

# *Remarks on Jones Slopes and Surfaces of Knots*

**Efstratia Kalfagianni**

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# Remarks on Jones Slopes and Surfaces of Knots

Efstratia Kalfagianni<sup>1</sup>

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## Abstract

We show that the strong slope conjecture implies that the degrees of the colored Jones knot polynomials detect the figure 8 knot. Furthermore, we propose a characterization of alternating knots in terms of the Jones period and the degree span of the colored Jones polynomial

**Keywords** Alternating knot · Colored jones polynomial · Figure 8 knot · Hyperbolic knot · Strong slope conjecture

**Mathematics Subject Classification (2010)** 57K10 · 57K32 · 57K36

## 1 Introduction

The colored Jones polynomial of a knot  $K \subset S^3$  is a collection of Laurent polynomials  $\{J_K(n) := J_K(n, t)\}_{n=1}^{\infty}$  in a variable  $t$ , such that  $J_K(1) = 1$  and  $J_K(2)$  is the classical Jones polynomial. In this note, we use the normalization

$$J_{\text{unknot}}(n) = \frac{t^{n/2} - t^{-n/2}}{t^{1/2} - t^{-1/2}}.$$

Let  $d_+[J_K(n)]$  and  $d_-[J_K(n)]$  denote the maximal and minimal degrees of  $J_K(n)$  in  $t$ , respectively. These degrees are quadratic quasi-polynomials in  $n$ . The strong slope conjecture asserts that they contain information about essential surfaces in knot exteriors. More specifically, the coefficients of the quadratic terms are *boundary slopes* of  $K$  and the linear terms encode information about the topology of essential surfaces that realize these boundary slopes.

In [17], we observed that the strong slope conjecture implies that  $d_+[J_K(n)]$  and  $d_-[J_K(n)]$  detect the unknot, and in [16] we show that they detect all the torus knots. In this note, we show the following:

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✉ Efstratia Kalfagianni  
kalfagia@math.msu.edu

<sup>1</sup> Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA

**Theorem 1.1** *Let  $K$  be a knot that satisfies the strong slope conjecture. If the degrees  $d_+[J_K(n)]$  and  $d_-[J_K(n)]$  are the same as those of the figure 8 knot then  $K$  is isotopic to the figure 8 knot.*

Theorem 1.1 implies that the degrees  $d_+[J_K(n)]$  and  $d_-[J_K(n)]$  detect the figure 8 knot within the classes of knots for which the strong slope conjecture is known (e.g., the class of adequate knots). The proof of the theorem relies on Gordon’s result [10] that gives bounds of the distance between boundary slopes of punctured tori in irreducible 3-manifolds with toroidal boundary.

We also observe that results on the strong slope conjecture, together with a result of Howie [14], suggest the following conjecture that proposes a characterization of alternating knots in terms of their colored Jones polynomial.

**Conjecture 1.2** *Given a knot  $K$ , let  $p_K$  denote the Jones period of  $K$ . Then,  $K$  is alternating if and only if we have*

$$p_K = 1 \text{ and } 2d_+[J_K(n)] - 2d_-[J_K(n)] = cn^2 + (2 - c)n - 2 \tag{1.1}$$

for some  $c \in \mathbb{Z}$ .

Alternating knots satisfy (1.1) with  $c = c(K)$ , the crossing number of  $K$ . Conversely, by [15] if  $K$  is a knot that satisfies (1.1) with  $c = c(K)$ , then  $K$  must be alternating. See Proposition 4.7. Conjecture 1.2 is seeking to remove the knot diagrammatic reference to crossing numbers and provide a characterization only in terms of properties of the degree of  $J_K(n)$ . The conjecture is known to be true for all the knots for which the strong slope conjecture holds. These include adequate knots, large classes of non-adequate Montesinos knots, graph knots, and knots obtained from these classes by certain satellite operations. See Section 2 for more details.

There are non-alternating knots with Jones period one. For instance, for any adequate knot  $K$ , we have  $p_K = 1$  but there exist also families of non-adequate knots that have this property. On the other hand, alternating knots are the only knots with zero Turaev genus and they form a sub-class of adequate knots. The degree span of the colored Jones polynomial of adequate knots is known to satisfy an analogue of (1.1) involving the Turaev genus. See (4.1) in Section 4. We show that this generalized equation, however, does not characterize adequate knots. See Proposition 4.8.

## 2 Background

### 2.1 Slope Conjectures

Garoufalidis [7] proved that the degrees  $d_+[J_K(n)]$  and  $d_-[J_K(n)]$  are quadratic quasi-polynomials: Given a knot  $K$ , there is  $n_K \in \mathbb{N}$  such that for all  $n > n_K$  we have

$$d_+[J_K(n)] = a(n)n^2 + b(n)n + c(n) \text{ and } d_-[J_K(n)] = a^*(n)n^2 + b^*(n)n + c^*(n),$$

where the coefficients are periodic functions from  $\mathbb{N}$  to  $\mathbb{Q}$ . For a sequence  $\{x_n\}$ , let  $\{x_n\}'$  denote the set of its cluster points.

**Definition 2.1** *The Jones period of  $K$ , denoted by  $p_K$ , is the least common multiple of the periods of these coefficient functions.*

The elements of the sets

$$j_{s_K} := \left\{ 4n^{-2}d_+[J_K(n)] \right\}' \quad \text{and} \quad j_{s_K^*} := \left\{ 4n^{-2}d_-[J_K(n)] \right\}'$$

are called *Jones slopes* of  $K$ .

Let  $\ell d_+[J_K(n)]$  and  $\ell d_-[J_K(n)]$  denote the linear terms of  $d_+[J_K(n)]$  and  $d_-[J_K(n)]$ , respectively. Now, we set

$$j_{x_K} := \left\{ 2n^{-1}\ell d_+[J_K(n)] \right\}' \quad \text{and} \quad j_{x_K^*} := \left\{ 2n^{-1}\ell d_-[J_K(n)] \right\}'.$$

Given a knot  $K \subset S^3$ , let  $n(K)$  denote a tubular neighborhood of  $K$  and let  $M_K := S^3 \setminus n(K)$  denote the exterior of  $K$ . Let  $\langle \mu, \lambda \rangle$  be the canonical meridian–longitude basis of  $H_1(\partial n(K))$ .

**Definition 2.2** A properly embedded surface  $(S, \partial S) \subset (M_K, \partial n(K))$ , is called *essential* if it is  $\pi_1$ -injective and it is not a boundary parallel annulus. An element  $\alpha/\beta \in \mathbb{Q} \cup \{1/0\}$ , where  $\alpha$  and  $\beta$  are relatively prime integers, is called a *boundary slope* of  $K$  if there is an essential surface  $(S, \partial S) \subset (M_K, \partial n(K))$ , such that each component of  $\partial S$  represents  $\alpha\mu + \beta\lambda \in H_1(\partial n(K))$ .

The longitude  $\lambda$  of every knot bounds an essential orientable surface in the exterior of  $K$ . Thus,  $0 = 0/1$  is a boundary slope of every knot in  $S^3$ . Hatcher showed that every knot  $K \subset S^3$  has finitely many boundary slopes [12].

For a surface  $(S, \partial S) \subset (M_K, \partial n(K))$  we will use the notation  $|\partial S|$  to denote the number of boundary components of  $S$ .

Garoufalidis conjectured [8, Conjecture 1.2] that the Jones slopes of any knot  $K$  are boundary slopes. The following statement, which is a refinement of the original conjecture, was stated by the author and Tran in [18, Conjecture 1.6].

**Conjecture 2.3** (Strong slope conjecture)

- Given a Jones slope  $a(n) = \alpha/\beta \in j_{s_K}$ , with  $\beta > 0$  and  $\gcd(\alpha, \beta) = 1$ , there is an essential surface  $S$  in  $M_K$  such that each component of  $\partial S$  has slope  $\alpha/\beta$  and we have  $2b(n) = \frac{\chi(S)}{|\partial S|\beta} \in j_{x_K}$ .
- Given a Jones slope  $a^*(n) = \alpha^*/\beta^* \in j_{s_K^*}$ , with  $\beta^* > 0$  and  $\gcd(\alpha^*, \beta^*) = 1$ , there is an essential surface  $S^*$  in  $M_K$  such that each component of  $\partial S^*$  has slope  $\alpha^*/\beta^*$  and we have  $2b^*(n) = -\frac{\chi(S^*)}{|\partial S^*|\beta^*} \in j_{x_K^*}$ .

**Remark 2.4** Strictly speaking in [18, Conjecture 1.6], we only required that  $\frac{\chi(S)}{|\partial S|\beta} \in j_{x_K}$  and  $-\frac{\chi(S^*)}{|\partial S^*|\beta^*} \in j_{x_K^*}$  without specifying that these values should correspond to points that correspond to the same values of  $n$  for which the slopes  $a(n)$  and  $a^*(n)$  occur. We do not know if the seemingly stronger version statement of [18, Conjecture 1.6] is stronger than Conjecture 2.3. A related point is that, at the moment, we do not know if there exist knots for which the sets  $j_{s_K}$  or  $j_{s_K^*}$  contain more than one point. In all cases for which the Jones slopes are computed, we have exactly one Jones slope in each of  $j_{s_K}$  or  $j_{s_K^*}$ .

## 2.2 Progress

Conjecture 2.3 is known for the following families of knots:

- Adequate knots and in particular alternating knots [4, 5].
- Iterated torus knots and iterated cables of adequate knots [1, 18, 26].
- Graph knots [2].
- Families of non-alternating 3-tangle pretzel knots [21, 22].
- Families of non-adequate Montesinos knots [9, 21, 23].
- Knots with up to 9 crossings [8, 13, 18].
- Near-alternating knots [20] constructed by taking Murasugi sums of an alternating diagram with a non-adequate diagram.
- Iterated untwisted generalized Whitehead doubles of adequate knots and torus knots [2].
- Knots obtained by any finite sequence of cabling, connect sums, and untwisted generalized Whitehead doubles of adequate knots and torus knots [1, 18, 26].

Under certain conditions, Conjecture 2.3 is known to be closed under cabling operations and Whitehead doubling operations [2, 18].

## 3 Exceptional Surgeries and the Figure 8 Knot

In [16], we noted that Conjecture 2.3 implies that the degrees of the colored Jones polynomial distinguish torus knots and in particular the unknot:

**Theorem 3.1** *Suppose that  $K$  is a knot that satisfies the strong slope conjecture and let  $T_{p,q}$  denote the  $(p, q)$ -torus knot. If  $d_+[J_K(n)] = d_+[J_{T_{p,q}}(n)]$  and  $d_-[J_K(n)] = d_-[J_{T_{p,q}}(n)]$  for all  $n$ , then, up to orientation change,  $K$  is isotopic to  $T_{p,q}$ .*

The proof of Theorem 3.1 begins with the observation that one of the Jones surfaces for  $T_{p,q}$  is an annulus (the cabling annulus). This implies that  $K$  also admits a Jones surface of zero Euler characteristic, which in turn implies that  $K$  must be a cable knot. The proof of the next theorem is similar in flavor as it begins with the observation that both the Jones surfaces of the figure 8 knot are punctured Klein bottles.

**Theorem 3.2** *Suppose that  $K$  is a knot that satisfies the strong slope conjecture and let  $F_8$  denote the figure 8 knot. If  $d_+[J_K(n)] = d_+[J_{F_8}(n)]$  and  $d_-[J_K(n)] = d_-[J_{F_8}(n)]$  for all  $n$ , then  $K$  is isotopic to  $F_8$ .*

*Proof* The degrees  $d_{\pm}[J_{F_8}(n)]$  are known (see, for example, [4, 8]). We have

$$d_-[J_K(n)] = d_-[J_{F_8}(n)] = -n^2 + \frac{1}{2}n + \frac{1}{2},$$

and

$$d_+[J_K(n)] = d_+[J_{F_8}(n)] = n^2 - \frac{1}{2}n - \frac{1}{2}.$$

Thus, we obtain

$$js_K := \left\{ 4n^{-2}d_+[J_K(n)] \right\}' = \{4\} \quad \text{and} \quad js_K^* := \left\{ 4n^{-2}d_-[J_K(n)] \right\}' = \{-4\},$$

and

$$jx_K := \left\{ 2n^{-1} \ell d_+[J_K(n)] \right\}' = \{-1\} \quad \text{and} \quad jx_K^* := \left\{ 2n^{-1} \ell d_-[J_K(n)] \right\}' = \{-1\}.$$

Since  $K$  satisfies the strong slope conjecture, we have essential surfaces  $S, S^*$  in the exterior of  $K$  such that

- (1) The boundary slope of  $S$  is 4 and  $\frac{\chi(S)}{|\partial S|} = -1$ , and
- (2) The boundary slope of  $S^*$  is  $-4$  and  $\frac{\chi(S^*)}{|\partial S^*|} = -1$ .

This implies that  $\chi(S) = -|\partial S|$  and  $\chi(S^*) = -|\partial S^*|$ . Thus,  $S, S^*$  are either punctured tori or punctured Klein bottles. By passing to the orientable double, we can assume that each component of  $S, S^*$  is a punctured torus. Thus, the knot exterior  $M_K$  contains punctured tori with boundary slopes  $s = 4$  and  $s^* = -4$ . Let  $i(s, s^*)$  denote the geometric intersection of  $s, s^*$  on  $\partial M_K$ . In this case, we have  $i(s, s^*) = 8$ . By a result of Gordon [10, Theorem 1.1], there are only two irreducible 3-manifolds  $M$  such that a toroidal component of  $\partial M$  contains two boundary slopes  $s, s^*$  of punctured tori with  $i(s, s^*) = 8$ . These are the lowest volume hyperbolic 3-manifolds with one cusp. From these two, the only one that is a knot complement in  $S^3$  is the complement of the figure 8 knot. Thus, we conclude that  $M_K$  is homeomorphic to the complement of  $F_8$ . By the Gordon-Luecke Knot Complement theorem [11],  $K$  has to be isotopic to  $F_8$ . □

To continue, we briefly recall the definition and some notation about adequate knots: Let  $D$  be a link diagram, and  $x$  be a crossing of  $D$ . Associated to  $D$  and  $x$  are two link diagrams, called the  $A$ -resolution and  $B$ -resolution of the crossing. See Fig. 1. A state on  $D$  is a function  $\sigma = \sigma(D)$  that assigns one of these two resolutions to each crossing of  $D$ . Applying the  $A$ -resolution (resp.  $B$ -resolution) to each crossing leads to a collection of disjointly embedded circles  $s_A(D)$  (resp.  $s_B(D)$ ).

**Definition 3.3** The diagram  $D$  is called  $A$ -adequate (resp.  $B$ -adequate) if for each crossing of  $D$  the two arcs of  $s_A(D)$  (resp.  $s_B(D)$ ) resulting from the resolution of the crossing lie on different circles. A knot diagram  $D$  is adequate if it is both  $A$ - and  $B$ -adequate. Finally, a knot that admits an adequate diagram is also called adequate.

Starting with a state  $\sigma = \sigma(D)$  we construct a state surface  $S_\sigma = S_\sigma(D)$  as follows: Each circle of  $\sigma(D)$  bounds a disk on the projection sphere  $S^2 \subset S^3$ . This collection of disks can be disjointly embedded in the ball below the projection sphere. At each crossing of  $D$ , we connect the pair of neighboring disks by a half-twisted band to construct a surface whose boundary is  $K$ . For details, see [4, 5].

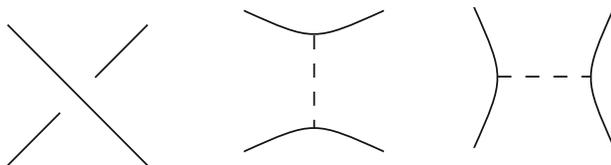


Fig. 1 From left to right: a crossing, the  $A$ -resolution and the  $B$ -resolution

The state surfaces corresponding to  $s_A(D)$  and  $s_B(D)$  are denoted by  $S_A(D)$  and  $S_B(D)$ , respectfully. In [27], Ozawa showed that the state surface  $S_A(D)$  is essential in the exterior of  $K$  if and only if  $D$  is an  $A$ -adequate diagram. Similarly,  $S_B(D)$  is essential in the exterior of  $K$  if and only if  $D$  is an  $B$ -adequate diagram. For a different proof of these results, see [5]. Thus, in particular, if  $D$  is an adequate diagram of a knot  $K$ , then  $S_A(D)$  and  $S_B(D)$  are essential surfaces in the exterior of  $K$ .

To continue, we recall the following well-known definition.

**Definition 3.4** A slope  $s$  for a hyperbolic knot is called *exceptional* if the 3-manifold obtained by filling  $M_K$  along  $s$  is not hyperbolic.

The proof of Theorem 3.1 shows that both the Jones slopes of the knot  $F_8$  are exceptional. Next we will see that  $F_8$  is the only adequate knot that has this property.

**Corollary 3.5** *Suppose that  $K$  is a hyperbolic adequate knot such that both the Jones slopes of  $K$  are exceptional. Then,  $K = F_8$ .*

*Proof* It is known that the number of negative crossings  $c_-(D)$  of an  $A$ -adequate knot diagram is a knot invariant. Similarly, the number of positive crossings  $c_+(D)$  of a  $B$ -adequate knot diagram is a knot invariant. In fact, if  $K$  is adequate, then the crossing number of  $K$  is realized by the adequate diagram; that is, we have  $c(K) = c(D) = c_-(D) + c_+(D)$  [24]. Let  $v_A(D)$  and  $v_B(D)$  be the numbers of circles in  $s_A(D)$  and  $s_B(D)$ , respectively. The boundary slope of  $S_A$  is  $-2c_-(D)$  and  $\chi(S_A) = v_A(D) - c(D)$ . The boundary slope of  $S_B$  is  $2c_+(D)$  and  $\chi(S_B) = v_B(D) - c(D)$ . By [4], the surfaces  $S_A = S_A(D)$  and  $S_B = S_B(D)$  satisfy the strong slope conjecture for  $K$ .

That is, we have

$$4d_-[J_K(n)] = -2c_-(D)n^2 + 2(c(D) - v_A(D))n + 2v_A(D) - 2c_+(D),$$

and

$$4d_+[J_K(n)] = 2c_+(D)n^2 + 2(v_B(D) - c(D))n + 2c_-(D) - 2v_B(D).$$

Thus, the distance of the two Jones slopes is  $i(2c_+(D), -2c_-(D)) = |2c_+(D) + 2c_-(D)| = 2c(D)$ .

Gordon's conjecture, proved by Lackenby and Meyerhoff [19], states that if  $s, s^*$  are exceptional boundary slopes for  $K$  then  $i(s, s^*) \leq 8$ . Thus, in order for  $s = 2c_+(D)$  and  $s^* = -2c_-(D)$  to be exceptional, we must have  $c(D) \leq 4$ . Since  $K$  is hyperbolic,  $K = F_8$ . □

*Example 3.6* Consider the 3-string pretzel knots  $K = P(r, s, t)$  such that  $r < 0 < s, t$  and  $-2r < s, t$ . It has exactly two Jones slopes with distance  $2(s + t)$  (see Proposition 4.8 below). Since by assumption  $s, t > 2$ , we cannot have  $2(s + t) \leq 8$ . Thus, not both of the Jones slopes can be exceptional.

As another example, let us mention the knot pretzel knot  $P(-2, 3, 7)$ , which is known to have seven exceptional slopes. The Jones slopes of  $P(-2, 3, 7)$  are  $\{\frac{37}{2}, 0\}$  and from these only  $\frac{37}{2}$  is exceptional.

*Question 3.7* Are there hyperbolic knots, other than the figure 8, that have more than one exceptional Jones slopes?

### 4 Characteristic Jones Surfaces and Alternating Knots

We begin by recalling from [17] that in all the cases where Conjecture 2.3 is proved, for each Jones slope, we can find a Jones surface where the number of sheets  $b|\partial S|$  divides the Jones period  $p_K$ . This observation led to the following definition [17, Definition 3.2].

**Definition 4.1** We call a Jones surface  $S$  of a knot  $K$  *characteristic* if the number of sheets of  $S$  divides the Jones period of  $K$ .

*Example 4.2* An adequate knot (and thus in particular an alternating knot) has Jones period  $p_K = 1$ , two Jones slopes and two corresponding Jones surfaces each with a single boundary component [4, 6]. Note that the characteristic Jones surfaces are spanning surfaces that are often non-orientable. In these cases, the orientable double cover is also a Jones surface but it is no longer characteristic since it has two boundary components.

*Question 4.3* Is it true that for every Jones slope of a knot  $K$  we can find a characteristic Jones surface?

If  $K$  is an alternating knot, then we have

$$p_K = 1 \text{ and } 2d_+[J_K(n)] - 2d_-[J_K(n)] = cn^2 + (2 - c)n - 2,$$

where  $c = c(K)$  is the crossing number of  $K$ . Thus, one direction Conjecture 1.2 is known. Furthermore, an alternating knot  $K$  satisfies the strong slope conjecture and every Jones slope is realized by a characteristic Jones surface. This follows, for example, from the discussion in the proof of Corollary 3.5. The state surfaces  $S_A(D), S_B(D)$  corresponding to any reduced alternating diagram  $D = D(K)$  are in fact the checkerboard surfaces of  $D$ .

We have the following converse:

**Theorem 4.4** *Suppose that  $K$  is a knot that satisfies the strong slope conjecture and such that every Jones slope is realized by a characteristic Jones surface. Suppose, moreover, that we have*

$$p_K = 1 \text{ and } 2d_+[J_K(n)] - 2d_-[J_K(n)] = cn^2 + (2 - c)n - 2$$

for some  $c \in \mathbb{Z}$ . Then,  $K$  is alternating and  $c$  is the crossing number of  $K$ .

*Proof* Since we have  $p_K = 1$ , for each of  $d_{\pm}[J_K(n)]$  we have exactly one Jones slope. That is, we have  $js_K = \{s\}$  and  $js_K^* = \{s^*\}$ . Furthermore, since knots of period one have integer Jones slopes ([8, Lemma 1.11], [17, Proposition 3.1]), both  $s$  and  $s^*$  are integers.

Since we assumed that each Jones slope of  $K$  is realized by a characteristic Jones surface, we conclude that we can take the Jones surfaces, say  $S, S^*$ , corresponding to  $s, s^*$ , respectively, to be spanning surfaces of  $K$ .

Finally, since we assumed that  $2d_+[J_K(n)] - 2d_-[J_K(n)] = cn^2 + (2 - c)n - 2$ , for some  $c \in \mathbb{Z}$ , we conclude that  $i(\partial S, \partial S^*) = s - s^* = 2c$ , where  $i(\partial S, \partial S^*)$  denotes the geometric intersection of the curves  $\partial S, \partial S^*$  on  $\partial M_K$ , and that  $\chi(S) + \chi(S^*) = 2 - c$ . Thus, in particular, we have

$$\chi(S) + \chi(S^*) + \frac{1}{2}i(\partial S, \partial S^*) = 2.$$

By Howie’s result [14, Theorem 2], it follows that  $K$  is alternating and in fact  $S, S^*$  are the checkerboard surfaces corresponding to an alternating diagram of  $K$ . But then  $c = c(K)$  by the discussion before the statement of the theorem.  $\square$

As a corollary of Theorem 4.4 we have the following.

**Corollary 4.5** *Suppose that  $K$  is an adequate knot. Then,  $K$  is alternating if and only if we have*

$$2d_+[J_K(n)] - 2d_-[J_K(n)] = cn^2 + (2 - c)n - 2$$

for some  $c \in \mathbb{Z}$ .

*Proof* Conjecture 2.3 has been proved for adequate knots [18]. Furthermore, as discussed earlier, adequate knots have period one and for every Jones slope we can find a characteristic Jones surface. Thus, the conclusion follows from Theorem 4.4.  $\square$

*Remark 4.6* Theorem 4.4 shows that the strong slope conjecture together with a positive answer to Question 4.3 implies Conjecture 1.2 stated in the Introduction. We also note that if Conjecture 1.2 is true then the degrees  $d_{\pm}[J_K(n)]$  would detect an alternating knot as long as they detect it among alternating knots with the same crossing number. That is, if we had a knot  $K$  such that  $d_{\pm}[J_K(n)] = d_{\pm}[J_{K'}(n)]$ , where  $K'$  is alternating, then by Theorem 4.4 we would conclude that  $K$  is also alternating with  $c(K) = c(K')$ . So if  $d_{\pm}[J_K(n)]$  distinguishes  $K'$  among alternating knots of the same crossing number, it will detect it among all knots. Given a prime reduced alternating diagram  $D = D(K)$ , the degrees  $d_{\pm}[J_K(n)]$  are completely determined by the quantities  $c_-(D), c_+(D), v_B(D), v_A(D)$ , introduced in the proof of Corollary 3.5. Thus, since  $v_B(D), v_A(D)$  are also the numbers of the checkerboard regions of  $D$ , given two reduced alternating diagrams of the same crossing number, one can decide whether they are distinguished by the degree of their colored Jones polynomial by a direct diagrammatic inspection. We illustrate these points with a few examples. We already know that Conjecture 2.3 implies that  $d_{\pm}[J_K(n)]$  detect the trefoil and figure 8 knots.

- If Conjecture 1.2 is true then the degrees  $d_{\pm}[J_K(n)]$ , would detect the  $5_2$  knot: suppose that for a knot  $K$  the degrees  $d_{\pm}[J_K(n)]$  are the same as those of the knot  $5_2$ . Then by Theorem 4.4  $K$  is alternating, and we have  $c(K) = 5$ . Thus  $K = T_{2,5}$  or  $K = 5_2$ . Since  $T_{2,5}$  is distinguished from  $5_2$  by the degrees of the colored Jones polynomial, we conclude that  $K = 5_2$ .
- Now, we discuss alternating knots with crossing number 6: The degrees  $d_{\pm}[J_K(n)]$  distinguish the knots  $6_1, 6_2, 6_3$  from each other. More specifically, the quantities  $c_-(D), c_+(D)$  (a.k.a. the Jones slopes) distinguish  $6_1$  from  $6_2$  and  $6_3$ , while the quantities  $v_B(D), v_A(D)$  distinguish  $6_2$  and  $6_3$  from each other. Hence, if Conjecture 1.2 is true then the degrees  $d_{\pm}[J_K(n)]$  would detect any of  $6_1, 6_2, 6_3$ .

The next proposition shows that in order to prove Conjecture 1.2, it is enough to show that if  $K$  is a knot that satisfies (1.1), then we must have  $c = c(K)$ .

**Proposition 4.7** *If  $K$  is a knot such that*

$$2d_+[J_K(n)] - 2d_-[J_K(n)] = c(K)n^2 + (2 - c(K))n - 2,$$

*then  $K$  is alternating.*

*Proof* Let  $D$  be a knot diagram of  $K$  that realizes  $c(K)$  and let  $g_T(D)$  denote the Turaev genus of  $D$  [3]. Then, we have

$$2d_+[J_K(n)] - 2d_-[J_K(n)] \leq c(K)n^2 + (2 - c(K) - 2g_T(D))n + 2g_T(K) - 2$$

for all  $n \in \mathbb{N}$ . See for example [15]. Thus, we must have  $2 - c(K) \leq 2 - c(K) - 2g_T(D)$ , which implies  $g_T(D) = 0$ . Now by [3, Corollary 4.6],  $D$  must be an alternating diagram.  $\square$

As mentioned above, alternating knots are the only knots that have Turaev genus zero [3, Corollary 4.6]. Thus, the degree span condition in the statement of Conjecture 1.2 can be reformulated to say

$$2d_+[J_K(n)] - 2d_-[J_K(n)] = cn^2 + (2 - 2g_T(K) - c)n + 2g_T(K) - 2, \tag{4.1}$$

where  $g_T(K)$  denotes the Turaev genus of  $K$  and  $c$  is an integer. By [15], adequate knots satisfy condition (4.1) above, and they have Jones period equal to 1. One can ask whether these conditions characterize adequate knots. The following proposition, shown to me by Christine Lee, shows that this is not the case.

**Proposition 4.8** *Consider any 3-string pretzel knot  $K = P(r, s, t)$  with  $r < 0 < s, t$  and  $-2r < s, t$ . Then, we have  $p_K = 1$  and*

$$2d_+[J_K(n)] - 2d_-[J_K(n)] = cn^2 + (2 - 2g_T(K) - c)n + 2g_T(K) - 2,$$

*but  $K$  is non-adequate.*

*Proof* The standard 3-string pretzel diagram  $D$  of  $K = P(r, s, t)$  has  $s + t - r$  crossings and by [25] this is the crossing number of  $K$ . That is  $c(K) = c(D) = s + t - r$ . The diagram  $D$  is also  $B$ -adequate with  $c_+(D) = c(D) = s + t - r$  and  $v_B(D) = -r + 1$ . Thus, we have

$$2d_+[J_K(n)] = (s + t - r)n^2 + (-s - t + 1)n + (r - 1).$$

On the other hand, Lee [20] shows that

- The Jones slope coming from  $d_+[J_K(n)]$  is equal to  $2c_-(D) - 2r = -2r$ ;
- It is realized by a Jones surface that is actually the state surface  $S_\sigma$  corresponding to the state  $\sigma$  that assigns the  $-r$  crossings the  $B$ -resolution and the  $s + t$  crossings the  $A$ -resolution.

Note that the hypothesis  $-2r < s, t$  is needed for these claims.

The number of state circles for above state  $\sigma$  is given by  $v_\sigma(D) = -r - 1 + s - 1 + t - 1 + 2 = -r + s + t - 1$ . We have

$$-\chi(S_\sigma) = -(v_\sigma(D) - c(D)) = -(-r + s + t - 1 - s - t + r) = 1,$$

and

$$2d_-[J_K(n)] = -rn^2 + n + (r - 1).$$

It follows that

$$2d_+[J_K(n)] - 2d_-[J_K(n)] = (s + t)n^2 - (s + t)n.$$

The Turaev genus of non-alternating 3-string pretzel knots is known to be one and hence  $2 - 2g_T(K) = 0$ . With this observation, we see that the last equation can be written in the form of (4.1) where  $c = s + t \in \mathbb{Z}$ . Finally since  $s + t < s + t - r = c(K)$ , the knot  $K$  is not adequate.  $\square$

**Remark 4.9** In [15], we show that if in (4.1) we require that the constant  $c$  is actually the crossing number of  $K$ , then  $K$  must be adequate. Proposition 4.8 and its proof show that the condition  $c = c(K)$  is necessary.

**Remark 4.10** The proof of Proposition 4.8 shows, in particular, that there are non-adequate knots  $K$  that admit spanning surfaces  $S, S^*$  such that

$$\chi(S) + \chi(S^*) + \frac{1}{2}i(\partial S, \partial S^*) = 2 - 2g_T(K).$$

This should be compared with the main result of [14] that states that for  $g_T(K) = 0$  this equation characterizes alternating knots and with [15, Problem 1.3].

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