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A functional non-central limit theorem for multiple-stable processes with long-range dependence

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Abstract

A functional limit theorem is established for the partial-sum process of a class of stationary sequences which exhibit both heavy tails and long-range dependence. The stationary sequence is constructed using multiple stochastic integrals with heavy-tailed marginal distribution. Furthermore, the multiple stochastic integrals are built upon a large family of dynamical systems that are ergodic and conservative, leading to the long-range dependence phenomenon of the model. The limits constitute a new class of self-similar processes with stationary increments. They are represented by multiple stable integrals, where the integrands involve the local times of intersections of independent stationary stable regenerative sets.

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1. Introduction

1.1. Background

The seminal work of Rosiński [47] revealed an intriguing connection between stationary stable processes and ergodic theory. Consider a stationary process in the form of

$$X_k = \int_E f(T^k x) M(dx), \quad k \in \mathbb{N}, \tag{1}$$

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where M is symmetric α-stable random measure on a measure space $(E, \mathcal{E}, \mu)$, $f : E \to \mathbb{R}$ is a measurable function and T is a measure-preserving transform from E to E. Then, many properties of the process X can be derived from the underlying dynamical system $(E, \mathcal{E}, \mu, T)$. Because of this connection, the process X is also referred to as *driven by the flow* T, and many developments on structures, representations, and ergodic properties of such processes have stemmed from this connection (see e.g., [24,42–44,50–53,56,63]; background to be reviewed in Section 4.1). In particular, it was argued by Samorodnitsky [52, Remark 2.5] that the case where T is conservative and ergodic is the most challenging to develop a satisfactory characterization of the ergodic properties of the process in terms of the underlying dynamical system.

While examples of stable processes driven by conservative and ergodic flows have been known for more than 20 years since [48], limit theorems for such processes have not been established until in very recent breakthroughs in a series of papers by Samorodnitsky and coauthors [32,40,41,55], all exhibiting phenomena of long-range dependence with new limit objects. Here, by long-range dependence, we mean generally that the partial-sum process $(S_{\lfloor nt \rfloor})_{t\in[0,1]}$, with $S_n := X_1 + \cdots + X_n$, scales to a non-degenerate stochastic process with a normalization that is different from the case when $(X_k)_{k\in\mathbb{N}}$ are i.i.d. We follow this point of view as in Samorodnitsky [53], and one could also consider limit theorems for other statistics; the key is always the abnormal normalization compared to the i.i.d. case.

The functional central limit theorem for stationary stable processes driven by a conservative and ergodic flow, established in [40], serves as our starting point and takes the following form. With f in (1) such that the support has finite μ-measure and $\mu(f) := \int_E f d\mu$ is finite and nonzero, it was shown that

$$\frac{1}{d_n}\left(S_{\lfloor nt \rfloor}\right)_{t\in[0,1]} \Rightarrow \mu(f)\left(\int_{\Omega'\times[0,\infty)} \mathcal{M}_\beta((t-v)_+, \omega')S_{\alpha,\beta}(d\omega', dv)\right)_{t\in[0,1]} \tag{2}$$

in $D([0, 1])$, where $\alpha \in (0, 2)$, $\beta \in (0, 1)$, and d_n is a regularly varying sequence with exponent $\beta + (1 - \beta)/\alpha$. (This was actually established in a slightly more general framework with M replaced by an infinitely-divisible random measure with heavy-tail index α.) Here, $(\Omega', \mathcal{F}', P')$ is a probability space separate from the one that carries the randomness of the stochastic integral itself, $S_{\alpha,\beta}$ is a symmetric α-stable (SαS) random measure on $\Omega' \times [0, \infty)$ with control measure $P' \times (1 - \beta)v^{-\beta}dv$, and $\mathcal{M}_\beta$ is the Mittag-Leffler process with index β, the inverse process of a β-stable subordinator, defined on $(\Omega', \mathcal{F}', P')$.

Here, $\beta \in (0, 1)$ is the memory parameter of an underlying dynamical system (see Section 4 and in particular how β characterizes the memory of T in terms of Assumption 1), and as $\beta \downarrow 0$ the limit process in (2) becomes an SαS Lévy process. At the core of this result, the appearance of the Mittag-Leffler process is established as a functional generalization of the one-dimensional Darling–Kac limit theorem in [1,11] for the underlying dynamical system, which is of independent interest in ergodic theory. Later developments [32,55] revealed that more essentially, stable regenerative sets [8] and their intersections play a fundamental role in describing the limit objects for a large family of processes driven by conservative and ergodic flows.

In this paper, as a generalization of (1) we consider the process defined in terms of multiple stochastic integrals in the form of

$$X_k = \int_{E^p}' f(T^k x_1, \ldots, T^k x_p)M(dx_1)\cdots M(dx_p), \quad k \in \mathbb{N}, \quad p \in \mathbb{N}, \tag{3}$$

where the prime mark $'$ indicates that the multiple integral is defined to *exclude the diagonals*, and this time f is a measurable function from E^p to $\mathbb{R}$. The definition of multiple stochastic integrals will be recalled in Section 3.1.

We restrict to the case of multiple integrals without the diagonals, in order to obtain limit processes in the form of *multiple stable integrals*, which we refer to as *multiple-stable processes*. Since the seminal works of Dobrushin and Major [14] and Taqqu [60], the processes in the form of multiple Gaussian integrals have frequently appeared in limit theorems under long-range dependence. For example, they were obtained as limits for partial sums [3,5,14,22,58,60]), for empirical processes [13,23,64] as well as for quadratic forms [18,61]. Such limit theorems are often referred to as *non-central limit theorems* and have found numerous applications to statistical theories for long-range dependent data (see, e.g., [7] and the references therein). Limit theorems with (non-Gaussian) multiple-stable processes as limits, to the best of our knowledge however, have been rarely considered so far in the literature of long-range dependence. Note that the exclusion of the diagonals is necessary to obtain multiple-stable processes with multiplicity $p \geq 2$: with the terms on the diagonal included, the case $p = 2$ has been partly considered in [39], and the limit is again a stable process.

1.2. Overview of main results

Our ultimate goal (Theorem 4.1) is to establish formally that

$$\frac{1}{d_n}\left(\sum_{k=1}^{\lfloor nt\rfloor} X_k\right)_{t\in[0,1]} \Rightarrow \left(Z_{\alpha,\beta,p}(t)\right)_{t\in[0,1]}$$

for a large family of (X_k) in (3), and the limit process has the representation

$$\begin{aligned}&\left(Z_{\alpha,\beta,p}(t)\right)_{t\geq 0}\\ &\quad\stackrel{d}{=}\left(\int'_{(\mathbf{F}\times[0,\infty))^p} L_t\left(\bigcap_{i=1}^p (R_i+v_i)\right) S_{\alpha,\beta}(dR_1,dv_1)\cdots S_{\alpha,\beta}(dR_p,dv_p)\right)_{t\geq 0},\end{aligned} \tag{4}$$

where $S_{\alpha,\beta}$ is an SαS random measure on $\mathbf{F}\times[0,\infty)$, with control measure $P_\beta\times(1-\beta)v^{-\beta}dv$, with P_β the probability measure on $\mathbf{F}\equiv\mathbf{F}([0,\infty))$, the space of closed subsets of $[0,\infty)$, induced by the law of a β-stable regenerative set, and L_t is the local-time functional for a $(p\beta-p+1)$-stable regenerative set [28].

An immediate observation is that for the right-hand side of (4) to be non-degenerate, we need $\bigcap_{i=1}^p (R_i+v_i)$ to be non-empty, with $(R_i)_{i=1,\dots,p}$ being i.i.d. β-stable regenerative sets. The key relation between the memory parameter β and the multiplicity p assumed throughout this paper is that

$$\beta\in(0,1),\quad p\in\mathbb{N}\quad\text{such that}\quad \beta_p := p\beta-p+1\in(0,1), \tag{5}$$

or equivalently $\beta\in(1-1/p,1)$. It is known (e.g., [55]) that this is exactly the case when $\bigcap_{i=1}^p (R_i+v_i)$ is a β_p-stable regenerative set with a random shift with probability one. When (5) is violated and v_i are all different, the intersection becomes an empty set with probability one and hence $Z_{\alpha,\beta,p}$ becomes degenerate. The limit theorem in such a case will be of a different nature and addressed in a separate paper.

Our theorem applies to a large family of dynamical systems, including in particular the shift transforms of certain null-recurrent Markov chains, and a class of transforms on the real line called the AFN-systems [65,66] often considered in the literature of infinite ergodic theory. Establishing the aforementioned convergence, however, turns out to be a completely different task from the one in [40], and the proof consists of two parts. The first part is devoted to the investigation of the integrand of the right-hand side of (4), which are local-time processes of intersections of stable regenerative sets (Section 2). Let $(R_i)_{i\in\mathbb{N}}$ be i.i.d. β-stable regenerative sets. To exploit a series representation of the multiple integral (4) (see (46)), we need to characterize the law of

$$L_{I,t} \equiv L_t\left(\bigcap_{i\in I}(R_i + v_i)\right) \text{ for all } I \subset \mathbb{N},\ |I| = p,\ t \ge 0,$$

jointly in I and t, governed by certain law on the shifts $(v_i)_{i\in I}$ independent from the regenerative sets. Marginally, for each I, $(L_{I,t})_{t\ge 0}$ has the law of a Mittag-Leffler process shifted in time with parameter β_p, up to a multiplicative constant [55]. In particular when $p = 1$ we have

$$(L_t(R_1 + v_1))_{t\ge 0} \stackrel{d}{=} c_\beta \left(\mathcal{M}_\beta((t - v_1)_+)\right)_{t\ge 0} \tag{6}$$

for some constant c_β. It is then a matter of convenience to work with either of the two representations in (6), and the right-hand side was used in [40]. However when $p \ge 2$, the information from the Mittag-Leffler process is only marginal, whereas we need to work with $L_{I,t}$ jointly in I, t. More precisely, we shall compute all their joint moments with appropriately randomized shifts. For this key calculation, we adapt the *random covering scheme* for constructing regenerative sets [15], to develop approximations of joint law of $L_{I,t}$ in Theorem 2.2.

The second part of the proof is devoted to the convergence of the partial-sum process to $Z_{\alpha,\beta,p}$. To illustrate the idea, assume for simplicity that $f(x_1, \ldots, x_p) = 1_A(x_1)\ldots 1_A(x_p)$, where A is a suitable finite-measure subset of E. To work with a series representation of the multiple integral (3) (see (61)), the key ingredient is to show the joint convergence after proper normalization, in I and t, of counting processes of simultaneous returns of i.i.d. dynamical systems, indexed by $i \in I$, in the form

$$\sum_{k=1}^{\lfloor nt\rfloor}\prod_{i\in I} 1_{\{T^k x_i \in A\}}, \tag{7}$$

where the starting points $x_i \in E$ are governed by i.i.d. infinite stationary distributions. For any individual I, our assumptions essentially entail that the simultaneous-return times behave like renewal times of a heavy-tailed renewal process, and then the above is known to converge to the local-time process $L_{I,t}(R^* + V^*)$ for β_p-stable regenerative set R^* with a random shift V^*. This certainly includes $p = 1$ as a special case ([11] and [40, Theorem 6.1]). The challenge lies in characterizing the joint limits for say $(I_j, t_j)_{j=1,\ldots,r}$. Theorem 5.2 is devoted to this task, showing that the limit of the above is $(L_{I_j,t_j})_{j=1,\ldots,r}$ (with respect to random shifts v_j). The proof is of combinatorial nature and by computing the asymptotic moments of (7). A delicate approximation scheme similar to Krickeberg [30] is then developed so that the asymptotic moment formula is extended to the case where the product in (7) is replaced by $f(T^k x_1, \ldots, T^k x_p)$ for a general class of functions of f.

We also mention that a simultaneous work [6] considers the case where the random measure M in (3) is replaced by a Gaussian one so that X_k has finite variance marginally. In that case, a functional non-central limit theorem is established with Hermite processes (e.g., [60]), a well-known class of processes represented by multiple Gaussian integrals, arising as limits. It is remarkable that the proof techniques of [6] exploit special properties of multiple Gaussian integrals, and in particular, the local-time processes and their approximations as we deal with here are not needed in [6]. On the other hand, however, the joint local-time processes are still intrinsically connected to the limit Hermite processes. As shown in the manuscript [4] after the present work, if the multiple-stable integrals in (4) are extended to the Gaussian case $\alpha = 2$, then they yield new representations for the Hermite processes.

The paper is organized as follows. Section 2 introduces the joint local-time processes, and establishes a formula for the joint moments by the random covering scheme. Section 3 reviews certain series representations of multiple integrals and defines formally the limit process $Z_{\alpha,\beta,p}$. Section 4 introduces our model of stationary processes in terms of multiple integrals with long-range dependence, and states the main non-central limit theorem. Section 5 is devoted to the proof of the main theorem. Throughout the paper, C and C_i denote generic positive constants which are independent of n and may change from line to line.

2. Local-time processes

2.1. Definitions and results

We start by recalling some facts about random closed sets on $[0,\infty)$, and in particular, stable regenerative sets. We refer the reader to [36] for more details. Let $\mathbf{F} \equiv \mathbf{F}([0,\infty))$ denote the collection of all closed subsets of $[0,\infty)$. We equip $\mathbf{F}$ with the Fell topology which is generated by the sets $\{F \in \mathbf{F} : F \cap G \neq \emptyset\}$ and $\{F \in \mathbf{F} : F \cap K = \emptyset\}$ for arbitrary open $G \subset [0,\infty)$ and compact $K \subset [0,\infty)$. A random closed set on $[0,\infty)$ is a Borel measurable random element taking values in $\mathbf{F}$. If the law of a random closed set R on $[0,\infty)$ is identical to that of the closed range of a subordinator [8], then R is said to be a regenerative set. The random set R is, in addition, said to be β-stable, $\beta \in (0,1)$, if the corresponding subordinator, say $(\sigma_t)_{t\geq 0}$, is β-stable; that is, $(\sigma_t)_{t\geq 0}$ is a non-decreasing Lévy process determined by

$$\mathbb{E}e^{-\lambda\sigma_t} = \exp(-t\lambda^{\beta}), \lambda \geq 0. \tag{8}$$

In this case, the associated Lévy measure of the regenerative set R is

$$\Pi_{\beta}(dx) = \frac{\beta}{\Gamma(1-\beta)} x^{-1-\beta} 1_{(0,\infty)}(x)dx, \tag{9}$$

which characterizes the law of R.

For our purposes, we shall work with a family of countably many independent stable regenerative sets with independent shifts, and we need in particular to describe their intersections. Let $(R_i)_{i\in\mathbb{N}}$ be i.i.d. β-stable regenerative sets and $(V_i)_{i\in\mathbb{N}}$ be independent random shifts with arbitrary laws, and the two sequences are independent. Under our assumption on β and p in (5), for every

$$I \in \mathcal{D}_p := \left\{ I = (i_1,\dots,i_p) \in \mathbb{N}^p :\ i_1 < \cdots < i_p \right\}, \tag{10}$$

we have

$$\bigcap_{i\in I} (R_i + V_i) \stackrel{d}{=} R_I + V_I, \tag{11}$$

where R_I is a β_p-stable regenerative set and V_I is an independent random variable. In words, the intersection of p independent randomly shifted β-stable regenerative set is β_p-stable regenerative with an independent random shift. This follows for example from the strong Markov property of the regenerative sets. See also [55, Appendix B].

There are multiple ways to construct the local time associated to a regenerative set [26, Chapter 12]. For the series representation of multiple integrals needed later, we use a construction due to Kingman [28] which treats the local time as a functional defined on $\mathbf{F}$. In particular, set

$$L = L^{(\beta_p)} : \mathbf{F} \to [0,\infty],\ \ L(F) := \limsup_{n\to\infty} \frac{1}{l_{\beta_p}(n)} \lambda\left(F + \left[-\frac{1}{2n}, \frac{1}{2n}\right]\right),$$

where λ is the Lebesgue measure, $F + [-1/2n, 1/2n] \equiv \cup_{x\in F}[x - 1/2n, x + 1/2n]$, and the normalization sequence

$$l_{\beta_p}(n) = \int_0^{1/n} \Pi_{\beta_p}((x,\infty))dx = \frac{n^{\beta_p - 1}}{\Gamma(2-\beta_p)},$$

where Π_β is as in (9). The exclusive choice of β_p as in (5) is due to the fact that we shall only deal with local times of shifted β_p-stable regenerative sets, obtained as the intersection of p independent stable regenerative sets. We then define

$$L_t(F) := L(F \cap [0,t]), \quad t \ge 0. \tag{12}$$

Lemma 2.1. *The functionals L and L_t are $\mathcal{B}(\mathbf{F})/\mathcal{B}([0,\infty])$-measurable, where $\mathcal{B}(\mathbf{F})$ and $\mathcal{B}([0,\infty])$ denote the Borel σ-fields on $\mathbf{F}$ and $[0,\infty]$ respectively.*

Proof. Direct sum and intersection are measurable operations for closed sets [36, Theorem 1.3.25]. The Lebesgue measure λ is also a measurable functional from $\mathbf{F}$ to $[0,\infty]$. Indeed, write $[0,\infty) = \cup_{n=0}^\infty K_n$ where $K_n = [n, n+1]$. Then $F \mapsto \lambda(F \cap K_n)$ is a measurable mapping from $\mathbf{F}$ to $[0,\infty]$ since it is upper semi-continuous [36, Proposition E.13]. Hence $F \to \lambda(F) = \sum_{n=0}^\infty \lambda(F \cap K_n)$ is measurable as well. □

From now on, we denote the local-time processes using the notation

$$L_{I,t} \equiv L_t\left(\bigcap_{i\in I}(R_i + V_i)\right),\ \ t \in [0,\infty),\ \ I \in \mathcal{D}_p. \tag{13}$$

In view of (11) and [28, Theorem 3] (conditioning on V_I in (11)), for each $I \in \mathcal{D}_p$, the finite-dimensional distributions of $(L_{I,t})_{t\ge 0}$ coincide with those of a randomly shifted β_p-Mittag-Leffler process, $(\mathcal{M}_{\beta_p}(t - V_I)_+)_{t\ge 0}$, where V_I is independent of $\mathcal{M}_{\beta_p}$. In particular, $(L_{I,t})_{t\ge 0}$ admits a version which has a non-decreasing and continuous path a.s.

The advantage of the above construction is that now for different I, t, the corresponding local times are constructed on a common probability space as measurable functions evaluated at intersections of independent shifted random regenerative sets. We shall develop the formula for their joint moments. We work with a specific choice of the random shifts: most of the time we assume in addition that $(V_i)_{i\in\mathbb{N}}$ are i.i.d. with the law

$$P(V_i \le v) = v^{1-\beta}, v \in [0,1]. \tag{14}$$

Remark 2.1. The law of the shift (14) will show up naturally in our limit theorem later. To understand the origin of (14), recall that a random closed set F on $[0,\infty)$ is said to be stationary, if its law is unchanged under the map $F \to (F \cap [x,\infty)) - x$ for any $x > 0$. While a β-stable regenerative set R_i itself is not stationary, it is known that with an independent shift V_i following an infinite law proportional to $v^{-\beta}dv$ on $\mathbb{R}_+$, the shifted random (with respect to an infinite measure) set $R_i + V_i$ is stationary ([32, Proposition 4.1], see also [17]). The law (14) is nothing but the normalized restriction to $[0,1]$ of this infinite law. As a consequence, one could derive that $\bigcap_{i\in I}(R_i + V_i) \equiv R_I + V_I$ is also stationary with respect to an infinite measure [55, Corollary B.3]. This is in accordance with the stationarity of the increments of the process $Z_{\alpha,\beta,p}$ in (4) (see Section 3.2).

From now on we fix $\beta \in (0, 1)$, $p \in \mathbb{N}$, such that (5) holds. Introduce for $q \geq 2$, a symmetric function $h_q{}^{(\beta)}$ on the off-diagonal subset of $(0, 1)^q$ determined by

$$h_q{}^{(\beta)}(x_1, \dots, x_q) = \Gamma(\beta)\Gamma(2-\beta)\prod_{j=2}^{q}(x_j - x_{j-1})^{\beta-1},\ 0 < x_1 < \cdots < x_q < 1. \tag{15}$$

Here and below, for any $q \in \mathbb{N}$, a q-variate function f is said to be symmetric, if $f(x_1, \dots, x_q) = f(x_{\sigma(1)}, \dots, x_{\sigma(q)})$ for any permutation σ of $\{1, \dots, q\}$. For a symmetric function on the off-diagonal set, we do not specify the values on the diagonal set $\{(x_1, \dots, x_q) \in (0, 1)^q : x_i = x_j \text{ for some i} \neq \text{j}\}$, which has zero Lebesgue measure and hence does not have any impact in our derivation. Introduce also $h_0{}^{(\beta)} := 1$ and $h_1{}^{(\beta)}(x) := \Gamma(\beta)\Gamma(2-\beta)$. The main result of this section is the following.

Theorem 2.2. *Let $(R_i)_{i\in\mathbb{N}}$ be i.i.d. β-stable regenerative sets and $(V_i)_{i\in\mathbb{N}}$ be i.i.d. with law (14), the two sequences being independent. Given a collection of $I_\ell \in \mathcal{D}_p$, $\ell = 1, \dots, r$, set $K = \max\left(\bigcup_{\ell=1}^r I_\ell\right)$. Then, for all $\boldsymbol{t} = (t_1, \dots, t_r) \in [0, 1]^r$,*

$$\mathbb{E}\left(\prod_{\ell=1}^{r} L_{I_\ell,t_\ell}\right) = \frac{1}{\Gamma(\beta_p)^r}\int_{\mathbf{0}<\boldsymbol{x}<\boldsymbol{t}}\prod_{i=1}^{K} h^{(\beta)}_{|\mathcal{I}(i)|}(\boldsymbol{x}_{\mathcal{I}(i)})\,d\boldsymbol{x} \tag{16}$$

with

$$\mathcal{I}(i) := \{\ell \in \{1, \dots, r\} : i \in I_\ell\},\ i = 1, \dots, K. \tag{17}$$

Above and below, we write $\boldsymbol{x} = (x_1, \dots, x_r), d\boldsymbol{x} = dx_1 \dots dx_r$, $\mathbf{0} = (0, \dots, 0), \mathbf{1} = (1, \dots, 1)$, and $\boldsymbol{x} < \boldsymbol{y}$ is understood in the coordinate-wise sense. Also, write

$$\boldsymbol{x}_{\mathcal{I}(i)} = (x_\ell)_{\ell\in\mathcal{I}(i)},$$

understood as the vector in $\mathbb{R}_+^{|\mathcal{I}(i)|}$. (Since each $h_{|\mathcal{I}(i)|}{}^{(\beta)}$ is a symmetric function, the order of coordinates of $\boldsymbol{x}_{\mathcal{I}(i)}$ is irrelevant here.)

Write $\boldsymbol{V}_I = (V_i)_{i\in I}$ and $\boldsymbol{R}_I = (R_i)_{i\in I}$. In view of (13), from now on we write explicitly $L_{I,t} \equiv L_{I,t}(\boldsymbol{R}_I, \boldsymbol{V}_I)$. We have, by Fubini's theorem,

$$\mathbb{E}\left(\prod_{\ell=1}^{r} L_{I_\ell,t_\ell}(\boldsymbol{R}_{I_\ell}, \boldsymbol{V}_{I_\ell})\right) = \int_{(0,1)^K}\mathbb{E}\left(\prod_{\ell=1}^{r} L_{I_\ell,t_\ell}(\boldsymbol{R}_{I_\ell}, \boldsymbol{v}_{I_\ell})\right)(1-\beta)^K\prod_{i=1}^{K} v_i^{-\beta}\,d\boldsymbol{v}.$$

We shall establish a formula for

$$\Psi(\boldsymbol{v}) := \mathbb{E}\left(\prod_{\ell=1}^{r} L_{I_\ell,t_\ell}(\boldsymbol{R}_{I_\ell}, \boldsymbol{v}_{I_\ell})\right),\ \text{for all } \boldsymbol{v} \in (0, 1)^K,$$

where the expectation is with respect to the randomness coming from $\boldsymbol{R}_{I_\ell}$, $\ell = 1, \dots, r$. At the core of our argument is the following proposition. Let g_q, $q \in \mathbb{N}$ be symmetric functions on the off-diagonal subset of $(0,1)^q$ such that

$$g_q^{(\beta)}(x_1,\dots,x_q) = \prod_{j=1}^{q}(x_j - x_{j-1})^{\beta-1}, \quad x_0 := 0 < x_1 < \cdots < x_q < 1, \tag{18}$$

and $g_0^{(\beta)} := 1$. We write $\max(\boldsymbol{v}_I) = \max_{i\in I} v_i$, and similarly for $\min(\boldsymbol{v}_I)$.

Proposition 2.3. *Under the assumption of Theorem 2.2,*

$$\Psi(\boldsymbol{v}) = \frac{1}{\Gamma(\beta_p)^r}\int_{\max(\boldsymbol{v}_{I_\ell})<x_\ell<t_\ell,\,\ell=1,\dots,r} \prod_{i=1}^{K} g^{(\beta)}_{|\mathcal{I}(i)|}(\boldsymbol{x}_{\mathcal{I}(i)} - v_i\mathbf{1})d\boldsymbol{x}. \tag{19}$$

In particular, $\Psi(\boldsymbol{v}) = 0$ *if* $\max(\boldsymbol{v}_{I_\ell}) \ge t_\ell$ *for some* $\ell = 1,\dots,r$.

The proof of the proposition is postponed to Section 2.2.

Proof of Theorem 2.2. We shall compute

$$(1-\beta)^K\int_{(0,1)^K}\Psi(\boldsymbol{v})\prod_{i=1}^{K} v_i^{-\beta}d\boldsymbol{v}. \tag{20}$$

We express the constraint $\max(\boldsymbol{v}_{I_\ell}) < x_\ell$, $\ell = 1,\dots,r$ in (19) as

$$v_i < \min(\boldsymbol{x}_{\mathcal{I}(i)}) =: m_i, \quad i = 1,\dots,K.$$

Then by Proposition 2.3, the expression in (20) becomes

$$\frac{(1-\beta)^K}{\Gamma(\beta_p)^r}\int_{\mathbf{0}<\boldsymbol{x}<\boldsymbol{t}}\int_{\mathbf{0}<\boldsymbol{v}<\boldsymbol{m}}\prod_{i=1}^{K}\left(g^{(\beta)}_{|\mathcal{I}(i)|}(\boldsymbol{x}_{\mathcal{I}(i)} - v_i\mathbf{1})v_i^{-\beta}\right)d\boldsymbol{v}d\boldsymbol{x}. \tag{21}$$

A careful examination shows that

$$g^{(\beta)}_{|\mathcal{I}(i)|}(\boldsymbol{x}_{\mathcal{I}(i)} - v_i\mathbf{1}) = \frac{1}{\Gamma(\beta)\Gamma(2-\beta)}(m_i - v_i)^{\beta-1}h^{(\beta)}_{|\mathcal{I}(i)|}(\boldsymbol{x}_{\mathcal{I}(i)}).$$

Then, (21) becomes

$$\begin{aligned}&\frac{1}{\Gamma(\beta_p)^r}\left(\frac{1}{\Gamma(\beta)\Gamma(1-\beta)}\right)^K\\ &\quad\times\int_{\mathbf{0}<\boldsymbol{x}<\boldsymbol{t}}\int_{\mathbf{0}<\boldsymbol{v}<\boldsymbol{m}}\prod_{i=1}^{K}\left((m_i - v_i)^{\beta-1}v_i^{-\beta}h^{(\beta)}_{|\mathcal{I}(i)|}(\boldsymbol{x}_{\mathcal{I}(i)})\right)d\boldsymbol{v}d\boldsymbol{x}\\ &\quad= \frac{1}{\Gamma(\beta_p)^r}\int_{\mathbf{0}<\boldsymbol{x}<\boldsymbol{t}}\prod_{i=1}^{K}h^{(\beta)}_{|\mathcal{I}(i)|}(\boldsymbol{x}_{\mathcal{I}(i)})d\boldsymbol{x},\end{aligned}$$

by integrating with respect to each v_i separately and applying the relation between beta and gamma functions. Then the desired result follows. □

In particular, we have the following.

Corollary 2.4. *Let $L_{I,t}$ be as in (13). Then for $0 \le s < t \le 1$,*

$$\mathbb{E}\left(L_{I,t} - L_{I,s}\right)^r = \mathbb{E}L^r_{I,t-s} = \frac{\Gamma(\beta)^p \Gamma(2-\beta)^p r!}{\Gamma(\beta_p)\Gamma((r-1)\beta_p + 2)} \cdot (t-s)^{(r-1)\beta_p+1}. \tag{22}$$

Proof. The second equality follows from (16) with $I_1 = \cdots = I_r = I$ and the following identity:

$$\int_{0<x_1<\cdots<x_r<1} \prod_{i=2}^{r} (x_i - x_{i-1})^\gamma d\boldsymbol{x} = \frac{\Gamma(\gamma+1)^{r-1}}{\Gamma(r(\gamma+1)-\gamma+1)} \text{ for all } \gamma > -1, r \ge 2,$$

which can be obtained by changes of variables and the relation between beta and gamma functions. The first equality can be either derived from (16) through an expansion, or from the fact that each underlying shifted β-stable regenerative set $R_i + V_i$ is stationary when restricted to the interval [0, 1] (Remark 2.1). □

Remark 2.2. As mentioned before Remark 2.1, when restricted to [0, 1], $L_{I,t} \stackrel{d}{=} M_{\beta_p}((t-V_I)_+)$ where V_I is a sub-random variable with density function $c_{\beta,p}(1-\beta_p)v^{-\beta_p}$ with $c_{\beta,p} = (\Gamma(\beta)\Gamma(2-\beta))^p/(\Gamma(\beta_p)\Gamma(2-\beta_p))$ [55, Eq.(B.9)]. Therefore, all the properties of $(L_{I,t})_{t\in[0,1]}$, for a single fixed I, can also be derived from the corresponding $M_{\beta_p}((t-V_I)_+)_{t\in[0,1]}$, where $\mathbb{P}(V_I \le v) = v^{1-\beta_p}$ and V_I is independent from M_{β_p}. For example, the rth moments of the latter have been known [39, bottom of page 77], and they entail (22) as an alternative proof.

2.2. Random covering scheme

To establish Proposition 2.3, we shall use a construction of local times motivated from the so-called random covering scheme, by first constructing a stable regenerative set as the set left uncovered by a family of random open intervals based on a Poisson point process (e.g. [9,16] and [8, Chapter 7]).

We shall work with a specific construction of $(R_i)_{i\in\mathbb{N}}$ as follows. Let $\mathcal{N} = \sum_{\ell\in\mathbb{N}} \delta_{(a_\ell,y_\ell,z_\ell)}$ be a Poisson point process on $[0, K) \times \mathbb{R}_+ \times \mathbb{R}_+$, $K \in \mathbb{N}$, with intensity measure $dadyz^{-2}dz$, where δ denotes the Dirac measure. Define

$$O_i := \bigcup_{\ell:a_\ell\in J_i} (y_\ell, y_\ell + z_\ell), \quad R_i := [0,\infty) \setminus O_i, \quad i = 1,\ldots,K,$$

where $J_i = [i-1, i-\beta)$. It is known that $(R_i)_{i=1,\ldots,K}$ constructed above are i.i.d. β-stable regenerative sets starting at the origin [16, Example 1]. In this section we shall work with deterministic shifts

$$\boldsymbol{v} = (v_1,\ldots,v_K) \in (0,1)^K.$$

Let

$$\mathcal{D}_p(m) := \{I \in \mathcal{D}_p : \max I \le m\}, \ m \in \mathbb{N}. \tag{23}$$

where $\mathcal{D}_p$ is as in (10). With the functional L_t in (12), consider

$$L_{I,t} \equiv L_t\left(\bigcap_{i\in I}(R_i + v_i)\right), \ I \in \mathcal{D}_p(K), \ t \ge 0, \tag{24}$$

where $(R_i)_{i\in\mathbb{N}}$ are as above. We emphasize that the notation in (24) is *strictly restricted to this section*, and in particular is different from our notation of $L_{I,t}$ in the other sections, where v_i will be replaced by random V_i.

Next, we consider the following approximations of $(R_i)_{i=1,\dots,K}$. For any $\epsilon > 0$, we set

$$O_i^{(\epsilon)} := \bigcup_{\ell: a_\ell \in J_i, z_\ell \geq \epsilon} (y_\ell, y_\ell + z_\ell), \quad R_i^{(\epsilon)} := [0,\infty) \setminus O_i^{(\epsilon)}, \quad i = 1,\dots,K.$$

Define

$$\widetilde{R}_i^{(\epsilon)} := R_i^{(\epsilon)} + v_i \quad \text{and} \quad \widetilde{R}_I^{(\epsilon)} := \bigcap_{i\in I} \widetilde{R}_i^{(\epsilon)}, \quad I \in \mathcal{D}_p(K).$$

Introduce then

$$L_{I,t}^{(\epsilon)} := \frac{1}{\Gamma(\beta_p)} \left(\frac{\epsilon}{e}\right)^{\beta_p - 1} \int_0^t 1_{\left\{x\in \widetilde{R}_I^{(\epsilon)}\right\}} dx \quad \text{and} \quad \Delta_{s,t}^{(\epsilon)}(I) := L_{I,t}^{(\epsilon)} - L_{I,s}^{(\epsilon)}, \tag{25}$$

for $0 < s < t$. Set also

$$\mathcal{N}_\epsilon := \sum_{\ell:\, z_\ell \geq \epsilon} \delta_{(a_\ell, y_\ell, z_\ell)}.$$

Below we begin with calculating certain asymptotic moments involving (25).

Lemma 2.5. *For any $I_\ell \in \mathcal{D}_p(K)$, $\boldsymbol{v} \in (0,1)^K$, and s_ℓ, t_ℓ satisfying $\max(\boldsymbol{v}_{I_\ell}) < s_\ell < t_\ell \leq 1$, $\ell = 1,\dots,r$, we have*

$$\lim_{\substack{s_\ell \downarrow \max(\boldsymbol{v}_{I_\ell}),\\ \ell=1,\dots,r}} \lim_{\epsilon\downarrow 0} \mathbb{E}\left(\prod_{\ell=1}^r \Delta_{s_\ell,t_\ell}^{(\epsilon)}(I_\ell)\right) \tag{26}$$

$$= \Gamma(\beta_p)^{-r} \int_{\max(\boldsymbol{v}_{I_\ell}) < x_\ell < t_\ell,\ \ell=1,\dots,r} \prod_{i=1}^K g_{|\mathcal{I}(i)|}^{(\beta)}(\boldsymbol{x}_{\mathcal{I}(i)} - v_i \mathbf{1}) d\boldsymbol{x}.$$

We start with a preparation. Define $g_{q,\epsilon}^{(\beta)}$ similarly as $g_q^{(\beta)}$ in (18) as the symmetric function determined by

$$g_{q,\epsilon}^{(\beta)}(x_1,\dots,x_q) = \prod_{j=1}^q f_\epsilon(x_j - x_{j-1}), \quad x_0 := 0 < x_1 < \cdots < x_q < 1,$$

where

$$f_\epsilon(y) := \left(e^{y/\epsilon - 1}\epsilon\right)^{\beta-1} 1_{\{y\leq\epsilon\}} + y^{\beta-1} 1_{\{y>\epsilon\}}, \quad y > 0. \tag{27}$$

We set also $g_{0,\epsilon}^{(\beta)} := 1$.

Proof of Lemma 2.5. First, we claim that if

$$(x_1,\dots,x_q) \in D_q := \{(x_1,\dots,x_q) \in (0,1)^q : x_i \neq x_j \text{ for } i \neq j\}, \ q \in \mathbb{N},$$

then for $\epsilon \in (0,1)$,

$$P\left(x_i \in R_1^{(\epsilon)},\ i = 1,\dots,q\right) = \left(\frac{e}{\epsilon}\right)^{q(\beta-1)} g_{q,\epsilon}^{(\beta)}(x_1,\dots,x_q). \tag{28}$$

For the proof, assume without loss of generality that $x_0 = 0 < x_1 < \cdots < x_q < 1$. Observe that the event in the probability sign in (28) occurs exactly when the Poisson point process $\mathcal{N}$

has no points in the following regions

$$\{(a, y, z) \in [0, 1-\beta) \times [x_{i-1}, x_i) \times \mathbb{R}_+ : y + z > x_i,\ z > \epsilon\}, \quad i = 1, \dots, q.$$

Therefore,

$$P\left(x_i \in R_1^{(\epsilon)},\ i = 1, \dots, q\right) = \prod_{i=1}^{q} \exp\left(-(1-\beta)\int_{x_{i-1}}^{x_i}\int_{\max\{x_i - y, \epsilon\}}^{\infty} \frac{1}{z^2} dz dy\right).$$

By elementary calculations,

$$\int_{x_{i-1}}^{x_i}\int_{\max\{x_i-y,\epsilon\}}^{\infty} \frac{1}{z^2} dz dy = \begin{cases} \dfrac{x_i - x_{i-1}}{\epsilon} & \text{if } x_i - x_{i-1} \le \epsilon; \\ \log\left(\dfrac{e}{\epsilon}(x_i - x_{i-1})\right) & \text{if } x_i - x_{i-1} > \epsilon. \end{cases}$$

Putting these together yields the desired result.

Now let us turn our attention to proving (26). We have, by (25) and Fubini,

$$\begin{aligned} \mathbb{E}\left(\prod_{\ell=1}^{r} \Delta_{s_\ell, t_\ell}^{(\epsilon)}(I_\ell)\right) &= \frac{1}{\Gamma(\beta_p)^r}\left(\frac{\epsilon}{e}\right)^{rp(\beta-1)} \mathbb{E}\left(\prod_{\ell=1}^{r}\int_{s_\ell}^{t_\ell} 1_{\left\{x \in \widetilde{R}_{I_\ell}^{(\epsilon)}\right\}} dx\right) \\ &= \frac{1}{\Gamma(\beta_p)^r}\left(\frac{\epsilon}{e}\right)^{rp(\beta-1)} \int_{\boldsymbol{s}<\boldsymbol{x}<\boldsymbol{t}} P\left(x_\ell \in \widetilde{R}_{I_\ell}^{(\epsilon)},\ \ell = 1, \dots, r\right) d\boldsymbol{x}. \end{aligned} \tag{29}$$

Notice that $\{x_\ell \in \widetilde{R}_{I_\ell}{}^{(\epsilon)}\} = \bigcap_{i:\ell \in \mathcal{I}(i)}\{x_\ell - v_i \in R_i{}^{(\epsilon)}\}$. Therefore by independence, we get

$$P\left(x_\ell \in \widetilde{R}_{I_\ell}^{(\epsilon)},\ \ell = 1, \dots, r\right) = \prod_{i=1}^{K} P\left(x_\ell - v_i \in R_i^{(\epsilon)},\ \ell \in \mathcal{I}(i)\right).$$

Note that the probability above is zero if one of $x_\ell - v_i$ is negative, $i = 1, \dots, K$. Hence by (28) and the fact $\sum_{i=1}^{K} |\mathcal{I}(i)| = rp$, we have

$$\prod_{i=1}^{K} P\left(x_\ell - v_i \in R_i^{(\epsilon)},\ \ell \in \mathcal{I}(i)\right) = \left(\frac{e}{\epsilon}\right)^{rp(\beta-1)} \prod_{i=1}^{K} g_{|\mathcal{I}(i)|,\epsilon}^{(\beta)}(\boldsymbol{x}_{\mathcal{I}(i)} - v_i \mathbf{1}) 1_{\{\boldsymbol{x}_{\mathcal{I}(i)} \ge v_i \mathbf{1}\}}.$$

Summing up, in view of (29), we claim that

$$\begin{aligned} &\lim_{\substack{s_\ell \downarrow \max(\boldsymbol{v}_{I_\ell}),\\ \ell=1,\dots,r}} \lim_{\epsilon \downarrow 0} \mathbb{E}\left(\prod_{\ell=1}^{r} \Delta_{s_\ell,t_\ell}^{(\epsilon)}(I_\ell)\right) \\ &= \Gamma(\beta_p)^{-r} \int_{\max(\boldsymbol{v}_{I_\ell}) < x_\ell < t_\ell,\ \ell=1,\dots,r} \prod_{i=1}^{K} g_{|\mathcal{I}(i)|}^{(\beta)}(\boldsymbol{x}_{\mathcal{I}(i)} - v_i \mathbf{1}) d\boldsymbol{x} \end{aligned} \tag{30}$$

where $g_q^{(\beta)}$ is as in (18). Indeed it is elementary to verify from (27) that as $\epsilon \downarrow 0$, we have $f_\epsilon(y) \uparrow y^{\beta-1}$ for any $y > 0$, and hence $g_{q,\epsilon}^{(\beta)} \uparrow g_q^{(\beta)}$ a.e. So (30) follows from the monotone convergence theorem. □

Next in order to establish Proposition 2.3, we need to identify an a.s. limit of $\lim_{\mathbb{Q} \ni s_\ell \downarrow \max(\boldsymbol{v}_{I_\ell})} \lim_{\epsilon \downarrow 0} \prod_{\ell=1}^{r} \Delta_{s_\ell,t_\ell}^{(\epsilon)}(I_\ell)$, together with an interchangeability between the limits and an expectation. To this aim we shall provide the following two lemmas. In the first lemma below, if $p = 1$, this is the same result as that in [9]. For general I the proof follows the same strategy.

Lemma 2.6. *For every $I \in \mathcal{D}_p(K)$ and s, t satisfying* $\max(\boldsymbol{v}_I) < s < t \le 1$,

$$\mathbb{E}\left(\Delta_{s,t}^{(\eta)}(I) \,\middle|\, \mathcal{N}_\epsilon\right) = \Delta_{s,t}^{(\epsilon)}(I) \text{ a.s. for all } 0 < \eta < \epsilon < s - \max(\boldsymbol{v}_I). \tag{31}$$

Proof. For $\eta \in (0, \epsilon)$, define

$$O_i^{(\eta,\epsilon)} = \bigcup_{\ell: a_\ell \in J_i, z_\ell \in [\eta,\epsilon)} (y_\ell, y_\ell + z_\ell), \quad R_i^{(\eta,\epsilon)} = [0,\infty) \setminus O_i^{(\eta,\epsilon)}, \quad i = 1, \dots, K,$$

and define

$$\widetilde{R}_i^{(\eta,\epsilon)} := R_i^{(\eta,\epsilon)} + v_i \quad \text{and} \quad \widetilde{R}_I^{(\eta,\epsilon)} := \bigcap_{i \in I} \widetilde{R}_i^{(\eta,\epsilon)}, \; I \in \mathcal{D}_p(K).$$

Then for $0 < \eta < \epsilon < s - \max(\boldsymbol{v}_I)$, by Fubini's theorem and the independence property of the Poisson point process, we have

$$\begin{aligned}\mathbb{E}\left(\int_s^t 1_{\left\{x \in \widetilde{R}_I^{(\eta)}\right\}} dx \,\middle|\, \mathcal{N}_\epsilon\right) &= \int_s^t \mathbb{E}\left(1_{\left\{x \in \widetilde{R}_I^{(\epsilon)}\right\}} 1_{\left\{x \in \widetilde{R}_I^{(\eta,\epsilon)}\right\}} \,\middle|\, \mathcal{N}_\epsilon\right) dx \\ &= \int_s^t P\left(x \in \widetilde{R}_I^{(\eta,\epsilon)}\right) 1_{\left\{x \in \widetilde{R}_I^{(\epsilon)}\right\}} dx.\end{aligned} \tag{32}$$

By a calculation similar to that in the proof of Lemma 2.5 (see also [9, page 10]), we have, for $w > \epsilon$,

$$P\left(w \in R_i^{(\eta,\epsilon)}\right) = \exp\left(-(1-\beta) \iint 1_{\{y<w<y+z,\ z \in [\eta,\epsilon)\}} \frac{1}{z^2} dz dy\right) = \left(\frac{\eta}{\epsilon}\right)^{1-\beta}.$$

Hence

$$P\left(x \in \widetilde{R}_I^{(\eta,\epsilon)}\right) = \prod_{i \in I} P\left(x - v_i \in R_i^{(\eta,\epsilon)}\right) = \left(\frac{\eta}{\epsilon}\right)^{p(1-\beta)} = \left(\frac{\eta}{\epsilon}\right)^{1-\beta_p}.$$

Plugging this back into (32), we obtain (31). □

This lemma says that $(\Delta_{s,t}{}^{(\epsilon)}(I))_{\epsilon \in (0, s - \max(\boldsymbol{v}_I))}$ is a martingale as $\epsilon \downarrow 0$ with respect to the filtration $(\sigma(\mathcal{N}_\epsilon))_{\epsilon>0}$. Since the convergence of the moments of $\Delta_{s,t}^{(\epsilon)}(I)$ as $\epsilon \downarrow 0$, was established in the proof of Lemma 2.5, by the martingale convergence theorem, we have for every $0 < s < t \le 1$,

$$\lim_{\epsilon \downarrow 0} \Delta_{s,t}^{(\epsilon)}(I) =: \Delta_{s,t}^*(I) \text{ a.s. and in } L^m \text{ for all } m \in \mathbb{N}. \tag{33}$$

Then there exists a probability-one set, on which the convergence in (33) holds for all $s \in \mathbb{Q} \cap (0, t)$. Since $\Delta_{s,t}^*(I)$ is non-increasing in $s \in \mathbb{Q} \cap (0, t)$, one can a.s. define

$$L_{I,t}^* := \begin{cases} \lim\limits_{\mathbb{Q} \ni s \downarrow \max(\boldsymbol{v}_I)} \Delta_{s,t}^*(I), & \text{if } \max(\boldsymbol{v}_I) < t, \\ 0 & \text{if } \max(\boldsymbol{v}_I) \ge t. \end{cases} \tag{34}$$

Lemma 2.7. *For any $0 < t \le 1$, $\boldsymbol{v} \in (0,1)^K$, and any $I \in \mathcal{D}_p(K)$, we have $L_{I,t} = L_{I,t}^*$ almost surely.*

Proof. First we write

$$\widetilde{R}_I = \bigcap_{i \in I} (R_i + v_i) = R_I + V_I$$

where $V_I := \inf \widetilde{R}_I$ and $R_I := (\widetilde{R}_I \cap [V_I, \infty)) - V_I$. (Note that even with all v_i fixed, V_I is still a non-degenerate random variable with probability one, unless $v_i = v$ for all $i \in I$.) In view of [55, Lemma 3.1], R_I is a β_p-stable regenerative set and $V_I \geq 0$ is a random shift independent of R_I. Observe that $L_{I,t} = L^*_{I,t} = 0$ for $t \in [0, V_I)$, so it suffices to show $L_{I,t+V_I} = L^*_{I,t+V_I}$ for any $t \geq 0$ a.s. By [28, Theorem 3], $L_{I,t+V_I} = L_t(R_I)$ is a version of the standard local time of R_I (or a standard β_p-Mittag-Leffler process). Here by "standard", we mean that $L_t(R_I)$ has the same law as the inverse of a standard β_p-stable subordinator satisfying (8) but with β there replaced by β_p. On the other hand, using Kolmogorov's criterion [25, Theorem 3.23] and the formula of moments in Lemma 2.5, one can verify that $\{L^*_{I,t}\}_{t\geq 0}$ admits a version which is continuous in t. It also follows from the construction that $L^*_{I,t+V_I}$ is additive and increases only over $t \in R_I$. Then by Maisonneuve [34, Theorem 3.1], for some constant $c > 0$, $L^*_{I,t+V_I} = cL_{I,t+V_I}$ almost surely for each $t \geq 0$.

We shall show that $c = 1$. Taking $t = 1$, $\mathbb{E}L_{I,1+V_I} = 1/\Gamma(\beta_p + 1)$ by our knowledge of Mittag-Leffler process (e.g. [11, Proposition 1(a)]). Now to show $c = 1$, it suffices to show that $\mathbb{E}L^*_{I,1+V_I} = 1/\Gamma(\beta_p + 1)$.

Let $(L^o_{I,t})_{t\geq 0}$ be $(L^*_{I,t})_{t\geq 0}$ in (34) but with $\boldsymbol{v}_I = \mathbf{0}$. From (19), one may verify that $\mathbb{E}L^o_{I,1} = 1/\Gamma(\beta_p + 1)$ (in fact, comparing all the moments leads to $L^o_{I,1} \stackrel{d}{=} L_{I,1+V_I}$). The proof is concluded by showing that

$$(L^*_{I,t+V_I})_{t\geq 0} \stackrel{d}{=} (L^o_{I,t})_{t\geq 0}. \tag{35}$$

This essentially follows from a strong regenerative property. Indeed, for fixed $\epsilon > 0$, let $\mathcal{G}^{(\epsilon)}_t$, $t \geq 0$, be the augmented filtration generated by the p-dimensional process $(D^{(\epsilon)}_{i,t},\ i \in I)_{t\geq 0}$, where $D_{i,t}{}^{(\epsilon)} = \inf(\widetilde{R}^{(\epsilon)}_i \cap (t, \infty))$. Note that for each $i \in I$, $\widetilde{R}_i{}^{(\epsilon)} = R_i{}^{(\epsilon)} + v_i$ is regenerative with respect to $(\mathcal{G}^{(\epsilon)}_t)_{t\geq 0}$ in the sense of [15, Definition 1.1]: this can be seen from the fact that $R^{(\epsilon)}_i$ is regenerative with respect to $(\mathcal{G}^{(\epsilon)}_{t+v_i})_{t\geq 0}$ (see e.g. [16, Eq.(6)]).

Next, consider the shift operator θ_t on $\mathbf{F}$ as $\theta_t F = (F \cap [t, \infty)) - t$, for $t \geq 0$. Write $V^{(\epsilon)}_I := \inf \widetilde{R}^{(\epsilon)}_I$, which is finite almost surely. Observe that $V_I{}^{(\epsilon)} = \inf\{t > 0 : D_{i,t-}{}^{(\epsilon)} = t,\ \text{for all } i \in I\}$, and hence it is an optional time with respect to $(\mathcal{G}_t{}^{(\epsilon)})_{t\geq 0}$. Note in addition that $V^{(\epsilon)}_I \in \widetilde{R}^{(\epsilon)}_i$ for all $i \in I$, and that $\theta_{V^{(\epsilon)}_I} \widetilde{R}^{(\epsilon)}_i$'s are conditionally independent given $\mathcal{G}^{(\epsilon)}_{V^{(\epsilon)}_I}$. So it follows from the strong regenerative property [15, Proposition (1.4)] that $\left(\theta_{V^{(\epsilon)}_I} \widetilde{R}^{(\epsilon)}_i\right)_{i\in I} \stackrel{d}{=} \left(R^{(\epsilon)}_i\right)_{i\in I}$. Therefore,

$$\left(\int_s^t 1_{\left\{x\in\theta_{V^{(\epsilon)}_I}\widetilde{R}^{(\epsilon)}_I\right\}} dx\right)_{0<s<t} \stackrel{d}{=} \left(\int_s^t 1_{\left\{x\in R^{(\epsilon)}_I\right\}} dx\right)_{0<s<t}.$$

Now, examining the construction starting from (25), we see that the relation above leads to (35). This completes the proof. □

By combining all the lemmas above, it is now straightforward to complete the proof of Proposition 2.3.

Proof of Proposition 2.3. In view of Lemmas 2.5 and 2.7, it suffices to show that

$$\lim_{\substack{\mathbb{Q}\ni s_\ell\downarrow\max(v_{I_\ell})\\ \ell=1,\ldots,r}} \lim_{\epsilon\downarrow 0} \mathbb{E}\left(\prod_{\ell=1}^r \Delta^{(\epsilon)}_{s_\ell,t_\ell}(I_\ell)\right) = \mathbb{E}\left(\prod_{\ell=1}^r L^*_{I_\ell,t_\ell}\right).$$

By the L^m convergence in (33),

$$\lim_{\epsilon\downarrow 0}\mathbb{E}\left(\prod_{\ell=1}^r \Delta^{(\epsilon)}_{s_\ell,t_\ell}(I_\ell)\right)=\mathbb{E}\left(\prod_{\ell=1}^r \Delta^{*}_{s_\ell,t_\ell}(I_\ell)\right).$$

It then remains to show that

$$\lim_{\substack{\mathbb{Q}\ni s_\ell\downarrow\max(v_{I_\ell})\\ \ell=1,\dots,r}}\mathbb{E}\left(\prod_{\ell=1}^r \Delta^{*}_{s_\ell,t_\ell}(I_\ell)\right)=\mathbb{E}\left(\prod_{\ell=1}^r L^{*}_{I_\ell,t_\ell}\right),$$

for which we have established the pointwise convergence in (34). To enhance to the convergence in expectation via uniform integrability, we need a uniform upper bound for $\mathbb{E}(\prod_{\ell=1}^r \Delta^{*}_{s_\ell,t_\ell}(I_\ell)^2)$ in terms of $\boldsymbol{s}$. This follows from a reexamination of (30). The proof is then completed. □

3. Stable-regenerative multiple-stable processes

3.1. Series representations for multiple integrals

We review the multilinear series representation of off-diagonal multiple integrals with respect to an infinitely divisible random measure without a Gaussian component. Our main reference is Szulga [59] and Samorodnitsky [53, Chapter 3].

Let $(E,\mathcal{E},\mu)$ be a measure space where μ is σ-finite and atomless. First we recall the infinitely divisible random measure without Gaussian component. Let $M(\cdot)$ be such a random measure with a control measure μ. Then, its law is determined by

$$\mathbb{E}e^{i\theta M(A)}=\exp\left(-\mu(A)\int_{\mathbb{R}}(1-\cos(\theta y))\rho(dy)\right),\ A\in\mathcal{E},\ \mu(A)<\infty,\ \theta\in\mathbb{R},$$

where ρ is a *symmetric* Lévy measure satisfying $\int_{\mathbb{R}}(1\wedge y^2)\rho(dy)\in(0,\infty)$ [53, Section 3.2]. We shall later on need a generalized inverse of the tail Lévy measure defined as

$$\rho^{\leftarrow}(y):=\inf\{x>0:\rho(x,\infty)\le y/2\},\quad y>0.$$

A special case of our interest is the symmetric α-stable (SαS) random measure on $(E,\mathcal{E})$, denoted by S_α ($\alpha\in(0,2)$), determined by $\mathbb{E}e^{iuS_\alpha(A)}=\exp(-|u|^\alpha\mu(A))$ for all $A\in\mathcal{E},\mu(A)<\infty$. In this case, the Lévy measure is

$$\rho(dy)=\frac{\alpha C_\alpha}{2}|y|^{-\alpha-1}1_{\{y\neq 0\}}dy\quad\text{with}\quad C_\alpha=\left(\int_0^\infty \sin(y)y^{-\alpha}dy\right)^{-1},\tag{36}$$

and $\rho^{\leftarrow}(y)=C_\alpha^{1/\alpha}y^{-1/\alpha}$, $y>0$. Throughout we shall work with the following assumption for ρ:

$$\rho((x,\infty))\in \mathrm{RV}_\infty(-\alpha),\alpha\in(0,2)\text{ and }\rho((x,\infty))=O(x^{-\alpha_0}),\ \alpha_0<2\text{ as }x\downarrow 0,\tag{37}$$

where $\mathrm{RV}_\infty(-\alpha)$ denotes the class of functions regularly varying with index $-\alpha$ at infinity [12].

Now we introduce the series representations for multiple integrals with respect to M. When working with series representations, we shall always treat integrands supported within a finite-measure subspace of E^p. In particular, fix an index set $\mathbb{T}$ and suppose $(f_t)_{t\in\mathbb{T}}$ is a family of product measurable symmetric functions from E^p to $\mathbb{R}$, such that $\cup_{t\in\mathbb{T}}\mathrm{supp}(f_t)\subset B^p$ for some $B\in\mathcal{E}$ with $\mu(B)\in(0,\infty)$, where $\mathrm{supp}(f_t):=\{x\in E^p:\ f_t(x)\neq 0\}$.

Now let $(\varepsilon_i)_{i\in\mathbb{N}}$ be i.i.d. Rademacher random variables, $(\Gamma_i)_{i\in\mathbb{N}}$ be consecutive arrival times of a standard Poisson process, and $(U_i)_{i\in\mathbb{N}}$ be i.i.d. random elements taking values in E with distribution $\mu(\cdot\cap B)/\mu(B)$, all assumed to be independent. Then for every $A\in\mathcal{E}$ with $A\subset B$, the series $M_0(A) := \sum_{i=1}^\infty \varepsilon_i \rho^{\leftarrow}(\Gamma_i/\mu(B))\delta_{U_i}(A)$ converges a.s. and $M_0 \stackrel{d}{=} M$ ([53, Theorem 3.4.3], see also [49]). Without loss of generality we shall make the identification $M = M_0$. Then the (off-diagonal) multiple integral of f_t with respect to M can be defined as

$$\left(\int_{B^p}' f_t(x_1,\dots,x_p)M(dx_1)\cdots M(dx_p)\right)_{t\in\mathbb{T}} \tag{38}$$
$$= \left(p!\sum_{I\in\mathcal{D}_p}\left(\prod_{i\in I}\varepsilon_i\rho^{\leftarrow}(\Gamma_i/\mu(B))\right)f_t(\boldsymbol{U}_I)\right)_{t\in\mathbb{T}},$$

where

$$\boldsymbol{U}_I \equiv (U_{i_1},\dots,U_{i_p}) \text{ for } I=(i_1,\dots,i_p)\in\mathcal{D}_p,$$

as long as the multilinear series in (38) converges a.s. It is known that the convergence holds if and only if

$$\sum_{I\in\mathcal{D}_p}\prod_{i\in I}\rho^{\leftarrow}(\Gamma_i/\mu(B))^2 f_t(\boldsymbol{U}_I)^2 < \infty \quad \text{a.s.}, \tag{39}$$

and in this case the convergence also holds unconditionally, namely, regardless of any deterministic permutation of its entries ([31] and [54, Remark 1.5]). On the other hand, a non-symmetric integrand, say g, can always be symmetrized without affecting the resulting multiple stochastic integral, by considering $(p!)^{-1}\sum_\sigma g(x_{\sigma(1)},\dots,x_{\sigma(p)})$, summing over all permutations of $\{1,\dots,p\}$.

The following lemma provides a condition to verify the convergence under (37).

Lemma 3.1. *Let $(\varepsilon_i)_{i\in\mathbb{N}}$ and $(\Gamma_i)_{i\in\mathbb{N}}$ be as above and let $f: E^p\to\mathbb{R}$ be a measurable symmetric function. For every $p\in\mathbb{N}, c>0$,*

$$\sum_{I\in\mathcal{D}_p}\left(\prod_{i\in I}\varepsilon_i\rho^{\leftarrow}(\Gamma_i/c)\right)f(\boldsymbol{U}_I)$$

converges almost surely and unconditionally, if $\mathbb{E}f(\boldsymbol{U}_I)^2<\infty$.

Proof. It suffices to prove for $c=1$, and in this case the convergence criterion (39) becomes

$$\sum_{I\in\mathcal{D}_p}\prod_{i\in I}\rho^{\leftarrow}(\Gamma_i)^2 f(\boldsymbol{U}_I)^2 < \infty \quad \text{a.s.} \tag{40}$$

Define

$$\mathcal{D}_{\le p}(M) := \{I\in\mathcal{D}_k :\ 0\le k\le p,\ \max I\le M\}, \tag{41}$$
$$\mathcal{H}(k,M) := \{I\in\mathcal{D}_k :\ \min I > M\},\ k=0,\dots,p, \tag{42}$$

for $M\in\mathbb{N}$, to be chosen later, where $\mathcal{D}_k$ is as in (10) with $\mathcal{D}_0=\emptyset$. Then the series in (40) is equal to

$$\sum_{I_1\in\mathcal{D}_{\le p}(M)}\left(\prod_{i\in I_1}\rho^{\leftarrow}(\Gamma_i)^2\right)\left[\sum_{I_2\in\mathcal{H}(p-|I_1|,M)}\left(\prod_{i\in I_2}\rho^{\leftarrow}(\Gamma_i)^2\right)f(\boldsymbol{U}_{I_1\cup I_2})^2\right]. \tag{43}$$

Note that $\mathcal{D}_{\le p}(M)$ is finite. Hence to prove the almost-sure convergence of the non-negative series, it suffices to show that for each $I_1 \in \mathcal{D}_{\le p}(M)$, the term in the bracket of (43) is finite almost surely. This follows, in view of (39), if we can show that

$$\sum_{I_2\in\mathcal{H}(k,M)} \mathbb{E}\left(\prod_{i\in I_2} \rho^{\leftarrow}(\Gamma_i)^2\right) \mathbb{E} f(\boldsymbol{U}_{I_1\cup I_2})^2 < \infty,\ k = 1,\dots,p.$$

From assumption (37), it follows that $\rho^{\leftarrow}(x) \in \mathrm{RV}_0(-1/\alpha)$, where the latter denotes the class of functions regularly varying at zero, and $\rho^{\leftarrow}(x) = O(x^{-1/\alpha_0})$ as $x \to \infty$. By Potter's bound and the fact that $\rho^{\leftarrow}$ is monotone, it then follows that there exists $C > 0$ and $\epsilon > 0$ such that

$$\rho^{\leftarrow}(x) \le C\left(x^{-1/\alpha_0} + x^{-(1/\alpha)-\epsilon}\right),\ \text{ for all } x > 0.$$

The following estimate can be obtained via Hölder's inequality as in [54, Eq.(3.2)]: given $\delta > 0$, there exists a constant $C > 0$, such that

$$\mathbb{E}\left(\prod_{i\in I_2} \Gamma_i^{-\delta}\right) \le C \prod_{i\in I_2} i^{-\delta} \quad \text{ for all } I_2 = (i_1,\dots,i_k) \in \mathcal{D}_k \text{ with } i_1 > \delta k. \tag{44}$$

It then follows that for all $\delta_1, \delta_2 > 0$,

$$\mathbb{E}\left(\prod_{i\in I_2} (\Gamma_i^{-\delta_1} + \Gamma_i^{-\delta_2})\right) \le C \prod_{i\in I_2} i^{-(\delta_1\wedge\delta_2)} \ \text{ for all } I_2 \in \mathcal{D}_k \text{ s.t.} \min I_2 > (\delta_1 \vee \delta_2)k.$$

Therefore, taking $M > 2p\max\{1/\alpha_0, (1/\alpha+\epsilon)\}$ and $\alpha^* := ((1/\alpha)+\epsilon)\wedge 1/\alpha_0 > 1/2$ we have,

$$\begin{aligned}\sum_{I\in\mathcal{H}(k,M)} \mathbb{E}\left(\prod_{i\in I} \rho^{\leftarrow}(\Gamma_i)^2\right) &\le C \sum_{I\in\mathcal{H}(k,M)} \prod_{i\in I} i^{-2\alpha^*}\\ &\le C \sum_{I\in\mathcal{D}_k} \prod_{i\in I} i^{-2\alpha^*} \le C\left(\sum_{i=1}^{\infty} i^{-2\alpha^*}\right)^k < \infty. \quad \square\end{aligned}$$

3.2. *Stable-regenerative multiple-stable process*

Recall our assumption on p, β and β_p in (5), and the local-time functional L_t in (12). We introduce the *stable-regenerative multiple-stable process of multiplicity* p, denoted throughout by $Z_{\alpha,\beta,p} \equiv (Z_{\alpha,\beta,p}(t))_{t\ge 0}$, $\alpha \in (0, 2)$, via the multiple integrals:

$$Z_{\alpha,\beta,p}(t) := \int_{(\mathbf{F}\times[0,\infty))^p} L_t\left(\bigcap_{i=1}^p (R_i + v_i)\right) S_{\alpha,\beta}(dR_1, dv_1)\cdots S_{\alpha,\beta}(dR_p, dv_p),\ \ t \ge 0. \tag{45}$$

where $S_{\alpha,\beta}(\cdot)$ is a SαS random measure on $\mathbf{F}\times[0,\infty)$ with control measure $P_\beta \times (1-\beta)v^{-\beta}dv$. Note that when $p = 1$, the process $Z_{\alpha,\beta,p}$ is represented as a stable integral, and in particular, is the same process known as the *β-Mittag-Leffler fractional SαS motion* introduced in [40]. The well-definedness of the multiple integral above when $t \in [0, 1]$ directly follows from Lemma 3.1 and Theorem 2.2, and can be similarly verified for $t > 1$ by a proper scaling. More specifically, if $t \in [0, 1]$, using the fact that L_t vanishes when any $v_i > 1$ in (45), the process $Z_{\alpha,\beta,p}(t)$ can be represented in the form of (45), with $\mathbf{F}\times[0,\infty)$ replaced by $\mathbf{F}\times[0, 1]$, and the control measure replaced by a probability measure $P_\beta \times (1-\beta)v^{-\beta}1_{\{v\in[0,1]\}}dv$. Then, as in (38), one can obtain the series representation

$$\left(Z_{\alpha,\beta,p}(t)\right)_{t\in[0,1]} \overset{f.d.d.}{=} \left(p!C_\alpha^{p/\alpha} \sum_{I\in\mathcal{D}_p} \left(\prod_{i\in I} \varepsilon_i \Gamma_i^{-1/\alpha}\right) L_t\left(\bigcap_{i\in I}(R_i+V_i)\right)\right)_{t\in[0,1]}, \tag{46}$$

where $f.d.d.$ stands for finite-dimensional distributions, C_α is as in (36), $(\varepsilon_i)_{i\in\mathbb{N}}$, $(\Gamma_i)_{i\in\mathbb{N}}$ are as in Section 3.1, $(R_i)_{i\in\mathbb{N}}$ are i.i.d. β-stable regenerative sets, $(V_i)_{i\in\mathbb{N}}$ are i.i.d. random variables with law (14), and the four sequences are independent from each other.

As a direct consequence of the functional limit theorem proved in Theorem 4.1 and Lamperti's theorem [33], the process $Z_{\alpha,\beta,p}$ turns out to be self-similar with Hurst index

$$H = \beta_p + \frac{1-\beta_p}{\alpha} = p\left(\frac{1}{\alpha}-1\right)(1-\beta)+1 \in (1/2,\infty),$$

that is,

$$(Z_{\alpha,\beta,p}(ct))_{t\geq 0} \overset{d}{=} c^H (Z_{\alpha,\beta,p}(t))_{t\geq 0} \text{ for all } c>0,$$

and have stationary increments. In view of self-similarity, we shall only work with $(Z_{\alpha,\beta,p}(t))_{t\in[0,1]}$ onward.

We conclude this section with a result on the path regularity of $Z_{\alpha,\beta,p}$.

Proposition 3.2. *The process $Z_{\alpha,\beta,p}$ admits a continuous version whose path is locally δ-Hölder continuous a.s. for any $\delta\in(0,\beta_p)$.*

Proof. We restrict $t\in[0,1]$ without loss of generality and work with the series representation (46). In view of independence, assume for convenience that the underlying probability space is the product space of $(\Omega_i,\mathcal{F}_i,P_i)$, $i=1,2$, where $(\epsilon_i)_{i\in\mathbb{N}}$ depends only on $\omega_1\in\Omega_1$ and $(\Gamma_i,R_i,V_i)_{i\in\mathbb{N}}$ depends only on $\omega_2\in\Omega_2$. The probability measures P_1 and P_2 are such that those random variables have the desired law, and P is the product measure of P_1 and P_2 on the product space. We also write $\mathbb{E}_i$ the integration with respect to P_i over Ω_i, $i=1,2$,

We shall work with the series representation in (46), where without loss of generality we replace $\overset{f.d.d.}{=}$ with $=$. Then as before, write $L_{I,t} = L_t(\bigcap_{i\in I}(R_i+V_i))$. Since $L_{I,t}(\omega_1,\omega_2)$ is a constant function of ω_1 with ω_2, I, t fixed, we write $L_{I,t}(\omega_1,\omega_2) = L_{I,t}(\omega_2)$ for the sake of simplicity. In addition, we shall identify $L_{I,t}$ with its continuous version, which exists in view of Corollary 2.4 and Kolmogorov's criterion.

Using a generalized Khinchine inequality for multilinear forms in Rademacher random variables ([29], see also [54, Theorem 1.3 (ii)]), for any $r>1$ and some constant $C>0$, we have for $0\leq s<t\leq 1$ that, writing $\boldsymbol{\omega}=(\omega_1,\omega_2)$,

$$\mathbb{E}_1|Z_{\alpha,\beta,p}(t)(\boldsymbol{\omega}) - Z_{\alpha,\beta,p}(s)(\boldsymbol{\omega})|^r \leq CY_{s,t}(\omega_2)$$

with

$$Y_{s,t}(\omega_2) := \left(\sum_{I\in\mathcal{D}_p}\left(\prod_{i\in I}\Gamma_i(\omega_2)^{-2/\alpha}\right)\left|L_{I,t}(\omega_2)-L_{I,s}(\omega_2)\right|^2\right)^{r/2}, 0\leq s<t\leq 1.$$

The two-parameter process $(Y_{s,t})_{0\leq s<t\leq 1}$ is finite P_2-almost surely in view of Lemma 3.1 and Corollary 2.4 (note that (37) is satisfied with $\alpha=\alpha_0$ in this case). Since $L_{I,t}$ is a shifted

β_p-Mittag-Leffler process, in view of [40, Lemma 3.4], the random variable

$$K_I(\omega_2) := \sup_{(s,t)\in D} \frac{|L_{I,t}(\omega_2) - L_{I,s}(\omega_2)|}{(t-s)^{\beta_p}|\log(t-s)|^{1-\beta_p}}$$

is P_2-a.s. finite, and has finite moments of all orders, where $D = \{(s,t) : \ 0 \le s < t \le 1, \ t-s < 1/2\}$. Hence for all $(s,t) \in D$, we have

$$\mathbb{E}_1|Z_{\alpha,\beta,p}(t)(\boldsymbol{\omega}) - Z_{\alpha,\beta,p}(s)(\boldsymbol{\omega})|^r \le C(t-s)^{r\beta_p}|\log(t-s)|^{r(1-\beta_p)}M(\omega_2),$$

where

$$M(\omega_2) = \left(\sum_{I\in\mathcal{D}_p}\left(\prod_{i\in I}\Gamma_i(\omega_2)^{-2/\alpha}\right)K_I(\omega_2)^2\right)^{r/2},$$

which is finite P_2-a.s.: this is a special case of (40), addressed in the proof of Lemma 3.1. Take r large enough so that $r\beta_p > 1$. Then by Kolmogorov's criterion, for any $\delta \in (0, \beta_p)$ and P_2-a.e. $\omega_2 \in \Omega_2$, $Z_{\alpha,\beta,p}(t)(\cdot,\omega_2)$ admits a version $Z^*_{\alpha,\beta,p}(t)(\cdot,\omega_2)$ under P_1 whose path is locally δ-Hölder continuous P_1-a.s. By Fubini, $Z^*_{\alpha,\beta,p}(t)(\boldsymbol{\omega})$ is also a version of $Z_{\alpha,\beta,p}(t)(\boldsymbol{\omega})$ under $P_1 \times P_2$ which has a locally δ-Hölder continuous path $(P_1 \times P_2)$-a.s. □

4. A functional non-central limit theorem

4.1. Infinite ergodic theory and Krickeberg's setup

We shall introduce some concepts in the infinite ergodic theory necessary for the formulation of our results. Our main reference is Aaronson [2]. Let $(E, \mathcal{E}, \mu)$ be a measure space where μ is a σ-finite measure satisfying $\mu(E) = \infty$. Suppose that $T : E \to E$ is a measure-preserving transform, namely, T is measurable and $\mu(T^{-1}B) = \mu(B)$ for all $B \in \mathcal{E}$. Let $\widehat{T}$ denote the *dual* (a.k.a. Perron–Frobenius, or transfer) operator of T, defined by

$$\widehat{T} : L^1(\mu) \to L^1(\mu), \quad \widehat{T}g := \frac{d\mu_g \circ T^{-1}}{d\mu},$$

where $\mu_g(B) = \int_B g d\mu$, $B \in \mathcal{E}$. It is also characterized by the relation

$$\int_E (\widehat{T}g)\cdot h d\mu = \int_E g\cdot(h\circ T)d\mu, \qquad \text{for all } g \in L^1(\mu),\ h \in L^\infty(\mu). \tag{47}$$

We always assume that T is *ergodic*, namely, $T^{-1}B = B \bmod \mu$ implies either $\mu(B) = 0$ or $\mu(B^c) = 0$, and that T is *conservative*, namely, for any $B \in \mathcal{E}$ with $\mu(B) > 0$, we have $\sum_{k=1}^\infty 1_B(T^k x) = \infty$ for a.e. $x \in B$. It is known that T is ergodic and conservative, if and only if for any $B \in \mathcal{E}$ with $\mu(B) > 0$, we have

$$\sum_{k=1}^\infty 1_B(T^k x) = \infty \qquad \text{for a.e. } x \in E,$$

or equivalently

$$\sum_{k=1}^\infty \widehat{T}^k g = \infty \qquad \text{a.e. for all } g \in L^1(\mu),\ g \ge 0,\ \text{a.e. and } \mu(g) > 0. \tag{48}$$

We shall, however, need a more quantitative description of the ergodic property of T, which provides information about the rate of divergence in (48). The following assumption is formulated in the spirits of Krickeberg [30] and Kesseböhmer and Slassi [27]. We shall use the following convention throughout: *any function defined on a subspace (e.g. A) will be extended to the full space (e.g. E) by assuming zero value outside the subspace, whenever necessary.*

Assumption 1. There exists $A \in \mathcal{E}$ with $\mu(A) \in (0, \infty)$ and A is a Polish space with $\mathcal{E}_A := \mathcal{E} \cap A$ being its Borel σ-field. In addition, there exists a positive rate sequence $(b_n)_{n\in\mathbb{N}}$ satisfying

$$(b_n) \in \mathrm{RV}_\infty(1-\beta), \quad \beta \in (0,1), \tag{49}$$

where $\mathrm{RV}_\infty(1-\beta)$ denotes the class of sequences regularly varying with index $1-\beta$ at infinity [12], so that

$$\lim_{n\to\infty} b_n \widehat{T}^n g(x) = \mu(g) \quad \text{uniformly for a.e. } x \in A \tag{50}$$

for all bounded and μ-a.e. continuous g on A.

Remark 4.1. The relation (50) was first explicitly formulated in [27] and termed as the *uniform return* condition. Due to the existence of weakly wandering sets [20], the relation (50) can fail even for a bounded integrable function g supported within A. To be able to treat a large family of integrands f in Theorem 4.1, we adopt an idea of [30]: we impose a topological structure on the subspace A, and retrain our attention to bounded and a.e. continuous functions supported within A. It is worth noting the resemblance of this approach to the theory of weak convergence of measures. See Section 4.3 for examples satisfying Assumption 1.

Remark 4.2. Assumption 1 has an alternative characterization in Proposition 4.2. Typically, the whole space E is Polish as well. Nevertheless, we stress that when a topological concept such as continuity, interior or boundary is mentioned, we solely refer to the Polish topology on the subspace A (or A^p in the context of product space).

Additionally, for A in Assumption 1, and $x \in E$, we define the *first entrance time*

$$\varphi(x) = \varphi_A(x) = \inf\{k \geq 1 : T^k x \in A\}, \tag{51}$$

and the *wandering rate* sequence

$$w_n = \mu(\varphi \leq n) = \mu\left(\bigcup_{k=1}^{n} T^{-k} A\right), \quad n \in \mathbb{N}, \tag{52}$$

which measures the amount of E which visits A up to time n. Kesseböhmer and Slassi [27, Proposition 3.1] proved that under Assumption 1,

$$b_n \sim \Gamma(\beta)\Gamma(2-\beta) w_n \tag{53}$$

as $n \to \infty$. In particular, $w_n \in \mathrm{RV}_\infty(1-\beta)$ (note that their β corresponds to our $1-\beta$, and their w_n corresponds to our w_{n+1}).

4.2. A non-central limit theorem

Let $(E, \mathcal{E}, \mu)$ be σ-finite infinite measure space and T a measure-preserving ergodic and conservative transform. We recall our model, a stationary sequence $(X_k)_{n\in\mathbb{N}}$ in (3), where M is the infinitely divisible random measure on $(E, \mathcal{E})$ with symmetric Lévy measure ρ and control measure μ as in Section 3.1.

We are now ready to state the main result of the paper. Below $\mu^{\otimes p}$ denotes the p-product measure of μ on the product σ-field $\mathcal{E}^p$.

Theorem 4.1. *Assume β, p and β_p are as in (5). For $(X_k)_{k\in\mathbb{N}}$ introduced in (3), suppose the following assumptions hold:*

(a) The Lévy measure ρ satisfies (37).
(b) There exists $A \in \mathcal{E}$ satisfying Assumption 1, and f is a bounded $\mu^{\otimes p}$-a.e. continuous function on A^p.

Then the stationary process $(X_k)_{k\in\mathbb{N}}$ in (3) is well-defined. Furthermore,

$$\left(\frac{1}{c_n}\sum_{k=1}^{\lfloor nt\rfloor} X_k\right)_{t\in[0,1]} \Rightarrow \Gamma(\beta_p) C_\alpha^{-p/\alpha} \mu^{\otimes p}(f) \cdot \big(Z_{\alpha,\beta,p}(t)\big)_{t\in[0,1]}, \tag{54}$$

in $D([0,1])$ with respect to the uniform metric as $n \to \infty$, where $Z_{\alpha,\beta,p}(t)$ is the stable-regenerative multiple-stable process defined in (45). Moreover,

$$c_n = n \cdot \left(\frac{\rho^{\leftarrow}(1/w_n)}{b_n}\right)^p \in \mathrm{RV}_\infty\left(\beta_p + \frac{1-\beta_p}{\alpha}\right), \tag{55}$$

where (w_n) is the wandering rate associated to A in (52) and C_α is as in (36).

The proof of Theorem 4.1 is carried out in Section 5.

Remark 4.3. Compared to the result for $p = 1$ established in [40], we assume the same assumption on ρ, but strictly stronger assumptions on the dynamical system and f. Indeed, weaker notions *Darling–Kac set* and *uniform set* were adopted in [40] instead of (50). For example, a set A is a *Darling–Kac set* if for some positive sequence $(a_n)_{n\in\mathbb{N}}$ tending to ∞,

$$\frac{1}{a_n}\sum_{k=1}^{n} \widehat{T}^k 1_A \to \mu(A) \quad \text{uniformly a.e. on } A, \tag{56}$$

which is a Cesáro average version of (50) when $g = 1_A$. See [27] for more discussions on the difference between uniform sets and uniformly returning sets. Also if $p = 1$, topologizing A as a Polish space is unnecessary since one can apply the powerful Hopf's ratio ergodic theorem in order to treat a general f (see the proof of Theorem 6.1 of [40]). The reason that we enforce a stronger assumption here is that for multiple integrals with $p \geq 2$, it is no longer clear how to write the statistic of interest in terms of a partial sum to which we can apply (56) (compare e.g. (74) below with [40, Eq. (6.10)]). It is unclear to us whether Theorem 4.1 continues to hold if Assumption 1 is relaxed to the Cesáro average version as in (56) or even to those in [40]. Nevertheless, Assumption 1 allows us to treat a sufficiently rich class of dynamical systems and functions f as exemplified in Section 4.3.

4.3. Examples

We shall provide two classes of examples regarding the assumptions involved in the main result Theorem 4.1, one about transforms on the interval [0, 1], and the other about Markov chains.

Example 4.1. The following example can be found in Thaler [62]. Let $(E,\mathcal{E}) = ([0,1], \mathcal{B}[0,1])$. Define a measure by

$$\mu_q(dx) = \left(\frac{1}{x^q} + \frac{1}{(1+x)^q}\right) 1_{(0,1]}(x)dx, \quad q > 1.$$

Define the transformation $T = T_q : E \to E$ by

$$T_q(x) := x\left(1 + \left(\frac{x}{1+x}\right)^{q-1} - x^{q-1}\right)^{1/(1-q)} \quad (\mathrm{mod}\ 1).$$

The transform T_q has an indifferent fixed point at $x = 0$, namely, $T_q(0) = 0$ and $T_q'(0+) = 1$, and the measure μ_q is infinite on any neighborhood of $x = 0$. Furthermore, T_q can be verified to be μ_q-preserving, conservative and ergodic.

If we choose $A = [\epsilon, 1]$, $\epsilon \in (0,1)$, then according to Thaler [62], any Riemann integrable function on A satisfies (49) and (50) with $\beta = 1/q$. In Theorem 4.1, we can take the p-variate function f to be any Riemman integrable function with support in A^p.

In fact, the example above belongs to the so-called AFN-systems, a well-known class of interval maps possessing indifferent fixed points and an infinite invariant measure. See Zweimüler [65,66] for the definitions. Recently for a large class of AFN-systems, Melbourne and Terhesiu [35, Theorem 1.1] and Gouëzel [19] established the uniform return relation (50) with (49) for Riemann integrable g on $A \subset [0,1]$ where A is a union of closed intervals which are away from the indifferent fixed points of T.

We state a primitive characterization of Assumption 1 which facilitates the discussion of the next example.

Proposition 4.2. *Let $(A, \mathcal{E}_A)$ be as in Assumption 1. Assumption 1 holds if and only if there exists a collection $\mathcal{C} \subset \mathcal{E}_A$ with the following properties:*

(a) $\mathcal{C}$ is a π-system containing A;
(b) $\mathcal{C}$ generates the Polish topology of A in the sense that for any open $G \subset A$ and any $x \in G$, there exists $U \in \mathcal{C}$ such that $x \in \mathring{U} \subset U \subset G$;
(c) Any set in $\mathcal{C}$ is μ-continuous;
(d) There exists a positive sequence $b_n \in RV_\infty(1-\beta)$, $0 < \beta < 1$, such that for any $B \in \mathcal{C}$,

$$b_n \widehat{T}^n 1_B(x) \to \mu(B) \quad \text{uniformly for a.e. } x \in A. \tag{57}$$

The proof of the proposition can be found in Section 5.1.

Example 4.2. Let S be a countably infinite state space. Consider an aperiodic irreducible and null-recurrent Markov chain $(Y_k)_{k\geq 0}$ on S, which has n-step transition probabilities $(p^{(n)}(i,j))_{i,j\in S}$ and an invariant measure π on S which satisfies $\pi_i > 0$ for any $i \in S$. Fix a state $o \in S$ and assume without loss of generality a normalization condition:

$$\pi_o = 1.$$

Consider the path space $E = \{x = (x(0), x(1), x(2), \dots) : \ x(k) \in S\}$ and let $\mathcal{E}$ be the cylindrical σ-field. Then one can define a σ-finite infinite measure μ on $(E,\mathcal{E})$ as

$$\mu(\cdot) = \sum_{i\in S} \pi_i P^i(\cdot),$$

where $P^i(\cdot)$ denotes the law $(Y_k)_{k\geq 0}$ starting at state $i \in S$ at time $k = 0$. Consider the measure preserving map of the left-shift

$$T : E \to E,\ T(x(0), x(1), x(2), \dots) = (x(1), x(2), \dots).$$

Due to the assumptions on the chain, the map T is ergodic and conservative [21], and each P^i can be verified to be atomless and thus so is μ.

Now let $A = \{x = (x(0), x(1), \dots) \in E : x(0) = o\}$. Consider the discrete topology on S induced by the metric $d(i, j) = 1_{\{i\neq j\}}$, $i, j \in S$. Then the product space A is known to be Polish with Borel σ-field $\mathcal{E}_A := \mathcal{E} \cap A$, and a topological basis of A is formed by

$$\mathcal{C} = \Big\{\{x \in E :\ x(0) = o,\ x(1) = s_1, \dots,\ x(m) = s_m\},\ m \in \mathbb{N},\ s_i \in S\Big\} \cup \{\emptyset,\ A\}.$$

See e.g. [37], Section 1A. Note that every set in $\mathcal{C}$ is both open and closed, so the boundary of each is empty. Therefore conditions (a)–(c) in Proposition 4.2 hold.

By [2, the last line of page 156], if $B = \{x \in A :\ x(1) = s_1, \dots,\ x(m) = s_m\} \in \mathcal{C}$, we have for $x = (o, x(1), x(2), \dots) \in A$ and $n > m$ that

$$(\widehat{T}^n 1_B)(x) = p(o, s_1)\cdots p(s_{m-1}, s_m)p^{(n-m)}(s_m, o) = \mu(B)p^{(n-m)}(s_m, o).$$

We claim that if we assume

$$p^{(n)}(o, o) \in \mathrm{RV}_\infty(\beta - 1), \tag{58}$$

then condition (d) of Proposition 4.2 holds with $b_n \sim 1/p^{(n)}(o, o)$ as $n \to \infty$. Indeed, this is the case if for any $m \in \mathbb{N}$ and $s \in S$, we have

$$\lim_n \frac{p^{(n-m)}(s, o)}{p^{(n)}(o, o)} = 1. \tag{59}$$

Condition (59) is essentially the *strong ratio limit property* in [38], and as shown there, it is equivalent to

$$\lim_n \frac{p^{(n+1)}(o, o)}{p^{(n)}(o, o)} = 1.$$

The last line follows from (58) and [12, Theorem 1.9.8].

In view of the topological basis $\mathcal{C}$, any function f on A^p which depends only on a finite number of coordinates of $(x_1, \dots, x_p) \in A^p$ can be verified to be continuous. On the other hand, a bounded continuous function on A^p depending on infinitely many coordinates can be constructed, for example, as $f(x_1, \dots, x_p) = \sum_{n=1}^{\infty} 2^{-n} \sum_{j=1}^{p} 1_{\{x_j(n)=o\}}$.

5. Proof of the non-central limit theorem

We first provide a summary of the proof. We prove our main Theorem 4.1 here by establishing the convergence of finite-dimensional distributions and the tightness in $D([0, 1])$ separately. We shall work with our series representation established in Section 3.1, and proceed by decomposing it into a leading term and a remainder term. Most of the effort is devoted to the convergence of the finite-dimensional distributions of the leading term. For this purpose, the key is Theorem 5.2 which concerns a convergence to the joint local-time processes introduced in Section 2. To prove Theorem 5.2, we shall apply the method of moments and make use of the moment formulas established in Theorem 2.2 for the joint local-time processes. To facilitate the moment computation, a delicate approximation scheme is developed in Section 5.1. The tightness in $D([0, 1])$ is also established with the aid of the aforementioned decomposition.

Finally we note that our proof techniques are essentially different from those in the case $p = 1$ considered in [40]. In the case $p = 1$, the proof in [40] relied heavily on the infinitely divisibility of the single stochastic integral and Hopf's ratio ergodic theorem. These ingredients are non-applicable for $p \geq 2$, and our proof strategy, instead, exploits the series representation of multiple stochastic integrals.

We now start by a series representation of the joint distribution of $(X_k)_{k=1,\ldots,n}$. For each fixed $n \in \mathbb{N}$, let $(U_i^{(n)})_{i\in\mathbb{N}}$ be i.i.d. taking values in E following the law

$$\mu_n(\cdot) := \frac{\mu(\cdot \cap \{\varphi \leq n\})}{\mu(\varphi \leq n)} = \frac{\mu(\cdot \cap \{\varphi \leq n\})}{w_n}, \tag{60}$$

where φ is the first entrance time to A as in (51). Let

$$T_p := T \times \cdots \times T : E^p \to E^p$$

be the product transform. For each fixed $n \in \mathbb{N}$, we apply the series representation (38) with $B = \{\varphi \leq n\}$, and obtain

$$(X_k)_{k=1,\ldots,n} \overset{d}{=} \left(p! \sum_{I\in\mathcal{D}_p} \left(\prod_{i\in I} \varepsilon_i \rho^{\leftarrow}(\Gamma_i/w_n) \right) f \circ T_p^k(\boldsymbol{U}_I^{(n)}) \right)_{k=1,\ldots,n}, n \in \mathbb{N}, \tag{61}$$

where $w_n = \mu(\varphi \leq n)$ is the wandering rate sequence as in (52), and $(\varepsilon_i)_{i\in\mathbb{N}}$, $(\Gamma_i)_{i\in\mathbb{N}}$ are as in Section 3.1 and are independent from $(U_i{}^{(n)})_{i\in\mathbb{N}}$. Recall the notation $\boldsymbol{U}_I{}^{(n)} = (U_{i_1}{}^{(n)}, \ldots, U_{i_p}{}^{(n)})$ with $I = (i_1, \ldots, i_p) \in \mathcal{D}_p$. For every n, the series representation converges almost surely by Lemma 3.1 since f is bounded.

Let

$$S_n(t) := \frac{1}{c_n} \sum_{k=1}^{\lfloor nt \rfloor} X_k \tag{62}$$

be the normalized partial sum of interest, with $c_n = n(\rho^{\leftarrow}(1/w_n)/b_n)^p$ as in (55). The proof consists of proving the convergence of finite-dimensional distributions and tightness.

5.1. An approximation scheme

Under the setup of Assumption 1, we introduce a class of functions useful for approximation purposes. Note that the product space A^p is also Polish with Borel σ-field $\mathcal{E}_A^p$.

Definition 5.1. A function $g : A^p \to \mathbb{R}$ is said to be an *elementary function*, if it is a finite linear combination of indicators of p-products of μ-continuity sets in $\mathcal{E}_A$, that is,

$$g(x_1, \ldots, x_p) = \sum_{m=1}^{M} b_m 1_{B_{1,m}\times\cdots\times B_{p,m}}(x_1, \ldots, x_p)$$

where $M \in \mathbb{N}$, b_m's are some real constants and $B_{j,m} \in \mathcal{E}_A$ with $\mu(\partial B_{j,m}) = 0$. A set $B \in \mathcal{E}_A^p$ is said to be an *elementary set*, if 1_B is an elementary function.

Lemma 5.1. *Let f be a bounded $\mu^{\otimes p}$-a.e. continuous function on A^p. Then for any $\epsilon > 0$, there exist elementary functions g_1, g_2 on A^p, such that $L(f) \leq g_1 \leq f \leq g_2 \leq U(f)$ and $|\mu^{\otimes p}(f) - \mu^{\otimes p}(g_i)| < \epsilon$, $i = 1, 2$, where $L(f) = \inf\{f(\boldsymbol{x}) : \boldsymbol{x} \in A^p\}$ and $U(f) = \sup\{f(\boldsymbol{x}) : \boldsymbol{x} \in A^p\}$.*

Proof. Suppose the Polish topology of A is induced by a metric d and let $N(x,\delta) = \{y \in A : d(x,y) < \delta\}$, $\delta > 0$. For any $\boldsymbol{x} = (x_1,\dots,x_p) \in A^p$ and $\delta > 0$, define the product neighborhood (corresponding to the uniform metric on A^p induced from d)

$$N_p(\boldsymbol{x},\delta) = N(x_1,\delta) \times \cdots \times N(x_p,\delta).$$

Let $C \subset A^p$ be the set of continuity points of f, and fix $\epsilon > 0$. For every $\boldsymbol{x} \in C$, when $\delta > 0$ is small enough and avoids a countable set of values, the set $N_p(\boldsymbol{x},\delta)$ can be made elementary (i.e., each $N(x_i,\delta)$ is μ-continuous, $i = 1,\dots,p$) and

$$\omega(\boldsymbol{x},\delta) := \sup\{|f(\boldsymbol{x}) - f(\boldsymbol{y})| : \boldsymbol{y} \in N_p(\boldsymbol{x},\delta)\} < \epsilon.$$

Next, note that the separable metric space A^p is second-countable and thus Lindelöf (every open cover has a countable subcover). Hence there exist $\delta_n > 0$ and $\boldsymbol{x}_n \in C$, such that $\cup_{n=1}^{\infty} N_p(\boldsymbol{x}_n,\delta_n) \supset C$, where each $N_p(\boldsymbol{x}_n,\delta_n)$ is elementary and $\omega(\boldsymbol{x}_n,\delta_n) < \epsilon$. For each $m \in \mathbb{N}$, set $C_m := \cup_{n=1}^{m} N_p(\boldsymbol{x}_n,\delta_n)$. This is an elementary set, and one can further choose m large enough so that $\mu^{\otimes p}(A^p \setminus C_m) = \mu^{\otimes p}(C \setminus C_m) < \epsilon$. One could further express C_m as a union of disjoint elementary sets $C_m = \cup_{n=1}^{m} D_n$ with $D_n := N_p(\boldsymbol{x}_n,\delta_n) \setminus (\cup_{i=1}^{n-1} N_p(\boldsymbol{x}_i,\delta_i))$. Then define

$$g_1(\boldsymbol{x}) := \sum_{n=1}^{m} \inf\{f(\boldsymbol{x}) : \boldsymbol{x} \in D_n\} 1_{D_n}(\boldsymbol{x}) + \inf\{f(\boldsymbol{x}) : \boldsymbol{x} \in A^p\} 1_{A^p \setminus C_m}(\boldsymbol{x})$$

and define g_2 with inf's replaced by sup's above. Then g_1 and g_2 are elementary functions satisfying $g_1 \le f \le g_2$, and

$$\mu^{\otimes p}(f-g_1) \wedge \mu_p(g_2-f) \ge 0, \quad \mu^{\otimes p}(f-g_1) \vee \mu^{\otimes p}(g_2-f) \le \epsilon(\mu^{\otimes p}(A^p) + 2\|f\|_\infty). \quad \square$$

Proof of Proposition 4.2. The "only if" part is immediate if $\mathcal{C}$ to consists of all μ-continuity sets in $\mathcal{E}_A$. We only need to show the "if" part.

Let $\mathcal{D}$ be the smallest class of subsets of A containing $\mathcal{C}$, which is also closed under (i) finite unions of disjoint sets and (ii) proper set differences. Then we apply a variant of Dynkin's π-λ theorem, where the σ-field is replaced by a field, and in the definition of a λ-system, the "countable disjoint union" is replaced by "finite disjoint union". This variant can be established using similar arguments as those in [45, Section 2.2.2]. Applying this we conclude that $\mathcal{D}$ is the smallest field containing $\mathcal{C}$. On the other hand, the class of μ-continuity subsets of A also forms a field, and so does $\mathcal{E}_A$. Hence any set in $\mathcal{D}$ is μ-continuous and $\mathcal{D} \subset \mathcal{E}_A$. Next, one can verify directly that the set operations (i) and (ii) mentioned above preserve (57), and hence the relation (57) holds for $B \in \mathcal{D}$.

Now note that μ restricted to Polish A is tight (see e.g. [10, Theorem 1.3]). Hence for any μ-continuity set $B \in \mathcal{E}_A$ and any $\epsilon > 0$, there exists a compact $K \subset \mathring{B}$, such that $\mu(B \setminus K) = \mu(\mathring{B} \setminus K) < \epsilon/2$. Due to the compactness and condition (b) of Proposition 4.2, there exists $D_1 \in \mathcal{D}$ which is a finite union of sets in $\mathcal{C}$, so that $K \subset D_1 \subset \mathring{B}$. This together with a similar argument with B replaced by $A \setminus B$ entails the existence of $D_1, D_2 \in \mathcal{D}$ satisfying $D_1 \subset B \subset D_2$ and $\mu(D_2) - \mu(D_1) < \epsilon$. Taking $n \to \infty$ in

$$b_n \widehat{T}^n 1_{D_1} \le b_n \widehat{T}^n 1_B \le b_n \widehat{T}^n 1_{D_2} \quad \text{a.e.,} \tag{63}$$

we see that (50) holds for $g = 1_B$. To obtain (50) in full generality, first observe that by linearity of $\widehat{T}$, the relation extends to g which is a finite linear combination of indicators of μ-continuity sets in $\mathcal{E}_A$. Then it extends to general bounded μ-a.e. continuous g by an approximation similar to (63) via Lemma 5.1 with $p = 1$. $\square$

5.2. Proof of convergence of finite-dimensional distributions

We proceed by first writing

$$\{S_n(t)\}_{t\in[0,1]} \overset{d}{=} \left\{S_{n,m}(t)+R_{n,m}(t)\right\}_{t\in[0,1]}, \tag{64}$$

for $m\in\mathbb{N}$ with

$$S_{n,m}(t) := \frac{1}{c_n}\sum_{k=1}^{\lfloor nt\rfloor} p! \sum_{I\in\mathcal{D}_p(m)}\left(\prod_{i\in I}\varepsilon_i\rho^{\leftarrow}(\Gamma_i/w_n)\right) f\circ T_p^k(\boldsymbol{U}_I^{(n)}),$$

where $\mathcal{D}_p(m)$ is as in (23). To show the convergence of finite-dimensional distributions, we shall show

$$S_{n,m}(t) \xrightarrow{f.d.d.} \Gamma(\beta_p)\cdot p!\cdot\mu^{\otimes p}(f)\sum_{I\in\mathcal{D}_p(m)}\left(\prod_{i\in I}\varepsilon_i\Gamma_i^{-1/\alpha}\right)L_t\left(\bigcap_{i\in I}(R_i+V_i)\right), \tag{65}$$

for all $m\in\mathbb{N}$ (compare it with (46)) and

$$\lim_{m\to\infty}\limsup_{n\to\infty} P(|R_{n,m}(t)|>\epsilon)=0,\ \text{ for all } t\in[0,1],\epsilon>0. \tag{66}$$

We prove the two claims separately.

Proof of (65)

Introduce

$$G_n(y) := \frac{\rho^{\leftarrow}(y/w_n)}{\rho^{\leftarrow}(1/w_n)}$$

and

$$L_{n,I,t} := \frac{b_n^p}{n}\sum_{k=1}^{\lfloor nt\rfloor} f\circ T_p^k(\boldsymbol{U}_I^{(n)}),\qquad I\in\mathcal{D}_p, t\geq 0, n\in\mathbb{N}, \tag{67}$$

and write

$$S_{n,m}(t) = p!\sum_{I\in\mathcal{D}_p(m)}\left(\prod_{i\in I}\varepsilon_i G_n(\Gamma_i)\right)L_{n,I,t}. \tag{68}$$

By the assumption $\rho((x,\infty))\in \mathrm{RV}_\infty(-\alpha)$ we have that

$$\lim_{n\to\infty} G_n(y) = y^{-1/\alpha},\qquad y>0.$$

Therefore, (65) follows from the following result.

Theorem 5.2. *With the notation above,*

$$\left(L_{n,I,t}\right)_{I\in\mathcal{D}_p,t\in[0,1]} \overset{f.d.d.}{\to} \mu^{\otimes p}(f)\Gamma(\beta_p)\left(L_t\left(\bigcap_{i\in I}(R_i+V_i)\right)\right)_{I\in\mathcal{D}_p,t\in[0,1]}.$$

Theorem 5.2 can be proved by a method of moments.

Proposition 5.3. *Let f be as in Theorem 4.1. Then for any $I_1, \ldots, I_r \in \mathcal{D}_p$, $t_1, \ldots, t_r \in [0, 1]$, we have*

$$\lim_{n\to\infty} \mathbb{E}\left(\prod_{\ell=1}^{r} L_{n,I_\ell,t_\ell}\right) = \mu^{\otimes p}(f)^r \int_{(\mathbf{0},\boldsymbol{t})} \prod_{i=1}^{K} h^{(\beta)}_{|\mathcal{I}(i)|}(\boldsymbol{x}_{\mathcal{I}(i)})\, d\boldsymbol{x}, \tag{69}$$

where $h_q^{(\beta)}$ is as in (15) and $K = \max(\bigcup_{\ell=1}^{r} I_\ell)$.

Proof. We may assume that $t_\ell > 0$ for all $\ell = 1, \ldots, r$, otherwise (69) trivially holds with both-hand sides being zeros. We then proceed as follows:

$$\begin{aligned}
\mathbb{E}\left(\prod_{\ell=1}^{r} L_{n,I_\ell,t_\ell}\right) &= \left(\frac{b_n^p}{n}\right)^r \mathbb{E}\left(\prod_{\ell=1}^{r}\sum_{k=1}^{\lfloor nt_\ell\rfloor} f\circ T_p^k(\boldsymbol{U}_{I_\ell}{}^{(n)})\right)\\
&= \left(\frac{b_n^p}{n}\right)^r \sum_{\mathbf{1}\le \boldsymbol{k}\le \lfloor n\boldsymbol{t}\rfloor} \mathbb{E}\left(\prod_{\ell=1}^{r} f\circ T_p^{k_\ell}(\boldsymbol{U}^{(n)}_{I_\ell})\right).
\end{aligned} \tag{70}$$

We claim that it is enough to prove (69) for function f of the form

$$f(\boldsymbol{x}) = \prod_{j=1}^{p} f_j(x_j), \qquad \text{with} \qquad f_j(x) = \mathbf{1}_{A_j}(x), \tag{71}$$

where each f_j is an indicator of a μ-continuity set $A_j \in \mathcal{E}_A$ satisfying the uniform return relation (49) and (50). Indeed, since f can always be written as a difference of two non-negative bounded $\mu^{\otimes p}$-a.e. continuous functions (e.g., $f = (f + \|f\|_\infty 1_{A^p}) - \|f\|_\infty 1_{A^p}$), so by an expansion of the product in (70), one may assume that $f \ge 0$. Next, in view of Lemma 5.1, Assumption 1 and an approximation argument exploiting monotonicity, it suffices to consider f which is elementary in the sense of Definition 5.1. By a further expansion of the product in (70), it suffices to focus on f with simple form (71).

From (71), we can rewrite using $I_\ell = (I_\ell(1), \ldots, I_\ell(p))$ with $I_\ell(1) < \cdots < I_\ell(p)$:

$$\begin{aligned}
\prod_{\ell=1}^{r} f\circ T_p^{k_\ell}(U^{(n)}_{I_\ell}) &= \prod_{\ell=1}^{r}\prod_{j=1}^{p} f_j\circ T^{k_\ell}(U^{(n)}_{I_\ell(j)})\\
&= \prod_{i=1}^{K}\prod_{\ell\in\mathcal{I}(i)} f_{\mathcal{K}(i,\ell)}\circ T^{k_\ell}(U_i^{(n)}),
\end{aligned}$$

where, for every $\ell \in \mathcal{I}(i) = \{\ell' \in \{1, \ldots, r\}, i \in I_{\ell'}\}$, $\mathcal{K}(i, \ell) \in \{1, \ldots, p\}$ is defined by the relation $I_\ell\big(\mathcal{K}(i, \ell)\big) = i$. Here and below, we follow the convention $\prod_{\ell\in\emptyset}(\cdot) \equiv 1$. Since $U_1^{(n)}, \ldots, U_K^{(n)}$ are i.i.d. following μ_n in (60), we have

$$\mathbb{E}\left(\prod_{i=1}^{K}\prod_{\ell\in\mathcal{I}(i)} f_{\mathcal{K}(i,\ell)}\circ T^{k_\ell}(U_i^{(n)})\right) = \prod_{i=1}^{K}\mu_n\left(\prod_{\ell\in\mathcal{I}(i)} f_{\mathcal{K}(i,\ell)}\circ T^{k_\ell}\right).$$

Then,

$$\mathbb{E}\left(\prod_{\ell=1}^{r} L_{n,I_\ell,t_\ell}\right) = \left(\frac{b_n^p}{n}\right)^r \sum_{\mathbf{1}\le \boldsymbol{k}\le \lfloor n\boldsymbol{t}\rfloor}\prod_{i=1}^{K}\mu_n\left(\prod_{\ell\in\mathcal{I}(i)} f_{\mathcal{K}(i,\ell)}\circ T^{k_\ell}\right). \tag{72}$$

Expressing the r-tuple sum over $\boldsymbol{k}$ above by an integral, we claim that

$$\mathbb{E}\left(\prod_{\ell=1}^{r} L_{n,I_\ell,t_\ell}\right) = b_n^{pr}\int_{(\mathbf{0},\lfloor n\boldsymbol{t}\rfloor/n)}\prod_{i=1}^{K}\mu_n\left(\prod_{\ell\in\mathcal{I}(i)} f_{\mathcal{K}(i,\ell)}\circ T^{\lfloor nx_\ell\rfloor+1}\right)d\boldsymbol{x}$$
$$\sim(\Gamma(\beta)\Gamma(2-\beta))^{pr}\int_{(\mathbf{0},\lfloor n\boldsymbol{t}\rfloor/n)}\prod_{i=1}^{K} w_n^{|\mathcal{I}(i)|-1}\mu\left(\prod_{\ell\in\mathcal{I}(i)} f_{\mathcal{K}(i,\ell)}\circ T^{\lfloor nx_\ell\rfloor}\right)d\boldsymbol{x}. \tag{73}$$

Indeed, in (73), we have used $\mu_n(\cdot)=\mu(\cdot\cap\{\varphi\le n\})/w_n$, the relation (53), and the fact that the functions $f_j\circ T^k$, $1\le k\le n$, are supported within $\{\varphi\le n\}$ and $\sum_{i=1}^{K}|\mathcal{I}(i)|=|I_1|+\cdots+|I_r|=pr$; we also drop the '+1' in the power of T, since T is measure-preserving with respect to μ.

To complete the proof, it remains to establish

$$\lim_{n\to\infty}\int_{(\mathbf{0},\lfloor n\boldsymbol{t}\rfloor/n)}\prod_{i=1}^{K} w_n^{|\mathcal{I}(i)|-1}\mu\left(\prod_{\ell\in\mathcal{I}(i)} f_{\mathcal{K}(i,\ell)}\circ T^{\lfloor nx_\ell\rfloor}\right)d\boldsymbol{x}$$
$$=\big(\Gamma(\beta)\Gamma(2-\beta)\big)^{-pr}\left(\prod_{i=1}^{K}\prod_{\ell\in\mathcal{I}(i)}\mu(f_{\mathcal{K}(i,\ell)})\right)\int_{(\mathbf{0},\boldsymbol{t})}\prod_{i=1}^{K} h_{|\mathcal{I}(i)|}^{(\beta)}(\boldsymbol{x}_{\mathcal{I}(i)})\,d\boldsymbol{x}. \tag{74}$$

Indeed, the desired convergence of moments (69) now follows from (72), (73), (74) and that

$$\prod_{i=1}^{K}\prod_{\ell\in\mathcal{I}(i)}\mu\left(f_{\mathcal{K}(i,\ell)}\right)=\left(\prod_{j=1}^{p}\mu(f_j)\right)^{r}=\mu^{\otimes p}(f)^r.$$

In order to show (74), we apply the dominated convergence theorem. To simplify the notation, we consider $q\in\{1,\dots,p\}$ and $f_1,\dots,f_q$ as in (71), and introduce

$$H_{n,q}(\boldsymbol{x}):=w_n^{q-1}\mu\left(\prod_{j=1}^{q} f_j\circ T^{\lfloor nx_j\rfloor}\right),\quad \boldsymbol{x}\in(0,1)^q.$$

A careful examination shows that (74) follows from the following two results:

$$\lim_{n\to\infty} H_{n,q}(\boldsymbol{x})=(\Gamma(\beta)\Gamma(2-\beta))^{-q}\left(\prod_{j=1}^{q}\mu(f_j)\right)h_q^{(\beta)}(\boldsymbol{x}),\ \text{for all } \boldsymbol{x}\in(\mathbf{0},\mathbf{1})_{\neq}, \tag{75}$$

and, for some $\eta\in(0,\beta)$,

$$H_{n,q}(\boldsymbol{x})\le Ch_q^{(\beta-\eta)}(\boldsymbol{x}),\ \text{for all } \boldsymbol{x}\in(\mathbf{0},\mathbf{1})_{\neq}. \tag{76}$$

(Recall $h_q{}^{(\beta)}$ in (15).) Note that we only need to consider the limit for $\boldsymbol{x}\in(\mathbf{0},\mathbf{1})_{\neq}:=\{\boldsymbol{y}\in(\mathbf{0},\mathbf{1}): y_\ell\neq y_{\ell'},\forall\ell\neq\ell'\}$. The product $\prod_{i=1}^{K}h_{|\mathcal{I}(i)|}^{(\beta-\eta)}(\boldsymbol{x}_{\mathcal{I}(i)})$ is integrable on $(\mathbf{0},\mathbf{1})_{\neq}$ since it is up to a multiplicative constant

$$\mathbb{E}\left(\prod_{\ell=1}^{r}\widetilde{L}_{I_\ell,t_\ell}\right)\le\frac{1}{r}\sum_{\ell=1}^{r}\mathbb{E}\widetilde{L}_{I_\ell,t_\ell}^{r},$$

where $\widetilde{L}_{I,t}$ is defined similarly as $L_{I,t}$, with the underlying β-stable regenerative sets replaced by $(\beta-\eta)$-stable regenerative sets (see (13)). Setting $\eta>0$ small enough so that $p(\beta-\eta)-p+1\in(0,1)$, the finiteness of the integration now follows from (22).

We now prove (75) and (76). Assume $q \geq 2$ below. The case $q = 1$ is similar and simpler and hence omitted. To show (75), it suffices to focus on the tetrahedron $(\mathbf{0}, \mathbf{1})_\uparrow := \{\boldsymbol{x} \in (0,1)^q : 0 < x_1 < \cdots < x_q < 1\}$. First write

$$\begin{aligned}\prod_{j=1}^q f_j \circ T^{\lfloor nx_j \rfloor} &= f_1 \circ T^{\lfloor nx_1 \rfloor} \times \left(\prod_{j=2}^q f_j \circ T^{\lfloor nx_j \rfloor - \lfloor nx_1 \rfloor} \right) \circ T^{\lfloor nx_1 \rfloor} \\ &= f_1 \circ T^{\lfloor nx_1 \rfloor} \times \left(\prod_{j=2}^q f_j \circ T^{\lfloor nx_j \rfloor - \lfloor nx_2 \rfloor} \right) \circ T^{\lfloor nx_2 \rfloor - \lfloor nx_1 \rfloor} \circ T^{\lfloor nx_1 \rfloor}.\end{aligned}$$

Then, by the measure-preserving property,

$$H_{n,q}(\boldsymbol{x}) = w_n^{q-1} \int_E f_1 \times \left(\prod_{j=2}^q f_j \circ T^{\lfloor nx_j \rfloor - \lfloor nx_2 \rfloor} \right) \circ T^{\lfloor nx_2 \rfloor - \lfloor nx_1 \rfloor} d\mu, \tag{77}$$

which, by duality (47), equals

$$w_n^{q-2} \frac{w_n}{w_{\lfloor nx_2 \rfloor - \lfloor nx_1 \rfloor}} \int_A w_{\lfloor nx_2 \rfloor - \lfloor nx_1 \rfloor} \left(\widehat{T}^{\lfloor nx_2 \rfloor - \lfloor nx_1 \rfloor} f_1 \right) \prod_{j=2}^q f_j \circ T^{\lfloor nx_j \rfloor - \lfloor nx_2 \rfloor} d\mu.$$

Due to the uniform convergence of a regularly varying sequence of positive index [46, Proposition 2.4], we have $\lim_{n\to\infty} w_{\lfloor nx_2 \rfloor - \lfloor nx_1 \rfloor}/w_n = (x_2 - x_1)^{1-\beta}$. In addition, using the uniform convergence in (50) and the relation (53), as $n \to \infty$,

$$H_{n,q}(\boldsymbol{x}) \sim \frac{\mu(f_1)}{\Gamma(\beta)\Gamma(2-\beta)} (x_2 - x_1)^{\beta-1} w_n^{q-2} \int_E \prod_{j=2}^q f_j \circ T^{\lfloor nx_j \rfloor - \lfloor nx_2 \rfloor} d\mu.$$

Repeating the arguments above yields (75).

We now prove (76). The situation is more delicate, and we shall introduce

$$D_{n,q} := \left\{ \boldsymbol{x} \in (\mathbf{0}, \mathbf{1})_\uparrow : \lfloor nx_i \rfloor \neq \lfloor nx_j \rfloor \text{ for all } i \neq j \right\}.$$

First assume that $\boldsymbol{x} \in D_{n,q}$, which implies $\lfloor nx_1 \rfloor < \lfloor nx_2 \rfloor$. By the Potter's bound [12, Theorem 1.5.6] and an elementary bound [5, Eq.(40)],

$$\frac{w_n}{w_{\lfloor nx_2 \rfloor - \lfloor nx_1 \rfloor}} \leq C_1 \left(\frac{\lfloor nx_2 \rfloor - \lfloor nx_1 \rfloor}{n} \right)^{\beta-1-\eta} \leq C_2 (x_2 - x_1)^{\beta-1-\eta}, \tag{78}$$

for all $n \in \mathbb{N}$, $\boldsymbol{x} \in D_{n,q}$, where recall that $\eta > 0$ is sufficiently small such that $\beta - \eta > 1 - 1/p$. In addition, the relations (50) and (53) imply

$$\sup_{\substack{0<x_1<x_2<1, y\in A \\ n:\lfloor nx_1 \rfloor < \lfloor nx_2 \rfloor}} w_{\lfloor nx_2 \rfloor - \lfloor nx_1 \rfloor} \left(\widehat{T}^{\lfloor nx_2 \rfloor - \lfloor nx_1 \rfloor} 1_A \right)(y) < \infty. \tag{79}$$

Applying these observations to (77), and bounding $|f_j|$'s by 1_A up to a constant almost everywhere, we get

$$H_{n,q}(\boldsymbol{x}) \leq C(x_2 - x_1)^{\beta-1-\eta} w_n^{q-2} \int_E 1_A \prod_{j=3}^q 1_A \circ T^{\lfloor nx_j \rfloor - \lfloor nx_2 \rfloor} d\mu.$$

Applying the bounds of the form (78) and (79) iteratively, we eventually get (76) for $\boldsymbol{x} \in D_{n,q}$.

Now we assume that $\boldsymbol{x} \in (\mathbf{0}, \mathbf{1})_{\uparrow} \setminus D_{n,q}$. Again in (77), we shall bound each $|f_j|$ by 1_A up to a constant almost everywhere. Assume first that only two of $\lfloor nx_i \rfloor$s are the same, and without loss of generality we consider $\lfloor nx_2 \rfloor = \lfloor nx_1 \rfloor$ and $\lfloor nx_j \rfloor \neq \lfloor nx_{j-1} \rfloor$ for $j = 3, \dots, q$. Then

$$\begin{aligned} H_{n,q}(\boldsymbol{x}) &\le C w_n^{q-1} \int_E \prod_{j=2}^{q} 1_A \circ T^{\lfloor nx_j \rfloor} d\mu \\ &= C w_n \cdot w_n^{q-2} \int_E 1_A \prod_{j=3}^{q} 1_A \circ T^{\lfloor nx_j \rfloor - \lfloor nx_2 \rfloor} d\mu. \end{aligned}$$

Handling the integral factor as in (78) and (79), we obtain

$$H_{n,q}(\boldsymbol{x}) \le C w_n \prod_{j=3}^{q} (x_j - x_{j-1})^{\beta-1-\eta} \tag{80}$$

Furthermore, since $\lfloor nx_2 \rfloor = \lfloor nx_1 \rfloor$ implies $x_2 - x_1 < 1/n$, under which $n^{\beta-1-\eta}(x_2 - x_1)^{\beta-1-\eta} > 1$. Inserting this into (80), it then follows that

$$H_{n,q}(\boldsymbol{x}) \le C w_n n^{\beta-1-\eta} h_q^{(\beta-\eta)}(\boldsymbol{x}).$$

Note that $w_n n^{\beta-1-\eta} \in \mathrm{RV}_\infty(-\eta)$ and thus converges to zero as $n \to \infty$. So the above satisfies what we need in (76). The case where $\boldsymbol{x} \in (\mathbf{0}, \mathbf{1})_{\uparrow} \setminus D_{n,q}$ with $\lfloor nx_i \rfloor = \lfloor nx_{i+1} \rfloor$ more than one value of $i = 1, \dots, q-1$ can be treated similarly. The proof is thus completed. □

Proof of Theorem 5.2. We have computed the joint moments of $(L_{I_\ell, t_\ell})_{\ell=1,\dots,r}$ in Theorem 2.2. On the other hand, we have established the convergence of the joint moments of $(L_{n, I_\ell, t_\ell})_{\ell=1,\dots,r}$ in Proposition 5.3. It remains to show that the law of $(L_{I_\ell, t_\ell})_{\ell=1,\dots,r}$ is uniquely determined by the joint moments, for every choice of $I_1, \dots, I_r, t_1, \dots, t_r$. Then, it suffices to check the multivariate Carleman condition [57, Theorem 1.12]

$$\sum_{k=1}^{\infty} \eta_{2k}^{-1/(2k)} = \infty, \qquad \text{with} \qquad \eta_{2k} := \sum_{\ell=1}^{r} \mathbb{E} L_{I_\ell, t_\ell}^{2k}. \tag{81}$$

In view of Corollary 2.4, we have $\eta_{2k} \le C^{2k}(2k)!/\Gamma(2k\beta_p - \beta_p + 2)$. By the Stirling's approximation, one can obtain the inequality $\eta_{2k}^{-1/(2k)} \ge C k^{\beta_p - 1}$. So (81) holds because $\beta_p > 0$. □

Proof of (66)

We shall need the following uniform control:

$$G_n(y) \equiv \frac{\rho^{\leftarrow}(y/w_n)}{\rho^{\leftarrow}(1/w_n)} \le C\left(y^{-1/\alpha_0} + y^{-(1/\alpha)-\epsilon}\right), \quad \text{for all } y > 0 \text{ and } n \in \mathbb{N}. \tag{82}$$

To see this, we first note that the assumptions on ρ in (37) imply that $\rho^{\leftarrow} \in \mathrm{RV}_0(-1/\alpha)$ and $\rho^{\leftarrow}(y) = O(y^{-1/\alpha_0})$ as $y \to \infty$. By Potter's bound [12, Theorem 1.5.6], for every $\epsilon > 0$ there exists a constant $A_\epsilon > 0$ such that If $y \le A_\epsilon w_n$, $G_n(y) \le 2y^{-(1/\alpha)-\epsilon}$. On the other hand, for $y > A_\epsilon w_n$, we have $\rho^{\leftarrow}(y/w_n) \le C(y/w_n)^{-1/\alpha_0}$ and $\rho^{\leftarrow}(1/w_n) \ge C(1/w_n)^{-(1/\alpha)+\epsilon}$, whence we have

$$G_n(y) \le C y^{-1/\alpha_0} w_n^{1/\alpha_0 - (1/\alpha) + \epsilon}, \text{ for all } y > A_\epsilon w_n, n \in \mathbb{N}.$$

(The constants C here and below depend on ϵ.) Now, note that for the second assumption on α_0 in (37), one could take α_0 arbitrarily close to and smaller than 2. Set also ϵ small so that $1/\alpha_0 - (1/\alpha) + \epsilon < 0$, so that the upper bound above becomes $G_n(y) \le C y^{-1/\alpha_0}$ for all $y > A_\epsilon w_n$. We have thus proved (82).

Fix a large M which will be specified later. In view of (64) and (68), we express

$$R_{n,m}(t) = \sum_{I_1 \in \mathcal{D}_{\le p-1}(M)} \left(\prod_{i \in I_1} \varepsilon_i G_n(\Gamma_i) \right) F(I_1, n, M, m)$$

where $\mathcal{D}_{\le p-1}(M)$ is as in (41), and

$$F(I_1, n, M, m) := \sum_{I_2 \in \mathcal{H}(p-|I_1|, M, m)} \left(\prod_{i \in I_2} \varepsilon_i G_n(\Gamma_i) \right) L_{n, I_1 \cup I_2, t},$$

with

$$\mathcal{H}(k, M, m) := \{ I \in \mathcal{D}_k : \ \min I > M, \ \max I > m \}.$$

(Compare it with $\mathcal{H}(k, M)$ in (42).) Observe that $\mathcal{D}_{\le p-1}(M)$ is finite and $\mathbb{E} | \prod_{i \in I_1} G_n(\Gamma_i)|^q < \infty$ for all $I_1 \in \mathcal{D}_{\le p-1}(M)$ when $q > 0$ is sufficiently small in view of (44) and (82). Hence by Hölder's inequality, it suffices to show for each $I_1 \in \mathcal{D}_{\le p-1}(M)$,

$$\lim_{m \to \infty} \sup_{n \in \mathbb{N}} \mathbb{E} F(I_1, n, M, m)^2 = 0. \tag{83}$$

For the above to hold we shall actually need M to be large enough, which will be determined at the end. Introduce

$$k := p - |I_1|.$$

We start by using the orthogonality $\mathbb{E}[(\prod_{i \in I} \varepsilon_i)(\prod_{i \in I'} \varepsilon_i)] = 1_{\{I = I'\}}$, $I, I' \in \mathcal{D}_k$ to obtain

$$\mathbb{E} F(I_1, n, M, m)^2 = \sum_{I_2 \in \mathcal{H}(k, M, m)} \mathbb{E} \left(\prod_{i \in I_2} G_n(\Gamma_i)^2 \right) \mathbb{E} L^2_{n, I_1 \cup I_2, t}.$$

Note that $\mathbb{E} L^2_{n, I_1 \cup I_2, t} = \mathbb{E} L^2_{n, I, t}$ for all $I \in \mathcal{D}_p$, which is convergent as $n \to \infty$ by Proposition 5.3 and hence uniformly bounded in I and n. Note also that $\mathcal{H}(k, M, m) \downarrow \emptyset$ as $m \to \infty$. Therefore, to show (83), by the dominated convergence theorem it suffices to find $g^* : \mathcal{H}(k, M) \to \mathbb{R}_+$ such that

$$g_n^*(I_2) := \mathbb{E} \left(\prod_{i \in I_2} G_n(\Gamma_i)^2 \right) \le g^*(I_2), \ \text{ for all } I_2 \in \mathcal{H}(k, M), n \in \mathbb{N}$$

and $\sum_{I_2 \in \mathcal{H}(k, M)} g^*(I_2) < \infty$. Setting $\gamma := \min\{1/\alpha_0, 1/\alpha + \epsilon\}$ and taking $M > 2\gamma k$, we have

$$\mathbb{E} \left(\prod_{i \in I_2} G_n(\Gamma_i)^2 \right) \le C \mathbb{E} \left(\prod_{i \in I_2} \left(\Gamma_i^{-1/\alpha_0} + \Gamma_i^{-(1/\alpha) - \epsilon} \right)^2 \right) \le C \prod_{i \in I_2} i^{-2\gamma} =: g^*(I_2), \tag{84}$$

where the first inequality follows from (82), and the second from (44). The bound g^* is summable over $\mathcal{H}(k, M)$ as

$$\sum_{I_2 \in \mathcal{H}(k, M)} g^*(I_2) \le C \left(\sum_{i=1}^{\infty} i^{-2\gamma} \right)^k,$$

and that $2\gamma > 1$. This completes the proof of (83) and hence (66).

5.3. Proof of tightness

Proposition 5.4. *Under the assumptions of Theorem 4.1, the laws of processes* $(S_n(t))_{t\in[0,1]}$, $n\in\mathbb{N}$ *are tight in the Skorokhod space* $D([0,1])$ *with respect to the uniform topology.*

Proof. Fix $m\in\mathbb{N}$ large enough specified later. Assume without loss of generality that $f\ge 0$, since a general f can be written as a difference of two non-negative bounded $\mu^{\otimes p}$-a.e. continuous functions on A^p. Recall the decomposition $S_n(t)=S_{n,m}(t)+R_{n,m}(t)$ as in (64). It suffices to check the tightness of $(S_{n,m})_{n\in\mathbb{N}}$ and $(R_{n,m})_{n\in\mathbb{N}}$ respectively. We start with $(S_{n,m})_{n\in\mathbb{N}}$. Let $L_{n,I,t}$ be as in (67). Recall that

$$S_{n,m}(t)=p!\sum_{I\in\mathcal{D}_p(m)}\left(\prod_{i\in I}\varepsilon_i G_n(\Gamma_i)\right)L_{n,I,t}.$$

By Theorem 5.2, the limit of each $L_{n,I,t}$ in finite-dimensional distribution is, up to a constant, the local time $L_t(\cap_{i\in I}(R_i+V_i))$ of the shifted β_p-stable regenerative set $\cap_{i\in I}(R_i+V_i)$, for which we shall work with its continuous version. Then for each fixed $I\in\mathcal{D}_p(m)$, the laws of the a.s. non-decreasing processes $(L_{n,I,t})_{t\in[0,1]}$, $n\in\mathbb{N}$ are tight [11, Theorem 3]. Furthermore, we have seen that $\prod_{i\in I}G_n(\Gamma_i)\to\prod_{i\in I}\Gamma_i^{-1/\alpha}$ as $n\to\infty$, and hence

$$\widetilde{G}_{n,I}:=\prod_{i\in I}\varepsilon_i G_n(\Gamma_i),\ n\in\mathbb{N}$$

is a tight sequence of random variables for every $I\in\mathcal{D}_p(m)$. For every fixed $m\in\mathbb{N}$, the tightness of $\{(S_{n,m}(t))_{t\in[0,1]},n\in\mathbb{N}\}$ then follows.

Next, we show the tightness of $(R_{n,m}(t))_{t\in[0,1]}$, $n\in\mathbb{N}$ for m fixed large enough. Write

$$R_{n,m}(t)=\sum_{I_1\in\mathcal{D}_{\le p-1}(m)}\widetilde{G}_{n,I_1}\sum_{I_2\in\mathcal{H}(p-|I_1|,m)}\widetilde{G}_{n,I_2}L_{n,I_1\cup I_2,t},$$

Since $\mathcal{D}_{\le p-1}(m)$ is finite, it suffices to prove, for fixed $I_1\in\mathcal{D}_{\le p-1}(m)$ and $k=p-|I_1|\ge 1$, the tightness of

$$A_n(t):=\sum_{I_2\in\mathcal{H}(k,m)}\widetilde{G}_{n,I_2}L_{n,I_1\cup I_2,t},\ t\in[0,1],n\in\mathbb{N}.$$

For this purpose, it is standard (e.g. [10, Theorem 13.5]) to show that for all $0\le s<t\le 1$, there exist constants $C>0$, $a>0$ and $b>1$, such that

$$\mathbb{E}|A_n(t)-A_n(s)|^a\le C\,(t-s)^b,\ \text{ for all } 0\le s<t\le 1, n\in\mathbb{N}. \tag{85}$$

For this purpose, we compute

$$\mathbb{E}(A_n(t)-A_n(s))^2=\sum_{I_2\in\mathcal{H}(k,m)}\mathbb{E}\left(\prod_{i\in I_2}G_n(\Gamma_i)^2\right)\mathbb{E}(L_{n,I_1\cup I_2,t}-L_{n,I_1\cup I_2,s})^2.$$

The first expectation is uniformly bounded by $g^*(I_2)$ as in (84) (assuming $m>2\gamma k$ in place of $M>2\gamma k$), which is summable over $\mathcal{H}(k,m)$. For the second, by first bounding f by 1_{A^p} up to a constant and then applying an argument similar to the proof of Proposition 5.3, in particular, using the bound (76), we have

$$\mathbb{E}(L_{n,I_1\cup I_2,t} - L_{n,I_1\cup I_2,s})^2 \le C \int_{\frac{\lfloor ns\rfloor}{n}<x_1<x_2<\frac{\lfloor nt\rfloor}{n}} (x_2 - x_1)^{p(\beta-1-\eta)} dx_1 dx_2$$
$$\le C\,(s-t)^{\beta_p+1-p\eta}\,,$$

where $\eta > 0$ is chosen sufficiently small so that $p\eta < \beta_p \in (0, 1)$. The proof of (85) is then completed. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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