)$.*

Proof. Formula (6.19) for λ has the form

$$\Big(\sum_{l\in I^{\max,j}_{\boldsymbol\lambda,i}} t_l^{(j)} - \sum_{l\in I^{\max,j+1}_{\boldsymbol\lambda,i}} t_l^{(j+1)}\Big)\,\mathrm{Sym}_{t^{(k)}}\dots\mathrm{Sym}_{t^{(N-1)}}\big(U_{I^{\max}_{\boldsymbol\lambda}}(\boldsymbol t)\big) \tag{6.20}$$
$$= \mathrm{Sym}_{t^{(k)}}\dots\mathrm{Sym}_{t^{(N-1)}}\Big(C_{\boldsymbol\lambda,i,j+1}\sum_{a=1}^{\lambda_i} U_{(I^{\max}_{\boldsymbol\lambda})'^{a}_{i,j+1}}(\boldsymbol t)\Big),$$

where $C_{\lambda,i,j+1}$ is the factor in the second and third lines of definition (4.5).

In addition to the variables $t = (t^{(k)},...,t^{(N)})$ appearing in formula (6.20), formula (6.19) for $\tilde\lambda$ contains the new variables $t_{\text{new}} = \big(t_1^{(k-1)},\, t^{(k)}_{1+\lambda^{(k)}},\, t^{(k+1)}_{1+\lambda^{(k+1)}},\dots,\, t^{(N-1)}_{1+\lambda^{(N-1)}},\, t^{(N)}_{1+\lambda^{(N)}}\big)$, and has the form

$$\Big(\sum_{l\in I^{\max,j}_{\tilde{\boldsymbol\lambda},i}} t_l^{(j)} - \sum_{l\in I^{\max,j+1}_{\tilde{\boldsymbol\lambda},i}} t_l^{(j+1)}\Big)\,\mathrm{Sym}_{\tilde t^{(k)}}\dots\mathrm{Sym}_{\tilde t^{(N-1)}}\big(U_{I^{\max}}(\tilde{\boldsymbol t})\big) \tag{6.21}$$

$$= \mathrm{Sym}_{\tilde t^{(k)}} \dots \mathrm{Sym}_{\tilde t^{(N-1)}} \Big(C_{\tilde{\boldsymbol\lambda},i,j+1} \sum_{a=1}^{\lambda_i} U_{(I^{\max}_{\tilde{\boldsymbol\lambda}})'^{a}_{i,j+1}} (\tilde{\boldsymbol t}) \Big),$$

where $\tilde{\boldsymbol t} = \boldsymbol t \cup \boldsymbol t_{\mathrm{new}} = (\tilde t^{(k-1)}, \tilde t^{(k)}, \dots, \tilde t^{(N)})$. It is easy to see from definition (3.6) that

$$U_{I^{\max}_{\tilde{\boldsymbol\lambda}}}(\tilde{\boldsymbol t}) \,=\, U_{I^{\max}_{\boldsymbol\lambda}}(\boldsymbol t)\, F(\tilde{\boldsymbol t}), \tag{6.22}$$

where $F(\tilde{\boldsymbol t})$ is the product of all factors appearing in (3.6) involving the interrelation of two variables at least one of those being from $\boldsymbol t_{\mathrm{new}}$. Moreover, $F(\tilde{\boldsymbol t})$ is symmetric in the variables $t^{(l)}$ for each $l = k, \dots, N$. Furthermore, since $I^{\max,j}_{\boldsymbol\lambda,i} = I^{\max,j}_{\tilde{\boldsymbol\lambda},i}$ and $I^{\max,j+1}_{\boldsymbol\lambda,i} = I^{\max,j+1}_{\tilde{\boldsymbol\lambda},i}$, the first factors in the left-hand sides of formulas (6.20) and (6.21) coincide.

By all these observations, to get formula (6.21) from (6.20), we need to verify that

$$\mathrm{Sym}_{\tilde t^{(k)}} \dots \mathrm{Sym}_{\tilde t^{(N-1)}} \Big(C_{\tilde{\boldsymbol\lambda},i,j+1} \sum_{a=1}^{\lambda_i} U_{(I^{\max}_{\tilde{\boldsymbol\lambda}})'^{a}_{i,j+1}} - F\, C_{\boldsymbol\lambda,i,j+1} \sum_{a=1}^{\lambda_i} U_{(I^{\max}_{\boldsymbol\lambda})'^{a}_{i,j+1}} \Big) \,=\, 0. \tag{6.23}$$

This equality follows from identity (3.12) for the variables $t^{(i-1)}_{1+\lambda^{(i-1)}}, t^{(i)}_{\lambda^{(i)}}, t^{(i)}_{1+\lambda^{(i)}}, \dots, t^{(j)}_{\lambda^{(j)}}$, $t^{(j)}_{1+\lambda^{(j)}}, t^{(j+1)}_{1+\lambda^{(j+1)}}$. Lemma 6.7 is proved.

Example. Let $N = 5$, $\boldsymbol\lambda = (0,0,1,0,0)$, $\tilde{\boldsymbol\lambda} = (1,0,1,0,0)$. For $i = 3$, $j = 3$, formulas (6.20) and (6.23) take the form $t^{(3)}_1 - t^{(4)}_1 = t^{(3)}_1 - t^{(4)}_1$ and

$$\begin{aligned}
&\mathrm{Sym}_{t^{(3)}_1, t^{(3)}_2} \mathrm{Sym}_{t^{(4)}_1, t^{(4)}_2} \Big((t^{(3)}_1 - t^{(4)}_1)(t^{(2)}_1 - t^{(3)}_1)(t^{(3)}_1 - t^{(4)}_2 - h)(t^{(3)}_2 - t^{(4)}_1) \\
&\qquad \times (t^{(4)}_1 - t^{(5)}_2 - h)(t^{(4)}_2 - t^{(5)}_1)\, \frac{t^{(3)}_2 - t^{(3)}_1 - h}{t^{(3)}_2 - t^{(3)}_1}\, \frac{t^{(4)}_2 - t^{(4)}_1 - h}{t^{(4)}_2 - t^{(4)}_1} \Big) \\
&= \mathrm{Sym}_{t^{(3)}_1, t^{(3)}_2} \mathrm{Sym}_{t^{(4)}_1, t^{(4)}_2} \Big((t^{(3)}_1 - t^{(4)}_2 - h)(t^{(2)}_2 - t^{(3)}_1)(t^{(3)}_2 - t^{(4)}_2)(t^{(3)}_1 - t^{(4)}_1) \\
&\qquad \times (t^{(4)}_1 - t^{(5)}_2 - h)(t^{(4)}_2 - t^{(5)}_1)\, \frac{t^{(3)}_2 - t^{(3)}_1 - h}{t^{(3)}_2 - t^{(3)}_1}\, \frac{t^{(4)}_2 - t^{(4)}_1 - h}{t^{(4)}_2 - t^{(4)}_1} \Big),
\end{aligned}$$

respectively. The last equality follows from identity (3.12) for the variables $t^{(2)}_1, t^{(3)}_1, t^{(3)}_2$, $t^{(4)}_2$.

For $i = 3$, $j = 4$, formulas (6.20) and (6.23) take the form $t^{(4)}_1 - t^{(5)}_1 = t^{(4)}_1 - t^{(5)}_1$ and

$$\begin{aligned}
&\mathrm{Sym}_{t^{(3)}_1, t^{(3)}_2} \mathrm{Sym}_{t^{(4)}_1, t^{(4)}_2} \Big((t^{(4)}_1 - t^{(5)}_1)(t^{(2)}_1 - t^{(3)}_1)(t^{(3)}_1 - t^{(4)}_2 - h)(t^{(3)}_2 - t^{(4)}_1) \\
&\qquad \times (t^{(4)}_1 - t^{(5)}_2 - h)(t^{(4)}_2 - t^{(5)}_1)\, \frac{t^{(3)}_2 - t^{(3)}_1 - h}{t^{(3)}_2 - t^{(3)}_1}\, \frac{t^{(4)}_2 - t^{(4)}_1 - h}{t^{(4)}_2 - t^{(4)}_1} \Big)
\end{aligned}$$

$$= \mathrm{Sym}_{t_1^{(3)},t_2^{(3)}} \mathrm{Sym}_{t_1^{(4)},t_2^{(4)}} \Big((t_1^{(2)} - t_2^{(3)} - h)(t_2^{(2)} - t_1^{(3)})(t_1^{(3)} - t_2^{(4)} - h)(t_2^{(4)} - t_1^{(5)}) \times (t_2^{(4)} - t_2^{(5)})(t_1^{(4)} - t_1^{(5)}) \frac{t_2^{(3)} - t_1^{(3)} - h}{t_2^{(3)} - t_1^{(3)}} \frac{t_2^{(4)} - t_1^{(4)} - h}{t_2^{(4)} - t_1^{(4)}} \Big),$$

respectively. The last equality follows from identity (3.12) for the variables $t_1^{(2)}, t_1^{(3)}, t_2^{(3)}, t_1^{(4)}, t_2^{(4)}, t_2^{(5)}$.

Lemma 6.8. *Assume that formula (6.19) holds for $\lambda = (0,\dots,0,\lambda_k,\dots,\lambda_N)$ with $k < i$ and $\lambda_k > 0$. Then formula (6.19) holds for $\tilde\lambda = (0,\dots,0,0,\lambda_k + 1,\dots,\lambda_N)$.*

Proof. The proof is completely similar to that of Lemma 6.7. The only change is that the new variables are $t_{\mathrm{new}} = \big(t^{(k)}_{1+\lambda^{(k)}}, t^{(k+1)}_{1+\lambda^{(k+1)}}, \dots, t^{(N-1)}_{1+\lambda^{(N-1)}}, t^{(N)}_{1+\lambda^{(N)}}\big)$.

Example. Let $N = 3$, $\lambda = (1,1,0)$, $\tilde\lambda = (2,1,0)$. For $i = 2$, $j = 2$, formula (6.23) proof follows from identity (3.12) for the variables $t_2^{(1)}, t_2^{(2)}, t_3^{(2)}, t_3^{(3)}$.

Lemmas 6.7 and 6.8 yield Proposition 6.6.

Theorem 4.4 for $i > 1$, arbitrary N, λ, and $I = I^{\max}$ is proved.

6.7. **Modification of the three-term relation.** For integers α,β, $1 \leqslant \alpha < \beta \leqslant N$, and $\boldsymbol\lambda \in \mathbb{Z}^N_{\geqslant 0}$, $|\boldsymbol\lambda| = n$, recall the notations $t^{\{\alpha,\beta\}}$, $t_{\{\alpha,\beta\}}$, $\lambda_{\alpha,\beta}$, in Sections 4.3 and 4.4.

Lemma 6.9. *For any $1 \leqslant \alpha < \beta \leqslant N$ and $I \in \mathcal{I}_{\lambda_{\alpha,\beta}}$, we have $\check d_{\{\alpha,\beta\}} U_I = c_{\alpha,\beta}\, \check d_{\{\alpha,\beta\}} W_I$, where*

$$c_{\alpha,\beta} = \frac{\prod_{i=\alpha}^{\beta-1} \lambda^{(i)}}{\prod_{i=1}^{N-1} \lambda^{(i)}!}.$$

Proof. Let

$$G_{\alpha,\beta}(\boldsymbol t, \boldsymbol z) = (t^{(\alpha-1)}_{\lambda^{(\alpha-1)}} - t^{(\alpha)}_{\lambda^{(\alpha)}} - h) \prod_{i=\alpha}^{\beta-1} \prod_{a=1}^{\lambda^{(i-1)}-1} (t^{(i-1)}_a - t^{(i)}_{\lambda^{(i)}} - h) \times \tag{6.24}$$
$$\times (t^{(\beta-1)}_{\lambda^{(\beta-1)}} - t^{(\beta)}_{\lambda^{(\beta)}}) \prod_{i=\alpha}^{\beta-1} \prod_{a=1}^{\lambda^{(i+1)}-1} (t^{(i)}_{\lambda^{(i)}} - t^{(i+1)}_a) \prod_{i=\alpha}^{\beta-1} \prod_{a=1}^{\lambda^{(i)}-1} \frac{t^{(i)}_{\lambda^{(i)}} - t^{(i)}_a - h}{t^{(i)}_{\lambda^{(i)}} - t^{(i)}_a}$$

be the product in the right-hand side of formula (4.5). Since $G_{\alpha,\beta}(t,z)$ is symmetric in the variables $t^{(i)}_{\{\alpha,\beta\}}$ for every $i = 1,\dots,N-1$, we can apply the symmetrization in those variables to $U_I(t_{\{\alpha,\beta\}},z)$ and divide the result by the order of the relevant product of the symmetric groups before doing the overall symmetrization in formula (4.8) for $\check d_{\{\alpha,\beta\}} U_I$.

This results in replacing $U_I(t_{\{\alpha,\beta\}},z)$ by $c_{\alpha,\beta}\, W_I(t_{\{\alpha,\beta\}},z)$, see formula (3.5).

Recall the operator $S_{i,i+1}$ acting on functions of $z_1,...,z_n$ given by formula (3.7).

Lemma 6.10. *For any $i = 1,...,n-1$, $1\ 6\ \alpha < \beta\ 6\ N$, and $I \in \mathcal{I}_{\lambda_{\alpha,\beta}}$, we have*

$$S_{i,i+1}(\check{d}_{\{\alpha,\beta\}}U_I) = \check{d}_{\{\alpha,\beta\}}U_{s_{i,i+1}(I)}\,. \tag{6.25}$$

Proof. The product $G_{\alpha,\beta}(t,z)$, see (6.24), is symmetric in $z_1,...,z_n$. Hence

$$\begin{aligned} S_{i,i+1}(\check{d}_{\{\alpha,\beta\}}U_I) &= c_{\alpha,\beta}\, S_{i,i+1}(\check{d}_{\{\alpha,\beta\}}W_I) = c_{\alpha,\beta}\, \check{d}_{\{\alpha,\beta\}}\big(S_{i,i+1}(W_I)\big) \\ &= c_{\alpha,\beta}\, \check{d}_{\{\alpha,\beta\}}W_{s_{i,i+1}(I)} = \check{d}_{\{\alpha,\beta\}}U_{s_{i,i+1}(I)}, \end{aligned}$$

by Lemmas 6.9 and 3.2.

6.8. **The end of the proof of Theorem 4.4.** Given l, $1\ 6\ l\ 6\ N-1$, we add formulas (4.14) for $i = 1,...,l$. The result is

$$\Big(\sum_{j=1}^{\lambda^{(l)}} t_j^{(l)} - \sum_{i=1}^{l}\sum_{a\in I_i} z_a\Big)W_I + h\sum_{i=1}^{l}\sum_{j=l+1}^{N}\sum_{m_1=1}^{\lambda_i}\sum_{\substack{m_2=1\\ \ell_{i,m_1}<\ell_{j,m_2}}}^{\lambda_j} W_{I_{i,j;m_1,m_2}} = \sum_{i=1}^{l}\sum_{j=l+1}^{N}\sum_{a=1}^{\lambda_i} \check{d}_{\{i,j\}}U_{(I)^a_{i,j}}\,, \tag{6.26}$$

To finish the proof of Theorem 4.4, we need to prove formula (6.26) for any I and any $i = 1,...,N-1$.

For any permutation σ, denote by $|\sigma|$ the length of σ. For any $J,J^0 \in \mathcal{I}_\lambda$, define the permutation σ_{J,J^0} as follows: if $J_m = \{j_{m,1} < \ldots < j_{m,\lambda_m}\}$, $J'_m = \{j'_{m,1} < \ldots < j'_{m,\lambda_m}\}$, then $\sigma_{J,J'}(j'_{m,l}) = j_{m,l}$. Set $\sigma_J = \sigma_{J,I^{\max}}$. The permutation σ_J has the minimal length amongst all permutations σ such that $\sigma(I^{\max}) = J$.

Lemma 6.11. *Assume that for $J \in \mathcal{I}_\lambda$ and a transposition $s_{i,i+1}$, we have $|s_{i,i+1}\sigma_J| < |\sigma_J|$.*

Then $s_{i,i+1}\sigma_J = \sigma_{s_{i,i+1}(J)}$.

We will prove formula (6.26) by induction with respect to the length of σ_I. For the base of induction $I = I^{\max}$, formula (6.26) is proved already.

Fix $I \in \mathcal{I}_\lambda$ and find m such that $|s_{m,m+1}\sigma_I| < |\sigma_I|$. Let p,r be such that $m \in I_p$ and $m+1 \in I_r$. Since $|s_{m,m+1}\sigma_I| < |\sigma_I|$, we have $p < r$.

Denote $\tilde{I} = s_{m,m+1}(I)$. Then $\tilde{I}_p = I_p - \{m\} \cup \{m+1\}$, $\tilde{I}_r = I_r - \{m+1\} \cup \{m\}$, and $\tilde{I}_c = I_c$, otherwise. And clearly, $I = s_{m,m+1}(\tilde{I})$.

Write formula (6.26) for $\tilde{I}$:

$$
(6.27)\qquad \Big(\sum_{j=1}^{\lambda^{(l)}} t_j^{(l)} - \sum_{i=1}^{l}\sum_{a\in\tilde I_i} z_a\Big) W_{\tilde I} + h\sum_{i=1}^{l}\sum_{j=l+1}^{N}\sum_{m_1=1}^{\lambda_i}\sum_{\substack{m_2=1\\ \tilde\ell_{i,m_1}<\tilde\ell_{j,m_2}}}^{\lambda_j} W_{\tilde I_{i,j;m_1,m_2}} = \sum_{i=1}^{l}\sum_{j=l+1}^{N}\sum_{a=1}^{\lambda_i} \check d_{\{i,j\}} U_{(\tilde I)^a_{i,j}} ,
$$

where $\tilde I_c = (\tilde\ell_{c,1},\dots,\tilde\ell_{c,\lambda_c})$. We will show that applying the operator $S_{m,m+1}$ to both sides of formula (6.27) transforms it to formula (6.26) for I.

To compare the right-hand sides, observe that $s_{m,m+1}\big((\tilde I)^a_{i,j}\big) = (I)^a_{i,j}$. Hence, Lemma 6.10 yields $S_{m,m+1}\big(\check d_{\{i,j\}} U_{(\tilde I)^a_{i,j}}\big) = \check d_{\{i,j\}} U_{(I)^a_{i,j}}$, that proves the desired assertion.

To compare the left-hand sides, observe first that

$$
s_{m,m+1}(\tilde I_{i,j;m_1,m_2}) = I_{i,j;s_{m,m+1}(m_1),s_{m,m+1}(m_2)}
$$

and

$$
S_{m,m+1}(W_{\tilde I_{i,j;m_1,m_2}}) = W_{I_{i,j;s_{m,m+1}(m_1),s_{m,m+1}(m_2)}}
$$

by Lemma 3.2. This proves the desired transformation of the second sum in the left-hand side of (6.27) term by term provided $p > l$ or $r \leqslant l$. If $p \leqslant l < r$, the matching between the terms of the second sums in (6.27) and (6.26) is not perfect and the sum in (6.26) contains one more term $hW_{I_{p,r;m,m+1}}$.

If $p > l$ or $r \leqslant l$, the sum $\sum_{i=1}^{l}\sum_{a\in\tilde I_i} z_a$ in formula (6.27) is symmetric in z_m, z_{m+1} and equals the sum $\sum_{i=1}^{l}\sum_{a\in I_i} z_a$ in formula (6.26). Thus

$$
S_{m,m+1}\Big(\Big(\sum_{j=1}^{\lambda^{(l)}} t_j^{(l)} - \sum_{i=1}^{l}\sum_{a\in\tilde I_i} z_a\Big) W_{\tilde I}\Big) = \Big(\sum_{j=1}^{\lambda^{(l)}} t_j^{(l)} - \sum_{i=1}^{l}\sum_{a\in I_i} z_a\Big) S_{m,m+1}(W_{\tilde I}) = \Big(\sum_{j=1}^{\lambda^{(l)}} t_j^{(l)} - \sum_{i=1}^{l}\sum_{a\in I_i} z_a\Big) W_I
$$

by Lemma 3.2. If $p \leqslant l < r$, then we have

$$
S_{m,m+1}\Big(\Big(\sum_{j=1}^{\lambda^{(l)}} t_j^{(l)} - \sum_{i=1}^{l}\sum_{a\in\tilde I_i} z_a\Big) W_{\tilde I}\Big) = \Big(\sum_{j=1}^{\lambda^{(l)}} t_j^{(l)} - \sum_{i=1}^{l}\sum_{a\in I_i} z_a\Big) S_{m,m+1}(W_{\tilde I}) + hW_{\tilde I} = \Big(\sum_{j=1}^{\lambda^{(l)}} t_j^{(l)} - \sum_{i=1}^{l}\sum_{a\in I_i} z_a\Big) W_I + hW_{I_{p,r;m,m+1}} ,
$$

since $\tilde{I} = I_{p,r;m,m+1}$. This shows that the operator $S_{m,m+1}$ transforms formula (6.27) to formula (6.26). This completes the induction step. Theorem 4.4 is proved.

Example. Let $N = 2$, $n = 3$, $\lambda = (2,1)$, $I = (\{1,3\},\{2\})$, $I^{\max} = (\{2,3\},\{1\})$, $\sigma_I = s_{1,2}$. Formula (6.26) is

$$(t_1^{(1)} + t_2^{(1)} - z_1 - z_3)\, W_{\{1,3\},\{2\}} + h W_{\{2,3\}\{1\}} = \check{d}_{\{1,2\}}\,(U_{\{1\},\{2,3\}} + U_{\{3\},\{1,2\}}), \tag{6.28}$$

formula (6.27) is

$$(t_1^{(1)} + t_2^{(1)} - z_2 - z_3)\, W_{\{2,3\},\{2\}} = \check{d}_{\{1,2\}}\,(U_{\{2\},\{1,3\}} + U_{\{3\},\{1,2\}}). \tag{6.29}$$

and the operator $S_{1,2}$ transforms formula (6.29) to formula (6.28).

7. Corollary of Theorems 4.3 and 4.4

Let $\boldsymbol{\lambda} \in \mathbb{Z}^N_{\geqslant 0}$, $|\boldsymbol{\lambda}| = n$, and $I \in \mathcal{I}_\lambda$. Recall the notations $(I)_{\alpha,\beta}^a$, $I_{i,j;m_1,m_2}$, see (4.11), (4.13), and the discrete differentials $d_{\{\alpha,\beta\}}\, g$, see (4.6). Define the discrete differential

$$D_{I,i} = \sum_{j=i+1}^{N} \sum_{a=1}^{\lambda_j} d_{\{j,i\}}\, U_{(I)_{j,i}^a} - \sum_{j=1}^{i-1} \sum_{a=1}^{\lambda_i} d_{\{i,j\}}\, U_{(I)_{i,j}^a}. \tag{7.1}$$

Corollary 7.1. *We have*

$$\Big(\sum_{j=1}^{\lambda^{(i)}} t_j^{(i)} - \sum_{j=1}^{\lambda^{(i-1)}} t_j^{(i-1)} - \sum_{a\in I_i} z_a\Big) W_I + h\Big(\sum_{j=1}^{i-1} \frac{q_i\,\lambda_j}{q_i - q_j} + \sum_{j=i+1}^{N} \frac{q_j\,\lambda_i}{q_i - q_j}\Big) W_I + \tag{7.2}$$
$$+\, h \sum_{\substack{j=1\\ j\neq i}}^{N} \sum_{m_1=1}^{\lambda_i} \Big(\sum_{\substack{m_2=1\\ \ell_{i,m_1} < \ell_{j,m_2}}}^{\lambda_j} W_{I_{i,j;m_1,m_2}} + \frac{q_j}{q_i - q_j} \sum_{m_2=1}^{\lambda_j} W_{I_{i,j;m_1,m_2}} \Big) = D_{I,i} .$$

Proof. Theorems 4.3 and 4.4, and formula (4.9) imply that

$$\Big(\sum_{j=1}^{\lambda^{(i)}} t_j^{(i)} - \sum_{j=1}^{\lambda^{(i-1)}} t_j^{(i-1)} - \sum_{a\in I_i} z_a\Big) W_I - h \sum_{j=1}^{i-1} \sum_{m_1=1}^{\lambda_i} \sum_{\substack{m_2=1\\ \ell_{i,m_1} > \ell_{j,m_2}}}^{\lambda_j} W_{I_{i,j;m_1,m_2}}$$
$$+\, h \sum_{j=i+1}^{N} \sum_{m_1=1}^{\lambda_i} \sum_{\substack{m_2=1\\ \ell_{i,m_1} < \ell_{j,m_2}}}^{\lambda_j} W_{I_{i,j;m_1,m_2}} + h \sum_{j=1}^{i-1} \frac{q_i}{q_i - q_j} \Big(\lambda_j W_I + \sum_{m_1=1}^{\lambda_i} \sum_{m_2=1}^{\lambda_j} W_{I_{i,j;m_1,m_2}} \Big)$$

$$+\,h\sum_{j=i+1}^{N}\frac{q_j}{q_i-q_j}\Big(\lambda_i W_I+\sum_{m_1=1}^{\lambda_i}\sum_{m_2=1}^{\lambda_j}W_{I_{i,j;m_1,m_2}}\Big)\;=\;D_{I,i}\,.$$

Formula (7.2) is obtained now by rearranging the terms in the left-hand side of this equality.

Recall the scalar master function $\Phi_\lambda(t;z;h;q)$ given by (4.1). Define

$$\Omega_\lambda(q)=\prod_{i=1}^{N-1}\prod_{j=i+1}^{N}(1-q_j/q_i)^{h\lambda_i/\kappa}. \tag{7.3}$$

Introduce the $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda}$-valued weight function

$$W_\lambda(t;z;h)=\sum_{I\in\mathcal{I}_\lambda}W_I(t;z;h)v_I. \tag{7.4}$$

Recall the dynamical Hamiltonians $X_i(z;h;q)$ defined in (2.1).

Theorem 7.2. *For every $i=1,\dots,N$, we have*

$$\Big(\kappa q_i\frac{\partial}{\partial q_i}-X_i(\boldsymbol z;h;\boldsymbol q)\Big)\,\Omega_{\boldsymbol\lambda}(\boldsymbol q)\,\Phi_{\boldsymbol\lambda}(t;z;h;q)W_\lambda(t;z)= \tag{7.5}$$

$$=\Omega_\lambda(q)\Phi_\lambda(t;z;h;q)\sum_{I\in\mathcal{I}_\lambda}D_{I,i}(t;z;h;q)v_I.$$

Proof. The statement is equivalent to Corollary 7.1.

8. Integral representations for solutions of dynamical equations

8.1. **Formal integrals.** Let $\boldsymbol\lambda\in\mathbb{Z}^N_{\geqslant0}$, $|\boldsymbol\lambda|=n$, and $\kappa\in\mathbb{C}^\times$. Consider the space of functions of the form $\Phi_\lambda(t;z;h;q)f(t;z;h;q)$, where $\Phi_\lambda(t;z;h;q)$ is the master function (4.1), and $f(t;z;h;q)$ is a polynomial in t and holomorphic function of z,h,q on some domain $L\subset\mathbb{C}^n\times\mathbb{C}\times\mathbb{C}^N$. Assume that we have a map $\mathcal{M}$ assigning to a function $\Phi_\lambda f$ a function $\mathcal{M}(\Phi_\lambda f)$ of variables z,h,q, holomorphic on L, such that:

(i) The map $\mathcal{M}$ is linear over the field of meromorphic on L functions in z,q,h,

$$\mathcal{M}\big(\Phi_{\boldsymbol\lambda}(g_1f_1+g_2f_2)\big)\;=\;g_1\,\mathcal{M}(\Phi_{\boldsymbol\lambda}f_1)+g_2\,\mathcal{M}(\Phi_{\boldsymbol\lambda}f_2) \tag{8.1}$$

for any meromorphic functions g_1,g_2 of z,h,q, such that g_1f_1 and g_2f_2 are holomorphic on L.

(ii) For any $i=1,\dots,N$, we have

(8.2)
$$\frac{\partial}{\partial q_i}\mathcal{M}(\Phi_{\boldsymbol{\lambda}} f) \,=\, \mathcal{M}\left(\frac{\partial}{\partial q_i}(\Phi_{\boldsymbol{\lambda}} f)\right).$$

(iii) If f is a discrete differential of a polynomial in t, then

(8.3)
$$\mathrm{M}(\Phi_\lambda f) = 0.$$

A map M is called a formal integral. We have the following corollary of Theorem 7.2.

Lemma 8.1. *If* M *is a formal integral, then the* $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol{\lambda}}$*-valued function*

$$F_{\mathrm{M}}(z;h;q) := \Omega_\lambda\, \mathrm{M}(\Phi_\lambda W_\lambda) = \Omega_\lambda \sum_{I\in \mathcal{I}_\lambda} \mathrm{M}(\Phi_\lambda W_I) v_I$$

holomorphic on L, is a solution of the dynamical differential equations (2.3)*.*

8.2. **Jackson integral.** Consider the space $\mathbb{C}^{\lambda^{\{1\}}}\times\mathbb{C}^n\times\mathbb{C}\times\mathbb{C}^N$ with coordinates t,z,h,q . The lattice $\kappa\mathbb{Z}^{\lambda^{\{1\}}}$ naturally acts on this space by shifting the t-coordinates.

Let $J = (J_1,\dots,J_N) \in \mathcal{I}_\lambda$. Recall the notation $\bigcup_{i=1}^k J_i \,=\, \{j_1^{(k)} < \ldots < j_{\lambda^{(k)}}^{(k)}\}$. Define $\Sigma_J \subset \mathbb{C}^{\lambda^{\{1\}}}\times \mathbb{C}^n\times \mathbb{C} \times \mathbb{C}^N$ by the equations:

(8.4)
$$t_i^{(k)} \,=\, z_{j_i^{(k)}}, \qquad k=1,\dots,N-1\,, \quad i=1,\dots,\lambda^{(k)},$$

and call it a *discrete cycle*.

For a function of t and a point $s \in \mathbb{C}^{\lambda^{\{1\}}}$, define $\mathrm{Res}_{t=s}$ to be the iterated residue,

$$\mathrm{Res}_{t=s} = \mathrm{Res}_{t_1^{(1)}=s_1^{(1)}} \cdots \mathrm{Res}_{t^{(1)}_{\lambda^{(1)}}=s^{(1)}_{\lambda^{(1)}}} \dots\, \mathrm{Res}_{t_1^{(N-1)}=s_1^{(N-1)}} \cdots \mathrm{Res}_{t^{(N-1)}_{\lambda^{(N-1)}}=s^{(N-1)}_{\lambda^{(N-1)}}}\,.$$

Let L^0 be the complement in $\mathbb{C}^n\times \mathbb{C}$ of the union of the hyperplanes

(8.5)
$$h = m\kappa, \qquad z_a - z_b = m\kappa, \qquad z_a - z_b + h = m\kappa,$$

for all $a,b = 1,\dots,n$, $a \neq b$, and all $m \in \mathbb{Z}$. Let $L^{00} \subset \mathbb{C}^N$ be the domain

(8.6)
$$|q_{i+1}/q_i| < 1, \qquad i = 1,\dots,N-1,$$

with additional cuts fixing a branch of $\log q_i$ for all $i = 1,\dots,N$. Set $L = L^0 \times L^{00} \subset \mathbb{C}^n\times \mathbb{C} \times \mathbb{C}^N$.

Let $f(t;z;h;q)$ be a polynomial in t and a holomorphic function of $z;h;q$ on L. For $(z;h;q) \in L$, define

(8.7)
$$\mathrm{M}_J(\Phi_\lambda f)(z;h;q) = \sum \mathrm{Res}_{t=\Sigma_J+r\kappa}\, \Phi_\lambda(t;z;h;q) f(t;z;h;q)\big).$$

$r\in\mathbb{Z}^{\lambda\{1\}}$

This sum is called the *Jackson integral over the discrete cycle* Σ_J.

Lemma 8.2. *The map* M_J *is a formal integral.*

Proof. Each term of the sum in formula (8.7) is a holomorphic function on L^0. Moreover, $\mathrm{Res}_{t=\Sigma_J+r\kappa}\,\Phi_\lambda(t;z;h;q)f(t;z;h;q) = 0$ if $r\notin\mathbb{Z}^{\lambda\{1\}}_{\leqslant 0}$. Hence, the sum over $\mathbb{Z}^{\lambda\{1\}}$ reduces to the sum over $\mathbb{Z}^{\lambda\{1\}}_{\leqslant 0}$. The result is similar to a multidimensional hypergeometric series multiplied by some fractional powers of $q_1,\dots,q_N$. The obtained sum converges if $|q_{i+1}/q_i| < 1$ for all $i = 1,\dots,N-1$, and gives a holomorphic function on L.

Properties (8.1)–(8.3) for the map M_J are clear. Lemma 8.2 is proved.

Lemma 8.3. *The function* $M_J(\Phi_\lambda f)$ *analytically continues to the hyperplanes* $h = m\kappa$ *for* $m\in\mathbb{Z}_{>0}$.

Proof. By the proof of Lemma 8.2,

(8.8) $$M_J(\Phi_\lambda f)(z;h;q) = \sum_{r\in\mathbb{Z}^{\lambda\{1\}}_{\leqslant 0}} \mathrm{Res}_{t=\Sigma_J+r\kappa}\,\Phi_\lambda(t;z;h;q)f(t;z;h;\boldsymbol{q})\,.$$

for $(z;h;q)\in L$. By inspection, if $h\to m\kappa$, $m\in\mathbb{Z}_{>0}$, and $r\in\mathbb{Z}^{\lambda\{1\}}_{\leqslant 0}$, then

$$\mathrm{Res}_{t=\Sigma_J+r\kappa}\,\Phi_\lambda(t;z;h;q)f(t;z;h;\boldsymbol{q}) \to \mathrm{Res}_{t=\Sigma_J+r\kappa}\,\Phi_\lambda(t;z;m\kappa;q)f(t;z;m\kappa;\boldsymbol{q}).$$

Hence

(8.9) $$M_J(\Phi_\lambda f)(z;h;q) \to \sum_{r\in\mathbb{Z}^{\lambda\{1\}}_{\leqslant 0}} \mathrm{Res}_{t=\Sigma_J+r\kappa}\,\Phi_\lambda(t;z;m\kappa;q)f(t;z;m\kappa;\boldsymbol{q}),$$

since the sum in the right-hand side converges if $|q_{i+1}/q_i| < 1$ for all $i = 1,\dots,N-1$.

Remark. For $m\in\mathbb{Z}_{>0}$, the sum $\sum_{r\in\mathbb{Z}^{\lambda\{1\}}} \mathrm{Res}_{t=\Sigma_J+r\kappa}\,\Phi_\lambda(t;z;m\kappa;q)f(t;z;m\kappa;\boldsymbol{q})$ diverges, and the function $M_J(\Phi_\lambda f)(z;m\kappa;q)$ is not given by formula (8.7).

Example. Let $N = 2$, $\lambda = (1,n-1)$, $J = (\{1\},\{2,3,\dots,n\})$. Then

$$\Phi_\lambda(t;z;h;\boldsymbol{q}) = \left(e^{\pi\sqrt{-1}}q_2\right)^{\sum_{a=1}^n z_a/\kappa}\left(e^{\pi\sqrt{-1}(n-2)}\frac{q_1}{q_2}\right)^{t_1^{(1)}/\kappa}\prod_{a=1}^n \Gamma\Big(\frac{t_1^{(1)}-z_a}{\kappa}\Big)\Gamma\Big(\frac{z_a-t_1^{(1)}+h}{\kappa}\Big),$$

and

(8.10) $$M_{(\{1\},\{2,3,\dots,n\})}(\Phi_\lambda f) = \sum \mathrm{Res}_{t_1^{(1)}=z_1+r\kappa}(\Phi_\lambda f).$$

$r\in\mathbb{Z}$

Nonzero contributions to the sum in the right-hand side of (8.10) come from the poles of $\Gamma\big((t_1^{(1)}-z_1)/\kappa\big)$. Explicitly, the answer is

$$\mathcal{M}_{(\{1\},\{2,3,\dots,n\})}(\Phi_{\boldsymbol\lambda} f) \,=\, \big(e^{\pi\sqrt{-1}\,(n-1)}q_1\big)^{z_1/\kappa}\,\big(e^{\pi\sqrt{-1}}\,q_2\big)^{\sum_{a=2}^n z_a/\kappa}\,\times$$
$$\times\;\kappa\,\Gamma\Big(\frac{h}{\kappa}\Big)\prod_{a=2}^n\Gamma\Big(\frac{z_1-z_a}{\kappa}\Big)\,\Gamma\Big(\frac{z_a-z_1+h}{\kappa}\Big)\,\times$$
$$\times\;\sum_{l=0}^\infty f(z_1-l\kappa\,;\boldsymbol z;h;\boldsymbol q)\,\Big(\frac{q_2}{q_1}\Big)^l\prod_{j=0}^{l-1}\Big(\frac{h+j\kappa}{\kappa+j\kappa}\prod_{a=2}^n\frac{z_a-z_1+h+j\kappa}{z_a-z_1+\kappa+j\kappa}\Big),$$

and the series converges if $|q_2/q_1|<1$.

8.3. **Solutions of dynamical equations.** Recall the $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda}$-valued weight function $W_\lambda(t;z)$, given by (7.4). For $J\in\mathcal{I}_\lambda$, define

$$\Psi_J(z;h;q) = \Omega_\lambda(q)\,\mathcal{M}_J(\Phi_\lambda W_\lambda)(z;h;q) = \Omega_\lambda(q)\sum_{I\in\mathcal{I}_\lambda}\mathcal{M}_J(\Phi_\lambda W_I)(z;h;q)\,v_I. \tag{8.11}$$

Theorem 8.4. *The function $\Psi_J(z;h;q)$ is a holomorphic $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda}$-valued function of z,h,q on the domain $L\subset\mathbb{C}^n\times\mathbb{C}\times\mathbb{C}^N$ such that*

$$h\notin\kappa\mathbb{Z}_{\le 0},\qquad z_a-z_b\notin\kappa\mathbb{Z},\qquad z_a-z_b+h\notin\kappa\mathbb{Z},$$

for all $a,b=1,\dots,n$, $a\neq b$,

$$|q_{i+1}/q_i|<1,\qquad i=1,\dots,N-1,$$

and a branch of $\log q_i$ *is fixed for each $i=1,\dots,N$. Furthermore, $\Psi_J(z;h;q)$ is a solution of the dynamical differential equations* (2.3).

Proof. The weight functions $W_I(t;z;h)$ are polynomials in t,z,h and do not depend on q. Hence, Theorem 8.4 follows from Lemmas 8.2, 8.3, and 8.1.

Theorem 8.5. *Under conditions of Theorem 8.4, the collection of $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda}$-valued functions $\big(\Psi_J(\boldsymbol z;h;\boldsymbol q)\big)_{J\in\mathcal{I}_\lambda}$ is a basis of solutions of the dynamical equations* (2.3).

Proof. By formulas (8.8), (8.11), if $|q_{i+1}/q_i|\to 0$ for all $i=1,\dots,N-1$, then

$$\Psi_J(z;h;q)\simeq\Omega_\lambda(q)\,\mathrm{Res}_{t=\Sigma_J}\Phi_\lambda(t;z;h;\boldsymbol q))\,W_{\boldsymbol\lambda}(\Sigma_J;\boldsymbol z;h) = \tag{8.12}$$
$$=\Omega_\lambda(q)\,\mathrm{Res}_{t=\Sigma_J}\Phi_\lambda(t;z;h;\boldsymbol q))\sum_{I\in\mathcal{I}_\lambda}W_I(\Sigma_J;\boldsymbol z;h)\,v_I.$$

By [RTV1, Lemma 3.1], the matrix $\big(W_I(\Sigma_J;\boldsymbol z;h)\big)_{I,J\in\mathcal{I}_\lambda}$ is triangular and the diagonal entries

$W_I(\Sigma_I;z;h)$ are nonzero if $h \neq 0$ and $z_a - z_b \neq 0$, $z_a - z_b + h \neq 0$, for all $a,b = 1,\dots,n$, $a \neq b$. Hence the vectors $W_\lambda(\Sigma_J)$, $J \in \mathcal{I}_\lambda$, form a basis of $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda}$ and the collection $\Psi_J(z;h;q))_{J\in\mathcal{I}_\lambda}$ is a basis of solutions of the dynamical equations (2.3). The functions $\Psi_J(z;h;q)$ were considered in [TV1]. It follows from [TV1, Theorem 1.5.2], cf. [TV4], that for every $J \in \mathcal{I}_\lambda$, the function $\Psi_J(z;h;q)$ is a solution of the qKZ equations (2.4).

Corollary 8.6. *The collection of* $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda}$*-valued functions* $\Psi_J(z;h;q))_{J\in\mathcal{I}_\lambda}$ *is a basis of solutions of both the dynamical and qKZ equations, see* (2.3)*,* (2.4)*, with values in* $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda}$*.*

Remark. The functions $\Psi_J(z;h;q)$ are called the *multidimensional q-hypergeometric solutions* of the dynamical equations. In [TV5], we constructed another type of solutions of the dynamical equations called the *multidimensional hypergeometric solutions*.

9. Equivariant quantum differential equations

9.1. **Partial flag varieties.** Let $\boldsymbol\lambda \in \mathbb{Z}^N_{\geqslant 0}$, $|\boldsymbol\lambda| = n$. Consider the partial flag variety $\mathcal{F}_\lambda$ parametrizing chains of subspaces

$$0 = F_0 \subset F_1 \subset \dots \subset F_N = \mathbb{C}^n$$

with $\dim F_i/F_{i-1} = \lambda_i$, $i = 1,\dots,N$. Denote by $T^*\mathcal{F}_\lambda$ the cotangent bundle of $\mathcal{F}_\lambda$ and

$$\mathcal{X}_n = \bigcup_{|\lambda|=n} T^*\mathcal{F}_\lambda .$$

Let $u_1,\dots,u_n$ be the standard basis of $\mathbb{C}^n$. For $I \in \mathcal{I}_\lambda$, let $x_I \in \mathcal{F}_\lambda$ be the point corresponding to the coordinate flag $F_1 \subset \dots \subset F_N$, where F_i is the span of the standard basis vectors $u_j \in \mathbb{C}^n$ with $j \in I_1 \cup \dots \cup I_i$. We embed $\mathcal{F}_\lambda$ in $T^*\mathcal{F}_\lambda$ as the zero section and consider the points x_I as points of $T^*\mathcal{F}_\lambda$.

9.2. **Equivariant cohomology.** Let $A \subset GL_n(\mathbb{C})$ be the torus of diagonal matrices and $T = A\times\mathbb{C}^\times$. The group A acts on $\mathbb{C}^n$ and hence on $T^*\mathcal{F}_\lambda$. Let the group $\mathbb{C}^\times$ act on $T^*\mathcal{F}_\lambda$ by multiplication in each fiber. We denote by $-h$ its $\mathbb{C}^\times$-weight.

We consider the equivariant cohomology algebras $H^*_T(T^*\mathcal{F}_\lambda;\mathbb{C})$ and

$$H^*_T(\mathcal{X}_n) = \bigoplus_{|\lambda|=n} H^*_T(T^*\mathcal{F}_\lambda;\mathbb{C}).$$

Denote by $\boldsymbol\Gamma_i = \{\gamma_{i,1},\dots,\gamma_{i,\lambda_i}\}$ the set of the Chern roots of the bundle over $\mathcal{F}_\lambda$ with fiber F_i/F_{i-1}. Let $\boldsymbol\Gamma = (\boldsymbol\Gamma_1;\dots;\boldsymbol\Gamma_N)$. Denote by $z = \{z_1,\dots,z_n\}$ the Chern roots corresponding to the factors of the torus A. Then

$$H_T^*(T^*\mathcal{F}_{\boldsymbol{\lambda}}) \,=\, \mathbb{C}[\boldsymbol{\Gamma}]^{S_{\lambda_1}\times\ldots\times S_{\lambda_N}} \otimes \mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}[h] \Big/ \Big\langle \prod_{i=1}^{N}\prod_{j=1}^{\lambda_i}(u-\gamma_{i,j}) \,=\, \prod_{a=1}^{n}(u-z_a)\Big\rangle . \tag{9.1}$$

The cohomology $H_T^*(T^*\mathcal{F}_\lambda)$ is a module over $H_T^*(pt;\mathbb{C}) = \mathbb{C}[z] \otimes \mathbb{C}[h]$.

Notice that

$$\prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{a=1}^{\lambda_i}\prod_{b=1}^{\lambda_j}(\gamma_{j,b}-\gamma_{i,a})(\gamma_{i,a}-\gamma_{j,b}-h) \tag{9.2}$$

is the equivariant total Chern class of the tangent bundle of $T^*\mathcal{F}_\lambda$ and

$$c_1(E_i) = \sum_{a=1}^{\lambda_i}\gamma_{i,a}\,, \qquad i = 1,\ldots,N\,, \tag{9.3}$$

is the equivariant first Chern class of the vector bundle E_i over $T^*\mathcal{F}_\lambda$ with fiber F_i/F_{i-1}.

For $i = 1,\ldots,N$, denote $\Theta_i = \{\theta_{i,1},\ldots,\theta_{i,\lambda^{(i)}}\}$ the Chern roots of the bundle $\mathcal{F}_i$ over $\mathcal{F}_\lambda$ with fiber F_i. Let $\boldsymbol{\Theta} = (\Theta_1,\ldots,\Theta_N)$. The relations

$$\prod_{a=1}^{\lambda^{(i)}}(u-\theta_{i,a}) = \prod_{j=1}^{i}\prod_{k=1}^{\lambda_j}(u-\gamma_{j,k}), \qquad i = 1,\ldots,N\,, \tag{9.4}$$

define the homomorphism

$$\mathbb{C}[\boldsymbol{\Theta}]^{S_{\lambda^{(1)}}\times\ldots\times S_{\lambda^{(N)}}} \otimes \mathbb{C}[\boldsymbol{z}] \otimes \mathbb{C}[h] \;\to\; H_T^*(T^*\mathcal{F}_{\boldsymbol{\lambda}}).$$

9.3. **Stable envelope map.** Recall the weight functions $\tilde W_I$ defined in Sections 3.1. Let $\check W_I(\boldsymbol{\Theta};z) \in H_T^*(T^*\mathcal{F}_\lambda)$ be the cohomology class represented by the polynomial $\check W_{\mathrm{id},I}(t;z)$ with the variables $t^{(i)}_a$ replaced by $\theta_{i,a}$ for all $i = 1,\ldots,N-1$, $a = 1,\ldots,\lambda^{(i)}$. Denote

$$c_\lambda(\boldsymbol{\Theta}) = \prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}}\prod_{b=1}^{\lambda^{(i)}}(\theta_{i,a}-\theta_{i,b}-h) \in H_T^*(T^*\mathcal{F}_\lambda).$$

Observe that $c_\lambda(\boldsymbol{\Theta})$ is the equivariant Euler class of the bundle $\bigoplus_{a=1}^{N-1}\mathrm{Hom}(\mathcal{F}_a,\mathcal{F}_a)$ if we make $\mathbb{C}^\times$ act on it with weight $-h$.

Theorem 9.1 ([RTV1, Theorem 4.1])**.** *For any λ and any $I \in \mathcal{I}_\lambda$, the cohomology class $\check W_I(\boldsymbol{\Theta};z) \in H^*_T(T^*\mathcal{F}_\lambda)$ is divisible by $c_\lambda(\boldsymbol{\Theta})$, that is, there exists a unique element* $\mathrm{Stab}_I \in H^*_T(T^*\mathcal{F}_\lambda)$ *such that*

$$[\check W_I(\boldsymbol{\Theta};z)] = c_\lambda(\boldsymbol{\Theta}) \cdot \mathrm{Stab}_I. \tag{9.5}$$

Define the *stable envelope map* by the rule

$$\mathrm{Stab} : (\mathbb{C}^N)^{\otimes n} \otimes \mathbb{C}[z] \otimes \mathbb{C}[h] \to H^*_T(\mathcal{X}_n), \qquad v_I \mapsto \mathrm{Stab}_I. \tag{9.6}$$

Remark. Stable envelope maps for Nakajima quiver varieties were introduced in [MO]. They were defined there geometrically in terms of the associated torus action. The map Stab given by formula (9.6) is the stable envelope map of [MO] for the Nakajima quiver variety $\mathcal{X}_n$, described in terms of the Chern roots $\boldsymbol{\Theta}, z, h$, see [RTV1].

Remark. After the substitution $h = 1$ the classes $\mathrm{Stab}_I \in H^*_T(T^*\mathcal{F}_\lambda)$ can be considered as elements of the equivariant cohomology $H^*_{(\mathbb{C}^\times)^n}(\mathcal{F}_\lambda)$ of the partial flag variety $\mathcal{F}_\lambda$ (and not of the cotangent bundle $T^*\mathcal{F}_\lambda$). These new classes are the equivariant Chern-SchwartzMacPherson classes (CSM classes) of the corresponding Schubert cells, see [RV].

Let $\mathbb{C}(z;h)$ be the algebra of rational functions in z, h. The map

$$\mathrm{Stab} : (\mathbb{C}^N)^{\otimes n} \otimes \mathbb{C}(z;h) \to H^*_T(\mathcal{X}_n) \otimes \mathbb{C}(z;h), \qquad v_I \mapsto \mathrm{Stab}_I, \tag{9.7}$$

is an isomorphism of $\mathbb{C}(z;h)$-modules by [RTV1, Lemma 6.7].

9.4. $H^*_T(T^*\mathcal{F}_\lambda)$**-valued weight function.** Define the $H^*_T(T^*\mathcal{F}_\lambda)$-valued function $\widehat W(t;\boldsymbol{\Gamma})$ as follows:

$$\widehat W(\boldsymbol{t};\boldsymbol{\Gamma}) = \mathrm{Sym}_{t^{(1)}_1,\dots,t^{(1)}_{\lambda^{(1)}}} \cdots \mathrm{Sym}_{t^{(N-1)}_1,\dots,t^{(N-1)}_{\lambda^{(N-1)}}} \widehat U(\boldsymbol{t};\boldsymbol{\Gamma}), \tag{9.8}$$

$$\widehat U(\boldsymbol{t};\boldsymbol{\Gamma}) = \prod_{j=1}^{N-1}\prod_{a=1}^{\lambda^{(j)}}\left(\prod_{c=1}^{a-1}(t^{(j)}_a - t^{(j+1)}_c - h)\prod_{d=a+1}^{\lambda^{(j+1)}}(t^{(j)}_a - t^{(j+1)}_d)\prod_{b=a+1}^{\lambda^{(j)}}\frac{t^{(j)}_a - t^{(j)}_b - h}{t^{(j)}_a - t^{(j)}_b}\right).$$

where $(t^{(N)}_1,\dots,t^{(N)}_n) = (\gamma_{1,1},\dots,\gamma_{1,\lambda_1},\gamma_{2,1},\dots,\gamma_{2,\lambda_2},\dots,\gamma_{N,1},\dots,\gamma_{N,\lambda_N})$, cf. formula (3.1) for $I = I^{\min} = (\{1,\dots,\lambda_1\},\dots,\{n-\lambda_N+1,\dots,n\})$.

Example. Let $N = 2$, $\lambda = (1,n-1)$. Then $\widehat W(\boldsymbol{t};\boldsymbol{\Gamma}) = \prod_{a=1}^{n-1}(t^{(1)}_1 - \gamma_{2,a})$.
Let $N = 3$, $\lambda = (1,1,1)$. Then

$$\begin{aligned}\widehat{W}(\boldsymbol{t};\boldsymbol{\Gamma}) \,=\, & (t_1^{(1)}-t_2^{(2)})\,(t_1^{(2)}-\gamma_{2,1})\,(t_1^{(2)}-\gamma_{3,1})\,(t_2^{(2)}-\gamma_{1,1}-h)\,(t_2^{(2)}-\gamma_{3,1})\,\frac{t_1^{(2)}-t_2^{(2)}-h}{t_1^{(2)}-t_2^{(2)}} \,+\\ & + (t_1^{(1)}-t_1^{(2)})\,(t_2^{(2)}-\gamma_{2,1})\,(t_2^{(2)}-\gamma_{3,1})\,(t_1^{(2)}-\gamma_{1,1}-h)\,(t_1^{(2)}-\gamma_{3,1})\,\frac{t_2^{(2)}-t_1^{(2)}-h}{t_2^{(2)}-t_1^{(2)}}\,.\end{aligned}$$

Define

$$Q(\boldsymbol{\Gamma}) = \prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{a=1}^{\lambda_i}\prod_{b=1}^{\lambda_j}(\gamma_{i,a}-\gamma_{j,b}-h) \in H_T^*(T^*\mathcal{F}_\lambda). \tag{9.9}$$

The image of the $(\mathbb{C}^N)^{\otimes n}_\lambda$-valued weight function $W_\lambda(t;z)$, see (7.4), is given by the next proposition.

Proposition 9.2. *We have*

$$\sum_{I\in\mathcal{I}_\lambda} W_I(t;z)\,\mathrm{Stab}_{\mathrm{id},I} = Q(\boldsymbol{\Gamma})\,\widehat{W}(t;\boldsymbol{\Gamma}). \tag{9.10}$$

Proof. Recall that $W_I(t;z) = (-h)^{-\lambda^{\{1\}}}\,\check{W}_{\sigma_0,I}(t;z)$, see (3.4). Recall the discrete cycle Σ_J given by (8.4). Let $\sigma^I\in S_n$ be a permutation such that $\sigma^I(I^{\min}) = I$. Then formula (9.10) is equivalent to the following equality

$$\sum_{I\in\mathcal{I}_\lambda} \check{W}_{\sigma_0,I}(t;z)\,\check{W}_I(\Sigma_J;z) = c_\lambda(\Sigma_J)\,Q(z_J)\,\check{W}_{\sigma_J,J}(t;z). \tag{9.11}$$

For the proof of formula (9.11), consider the function

$$Z(t;\tilde t;z) = \sum_{I\in\mathcal{I}_\lambda} \check{W}_{\sigma_0,I}(t;z)\,\check{W}_I(\tilde t;z).$$

Here $\tilde t$ is an additional set of variables similar to t. Then formula (9.11) reads

$$Z(t;\Sigma_J;z) = c_\lambda(\Sigma_J)\,Q(z_J)\,\check{W}_{\sigma_J,J}(t;z). \tag{9.12}$$

Three-term relations (3.3) imply that for any $\sigma\in S_n$, we have

$$Z(t;\tilde t;z) = \sum \check{W}_{\sigma,I}(t;z)\,\check{W}_{\sigma\sigma_0,I}(\tilde t;z). \tag{9.13}$$

$I\in\mathcal{I}_\lambda$

By [RTV1, Lemma 3.2], we have $\check W_{\sigma_J\sigma_0,I}(\Sigma_J,z) = c_\lambda(z_J)\,Q(z_J)\,\delta_{I,J}$. Thus taking $\sigma=\sigma_J$, $\tilde t=\Sigma_J$ in formula (9.13), we get equality (9.12). Proposition 9.2 is proved. Define the cohomology classes

$$R(\boldsymbol{\Gamma}) = \prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{a=1}^{\lambda_i}\prod_{b=1}^{\lambda_j}(\gamma_{i,a}-\gamma_{j,b}) \tag{9.14}$$

and

$$R_I(\boldsymbol{\Gamma};z) = \prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{a=1}^{\lambda_i}\prod_{b\in I_j}(\gamma_{i,a}-z_b), \qquad I\in\mathcal{I}_\lambda . \tag{9.15}$$

Notice that $R_I(z_J;z)=R(z_J)\,\delta_{I,J}$.

Proposition 9.3. *For any $K\in\mathcal{I}_\lambda$, we have*

$$\sum_{I\in\mathcal{I}_\lambda} W_I(\Sigma_K;z)\,\mathrm{Stab}_{\mathrm{id},I} = (-h)^{-\lambda^{\{1\}}}c_\lambda(\boldsymbol{\Theta})\,R_K(\boldsymbol{\Gamma};z)\,Q(\boldsymbol{\Gamma}). \tag{9.16}$$

Proof. Formula (9.16) is equivalent to the equality

$$\sum_{I\in\mathcal{I}_\lambda}\check W_{\sigma_0,I}(\Sigma_K;z)\,\check W_I(\Sigma_J;z) = \big(c_\lambda(\Sigma_J)\big)^2 R(z_J)\,Q(z_J)\,\delta_{J,K}. \tag{9.17}$$

By [RTV1, Lemma 3.2], we have $\check W_{\sigma_J,J}(\Sigma_K,z;h) = c_\lambda(z_J)R(z_J)\delta_{J,K}$. Thus taking $\tilde t=\Sigma_K$ in formula (9.11), we get equality (9.17).

Formula (9.17) also follows from [RTV1, Lemma 3.4]. Proposition 9.3 is proved.

9.5. **Quantum multiplication by divisors on $H_T^*(T^*\mathcal{F}_\lambda)$.** The quantum multiplication by divisors on $H_T^*(T^*\mathcal{F}_\lambda)$ is described in [MO]. The fundamental equivariant cohomology classes of divisors on $T^*\mathcal{F}_\lambda$ are linear combinations of $D_i=\gamma_{i,1}+\dots+\gamma_{i,\lambda_i}$, $i=1,\dots,N$.

The quantum multiplication $D_i*_{\tilde q}: H_T^*(T^*\mathcal{F}_\lambda)\to H_T^*(T^*\mathcal{F}_\lambda)$ by the divisor D_i depends on parameters $\tilde q=(\tilde q_1,\dots,\tilde q_N)\in(\mathbb{C}^\times)^N$ and is given in [MO, Theorem 10.2.1].

The quantum connection $\nabla^{\mathrm{quant}}_{\lambda,\tilde q,\tilde\kappa}$ on $H_T^*(T^*\mathcal{F}_\lambda)$ is defined by the formula

$$\nabla^{\mathrm{quant}}_{\lambda,\tilde q,\tilde\kappa,i} = \tilde\kappa\,\tilde q_i\frac{\partial}{\partial\tilde q_i} - D_i*_{\tilde q}, \qquad i=1,\dots,N,$$

where $\tilde\kappa \in \mathbb{C}^\times$ is the parameter of the connection, see [BMO]. The system of equations for flat sections of the quantum connection is called the system of the *equivariant quantum differential equations*.

The isomorphism Stab allows us to compare the operators $\nabla_{\boldsymbol\lambda,\boldsymbol q,\kappa,i} := \nabla_{\boldsymbol q,\kappa,i}\big|_{(\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda}}$ of the dynamical connection on $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda}$, see (2.2), and the operators $\nabla^{\mathrm{quant}}_{\lambda,\tilde q,\tilde\kappa,i}$ of the quantum connection on $H^*_T(T^*\mathcal{F}_\lambda)$.

Recall the dynamical Hamiltonians $X_i(z;h;q)$, see (2.1). Define the modified dynamical Hamiltonians

$$X^-_{\boldsymbol\lambda,i}(\boldsymbol z;h;\boldsymbol q) = \tag{9.18}$$
$$= X_i(\boldsymbol z;h;\boldsymbol q)\big|_{(\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda}} - h\sum_{j=1}^{i-1}\frac{q_i}{q_i-q_j}\,\min(\lambda_i,\lambda_j) - h\sum_{j=i+1}^{N}\frac{q_j}{q_i-q_j}\,\min(\lambda_i,\lambda_j).$$

The modified dynamical connection on $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda}$ is

$$\nabla^-_{\boldsymbol\lambda,\boldsymbol q,\kappa,i} = \kappa\, q_i\frac{\partial}{\partial q_i} - X^-_{\boldsymbol\lambda,i}(\boldsymbol z;h;\boldsymbol q)\,, \qquad i=1,\dots,N. \tag{9.19}$$

see [GRTV, Section 3.4]. Recall that $h_{\mathrm{GRTV}} = -h$.

Theorem 9.4 ([RTV1, Corollary 7.6]). *The isomorphism* Stab *identifies the operators $D_i *_{\tilde q}$ of quantum multiplication by D_i on $H^*_T(T^*\mathcal{F}_\lambda)$ with the action of the modified dynamical Hamiltonians $X^-_{\boldsymbol\lambda,i}(\boldsymbol z;h;\tilde{\boldsymbol q}^{-1})$ on $(\mathbb{C}^N)^{\otimes n}_{\boldsymbol\lambda}$, where $\tilde q^{-1} = (\tilde q_1^{-1},\dots,\tilde q_N^{-1})$. Consequently, the differential operators $\nabla^{\mathrm{quant}}_{\lambda,\tilde q,\tilde\kappa,i}$ are identified with the differential operators $\nabla^-_{\lambda,\tilde q^{-1},-\kappa,i}$.*

See also [RTV1, Theorem 7.5].

Set

$$\widehat\Omega_\lambda(\tilde q;\tilde\kappa) = \prod_{i=1}^{N-1}\prod_{j=i+1}^{N}(1-\tilde q_i/\tilde q_j)^{h\min(0,\lambda_j-\lambda_i)/\tilde\kappa}. \tag{9.20}$$

Set $\lambda^{\{2\}} = \sum_{1\leqslant i<j\leqslant N}\lambda_i\lambda_j$. For any $I\in\mathcal{I}_\lambda$, define

$$\widehat\Psi_I(\boldsymbol z;h;\tilde{\boldsymbol q};\tilde\kappa) = \tilde\kappa^{-\lambda^{\{1\}}-2\lambda_{\{2\}}}(-1)^{\lambda^{\{1\}}+\lambda_{\{2\}}}\big(\Gamma(-h/\tilde\kappa)\big)^{-\lambda^{\{1\}}}\,\widetilde\Psi_I(\boldsymbol z;h;\tilde{\boldsymbol q};\tilde\kappa), \tag{9.21}$$
$$\widetilde\Psi_I(\boldsymbol z;h;\tilde{\boldsymbol q};\tilde\kappa) = \widehat\Omega_{\boldsymbol\lambda}(\tilde{\boldsymbol q};\tilde\kappa)\sum_{\boldsymbol r\in\mathbb{Z}^{\lambda^{\{1\}}}_{\geqslant 0}}\operatorname{Res}_{t=\Sigma_I+\boldsymbol r\tilde\kappa}\Phi_\lambda(t;z;\tilde q^{-1};-\tilde\kappa)\big)\,Q(\boldsymbol\Gamma)\,\widehat W(\Sigma_I+\boldsymbol r\tilde\kappa;\boldsymbol\Gamma).$$

$\widehat\Psi_I(\boldsymbol z;h;\tilde{\boldsymbol q};\tilde\kappa)$ belongs to the extension of $H^*_T(T^*\mathcal{F}_\lambda)$ by functions in $z,h,\tilde q,\tilde\kappa$ holomorphic on the domain $L\subset\mathbb{C}^n\times\mathbb{C}\times\mathbb{C}^N$ such that

$$h\notin\tilde\kappa\mathbb{Z}_{>0}, \qquad z_a-z_b\notin\tilde\kappa\mathbb{Z}, \qquad z_a-z_b+h\notin\tilde\kappa\mathbb{Z}, \tag{9.22}$$

for all $a,b = 1,\dots,n$, $a \neq b$,

$$|\tilde q_i/\tilde q_{i+1}| < 1, \qquad i = 1,\dots,N-1,$$

and a branch of $\log \tilde q_i$ is fixed for each $i = 1,\dots,N$.

Example. Let $N = 2$, $n = 2$, $\lambda = (1,1)$. Recall the Gauss hypergeometric series

$$ {}_2F_1(a,b;c;x) = \sum_{m=0}^{\infty} \frac{(a)_m\,(b)_m}{(c)_m}\,\frac{x^m}{m!},$$

where $(u)_m = u(u-1)\dots(u-m+1)$. Set

$$F(z_1,z_2;h;\tilde\kappa;x) = {}_2F_1\Big(-\frac{h}{\tilde\kappa}, \frac{z_1-z_2-h}{\tilde\kappa}; 1+\frac{z_1-z_2}{\tilde\kappa}; x\Big)$$

and

$$F'(z_1,z_2;h;\tilde\kappa;x) = \frac{\partial F}{\partial x}(z_1,z_2;h;\tilde\kappa;x).$$

Then

$$\begin{aligned}&\widehat\Psi_{(\{1\},\{2\})}(z_1,z_2;h;\tilde q_1,\tilde q_2;\tilde\kappa) =\\ &= \tilde\kappa^{-2}\,(e^{-\pi\sqrt{-1}}\,\tilde q_1)^{z_1/\tilde\kappa}\,(e^{-\pi\sqrt{-1}}\,\tilde q_2)^{z_2/\tilde\kappa}\,\Gamma\Big(\frac{z_2-z_1}{\tilde\kappa}\Big)\,\Gamma\Big(\frac{z_1-z_2-h}{\tilde\kappa}\Big)\\ &\quad\times(\gamma_{1,1}-\gamma_{2,1}-h)\big((\gamma_{2,1}-z_1)\,F(z_1,z_2;h;\tilde\kappa;\tilde q_1/\tilde q_2) - \tilde\kappa\,(\tilde q_1/\tilde q_2)\,F'(z_1,z_2;h;\tilde\kappa;\tilde q_1/\tilde q_2)\big)\end{aligned}$$

and $\widehat\Psi_{(\{2\},\{1\})}(z_1,z_2;h;\tilde q_1,\tilde q_2;\tilde\kappa) = \widehat\Psi_{(\{1\},\{2\})}(z_2,z_1;h;\tilde q_1,\tilde q_2;\tilde\kappa)$.

Theorem 9.5. *The collection of functions* $(\widehat\Psi_I(\boldsymbol z;h;\tilde{\boldsymbol q};\tilde\kappa))_{I\subset\mathcal I_\lambda}$ *is a basis of solutions of both the quantum differential equations* $\nabla^{\mathrm{quant}}_{\lambda,\tilde q,\tilde\kappa,i} f = 0$, $i = 1,\dots,N$, *and the associated qKZ difference equations.*

Proof. The statement follows from Theorems 8.4, 8.5, 9.4, and Proposition 9.2, see Corollary 8.6.

Remark. The integral representations for solutions of the equivariant quantum differential equations is a manifestation of a version of mirror symmetry. The basis of solutions given by Theorem 9.5 is an analog of Givental's *J*-function.

For $\tilde q_i/\tilde q_{i+1} \to 0$ for all $i = 1,\dots,N-1$, the leading term of the asymptotics of $\widehat\Psi_I(z;h;\tilde q;\tilde\kappa)$ is given by taking the residue at $t = \Sigma_I$.

Theorem 9.6. *Assume that* $\tilde q_i/\tilde q_{i+1} \to 0$ *for all* $i = 1,\dots,N-1$*. Then*

$$\widehat{\Psi}_I(\boldsymbol{z};h;\tilde{\boldsymbol{q}};\tilde\kappa) = \prod_{i=1}^{N}\left(e^{\pi\sqrt{-1}\,(\lambda_i-n)}\,\tilde q_i\right)^{\sum_{a\in I_i} z_a/\tilde\kappa}$$
$$\times \prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{a\in I_i}\prod_{b\in I_j}\Gamma\Big(1+\frac{z_b-z_a}{\tilde\kappa}\Big)\,\Gamma\Big(1+\frac{z_a-z_b-h}{\tilde\kappa}\Big)$$
$$\times\Big(\Delta_I + \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^{N-1}_{\geq 0}\\ \boldsymbol{m}\neq 0}} \widehat{\Psi}_{I,\boldsymbol{m}}(\boldsymbol{z};h;\tilde\kappa)\prod_{i=1}^{N-1}\Big(\frac{\tilde q_i}{\tilde q_{i+1}}\Big)^{m_i}\Big)\,, \tag{9.23}$$

where $\Delta_I(\boldsymbol{\Gamma},z) = R_I(\boldsymbol{\Gamma};z)/R(z_I)$ *is the cohomology class such that* $\Delta_I(z_J;z) = \delta_{I,J}$, *and the classes* $\widehat{\Psi}_{I,\boldsymbol{m}}(\boldsymbol{z};h;\tilde\kappa)$ *are rational functions in* $z,h,\tilde\kappa$, *regular on the domain* $h \notin \tilde\kappa\mathbb{Z}_{>0}$, $z_a - z_b \notin \tilde\kappa\mathbb{Z}$, $z_a - z_b + h \notin \tilde\kappa\mathbb{Z}$, *for all* $a,b = 1,\dots,n$, $a \neq b$.

Proof. The statement follows from formula (9.21) and Propositions 9.2, 9.3.

Example. Let $N = 2$, $n = 2$, $\lambda = (1,1)$. As $\tilde q_1/\tilde q_2 \to 0$, the leading term of the solution $\widehat{\Psi}_{(\{1\},\{2\})}(z_1,z_2;h;\tilde q_1,\tilde q_2;\tilde\kappa)$ is the cohomology class

$$(e^{-\pi\sqrt{-1}}\,\tilde q_1)^{z_1/\tilde\kappa}\,(e^{-\pi\sqrt{-1}}\,\tilde q_2)^{z_2/\tilde\kappa}\,\Gamma\Big(1+\frac{z_2-z_1}{\tilde\kappa}\Big)\,\Gamma\Big(1+\frac{z_1-z_2-h}{\tilde\kappa}\Big)\,\Delta_{(\{1\},\{2\})}$$

and the leading term of the solution $\widehat{\Psi}_{(\{2\},\{1\})}(z_1,z_2;h;\tilde q_1,\tilde q_2;\tilde\kappa)$ is the cohomology class

$$(e^{-\pi\sqrt{-1}}\,\tilde q_1)^{z_2/\tilde\kappa}\,(e^{-\pi\sqrt{-1}}\,\tilde q_2)^{z_1/\tilde\kappa}\,\Gamma\Big(1+\frac{z_1-z_2}{\tilde\kappa}\Big)\,\Gamma\Big(1+\frac{z_2-z_1-h}{\tilde\kappa}\Big)\,\Delta_{(\{2\},\{1\})}.$$

10. Quantum Pieri rules

10.1. **Quantum equivariant cohomology algebra** $\mathcal{H}^{\tilde q}_T(T^*\mathcal{F}_\lambda)$**.** Let $\tilde q = (\tilde q_1,\dots,\tilde q_N) \in (\mathbb{C}^\times)^N$ have distinct coordinates. The *quantum equivariant cohomology algebra* $\mathcal{H}^{\tilde q}_T(T^*\mathcal{F}_\lambda)$ is the algebra generated by the operators $D_i *_{\tilde q} : H^*_T(T^*\mathcal{F}_\lambda) \to H^*_T(T^*\mathcal{F}_\lambda)$ of quantum multiplication by the divisors D_i, $i = 1,\dots,N$, see details in [MO, GRTV]. The algebra can be defined by generators and relations as follows.

Introduce the variables $\tilde\gamma_{i,1},\dots,\tilde\gamma_{i,\lambda_i}$ for $i = 1,\dots,N$. Set

$$W^{\tilde q}(u) = \det\Big(\tilde q_i^{\,j-1}\prod_{k=1}^{\lambda_i}\big(u-\tilde\gamma_{i,k}-h(i-j)\big)\Big)_{i,j=1}^{N}. \tag{10.1}$$

Theorem 10.1. *The quantum equivariant cohomology algebra* $\mathcal{H}^{\tilde q}_T(T^*\mathcal{F}_\lambda)$ *is isomorphic to the algebra*

$$\mathbb{C}[\tilde{\boldsymbol{\Gamma}}]^{S_{\lambda_1}\times\dots\times S_{\lambda_N}}\otimes\mathbb{C}[z]\otimes\mathbb{C}[h]\Big/\Big\langle W^{\tilde q}(u) = \prod_{1\leq i<j\leq N}(\tilde q_j-\tilde q_i)\prod_{a=1}^{n}(u-z_a)\Big\rangle \tag{10.2}$$

$$D_i *_{\tilde q} \mapsto \Big[\sum_{k=1}^{\lambda_i} \tilde\gamma_{i,k} - h\sum_{j=1}^{i-1} \frac{\tilde q_j}{\tilde q_j - \tilde q_i}\min(0,\lambda_j-\lambda_i) - h\sum_{j=i+1}^{N} \frac{\tilde q_i}{\tilde q_j - \tilde q_i}\min(0,\lambda_j-\lambda_i)\Big]$$

where $\tilde\Gamma = (\tilde\gamma_{1,1},\dots,\tilde\gamma_{1,\lambda_1},\dots,\tilde\gamma_{N,1},\dots,\tilde\gamma_{N,\lambda_N})$, *and the correspondence is*

.

This theorem follows from [GRTV, Theorems 6.5, 7.10, and Lemma 6.10], see also [MTV2]. Notice that the parameters in this paper and in [GRTV] are related as follows: $h = -h_{\mathrm{GRTV}}$, $\tilde q_i = q_i^{-1}$, $i = 1,\dots,N$.

Example. Let $N = 2$, $n = 2$, $\lambda = (1,1)$. Then $D_i *_{\tilde q} \mapsto \tilde\gamma_{i,1}$, $i = 1,2$, and the relations are

(10.3) $\quad \tilde\gamma_{1,1} + \tilde\gamma_{2,1} = z_1 + z_2, \qquad \tilde\gamma_{1,1}\tilde\gamma_{2,1} - h\,\dfrac{\tilde q_1}{\tilde q_2 - \tilde q_1}(\tilde\gamma_{1,1} - \tilde\gamma_{2,1} - h) = z_1z_2$.

It is easy to see that the algebra $H_T^{\tilde q}(T^*\mathcal F_\lambda)$ does not change if all $\tilde q_1,\dots,\tilde q_N$ are multiplied by the same number. In the limit $\tilde q_i/\tilde q_{i+1} \to 0$, $i = 1,\dots,N-1$, the relations in $H_T^{\tilde q}(T^*\mathcal F_\lambda)$ turn into the relations in $H^*_T(T^*\mathcal F_\lambda)$, see formula (9.1).

10.2. **Quantum equivariant Pieri rules.** Recall the weight functions $W_I(t;z)$, see (3.5).

Introduce the variables $\tilde\Theta_i = \{\tilde\theta_{i,1},\dots,\tilde\theta_{i,\lambda^{(i)}}\}$, $\tilde\Theta = (\tilde\Theta_1,\dots,\tilde\Theta_N)$. Let $W_I(\tilde\Theta;z)$ be the polynomial $W_I(t;z)$ with the variables $t^{(i)}_a$ replaced by $\tilde\theta_{i,a}$ for all $i = 1,\dots,N-1$, $a = 1,\dots,\lambda^{(i)}$. For any $m = 1,\dots,N-1$, the relation

(10.4) $$\det\Big(\tilde q_i^{\,j-1}\prod_{k=1}^{\lambda_i}\big(u - \tilde\gamma_{i,k} - h(i-j)\big)\Big)_{i,j=1}^{m} = \prod_{1\leqslant i<j\leqslant m}(\tilde q_j - \tilde q_i)\prod_{a=1}^{\lambda^{(m)}}(u - \tilde\theta_{m,a})$$

allows us to express the elementary symmetric functions in the variables $\tilde\Theta_m$ in terms of the elementary symmetric functions in the variables $\tilde\Gamma_i = (\tilde\gamma_{i,1},\dots,\tilde\gamma_{i,\lambda_i})$ with $i = 1,\dots,m$. For example,

(10.5) $$\sum_{a=1}^{\lambda^{(m)}}\tilde\theta_{\ell,a} = \sum_{i=1}^{m}\sum_{k=1}^{\lambda_i}\tilde\gamma_{i,k} - h\sum_{1\leqslant i<j\leqslant m}(\lambda_i-\lambda_j)\frac{\tilde q_i}{\tilde q_i - \tilde q_j}, \qquad m = 1,\dots,N.$$

Relations (10.4) define a homomorphism

$$\mathbb C[\tilde\Theta]^{S_{\lambda^{(1)}}\times\dots\times S_{\lambda^{(N)}}}\otimes\mathbb C[z]\otimes\mathbb C[h] \to H_T^{\tilde q}(T^*\mathcal F_\lambda).$$

Let $\{W_I\} \in H_T^{\tilde q}(T^*\mathcal F_\lambda)$ be the cohomology class represented by the image of $W_I(\tilde\Theta;z)$.

Theorem 10.2. *For any $i = 1,\dots,N$ and $I \in \mathcal{I}_\lambda$, the following relation in $\mathcal{H}^{\tilde q}_T(T^*\mathcal{F}_\lambda)$ holds:*

$$\left(\sum_{k=1}^{\lambda_i} \tilde\gamma_{i,k} - \sum_{a\in I_i} z_a\right)\{W_I\} = h\,\lambda_i\left(\sum_{j=1}^{i-1}\frac{\tilde q_j}{\tilde q_i-\tilde q_j} + \sum_{j=i+1}^{N}\frac{\tilde q_i}{\tilde q_i-\tilde q_j}\right)\{W_I\} + \tag{10.6}$$
$$- h\sum_{\substack{j=1\\ j\neq i}}^{N}\sum_{m_1=1}^{\lambda_i}\left(\sum_{\substack{m_2=1\\ \ell_{i,m_1}<\ell_{j,m_2}}}^{\lambda_j}\{W_{I_{i,j;m_1,m_2}}\} - \frac{\tilde q_i}{\tilde q_i-\tilde q_j}\sum_{m_2=1}^{\lambda_j}\{W_{I_{i,j;m_1,m_2}}\}\right),$$

where ℓ_{i,m_1}, ℓ_{j,m_2}, $I_{i,j;m_1,m_2}$ are defined in Section 4.5.

Theorem 10.2 is proved in Section 10.4.

Example. Let $N = 2$, $n = 2$, $\lambda = (1,1)$. Then

$$\{W_{(\{1\},\{2\})}\} = \tilde\gamma_{1,1} - z_2 - h, \qquad \{W_{(\{2\},\{1\})}\} = \tilde\gamma_{1,1} - z_1,$$

and the quantum Pieri rules take the form

$$(\tilde\gamma_{1,1}-z_1)\{W_{(\{1\},\{2\})}\} = h\,\frac{\tilde q_1}{\tilde q_1-\tilde q_2}\left(\{W_{(\{1\},\{2\})}\}+\{W_{(\{2\},\{1\})}\}\right) - h\{W_{(\{2\},\{1\})}\}, \tag{10.7}$$
$$(\tilde\gamma_{1,1}-z_2)\{W_{(\{2\},\{1\})}\} = h\,\frac{\tilde q_1}{\tilde q_1-\tilde q_2}\left(\{W_{(\{2\},\{1\})}\}+\{W_{(\{1\},\{2\})}\}\right).$$

These are the same relations as in formula (10.3).

10.3. **Bethe ansatz equations.** The Bethe ansatz equations are the following system of algebraic equations with respect to the variables t:

$$\prod_{k=1}^{\lambda^{(i-1)}}\frac{t^{(i-1)}_k-t^{(i)}_j-h}{t^{(i-1)}_k-t^{(i)}_j}\prod_{k=1}^{\lambda^{(i+1)}}\frac{t^{(i)}_j-t^{(i+1)}_k}{t^{(i)}_j-t^{(i+1)}_k-h}\prod_{\substack{k=1\\k\neq j}}^{\lambda^{(i)}}\frac{t^{(i)}_j-t^{(i)}_k-h}{t^{(i)}_j-t^{(i)}_k+h} = \frac{q_{i+1}}{q_i}, \tag{10.8}$$

for $i = 1,\dots,N-1$, $j = 1,\dots,\lambda^{(i)}$. This system can be reformulated as the system of equations:

$$\lim_{\kappa\to 0}\frac{\Phi_{\boldsymbol\lambda}(\dots,t^{(i)}_j+\kappa,\dots;\boldsymbol z;h;\boldsymbol q)}{\Phi_\lambda(t;z;h;q)} = 1, \qquad i = 1,\dots,N-1, \qquad j = 1,\dots,\lambda^{(i)}, \tag{10.9}$$

see [TV1, MTV1].

Lemma 10.3. *For $I \in \mathcal{I}_\lambda$ and $i = 1,\dots,N-1$, let $D_{I,i}(t;z;h;q)$ be the function defined in (7.1). Let $\check t$ be a solution of the Bethe ansatz equations (10.8). Then $D_{I,i}(\check t;z;h;q) = 0$ and the right-hand side of formula (7.2) equals zero at $t = \check t$.*

Proof. If $\check t$ is a solution of equations (10.8), then the second of the two factors in the righthand side of formula (4.3) equals zero at $t = \check t$.

10.4. **Proof of Theorem 10.2.** We have the following theorem.

Theorem 10.4. *Let* $\check t$ *be a solution of the Bethe ansatz equations* (10.8)*. Then there exist unique polynomials* $\prod_{k=1}^{\lambda_i}(u-\tilde\gamma_{i,k}) \in \mathbb{C}[u]$, $i=1,\dots,N$*, such that*

(10.10) $$\det\Big(q_i^{m-j}\prod_{k=1}^{\lambda_i}\big(u-\tilde\gamma_{i,k}-h(i-j)\big)\Big)_{i,j=1}^{m} = \prod_{1\leqslant i<j\leqslant m}(q_i-q_j)\prod_{a=1}^{\lambda^{(m)}}(u-\tilde t_a^{(i)})$$

for $i=1,\dots,N-1$*, and*

(10.11) $$\det\Big(q_i^{N-j}\prod_{k=1}^{\lambda_i}\big(u-\tilde\gamma_{i,k}-h(i-j)\big)\Big)_{i,j=1}^{N} = \prod_{1\leqslant i<j\leqslant N}(q_i-q_j)\prod_{a=1}^{N}(u-z_a).$$

This is [MTV2, Theorem 7.2], which is [MV2, Proposition 7.6], which in its turn is a generalization of [MV1, Lemma 4.8].

Proof of Theorem 10.2. Formula (10.6) is obtained from formula (7.2) by several substitutions. First take $q_i=\tilde q_i^{-1}$ for all $i=1,\dots,N$, substitute the variables $t_j^{(i)}$ by $\tilde\theta_{i,j}$, and replace the term $D_{I,i}$ by zero. Then write symmetric functions in the variables $\boldsymbol{\Theta}_m$ via symmetric functions in the variables $\tilde{\boldsymbol{\Gamma}}_i$, $i=1,\dots,m$. As a result, the expression $\sum_{j=1}^{\lambda^{(i)}} t_j^{(i)} - \sum_{j=1}^{\lambda^{(i-1)}} t_j^{(i-1)}$ becomes $\sum_{j=1}^{\lambda_i}\tilde\gamma_{i,j} - h\sum_{j=1}^{i-1}(\lambda_i-\lambda_j)\,\tilde q_j/(\tilde q_i-\tilde q_j)$ according to formula (10.5).

Lemma 10.3 and Theorem 10.4 mean that formula (10.6) holds for those values of $\boldsymbol{\Gamma}_1,\dots,\boldsymbol{\Gamma}_{N-1}$ that come from solutions $\check t$ of the Bethe ansatz equations (10.8). By [MTV2, Theorem 7.3] of completeness of the Bethe ansatz, such values of $\boldsymbol{\Gamma}_1,\dots,\boldsymbol{\Gamma}_{N-1}$ form a Zariski open subset of all values of $\ddot{\boldsymbol{\Gamma}}_1,\dots,\ddot{\boldsymbol{\Gamma}}_{N-1}$ satisfying defining relations of the algebra $H_T^{\tilde q}(T^*\mathcal{F}_\lambda)$, see (10.2). This proves Theorem 10.2.

10.5. **Limit** $\tilde q_i/\tilde q_{i+1}\to 0$**,** $i=1,\dots,N-1$**, and CSM classes of Schubert cells.** In the limit $\tilde q_i/\tilde q_{i+1}\to 0$ for all $i=1,\dots,N-1$, the algebra $\mathcal{H}_T^{\tilde q}(T^*\mathcal{F}_\lambda)$ turns into the algebra $H_T^*(T^*\mathcal{F}_\lambda)$ and the classes $\{W_I\}\in H_T^{\tilde q}(T^*\mathcal{F}_\lambda)$ become the classes $[W_I]\in H_T^*(T^*\mathcal{F}_\lambda)$. Then formula (10.6) takes the form

(10.12) $$\Big(\sum_{k=1}^{\lambda_i}\gamma_{i,k}-\sum_{a\in I_i}z_a\Big)[W_I] = h\sum_{j=1}^{i-1}\sum_{m_1=1}^{\lambda_i}\sum_{\substack{m_2=1\\ \ell_{i,m_1}>\ell_{j,m_2}}}^{\lambda_j}[W_{I_{i,j;m_1,m_2}}] - h\sum_{j=i+1}^{N}\sum_{m_1=1}^{\lambda_i}\sum_{\substack{m_2=1\\ \ell_{i,m_1}<\ell_{j,m_2}}}^{\lambda_j}[W_{I_{i,j;m_1,m_2}}].$$

In particular, identities in (10.7) turn into the identities

(10.13) $(\gamma_{1,1} - z_1)[W_{(\{1\},\{2\})}] = -h[W_{(\{2\},\{1\})}], \qquad (\gamma_{1,1} - z_2)[W_{(\{2\},\{1\})}] = 0.$

Remark. After the substitution $h = 1$, the classes $[W_I] \in H^*_T(T^*\mathcal{F}_\lambda)$ can be considered as elements of the equivariant cohomology $H^*_{(\mathbb{C}^\times)^n}(\mathcal{F}_\lambda)$. By [RV] these new classes $[W_I]_{h=1}$ are proportional to the CSM classes κ_I of the corresponding Schubert cells with the coefficient of proportionality independent of the index I. Hence formula (10.12) induces the equivariant Pieri rules for the equivariant CSM classes:

(10.14)
$$= \Big(\sum_{k=1}^{\lambda_i} \gamma_{i,k} - \sum_{a\in I_i} z_a\Big)\kappa_I$$
$$= h\sum_{j=1}^{i-1}\sum_{m_1=1}^{\lambda_i}\sum_{\substack{m_2=1\\ \ell_{i,m_1}>\ell_{j,m_2}}}^{\lambda_j} \kappa_{I_{i,j;m_1,m_2}} - h\sum_{j=i+1}^{N}\sum_{m_1=1}^{\lambda_i}\sum_{\substack{m_2=1\\ \ell_{i,m_1}<\ell_{j,m_2}}}^{\lambda_j} \kappa_{I_{i,j;m_1,m_2}},$$

see detailed definitions of the CSM classes in [RV].

11. Solutions of quantum differential equations and equivariant K-theory

11.1. **Solutions and equivariant K-theory.** Introduce more variables: $y = e^{2\pi\sqrt{-1}h/\tilde\kappa}$, $\acute t^{(i)}_j = e^{2\pi\sqrt{-1}\,t^{(i)}_j/\tilde\kappa}$, $\acute z_i = e^{2\pi\sqrt{-1}\,z_i/\tilde\kappa}$, $\acute\gamma_{i,j} = e^{2\pi\sqrt{-1}\,\gamma_{i,j}/\tilde\kappa}$, etc. We will use the acute superscript also for the corresponding collections of those variables like $\acute{\boldsymbol\Gamma}$, $\acute t, \acute z$. We will write $\acute{\boldsymbol\Gamma}^{\pm1}$, $\acute t^{\pm1}$, $\acute z^{\pm1}$ for the collections extended by the inverse variables, for instance, $\acute z^{\pm1} = (\acute z_1^{\pm1},\dots,\acute z_n^{\pm1})$.

Let P be a Laurent polynomial in the variables $\acute t, \acute z, y$, symmetric in $\acute t^{(i)}_1,\dots,\acute t^{(i)}_{\lambda^{(i)}}$ for each $i = 1,\dots,N-1$. Define

(11.1)
$$\widehat\Psi_P(z;h;\tilde q;\tilde\kappa) = \sum_{I\in\mathcal{I}_\lambda} P(\acute\Sigma_I,\acute z,y)\,\widehat\Psi_I(z;h;\tilde q;\tilde\kappa),$$

where $\widehat\Psi_I(\boldsymbol z;h;\tilde{\boldsymbol q};\tilde\kappa)$ are given by (9.21).

Lemma 11.1. *The function $\widehat\Psi_P(\boldsymbol z;h;\tilde{\boldsymbol q};\tilde\kappa)$ is a solution of both the quantum differential equations $\nabla^{\mathrm{quant}}_{\boldsymbol\lambda,\tilde{\boldsymbol q},\tilde\kappa,i} f = 0$, $i = 1,\dots,N$, and the associated qKZ difference equations.*

Proof. The statement follows from Theorem 9.5.

Lemma 11.2. *For a Laurent polynomial P in t',z',y symmetric in $t^{(i)}_1,\dots,t^{(i)}_{\lambda^{(i)}}$ for each $i = 1,\dots,N-1$, the function $\widehat{\Psi}_P(\boldsymbol{z};h;\bar{\boldsymbol{q}};\tilde\kappa)$ is holomorphic in $\tilde q$ on the domain $L^{00}\subset\mathbb{C}^N$ such that $|\tilde q_i/\tilde q_{i+1}| < 1$, $i = 1,\dots,N-1$, and a branch of $\log\tilde q_i$ is fixed for each $i = 1,\dots,N$, and $\widehat{\Psi}_P(z;h;\tilde q;\tilde\kappa)$ is holomorphic in z,h on the domain $L^{000}\subset\mathbb{C}^n\times\mathbb{C}$ such that*

$$(11.2)\qquad h\notin\tilde\kappa\mathbb{Z}_{>0},\quad z_a-z_b+h\notin\tilde\kappa\mathbb{Z},\qquad a,b=1,\dots,n,\qquad a\neq b.$$

Proof. By the properties of $\widehat{\Psi}_I(\boldsymbol{z};h;\bar{\boldsymbol{q}};\tilde\kappa)$, see (9.22), we need only to show that the function $\widehat{\Psi}_P(\boldsymbol{z};h;\bar{\boldsymbol{q}};\tilde\kappa)$ is regular at the hyperplanes $z_a-z_b\in\tilde\kappa\mathbb{Z}$. This will be done in Section 11.2 below.

Consider the equivariant K-theory algebra, see [RTV2, Section 2.3], [RTV3, Section 4.4],

$$(11.3)\quad K_T(T^*\mathcal{F}_{\boldsymbol\lambda}) = \mathbb{C}[\acute{\boldsymbol\Gamma}^{\pm1}]^{S_{\lambda_1}\times\dots\times S_{\lambda_N}}\otimes\mathbb{C}[\acute{\boldsymbol z}^{\pm1}]\otimes\mathbb{C}[y^{\pm1}]\Big/\Big\langle\prod_{i=1}^{N}\prod_{j=1}^{\lambda_i}(u-\acute\gamma_{i,j}) = \prod_{a=1}^{n}(u-\acute z_a)\Big\rangle,$$

cf. (9.1). Introduce the variables $\acute\theta_{i,a}$, $i=1,\dots,N-1$, $a=1,\dots,\lambda^{(i)}$. The relations

$$(11.4)\qquad \prod_{a=1}^{\lambda^{(i)}}(u-\acute\theta_{i,a}) = \prod_{j=1}^{i}\prod_{k=1}^{\lambda_j}(u-\acute\gamma_{j,k}),\quad i=1,\dots,N,$$

define the epimorphism $\mathbb{C}[\acute{\boldsymbol\Theta}^{\pm1}]^{S_{\lambda^{(1)}}\times\dots\times S_{\lambda^{(N-1)}}}\otimes\mathbb{C}[\acute{\boldsymbol z}^{\pm1}]\otimes\mathbb{C}[y^{\pm1}]\to K_T(T^*\mathcal{F}_{\boldsymbol\lambda})$. Thus the assignment $P\mapsto\widehat{\Psi}_P$ defines a map from $K_T(T^*\mathcal{F}_\lambda)$ to the space of solutions of the quantum differential equations and the associated *qKZ* difference equations with values in $H^*_T(T^*\mathcal{F}_\lambda)$ extended by functions in $z,h,\tilde q$ holomorphic in the domain $L^{000}\times L^{00}$. We evaluate below the determinant of this map.

The cohomology algebra $H^*_T(T^*\mathcal{F}_\lambda)$ is a free module over $H^*_T(pt;\mathbb{C}) = \mathbb{C}[z]\otimes\mathbb{C}[h]$, with a basis given by the classes of Schubert polynomials

$$(11.5)\qquad Y_I(\boldsymbol\Gamma) = A_{\sigma_I}(\gamma_{1,1},\dots,\gamma_{1,\lambda_1},\gamma_{2,1},\dots,\gamma_{2,\lambda_2},\dots,\gamma_{N,1},\dots,\gamma_{N,\lambda_N}),\qquad I\in\mathcal{I}_\lambda.$$

Similarly, the algebra $K_T(T^*\mathcal{F}_\lambda)$ is a free module over $K_T(pt;\mathbb{C}) = \mathbb{C}[z'^{\pm1}]\otimes\mathbb{C}[y^{\pm1}]$, with a basis given by the classes of Schubert polynomials

$$(11.6)\qquad \widehat{Y}_I(\acute{\boldsymbol\Gamma}) = A_{\sigma_I}(\acute\gamma_{1,1},\dots,\acute\gamma_{1,\lambda_1},\acute\gamma_{2,1},\dots,\acute\gamma_{2,\lambda_2},\dots,\acute\gamma_{N,1},\dots,\acute\gamma_{N,\lambda_N}),\qquad I\in\mathcal{I}_{\boldsymbol\lambda}.$$

Both assertions are clear from Proposition A.7.

Expand solutions of the quantum differential equation using those Schubert bases:

$$\widehat{\Psi}_{\widehat{Y}_I}(\boldsymbol{\Gamma};\boldsymbol{z};h;\tilde{\boldsymbol{q}};\tilde{\kappa}) \,=\, \sum_{J\in\mathcal{I}_{\boldsymbol{\lambda}}} \widehat{\Psi}_{I,J}(\boldsymbol{z};h;\tilde{\boldsymbol{q}};\tilde{\kappa})\,Y_J(\boldsymbol{\Gamma}) \tag{11.7}$$

Theorem 11.3. *Let* $n > 2$. *We have*

$$\det\bigl(\widehat{\Psi}_{I,J}(\boldsymbol{z};h;\tilde{\boldsymbol{q}};\tilde{\kappa})\bigr)_{I,J\in\mathcal{I}_{\boldsymbol{\lambda}}} = \prod_{i=1}^{N-1}\prod_{j=i+1}^{N}(1-\tilde{q}_i/\tilde{q}_j)^{h\min(\lambda_i,\lambda_j)/\tilde{\kappa}}\,\prod_{i=1}^{N}\tilde{q}_i^{\,d^{(1)}_{\boldsymbol{\lambda},i}\sum_{a=1}^n z_a/\tilde{\kappa}}\;\times \tag{11.8}$$
$$\times\prod_{a=1}^{n}\prod_{\substack{b=1\\ b\neq a}}^{n}\Bigl(2\pi\sqrt{-1}\;\Gamma\Bigl(1+\frac{z_a-z_b-h}{\tilde{\kappa}}\Bigr)\Bigr)^{d^{(2)}_{\boldsymbol{\lambda}}},$$

where

$$d^{(1)}_{\boldsymbol{\lambda},i} \,=\, \frac{\lambda_i\,(n-1)!}{\lambda_1!\dots\lambda_N!}\,,\qquad d^{(2)}_{\boldsymbol{\lambda}} \,=\, \frac{(n-2)!}{\lambda_1!\dots\lambda_N!}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\lambda_i\lambda_j \tag{11.9}$$

Proof. By Lemma 11.1, the left-hand side of (11.8) solves the differential equations

$$\Bigl(\tilde{\kappa}\,\tilde{q}_i\frac{\partial}{\partial\tilde{q}_i}\,-\,\mathrm{tr}\bigl(X^{-}_{\boldsymbol{\lambda},i}(\boldsymbol{z};-h;\tilde{\boldsymbol{q}}^{-1})\bigr)\Bigr)\,\det\bigl(\widehat{\Psi}_{I,J}(\boldsymbol{z};h;\tilde{\boldsymbol{q}};\tilde{\kappa})\bigr)_{I,J\in\mathcal{I}_{\boldsymbol{\lambda}}} \,=\, 0\,,\qquad i=1,\dots,N,$$

where $X^{-}_{\boldsymbol{\lambda},i}$ are the modified dynamical Hamiltonians (9.18). Thus $\det(\widehat{\Psi}_{I,J})$ equals the first two products in the right-hand side of (11.8) multiplied by a factor that does not depend on $\tilde{q}$. The remaining factor is found by taking the limit $\tilde{q}_i/\tilde{q}_{i+1}\to 0$ for all $i = 1,\dots,N-1$, and applying Theorem 9.6 and Proposition A.9.

Corollary 11.4. *The collection of functions* $\bigl(\widehat{\Psi}_{\widehat{Y}_I}(\boldsymbol{z};h;\tilde{\boldsymbol{q}};\tilde{\kappa})\bigr)_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}$ *is a basis of solutions of both the quantum differential equations* $\nabla^{\mathrm{quant}}_{\boldsymbol{\lambda},\tilde{\boldsymbol{q}},\tilde{\kappa},i}\,f \,=\, 0\,,\ \ i=1,\dots,N$, *and the associated qKZ difference equations.*

11.2. End of proof of Lemma 11.2.

It is enough to show the regularity of $\widehat{\Psi}_P(\boldsymbol{z};h;\tilde{\boldsymbol{q}};\tilde{\kappa})$ at the hyperplanes $z_a - z_b \in \tilde{\kappa}\mathbb{Z}$ assuming that $h/\tilde{\kappa}$ is real negative and sufficiently large.

For a number A, let $C(A) \subset \mathbb{C}$ be a parabola with the following parametrization:

$$C(A) \,=\, \{\,\tilde{\kappa}\,(A+s^2-s\sqrt{-1}\,)\ \mid\ s\in\mathbb{R}\,\}. \tag{11.10}$$

Given $\boldsymbol{z}$, $\tilde{\kappa}$, take A such that all the points $z_1,\dots,z_n$ are inside $C(A+N-2)$. Suppose $h/\tilde{\kappa}$ is a sufficiently large negative real so that all the points $z_1+h,\dots,z_n+h$ are outside $C(A)$. Set

$$\mathcal{C}_{\boldsymbol{\lambda}}(\boldsymbol{z}) = \bigl(C(A)\bigr)^{\times\lambda^{(1)}}\times\ldots\times\bigl(C(A+N-2)\bigr)^{\times\lambda^{(N-1)}} \tag{11.11}$$

indicating the dependence on λ and z explicitly. The integral (11.12) below does not depend on a particular choice of A.

Lemma 11.5. *For a Laurent polynomial P in t', z', y symmetric in $t^{(i)}_1, \dots, t^{(i)}_{\lambda^{(i)}}$ for each $i = 1,\dots,N-1$, we have*

$$(11.12)\quad \widehat\Psi_P(\boldsymbol z; h; \tilde{\boldsymbol q}; \tilde\kappa) = \tilde\kappa^{-\lambda^{\{1\}}-2\lambda_{\{2\}}} (-1)^{\lambda^{\{1\}}+\lambda_{\{2\}}} \left(2\pi\sqrt{-1}\,\Gamma(-h/\tilde\kappa)\right)^{-\lambda^{\{1\}}} \widetilde\Psi_P(\boldsymbol z; h; \tilde{\boldsymbol q}; \tilde\kappa),$$

$$\widetilde\Psi_P(\boldsymbol z; h; \tilde{\boldsymbol q}; \tilde\kappa) = \frac{\widehat\Omega_{\boldsymbol\lambda}(\tilde{\boldsymbol q}; \tilde\kappa)}{\lambda^{(1)}! \dots \lambda^{(N-1)}!} \int_{\mathcal C_{\boldsymbol\lambda}(\boldsymbol z)} P(t';z';y)\Phi_\lambda(t;z;\tilde q^{-1};-\tilde\kappa)Q(\boldsymbol\Gamma)W_{\mathrm b}(t;\boldsymbol\Gamma)\, d^{\lambda^{\{1\}}}t.$$

Proof. The integral converges provided $|\tilde q_i/\tilde q_{i+1}| < 1$ for all $i = 1,\dots,N-1$, and a branch of $\log \tilde q_i$ is fixed for each $i = 1,\dots,N$. Evaluate the integral by residues in the following way: replace $\mathcal C_\lambda(z)$ by

$$\left(C(A+B)\right)^{\times\lambda^{(1)}} \times \dots \times \left(C(A+B+N-2)\right)^{\times\lambda^{(N-1)}},$$

where $B \in \mathbb R_{>0}$, and send B to infinity. Then by (9.21), the resulting series yields formula (11.1).

The integrand in formula (11.12) is regular at the hyperplanes $z_a - z_b \in \tilde\kappa\mathbb Z$, and so does the function $\widehat\Psi_P(\boldsymbol z; h; \tilde{\boldsymbol q}; \tilde\kappa)$. Lemma 11.2 is proved.

11.3. **The homogeneous case** $z = 0$**.** The quantum differential equations $\nabla^{\text{quant}}_{\boldsymbol\lambda,\tilde{\boldsymbol q},\tilde\kappa,i} f = 0$ depend on z as a parameter and are well defined at $z = 0$.

For any Laurent polynomial P in t', y, symmetric in $t^{(i)}_1, \dots, t^{(i)}_{\lambda^{(i)}}$ for each $i = 1,\dots,N-1$, the function $\widehat\Psi_P(0; h; \tilde q; \tilde\kappa)$ is a solution of the quantum differential equations $\nabla^{\text{quant}}_{\boldsymbol\lambda,\tilde{\boldsymbol q},\tilde\kappa,i,\boldsymbol z=0} f = 0,\ i = 1,\dots,N$, see Lemma 11.1.

Lemma 11.6. *The function $\widehat\Psi_P(0; h; \tilde q; \tilde\kappa)$ is holomorphic in $\tilde q, h$ provided $|\tilde q_i/\tilde q_{i+1}| < 1$, $i = 1,\dots,N-1$, a branch of $\log \tilde q_i$ is fixed for each $i = 1,\dots,N$, and $h \notin \tilde\kappa\mathbb Z_{>0}$.*

Proof. By Lemma 11.2, we need only to show that $\widehat\Psi_P(0; h; \tilde q; \tilde\kappa)$ is regular if $h \in \tilde\kappa\mathbb Z_{<0}$. We will prove that $\widetilde\Psi_P(0;h;\tilde q;\tilde\kappa)$ is regular if $h/\tilde\kappa \in \mathbb R_{<0}$.

If $h/\tilde\kappa$ is a sufficiently large negative real, write $\widehat\Psi_P(0; h; \tilde q; \tilde\kappa)$ by formula (11.12). Then one can replace the integration contour $\mathcal C_\lambda(0)$ by the contour

$$\mathcal C'_{\boldsymbol\lambda}(h,\tilde\kappa) = \left(C((N-1)\varepsilon)\right)^{\times\lambda^{(1)}} \times \left(C((N-2)\varepsilon)\right)^{\times\lambda^{(2)}} \times \dots \times \left(C(\varepsilon)\right)^{\times\lambda^{(N-1)}},$$

where $\varepsilon = h/(N\tilde\kappa)$, without changing the integral. With the integration over $\mathcal C'_{\boldsymbol\lambda}(h,\tilde\kappa)$, it is clear that $\widetilde\Psi_P(0;h;\tilde q;\tilde\kappa)$ continues to a function regular for all negative real $h/\tilde\kappa$. Consider the algebras

$$H^*_{\mathbb C^\times}(T^*\mathcal F_{\boldsymbol\lambda}) = H^*_T(T^*\mathcal F_{\boldsymbol\lambda})/\langle \boldsymbol z = 0\rangle, \qquad K_{\mathbb C^\times}(T^*\mathcal F_{\boldsymbol\lambda}) = K_T(T^*\mathcal F_{\boldsymbol\lambda})/\langle \acute{\boldsymbol z} = (1,\dots,1)\rangle.$$

The algebra $H^*_{\mathbb{C}^\times}(T^*\mathcal{F}_\lambda)$ is a free module over $\mathbb{C}[h]$ and the algebra $K_{\mathbb{C}^\times}(T^*\mathcal{F}_\lambda)$ is a free module over $\mathbb{C}[y^{\pm1}]$, with bases given by the respective classes of Schubert polynomials, see (11.5), (11.6).

Expand solutions of the quantum differential equation at $z = 0$ using those Schubert bases:

$$\widehat{\Psi}_{\widehat{Y}_I}(\mathbf{\Gamma};0;h;\tilde{\boldsymbol{q}};\tilde{\kappa}) = \sum_{J\in\mathcal{I}_\lambda} \widehat{\Psi}_{I,J}(0;h;\tilde{\boldsymbol{q}};\tilde{\kappa})\, Y_J(\mathbf{\Gamma}) \tag{11.13}$$

Let $d_\lambda = n!/(\lambda_1!\dots\lambda_N!)$. Formula (11.8) at $z = 0$ takes the form

$$\det\bigl(\widehat{\Psi}_{I,J}(0;h;\tilde{\boldsymbol{q}};\tilde{\kappa})\bigr)_{I,J\in\mathcal{I}_\lambda} = \tag{11.14}$$
$$= \Bigl(2\pi\sqrt{-1}\,\Gamma\Bigl(1-\frac{h}{\tilde{\kappa}}\Bigr)\Bigr)^{d_\lambda\sum_{1\leqslant i<j\leqslant N}\lambda_i\lambda_j} \prod_{i=1}^{N-1}\prod_{j=i+1}^{N}(1-\tilde{q}_i/\tilde{q}_j)^{h\min(\lambda_i,\lambda_j)/\tilde{\kappa}}.$$

Corollary 11.7. *The collection of functions* $\bigl(\widehat{\Psi}_{\widehat{Y}_I}(0;h;\tilde{\boldsymbol{q}};\tilde{\kappa})\bigr)_{I\in\mathcal{I}_\lambda}$ *is a basis of solutions of the quantum differential equations* $\nabla^{\text{quant}}_{\lambda,\tilde{\boldsymbol{q}},\tilde{\kappa},i,\boldsymbol{z}=0} f = 0,\ i = 1,\dots,N$.

11.4. **The limit** $h \to \infty$**.** Suppose that $\tilde{q}_i/\tilde{q}_{i+1} = (-h)^{-\lambda_i-\lambda_{i+1}}p_i/p_{i+1}$, $i = 1,\dots,N-1$, and $\tilde{q}_N = p_N$, where $p_1,\dots,p_N$ are new variables. The limit $h \to \infty$ keeping $p_1,\dots,p_N$ fixed corresponds to replacing the cotangent bundle $T^*\mathcal{F}_\lambda$ by the partial flag variety $\mathcal{F}_\lambda$ itself, the algebras $H^*_T(T^*\mathcal{F}_\lambda)$, $K_T(T^*\mathcal{F}_\lambda)$ by the respective algebras $H^*_A(\mathcal{F}_\lambda)$, $K_A(\mathcal{F}_\lambda)$, where $A \subset GL_n(\mathbb{C})$ is the torus of diagonal matrices, and the equivariant quantum differential equations for $T^*\mathcal{F}_\lambda$ by the analogous equations for $\mathcal{F}_\lambda$, cf. [BMO, Sections 7,8]. We will discuss this limit in detail in a separate paper making here only a few remarks.

We identify $H^*_A(\mathcal{F}_\lambda)$ with the subalgebra in $H^*_T(T^*\mathcal{F}_\lambda)$ of h-independent elements, and $K_A(\mathcal{F}_\lambda)$ with the subalgebra in $K_T(T^*\mathcal{F}_\lambda)$ of y-independent elements.

The discussion of the limit $h \to \infty$ is based on Stirling's formula

$$\frac{\Gamma(\alpha-h/\tilde{\kappa})}{\Gamma(\beta-h/\tilde{\kappa})} \sim (-h/\tilde{\kappa})^{\alpha-\beta}, \qquad h\to\infty \tag{11.15}$$

For $\mathcal{F}_\lambda$, we have the following counterparts of the master function

$$\Phi^\circ_\lambda(\boldsymbol{t};\boldsymbol{z};\boldsymbol{p};\tilde{\kappa}) = \bigl(e^{\pi\sqrt{-1}(\lambda_N-n)}p_N\bigr)^{\sum_{a=1}^n z_a/\tilde{\kappa}} \prod_{i=1}^{N-1}\Bigl(\frac{e^{\pi\sqrt{-1}(\lambda_i-\lambda_{i+1})}}{\tilde{\kappa}^{\lambda_i+\lambda_{i+1}}}\,\frac{p_i}{p_{i+1}}\Bigr)^{\sum_{j=1}^{\lambda^{(i)}} t^{(i)}_j/\tilde{\kappa}} \times \tag{11.16}$$
$$\times \prod_{i=1}^{N-1}\prod_{a=1}^{\lambda^{(i)}}\Bigl(\prod_{\substack{b=1\\ b\neq a}}^{\lambda^{(i)}} \frac{1}{\Gamma\bigl((t^{(i)}_b-t^{(i)}_a)/\tilde{\kappa}\bigr)} \prod_{c=1}^{\lambda^{(i+1)}}\Gamma\bigl((t^{(i+1)}_c-t^{(i)}_a)/\tilde{\kappa}\bigr)\Bigr),$$

the weight function

$N-1 \quad N \quad \lambda^{(i)} \quad \lambda^{(i+1)}$

$$W_{\mathbf{b}}^{\circ}(t,\boldsymbol{\Gamma}) = \prod_{i=1}\prod_{j=i+1}\prod_{a=1}\prod_{b=\lambda^{(i)}+1}(t_{a^{(i)}}-\gamma_{j,b}), \tag{11.17}$$

and solutions of the quantum differential equations

$$\widehat{\Psi}^{\circ}_{I}(z;p;\tilde\kappa) = (-\tilde\kappa)^{-\lambda^{\{1\}}-\lambda_{\{2\}}}\,\tilde\kappa^{\sum_{i=1}^{N-1}\sum_{a\in I^i}(\lambda_i+\lambda_{i+1})z_a/\tilde\kappa}\,\widetilde{\Psi}^{\circ}_{I}(z;p;\tilde\kappa), \tag{11.18}$$

$$\widetilde{\Psi}^{\circ}_{I}(z;p;\tilde\kappa) = \sum_{r\in\mathbb{Z}^{\lambda^{\{1\}}}_{\geqslant 0}} \operatorname{Res}_{t=\Sigma_I+r\tilde\kappa}\bigl(\Phi^{\circ}_{\boldsymbol\lambda}(t;z;p;\tilde\kappa)\bigr)\,\widehat{W}^{\circ}(\Sigma_I+\boldsymbol{r}\tilde\kappa;\boldsymbol\Gamma),$$

where $I^i = \bigcup_{j=1}^{i} I_j$ and $\lambda_{\{2\}} = \sum_{1\leqslant i<j\leqslant N}\lambda_i\lambda_j$. The series converges and defines a holomorphic function $\Psi^{\circ}_{I}(z;p;\tilde\kappa)$ of z,p on the domain in $\mathbb{C}^n\times\mathbb{C}^N$ such that a branch of $\log p_i$ is fixed for each $i=1,\dots,N$, and $z_a-z_b\notin\tilde\kappa\mathbb{Z}$ for all $a,b=1,\dots,n$, $a\neq b$.

Set $\lambda^{\{2\}} = \sum_{i=1}^{N-1}\bigl(\lambda^{(i)}\bigr)^2$. As $h\to\infty$, we have $(-h)^{\lambda^{\{1\}}-\lambda_{\{2\}}}W_{\mathbf b}(t,\boldsymbol\Gamma)\to W_{\mathbf b}^{\circ}(t,\boldsymbol\Gamma)$,

$$\frac{(-h)^{\lambda^{\{2\}}-\lambda^{\{1\}}}(-h/\tilde\kappa)^{(n-\lambda_N)\sum_{a=1}^{n}z_a/\tilde\kappa}}{\bigl(\Gamma(-h/\tilde\kappa)\bigr)^{\lambda^{\{1\}}+\lambda_{\{2\}}}}\,\Phi_{\boldsymbol\lambda}(t;z;h;\tilde{\boldsymbol q}^{\,1};-\tilde\kappa)\;\to\;\Phi^{\circ}_{\boldsymbol\lambda}(t;z;p;\tilde\kappa),$$

and

$$\frac{\tilde\kappa^{\sum_{i=1}^{N-1}\sum_{a\in I^i}(\lambda_i+\lambda_{i+1})z_a/\tilde\kappa}(-h/\tilde\kappa)^{(n-\lambda_N)\sum_{a=1}^{n}z_a/\tilde\kappa}}{\bigl(\Gamma(1-h/\tilde\kappa)\bigr)^{\lambda^{\{2\}}}}\,\widehat{\Psi}_{I}(z;h;\tilde{\boldsymbol q};\tilde\kappa)\;\to\;\widehat{\Psi}^{\circ}_{I}(z;p;\tilde\kappa).$$

If $p_i/p_{i+1}\to 0$ for all $i=1,\dots,N-1$, then similarly to (9.23),

$$\widehat{\Psi}^{\circ}_{I}(z;p;\tilde\kappa) = \prod_{i=1}^{N}\bigl(e^{\pi\sqrt{-1}(\lambda_i-n)}p_i\bigr)^{\sum_{a\in I_i}z_a/\tilde\kappa}\prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{a\in I_i}\prod_{b\in I_j}\Gamma\Bigl(1+\frac{z_b-z_a}{\tilde\kappa}\Bigr)\times \tag{11.19}$$

$$\times\Bigl(\Delta_I+\sum_{\substack{\boldsymbol m\in\mathbb{Z}^{N-1}_{\geqslant 0}\\ \boldsymbol m\neq 0}}\widehat{\Psi}^{\circ}_{I,\boldsymbol m}(\boldsymbol z;\tilde\kappa)\prod_{i=1}^{N-1}\Bigl(\frac{p_i}{p_{i+1}}\Bigr)^{m_i}\Bigr),$$

where $\Delta_I(\boldsymbol\Gamma,z) = R_I(\boldsymbol\Gamma;z)/R(z_I)$ is the cohomology class such that $\Delta_I(z_J;z)=\delta_{I,J}$, and the classes $\Psi^{\circ}_{I,\boldsymbol m}(\boldsymbol z;\tilde\kappa)$ are rational functions in $z,\tilde\kappa$, regular if $z_a-z_b\notin\tilde\kappa\mathbb{Z}$ for all $a,b=1,\dots,n$, $a\neq b$.

Recall the contour $C_{\boldsymbol\lambda}(z)$, see (11.11). Given a Laurent polynomial P in the variables t',z', symmetric in $t^{(i)}_1,\dots,t^{(i)}_{\lambda^{(i)}}$ for each $i=1,\dots,N-1$, define

$$(11.20)\qquad \widehat{\Psi}^{\circ}_{P}(z;p;\tilde\kappa) \,=\, \frac{(-\tilde\kappa)^{-\lambda^{\{1\}}-\lambda_{\{2\}}}\ \tilde\kappa^{\sum_{i=1}^{N-1}\sum_{a\in I^i}(\lambda_i+\lambda_{i+1})z_a/\tilde\kappa}}{(2\pi\sqrt{-1})^{\lambda^{\{1\}}}}\ \widetilde{\Psi}^{\circ}_{P}(z;p;\tilde\kappa),$$

$$\widetilde{\Psi}^{\circ}_{P}(z;p;\tilde\kappa) \,=\, \frac{1}{\lambda^{(1)}!\dots\lambda^{(N-1)}!}\int_{\mathcal{C}_{\boldsymbol\lambda}(z)} P(\boldsymbol t;z)\,\Phi^{\circ}_{\boldsymbol\lambda}(\boldsymbol t;z;p;\tilde\kappa)\,\widehat{W}^{\circ}(\boldsymbol t;\boldsymbol\Gamma)\,d^{\lambda^{\{1\}}}\boldsymbol t,$$

cf. (11.18). The integral converges and defines a holomorphic function $\widehat{\Psi}^{\circ}_{P}(z;p;\tilde\kappa)$ of z,p on the domain in $\mathbb{C}^n\times\mathbb{C}^N$ such that a branch of $\log p_i$ is fixed for each $i=1,\dots,N$. Furthermore,

$$(11.21)\qquad \widehat{\Psi}^{\circ}_{P}(z;p;\tilde\kappa) = \sum_{I\in\mathcal{I}_\lambda} P(\Sigma_I,z)\,\widehat{\Psi}^{\circ}_{I}(z;p;\tilde\kappa),$$

and the assignment $P\mapsto\widehat{\Psi}^{\circ}_{P}$ defines a map from $K_A(\mathcal{F}_\lambda)$ to the space of solutions of the quantum differential equations with values in $H^*_A(\mathcal{F}_\lambda)$.

Consider the classes $Y_I(\Gamma)$, $\widehat{Y}_I(\Gamma)$ given by (11.5), (11.6), and write

$$(11.22)\qquad \widehat{\Psi}^{\circ}_{\widehat{Y}_I}(\boldsymbol\Gamma;z;p;\tilde\kappa) \,=\, \sum_{J\in\mathcal{I}_{\boldsymbol\lambda}}\widehat{\Psi}^{\circ}_{I,J}(z;p;\tilde\kappa)\,Y_J(\boldsymbol\Gamma),$$

cf. (11.7). Taking the limit $h\to\infty$ in formula (11.8) yields

$$(11.23)\qquad \det\bigl(\widehat{\Psi}^{\circ}_{I,J}(z;p;\tilde\kappa)\bigr)_{I,J\in\mathcal{I}_{\boldsymbol\lambda}} \,=\, (2\pi\sqrt{-1})^{d_{\boldsymbol\lambda}\sum_{1\leqslant i<j\leqslant N}\lambda_i\lambda_j}\ \prod_{i=1}^{N} p_i^{d^{(1)}_{\boldsymbol\lambda,i}\sum_{a=1}^{n} z_a/\tilde\kappa},$$

where

$$d_{\boldsymbol\lambda} \,=\, \frac{n!}{\lambda_1!\dots\lambda_N!}\,,\qquad d^{(1)}_{\boldsymbol\lambda,i} \,=\, \frac{\lambda_i\,(n-1)!}{\lambda_1!\dots\lambda_N!}\,.$$

Therefore, the collection of functions $\bigl(\widehat{\Psi}^{\circ}_{\widehat{Y}_I}(z;p;\tilde\kappa)\bigr)_{I\in\mathcal{I}_{\boldsymbol\lambda}}$ is a basis of solutions of both the quantum differential equations with values in $H^*_A(\mathcal{F}_{\boldsymbol\lambda})$ and the associated *qKZ* difference equations.

Appendix A. Basics on Schubert polynomials

For references regarding Schubert polynomials, see for example [L, M].

Let $D_1,\dots,D_{n-1}$ be the divided difference operators acting on functions of $x_1,\dots,x_n$:

$$D_i f(x_1,\dots,x_n) \,=\, \frac{f(x_1,\dots,x_n)-f(x_1,\dots,x_{i+1},x_i,\dots,x_n)}{x_i-x_{i+1}}\,,$$

cf. (3.7). They satisfy the nil-Coxeter algebra relations,

$$(A.1)\qquad (D_i)^2=0,\qquad D_iD_{i+1}D_i=D_{i+1}D_iD_{i+1}\,,\qquad D_iD_j=D_jD_i\,,\qquad |i-j|>1.$$

Given $\sigma \in S_n$ with a reduced decomposition $\sigma = s_{i_1,i_1+1} \dots s_{i_j,i_j+1}$, define $D_\sigma = D_{i_1} \dots D_{i_j}$. For instance, D_{id} is the identity operator and $D_{s_{i,i+1}} = D_i$. Due to relations (A.1), the operator D_σ does not depend on the choice of a reduced decomposition. Moreover,

$$D_\sigma D_\tau = D_{\sigma\tau}, \qquad \text{if } |\sigma| + |\tau| = |\sigma\tau|, \qquad D_\sigma D_\tau = 0, \qquad \text{otherwise}.$$

Here $|\sigma|$ is the length of σ. Denote $x_\sigma = (x_{\sigma(1)},\dots,x_{\sigma(n)})$. Let σ_0 be the longest permutation, $\sigma_0(i) = n + 1 - i$, $i = 1,\dots,n$. Then

$$D_{\sigma_0} f(x) = \prod_{1 \leqslant i<j \leqslant n} (x_i - x_j)^{-1} \sum_{\sigma \in S_n} (-1)^\sigma f(x_\sigma).$$

The Schubert polynomials $A_\sigma(x)$, $\sigma \in S_n$, are defined by the rule

$$A_\sigma(\boldsymbol{x}) \;=\; D_{\sigma^{-1}\sigma_0}(x_1^{n-1} x_2^{n-2} \dots x_{n-1}). \tag{A.2}$$

In particular, $A_{\sigma_0} = x_1^{n-1} x_2^{n-2} \dots x_{n-1}$ and $A_{\mathrm{id}} = 1$.

Proposition A.1. *For any $\sigma,\tau \in S_n$,*

$$D_{\sigma_0}\big(A_\sigma(\boldsymbol{x})\, A_{\tau\sigma_0}(\boldsymbol{x}_{\sigma_0})\big) \;=\; (-1)^{\sigma\sigma_0}\, \delta_{\sigma,\tau}. \tag{A.3}$$

Proposition A.2. *Cauchy formula holds,*

$$\sum_{\sigma \in S_n} (-1)^\sigma A_\sigma(x) A_{\sigma\sigma_0}(y) = \prod_{i=1}^{n-1} \prod_{j=1}^{n-i} (y_i - x_j). \tag{A.4}$$

For any $f \in \mathbb{C}[x]$ and $\sigma \in S_n$, define $f_{\langle\sigma\rangle} \in \mathbb{C}[x]^{S_n}$ by the rule

$$f_{\langle\sigma\rangle}(\boldsymbol{x}) \;=\; (-1)^{\sigma\sigma_0} D_{\sigma_0}\big(f(\boldsymbol{x})\, A_{\sigma\sigma_0}(\boldsymbol{x}_{\sigma_0})\big). \tag{A.5}$$

Proposition A.3. *For any $f \in \mathbb{C}[x]$,*

$$f(x) = \sum_{\sigma \in S_n} f_{\langle\sigma\rangle}(x) A_\sigma(x). \tag{A.6}$$

Thus $\mathbb{C}[x]$ is a free module over $\mathbb{C}[x]^{S_n}$ of rank $n!$ with a basis given by Schubert polynomials.

Recall the notation from Section 2.1, and $I^{\min}, I^{\max} \in \mathcal{I}_\lambda$,

$$\begin{aligned} I^{\min} &= \big(\{1,\dots,\lambda_1\},\dots,\{n-\lambda_N+1,\dots,n\}\big) \\ I^{\max} &= \big(\{n-\lambda_1+1,\dots,n\},\dots,\{1,\dots,\lambda_N\}\big). \end{aligned}$$

For $I = (I_1,\dots,I_N) \in \mathcal{I}_\lambda$, $I_j = \{i_{j,1} < \dots < i_{j,\lambda_j}\}$, define the permutations σ_I,

$$\sigma_I(k) = i_{j,k-\lambda^{(j-1)}}, \qquad k \in I_j^{\min}, \qquad j = 1,\dots,N,$$

and $\sigma_I = \sigma^I(\sigma^{I^{\max}})^{-1}$. Then $\sigma_I(I^{\min}) = \sigma^I(I^{\max}) = I$.

Let $S_{\lambda_1} \times \ldots \times S_{\lambda_N} \subset S_n$ be the isotropy subgroup of $I^{\min}$.

Lemma A.4. *For any $I \in \mathcal{I}_\lambda$, we have $A_{\sigma^I}(x) \in \mathbb{C}[x]^{S_{\lambda_1}\times\ldots\times S_{\lambda_N}}$.*

For example, $A_{\sigma^{I^{\max}}}(\boldsymbol{x}) = \prod_{a=1}^{N-1} \prod_{i\in I_a^{\min}} x_i^{N-a}$.

Proposition A.5. *For any $I,J \in \mathcal{I}_\lambda$,*

$$D_{\sigma^{I^{\max}}}\big(A_{\sigma^I}(\boldsymbol{x})\, A_{\sigma_J}(\boldsymbol{x}_{\sigma_0})\big) = (-1)^{\sigma_I}\,\delta_{I,J}. \tag{A.7}$$

Proposition A.6. *We have,*

$$\sum_{I\in\mathcal{I}_\lambda} (-1)^{\sigma_I} A_{\sigma^I}(x) A_{\sigma^I}(y_{\sigma_0}) = \prod_{1\leqslant a<b\leqslant N}\ \prod_{i\in I_a^{\min}}\ \prod_{j\in I_b^{\min}} (y_j - x_i). \tag{A.8}$$

Proposition A.7. *For any $f \in \mathbb{C}[x]^{S_{\lambda_1}\times\ldots\times S_{\lambda_N}}$, we have*

$$f(x) = \sum_{I\in\mathcal{I}_\lambda} f_{\langle\sigma^I\rangle}(x) A_{\sigma^I}(x), \tag{A.9}$$

that is, in formula (A.6), $f_{\langle\sigma\rangle} = 0$ *unless $\sigma = \sigma^I$ for some $I \in \mathcal{I}_\lambda$, and*

$$f_{\langle\sigma^I\rangle}(\boldsymbol{x}) = (-1)^{\sigma_I} D_{\sigma^{I^{\max}}}\big(f(\boldsymbol{x})\, A_{\sigma_I}(\boldsymbol{x}_{\sigma_0})\big). \tag{A.10}$$

Define

$$R_\lambda(x) = \prod_{1\leqslant a<b\leqslant N}\ \prod_{i\in I_a^{\min}}\ \prod_{j\in I_b^{\min}} (x_i - x_j).$$

Proposition A.8. *For any $f \in \mathbb{C}[x]^{S_{\lambda_1}\times\ldots\times S_{\lambda_N}}$, we have*

$$D_{\sigma^{I^{\max}}} f(\boldsymbol{x}) = \sum_{I\in\mathcal{I}_\lambda} \frac{f(\boldsymbol{x}_{\sigma^I})}{R_\lambda(\boldsymbol{x}_{\sigma^I})}. \tag{A.11}$$

Proposition A.9. *Let $n > 2$. Then*

$$\det\big(A_{\sigma^I}(\boldsymbol{x}_{\sigma^J})\big)_{I,J\in\mathcal{I}_\lambda} = \prod_{1\leqslant i<j\leqslant n} (x_j - x_i)^{m_\lambda}, \tag{A.12}$$

where $(n-2)!\quad {}^{N-1}\quad {}^{N}$

$$m_{\boldsymbol{\lambda}} = \frac{}{\lambda_1!\dots\lambda_N!}\sum_{i=1}\sum_{j=i+1}\lambda_i\lambda_j\,.$$

Appendix B. aLeading Terms of Solutions and Gamma Conjecture

The formula (9.23) for the asymptotics of solutions $(\widetilde{\Psi}_I(\boldsymbol{z};h;\tilde{\boldsymbol{q}};\tilde{\kappa}))_{I\in\mathcal{I}_{\boldsymbol{\lambda}}}$ to the joint system of the quantum differential equations and associated qKZ difference equations reminds the statement of the gamma conjecture, see [D1, D2, KKP, GGI, GI, GZ].

The gamma conjecture [D2, GGI] is a conjecture relating the quantum cohomology of a Fano manifold X with its topology. The quantum cohomology of X defines a flat quantum connection over $\mathbb{C}^\times$ in the direction of the first Chern class $c_1(X)$. The connection has a regular singular point at $t = 0$ and an irregular singular point at $t = \infty$. The connection has a distinguished (multivalued) flat section $J_X(t)$ defined by Givental in [Gi1] and called the J-function. Under certain assumptions, the limit of the J-function:

$$A_X := \lim_{t\to\infty}\frac{J_X(t)}{\langle[\mathrm{pt}],J_X(t)\rangle} \in H^*(x)$$

exists and defines the *principal asymptotic class* A_X of X. The gamma conjecture says that A_X equals the gamma class $\hat{\Gamma}_X$ of the tangent bundle of X.

The gamma class of a holomorphic vector bundle E over a topological space X is the multiplicative characteristic class, in the sense of Hirzebruch, associated to the power series expansion $\Gamma(1+x) = 1-\gamma x+\frac{\gamma^2+\zeta(2)}{2}x^2+\dots$ of the gamma function at 1, where γ is the Euler constant and $\zeta(2)$ is the value at 2 of the zeta function. In other words, the gamma class is the function that associates to a holomorphic bundle E over X the cohomology class $\hat{\Gamma}(E) = \prod_i\Gamma(1+\tau_i) \in H^*(X;\mathbb{R})$, where the total Chern class of E has the formal factorization $c(E) = \prod_i(1+\tau_i)$ with the Chern roots τ_i of degree 2. If E is the tangent bundle of X, we write $\hat{\Gamma}_X$ for $\hat{\Gamma}(E)$. Its terms of degree $\leqslant 3$ are given by the formula

$$\hat{\Gamma}(E) = 1-\gamma c_1+\big(-\zeta(2)c_2+\frac{\zeta(2)+\gamma^2}{2}c_1^2\big) + \big(-\zeta(3)c_3+(\zeta(3)+\gamma\zeta(2))c_1c_2-\frac{2\zeta(3)+3\gamma\zeta(2)+\gamma^3}{6}c_1^3\big)+\dots, \tag{B.1}$$

see [GZ].

Consider the equivariant gamma class of $T^*\mathcal{F}_\lambda$,

$$\hat{\Gamma}_{T^*\mathcal{F}_\lambda} = \prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{a=1}^{\lambda_i}\prod_{b=1}^{\lambda_j}\Gamma(1+\gamma_{j,b}-\gamma_{i,a})\Gamma(1+\gamma_{i,a}-\gamma_{j,b}-h).$$

cf. (9.2), and the equivariant first Chern classes $c_1(E_i) = \sum_{a=1}^{\lambda_i} \gamma_{i,a}$, $i = 1,\dots,N$, of the vector bundles E_i over $T^*\mathcal{F}_\lambda$ with fibers F_i/F_{i-1}, see (9.3). Theorem 9.6 can be reformulated as follows.

Theorem B.1 (Gamma theorem for $T^*\mathcal{F}_\lambda$)**.** *For $\kappa = 1$, the leading term of the asymptotics of the q-hypergeometric solutions $(\Psi_I(z;h;\tilde q;\tilde\kappa))_{I\in\mathcal{I}_\lambda}$ in (9.23) is the product of the equivariant gamma class of $T^*\mathcal{F}_\lambda$ and the exponentials of the equivariant first Chern classes of the associated vector bundles $E_1,\dots,E_N$:*

$$\hat\Gamma_{T^*\mathcal{F}_\lambda} \prod_{l=1}^{N} \left(e^{\pi\sqrt{-1}\,(\lambda_i - n)}\,\tilde q_i\right)^{c_1(E_i)} \tag{B.2}$$

Similarly formula (11.19) can be reformulated as follows.

Theorem B.2 (Gamma theorem for $\mathcal{F}_\lambda$)**.** *For $\kappa = 1$, the leading term of the asymptotics of the q-hypergeometric solutions $(\Psi^{\flat}_I(z;p;\tilde\kappa))_{I\in\mathcal{I}_\lambda}$ in (11.18) is the product of the equivariant gamma class of $\mathcal{F}_\lambda$ and the exponentials of the equivariant first Chern classes of the associated vector bundles $E_1,\dots,E_N$:*

$$\hat\Gamma_{\mathcal{F}_\lambda} \prod_{l=1}^{N} \left(e^{\pi\sqrt{-1}\,(\lambda_i - n)}\,p_i\right)^{c_1(E_i)} \tag{B.3}$$

Example. Let $N = 2$, $n = 2$, $\lambda = (1,1)$. For $\tilde\kappa = 1$, the leading term of the asymptotics of the q-hypergeometric solutions for $T^*\mathbb{P}^1$ is the class

$$(e^{-\pi\sqrt{-1}}\,\tilde q_1)^{\gamma_{1,1}}\,(e^{-\pi\sqrt{-1}}\,\tilde q_2)^{\gamma_{2,1}}\,\Gamma(1+\gamma_{2,1}-\gamma_{1,1})\,\Gamma(1+\gamma_{1,1}-\gamma_{2,1}-h)$$

and the leading term of the asymptotics of the q-hypergeometric solutions for $\mathbb{P}^1$ is the class

$$(e^{-\pi\sqrt{-1}}\,p_1)^{\gamma_{1,1}}\,(e^{-\pi\sqrt{-1}}\,p_2)^{\gamma_{2,1}}\,\Gamma(1+\gamma_{2,1}-\gamma_{1,1}).$$

References

[BMO] A.Braverman, D.Maulik, A.Okounkov, *Quantum cohomology of the Springer resolution*, Adv. Math. **227** (2011), no. 1, 421–458, DOI: 10.1016/j.aim.2011.01.021, arXiv:1001.0056

[D1] B.A.Dubrovin, *Geometry and analytic theory of Frobenius manifolds*, In Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998), 315–

326, arXiv:math/9807034

[D2] B.A.Dubrovin, *Quantum Cohomology and Isomonodromic Deformation*. Lecture at Recent Progress in the Theory of Painlev´e Equations: Algebraic, asymptotic and topological aspects, Strasbourg, November 2013.

[FR] I.Frenkel, N.Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Comm. Math. Phys. **146** (1992), no. 1, 1–60

[GGI] S.Galkin, V.Golyshev, H.Iritani, *Gamma classes and quantum cohomology of Fano manifolds: gamma conjectures*, Duke Math. J. **165** (2016), no. 11, 2005–2077, arXiv:1404.6407

[GI] S.Galkin, H.Iritani, *Gamma conjecture via mirror symmetry*, arXiv:1508.00719, 1– 43

[Gi1] A.Givental, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices (1996), no. 13, 613–663

[Gi2] A.Givental, *Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture*, Topics in singularity theory, AMS Transl. Ser. 2 **180** (1997), 103–115

[Gi3] A.Givental, *Gromov-Witten invariants and quantization of quadratic Hamiltonians*, Mosc. Math. J. **1** (2001), no. 4, 551–568

[GRTV] V.Gorbounov, R.Rima´nyi, V.Tarasov, A.Varchenko, *Quantum cohomology of the cotangent bundle of a flag variety as a Yangian Bethe algebra*, Journal of Geometry and Physics (2013), pp. 56–86, DOI: 10.1016/j.geomphys.2013.07.006

[GZ] V.Golyshev, D.Zagier, *Proof of the gamma conjecture for Fano manifolds of Picard rank 1*, preprint at http://people.mpim-bonn.mpg.de/zagier/files/tex/GammaConjecture/GammaConj.pdf

[KKP] L.Katzarkov, M.Kontsevich, T.Pantev, *Hodge theoretic aspects of mirror symmetry*, Proc. Sympos. Pure Math. **78**, Amer. Math. Soc., Providence, RI, 2008, pp. 87– 174, arXiv:0806.0107, 1–124

[L] A.Lascoux, *Polynomials*, 2013, http://www-igm.univ-mlv.fr/ al/ARTICLES/CoursYGKM.pdf
~

[M] I.Macdonald, *Notes on Schubert polynomials* D´epartement de math´ematiques et d'informatique, Universit´e de Qu´ebec a´ Montr´eal, 1991, ISBN 2892760860

[MO] D.Maulik, A.Okounkov, *Quantum Groups and Quantum Cohomology*, Preprint (2012), 1–276, arXiv:1211.1287

[MTV1] E.Mukhin, V.Tarasov, A.Varchenko, *Bethe eigenvectors of higher transfer matrices*, J. Stat. Mech. (2006), no. 8, P08002, 1–44

[MTV2] E.Mukhin, V.Tarasov, A.Varchenko, *Spaces of quasi-exponentials and representations of the Yangian Y* ($\mathfrak{gl}_N$), Transf. Groups, **19** (2014), no. 2, 1–27, DOI: 10.1007/s00031-014-9275-8, arXiv:1303.1578

[MV1] E.Mukhin, A.Varchenko, *Solutions to the XXX type Bethe ansatz equations and flag varieties*, Cent. Eur. J. Math., **1**, no. 2 (2003), 238–271

[MV2] E.Mukhin, A.Varchenko, *Quasi-polynomials and the Bethe Ansatz*, Geometry & Topology Monographs **13**, Groups, homotopy and configuration spaces (Tokyo 2005), in honour of Fred Cohen, (2008), 385–420

[N1] H.Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), no. 2, 365–416

[N2] H.Nakajima, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. **91** (1998), no. 3, 515–560

[RTV1] R.Rimanyi, V.Tarasov, A.Varchenko, *Partial flag varieties, stable envelopes and weight functions*, Quantum Topology **6** (2015), no. 2, 333–364, DOI: 10.4171/QT/64

[RTV2] R.Rimanyi, V.Tarasov, A.Varchenko, *Trigonometric weight functions as Ktheoretic stable envelope maps for the cotangent bundle of a flag variety*, Journal of Geometry and Physics, **94** (2015), 81–119, DOI: 10.1016/j.geomphys.2015.04.002, arXiv:1411.0478

[RTV3] R.Rimanyi, V.Tarasov, A.Varchenko, *Elliptic and K-theoretic stable envelopes and Newton polytopes*, arXiv:1705.09344, 1–35

[RV] R.Rimanyi, A.Varchenko, *Equivariant Chern-Schwartz-MacPherson classes in partial flag varieties: interpolation and formulae*, arXiv:1509.09315, 1–9

[TV1] V.Tarasov, A.Varchenko, *Jackson Integral Representations for Solutions to the Quantized Knizhnik-Zamolodchikov Equation*, St.Petersb. Math. J. **6**, no. 2 (1994), 275–313

[TV2] V.Tarasov and A.Varchenko, *Duality for Knizhnik-Zamolodchikov and dynamical equations*, Acta Appl. Math. **73** (2002), no. 1-2, 141–154

[TV3] V.Tarasov and A.Varchenko, *Geometry of q-hypergeometric functions as a bridge between Yangians and quantum affine algebras*, Invent. Math. **128** (1997), no. 3, 501–588

[TV4] V.Tarasov and A.Varchenko, *Combinatorial formulae for nested Bethe vectors*, SIGMA Symmetry Integrability Geom. Methods Appl. **9** (2013), Paper 048, 1–28

[TV5] V.Tarasov and A.Varchenko, *Hypergeometric solutions of the quantum differential equation of the cotangent bundle of a partial flag variety*, Cent. Eur. J. Math. **12** (2014), no. 5, 694–710